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OPTIMIZATION PROBLEM FOR KLEIN-GORDON EQUATION

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DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My Parents

Jinniu Luo and Baozhen Shen

For

Encouraging Me to Pursue My Dreams

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# Abstract

In the thesis we consider the damped Klein-Gordon equation with a variable diffusion coefficient:

$$u_{tt}(t,x) + \alpha u_t(t,x) - \nabla(\beta(x)\nabla u(t,x)) + \delta g(u(t,x)) = f(t,x)$$
$$u(t,x)|_{x\in\Gamma} = 0, \quad t \in (0,T)$$
$$u(0,x) = y_0(x), \quad u_t(0,x) = y_1(x), \quad x \in \Omega,$$

where the nonlinear term is $g(u) = |u|^{\gamma}u$ with the constant $\gamma$ satisfying

$$\begin{cases} 0 \le \gamma < \infty & \text{if} \quad n = 1,2, \\ 0 \le \gamma \le 2 & \text{if} \quad n = 3, \\ \gamma = 0 & \text{if} \quad n \ge 4. \end{cases}$$

The goal is to derive necessary conditions for the optimal set of parameters $q^* = (\alpha, \beta, \gamma) \in P$ minimizing the objective function $J(q) = \|u(q) - z_d\|^2_{L^2(0,T;H)}$. First, we study the nonlinear term $g(u)$ for the different cases of $\gamma$, and derive its properties which are crucial to the entire research. Then we show that the solution maps $q \to u(q)$: $P \to L^2(0,T;V)$ and $q \to u'(q)$: $P \to L^2(0,T;H)$ are continuous. Furthermore, the solution map is shown to be weakly Gâteaux differentiable on the admissible set $P$, implying the Gâteaux differentiability of the objective function. Finally we study the Fréchet differentiability of $J$ and optimal parameters for these problems. Unlike the sine-Gordon equation, which

has a bounded nonlinear term, Klein-Gordon equation requires stronger assumptions on the initial data. The further difficulties in mathematical analysis of the equation arise from the unbounded nonlinear term $g(u) = |u|^\gamma u$ and the variable diffusion coefficient $\beta(x)$.

# Chapter 1

# Introduction

In the thesis, we study a damped Klein-Gordon equation with a variable diffusion coefficient. The goal is to derive necessary conditions for the optimal set of parameters minimizing the objective function $J$.

Klein-Gordon equation has many different forms. Its original form is

$$\frac{1}{c^2}u_{tt} - \Delta u + \frac{m^2c^2}{h^2}u = 0.$$

The equation was named after the physicists Oskar Klein and Walter Gordon, who in 1926 proposed that it describes relativistic electrons. The Klein-Gordon equation is considered a relativistic version of the Schrödinger equation (see [36], [37], and [38]). It is the equation of motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles (see [10], [44]).

Klein-Gordon equation is also used to model the propagation of dislocations in crystals, the behavior of elementary particles, Josephson junctions (see [12], Chapter 8.2 for details), and others. It has been studied extensively. More details of the field are reviewed in the next chapter.

The Klein-Gordon equation studied here is the following.

Let $\Omega$ be an open bounded set in $\mathbb{R}^n$ with a sufficiently regular boundary $\Gamma$. The damped Klein-Gordon equation with a damping coefficient $\alpha$, a variable diffusion coefficient $\beta(x)$, and a magnitude of the nonlinear term $\delta \geq 0$ is

$$u_{tt}(t,x) + \alpha u_t(t,x) - \nabla(\beta(x)\nabla u(t,x)) + \delta g(u(t,x)) = f(t,x) \quad (1.1)$$
$$u(t,x)|_{x\in\Gamma} = 0, \quad t \in (0,T)$$
$$u(0,x) = y_0(x), \quad u_t(0,x) = y_1(x), \quad x \in \Omega,$$

where $T > 0$, $(t,x) \in Q = (0,T) \times \Omega$. The nonlinear term is assumed to be of the form $g(u) = |u|^\gamma u$ with the constant $\gamma$ satisfying

$$\begin{cases} 0 \leq \gamma < \infty & \text{if} \quad n = 1,2, \\ 0 \leq \gamma \leq 2 & \text{if} \quad n = 3, \\ \gamma = 0 & \text{if} \quad n \geq 4. \end{cases} \quad (1.2)$$

The precise statement of the problem will be given in Chapter 3. The Lipschitz continuous diffusion coefficient $\beta(x)$ is assumed to be in $\mathcal{B} \subset C(\Omega)$ where

$$\mathcal{B} = \{\beta(x) : \; 0 < \nu \leq \beta(x) \leq \mu; \; |\beta(x) - \beta(y)| \leq C|x-y|, \; x,y \in \Omega\} \quad (1.3)$$

for some positive constants $\nu$, $\mu$ and $C$. The identification problem for (1.1) is to find the parameters $\alpha, \beta(x)$, and $\delta$ such that the solution of (1.1) exhibits the desired behavior. More precisely, let

$$P = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times \mathcal{B} \times [\delta_{min}, \delta_{max}]\}. \quad (1.4)$$

Define the cost functional $J(q)$ by

$$J(q) = \int_Q [u(q;t,x) - z_d(t,x)]^2 dxdt, \quad q \in P, \tag{1.5}$$

where $z_d$ is a given function in $L^2(Q)$. The data $z_d$ can be thought of as the targeted behavior of (1.1). The parameter identification problem for (1.1) with the objective function (1.5) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in P$ satisfying

$$J(q^*) = \inf_{q \in P} J(q). \tag{1.6}$$

One of the main results of this thesis is that we prove the existence and uniqueness of the weak solution $u$ of the Klein-Gordon equation,

$$\begin{aligned} &u'' + \alpha u' + A_\beta u + \delta g(u) = f, \quad \text{in } V' \quad a.e. \quad on \quad [0,T], \\ &u(0) = y_0 \in V, \quad u'(0) = y_1 \in H. \end{aligned}$$

Because $\beta(x)$ is not a constant and the operator $A_\beta$ relies on $\beta$, we study the eigenvalues and eigenfunctions of the operator $A_\beta$, and prove that the mapping $\beta \to A_\beta v$ from $\mathcal{B}$ into $V'$ is continuous. Then we establish the continuity of the solution maps $q \to u(q)$: $P \to C([0,T];V)$ and $q \to u'(q)$: $P \to C([0,T];H)$. Based on the continuity, we prove the Gâteaux differentiability of the solution map in $P$. Hence we obtain that the objective function $J(q) = \|u(q) - z_d\|^2_{L^2(0,T;H)}$ is also Gâteaux differentiable. Furthermore, we prove the Fréchet differentiability of the functional $J$ with respect to the parameters $q \in P$, and give a variational characterization for the minimizers, expressed through the solutions of the state and adjoint systems.

The thesis is organized as follows.

In Chapter 2 we review the research has been done in studying Klein-Gordon equations. We start with how the original Klein-Gordon equation was introduced and its significance in physics. Then we review the research on different forms of nonliear Klein-Gordon equations with and without a damping term. Also we review the research on optimization problems of nonlinear hyperbolic wave equations. In the end we introduce the research that this thesis is mainly based on.

In Chapter 3 we introduce appropriate function spaces with their respective inner products and norms. Because, in general, the Klein-Gordon equation (1.1) does not have a classical solution, we define the weak solution of the equation in an appropriate function space.

In Chapter 4 we study the nonlinear term $g(u) = |u|^\gamma u$. Unlike the bounded nonlinear terms $\sin u$ in sine-Gordon equations, the nonlinear term $|u|^\gamma u$ in the Klein-Gordon equaiton is unbouded. This brings a lot difficulty to the problem. Therefore, we investigate properties of the term which are crucial to the whole problem. The properties of $g(u)$ we derive in Chapter 4 are important for later chapters when we discuss the solution of the Klein-Gordon equation, the continuity and differentiability of the solution map.

In Chapter 5 we derive the energy estimates of the solution of the Klein-Gordon equation (1.1) and hence prove the uniqueness of the solution. The existence of the solution is established by standard Galerkin method.

In Chapter 6 we study the eigenvalues and eigenfunctions of the operator $A_\beta$, and the mapping $\beta \to A_\beta v$ from $\mathcal{B}$ into $V'$. Then, we derive the weak solution for more regular initial conditions of equation (1.1).

In Chapter 7 we prove that the solution maps $q \to u(q)$: $P \to C([0,T];V)$ and $q \to u'(q)$: $P \to C([0,T];H)$ are continuous.

In Chapter 8 we define the weak Gâteaux derivative of the solution map $q \to u(q)$ at $q^*$ in the direction $q - q^*$. Because the diffusion coefficient $\beta(x)$ is not a constant, the standard variational method cannot be used to show the Gâteaux differentiability of the solution map. Instead we show that the weak Gâteaux derivative $z = Du(q^*; q - q^*) \in L^2(0, T; H)$ exists and it is the unique weak solution of the problem

$$z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + \delta^* g'(u(t; q^*)) z(t) = f_0(t),$$
$$z(0) = 0,\ z'(0) = 0,\ t \in (0, T),$$

where $f_0(t) = (\alpha^* - \alpha) u'(t; q^*) + (A_{\beta^*} - A_\beta) u(t; q^*) + (\delta^* - \delta) g(u(t; q^*))$.

In Chapter 9 we show that the objective function $J(q) = \|u(q) - z_d\|^2_{L^2(0,T;H)}$ is also Gâteaux and Fréchet differentiable on P. In the end, we derive the equations to optimize parameters.

In Chapter 10 we summarize the main results of the thesis and propose our future work for the topic.

# Chapter 2

## Review of the field

In the early 20th century, Max Planck, Albert Einstein, De Broglie, Niels Bohr, Erwin Schroedinger, Heisenberg and others achieved a highly successful breakthrough in the world of physics. At that time many physical phenomena, such as photoeletric effect, blackbody radiation, the Zeeman effect, Stark effect, could not be explained by the laws of classical physics. The breakthrough came with the invention of Quantum Mechanics.

Quantum mechanics is very successful in explaining these phenomena, so the theory can model the atom by matching the experimental results. In addition, with the help of Einstein's Special Relativity, quantum mechanics can also explain the phenomenon of nuclear physics, elementary particles, and other physical phenomena which exhibit the small size of the object relative to the displacement. Quantum mechanics with special relativity is known as relativistic quantum mechanics.

Equation of relativistic quantum mechanics called the Klein-Gordon equation is named after Oskar Klein and Walter Gordon, who in 1926 tried to explain properties of electrons using the special relativity theory. Unfortunately, because the electron has spin 1/2, the results were not satisfactory to explain properties of the electron. Nevertheless, the Klein-Gordon equation describes the electron particles without spin or other particles that have integer spins.

Its original form is

$$\frac{1}{c^2}u_{tt} - \Delta u + \frac{m^2c^2}{h^2}u = 0,$$

which is a linear wave equation.

Klein-Gordon equation has many different forms, such as:

$$u_{tt} - \Delta u + m^2 u + g(u) = 0, \tag{2.1}$$

where $g$ is some nonlinear real-valued equation. It is called nonlinear Klein-Gordon equation. And the equation

$$u_{tt} + u_t - \Delta u + m^2 u + g(u) = 0, \tag{2.2}$$

which has the term $u_t$, is called nonlinear damped Klein-Gordon equation.

Klein-Gordon equation is used to model many natural phenomena, such as the propagation of dislocations in crystals, the behavior of elementary particles, Josephson junctions (see [12], Chapter 8.2 for details), and others. Mathematically, to study local and long term behavior of a hyperbolic wave equation could be very interesting and challenging. The equation has been studied extensively.

The local existence of nonlinear Klein-Gordon equation has been studied in Nakanishi [41], Huang and Zhang [29], Grundland and Infeld [20], Guan [16], Shatah [46], Duncan [13], and Park and Jeong [42]. The global existence existence and blowing up of the solutions has been studied in Ha and Park [28], Ginibre and Velo [18], [19], Brenner [6], and Gan, Guo and Zhang [15]. The existence of the soliton solutions has been studied in Benci and Fortunato [5], and Bellazzini, Benci and Bonanno [4]. A large amount of work has been devoted to the study of the Cauchy problem for the nonlinear Klein-Gordon equation (see [39], [42], [3],

[30], [31], [45], [26] and [27]). In numerical aspect, a lot of work also have been done (see [7], [8], [11], [33], [43], [47], [49]).

Recently many works are devoted to study the control and parameter estimation problems for hyperbolic equations (see Ahmed [1],[2], and Lions [34]). However there are not many results on the optimal control theory for the nonlinear damped hyperbolic equations. In Ha and Nakagiri [25],[23],[27], [40] and Ha and Gutman [24], the authors study the optimal control problems and the numerical analysis for the nonlinear control systems described by damped second order equations in a Hilbert space $H$. For all the equations studied, their nonlinear terms $\sin y$ and $e^{-ay}$ are bounded and the associated energy estimates of solutions are rather easily obtained compared with the unbounded nonlinear terms such as polynomials. Since the estimates and differentiations of solutions with respect to control variables are essential in deriving the existence of optimal controls and the necessary optimality conditions, the control problems for unbounded nonlinearities become more difficult than those for bounded nonlinearities.

The thesis is mainly based on the work of J.L. Lions (see [34], [35]), Roger Temam (see [48]), Semion Gutman (see [24],[21]), and Junhong Ha and Shin-ichi Nakagiri (see [26], [41]). Lions studied linear and nonlinear boundary value problems by using the theory of distributions, and then find ways to numerically approximate their solutions. He developed standard ways to study existence and uniqueness of solutions of partial differential equations with initial/boundary conditions. In [48] (Chapter 4, Section 3), Temam studied the nonlinear wave equation of relativistic quantum mechanics of the form:

$$u_{tt} + \alpha u_t - \Delta u + g(u) = f \quad in \quad \Omega \times R_+,$$

where $g$ is a $\mathcal{C}^2$ function from $R$ to $R$ satisfying some assumptions. Ha and Nakagiri [26] studied identification problem for the damped Klein-Gordon equation with constant parameters. Gutman [24],[21] studied the optimization problem for the damped sine-Gordon equation with a variable diffusion coefficient. In the thesis, we follow the similar approach to the Klein-Gordon equation as in Gutman [24],[21] for sine-Gordon equation. The further difficulties in mathematical analysis arise from the unbounded nonlinear term $g(u) = |u|^\gamma u$ and the variable diffusion coefficient $\beta(x)$.

# Chapter 3

# Problem Setup

Let the Hilbert space $H = L^2(\Omega)$ have the norm $|u|$ and the inner product $(u, v)$. Let the Hilbert space $V = H_0^1(\Omega)$ have the norm $\|u\|$ and the inner product $(\nabla u, \nabla v)$. The dual $H'$ is identified with $H$ leading to $V \subset H \subset V'$ with compact, continuous and dense injections. Hence, there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1 \|w\|, \quad \text{for any} \quad w \in V. \tag{3.1}$$

Let $\langle u, v \rangle$ denote the duality pairing between $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$.

Given $\beta \in \mathcal{B}$, we define the bilinear, continuous and coercive form $a_\beta$ on $V \times V$, and the associated linear operator $A_\beta$ from $V$ to $V'$ by

$$a_\beta(u, v) = \int_\Omega \beta(x) \nabla u(x) \nabla v(x) dx = \langle A_\beta u, v \rangle. \tag{3.2}$$

Then $a_\beta(u, u) \geq \nu \|u\|^2$ for any $u \in V$. The bilinear form $a_\beta(u, v)$ is an equivalent inner product on $V$. Let $V_\beta$ denote $V$ with the inner product $((u, v))_\beta = a_\beta(u, v)$ and the norm $\|v\|_\beta^2 = a_\beta(v, v)$. The domain of $A_\beta$ is $D(A_\beta) = \{v \in V \ : \ A_\beta v \in H\}$. Now we prove that $D(A_\beta) = H^2(\Omega) \cap V$ for any $\beta \in \mathcal{B}$. We need a result from [17] on the regularity of solutions of linear second order equations.

**Theorem 3.1.** *Let $\Omega$ be an open bounded set in $\mathbb{R}^n$ with $C^2$ boundary $\partial\Omega$. Let*

$$Lu = D_i(a^{i,j}(x) D_j u) + b^i(x) u) + c^i(x) D_i u + d(x) u,$$

*where the coefficients* $a^{i,j},\ b^i,\ i,j = 1,...,n$ *are uniformly Lipschitz continuous in* $\Omega$, *and* $\sum a^{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, *for any* $x \in \Omega,\ \xi \in R^n$. *Suppose that the coefficients* $c^i, d, i = 1,...n$ *are essentially bounded in* $\Omega$, *and* $f \in L^2(\Omega)$. *Then the weak solution* $u \in W_0^{1,2}(\Omega)$ *of the equation* $Lu = f$ *in* $\Omega$ *satisfies* $u \in W^{2,2}(\Omega)$ *and*

$$\|u\|_{W^{2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}), \tag{3.3}$$

*where* $C = C(n, \lambda, K, \partial\Omega)$, *and* $K = max\,\{\|a^{i,j}, b^i\|_{C^{0,1}(\bar{\Omega})}, \|c^i, d\|_{L^\infty(\Omega)}\}$.

*Proof.* Operator $L$ is strictly elliptic since $\sum a^{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for any $x \in \Omega,\ \xi \in R^n$. The result follows from [17], Theorem 8.12. □

**Theorem 3.2.** *Let* $\beta \in \mathcal{B}$, *then* $D(A_\beta) = H^2(\Omega) \cap V$.

*Proof.* Let $u \in D(A_\beta)$. This means that $A_\beta u = -\sum_{i=1}^n (\beta(x)u_{x_i})_{x_i} \in H$. Let $Lu = -A_\beta u$, and $f = \sum_{i=1}^n (\beta(x)u_{x_i})_{x_i}$. Thus $Lu = f$. For the coefficients of $L$ we have $a^{i,i}(x) = \beta(x),\ a^{i,j}(x) = 0$ for $i \neq j$, $0 < \nu \leq \beta(x) \leq \mu$, and $\beta(x)$ is uniformly Lipschitz continuous in $\Omega$. Then by Theorem 3.1, $u \in W^{2,2}(\Omega) = H^2(\Omega)$. Therefore $D(A_\beta) \subseteq H^2(\Omega) \cap V$.

For the other direction notice that $\beta(x)$ is Lipschitz continuous. Then by Theorem 5.8.4 in [14], $\beta(x) \in W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$. If $u \in H^2(\Omega) \cap V$, then

$$A_\beta u = -\sum_{i=1}^n (\beta(x)u_{x_i})_{x_i} = \nabla\beta(x) \cdot \nabla u + \beta(x)\Delta u \in H.$$

This implies $H^2(\Omega) \cap V \subset D(A_\beta)$. Therefore, $D(A_\beta) = H^2(\Omega) \cap V$. □

By Theorem 3.2 the domain $D(A_\beta) = H^2(\Omega) \cap V$ does not depend on $\beta(x)$. Accordingly, we use the notation $D(A)$ instead of $D(A_\beta)$ for any $\beta \in \mathcal{B}$.

Let

$$W(0,T) = \{u \ : \ u \in L^2(0,T;V),\ u' \in L^2(0,T;H),\ u'' \in L^2(0,T;V')\}, \quad (3.4)$$

and $f \in L^2(0,T;H)$. Function $u \in L^\infty(0,T;V) \cap W(0,T)$ is called a weak solution of the damped Klein-Gordon equation (1.1) if

$$\begin{aligned} & u'' + \alpha u' + A_\beta u + \delta g(u) = f, \quad \text{in } V' \quad a.e. \quad on \quad [0,T], \\ & u(0) = y_0 \in V, \quad u'(0) = y_1 \in H, \end{aligned} \quad (3.5)$$

In the sequel, the solution of the Klein-Gordon equation means the weak solution. In this problem the derivatives are understood in the sense of distributions with the values in $V'$, see [35], [9]. A weak solution $u$ of (3.5) is called simply a solution in what follows.

We use the following Lemmas established in [48].

**Lemma 3.3.** *Let $\beta \in \mathcal{B}$, $w \in L^2(0,T;V)$, $w' \in L^2(0,T;H)$ and $w'' + A_\beta w \in L^2(0,T;H)$. Then, after a modification on a set of measure zero, $w \in C([0,T];V)$, $w' \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$ one has*

$$(w'' + A_\beta w, w') = \frac{1}{2}\frac{d}{dt}\{|w'|^2 + \|w\|_\beta^2\}. \quad (3.6)$$

**Lemma 3.4.** *Let $\beta \in \mathcal{B}$, $w \in L^2(0,T;D(A))$, $w' \in L^2(0,T;V)$ and $w'' + A_\beta w \in L^2(0,T;V)$. Then, after a modification on a set of measure zero, $w \in C([0,T];D(A))$, $w' \in C([0,T];V)$ and, in the sense of distributions on $(0,T)$ one has*

$$((w'' + A_\beta w, w'))_\beta = \frac{1}{2}\frac{d}{dt}\{\|w'\|_\beta^2 + |A_\beta w|^2\}. \quad (3.7)$$

*Proof.* Use Lemma 3.3 with the triple $V \subset H = H' \subset V'$ replaced by $D(A_\beta) \subset V_\beta = V'_\beta \subset H$, see [48], Section 2.4.2. □

**Lemma 3.5.** *Let $w \in W_r(0,T)$ , where*

$$W_r(0,T) = \{u \;:\; u \in L^2(0,T;V),\; u' \in L^2(0,T;V),\; u'' \in L^2(0,T;V')\}. \quad (3.8)$$

*Then, after a modification on a set of measure zero, $w \in C([0,T];V)$, $w' \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$ one has*

$$\frac{d}{dt}||w||^2_\beta = 2a_\beta(w',w) = 2\langle A_\beta w, w'\rangle, \quad \text{and} \quad \frac{d}{dt}|w'|^2 = 2\langle w'', w'\rangle. \quad (3.9)$$

*Proof.* According to ([48], Lemma 2.3.2), if $u \in L^2(0,T;V)$ and its derivative $u' \in L^2(0,T;V')$, then $u \in C([0,T];H)$ after a modification on a set of measure zero, and it satisfies $d/dt|u|^2 = 2\langle u', u\rangle$. Let $w \in W_r(0,T)$, and $u = w'$. Then $u \in L^2(0,T;V)$ and $w' \in L^2(0,T;V')$. Therefore we have $w' = u \in C([0,T];H)$, and the second equality in (3.9). For the first equality in (3.9) we can use ([48], Lemma 2.3.2) with $V = H = V'$, since $\|w\|_\beta$ is an equivalent norm in $V$. □

# Chapter 4

# Properties of the Nonlinear Mapping $g(u)$

Let $g(t) = |t|^\gamma t$, where $t \in R$ and $\gamma \geq 0$. Notice that $g'(t) = (\gamma+1)|t|^\gamma$. We will use the following elementary inequality

$$|g(t) - g(s)| = ||t|^\gamma t - |s|^\gamma s| \leq (\gamma+1)(|t| + |s|)^\gamma |t - s|. \tag{4.1}$$

Indeed, by the Mean Value Theorem,

$$||t|^\gamma t - |s|^\gamma s| \leq (\gamma+1)|\theta t + (1-\theta)s|^\gamma |t-s|,$$

and $|\theta t + (1-\theta)s| \leq |t| + |s|, \quad 0 \leq \theta \leq 1$.

**Lemma 4.1.** *If $u \in V$, then $|g(u)| \leq c\|u\|^{\gamma+1}$. Mapping $g : V \to H$ is locally Lipschitz on $V$.*

*Proof.* We recall the Sobolev embedding: $V \subset L^q(\Omega)$, for any $1 \leq q < \infty$ if $n = 1, 2$; $V \subset L^q(\Omega)$, for $1 \leq q \leq 6$ if $n = 3$ (see [48], II (1.15) and [14], 5.6 Theorem 2). Hence, for $u \in V$, $|g(u)|^2 = \int_\Omega |u|^{2\gamma+2}dx = \|u\|^{2\gamma+2}_{L^{2\gamma+2}(\Omega)} \leq c\|u\|^{2(\gamma+1)}$. For the case $n \geq 4$, $|g(u)| = |u| \leq c\|u\|$.

Now we prove the Lipschitz continuity of $g$. Assume $u, v \in V$. For $n \geq 4$, the case is trivial. Let $n \leq 3$. By (4.1),

$$|g(u) - g(v)|^2 = \int_\Omega |g(u) - g(v)|^2 dx \leq (\gamma + 1)^2 \int_\Omega (|u| + |v|)^{2\gamma} |u - v|^2 dx$$
$$\leq C\Big(\int_\Omega (|u| + |v|)^{3\gamma} dx\Big)^{\frac{2}{3}} \Big(\int_\Omega |u - v|^6 dx\Big)^{\frac{1}{3}}$$
$$\leq C\Big(\int_\Omega (|u| + |v|)^{3\gamma} dx\Big)^{\frac{2}{3}} \|u - v\|^2.$$

If $0 \leq 3\gamma < 1$, then

$$\int_\Omega (|u| + |v|)^{3\gamma} dx \leq \int_\Omega (1 + |u| + |v|)^2 dx \leq C \int_\Omega (1 + |u|^2 + |v|^2) dx$$
$$\leq C(1 + |u|^2 + |v|^2) \leq C(1 + \|u\|^2 + \|v\|^2) \leq C(1 + \|u\| + \|v\|)^2.$$

If $3\gamma \geq 1$, then

$$\int_\Omega (|u| + |v|)^{3\gamma} dx \leq C \int_\Omega (|u|^{3\gamma} + |v|^{3\gamma}) dx = C(\|u\|^{3\gamma}_{L^{3\gamma}(\Omega)} + ||v||^{3\gamma}_{L^{3\gamma}(\Omega)})$$
$$\leq C(\|u\|^{3\gamma} + ||v||^{3\gamma}) \leq C(||u|| + ||v||)^{3\gamma}.$$

Combining these results, we get
$|g(u) - g(v)|^2 \leq C\Big(\int_\Omega (|u| + |v|)^{3\gamma} dx\Big)^{\frac{2}{3}} \|u - v\|^2 \leq C(1 + \|u\| + \|v\|)^{2\gamma+2} \|u - v\|^2.$

□

**Lemma 4.2.** *If $\beta(x) \in \mathcal{B}$ and $u \in D(A)$, then $\|g(u)\| \leq C\|u\|^{\gamma}_{H^2}\|u\|$, where $C$ depends only on $\Omega$ .*

*Proof.* By the Sobolev embeddings Theorem (see [48], II (1.12)), $H^2(\Omega) \subset C(\Omega)$ continuously for $n \leq 3$ . Therefore, for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$, $\|u\|_{C(\Omega)} \leq$

$C\|u\|_{H^2}$, for some constant $C$ that depends only on $\Omega$. Hence,

$$\|g(u)\| = \int_\Omega |\nabla g(u)|^2 dx = \int_\Omega |g'(u)\nabla u|^2 dx = (\gamma+1)^2 \int_\Omega |u|^{2\gamma}|\nabla u|^2 dx$$
$$\leq C\|u\|_{H^2}^{2\gamma} \int_\Omega |\nabla u|^2 dx = C\|u\|_{H^2}^{2\gamma}\|u\|.$$

The case of $n \geq 4$ is evident. □

**Theorem 4.3.** *If $u_n \to u$ pointwise a.e. on $\Omega$, and $\|u_n\|, \|u\| \leq C$, for some constant $C$, then $g(u_n) \rightharpoonup g(u)$ weakly in $H$, as $n \to \infty$.*

*Proof.* Since $u_n \to u$ pointwise a.e. on $\Omega$, then $g(u_n) \to g(u)$ pointwise a.e. on $\Omega$. By Egoroff's theorem, for any $\epsilon > 0$ there exists a measurable set $E \subset \Omega$ such that $m(E) < \epsilon$ and the convergence $g(u_n)(x) \to g(u)(x)$ is uniform for $x \in \Omega \setminus E$. Therefore, for large $n$

$$\int_\Omega |g(u_n) - g(u)|dx = \int_E + \int_{\Omega\setminus E} \leq \int_E |g(u_n)|dx + \int_E |g(u)|dx + \epsilon.$$

By Lemma 4.1, $|g(v)| \leq c\|v\|^{\gamma+1}$. Hence, if $v \in V$, and $\|v\| \leq C$, then

$$\int_E |g(v)(x)|dx \leq |g(v)|\sqrt{\int_E dx} \leq c\|v\|^{\gamma+1}\sqrt{m(E)} < cC^{\gamma+1}\sqrt{\epsilon}.$$

Therefore,

$$\int_\Omega |g(u_n) - g(u)|dx \leq 2cC^{\gamma+1}\sqrt{\epsilon} + \epsilon.$$

Thus,

$$\|g(u_n) - g(u)\|_{L^1(\Omega)} \to 0, \qquad n \to \infty.$$

By Lemma 4.4, it is enough to check the convergence $(g(u_n), p) \to (g(u), p)$ for any bounded $p \in H$. For such a $p \in H$ we have

$$
\begin{aligned}
|(g(u_n), p) - (g(u), p)| &\le \int_\Omega |g(u_n)(x) - g(u)(x)||p(x)|dx \\
&\le \|p\|_\infty \int_\Omega |g(u_n) - g(u)|dx \to 0
\end{aligned}
$$

as $n \to \infty$. □

**Lemma 4.4.** *If $|f_n|, |f| < C$ and $(f_n, p) \to (f, p)$ for any bounded function $p \in H$, then $f_n \rightharpoonup f$ weakly in $H$.*

*Proof.* Let $h \in H$. For each $N \in \mathbb{N}$ define function $h_N \in H$ by $h_N(x) = h(x)$ for $x \in \Omega$ such that $|h(x)| \le N$, and $h_N(x) = 0$ otherwise. Functions $h_N$ are bounded on $\Omega$, and converge to $h$ in $H$ by Lebesgue's Dominated Convergence theorem. This shows that bounded functions are dense in $H$.

Let $\epsilon > 0$. Given $h \in H$ choose a bounded function $p \in H$ such that $|h - p| < \epsilon$. Then, for large $n$

$$|(f_n, h) - (f, h)| = |(f_n - f, h)| \le |(f_n - f, p)| + |(f_n - f, h - p)|$$

$$\le \epsilon + |f_n - f||h - p| \le \epsilon + 2C\epsilon.$$

□

**Theorem 4.5.** *If $u_n \to u$ in $H$ and $\|u_n\|, \|u\| \le C$ for some constant $C$, then $g(u_n) \rightharpoonup g(u)$ weakly in $H$.*

*Proof.* Suppose that $g(u_n) \not\rightharpoonup g(u)$ weakly in $H$. Then there exist $\epsilon > 0$, a subsequence $u_k$ of $u_n$, and an $h \in H$ such that $|(g(u_k), h) - (g(u), h)| \ge \epsilon$ for any $k$. On the other hand, since $u_n \to u$ in $H$, then the same is true for the subsequence

$u_k$. The convergence in $H$ implies the convergence in measure $u_k \to u$ on $\Omega$. This, in turn, implies that there is a subsequence of $u_k$ (call it $u_k$ again) that converges to $u$ pointwise a.e. on $\Omega$. By Theorem 4.3 we have $g(u_k) \rightharpoonup g(u)$ weakly in $H$ contradicting the assumption. □

**Theorem 4.6.** *If $u_n \to u$ in $L^2(0,T;H)$ and $\|u_n(t)\|, \|u(t)\| \le C$ for some constant $C$ and any $t \in [0,T]$, then $g(u_n) \rightharpoonup g(u)$ weakly in $L^2(0,T;H)$.*

*Proof.* Suppose that $g(u_n) \not\rightharpoonup g(u)$ weakly in $L^2(0,T;H)$. This means that there exist $\epsilon > 0$, a subsequence $u_k$ of $u_n$, and an $h \in L^2(0,T;H)$ such that $|(g(u_k) - g(u), h)_{L^2(0,T;H)}| \ge \epsilon$ for any $k$.

On the other hand, $u_n \to u$ in $L^2(0,T;H)$, as $n \to \infty$. The same is true for the subsequence $u_k$, i.e. $\int_0^T |u_k(t) - u(t)|^2 dt \to 0$. This implies that $u_k$ converges to $u$ in measure on $[0,T]$ in the space $H = L^2(\Omega)$. This, in turn, implies that there is a subsequence $u_m$ such that $|u_m(t) - u(t)| \to 0$ pointwise a.e. on $[0,T]$. In other words, $u_m(t) \to u(t)$ in $H$, a.e on $[0,T]$. By Theorem 4.5 we have $g(u_m)(t) \rightharpoonup g(u)(t)$ weakly in $H$ for any $t \in [0,T]$. This implies that $(g(u_m)(t) - g(u)(t), h(t)) \to 0$ pointwise a.e. on $[0,T]$.
Let $F_n(t) = (g(u_m)(t) - g(u)(t), h(t))$, then

$$|F_n(t)| \le |g(u_m)(t) - g(u)(t)||h(t)| \le (|g(u_m)(t)| + |g(u)(t))||h(t)| \le C|h(t)|.$$

Here we used the fact that $|g(u_m)(t)| + |g(u)(t)| \le c(\|u_m(t)\|^{\gamma+1} + \|u(t)\|^{\gamma+1}) \le 2cC^{\gamma+1}$. Function $|h|$ is integrable on $[0,T]$, since $|h| \in L^2(0,T)$. Hence

$$\Big(g(u_m) - g(u), h\Big)_{L^2(0,T;H)} \to 0,$$

by the Lebesgue's Dominated Convergence Theorem. This contradicts the assumption. $\square$

For convenience we summarize the results of this chapter.

**Theorem 4.7.** *Let $g(u) = |u|^\gamma u$ and $\gamma$ is assumed to satisfy*

$$\begin{cases} 0 \leq \gamma < \infty & \text{if} \quad n = 1, 2, \\ 0 \leq \gamma \leq 2 & \text{if} \quad n = 3, \\ \gamma = 0 & \text{if} \quad n \geq 4. \end{cases} \tag{4.2}$$

*Then, we have $g'(u) = (\gamma + 1)|u|^\gamma$, and*

*(i) If $n = 1, 2$, then we have the Sobolev embedding $V \subset L^q(\Omega)$, where $1 \leq q < \infty$. If $n = 3$, then we have the Sobolev embedding $V \subset L^q(\Omega)$, where $1 \leq q \leq 2\gamma + 2$. Therefore, we have, for $n = 1, 2, 3$ and $u \in V$, the Sobolev inequality*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|, \tag{4.3}$$

*where $1 \leq q \leq 2\gamma + 2$.*

*(ii) If $u \in V$, then $|g(u)| \leq C\|u\|^{\gamma+1}$ and $g : V \to H$ is locally Lipschitz with*

$$|g(u) - g(v)| \leq C(1 + \|u\| + \|v\|)^{\gamma+1}\|u - v\|, \tag{4.4}$$

*where $u, v \in V$.*

*(iii) If $\beta(x) \in \mathcal{B}$ and $u \in D(A)$, then $\|g(u)\| \leq C\|u\|^\gamma_{H^2(\Omega)}\|u\|$, where $C$ depends only on $\Omega$.*

*(iv) If $u_n \to u$ in $L^2(0, T; H)$ and $\|u_n(t)\|, \|u(t)\| \leq C$ for some constant $C$ and any $t \in [0, T]$, then $g(u_n) \rightharpoonup g(u)$ weakly in $L^2(0, T; H)$.*

# Chapter 5

## Solutions of the Klein-Gordon Problem

The existence and the uniqueness of the solutions for (3.5) is established using standard Galerkin method. Our goal here is to establish the convergence estimate (5.14). We also ascertain the uniformity of the estimates with respect to the parameters $q \in P$. Everywhere in the sequel $c > 0$ denotes various constants that depend only on the bounds of the admissible set $P$.

**Lemma 5.1.** *Let $u \in W(0,T) \cap L^\infty(0,T;V)$ be a solution of the damped Klein-Gordon problem*

$$u'' + \alpha u' + A_\beta u + \delta g(u) = f, \quad f \in L^2(0,T;H) \tag{5.1}$$

$$u(0) = y_0 \in V, \quad u'(0) = y_1 \in H.$$

*Then*

$$|u'(t)|^2 + \|u(t)\|^2 \leq cI, \tag{5.2}$$

*for any $t \in [0,T]$, where*

$$I = |y_1|^2 + \|y_0\|^2 + \|y_0\|^{\gamma+2} + \|f\|^2_{L^2(0,T;H)}. \tag{5.3}$$

*Proof.* By the definition of the weak solution we have $u \in L^\infty(0,T;V)$. Therefore,

$g(u) \in L^\infty(0,T;H) \subset L^2(0,T;H)$ by Theorem 4.1. Then

$$u'' + A_\beta u = f - \alpha u' - \delta g(u) \in L^2(0,T;H).$$

Multiply both sides by $u'$ to get

$$(u'' + A_\beta u, u') + \delta(|u|^\gamma u, u') = (f, u') - \alpha(u', u').$$

By Lemma 3.3 and $\int_\Omega |u|^\gamma u u' dx = \frac{1}{\gamma+2}\frac{d}{dt}\int_\Omega |u|^{\gamma+2} dx$ , we have

$$\frac{1}{2}\frac{d}{dt}\{|u'|^2 + \|u\|_\beta^2\} + \frac{\delta}{\gamma+2}\frac{d}{dt}\int_\Omega |u|^{\gamma+2} dx = (f, u') - \alpha|u'|^2.$$

Integrating both sides from 0 to $t$ we get

$$|u'(t)|^2 + \|u(t)\|_\beta^2 + \frac{2\delta}{\gamma+2}||u(t)||_{L^{\gamma+2}(\Omega)}^{\gamma+2} + 2\alpha\int_0^t |u'(s)|^2 ds$$
$$= |y_1|^2 + \|y_0\|_\beta^2 + \frac{2\delta}{\gamma+2}\|y_0\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} + 2\int_0^t (f(s), u'(s)) ds.$$

It follows readily from the inequality

$$2\int_0^t (f(s), u'(s)) ds \le \int_0^t |f(s)|^2 ds + \int_0^t |u'(s)|^2 ds$$

that

$$|u'(t)|^2 + \|u(t)\|^2 \le C\Big(|y_1|^2 + \|y_0\|^2 + \|y_0\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} + \|f\|_{L^2(0,T;H)}^2 + \int_0^t |u'(s)|^2 ds\Big).$$

By the Sobolev inequality (4.3), we have $\|y_0\|_{L^{\gamma+2}(\Omega)} \le c\|y_0\|$. Therefore, by

Gronwall's Lemma (see [14], section B.2)

$$|u'(t)|^2 + \|u(t)\|^2 \le c\big(|y_1|^2 + \|y_0\|^2 + \|y_0\|^{\gamma+2} + \|f\|^2_{L^2(0,T;H)}\big) = cI.$$

□

**Lemma 5.2.** *(i). Let $u_i,\ i = 1, 2$ be two solutions of the damped Klein-Gordon problem*

$$u_i'' + \alpha u_i' + A_\beta u_i + \delta g(u_i) = f_i, \quad f_i \in L^2(0, T; H) \tag{5.4}$$
$$u_i(0) = y_{0,i} \in V, \quad u_i'(0) = y_{1,i} \in H.$$

*Then*

$$|u_2'(t) - u_1'(t)|^2 + \|u_2(t) - u_1(t)\|^2$$
$$\le C(I_1, I_2)(\|y_{0,2} - y_{0,1}\|^2 + |y_{1,2} - y_{1,1}|^2 + \|f_2 - f_1\|^2_{L^2(0,T;H)}) \tag{5.5}$$

*for any $t \in [0, T]$, where $I_i = |y_{1,i}|^2 + \|y_{0,i}\|^2 + \|y_{0,i}\|^{\gamma+2} + \|f\|^2_{L^2(0,T;H)}$, and $i = 1, 2$.*

*(ii). The solution of the damped Klein-Gordon problem (5.1) is unique.*

*Proof.* Let $u$ be a solution of the equation (5.1). Then $\|u(t)\| \le c\sqrt{I}$, where $c\sqrt{I}$ is not dependent on $t$ by Lemma 5.1.

To show (i), notice that the difference $w = u_2 - u_1$ satisfies

$$w'' + A_\beta w = -\alpha w' - \delta(g(u_2) - g(u_1)) + f_2 - f_1 \in L^2(0, T; H).$$

Multiplying both sides by $w'$ and using Lemma 3.3, we get

$$\frac{1}{2}\frac{d}{dt}\{|w'|^2 + \|w\|_\beta^2\} = -\alpha|w'|^2 - \delta(g(u_2) - g(u_1), w') + (f_2 - f_1, w'). \tag{5.6}$$

Integrating over $[0,t],\ 0 < t \le T$ and using estimate (3.1) gives

$$|w'(t)|^2 + \|w(t)\|_\beta^2 = |w'(0)|^2 + \|w(0)\|_\beta^2 - 2\alpha\int_0^t |w'(s)|^2 ds$$
$$- 2\delta\int_0^t (g(u_2(s)) - g(u_1(s)), w'(s))ds + 2\int_0^t (f_2(s) - f_1(s), w'(s))ds.$$

The Lipschitz continuity of $g(u)$ and (4.4) imply

$$|w'(t)|^2 + ||w(t)||_\beta^2 \le |w'(0)|^2 + ||w(0)||_\beta^2 + C_1\int_0^t |w'(s)|ds$$
$$+ C_2\int_0^t (1 + \|u_1\| + \|u_2\|)^{2\gamma+2}||u_1 - u_2||^2 ds + \int_0^t |f_2 - f_1|^2 ds.$$

By Lemma 5.1

$$|w'(t)|^2 + ||w(t)||_\beta^2 \le |w'(0)|^2 + ||w(0)||_\beta^2 + C_1\int_0^t |w'(s)|ds$$
$$+ C_2'(1 + I_1 + I_2)^{\gamma+1}\int_0^t ||u_1 - u_2||^2 ds + \int_0^t |f_2 - f_1|^2 ds.$$

Gronwall's Lemma ([14], Section B.2) implies (5.5).

Part (ii) follows from (5.5), since in this case the initial conditions are the same, and $f_1 = f_2$. □

Now we establish the existence of a weak solution for Klein-Gordon equation (3.5). Let $\{\lambda_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ be the eigenvalues and the eigenfunctions of the operator $A_\beta$ in $V$, such that $\{w_k\}_{k=1}^{\infty}$ form an orthonormal basis in $H$. Then $\{w_k/\sqrt{\lambda_k}\}_{k=1}^{\infty}$ form an orthonormal basis in $V_\beta$, see [14], Chap. 6.

Fix $m \in \mathbb{N}$ and let $V_m = span\{w_1, ..., w_m\}$. Define

$$P_m h = \sum_{k=1}^{m} (h, w_k) w_k, \quad h \in H. \tag{5.7}$$

Then $P_m : H \to V_m$ is an orthogonal projection in $H$ with $|P_m h| \leq |h|$ for any $h \in H$. It is also an orthogonal projection in $V_\beta$ with $\|P_m v\|_\beta \leq \|v\|_\beta, \ v \in V$. Note that $P_m$ is dependent on $\beta$.

The approximate solution of (3.5) is defined to be a function $u_m \in W(0,T) \cap L^\infty(0,T;V)$ that satisfies

$$\begin{aligned} &u_m'' + \alpha u_m' + A_\beta u_m + \delta P_m g(u_m) = P_m f, \quad \text{in } V', \\ &u_m(0) = P_m y_0, \quad u_m'(0) = P_m y_1. \end{aligned} \tag{5.8}$$

**Lemma 5.3.** *Equation* (5.8) *has a unique solution* $u_m$ *satisfying* $u_m(t) \in V_m$, $u_m, u_m' \in C([0,T];V)$, *and*

$$\max_{0 \leq t \leq T} \left( \|u_m(t)\|^2 + |u_m'(t)|^2 \right) + \|u_m''(\cdot)\|^2_{L^2(0,T;V')} \leq c(I + I^{\gamma+1}), \tag{5.9}$$

*where* $I$ *is defined as in (5.3).*

*Proof.* The uniqueness is established as in Lemma 5.2. Let $z_m(t) = \sum_{k=1}^{m} g_{km}(t) w_k$

satisfy

$$(z''_m, w_k) + \alpha(z'_m, w_k) + a_\beta(z_m, w_k) + \delta(g(z_m), w_k) = (f, w_k), \tag{5.10}$$
$$z_m(0) = P_m y_0, \quad z'_m(0) = P_m y_1$$

for $1 \leq k \leq m$. For each $m \in \mathbb{N}$ this is a Cauchy problem for the system of ordinary differential equations that has a unique solution $z_m(t)$ with $z_m,\ z'_m \in C([0,T]; V)$ and $z''_m \in L^2([0,T]; V)$. To see that the solution $z_m(t)$ also satisfies (5.8) it is enough to establish that

$$\langle z''_m + \alpha z'_m + A_\beta z_m + \delta P_m g(z_m), w_k\rangle = \langle P_m f, w_k\rangle \tag{5.11}$$

for any $k \in \mathbb{N}$. But for $1 \leq k \leq m$, equations (5.11) are the same as (5.10), and for $k > m$ equations (5.11) are reduced to $0 = 0$, because $w_k$ are the eigenfunctions of the operator $A_\beta$. The uniqueness of $u_m$ implies $u_m = z_m$.

To obtain estimate (5.9) multiply both sides in (5.8) by $u'_m$ to get

$$(u''_m, u'_m) + (A_\beta u_m, u'_m) + \delta(P_m g(u_m), u'_m) = -\alpha(u'_m, u'_m) + (P_m f, u'_m).$$

Notice that $(P_m g(u_m), u'_m) = (g(u_m), P_m u'_m) = (g(u_m), u'_m)$. Similarly to the proof of Lemma 5.1, we have

$$\|u_m(t)\|^2 + |u'_m(t)|^2 \leq c I_m,$$

where $I_m = |P_m y_1|^2 + \|P_m y_0\|^2 + \|P_m y_0\|^{\gamma+2} + \|P_m f\|^2_{L^2(0,T;H)}$. Notice that $I_m \leq I$

for any $m$. Hence, we obtain

$$\max_{0\leq t\leq T}\left(\|u_m(t)\|^2+|u_m'(t)|^2\right)\leq cI, \tag{5.12}$$

for any $m$. Let $v\in V$ with $\|v\|\leq 1$. Then

$$\langle u_m'', v\rangle = -\alpha(u_m', v) - \langle A_\beta u_m, v\rangle - \delta(P_m g(u_m), v) + (P_m f, v).$$

Using $|v|\leq K_1\|v\|=K_1$ we get

$$\begin{aligned}|\langle u_m'', v\rangle| &\leq K_1 max\{|\alpha_{min}|, |\alpha_{max}|\}|u_m'| + \mu\|u_m\| + K_1\delta_{max}\|u_m\|_{L^{2\gamma+2}}^{\gamma+1} + K_1|f|\\ &\leq K_1 max\{|\alpha_{min}|, |\alpha_{max}|\}|u_m'| + \mu\|u_m\| + CK_1\delta_{max}\|u_m\|^{\gamma+1} + K_1|f|.\end{aligned}$$

Therefore

$$\|u_m''\|_{V'}^2\leq c(|f|^2+|u_m'|^2+\|u_m\|^2+\|u_m\|^{2\gamma+2}),$$

and the energy estimate (5.9) follows from (5.12). □

**Theorem 5.4.** *Let $q\in P,\ y_0\in V,\ y_1\in H,\ f\in L^2(0,T;H)$, and $I$ is defined as in (5.3). Then*

*(i). There exists a unique solution $u(t)=u(t;q)$ of (3.5). This solution satisfies $u\in C([0,T];V)\cap W(0,T)$, $u'\in C([0,T];H)$, and*

$$\max_{0\leq t\leq T}\left(\|u(t)\|^2+|u'(t)|^2\right)+\|u''(t)\|_{L^2(0,T;V')}^2\leq c(I+I^{\gamma+1}). \tag{5.13}$$

*(ii). The solution $u(t)=u(t;q)$ and its approximations $u_m(t)=u_m(t;q)$ satisfy*

*the following convergence estimate*

$$|u'(t) - u'_m(t)|^2 + \|u(t) - u_m(t)\|^2 \leq c(I)\Big(|y_1 - P_m y_1|^2 + \|y_0 - P_m y_0\|^2$$
$$+\|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |g(u(s;q)) - P_m g(u(s;q))|^2 ds\Big). \tag{5.14}$$

*(iii). Furthermore,* $u_m \to u$ *in* $C([0,T];V)$*, and* $u'_m \to u'$ *in* $C([0,T];H)$ *as* $m \to \infty$*.*

*Proof.* The uniqueness of the solutions for (3.5) is already established in Lemma 5.2. The existence is proved using standard methods, see [48], Section 4.4.4. That is, the energy estimate (5.9) shows that the approximate solutions $u_m,\ m \in \mathbb{N}$ remain within a bounded ball of the Hilbert space $W(0,T)$. Therefore we can choose a subsequence $u_{m_k}$ such that

$$u_{m_k} \rightharpoonup z \text{ in } L^2(0,T;V), \quad u'_{m_k} \rightharpoonup z' \text{ in } L^2(0,T;H), \quad u''_{m_k} \rightharpoonup z'' \text{ in } L^2(0,T;V'), \tag{5.15}$$

as $m_k \to \infty$, where $\rightharpoonup$ denotes the weak convergence. Furthermore, estimate (5.9) shows that we can also assume that $u_{m_k} \rightharpoonup z$ weak-star in $L^\infty(0,T;V)$ and $u'_{m_k} \rightharpoonup z'$ weak-star in $L^\infty(0,T;H)$. Since $V$ is compactly embedded in $H$, $u_{m_k} \to z$ in $L^2(0,T;H)$. By Theorem (4.6), $g(u_{m_k}) \rightharpoonup g(z)$ weakly in $L^2(0,T;H)$. Therefore, for any $h \in L^2(0,T;V)$, we have

$$\langle u''_{m_k}, h\rangle + \alpha\langle u'_{m_k}, h\rangle + \langle A_\beta u_{m_k}, h\rangle + \delta\langle g(u_{m_k}), h\rangle$$
$$\longrightarrow \quad \langle z'', h\rangle + \alpha\langle z, h\rangle + \langle A_\beta z, h\rangle + \delta\langle g(z), h\rangle,$$

as $m_k \to \infty$.

That is $z'' + \alpha z' + A_\beta z + \delta g(z) = f$ in $L^2(0,T;V')$. Furthermore, we have

$z(0) = y_0$ and $z'(0) = y_1$. For the details of the proof see [14, 48].

Thus $z$ is a solution of (3.5). Since the solution of (3.5) is unique, we have $u_m \to z$ for the entire sequence $u_m$ and not just for its subsequence $u_{m_k}$.

Rewrite (3.5) as $u'' + A_\beta u = f - \alpha u' - \delta g(u)$. Hence $u'' + A_\beta u \in L^2(0,T;H)$. Rewrite (5.8) as $u_m'' + A_\beta u_m = P_m f - \alpha u_m' - \delta P_m g(u_m)$. Hence $u_m'' + A_\beta u_m \in L^2(0,T;H)$. For the difference $u - u_m$ we have

$$
\begin{aligned}
&(u-u_m)'' + A_\beta(u-u_m) \\
&\qquad = f - P_m f - \alpha(u-u_m)' - \delta(g(u) - P_m g(u_m)) \in L^2(0,T;H). \quad (5.16)
\end{aligned}
$$

By Lemma 3.3,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{|u'-u_m'|^2 + a_\beta(u-u_m, u-u_m)\} \\
&\qquad = (f - P_m f, (u-u_m)') - \alpha|(u-u_m)'|^2 - \delta\Big(g(u) - P_m g(u_m), u' - u_m'\Big).
\end{aligned}
$$

Integration on $[0,t]$ gives

$$
\begin{aligned}
|u'(t) - u_m'(t)|^2 &+ a_\beta(u(t) - u_m(t), u(t) - u_m(t)) \le c\Big(|y_1 - P_m y_1|^2 \\
&+ a_\beta(y_0 - P_m y_0, y_0 - P_m y_0) + \|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |u' - u_m'|^2 ds \\
&+ \int_0^t |g(u) - P_m g(u)|^2 ds + \int_0^t |P_m(g(u) - g(u_m))|^2 ds\Big).
\end{aligned}
$$

Furthermore, using $|v| \le K_1\|v\|$ and

$$
|g(u) - g(u_m)|^2 \le C(1 + \|u\| + \|u_m\|)^{2\gamma+2}\|u - u_m\|^2,
$$

we get

$$|u'(t) - u'_m(t)|^2 + \|u(t) - u_m(t)\|^2 \leq c(I)\Big(|y_1 - P_m y_1|^2$$
$$+ \|y_0 - P_m y_0\|^2 + \|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |g(u) - P_m g(u)|^2 ds$$
$$+ \int_0^t |u'(s) - u'_m(s)|^2 ds + \int_0^t \|u(s) - u_m(s)\|^2 ds\Big).$$

Now the Gronwall's Lemma gives the convergence estimate (5.14).

By the Lebesgue Dominated Convergence Theorem, the right side of (5.14) approaches zero as $m \to \infty$. This implies that $u_m \to u$ in $L^\infty(0,T;V)$ and $u'_m \to u'$ in $L^\infty(0,T;H)$. Since $u_m,\ u'_m \in C([0,T];V)$ we conclude that $u \in C([0,T];V)$, and $u' \in C([0,T];H)$ after a modification on a set of measure zero in $[0,T]$. □

# Chapter 6

# Estimates for Eigenvalues and Eigenfunctions

**Definition 6.1.** Let $\beta \in \mathcal{B}$, and $\lambda_k(\beta)$ and $w_k(\beta),\ k \in \mathbb{N}$ be the eigenvalues and the normalized in $H$ eigenfunctions of the operator $A_\beta$ defined in (3.2).

Denote by $\Lambda_k,\ k \in \mathbb{N}$ the eigenvalues of the negative Dirichlet Laplacian $-\Delta = A_\beta$ with $\beta = 1$.

The following four lemmas are from [22].

**Lemma 6.2.** *Let $\beta \in \mathcal{B}$. Then*

$$\nu\Lambda_k \leq \lambda_k(\beta) \leq \mu\Lambda_k \tag{6.1}$$

*for any $k \in \mathbb{N}$.*

*Proof.* Recall the Courant Minimax Principle

$$\lambda_k(\beta) = \min_{V_k} \max_{v \in V_k} \frac{a_\beta(v,v)}{\|v\|^2},$$

where $V_k$ varies over all $k$-dimensional subspaces of $V$, see [32], Chapter 6. Since $\nu \leq \beta \leq \mu$ for any $\beta \in \mathcal{B}$, the Minimax Principle implies that $\lambda_k(\nu) \leq \lambda_k(\beta) \leq \lambda_k(\mu)$. But $\nu = const$, so $\lambda_k(\nu) = \nu\Lambda_k$. Similarly, $\lambda_k(\mu) = \mu\Lambda_k$. □

**Lemma 6.3.** *Let $v \in D(A)$, $\beta \in \mathcal{B}$, then*

$$\|v\|_\beta \leq \frac{1}{\sqrt{\nu\Lambda_1}}\ |A_\beta v|, \quad \text{and} \quad \|v\| \leq \frac{1}{\nu\sqrt{\Lambda_1}}\ |A_\beta v|. \tag{6.2}$$

*Proof.* We have $v = \sum_{k=1}^{\infty}(v, w_k(\beta))w_k(\beta)$ in $V$. Therefore

$$\|v\|_\beta^2 = \sum_{k=1}^{\infty} \lambda_k(\beta)|(v, w_k(\beta))|^2. \tag{6.3}$$

By [48], Section 2.2.1,

$$|A_\beta(v)|^2 = \sum_{k=1}^{\infty} \lambda_k^2(\beta)|(v, w_k(\beta))|^2. \tag{6.4}$$

Using Lemma 6.2, and $\lambda_{m+1}(\beta) \leq \lambda_k(\beta)$ for $k \geq m+1$ we get

$$\nu\Lambda_1\|v\|_\beta^2 \leq \lambda_1(\beta)\sum_{k=1}^{\infty} \lambda_k(\beta)|(v, w_k(\beta))|^2 \leq |A_\beta v|^2, \tag{6.5}$$

Since $\nu\|v\|^2 \leq \|v\|_\beta^2$, we establish (6.2). □

Recall that $\mathcal{B}$ is equipped with the $C(\bar{\Omega})$ topology.

**Lemma 6.4.** *Let $v \in V$. Then the mapping $\beta \to A_\beta v$ from $\mathcal{B}$ into $V'$ is continuous.*

*Proof.* Suppose that $\beta_n \to \beta$ in $\mathcal{B}$ as $n \to \infty$. We denote $A = A_\beta$ and $A_n = A_{\beta_n}$. Our goal is to show that $\|(A_n - A)v\|_{V'} \to 0$ as $n \to \infty$. Let $w \in V$ with $\|w\| \leq 1$. Then

$$\begin{aligned} |\langle (A_n - A)v, w\rangle|^2 &\leq \left(\int_\Omega |\beta_n(x) - \beta(x)||\nabla v(x)||\nabla w(x)|dx\right)^2 \\ &\leq \|\beta_n(x) - \beta(x)\|_\infty^2\|v\|^2 \to 0, \quad as \quad n \to \infty. \end{aligned} \tag{6.6}$$

□

**Lemma 6.5.** *Suppose that $\beta_n \to \beta$ in $\mathcal{B}$, and $v_n \rightharpoonup v$ weakly in $V$, as $n \to \infty$. Then $A_n v_n \rightharpoonup Av$ weakly in $V'$.*

*Proof.* Let $w \in V$, then

$$|\langle A_n v_n, w\rangle - \langle Av, w\rangle| = |\langle A_n w, v_n\rangle - \langle Aw, v\rangle| \tag{6.7}$$
$$\leq |\langle (A_n - A)w, v_n\rangle| + |\langle Aw, v_n - v\rangle|.$$

Since a weakly convergent sequence is bounded, for the first term we have

$$|\langle (A_n - A)w, v_n\rangle| \leq \|A_n w - Aw\|_{V'}\|v_n\| \leq c\|A_n w - Aw\|_{V'} \to 0$$

as $n \to \infty$ by Lemma 6.4. The second term $|\langle Aw, v_n - v\rangle| \to 0$ since $v_n \rightharpoonup v$ weakly in $V$, as $n \to \infty$. □

An additional regularity of the weak solution $u$ of (3.5) is obtained under more restrictive conditions on $y_0, y_1$ and $f$.

**Theorem 6.6.** *If $f \in L^2(0,T;V)$, $y_0 \in D(A)$, and $y_1 \in V$, then the solution $u$ of (3.5) satisfies $u \in C([0,T];D(A))$, $u' \in C([0,T];V)$, and*

$$\max_{0\leq t\leq T}(|A_\beta u(t)|^2 + \|u'(t)\|^2) \leq C(I)\Big(|A_\beta y_0|^2 + \|y_1\|^2 + \int_0^T \|f(s)\|^2 ds\Big). \tag{6.8}$$

*Furthermore, the solution $u$ and its approximations $u_m$ satisfy the following convergence estimate*

$$\max_{0\leq t\leq T}(|A_\beta(u(t) - u_m(t))|^2 + \|u'(t) - u'_m(t)\|^2) \leq C(I)\Big(|A_\beta(y_0 - P_m y_0)|^2$$
$$+ \|y_1 - P_m y_1\|^2 + \|f - P_m f\|^2_{L^2(0,T;V)} + \int_0^t \|g(u(s;q)) - P_m g(u(s;q))\|^2_\beta ds\Big). \tag{6.9}$$

*Proof.* Equality

$$((u_m'' + A_\beta u_m, u_m'))_\beta = \frac{1}{2}\frac{d}{dt}\left[|A_\beta u_m|^2 + \|u_m'\|_\beta^2\right] \tag{6.10}$$

is verified by substituting $u_m(t) = \sum_{k=1}^m g_{km}(t)w_k$ into it, and recalling that the eigenfunctions $\{w_k/\sqrt{\lambda_k}\}_{k=1}^\infty$ form an orthonormal basis in $V_\beta$. It also follows from Lemma 3.4.

Take the $((\cdot,\cdot))_\beta$ inner product of both sides of (5.8) with $u_m'$, use equality (6.10), and integrate the result on interval $[0,t]$ to obtain

$$\begin{aligned}|A_\beta u_m|^2 + \|u_m'\|_\beta^2 &= |A_\beta P_m y_0|^2 + \|P_m y_1\|_\beta^2\\ &+ 2\int_0^t [-\alpha\|u_m'(s)\|_\beta^2 - 2\delta((g(u_m(s)), u_m'(s)))_\beta + ((f(s), u_m'(s)))_\beta]ds.\end{aligned} \tag{6.11}$$

Since $A_\beta P_m = P_m A_\beta$, $\|P_m v\|_\beta \le \|v\|_\beta$, similarly to Lemma 4.2 we have $\|g(u_m)\|_\beta \le C\|u_m\|_{H^2}^\gamma \|u_m\|_\beta$. By (3.3) and energy estimate (5.9), we have $\|u_m\|_{H^2} \le C(I)$, where $C(I)$ does not depend on $m$. Hence,

$$\begin{aligned}|((P_m g(u_m), u_m'))_\beta| &\le ||P_m g(u_m)||_\beta ||u_m'||_\beta\\ &\le ||g(u_m)||_\beta ||u_m'||_\beta \le C(I)(||u_m||_\beta^2 + ||u_m'||_\beta^2).\end{aligned}$$

Therefore, we get

$$\begin{aligned}|A_\beta u_m|^2 + \|u_m'\|^2 \le C(I)\Big(&|A_\beta y_0|^2 + \|y_1\|^2\\ &+ \int_0^t [\|u_m'(s)\|^2 + \|u_m(s)\|^2]ds + \int_0^t \|f(s)\|^2 ds\Big).\end{aligned} \tag{6.12}$$

Use (6.2) to estimate $\|u_m(s)\|^2 \leq c|A_\beta u_m|^2$. Then Gronwall's inequality gives

$$\max_{0\leq t\leq T}(|A_\beta u_m(t)|^2 + \|u_m'(t)\|^2) \leq C(I)\Big(|A_\beta y_0|^2 + \|y_1\|^2 + \int_0^T \|f(s)\|^2 ds\Big) \quad (6.13)$$

for any $t \in [0,T]$. Estimate (6.13) shows that $\{u_m, u_m'\}$, $m \in \mathbb{N}$ remains within a bounded ball in $L^2(0,T;D(A)) \times L^2(0,T;V)$. Let $\{z, z'\}$ be a weak limit point of this sequence in $L^2(0,T;D(A)) \times L^2(0,T;V)$. By Theorem 5.4, $u_m \to u$ in $C([0,T];V)$, and $u_m' \to u'$ in $C([0,T];H)$ as $m \to \infty$. Therefore $\{z, z'\} = \{u, u'\}$. Thus $u$ satisfies (6.8).

The difference $u - u_m$ satisfies

$$\begin{aligned}(u-u_m)'' &+ A_\beta(u-u_m) \\ &= f - P_m f - \alpha(u-u_m)' - \delta(g(u) - P_m g(u_m)) \in L^2(0,T;V). \quad (6.14)\end{aligned}$$

Multiplication of (6.14) by $u' - u_m' \in L^2(0,T;V)$, Lemma 3.4, and Gronwall's inequality give the convergence estimate (6.9). Now,

$$|A_\beta(y_0 - P_m y_0)| = |(I - P_m)A_\beta(y_0)|, \quad \textit{and} \quad A_\beta(y_0) \in H$$

Therefore this equality and (6.9) imply that $u_m \to u$ in $C([0,T];D(A))$, and $u_m' \to u'$ in $C([0,T];V)$ as $m \to \infty$. The Theorem follows. □

# Chapter 7

# Continuity of the Solution Maps

The main goal of this chapter is to prove the continuity of the mapping $q \to u(q)$ for $q \in P$. Recall that $\mathcal{B}$ is equipped with the $C(\bar{\Omega})$ topology.

**Theorem 7.1.** *Let $u(q)$, $q \in P$ be the solution of the Klein-Gordon problem* (3.5)*. Then the solution maps $q \to u(q) : P \to C([0,T];V)$ and $q \to u'(q) : P \to C([0,T];H)$ are continuous.*

*Proof.* Let $q = (\alpha, \beta, \delta) \in P$, and $q_n = (\alpha_n, \beta_n, \delta_n) \in P$. Denote $A_n = A_{\beta_n}$. Suppose that $q_n \to q$ in $P$, as $n \to \infty$. First, assume that $f \in L^2(0,T;V)$, $y_0 \in D(A)$ and $y_1 \in V$. Consider Klein-Gordon problems

$$\begin{aligned} &v''(q) + \alpha v'(q) + A_\beta v(q) + \delta g(v(q)) = f \\ &v(0;q) = y_0, \quad v'(0;q) = y_1, \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} &v''(q_n) + \alpha_n v'(q_n) + A_n v(q_n) + \delta_n g(v(q_n)) = f \\ &v(0;q_n) = y_0, \quad v'(0;q_n) = y_1. \end{aligned} \tag{7.2}$$

Let $w = v(q_n) - v(q)$, then

$$w'' + A_n w = (A_\beta - A_n)v(q) - \alpha_n w' + (\alpha - \alpha_n)v'(q) \\ - \delta_n(g(v(q_n)) - g(v(q))) + (\delta - \delta_n)g(v(q)), \quad (7.3)$$

with $w(0) = 0$ and $w'(0) = 0$. By Theorem 5.8.4 in [14], we have $\beta_n(x) \in W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$. Also $y_0 \in D(A) \subset H^2(\Omega)$. Therefore, we get

$$|A_n y_0|^2 = \int_\Omega |\nabla \beta_n(x) \cdot \nabla y_0 + \beta_n(x) \triangle y_0|^2 dx \leq L \quad (7.4)$$

for some constant $L$ independent on $\beta$, but dependent on $y_0$. By estimate (6.8) of Theorem 6.6 and estimate (7.4) we get

$$\begin{aligned} \|w'(t)\|^2 &= ||v'(t, q_n) - v'(t, q)||^2 \leq 2(||v'(q_n)||^2 + ||v'(q)||^2) \quad (7.5) \\ &\leq C(I)\Big(|A_n y_0|^2 + |Ay_0|^2 + ||y_1||^2 + \int_0^T \|f(s)\|^2 ds\Big) \\ &\leq C(I)\Big(L^2 + \|y_1\|^2 + \int_0^T \|f(s)\|^2 ds\Big). \end{aligned}$$

By energy estimate (6.8) we have $v(q), v(q_n) \in W_r(0, T)$. Therefore $w \in W_r(0, T)$, and we can use Lemma 3.5 to derive

$$\langle w'', w' \rangle = \frac{1}{2}\frac{d}{dt}|w'|^2, \quad \text{and} \quad \langle A_n w, w' \rangle = \frac{1}{2}\frac{d}{dt}\|w\|_n^2.$$

By (4.4) of Theorem 4.7 and energy estimate (5.13), we have

$$|g(v(q_n)) - g(v(q))|^2 \\ \leq C(\|v(q_n)\| + \|v(q)\|)^{2\gamma}\|u(q_n) - v(q)\|^2 \leq C(I)\|v(q_n) - v(q)\|^2.$$

Multiply (7.3) by $w' \in L^2(0,T;V)$, and integrate it on $[0,t]$ to get

$$\int_0^t \langle w'', w' \rangle ds + \int_0^t \langle A_n w, w' \rangle ds$$
$$= \int_0^t \langle (A_n - A)v(q), w' \rangle ds + (\alpha - \alpha_n) \int_0^t \langle v'(q), w' \rangle ds$$
$$+ \delta_n \int_0^t \langle (g(v(q_n)) - g(v(q)), w' \rangle ds + (\delta - \delta_n) \int_0^t \langle g(v(q)), w' \rangle ds. \quad (7.6)$$

By Lemma 4.1 and estimate (7.6), we get

$$|w'(t)|^2 + \|w(t)\|^2 \leq C(I)\Big( \int_0^t \|(A - A_n)v(s;q)\|_{V'} \|w'(s)\| ds$$
$$+ |\alpha - \alpha_n| \int_0^t |v'(s;q)|^2 ds + |\delta - \delta_n| \int_0^t \|v(s;q)\|^2 ds + \int_0^t |w'(s)|^2 ds + \int_0^t \|w(s)\|^2 ds \Big). \quad (7.7)$$

By energy estimate (6.8) and by (7.5)

$$|w'(t)|^2 + \|w(t)\|^2 \leq C(I)\Big( L \int_0^t \|(A - A_n)v(s;q)\|_{V'} ds + |\alpha - \alpha_n|$$
$$+ |\delta - \delta_n| + \int_0^t |w'(s)|^2 ds + \int_0^t \|w(s)\|^2 ds \Big). \quad (7.8)$$

Now the Gronwall's inequality gives

$$|v'(t;q_n) - v'(t;q)|^2 + \|v(t;q_n) - v(t;q)\|^2$$
$$\leq C(I)\Big( L \int_0^T \|(A - A_n)v(s;q)\|_{V'} ds + |\alpha - \alpha_n| + |\delta - \delta_n| \Big), \quad (7.9)$$

for any $t \in [0,T]$.

By the assumption $q_n \to q$ in $P$, that is $\alpha_n \to \alpha$, $\delta_n \to \delta$ and $\beta_n \to \beta$ in $\mathcal{B}$ as $n \to \infty$. The integral term in the right hand side of (7.9) approaches zero by Lemma 6.4 and the Lebesgue's Dominated Convergence Theorem. Therefore the convergence $u(q_n) \to u(q)$ in $C([0,T];V)$, and $u'(q_n) \to u'(q)$ in $C([0,T];H)$ follows.

Now we prove the general case $f \in L^2(0,T;H)$, $y_0 \in V$, and $y_1 \in H$. Given $0 < \epsilon < 1$, one can find $h \in L^2(0,T;V)$, $z_0 \in D(A)$, and $z_1 \in V$, such that $\|f-h\|_{L^2(0,T;H)} < \epsilon$, $\|y_0 - z_0\| < \epsilon$, and $|y_1 - z_1| < \epsilon$.

Let $u(t;q_n)$ and $u(t;q)$ be solutions for equations with $f \in L^2(0,T;H)$, $y_0 \in V$, and $y_1 \in H$, and $v(t;q_n)$ and $v(t;q)$ be solutions for equations with $h \in L^2(0,T;V)$, $z_0 \in D(A)$, and $z_1 \in V$.

We have

$$\begin{aligned} |u'(t;q_n) - u'(t;q)| &= |u'(t;q_n) - v'(t;q_n) + v'(t;q_n) - v'(t;q) + v'(t;q) - u'(t;q)| \\ &\le |u'(t;q_n) - v'(t;q_n)| + |v'(t;q_n) - v'(t;q)| + |v'(t;q) - u'(t;q)|. \end{aligned}$$

By estimate (5.5), we conclude that $|u'(t;q_n) - v'(t;q_n)| \le C(I^{|u|}, I^{|v|})\epsilon$, $|v'(t;q) - u'(t;q)| \le C(I^{|u|}, I^{|v|})\epsilon$ for $t \in [0,T]$, where

$$\begin{aligned} I^{|u|} &= |y_1|^2 + \|y_0\|^2 + \|y_0\|^{\gamma+2} + \|f\|^2_{L^2(0,T;H)}, \\ I^{|v|} &= |z_1|^2 + \|z_0\|^2 + \|z_0\|^{\gamma+2} + \|h\|^2_{L^2(0,T;H)}. \end{aligned}$$

For $\epsilon < 1$, we can find a constant $\hat{c}$ such that $C(I^{|u|}, I^{|v|}) \le \hat{c}$, and $\hat{c}$ only depends on $f$, $y_0$, $y_1$, and the bounds of parameters in $P$.

Also, we have $|v'(t;q_n) - v'(t;q)| < \epsilon$ for a sufficiently large $n$, and for any $t \in [0,T]$. Therefore, we have $|u'(t;q_n) - u'(t;q)| \le (2\hat{c}+1)\epsilon$. Since $\epsilon$ could be arbitrary small, we get $|u(t;q_n) - u(t;q)| \to 0$ as $n \to 0$.

Similarly, we obtain $\|u(t;q_n) - u(t;q)\| \to 0$ as $n \to 0$.

$\square$

# Chapter 8

## Gâteaux Differentiability of the Solution Map in $P$

The goals of this chapter are to derive the existence of Gâteaux derivatives and to give a characterization of the weak right Gâteaux derivative of the solution map. To avoid confusion of notation, in this chapter, we use the notation $|.|_H$ for the norm in $H$, and $|.|$ simply for the absolute value.

**Definition 8.1.** Let $q^*, q \in P$. The solution map $q \to u(q)$ of $P$ into $L^2(0,T;H)$ is said to be weakly (right) Gâteaux differentiable at $q^*$ in the direction $q - q^*$ if there exists a function $Du(q^*; q - q^*) \in L^2(0,T;H)$ such that

$$\lim_{\lambda \to 0+} \left( \frac{u(q^* + \lambda(q - q^*)) - u(q^*)}{\lambda}, v \right) = (Du(q^*; q - q^*), v) \tag{8.1}$$

for any $v \in L^2(0,T;H)$.

For convenience the word "right" is omitted in the sequel. Also, since $q^*, q \in P$, the above definition is applied to a possibly restricted set of directions in the convex admissible set $P$.

Let $q^* = (\alpha^*, \beta^*, \delta^*)$. We will show that the weak Gâteaux derivative of the solution map at $q^*$ in the direction $q - q^*$, $z = Du(q^*; q - q^*)$ exists and satisfies the *linear* equation

$$z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + \delta^* h(t) z(t) = \tilde{f}(t), \quad t \in (0,T) \tag{8.2}$$

with some $h \in L^2(0,T;H)$ and $\tilde{f} \in L^2(0,T;H)$.

Now we proceed with the determination of the weak Gâteaux derivative $z = Du(q^*; q - q^*)$. Fix $q^*, q \in P$. Let $\lambda \in (0,1]$. For simplicity we write
$q_\lambda = q^* + \lambda(q - q^*)$.

Let $u(q_\lambda)$ and $u(q^*)$ be the weak solutions of the equations

$$u''(q_\lambda) + \alpha_\lambda u'(q_\lambda) + A_{\beta_\lambda} u(q_\lambda) + \delta_\lambda g(u(q_\lambda)) = f \in L^2(0,T;V), \quad (8.3)$$
$$u_\lambda(0;q_\lambda) = y_0 \in D(A), \quad u'_\lambda(0;q_\lambda) = y_1 \in V$$

and

$$u''(q^*) + \alpha^* u'(q^*) + A_{\beta^*} u(q^*) + \delta^* g(u(q^*)) = f \in L^2(0,T;V), \quad (8.4)$$
$$u(0;q^*) = y_0 \in D(A), \quad u'(0;q^*) = y_1 \in V$$

correspondingly.

**Lemma 8.2.** *Let*

$$B_\lambda(t,x) = B(u(t,x;q_\lambda), u(t,x;q^*)) = \frac{g(u(t,x;q_\lambda)) - g(u(t,x;q^*))}{u(t,x;q_\lambda) - u(t,x;q^*)}.$$

*Then we have* $|B_\lambda(t,x)| \le c$ *on* $[0,T]\times\Omega$ *for some constant* $c$ *that does not depend on* $\lambda$*. Also,*

$$|B_\lambda(t) - g'(u(t;q^*))|_H \to 0, \quad (8.5)$$

*for any fixed* $t \in [0,T]$*, as* $\lambda \to 0+$.

*Proof.* For $n \ge 4$, we have $B_\lambda(t,x) = 1$. The case is trivial.

Now we prove the case of $n \le 3$.

By Theorem 6.6, $u(t;q_\lambda)$, $u(t;q^*) \in D(A)$ for any $t \in [0,T]$ . By the Sobolev embeddings Theorem (see [48], II(1.12)), $H^2(\Omega) \subset C(\Omega)$ continuously for $n \leq 3$. Hence,

$$\|u(t;q_\lambda)\|_{C(\Omega)} \leq c\|u(t;q_\lambda)\|_{H^2(\Omega)} \leq \hat{c},$$

$$\|u(t;q^*)\|_{C(\Omega)} \leq c\|u(t;q^*)\|_{H^2(\Omega)} \leq \hat{c},$$

where $\hat{c}$ depends only on the $\|f\|_{L^2(0,T;H}$, $\|y_0\|$, $|y_1|$, and the bounds of three parameters by inequalities (3.3), (5.13) and (6.8).

Therefore,

$$|u(t,x;q_\lambda)|, \quad |u(t,x;q^*)| \leq \hat{c}, \tag{8.6}$$

on $[0,T] \times \Omega$. Hence,

$$|B_\lambda(t,x)| = |B(u(t,x;q_\lambda), u(t,x;q^*))| = |g'(\theta u(t,x;q_\lambda) + (1-\theta)u(t,x;q^*))|$$

$$= (\gamma+1)|\theta u(t,x;q_\lambda) + (1-\theta)u(t,x;q^*)|^\gamma$$

$$\leq (\gamma+1)(|u(t,x;q_\lambda)| + |u(t,x;q^*)|)^\gamma \leq (\gamma+1)(2\hat{c})^\gamma = c,$$

on $[0,T] \times \Omega$, and $c$ does not depend on $\lambda$.

For any fixed $(t,x) \in [0,T] \times \Omega$, we have

$$|B_\lambda(t,x) - g'(u(t,x;q^*))| \leq |g''(\tau)||u(t,x;q_\lambda) - u(t,x;q^*)|$$

$$= (\gamma+1)\gamma|\tau|^{\gamma-1}|u(t,x;q_\lambda) - u(t,x;q^*)| \leq c|u(t,x;q_\lambda) - u(t,x;q^*)|,$$

where $\tau$ is between $u(t,x;q_\lambda)$ and $u(t,x;q^*)$. Notice that $c$ does not depend on $t$ and $x$.

Therefore, we have

$$|B_\lambda(t) - g'(u(t;q^*))|_H \leq c|u(t;q_\lambda) - u(t;q^*)|_H \to 0,$$

as $\lambda \to 0+$, since the solution map $q \to u(q) : P \to C(0,T;V)$ is continuous by Theorem 7.1.

□

**Theorem 8.3.** *Let $q = (\alpha, \beta, \delta)$, $q^* = (\alpha^*, \beta^*, \delta^*) \in P$. Let function $u(q^*)$ be the weak solution of the equation*

$$u''(q^*) + \alpha^* u'(q^*) + A_{\beta^*} u(q^*) + \delta^* g(u(q^*)) = f \in L^2(0,T;V),$$
$$u(0;q^*) = y_0 \in D(A), \quad u'(0;q^*) = y_1 \in V.$$

*Then the weak Gâteaux derivative $z = Du(q^*; q - q^*) \in L^2(0,T;H)$ at $q^* \in P$ in the direction $q - q^*$ exists and is the unique weak solution of the problem*

$$z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + \delta^* g'(u(t;q^*)) z(t) = f_0(t), \ t \in (0,T) \quad (8.7)$$
$$z(0) = 0, \ z'(0) = 0,$$

*where $f_0(t) = (\alpha^* - \alpha) u'(t;q^*) + (A_{\beta^*} - A_\beta) u(t;q^*) + (\delta^* - \delta) g(u(t;q^*))$.*

*Proof.* First, we prove the case of $n \leq 3$. Case $n \geq 4$ is proved in Lemma 8.4.

Let $q_\lambda = q^* + \lambda(q - q^*) = (\alpha_\lambda, \beta_\lambda, \delta_\lambda)$. Functions $u(q_\lambda)$ and $u(q^*)$ are the weak solutions of the equations

$$u''(q_\lambda) + \alpha_\lambda u'(q_\lambda) + A_{\beta_\lambda} u(q_\lambda) + \delta_\lambda g(u(q_\lambda)) = f \in L^2(0,T;V), \quad (8.8)$$
$$u_\lambda(0;q_\lambda) = y_0 \in D(A), \quad u'_\lambda(0;q_\lambda) = y_1 \in V$$

and

$$u''(q^*) + \alpha^* u'(q^*) + A_{\beta^*} u(q^*) + \delta^* g(u(q^*)) = f \in L^2(0,T;V), \qquad (8.9)$$
$$u(0;q^*) = y_0 \in D(A), \quad u'(0;q^*) = y_1 \in V$$

correspondingly.

Then, in the distribution sense, the quotient $z_\lambda = (u(q_\lambda) - u(q^*))/\lambda$ satisfies

$$\begin{aligned} & z_\lambda'' + \alpha^* z_\lambda' + A_{\beta^*} z_\lambda + \delta^* \frac{g(u(q_\lambda)) - g(u(q^*))}{\lambda} \\ & = (\alpha^* - \alpha) u'(q_\lambda) + (A_{\beta^*} - A_\beta) u(q_\lambda) + (\delta^* - \delta) g(u(q_\lambda)), \\ & z_\lambda(0) = 0, \quad z_\lambda'(0) = 0. \end{aligned}$$

Let

$$f_\lambda(t) = (\alpha^* - \alpha) u'(t;q_\lambda) + (A_{\beta^*} - A_\beta) u(t;q_\lambda) + (\delta^* - \delta) g(u(t;q_\lambda)).$$

Let $B_\lambda(t) = B(u(t;q_\lambda), u(t;q^*))$ be defined as in Lemma 8.2.

Then

$$z_\lambda'' + \alpha^* z_\lambda' + A_{\beta^*} z_\lambda + \delta^* B_\lambda z_\lambda = f_\lambda, \qquad (8.10)$$
$$z_\lambda(0) = 0, \quad z_\lambda'(0) = 0.$$

By Lemma 4.1 we have

$$|g(u(t;q_\lambda))|_H = \|u(t;q_\lambda)\|_{L^{2\gamma+2}}^{\gamma+1} \le C \|u(t;q_\lambda)\|^{\gamma+1}.$$

Therefore one can estimate

$$|f_\lambda(t)|_H \le |\alpha^* - \alpha||u'(t;q_\lambda)| + 2\mu K_1\|u(t;q_\lambda)\| + CK_1^{\gamma+1}|\delta^* - \delta|\|u(t;q_\lambda)\|^{\gamma+1}).$$

Now the energy estimate (5.13) shows that there exists $c \ge 0$ independent of $q \in P$ such that

$$\|f_\lambda\|_{L^2(0,T;H)} \le c \tag{8.11}$$

for all $\lambda \in (0,1]$.

Since $u(t;q_\lambda), u(t;q^*) \in W(0,T)$, $z_\lambda \in W(0,T)$. Then

$$z''_\lambda + A_{\beta^*} z_\lambda = f_\lambda - \alpha^* z'_\lambda - \delta^* B_\lambda z_\lambda \in L^2(0,T;H)$$

Multiply both sides by $z'_\lambda$ to get

$$(z''_\lambda + A_{\beta^*} z_\lambda, z'_\lambda) = (f_\lambda, z'_\lambda) - \alpha^*(z'_\lambda, z'_\lambda) - \delta^*(B_\lambda z_\lambda, z'_\lambda).$$

By Lemma 3.3, we have

$$\frac{1}{2}\frac{d}{dt}\{|z'_\lambda|_H^2 + \|z_\lambda\|_\beta^2\} = (f_\lambda, z'_\lambda) - \alpha^*(z'_\lambda, z'_\lambda) - \delta^*(B_\lambda z_\lambda, z'_\lambda)$$

Integrating both sides from 0 to $t$ we get

$$\begin{aligned}&|z'_\lambda(t)|_H^2 + \|z_\lambda(t)\|_\beta^2 \\ &= 2\int_0^t (f_\lambda(s), z'_\lambda(s))ds - 2\alpha^*\int_0^t (z'_\lambda(s), z'_\lambda(s))ds - 2\delta^*\int_0^t (B_\lambda(s)z_\lambda(s), z'_\lambda(s))ds.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&|z'_\lambda(t)|_H^2 + \|z_\lambda(t)\|_\beta^2 \\
&\leq \int_0^t |f_\lambda(s)|_H^2 ds + \int_0^t |z'_\lambda(s)|_H^2 ds + 2\alpha^* \int_0^t |z'_\lambda(s)|_H^2 ds \\
&+ 2c\delta^* (\int_0^t |z_\lambda(s)|_H^2 ds + \int_0^t |z'_\lambda(s)|_H^2) ds) \\
&\leq \|f_\lambda\|_{L^2(0,T;H)} + c(\int_0^t |z'_\lambda(s)|_H^2 ds + \int_0^t \|z_\lambda(s)\|^2 ds).
\end{aligned}
$$

Then,

$$
|z'_\lambda(t)|_H^2 + \|z_\lambda(t)\|^2 \leq c\|f_\lambda\|_{L^2(0,T;H)} + c(\int_0^t |z'_\lambda(s)|_H^2 ds + \int_0^t \|z_\lambda(s)\|^2 ds).
$$

Therefore, Gronwall's inequality implies

$$
|z'_\lambda(t)|_H^2 + \|z_\lambda(t)\|^2 \leq c\|f_\lambda\|_{L^2(0,T;H)},
$$

where $c$ does not depend on $\lambda$ and $t$.

Hence, by (8.11), we get

$$
\max_{0\leq t\leq T} \left( |z'_\lambda(t)|_H^2 + \|z_\lambda(t)\|^2 \right) \leq c, \tag{8.12}
$$

where $c$ does not depend on $\lambda$.

Let $v \in V$ with $\|v\| \leq 1$. Then

$$
\langle z''_\lambda, v\rangle = -\alpha^*(z'_\lambda, v) - \langle A_{\beta^*} z_\lambda, v\rangle - \delta^*(B_\lambda z_\lambda, v) + (f_\lambda, v).
$$

Using $|v| \leq K_1\|v\| = K_1$ we get

$$|\langle z''_\lambda(t), v\rangle| \leq K_1|\alpha^*||z'_\lambda(t)|_H + \mu\|z_\lambda(t)\| + cK_1\delta^*\|z_\lambda(t)\| + K_1|f_\lambda(t)|_H.$$

Therefore,

$$\|z''_\lambda(t)\|_{V'} \leq K_1|\alpha^*||z'_\lambda(t)|_H + \mu\|z_\lambda(t)\| + cK_1\delta^*\|z_\lambda(t)\| + K_1|f_\lambda(t)|_H.$$

By (8.11) and (8.12), we get

$$\|z''_\lambda\|_{L^2(0,T;V')} \leq c, \tag{8.13}$$

where $c$ does not depend on $\lambda$.

Function $z_\lambda \in W(0,T)$ and it is bounded in $W(0,T)$ by (8.12) and (8). Therefore one can extract a subsequence $z_{\lambda_k}$, $\lambda_k \to 0+$, and find $z \in W(0,T)$ such that $z_{\lambda_k} \rightharpoonup z$ weakly in $L^2(0,T;V)$, $z'_{\lambda_k} \rightharpoonup z'$ weakly in $L^2(0,T;H)$, and $z''_{\lambda_k} \rightharpoonup z''$ weakly in $L^2(0,T;V')$.

Now let us prove that $B_{\lambda_k} z_{\lambda_k} \rightharpoonup g'(u(q^*))z$ weakly in $V'$ as $\lambda_k \to 0+$.

Let $B_0 = g'(u(q^*))$. Then, by Lemma 8.2 and inequality (8.6), we have $\|B_{\lambda_k} z_{\lambda_k} - B_0 z\|_{V'} \leq c$, where $c$ does not depend on $\lambda_k$.

Suppose that $D$ is a dense subset of $V$, then for any $v \in V$, and any given

$\epsilon > 0$, one can find a $v_0 \in D$ such that $\|v - v_0\| \leq \epsilon$. Then

$$
\begin{aligned}
&|\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v \rangle| = |\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v - v_0 + v_0 \rangle| \\
&\leq |\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v - v_0 \rangle| + |\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v_0 \rangle| \\
&\leq \|B_{\lambda_k} z_{\lambda_k} - B_0 z\|_{V'} \|v - v_0\|_V + |\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v_0 \rangle| \\
&\leq c\epsilon + |\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v_0 \rangle|.
\end{aligned}
$$

Therefore, to prove the weak convergence, it is enough to show that $|\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v_0 \rangle| \to 0$ for any $v_0 \in D$.

Since $C_0^\infty(\Omega)$ is dense in $V$, for any $v_0 \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
&|\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v_0 \rangle| = |\langle B_{\lambda_k} z_{\lambda_k} - B_0 z_{\lambda_k} + B_0 z_{\lambda_k} - B_0 z, v_0 \rangle| \\
&\leq |\langle (B_{\lambda_k} - B_0) z_{\lambda_k}, v_0 \rangle| + |\langle B_0 (z_{\lambda_k} - z), v_0 \rangle| \\
&\leq \int_\Omega |B_{\lambda_k} - B_0| |z_{\lambda_k}| |v_0| dx + \int_\Omega |B_0| |z_{\lambda_k} - z| |v_0| dx \\
&\leq |v_0|_\infty \int_\Omega |B_{\lambda_k} - B_0| |z_{\lambda_k}| dx + |v_0|_\infty \int_\Omega |B_0| |z_{\lambda_k} - z| dx \\
&\leq |v_0|_\infty |B_{\lambda_k} - B_0|_H |z_{\lambda_k}|_H + |v_0|_\infty |B_0|_H |z_{\lambda_k} - z|_H
\end{aligned}
$$

By Lemma 8.2, we have $|B_{\lambda_k} - B_0|_H \to 0$ for any fixed $t \in [0, T]$ as $\lambda_k \to 0+$. Also $|z_{\lambda_k} - z|_H \to 0$ for any fixed $t \in [0, T]$ as $\lambda_k \to 0+$. Hence, we have $\langle B_{\lambda_k} z_{\lambda_k} - B_0 z, v_0 \rangle \to 0$ for any fixed $t \in [0, T]$ as $\lambda_k \to 0+$.

Now we have $z_{\lambda_k} \rightharpoonup z$ weakly in $L^2(0, T; V)$, $z'_{\lambda_k} \rightharpoonup z'$ weakly in $L^2(0, T; H)$, $z''_{\lambda_k} \rightharpoonup z''$ weakly in $L^2(0, T; V')$, and $B_{\lambda_k} z_{\lambda_k} \rightharpoonup g'(u(q^*))z$ weakly in $V'$ as $\lambda_k \to 0+$.

Therefore, for any $h \in L^2(0,T;V)$, we have

$$\begin{aligned} & \langle z''_{\lambda_k}, h\rangle + \alpha^* \langle z'_{\lambda_k}, h\rangle + \langle A_{\beta^*} z_{\lambda_k}, h\rangle + \delta^* \langle B_{\lambda_k} z_{\lambda_k}, h\rangle \\ \longrightarrow \quad & \langle z'', h\rangle + \alpha^* \langle z, h\rangle + \langle A_{\beta^*} z, h\rangle + \delta^* \langle g'(u(q^*) z, h\rangle, \end{aligned}$$

as $\lambda_k \to 0+$.

Let

$$f_0(t) = (\alpha^* - \alpha) u'(t; q^*) + (A_{\beta^*} - A_\beta) u(t; q^*) + (\delta^* - \delta) g(u(t; q^*)). \tag{8.14}$$

From Theorem 7.1 we get $u(q_\lambda) \to u(q^*)$ in $L^2(0,T;V)$, and $u'(q_\lambda) \to u'(q^*)$ in $L^2(0,T;V)$, and $g(u(t;q_\lambda)) \to g(u(t;q^*))$ in $L^2(0,T;H)$, as $\lambda \to 0+$. And $(A_{\beta^*} - A_\beta) u(t; q_\lambda) \rightharpoonup (A_{\beta^*} - A_\beta) u(t; q^*)$ weakly in $L^2(0,T;V')$ by Lemma 6.5. Therefore $f_\lambda \rightharpoonup f_0$ weakly in $L^2(0,T;H)$.

Now we can pass to the limit as $\lambda_k \to 0+$ in (8.10) and conclude that

$$\begin{aligned} & z'' + \alpha^* z' + A_{\beta^*} z + \delta^* g'(u(t;q^*)) z = f_0, \\ & z(0) = 0, \quad z'(0) = 0. \end{aligned}$$

Since $f_0 \in L^2(0,T;H)$, we can prove that the solution $z$ is unique in $W(0,T)$ by following the same approach as in Chapter 5. Hence the entire sequence $z_\lambda$ is convergent to $z$ as $\lambda \to 0+$. This proves that $z$ is the weak Gâteaux derivative $Du(q^*; q - q^*)$ of the map $q \to u(q)$ as claimed in the Theorem. □

**Lemma 8.4.** *Let $n \geq 4$, $q = (\alpha, \beta, \delta)$, $q^* = (\alpha^*, \beta^*, \delta^*) \in P$. Let function $u(q^*)$*

*be the weak solution of the equation*

$$u''(q^*) + \alpha^* u'(q^*) + A_{\beta^*} u(q^*) + \delta^* u(q^*) = f \in L^2(0,T;V),$$
$$u(0;q^*) = y_0 \in D(A), \quad u'(0;q^*) = y_1 \in V.$$

*Then the weak Gâteaux derivative* $z = Du(q^*; q - q^*) \in L^2(0,T;H)$ *at* $q^* \in P$ *in the direction* $q - q^*$ *exists and is the unique weak solution of the problem*

$$z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + \delta^* z(t) = f_0(t),\ t \in (0,T)$$
$$z(0) = 0,\ z'(0) = 0,$$

*where* $f_0(t) = (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*) + (\delta^* - \delta)u(t;q^*)$.

*Proof.* The proof of this lemma follows as the proof of Theorem 8.3 with $g(u) = u$ and $B_\lambda = 1$. In this case, we have

$$z_\lambda'' + \alpha^* z_\lambda' + A_{\beta^*} z_\lambda + \delta^* z_\lambda = f_\lambda, \tag{8.15}$$
$$z_\lambda(0) = 0, \quad z_\lambda'(0) = 0,$$

where $f_\lambda(t) = (\alpha^* - \alpha)u'(t;q_\lambda) + (A_{\beta^*} - A_\beta)u(t;q_\lambda) + (\delta^* - \delta)u(t;q_\lambda)$.

Similar to the proof of Theorem 8.3, we can show that

$$\|f_\lambda\|_{L^2(0,T;H)} \le c,$$

$$\max_{0\le t\le T} \left( |z_\lambda'(t)|_H^2 + \|z_\lambda(t)\|^2 \right) \le c,$$

$$\|z_\lambda''\|_{L^2(0,T;V')} \le c,$$

for all $\lambda \in (0,1]$.

Similarly, one can extract a subsequence $z_{\lambda_k} \in W(0,T)$ such that as $\lambda_k \to 0+$ we get

$$
\begin{aligned}
&z'' + \alpha^* z' + A_{\beta^*} z + \delta^*(u(t;q^*)z = f_0,\\
&z(0) = 0, \quad z'(0) = 0.
\end{aligned}
$$

where $f_0(t) = (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*) + (\delta^* - \delta)u(t;q^*)$, and $z \in W(0,T)$.

Since $f_0 \in L^2(0,T;H)$, we can prove that the solution $z$ is unique in $W(0,T)$ by following the same approach as in Chapter 5. Hence $z_\lambda$ is convergent to $z$ as $\lambda \to 0+$. This proves that $z$ is the weak Gâteaux derivative $Du(q^*; q - q^*)$ of the map $q \to u(q)$ as claimed in the Theorem. $\square$

# Chapter 9

# Gâteaux and Fréchet Differentiability of the Objective Function

We can show that the objective function $J(q) = \|u(q)-z_d\|^2_{L^2(0,T;H)}$ has practically the same differentiability properties as the ones established in [21] in the case of the sine-Gordon equation. Here we follow the results from [21].

The objective function $J(q)$ is Gâteaux differentiable. Indeed, by Theorem 8.3 the map $q \to u(q)$ is weakly Gâteaux differentiable at any $q^* \in P$ in any direction of $q - q^*$ for $q \in P$, and its weak Gâteaux derivative $z(t,x) = Du(q^*; q-q^*)(t,x)$ can be described by the weak solution of equation (8.7). From the definition of the functional $J(q) = \|u(q) - z_d\|^2_{L^2(0,T;H)}$ we get

$$\begin{aligned} &DJ(q^*; q - q^*) = 2(u(q^*) - z_d, Du(q^*; q - q^*)) \\ &= 2\int_Q [u(q^*; t, x) - z_d(t, x)] z(t, x) dx dt. \end{aligned} \tag{9.1}$$

Let the adjoint state $p(q^*)$ is defined as the weak solution of the linear terminal value problem

$$\begin{aligned} &p'' - \alpha^* p' + A_{\beta^*} p + \delta^* g'(u(q^*) p = u(q^*) - z_d, \\ &p(T) = 0,\ p'(T) = 0. \end{aligned}$$

Since $u(q^*) - z_d \in L^2(0,T;H)$, one can show that the solution $p(q^*)$ exists and is unique in $W(0,T)$ by following the similar approach as in Chapter 5.

Therefore, expression (9.1) becomes

$$
\begin{aligned}
&DJ(q^*; q-q^*) = 2\int_0^T (z(t), p'' - \alpha^* p' + A_{\beta^*}p + \delta^* g'(u(q^*)p))dt \\
&= 2\int_0^T (z''(t) + \alpha^* z'(t) A_{\beta^*} z + \delta^* g'(u(q^*))z, p(t;q^*))dt \\
&= 2(\alpha^* - \alpha)\int_0^T (u'(t;q^*), p(t;q^*))\,dt + 2\int_0^T (A_{\beta^*} - A_\beta)u(t;q^*), p(t;q^*))\,dt \\
&+2(\delta^* - \delta)\int_0^T (g(u(t;q^*)), p(t;q^*))\,dt.
\end{aligned}
$$

Thus we obtain the following result

**Theorem 9.1.** *Let $q, q^* \in P$. Then the Gâteaux derivative $DJ(q^*; q-q^*)$ of the objective function $J(q)$ at $q^*$ in the direction $q-q^*$ has the following representation*

$$
DJ(q^*; q-q^*) = (\alpha^* - \alpha)a(q^*) + \int_\Omega (\beta^*(x) - \beta(x))G(x;q^*)dx + (\delta^* - \delta)c(q^*), \tag{9.2}
$$

*where*

$$
a(q^*) = 2\int_Q u_t(t,x;q^*)p(t,x;q^*)\,dxdt. \tag{9.3}
$$

$$
c(q^*) = 2\int_Q g(u(t,x;q^*))p(t,x;q^*)\,dxdt. \tag{9.4}
$$

*and*

$$
G(x;q^*) = 2\int_0^T \nabla u(t,x;q^*)\nabla p(t,x;q^*)dt, \quad x \in \Omega. \tag{9.5}
$$

Note that $G \in L^1(\Omega)$.

Our main goal is to prove that the objective function $J(q)$ is Fréchet differentiable. For this purpose we will consider the interior *int* $P$ of the admissible set $P$ defined in (1.4) as an open subset of the Banach space $X = \mathbb{R} \times L^\infty(\Omega) \times \mathbb{R}$.

The norm of $(\alpha, \beta, \delta) \in X$ is defined by

$$\|(\alpha, \beta, \delta)\|_X = \max\{|\alpha|, \|\beta\|_{C(\Omega)}, |\delta|\}.$$

**Definition 9.2.** Function $J(q)$ is called Fréchet differentiable at $q^* \in int\, P$, if there exists a bounded linear functional $DJ(q^*) : X \to \mathbb{R}$ such that

$$\lim_{q \to q^*} \frac{|J(q) - J(q^*) - DJ(q^*)(q - q^*)|}{\|q - q^*\|_X} = 0, \quad q \in int\, P. \tag{9.6}$$

**Theorem 9.3.** *Objective function $J(q)$ is Fréchet differentiable at any $q^* \in int\, P$. Let $a(q^*), c(q^*)$ and $G(q^*)$ be defined by (9.3), (9.4) and (9.5). Then the Fréchet derivative $DJ(q^*) \in X'$ is the bounded linear functional defined on $q - q^* \in X$ by*

$$DJ(q^*)(q-q^*) = (\alpha^*-\alpha)a(q^*)+\int_\Omega(\beta^*(x)-\beta(x))G(x;q^*)dx+(\delta^*-\delta)c(q^*), \tag{9.7}$$

*where $q = (\alpha, \beta, \delta) \in int\, P$.*

*Proof.* We follow the Calculus argument that for a function of several variables the continuity of its partial derivatives implies the differentiability.

Fix $q, q^* \in int\, P$. Then $u(q^*), p(q^*) \in L^2(0, T; V)$ by the results of Chapter 5. Therefore $DJ(q^*)$ defined in (9.7) is a bounded linear functional on $X$.

Define the real valued function

$$F(t) = J(q^* + t(q - q^*)), \quad t \in \mathbb{R}.$$

Then $F$ is defined on an open interval containing $[0, 1]$. It is continuous on it by Theorem 7.1. Moreover $F'(0+) = DJ(q^*; q - q^*)$ and $F'(0-) = -DJ(q^*; -(q - q^*))$. According to (9.2), $F'(0+) = F'(0-)$. Therefore $F$ is differentiable at $t = 0$.

Clearly, the same argument can be applied at any $t \in [0,1]$. The conclusion is that one can apply the Mean-Value Theorem to $F$ on $[0,1]$.

Define the mappings $q \to a(q)$, $q \to c(q)$ and $q \to G(q)$ from $P$ into $\mathbb{R}$, $\mathbb{R}$, and $L^1(\Omega)$, by (9.3), (9.4) and (9.5), respectively with $q^* \in P$ being replaced by $q = (\alpha, \beta, \delta) \in P$. These mappings are continuous by Theorem 7.1. Since $q^* \in int\ P$, for a given $\epsilon > 0$ there exists a convex neighborhood $U \subset int\, P$ of $q^*$ such that

$$|a(p) - a(q^*)| < \epsilon, \quad |c(p) - c(q^*)| < \epsilon, \quad \|G(p) - G(q^*)\|_{L^1(\Omega)} < \epsilon \quad \text{for } p \in U.$$

By the Mean-Value Theorem there exists $\tau \in (0,1)$ such that $J(q) - J(q^*) = DJ(q_\tau; q - q^*)$, where $q_\tau = q^* + \tau(q - q^*)$. If $q \in U$, then $q_\tau \in U$ by the convexity of $U$. Thus

$$\begin{aligned}
&|J(q) - J(q^*) - DJ(q^*)(q - q^*)| = |DJ(q_\tau; q - q^*) - DJ(q^*)(q - q^*)| \\
&\le \left( |a(q_\tau) - a(q^*)| + \int_\Omega |G(q_\tau) - G(q^*)|(x)\ dx + |c(q_\tau) - c(q^*)| \right) \|q - q^*\|_X \\
&< 3\epsilon \|q - q^*\|_X
\end{aligned}$$

for $q \in U$, and the result follows. □

A corollary of Theorem 9.3 is

**Theorem 9.4.** *Consider Klein-Gordon equation (1.1) with constant diffusion coefficients $\beta$. Let the admissible set be*

$$P = [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]$$

*with $\beta_{min} > 0$.*

*Let the objective function be defined by* $J(q) = \|y(q) - z_d\|_{L^2(0,T;H)}$. *Then the mapping* $q \to J(q)$ *from* $\mathrm{int}\, P \subset \mathbb{R}^3$ *into* $\mathbb{R}$ *is differentiable. Its gradient* $\nabla J(q) = (a(q), b(q), c(q))$, *where* $b(q) = \int_\Omega G(x; q)dx$, *and* $a(q), G(x; q), c(q)$ *are defined in (9.3), (9.5), and (9.4).*

Now assume that $q^* \in P$ is an optimal parameter for (1.5), that is

$$J(q^*) = \inf_{q \in P} J(q). \tag{9.8}$$

The necessary optimality condition for $q^*$ is $DJ(q^*; q - q^*) \geq 0$ for any $q \in P$. According to Theorem 9.1 it takes the form

$$(\alpha^* - \alpha)a(q^*) + \int_\Omega (\beta^*(x) - \beta(x))G(x; q^*)dx + (\delta^* - \delta)c(q^*) \geq 0 \tag{9.9}$$

for any $q = (\alpha, \beta, \delta) \in P$.

Let us analyze condition (9.9) for the optimal parameter $q^* \in P$, where

$$P = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times \mathcal{B} \times [\delta_{min}, \delta_{max}]\} \tag{9.10}$$

and

$$\mathcal{B} = \{\beta \in L^\infty(\Omega) \ : \ 0 < \nu \leq \beta(x) \leq \mu \ \ a.e. \ \text{ on } \ \Omega\} \tag{9.11}$$

for some positive constants $\nu$ and $\mu$.

Choose $q = (\alpha, \beta^*, \delta^*) \in P$. Then (9.9) becomes $(\alpha^* - \alpha)a(q^*) \geq 0$ for all $\alpha \in [\alpha_{min}, \alpha_{max}]$. If $\alpha^* \in (\alpha_{min}, \alpha_{max})$ then we must have $a(q^*) = 0$. If $a(q^*) > 0$ then $\alpha^* = \alpha_{max}$. If $a(q^*) < 0$ then $\alpha^* = \alpha_{min}$. The case $a(q^*) \neq 0$ can be compactly written as

$$\alpha^* = \frac{1}{2}\{\text{sign}(a(q^*)) + 1\}\alpha_{max} - \frac{1}{2}\{\text{sign}(a(q^*)) - 1\}\alpha_{min}. \tag{9.12}$$

Similarly to the previous case, if $\delta^* \in (\delta_{min}, \delta_{max})$ then we must have $c(q^*) = 0$. If $c(q^*) > 0$ then $\delta^* = \delta_{max}$. If $c(q^*) < 0$ then $\delta^* = \delta_{min}$. The case $c(q^*) \neq 0$ can be compactly written as

$$\delta^* = \frac{1}{2}\{\text{sign}(c(q^*)) + 1\}\delta_{max} - \frac{1}{2}\{\text{sign}(c(q^*)) - 1\}\delta_{min}. \tag{9.13}$$

Next we consider the implications for $\beta^* \in \mathcal{B}$.

Suppose that $\beta^* \in int\,\mathcal{B}$, i.e. $\nu < \operatorname{ess\,inf} \beta^*(x) \leq \operatorname{ess\,sup} \beta^*(x) < \mu$. Then for a sufficiently small $r > 0$ we have $\beta^* + \gamma \in \mathcal{B}$ for any $\gamma \in L^\infty(\Omega)$ with $\|\gamma\|_\infty \leq r$. Choose $q = (\alpha^*, \beta^*(x) - \gamma(x), \delta^*)$. Then (9.9) becomes

$$\int_\Omega \gamma(x) G(x; q^*) dx \geq 0.$$

Choosing $q = (\alpha^*, \beta^*(x) + \gamma(x), \delta^*)$ gives

$$\int_\Omega \gamma(x) G(x; q^*) dx \leq 0.$$

Thus

$$\int_\Omega \gamma(x) G(x; q^*) dx = 0. \tag{9.14}$$

for any $\gamma \in L^\infty(\Omega)$. We conclude that $\beta^* \in int\,\mathcal{B}$ implies $G(x; q^*) = 0$ a.e. in $\Omega$.

Let

$$\Omega_+ = \{x \in \Omega \ : \ G(x; q^*) > 0\}$$

defined up to a set of measure zero. Then we must have $\beta(x) = \mu$ for a.e. $x \in \Omega_+$.

Let

$$\Omega_- = \{x \in \Omega \ : \ G(x; q^*) < 0\}$$

defined up to a set of measure zero. Then we must have $\beta(x) = \nu$ for a.e. $x \in \Omega_-$. This analysis shows that the optimal coefficient $q^*$ satisfies a bang bang control law. Its other consequence is summarized in the following Theorem.

**Theorem 9.5.** *If the optimal coefficient $q^*$ is located in the interior $int\, P$ of the admissible set $P$, then*

$$a(q^*) = 0, \quad c(q^*) = 0, \quad \text{and} \quad G(x; q^*) = 0 \quad \text{a.e. in } \Omega.$$

In the case of constant diffusion coefficients $\beta$ in (1.1) the gradient $\nabla J(q) = (a(q), b(q), c(q))$ of the objective function was obtained in Theorem 9.3. Combining this result with Theorem 9.5 gives

**Theorem 9.6.** *Consider the Klein-Gordon equation (1.1) with constant diffusion coefficients $\beta$. Let the admissible set be*

$$P = [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]$$

*with $\beta_{min} > 0$. Let the objective function be defined by $J(q) = \|y(q) - z_d\|_{L^2(0,T;H)}$. If the parameter $q^* \in int\, P$ is optimal, then $\nabla J(q^*) = 0$.*

# Chapter 10

# Conclusions and Future Work

Because of its significance in physics, Klein-Gordon equation has been studied extensively from different perspectives. In the thesis, we study the optimization problem of a nonlinear damped Klein-Gordon equation with a variable diffusion coefficient. Our work is mainly based on the work of J.L. Lions (see [34], [35]), Roger Temam (see [48]), Semion Gutman (see [24],[21]), and Junhong Ha and Shin-ichi Nakagiri (see [26], [41]. We follow as in Gutman [24],[21] for sine-Gordon equation. Further difficulties in mathematical analysis arise from the unbounded nonlinear term $g(u) = |u|^\gamma u$, and the variable diffusion coefficient $\beta(x)$.

The summary of the research is as follows.

We studied the weak solution of the damped Klein-Gordon equation

$$\begin{aligned} &u'' + \alpha u' + A_\beta u + \delta g(u) = f, \quad \text{in } V' \quad a.e. \quad on \quad [0,T], \\ &u(0) = y_0 \in V, \quad u'(0) = y_1 \in H, \end{aligned}$$

where the nonlinear term is $g(u) = |u|^\gamma u$ with the constant $\gamma$ satisfying

$$\begin{cases} 0 \le \gamma < \infty & \text{if} \quad n = 1, 2, \\ 0 \le \gamma \le 2 & \text{if} \quad n = 3, \\ \gamma = 0 & \text{if} \quad n \ge 4. \end{cases}$$

We carefully studied the nonlinear term $g(u)$ for the different cases of $\gamma$, and derived its desirable properties, which are crucial to the entire work, as following:

(i) If $n = 1, 2, 3$ and $u \in V$, the we have the Sobolev inequality

$$\|u\|_{L^q(\Omega)} \leq C\|u\|,$$

where $1 \leq q \leq 2\gamma + 2$.

(ii) If $u \in V$, then $|g(u)| \leq C\|u\|^{\gamma+1}$ and $g : V \to H$ is locally Lipschitz with

$$|g(u) - g(v)| \leq C(1 + \|u\| + \|v\|)^{\gamma+1}\|u - v\|,$$

where $u, v \in V$.

(iii) If $\beta(x) \in \mathcal{B}$ and $u \in D(A)$, then $\|g(u)\| \leq C\|u\|^{\gamma}_{H^2(\Omega)}\|u\|$, where $C$ depends only on $\Omega$.

(iv) If $u_n \to u$ in $L^2(0, T; H)$ and $\|u_n(t)\|, \|u(t)\| \leq C$ for some constant $C$ and any $t \in [0, T]$, then $g(u_n) \rightharpoonup g(u)$ weakly in $L^2(0, T; H)$.

We proved existence and uniqueness of the weak solution of the Klein-Gordon equation by using energy estimates and standard Galerkin method. We obtained that, for $q \in P,\ y_0 \in V,\ y_1 \in H,\ f \in L^2(0, T; H)$, the unique solution $u(t) = u(t; q)$ satisfies $u \in C([0, T]; V) \cap W(0, T)$, $u' \in C([0, T]; H)$, and

$$\max_{0\leq t\leq T} \left(\|u(t)\|^2 + |u'(t)|^2\right) + \|u''(t)\|^2_{L^2(0,T;V')} \leq c(I + I^{\gamma+1}).$$

where $c$ is only depends on the bounds of the admissible set $P$, and

$$I = |y_1|^2 + \|y_0\|^2 + \|y_0\|^{\gamma+2} + \|f\|^2_{L^2(0,T;H)}.$$

We showed that the solution maps $q \to u(q)$: $P \to C([0,T];V)$ and $q \to u'(q)$: $P \to C([0,T];H)$ are continuous. We established the weak Gâteaux differentiability of the solution map and showed that the weak Gâteaux derivative $z = Du(q^*; q - q^*) \in L^2(0,T;H)$ at $q^* \in P$ in the direction $q - q^*$ is the unique weak solution of the problem

$$
\begin{aligned}
&z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + \delta^* g'(u(t;q^*))z(t) = f_0(t) \ t \in (0,T) \\
&z(0) = 0,\ z'(0) = 0,
\end{aligned}
$$

where $f_0(t) = (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*) + (\delta^* - \delta)g(u(t;q^*))$.

Then the Gâteaux differentiability of the objective function $J(q) = \|u(q) - z_d\|^2_{L^2(0,T;H)}$ is followed directly, and the Gâteaux derivative $DJ(q^*; q-q^*)$ of $J(q)$ at $q^*$ in the direction $q - q^*$ has the following representation

$$
DJ(q^*; q - q^*) = (\alpha^* - \alpha)a(q^*) + \int_\Omega (\beta^*(x) - \beta(x))G(x;q^*)dx + (\delta^* - \delta)c(q^*),
$$

where

$$
a(q^*) = 2\int_Q u_t(t,x;q^*)p(t,x;q^*)\ dxdt.
$$

$$
c(q^*) = 2\int_Q g(u(t,x;q^*))p(t,x;q^*)\ dxdt.
$$

and

$$
G(x;q^*) = 2\int_0^T \nabla u(t,x;q^*)\nabla p(t,x;q^*)dt, \quad x \in \Omega.
$$

Finally, we showed that the objective function $J(q)$ is Fréchet differentiable on $P$, which allow us conclude that if the optimal coefficient $q^*$ is located in the interior $int\, P$ of the admissible set $P$, then necessary conditions for the optimal set of parameters $q^* = (\alpha, \beta, \gamma) \in P$ minimizing the objective function $J(q)$ are

$$a(q^*) = 0, \quad c(q^*) = 0, \quad \text{and} \quad G(x; q^*) = 0 \quad \text{a.e. in } \Omega.$$

I plan to continue my research on nonlinear wave equations such as sine-Gordon equation and Klein-Gordon equation. These equations are examples of infinite dimensional dynamical systems. The goal is to investigate their solutions (trajectories). It is known that such trajectories can exhibit chaotic behaviors. A chaotic behavior of a dissipative dynamical system can be explained by the existence of a complicated attractor $\mathcal{A}$ to which the trajectories converge as $t \to \infty$. The plan is to study theoretical properties and computational methods for the solutions of these nonlinear equations.

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