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THIS DISSERTATION IS DEDICATED

TO

My sister

Julie Ann Mecham

Until we meet again.

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Chapter 1

Introduction

A group is said to be *as large as* F_2 [2], if it has a finite index subgroup which maps onto a free group of rank 2. Following Gromov [25], we will abbreviate this to *large*. In other words, a group is said to be *large* if some finite index subgroup has a non-abelian free homomorphic image.

There are other group theoretic properties, called large properties, that are related to the largeness of a group. A group theoretic property $\mathbb{P}$ is said to be a *large property* if it satisfies the following four conditions [22], [23] :

1. $\mathbb{P}$ is closed under taking pre-images;

2. if H is a (normal) subgroup of finite index in G, then G has $\mathbb{P}$ if and only if H has $\mathbb{P}$;

3. if G has a finite normal subgroup N, then G has $\mathbb{P}$ if and only if G/N has $\mathbb{P}$;

4. the trivial group does not have $\mathbb{P}$.

Being large is an example of a large property: other examples of large properties include:

- being SQ-universal;
- having super-exponential subgroup growth;
- having infinite first virtual Betti-number.

When a group is shown to be large, it can be shown that the group has these other large properties as well.

Large groups have been studied by many people including B. Baumslag and S. Pride [2], [3], G. Baumslag and D. Solitar, [5], and M. Edjvet [22]. More recently, ground breaking results were obtained by Marc Lackenby [30], [31], [32], [33], [35]. Jack Button [13], [14], [15], [17], [18], [16] has also produced several results that are related to the study of large groups.

1.1 SQ-universal Groups

A countable group is said to be *SQ-universal* if every countable group can be embedded in one of its quotient groups. The first known example of an SQ-universal group, F_2, was given by Higman, Neumann, and Neumann in their landmark paper on embedding theorems [29]. This paper introduced the HNN-extension construction which is now widely used.

In [41], B. Neumann showed that there are uncountably many non-isomorphic two generator groups. This result was also shown by Bowditch using quasi-geometric methods [8]. Only countably many different two generator groups can be embedded in a single quotient group of a group G, so one consequence of a group being SQ-universal is that it contains uncountably many non-isomorphic, infinite quotient groups.

It is easy to see that every large group is virtually SQ-universal. A lot more work is required to show that every virtually SQ-universal group is SQ-universal, [42]. Thus every large group is SQ-universal, but it is not true that every SQ-universal group is large. One example is the Higman group [28] with presentation,

$$H = \langle a, b, c, d \mid a^b = a^2,\, b^c = b^2,\, c^d = c^2,\, d^a = d^2 \rangle.$$

Schupp [44] used a particular property of the free product with amalgamation description of H,

$$\langle a, b, c \mid a^b = a^2,\, b^c = b^2 \rangle *_{\langle a,d \rangle} \langle b, c, d \mid b^c = b^2,\, c^d = c^2 \rangle,$$

which he called bounding pairs, to show that H is SQ-universal.

On the other hand, Higman showed that this group has no proper finite index subgroups. The proof of this relied on the following fact from number theory: *If n is an integer greater than 1, the least prime factor of n is less than the least prime factor of $2^n - 1$.* Using this fact, he showed that the only way for the relations of H to hold in a finite group was for all the generators to be the identity element. If one considers the group with analogous defining relations and $m > 4$ generators, a similar argument will show that this group also has no proper finite index subgroups.

Suppose now that H is large. Since H has no finite index subgroups, there must exist a map $H \twoheadrightarrow F_2$. However, $F_2 \twoheadrightarrow \mathbb{Z}_2$, and the composition of these maps gives a finite quotient of H, a contradiction.

1.2 Subgroup Growth

The *subgroup growth function* of a group G [37] is defined to be the function $n \mapsto a_n(G)$ where $a_n(G)$ is the number of subgroups of index n in G. The study of subgroup growth can be seen as an attempt to arrange all residually finite groups into a spectrum. Recall that a group is said to be *residually finite* if and only if the intersection of all it's finite index subgroups is the trivial group. To be residually finite therefore can be interpreted to mean that there are enough finite index subgroups that their intersection is trivial. Some residually finite groups, those with slow growth, will have only just enough subgroups while those with very fast subgroup growth will have more than enough.

Free groups have the fastest rate of subgroup growth. In the first modern paper to deal with subgroup counting, Hall [26] gave a recursive formula for the subgroup growth of non-abelian free groups. A corollary to this theorem given in Chapter 2 of [37] is that for a free group F on $d \geq 2$ generators, $a_n(F) \geq n!^{d-1}$ for every n. Since large groups have finite index subgroups that map onto free groups, they also have a super-exponential subgroup growth rate.

One can also consider normal subgroup growth (the number of normal subgroups of a certain index). In [14], Button gives a formula for the normal subgroup growth of Baumslag-Solitar groups $BS(p,q)$ when p and q are relatively prime. This formula is of interest since it can be used to distinguish different Baumslag-Solitar groups whereas the regular subgroup growth formula can not distinguish them.

1.3 Infinite Virtual First Betti-Numbers

The first Betti-number of group G, $b_1(G)$, is defined to be the rank of the first homology group or abelianization of G. The *virtual first Betti-number* of a group G is defined as $vb_1(G) = \sup\{b_1(H)\colon H \text{ is a finite index subgroup of } G\}$. A group G which has infinite virtual first Betti-number has finite index subgroups with arbitrarily high rank abelianizations. Since F_2 has infinite first virtual Betti-number, large groups also have infinite first virtual Betti-numbers. However, the converse is not true. One well known example is the wreath product, $\mathbb{Z} \wr \mathbb{Z}$. This group has the following presentation:

$$\langle \ldots, a_{-1}, a_0, a_1, \ldots, t \mid t a_i t^{-1} = a_{i+1},\ [a_i, a_j];\ i, j \in \mathbb{Z} \rangle$$

which is a semi-direct product of an infinite sum of cyclic groups, generated by $\{a_i\}_{i \in \mathbb{Z}}$, with the cyclic group generated by t. This group is finitely generated with presentation

$$\langle a_0, t \mid [t^i a_0 t^{-i}, t^j a_0 t^{-j}];\ i, j \in \mathbb{Z} \rangle,$$

but it is not finitely presentable [4]. The subgroup generated by $a_0, a_1, \ldots, a_k, t^{k+1}$ is index $(k+1)$ in $\mathbb{Z} \wr \mathbb{Z}$ and has abelianization $\mathbb{Z}^{k+2}$. Thus $\mathbb{Z} \wr \mathbb{Z}$ has infinite virtual first Betti-number.

To see that $\mathbb{Z} \wr \mathbb{Z}$ is not large we consider the short exact sequence

$$1 \to \bigoplus_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z} \wr \mathbb{Z} \to \mathbb{Z} \to 1. \tag{1.1}$$

Let σ be the map from $\mathbb{Z} \wr \mathbb{Z}$ onto $\mathbb{Z}$. If $\mathbb{Z} \wr \mathbb{Z}$ is large it will contain a free group of rank 2 as a subgroup. F_2 cannot be contained in the kernel of σ since this kernel is abelian. Let K be the kernel of the map $\sigma|_{F_2} : F_2 \to \mathbb{Z}$. Now K is contained in $ker(\sigma)$, but K is free of infinite rank and cannot be contained in an abelian group.

There are many conjectures from the study of 3-manifolds that are closely related to large groups and virtual first Betti-numbers, one of which is the following [34]:

Conjecture 1.1. *Let $G = \pi_1(M)$ for M a finite volume hyperbolic 3-manifold. Then there is a finite index subgroup G' of G such that :*

1. *G' is a free product with amalgamation or an HNN extension;*

2. *$b_1(G') > 0$;*

3. *$b_1(G')$ is arbitrarily large;*

4. *G' has a free non-abelian quotient.*

This is still an open question for closed hyperbolic manifolds but in [20], Cooper, Long, and Reid showed that if M is a finite volume hyperbolic 3-manifold with non-empty boundary, then $\pi_1(M)$ is large.

1.4 Recent Developments

Traditional coarse geometric techniques do not interact well with the theory of finite index subgroups for the following reasons:

1. Any finite index subgroup of a group G is quasi-isometric to G, so we can't distinguish a group from any of its finite index subgroups.

2. (Burger-Mozes)[12] There exist quasi-isometric groups G and G' such that

(a) G is simple (has no proper, finite index subgroups), and G is not large;

(b) G' is residually finite and large, indeed, G' is a direct product of finite rank free groups.

As we will see, there are still many geometric, topological, and algebraic methods that can be used to obtain largeness results. The deficiency of a group or a group presentation plays an important role in many of the following results concerning large groups.

Definition 1.1. Let G be a group with a finite presentation $\mathcal{P}$ of g generators and r relators. The *deficiency of* $\mathcal{P}$*,* $def(\mathcal{P})$*,* is defined to be $(g-r)$, and the *deficiency of* G*,* $def(G)$*,* is the maximum deficiency of all presentations $\mathcal{P}$ of G.

Remark 1.2. Other sources such as [11] use $(r-g)$ to define deficiency. In this case the deficiency of a group would be the minimum deficiency of all presentations for the group.

Although the deficiency of a presentation is very simple to calculate, the deficiency of a group is not; it is however bounded above by the torsion free rank of the abelianization of the group. Bieri gave the following corollary to his Theorem 1 in [7]: *If* G *contains a finitely generated normal subgroup* N *with* G/N *infinite abelian, then* $def(G) \leq 1$.

In [3] Baumslag and Pride proved the following theorem relating largeness to the deficiency of a group presentation.

Theorem 1.3. *Suppose that a group* G *admits a presentation with* $n \geq 2$ *generators and at most* $n-2$ *relations. Then* G *is large.*

They also conjectured that a group with a deficiency 1 presentation such that one of the relations is a proper power is large [2]. This conjecture was proved independently by Gromov [25] using bounded cohomology and by Stöhr [47] using an algebraic argument.

In 1984, Edjvet proved the following two theorems about groups with deficiency 0 presentations:

Theorem 1.4. *[22] If G has a deficiency 0 presentation in which two relators are the p-th powers for some prime p and the abelianization of G is infinite, then G is large.*

Theorem 1.5. *[22] If the abelianization of G is finite, and G has a deficiency 0 presentation in which either three relators are proper powers, or two relators are proper powers with at least one being a third or higher power, then G is large.*

More recently, Marc Lackenby [30] gave a characterization of finitely presented large groups in terms of the existence of certain normal series in G.

Theorem 1.6 (Lackenby)**.** *Let G be a finitely presented group. Then the following are equivalent:*

1. *some finite index subgroup of G admits a surjective homomorphism onto a non-abelian free group;*
2. *there exists a sequence $G_1 \geq G_2 \geq \dots$ of finite index subgroups of G, each normal in G_1, such that*
   (a) *G_i/G_{i+1} is abelian for all $i \geq 1$;*
   (b) *$\lim_{i\to\infty}((\log[G_i : G_{i+1}])/[G : G_i]) = \infty$;*
   (c) *$\limsup_i(d(G_i/G_{i+1})/[G : G_i]) > 0$ where $d(\star)$ is the minimum number of generators for the group.*

The geometry and topology of finite Cayley graphs play a central role in the proof. Theorem 1.3 and the conjecture of Baumslag and Pride are both straightforward consequences of Theorem 1.6.

Another theorem of Lackenby's gave conditions under which relations could be added to a large group, resulting in another large group [32].

Theorem 1.7 (Lackenby)**.** *Let G be a finitely generated, large group and let $g_1, \ldots, g_r$ be a collection of elements of G. Then for infinitely many integers n, $G/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is also large.*

Theorem 1.7 has interesting applications to Dehn surgery on 3-manifolds and to Conjecture 1.1. The proof of the theorem uses deep theory related to property (τ) [38] and homology growth developed by Lackenby in his previous papers. Olshanskii and Osin [43] used Theorem 1.3 to give a short, group theoretic proof of this result.

In [33], Lackenby provided some new methods for detecting large groups. These methods are related to group properties such as profinite and pro-p completions, first L_2-Betti-number, and the existence of certain finite index subgroups.

Jack Button has also been very active in this area of late. Besides his work mentioned in Section 1.2, he gave the following theorems in [16] for subgroup separable (LERF) groups. A group is *subgroup separable* if every finitely generated subgroup is an intersection of finite index subgroups.

Theorem 1.8 (Button)**.** *If G is LERF and of deficiency 1 then either G is large or G is of the form $F_n \rtimes \mathbb{Z}$.*

Theorem 1.9 (Button)**.** *If G is LERF, of deficiency 1 and has a finite index subgroup with first Betti-number at least 2 then G is large or $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.*

In the same paper, he gave the following corollary to his Theorem 4.1 which gives a largeness result for certain deficiency 1 groups.

Corollary 1.10. *If $G = \langle a, t \mid w \rangle$ where w has t-exponent sum equal to 1, then G is large if and only if there exists a finite index subgroup H of G such that the minimum number of generators of the abelianization of H is as least 3.*

Remark 1.11. The generalized Baumslag-Solitar groups described in Subsection 2.2.2 all have presentations of deficiency 1. There is however, very little overlap with these groups and groups which are LERF. See [39].

In [18], Button gave the following result for free-by-cyclic groups.

Theorem 1.12. *If G is a finitely generated group which is F-by-$\mathbb{Z}$ for F free, then G is large if F is infinitely generated or if G contains $\mathbb{Z} \times \mathbb{Z}$, with the sole exception of $F = \mathbb{Z}$ and $G = \langle x, y \mid xyx^{-1} = y^{\pm 1} \rangle$.*

Thus any non-hyperbolic free-by-cyclic group, with the noted exceptions, is large.

In [18], he also gave an algorithm for determining the largeness of any finitely presented group. The algorithm takes a finite presentation as input and converts relevant criteria into a statement about the Alexander polynomial of the group. The algorithm will terminate with an answer yes if and only if the group is large. If the group is not large the algorithm will generally not terminate, however the algorithm has a particularly nice form for groups of deficiency 1. In [17] he gave some tables of outcomes for specific 1-relator group presentations which have been generated by a program written in Magma.

1.5 Examples

Before we consider graphs of abelian groups, let's examine some examples of large groups. We'll begin with right-angled Artin groups [6]. Presentations of these groups are fairly easy to understand, and it is easy to determine which are large.

Example 1. Let Λ be a graph on n vertices labeled 1 through n. Such a graph will be associated to a right-angled Artin group, $A(\Lambda)$ with generators $x_1, x_2, \ldots, x_n$ and relations $x_i x_j = x_j x_i$ when the vertices i and j are connected by an edge in Λ. In Figure 1.1 we see an example of such a graph Λ. The group $A(\Lambda)$ has presentation $\langle x_1, x_2, x_3 \,|\, x_1x_2 = x_2x_1,\, x_2x_3 = x_3x_2\rangle$. The map $f : A(\Lambda) \to F_{a,b}$ defined by $x_1 \mapsto a$, $x_2 \mapsto 1$, and $x_3 \mapsto b$, is a surjective homomorphism.

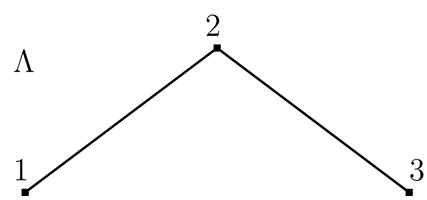


Figure 1.1: This graph represents a right-angled Artin group $A(\Lambda)$.

If the graph Λ has no edges, $A(\Lambda)$ is a free group on n generators. If Λ is complete, $A(\Lambda)$ is the free abelian group $\mathbb{Z}^n$. It is not difficult to see that if the graph Λ has 2 or more vertices and is not complete, then the map defined by sending any two non-adjacent generators of $A(\Lambda)$ to the generators of F_2 and all other generators of $A(\Lambda)$ to the identity element defines a homomorphism onto F_2. Hence a right-angled Artin group $A(\Lambda)$ is large if and only if Λ has 2 or more vertices and is not a complete graph.

Example 2. For our second example, we'll look at a group with first Betti-number 0 which is also large. Consider the free product of two copies of the alternating group A_5 of order 60. The group, A_5, is a perfect group, so $b_1(A_5) = 0$. It is also simple so it has no non-trivial subgroups. The free product $A_5 * A_5$ also has first Betti-number zero, but as we shall see, it is large and therefore has infinite first virtual Betti-number. Let X be a space such that $\pi_1(X, x) = A_5$. Let Y be obtained from the disjoint union $(X_1 \amalg X_2 \amalg [0, 1])$ by identifying 0 with the basepoint $x_1 \in X_1$ and 1 with the basepoint $x_2 \in X_2$. Then $\pi_1(Y) = A_5 * A_5$. Consider the map $f : A_5 * A_5 \twoheadrightarrow A_5$ which is an isomorphism of each factor of the free product. Let K be the kernel of this map. K is an index 60 subgroup of $A_5 * A_5$ and corresponds to a 60-fold cover Y' of Y. See Figure 1.2. Y' is composed of simply-connected covers of X_1 and X_2 with 60 copies of the interval $[0, 1]$ connecting pre-images of the basepoints x_1 and x_2. Since the X_i' are simply connected, $K \cong \pi_1(Y') \cong F_{59}$, so $b_1(K) = 59$, and $A_5 * A_5$ is large.

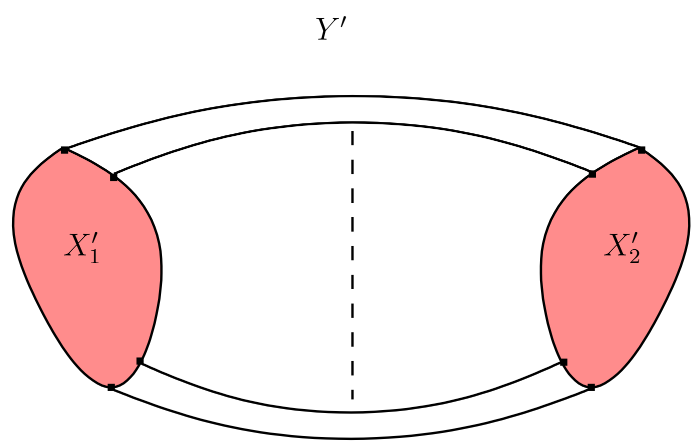


Figure 1.2: The 60-fold cover Y' of Y. $\pi_1(Y') \cong F_{59}$.

Chapter 2

Background and Definitions

Informally, a graph of groups is a collection of groups which is indexed by the vertex and edge sets of some graph. Additional information, injective group homomorphisms from edge groups to adjacent vertex groups, tell us how to build up a group from the collection of edge and vertex groups. This group which is built is called the fundamental group of the graph of groups. See Definition 2.3.

There is a natural epimorphism from the fundamental group of a graph of groups to the fundamental group of its underlying graph. Thus if the rank of the underlying graph is greater than or equal to 2, the fundamental group of the graph of groups is easily seen to be large.

Suppose we are given a graph of groups where the underlying graph has rank less than 2. The strategy will be to study finite index subgroups of the fundamental group of the given graph of groups. This subgroup will inherit a graph of groups structure, and if we are careful, it is sometimes possible to ensure that the rank of the underlying graph will be at least 2.

Before we do this, we need to review the notion of a graph of groups and to study the graph of groups structure inherited by finite index subgroups.

2.1 Graphs of Groups

The following definitions are taken from Scott and Wall's article [45] and use notation introduced by Serre [46].

Definition 2.1. A graph Γ is a pair of sets $(V(\Gamma), E(\Gamma))$ with maps $\partial_0, \partial_1 : E(\Gamma) \to V(\Gamma)$ and an involution $e \mapsto \bar{e}$ (for $e \in E(\Gamma)$), such that $\partial_i(e) = \partial_{1-i}(\bar{e})$ and $e \neq \bar{e}$ for all e. An element $e \in E(\Gamma)$ is to be thought of as an oriented edge with initial vertex $\partial_0(e)$ and terminal vertex $\partial_1(e)$. We denote by $E_0(v)$ the set of all edges having initial vertex v.

We define $\partial_1(e) = \partial_0(\bar{e})$ and say that e joins $\partial_0(e)$ to $\partial_1(e)$. An edge e is a loop if $\partial_0(e) = \partial_1(e)$.

The geometric realization of Γ, denoted by $|\Gamma|$, is a 1-dimensional CW-complex [27]. The 0-skeleton of the complex is in one to one correspondence with $V(\Gamma)$ and the 1-cells correspond to pairs $(e, \bar{e})$. When we say that Γ has some topological property, we mean that $|\Gamma|$ has that property.

Definition 2.2 (Graph of Groups)**.** [Scott-Wall [45]] A *graph of groups*, $\mathcal{G}$, consists of an abstract graph Γ, which we will assume to be connected, together with a function $\mathcal{G}$ assigning to each vertex v of Γ a group G_v and to each edge e a group G_e with $G_e = G_{\bar{e}}$, and an injective homomorphism $\sigma_e : G_e \to G_{\partial_0(e)}$.

We further write $G_{v/e} = G_v/\sigma_e(G_e)$ when $\partial_0(e) = v$ to represent the (left) coset space of a vertex group by the image of an edge group, and we define $i(e) = [G_v : \sigma_e(G_e)]$.

It is important to note that groups in general do not have a unique graph of groups representation, and non-isomorphic groups can have the same underlying graph Γ. See Figure 2.1. The underlying graphs (a) and (b) in the figure are

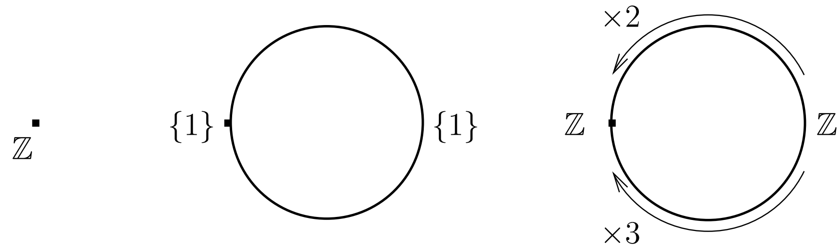


Figure 2.1: Examples of graphs of groups.

different, but the fundamental group of each graph of groups is $\mathbb{Z}$. Graph (c) has the same underlying graph as (b), but the fundamental group of this graph of groups is the Baumslag-Solitar group $BS(2,3) \ncong \mathbb{Z}$.

We will consider the underlying graph Γ of a graph of groups $\mathcal{G}$ to be an *edge-labeled graph* where the injective homomorphism, σ_e, is assigned to each edge e in Γ. Thus, for graph (c) in Figure 2.1, the edge labels are $\sigma_e : \mathbb{Z} \to \mathbb{Z}$ where $x \mapsto 2x$, and $\sigma_{\bar{e}} : \mathbb{Z} \to \mathbb{Z}$ where $x \mapsto 3x$.

Given a graph of groups $\mathcal{G}$ we can define a graph $\mathcal{X}$ of topological spaces (with preferred basepoints). $\mathcal{X}$ will have the same underlying graph Γ as $\mathcal{G}$, and $\mathcal{X}$ will assign to each vertex v and each edge e connected $K(G,1)$ spaces X_v and X_e such that $\pi_1(X_v) = G_v$ and $\pi_1(X_e) = G_e$. There exist basepoint preserving continuous maps $f_e : X_e \to X_{\partial_0(e)}$, which will induce the injective homomorphisms $(f_e)_* = \sigma_e$. The total space X_Γ of a given graph $\mathcal{X}$ can be defined as the quotient of

$$\bigcup \{X_v : v \in V(\Gamma)\} \cup \bigcup \{X_e \times I : e \in E(\Gamma)\}$$

by the identifications:

$$X_e \times I \to X_{\bar{e}} \times I \text{ by } (x,t) \mapsto (x, 1-t)$$

and

$$X_e \times 0 \to X_{\partial_0}(e) \text{ by } (x, 0) \mapsto f_e(x).$$

Definition 2.3. [Scott-Wall [45]] The fundamental group G_Γ of a graph of groups $\mathcal{G}$ is defined to be the fundamental group of the total space X_Γ.

Remark 2.4. Scott and Wall showed that the fundamental group of a graph of groups does not depend upon the choice of $K(G, 1)$ spaces or injective maps f_e since the total space will be determined up to homotopy, and thus the fundamental group is unique up to isomorphism.

Remark 2.5. There is an algebraic formulation for the fundamental group of a graph of groups [46], but since our methods of proof rely on topological properties, we will use this topological definition.

Remark 2.6. There is a map $r : X_\Gamma \twoheadrightarrow \Gamma$ which is a retraction of the total space onto the underlying graph. The map r restricted to a vertex space will be the constant map taking the entire vertex space to its basepoint. When restricted to $X_e \times I$, it will be defined by $(x, t) \mapsto (x', t)$ where x' is the basepoint of X_e. The points $(x, 0)$ and $(x, 1)$ are identified with the basepoints of their respective vertex spaces, so this map is well defined and the following diagram commutes.

$$\begin{array}{ccc} X_\Gamma & \xrightarrow{\;r\;} & \Gamma \\ {\scriptstyle\iota}\uparrow & \nearrow_{id_\Gamma} & \\ \Gamma & & \end{array}$$

Since $r \circ \iota = id_\Gamma$, we see that r is a retraction onto Γ. On the level of groups we get

$$\begin{array}{ccc} \pi_1(X_\Gamma) & \xrightarrow{r_*} & \pi_1(\Gamma) \\ \iota_* \big\uparrow & \nearrow_{id_{\pi_1(\Gamma)}} & \\ \Gamma & & \end{array}$$

which also commute, so there is a surjection from the fundamental group of the graph of groups to the fundamental group of the underlying graph. If the underlying graph has rank greater than one, then the fundamental group of the graph of groups is large.

2.2 Examples

In this section we describe the two main types of graphs of abelian groups that we investigate in this thesis. In Subsection 2.2.1 the edge groups and vertex groups are all finitely generated infinite abelian, and in Subsection 2.2.2 the edge and vertex groups are all cyclic. There is some overlap between these types of graphs of groups, but we can get more complete results in the cyclic groups case.

2.2.1 Graphs of Finitely Generated Infinite Abelian Groups

A finitely generated infinite abelian group G is the direct sum of a free abelian group $\mathbb{Z}^k$ of finite rank k and a finite abelian group T, called the torsion subgroup of G. The first Betti-number of G, $b_1(\mathbb{Z}^k \oplus T) = k$ is also called the free abelian rank of G.

For now, let us assume that $i(e) < \infty$ for all $e \in \mathcal{G}$, a graph of infinite abelian groups. In this case, the free abelian rank of all the edge and vertex groups must be the same. This is a consequence of the the fact that any finite index subgroup of the free abelian group of finite rank k, also has rank k. In this thesis we will focus on graphs of finitely generated free abelian groups. In this case, the injective homomorphisms from the edge groups to the vertex groups $\sigma_e : \mathbb{Z}^k \to \mathbb{Z}^k$ can be represented by $(k \times k)$ matrices. Figure 2.2 shows a graph of $\mathbb{Z}^2$ groups. It is indexed by (2×2) matrices which indicate the image of the generators of the edge $\mathbb{Z}^2$ in the vertex groups. The fundamental group of this graph of groups is:

$$\langle x_1, y_1, x_2, y_2 \,|\, [x_1, y_1] = [x_2, y_2] = 1,\ x_1^2 = x_2,\ x_1 y_1 = y_2^3 \rangle.$$

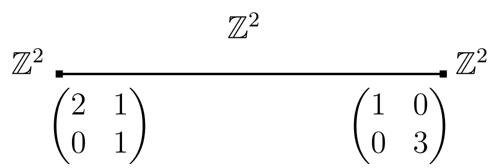


Figure 2.2: A graph of $\mathbb{Z}^2$ groups.

Graphs of infinite abelian groups with $i(e) = \infty$ for some edge e will be discussed in more detail in Chapter 6. Examples of these groups include tubular groups which include graphs of groups where the vertex groups are $\mathbb{Z}^2$ and the edge groups are $\mathbb{Z}$. Tubular groups were introduced in [9] and were further investigated in [10] and [19].

2.2.2 Graphs of Cyclic Groups

In this section we will discuss graphs of finite cyclic groups and infinite cyclic groups. If our vertex groups are all finite cyclic groups, then the edge groups must also be finite cyclic. Since the edge groups inject into the vertex groups, if $e \in E_0(v)$, then $|G_e|$ divides $|G_v|$. Figure 2.3 shows a graph of finite cyclic groups with presentation:

$$\langle a, b, c \mid a^6 = b^{10} = c^{15} = 1,\ a^3 = b^5,\ b^2 = c^3 \rangle.$$

If a vertex group is infinite cyclic, then any edge adjacent to that vertex must either represent an infinite cyclic group, in which case $i(e) < \infty$, or the trivial group, in which case $i(e) = \infty$. We may also have graphs of cyclic groups which have a mix of finite and infinite cyclic groups for vertex and edge groups.

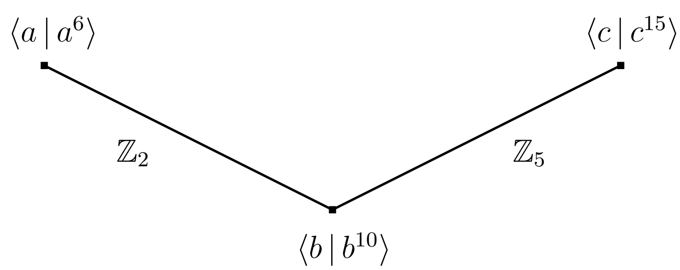


Figure 2.3: A graph of finite cyclic groups.

Graphs of infinite cyclic groups are a well studied subclass of the groups introduced in subsection 2.2.1. These are called Generalized Baumslag-Solitar groups.

Definition 2.7 (Generalized Baumslag-Solitar Group)**.** A Generalized Baumslag-Solitar, (GBS) group is a group which admits a graph of groups representation where all vertex groups and edge groups are infinite cyclic. We will call such a graph of groups a GBS-graph.

This is equivalent to other definitions of GBS groups given in [24] and [36]. These references define GBS groups as arising from G-trees whose vertex and edge groups are all infinite cyclic. The quotient graph of such a G-tree will result

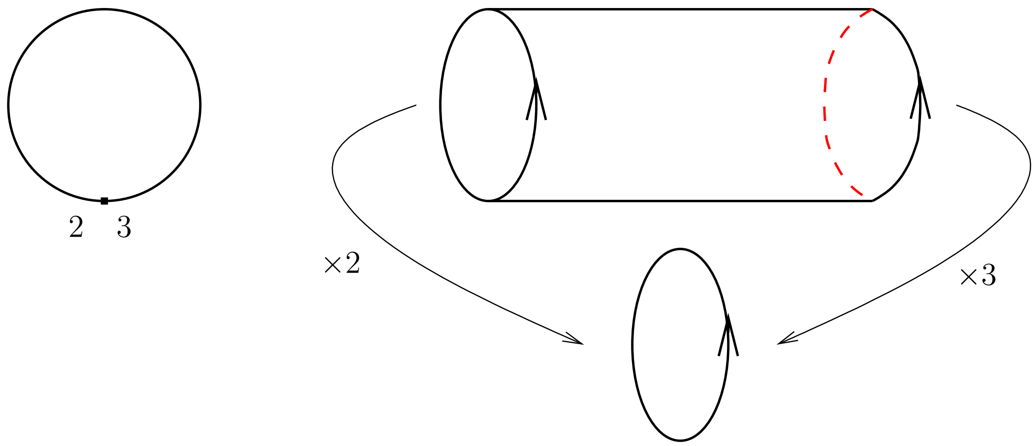


Figure 2.4: The graph of groups and total space of the Baumslag-Solitar group $BS(2,3)$.

in a graph of groups as described in Definition 2.7. Well known examples of GBS groups include Baumslag-Solitar groups [5], torus knot and link groups, and finite index subgroups of these groups. See Figures 2.4 and 2.5.

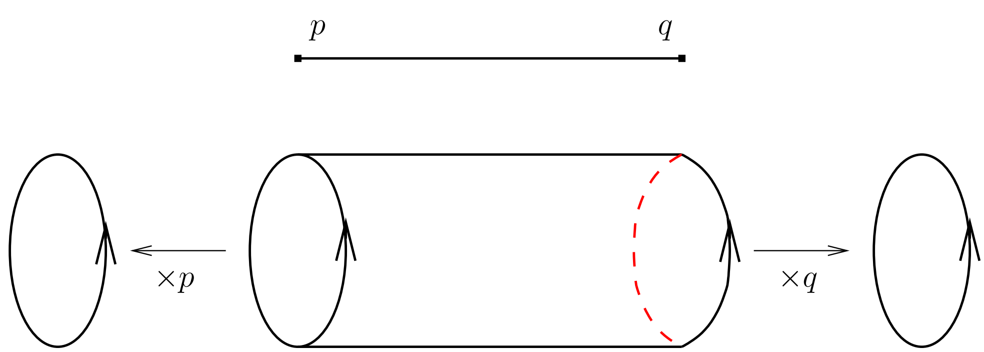


Figure 2.5: The graph of groups and total space of the torus knot group $T_{(p,q)}$.

If a graph of groups is known to be a GBS-graph, we know that $G_v = \mathbb{Z}$ for all $v \in V(\Gamma)$ and $G_e = \mathbb{Z}$ for all $e \in E(\Gamma)$, so we need no special encoding on Γ for the groups. The remaining information, namely the injective homomorphisms from the edge groups into the vertex groups, will be given as non-zero integers

(1 × 1 matrices), labeling the initial and terminal ends of each edge e chosen from the pairs $(e, \bar{e})$ of the graph of groups. Since vertex and edge groups are infinite cyclic, a label of m indicates the $\times m$ map. In this case, the graph Γ is the underlying graph of a graph of spaces, and the label m is the degree of the map from the edge circle onto the vertex circle. A graph of groups representation and the total space for $BS(2,3)$ are shown in Figure 2.4.

As stated before, a graph of groups representation need not be unique. In the case of GBS groups, the graph of groups description can be manipulated using what are called *elementary moves.* These moves, called *collapse, expansion, slide*, and *induction*, are described in [24] .

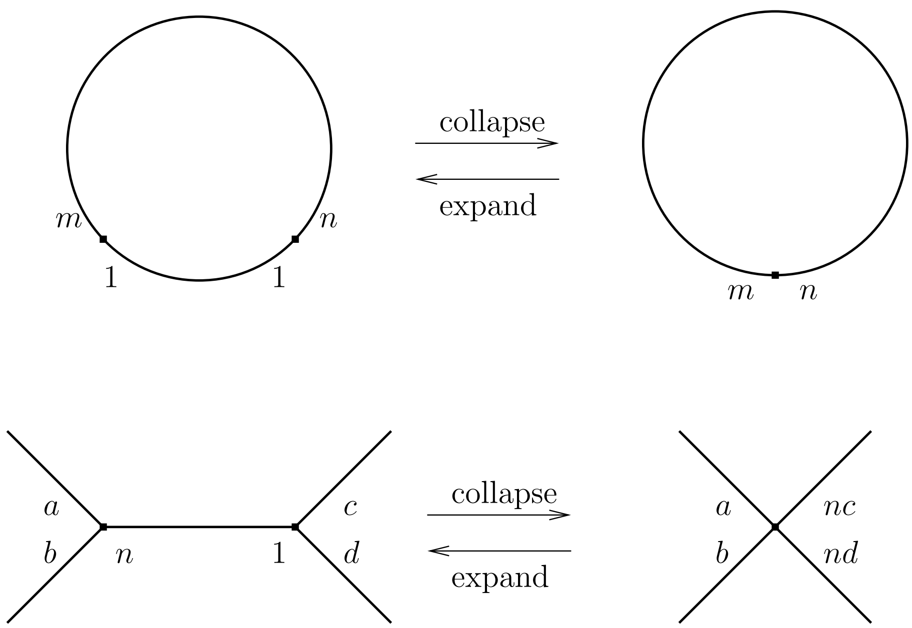


Figure 2.6: Expansion and collapse moves are shown for GBS graphs.

In [24], it is shown that these moves do not change the homotopy type of the total space, and hence have no effect on the fundamental group of the graph of groups. The collapse and expansion moves will be useful in proving some of the results that follow. They are shown in their most used forms in Figure 2.6. The collapse of an edge labeled on both ends by a 1 is equivalent to collapsing a cylinder in the total space, which will not change the homotopy type, so the process is reversible and an expansion move adds a collapsible cylinder to the total space. Changing the signs of all labels at a single vertex simultaneously or changing the sign on both labels of a single edge also preserves homotopy type.

2.3 Half-edges and Half-spaces

To facilitate the proofs in the following chapters, we will need the notion of open half-spaces for a graph of groups. These open half-spaces will be defined by removing either a single vertex or a pair of vertices from the underlying graph, the removal of which will separate the graph into disjoint components.

Recall that $|\Gamma|$ is a 1-dimensional CW-complex with the 0-skeleton of the complex in one to one correspondence with $V(\Gamma)$ and the 1-skeleton corresponding to pairs $(e, \bar{e})$ in $E(\Gamma)$. The characteristic map of a 1-cell e of $|\Gamma|$, $\chi_e : I_e \to |\Gamma|$ extends the attaching map of the 1-cell and is a homeomorphism from the open interval onto the interior of each edge of $|\Gamma|$. We will use these characteristic maps and the barycenter of the unit interval to define open half-edges in $|\Gamma|$.

Definition 2.8 (Open Half-edge)**.** Let Γ be the underlying (labeled) graph for a graph of groups $\mathcal{G}$. For an edge $e \in E(\Gamma)$ an *open half-edge of* Γ is defined to be $\chi_e((0, \frac{1}{2})) \subset |\Gamma|$. We will use the edge label σ_e to refer to $\chi_e((0, \frac{1}{2}))$ and $|\sigma_e| = [G_{\partial_0(e)} : \sigma_e(G_e)]$.

Remark 2.9. We will use the symbol v_{σ_e} to refer to the vertex in $V(\Gamma)$ which is in the closure of σ_e. Two half-edges σ_{e_1} and σ_{e_2} will be said to be *adjacent* if $v_{\sigma_{e_1}} = v_{\sigma_{e_2}}$.

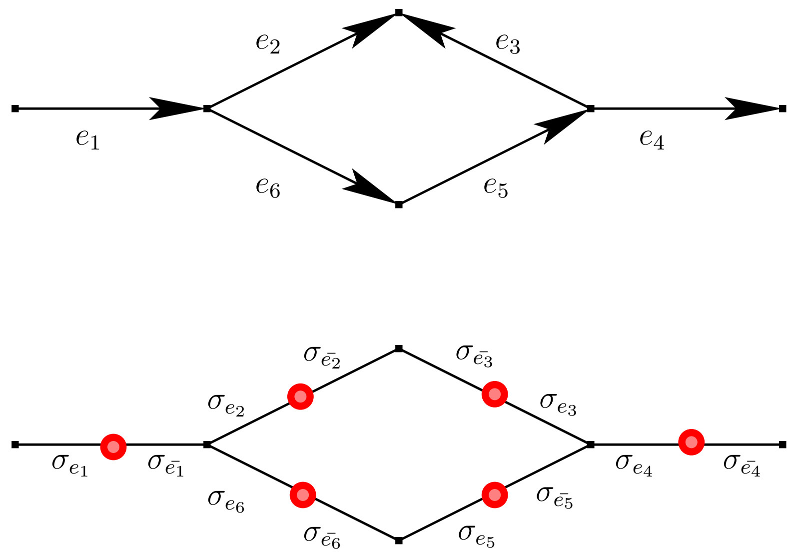


Figure 2.7: A graph with oriented edges and the same graph showing the edge labels representing the open half-edges of the graph. The large circles are the barycentric subdivision of the graph.

In the definition of a graph we used the symbol $E_0(v)$ to represent all edges with initial vertex v. We denote by $Star(v)$ or $Star^{\Gamma}(v)$ the set of all open half-edges σ_e which contain v in their closure, so σ_{e_1} is adjacent to σ_{e_2} if and only if $\sigma_{e_1} \in Star(v_{\sigma_{e_2}})$.

Definition 2.10 (Index of a vertex)**.** For a vertex $v \in \Gamma$ we define *the index of* v to be:

$$i(v) = \sum_{\sigma_e \in Star(v)} |\sigma_e|. \tag{2.1}$$

An open half-edge σ_e will be called *separating* if $\Gamma \setminus \sigma_e$ is disconnected; otherwise we will call σ_e *non-separating*. An open half-edge that contains a valence 1 vertex in its closure will be called a *terminal half-edge*.

Remark 2.11. When working with GBS graphs, each half-edge will be labeled by a non-zero integer m. The symbol $|m|$ will be the absolute value of m since this is the index of $m\mathbb{Z}$ in $\mathbb{Z}$. Likewise, when working with graphs of $\mathbb{Z}^k$ groups, each open half-edge will be labeled by a $(k \times k)$ matrix $\mathbf{H}$, and $|\mathbf{H}|$ will be $|det(\mathbf{H})|$.

The notion of a half-edge is in defining a half-space for a graph. Since there are two types of half-edges, we will have two types of half-spaces. Type one will be defined for a single separating half-edge, and type two will be defined for a non-separating half-edge with respect to another non-separating half-edge.

Definition 2.12 (Half-Space)**.** For an open half-edge $\sigma_e \subset |\Gamma|$ with the property that $|\Gamma| \setminus \sigma_e$ is disconnected, i.e. σ_e is a separating half-edge, we define the *open half-space of* σ_e, $\mathcal{H}_{\sigma_e}$, to be the connected component of $|\Gamma| \setminus \{v_{\sigma_e}\}$ which contains σ_e. $\overline{\mathcal{H}_{\sigma_e}}$ will denote the closure and $\mathcal{H}^c_{\sigma_e}$ will denote the complement.

Figure 2.8 shows an open half-space for a GBS-graph. The open half-space of m, $\mathcal{H}_m$, is shown in bold. Note that $\mathcal{H}_p$ and $\mathcal{H}_q$ are both contained in $\mathcal{H}_m$ while $\mathcal{H}_r$ and $\mathcal{H}_s$ have no intersection with $\mathcal{H}_m$. $\mathcal{H}_n$ has a non-trivial intersection with $\mathcal{H}_m$, but $\mathcal{H}_m \cap \mathcal{H}_n$ is not itself an open half-space.

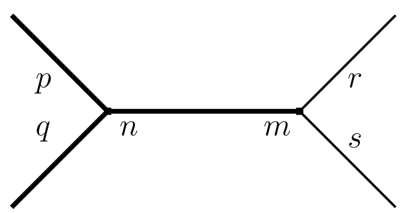


Figure 2.8: The open half-space of m, $\mathcal{H}_m$, for GBS-graph is shown in bold.

For any pair of open half-edges σ_{e_1} and σ_{e_2} of $|\Gamma|$ with the property that both $|\Gamma| \setminus \sigma_{e_1}$ and $|\Gamma| \setminus \sigma_{e_2}$ are connected but $|\Gamma| \setminus (\sigma_{e_1} \cup \sigma_{e_2})$ is not connected, we define the *open half-space of* σ_{e_1} *with respect to* σ_{e_2}, $\mathcal{H}_{\sigma_{e_1},\sigma_{e_2}}$, to be the connected component of $|\Gamma| \setminus \{v_{\sigma_{e_1}}, v_{\sigma_{e_2}}\}$ which contains σ_{e_1}. In the case that $\sigma_{e_2} \subset \mathcal{H}_{\sigma_{e_1},\sigma_{e_2}}$, we have $\mathcal{H}_{\sigma_{e_1},\sigma_{e_2}} = \mathcal{H}_{\sigma_{e_2},\sigma_{e_1}}$. As above, $\overline{\mathcal{H}_{\sigma_{e_1},\sigma_{e_2}}}$ will denote the closure, and $\mathcal{H}^c_{\sigma_{e_1},\sigma_{e_2}}$ will denote the complement. Figure 2.9 shows an example for a GBS-graph. The half-space $\mathcal{H}_{m,n} = \mathcal{H}_{n,m}$ is shown in bold. Note that $\mathcal{H}_{m,n} = \mathcal{H}_{m,q}$ since $v_n = v_q$, but $\mathcal{H}_{m,q} \neq \mathcal{H}_{q,m}$ since q is not contained in $\mathcal{H}_{m,q}$

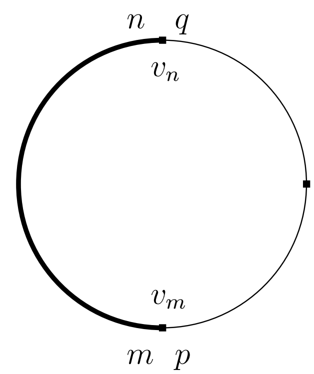


Figure 2.9: The half-space $\mathcal{H}_{m,n} = \mathcal{H}_{n,m}$ is shown in bold for this GBS-graph.

Remark 2.13. It is possible that when removing a vertex or pair of vertices that there will be only one connected component, say if v_{σ_e} is a valence 1 vertex or if $v_{\sigma_{e_1}} = v_{\sigma_{e_2}}$. It is also possible that there will be more than two components when removing a vertex or pair of vertices.

2.4 Coverings

Given a graph of groups $\mathcal{G}$ with an underlying graph of rank less than 2, we will seek a finite index subgroup of $\pi_1(\mathcal{G})$ which maps onto a non-abelian free group. Since finite index subgroups of $\pi_1(\mathcal{G})$ correspond to finite covers of the total space X_Γ, we will try to find such covers which retract onto higher rank graphs.

A finite cover of X_Γ will consist of copies of finite covers of each vertex space X_v and each edge space X_e in the total space. This covering space can be described as a graph of spaces with underlying graph Γ'. From this, we get a graph of groups, $\mathcal{G}'$ such that $\pi_1(\mathcal{G}')$ is a finite index subgroup of $\pi_1(\mathcal{G})$. There is a notion of *covering* for graphs of groups which corresponds to finite covers of the total space. We will first define a graph morphism and then give the definition of a covering for graphs of groups.

Definition 2.14 (Graph Morphisms)**.** (Bass [1]) A *morphism* $\phi : \Gamma' \to \Gamma$ is a map between graphs that carries vertices to vertices, edges to edges, and for $e \in E(\Gamma')$, $\phi(\partial_i(e)) = \partial_i(\phi(e))$, $(i = 0, 1)$ and $\phi(\bar{e}) = \overline{\phi(e)}$. For $v' \in \Gamma'$ we then have the local map

$$\phi_{(v')} : Star^{\Gamma'}(v') \to Star^{\Gamma}(\phi(v')).$$

If a total space $X_{\Gamma'}$ is a finite cover of the total space X_Γ, then the graph morphism $\phi : \Gamma' \to \Gamma$ of underlying graphs will be locally surjective and hence surjective.

Definition 2.15 (Covering)**.** (Bass [1]) $\Phi : \mathcal{G}' \to \mathcal{G}$ is a *covering* if

1. each $\Phi_{v'} : G'_{v'} \to G_{\phi(v')}$ is injective, and

2. $\Phi_{v'/e} : (\coprod_{e' \in \phi_{v'}^{-1}(e)} G'_{v'/e'}) :\to G_{\phi(v')/e}$ is bijective.

This definition is equivalent to certain conditions on open half-edges. These conditions are given in terms of edges in [1]. If Φ represents a covering map of graphs of groups and $v' \in \phi^{-1}(v)$, then $i(v') = i(v)$. In particular, if $\sigma \subset \Gamma$ is a half-edge with v in it's closure and $\sigma_1, \sigma_2, ...\sigma_k$ are half-edges in Γ' such that $\phi(\sigma_j) = \sigma$ with v' in the closure of each σ_j, then

$$\sum_{j=1}^{k} |\sigma_j| = |\sigma|. \tag{2.2}$$

In our construction of coverings, we will be focusing on two, (or possibly three) half-edges. For example, if a particular open half-edge of a GBS-graph is labeled by m we will want all pre-images $v'_m \in \Gamma'$ of v_m to represent $|m|$-fold covers of the vertex space X_v. This means at v'_m we will see open half-edges which will represent 1-fold covers of the half-edge space represented by m. It is possible however that there will be other half-edges $p_i, ..., p_k$ of v_m in Γ and that the covers of these half-edge spaces will in general not be 1-fold covers. This situation has the potential to cause problems in our construction.

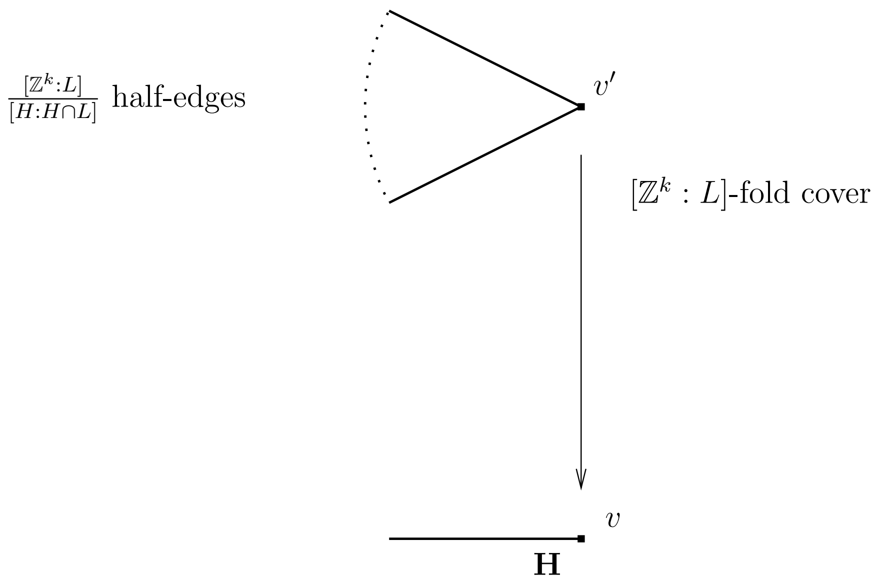


Figure 2.10: The L cover of a vertex and half-edge.

Let's look at an example from a graph of finitely generated infinite abelian groups where all edge groups are finite index in the vertex groups. Let $v \in \Gamma$ with $\mathbf{H} \in Star(v)$ and H the subgroup of $\mathbb{Z}^k$ represented by $\mathbf{H}$. Now consider $v' \in \Gamma'$ a finite covering of Γ, The covering map Φ when restricted to v' will correspond to some subgroup L of $\mathbb{Z}^k$ and will constitute a $[\mathbb{Z}^k : L]$-fold cover of X_v. Each open half-edge which will cover $\mathbf{H}$ will be the $H \cap L$ covering of $\mathbf{H}$. This means the number of branches covering $\mathbf{H}$ will be $\frac{[\mathbb{Z}^k:L]}{[H:H\cap L]}$ and each will constitute an $[H : H \cap L]$-fold cover. Results from the study of double cosets gives us $\frac{[\mathbb{Z}^k:L]}{[H:H\cap L]} = 1$ if and only if $HL = \mathbb{Z}^k$. See Figure 2.10.

The advantage of looking at graphs of abelian groups is that all subgroups are normal. In the general case, the cover of a vertex v corresponding to a subgroup L of the vertex group G_v will induce covers of a half-edge σ_e, which are conjugates of the subgroup $H \cap L$, where $H = \sigma_e(G_v)$.

Remark 2.16. For nonzero integers m and n, we will use the notation (m, n) to refer to the *greatest common denominator of* m *and* n and $[m, n]$ to refer to the *least common multiple of* m *and* n.

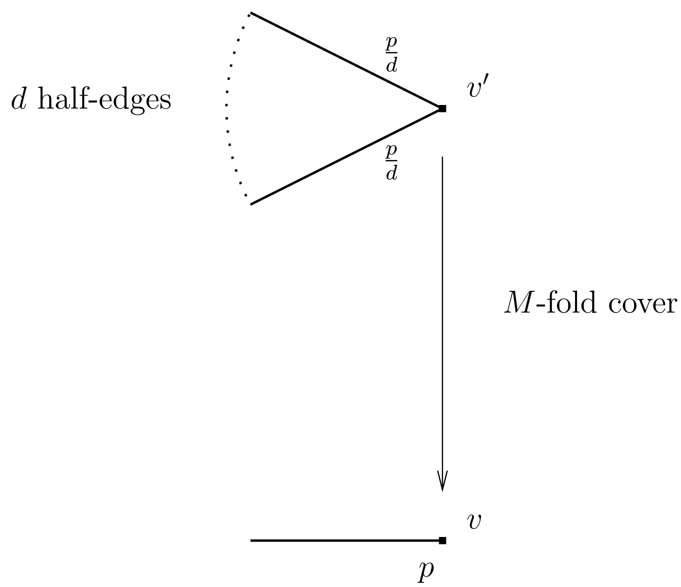


Figure 2.11: This figure shows the branching that will occur for an M-fold cover of the half-edge p. Here $d = (M, |p|)$

Now let's consider the case of a GBS group. Let $v \in \Gamma$ with $Star(v) = \{p_1, ..., p_k\}$ and consider $v' \in \Gamma'$ a finite cover of Γ. The covering map Φ when restricted to v' will represent an M-fold cover of the circle represented by v. This means that on the level of groups, for each half-edge p_i, we have an injective map $\frac{Mp_i}{d_i}\mathbb{Z} \hookrightarrow p_i\mathbb{Z}$ where $d_i = (M, p_i)$. Since $(\bigcup \phi^{-1}(p_i) \in \Gamma') \subset Star(v')$ must also constitute an M-fold cover of p_i. In order for this to be the case, there must be d_i half-edges each labeled by the integer $\frac{p_i}{d_i}$ and each will be a $\frac{M}{d_i}$-fold cover of p_i. See Figure 2.11.

Chapter 3

Results for General Graphs of Groups

Although this thesis is primarily concerned with graphs of abelian groups, we can show some results in a more general setting. We begin with a simple yet important proposition.

Proposition 3.1. *Let G be any group. If G admits a graph of groups $\mathcal{G}$ with underlying graph Γ of rank greater than 1, then G is large.*

Proof. Recall that Γ will also be the underlying graph for the graph of spaces $\mathcal{X}$ from which we construct the total space X_Γ. From Remark 2.6, there is an inclusion map $i : \Gamma \hookrightarrow X_\Gamma$ and a retraction $r : X_\Gamma \twoheadrightarrow \Gamma$. On the level of groups we have the induced map $r_* : \pi_1(X_\Gamma) \twoheadrightarrow \pi_1(\Gamma)$, so if Γ has rank greater than 1, this is a map of G onto a non-abelian free group.

□

Proposition 3.1 is essential to proving most of our results. When we have a group that does not admit a higher rank graph of groups, we try to find a finite index subgroup that does admit such a graph of groups. Alternatively, we show that no normal finite index subgroups admit such a graph of groups and conclude, with additional arguments, that the group is not large.

Remark 3.2. Given a graph of groups $\mathcal{G}$ with underlying graph Γ, we can assume that if σ_e is a terminal half-edge, then $|\sigma_e| > 1$. If $|\sigma_e| = 1$, then the map $f_e : X_e \to X_{v_{\sigma_e}}$ is a homotopy equivalence. Therefore, X_e is a deformation retract of the mapping cylinder M_{f_e}. Furthermore, $M_{f_{\bar{e}}}$ deformation retracts onto $X_{v_{\sigma_{\bar{e}}}}$. Combining these two deformation retracts, any edge with a terminal half-edge σ_e, such that $|\sigma_e| = 1$, may be collapsed in the graph of groups (or graph of spaces) without changing the homotopy type of the corresponding total space.

Theorem 3.3. *Let G be the fundamental group of a rank 0 graph of groups $\mathcal{G}$ which contains a pair of terminal half-edges σ_{e_1} and σ_{e_2} with $2 \leq |\sigma_{e_1}| < \infty$ and $3 \leq |\sigma_{e_2}| < \infty$, then G is large.*

Proof. To simplify notation, let $v_i := v_{\sigma_i}$. We will begin building a cover $\widehat{X_\Gamma}$ of the total space X_Γ with $|\sigma_{e_2}|$ copies of $\widehat{X_{v_1}}$, the $|\sigma_{e_1}|$-fold cover of the vertex space X_{v_1} which has fundamental group $\sigma_{e_1}(G_e)$. Similarly we will take $|\sigma_{e_1}|$ copies of $\widehat{X_{v_2}}$, the $|\sigma_{e_2}|$-fold cover of the vertex space X_{v_2} which has fundamental group $\sigma_{e_2}(G_e)$.

The rest of the cover will consist of $(|\sigma_{e_1}||\sigma_{e_2}|)$ copies of $X_\Gamma \setminus (X_{v_1} \cup X_{v_2})$. We attach $|\sigma_{e_1}|$ of these copies to each copy of $\widehat{X_{v_1}}$ via the unique lifts of f_{e_1} corresponding to each of the $|\sigma_{e_1}|$ pre-images of the basepoint of X_{v_1}. Likewise, $|\sigma_{e_2}|$ copies are attached to each copy of $\widehat{X_{v_2}}$ via the unique lifts of f_{e_2} corresponding to each of the $|\sigma_{e_2}|$ pre-images of the basepoint of X_{v_2}. These lifts of f_{e_i} are homotopy equivalences.

The cover $\widehat{X_\Gamma}$ is a $(|\sigma_{e_1}||\sigma_{e_2}|)$-fold cover of X_Γ and is a total space of a graph of spaces with an underlying graph of rank $(|\sigma_{e_1}||\sigma_{e_2}|) - (|\sigma_{e_1}| + |\sigma_{e_2}|) + 1$. Thus $\pi_1(\widehat{X_\Gamma})$ is an index $(|\sigma_{e_1}||\sigma_{e_2}|)$ subgroup of G and, by Proposition 3.1, G is large.

□

Example 3. Figure 3.1 shows a 6-fold covering for the GBS-group $T_{(2,3)}$. The circular vertices in the covering represent 2-fold covers, and the triangular vertices in the covering represent 3-fold covers.

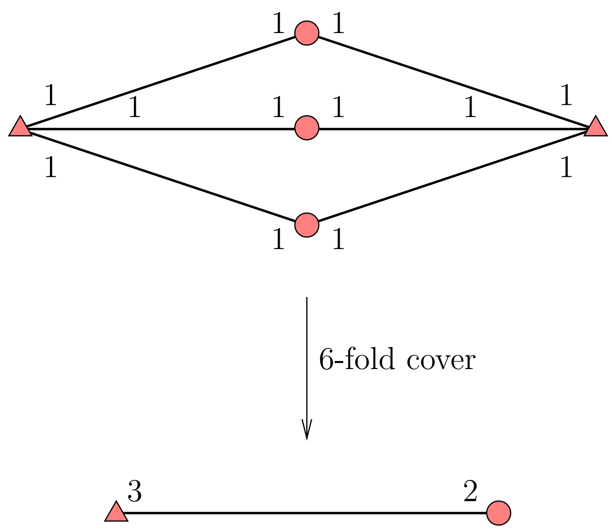


Figure 3.1: The 6-fold covering of the GBS graph of groups for $T_{(2,3)}$.

Theorem 3.4. *Let G be the fundamental group of a graph of groups $\mathcal{G}$ of rank 1 which contains a terminal half-edge σ_e with $2 \leq |\sigma_e| < \infty$, then G is large.*

Proof. As in Theorem 3.3, we will construct a cover of the total space X_Γ. Let $v := v_{\sigma_e}$. We begin with a single copy of $\widehat{X_v}$, the $|\sigma_e|$-fold cover of X_v such that $\pi_1(\widehat{X_v}) = \sigma_e(G_e) < G_v$. We then take $|\sigma_e|$ copies of $X_\Gamma \setminus X_v$ and attach each copy by continuous maps which are the unique lifts of f_e corresponding to each pre-image of the basepoint of X_v in the the cover $\widehat{X_v}$. The cover $\widehat{X_\Gamma}$ is a $|\sigma_e|$-fold cover of X_Γ, and its underlying graph has rank $|\sigma_e|$; therefore, by Proposition 3.1, G is large.

□

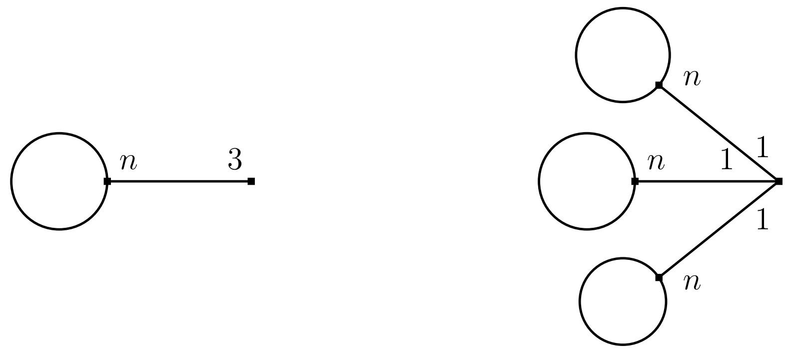


Figure 3.2: The GBS-graph on the right represents a 3-fold covering of the GBS-graph on the left.

Using the result of Theorem 3.4, we now prove another result for rank 0 graphs of groups.

Theorem 3.5. *Let G be the fundamental group of a rank 0 graph of groups $\mathcal{G}$. If $\mathcal{G}$ contains three terminal half-edges σ_{e_1}, σ_{e_2}, and σ_{e_3} such that $2 \leq |\sigma_{e_i}| < \infty$ for $i = 1, 2, 3$, then G is large.*

Proof. If $|\sigma| > 2$ for any terminal half-edge σ, then by Theorem 3.3, G is large, so assume that $|\sigma_{e_i}| = 2$ for $i = 1, 2, 3$. Let $v_i := v_{\sigma_{e_i}}$, and let $H_i := \sigma_{e_i}(X_{e_i})$ for $i = 1, 2, 3$. We'll begin with a 2-fold cover $\widehat{X_{v_1}}$ of X_{v_1} such that $\pi_1(\widehat{X_{v_1}}) = H_1$ and a 2-fold cover $\widehat{X_{v_2}}$ of X_{v_2} such that $\pi_1(\widehat{X_{v_2}}) = H_2$. The rest of the cover $\widehat{X_\Gamma}$ of X_Γ will be made of two copies of $X_\Gamma \setminus (X_{v_1} \cup X_{v_2})$. These copies will be attached to $\widehat{X_{v_1}}$ and $\widehat{X_{v_2}}$ via the unique lifts of f_{e_1} and f_{e_2} corresponding to the pre-images of the basepoints of X_{v_1} and X_{v_2} respectively. Therefore $\widehat{X_\Gamma}$ will be a 2-fold cover of X_Γ which retracts onto a rank 1 graph $\widehat{\Gamma}$. The graph of groups corresponding to this cover will have two terminal half-edges ϕ_1 and ϕ_2 (corresponding to covers of σ_{e_3}) such that $|\phi_1| = |\phi_2| = 2$, so by Theorem 3.4, $\pi_1(\widehat{X_\Gamma})$ is large, and consequently G is large. $\square$

The last proof in this section is a special case of rank 1 graphs of groups.

Theorem 3.6. *Let G be the fundamental group of a rank 1 graph of groups $\mathcal{G}$. Suppose $\mathcal{G}$ contains two half-edges σ_{e_1} and σ_{e_2} with the following properties:*

1. *σ_{e_1} and σ_{e_2} share a common vertex $v = v_{\sigma_{e_1}} = v_{\sigma_{e_1}}$;*
2. *$H := \sigma_{e_1}(X_{e_1})$ and $K := \sigma_{e_2}(X_{e_2})$ are both contained in a non-trivial finite index normal subgroup $L \lhd G_v$.*

Then G is large.

Proof. Recall that if there is a terminal half-edge σ then $|\sigma| \geq 2$ and G is large by Theorem 3.4. Therefore, we will assume that there are no terminal half-edges.

We'll begin building a cover for X_Γ with a $[G_v : L]$-fold cover $\widehat{X_v}$ of X_v such that $\pi_1(\widehat{X_v}) = L$. The rest of the cover will be made of $[G_v : L]$ copies of $X_\Gamma \setminus X_v$. These copies will be attached to $\widehat{X_v}$ by the unique lifts, corresponding to the pre-image of the basepoint of X_v, of f_{e_1} and f_{e_2} respectively. The underlying graph of the total space $\widehat{X_\Gamma}$ will have rank $[G_v : L]$, which is at least 2, so by Proposition 3.1, G is large. □

Remark 3.7. A more general result than that given in Theorem 3.6 can be shown. It is sufficient to assume that for $\{t_1, \ldots, t_n\}$ a transversal of L that $H^{t_i} \subseteq L$ and $K^{t_i} \subseteq L$ for all $i = 1, \ldots, n$.

Chapter 4

Largeness of Graphs of Abelian Groups

This chapter contains results for the largeness of graphs of free abelian groups. Before these results can be shown, we must have a uniform way of representing subgroups of $\mathbb{Z}^k$. The following section describes a way of representing subgroups of $\mathbb{Z}^k$ with unique $(k \times k)$ matrices.

4.1 Canonical Subgroup Matrices

Let $H \leq \mathbb{Z}^k$ be a subgroup of finite index. There exists a free abelian basis $\{h_1, \ldots, h_k\}$ of H. Writing each h_i as a column vector in $\mathbb{Z}^k$, we can see that H can be encoded by a $(k \times k)$ integral matrix $\mathbf{H}$ with column vectors $\{h_1, \ldots, h_k\}$. An element of H is represented by a column vector which is a unique integral linear combination of the columns of $\mathbf{H}$. Since a matrix $\mathbf{H}$ will be a half-edge in a graph of groups we have $|\mathbf{H}| = |det(\mathbf{H})| = [\mathbb{Z}^k : H]$.

When we have a graph of $\mathbb{Z}^k$ groups, the half-edges will be indexed by $(k \times k)$ integral matrices that represent the maps from the edge $\mathbb{Z}^k$ groups into the vertex $\mathbb{Z}^k$ groups. Thus, the matrices can also be considered representatives of the subgroup images of these maps. The work of Davies, Dirl, and Goldsmith [21] provides us with a way to determine when two matrices represent the same subgroup and algorithms for finding the intersection and union of two subgroups.

Given a $(k \times k)$ matrix $\mathbf{H}$ representing the subgroup $H < \mathbb{Z}^k$, we can use elementary column operations to get a canonical triangular integral matrix $\star\mathbf{H}$ which is column equivalent to the matrix $\mathbf{H}$ and has the form:

$$\star\mathbf{H} = \begin{pmatrix} d_k & \cdots & 0 & 0 & 0 \\ a_{k-1,k} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{2,k} & \cdots & a_{2,3} & d_2 & 0 \\ a_{1,k} & \cdots & a_{1,3} & a_{1,2} & d_1 \end{pmatrix}$$

where $1 \leq d_j$, for $j = k, k-1, \ldots, 2, 1$, and $0 \leq a_{j,i} \leq d_j - 1$, for $i = k, k-1, \ldots, j+1$ and $j = k-1, k-2, \ldots, 2, 1$.

The column operations used are equivalent to changing the basis for H. In Section 2.2 of [21], it is shown that $\star\mathbf{H}$ is unique for any finite index subgroup $H \leq \mathbb{Z}^k$, thus two subgroups $H, K \leq \mathbb{Z}^k$ are equal if and only if $\star\mathbf{H} = \star\mathbf{K}$. For a matrix representative of a subgroup $H \leq \mathbb{Z}^k$ we write $\mathbf{H} \sim_{col} \star\mathbf{H}$.

The results for graphs of $\mathbb{Z}^k$ groups will rely on the algorithm given in Section 3.2 of [21] for finding the intersection of two subgroups $H, K \leq \mathbb{Z}^k$. The subgroup $H \cap K$ will also have a canonical matrix representative which we denote by $\mathbf{H} \wedge \mathbf{K}$.

4.2 Results for Graphs of $\mathbb{Z}^k$ Groups

Theorem 4.1. *Let $k \geq 1$ and let G be the fundamental group of a rank 0 graph of groups $\mathcal{G}$ where all vertex and edge groups are $\mathbb{Z}^k$. Then G is large if and only if it is not $\mathbb{Z}^k$ or virtually $\mathbb{Z}^{k+1}$.*

Proof. Theorem 3.3 covers graphs of groups with two terminal half edges, one of which has index greater than two, and Theorem 3.5 covers graphs of groups with more than two terminal half-edges. If $\mathcal{G}$ has no terminal half-edges then it is a single vertex and G is $\mathbb{Z}^k$. The only remaining case is if $\mathcal{G}$ has exactly two terminal half-edges, both of which have index 2. Starting with a terminal vertex, we can label the vertices sequentially from 0 to n and the half-edges as in Figure 4.1. We may also assume that for $i = 1, \dots n-1$, $|\mathbf{A}_i| \geq 2$ and $|\mathbf{B}_i| \geq 2$. The terminal half-edges, $\mathbf{B}_0$ and $\mathbf{A}_n$ will represent index 2 subgroups of $\mathbb{Z}^k$.

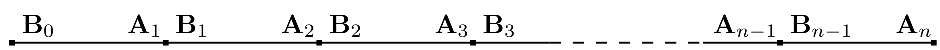


Figure 4.1: A graph of abelian groups as described in Theorem 4.1.

We will first construct a 2-fold cover of X_Γ as in Theorem 3.5. The graph of groups, $\mathcal{G}'$, corresponding to this cover will have the form shown in Figure 4.2 where $\mathbf{I}$ represents the identity matrix. If $n = 1$ then G is virtually $\mathbb{Z}^{k+1}$, and we are done. For $n > 1$, we will construct a cover of the total space X'_Γ corresponding to the graph of groups $\mathcal{G}'$.

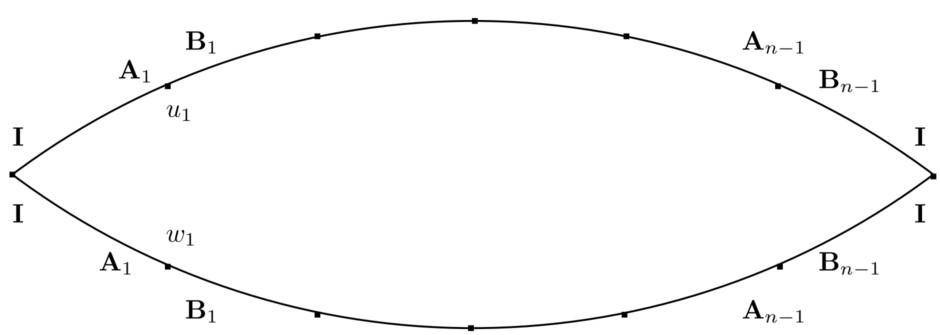


Figure 4.2: A 2-fold cover of the graph in Figure 4.1.

Let u_1 and w_1 in $\mathcal{G}'$ be the pre-images of the vertex v_1. Let $\widehat{X_{u_1}}$ and $\widehat{X_{w_1}}$ be the covers of X_{v_1} and X_{w_1} such that $\pi_1(\widehat{X_{u_1}}) \cong \pi_1(\widehat{X_{w_1}}) \cong B_1$. We will then take

$|\mathbf{B}_1|$ copies of the total space of the closure of the half-space defined by the two half-edges labeled by $\mathbf{B}_1$ and attach these to $\widehat{X_{u_1}}$ and $\widehat{X_{w_1}}$ by the unique lifts of their respective continuous maps.

We will now consider the total space of the closure of the half-space defined by the two half-edges labeled by $\mathbf{A}_1$. We know that there is a cover of the closure of each half-edge corresponding to the covers of X_{u_1} and X_{w_1} respectively. These covers may or may not branch depending upon the group A_1, but they will be homeomorphic to each other. Thus, we can construct a complete cover $\widehat{X_\Gamma}$ of X'_Γ which has rank greater than or equal to 2, so G is large. □

Before providing results for the rank 1 graph of groups case, we will need the following lemmas. These lemmas will be used to show that the fundamental groups of certain rank 1 graphs of abelian groups are not large.

Lemma 4.2. *Let G be a large group. For all $m > 1$ there exists a normal finite index subgroup $N \lhd G$ such that $N \twoheadrightarrow F_m$.*

Proof. Since G is large, there is a finite index subgroup $H < G$ such that $f : H \twoheadrightarrow F_2$. There is a finite index subgroup isomorphic to F_m contained in F_2, so $K = f^{-1}(F_m)$ is finite index in H. There exists a finite index subgroup N of K which is normal in G such that $[G : N] \leq [G : K]!$. The image of N, $f(N)$ is a finite index subgroup of $f(K) = F_m$, so $f(N) \cong F_n$ for some $n \geq m$. Since $F_n \twoheadrightarrow F_m$, there exists a surjective map $N \twoheadrightarrow F_m$. □

Lemma 4.3. *Let $k \geq 1$, and let G be the fundamental group of a graph of groups $\mathcal{G}$ with all edge and vertex groups isomorphic to $\mathbb{Z}^k$. Furthermore, suppose that the underlying graph Γ consists of a single cycle. Then $b_1(G)$ is at most $k + 1$.*

Proof. Let Γ have n vertices and let the half-edges of Γ be labeled cyclicly by $(k \times k)$ matrices $\mathbf{A}_i$ and $\mathbf{B}_i$, $i = 1, \ldots, n$, as in Figure 4.3. Assume without loss of generality that each matrix $\mathbf{A}_i$ is given in canonical form, and let X_Γ be the total space.

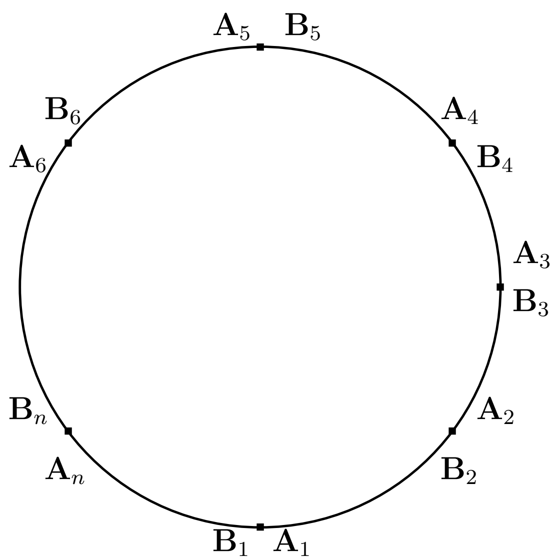


Figure 4.3: A graph of groups as described Lemma 4.3.

We can read $H_1(X_\Gamma)$ from the finite 2-skeleton of X_Γ, so we have the following exact sequence:

$$0 \longrightarrow C_2(X_\Gamma^{(2)}) \xrightarrow{\partial_2} C_1(X_\Gamma^{(2)}) \xrightarrow{\partial_1} C_0(X_\Gamma^{(2)}) \longrightarrow 0$$

Let $Z_1 := ker(\partial_1)$ and let $B_2 := im(\partial_2)$. Then $H_1(X_\Gamma) = Z_1/B_2$. There are k 1-cells from each vertex space and there is a single 1-cell from each $X_e \times I$, so $C_1(X_\Gamma^{(2)}) = \mathbb{Z}^{nk} \oplus \mathbb{Z}^n \cong \mathbb{Z}^{nk} \oplus \mathbb{Z} \oplus \mathbb{Z}^{n-1}$. By inspection of the 1 skeleton of X_Γ,we get $Z_1 \cong \mathbb{Z}^{nk} \oplus \mathbb{Z}$; see Figure 4.4. The transition matrix for B_2 is given by the

$(nk \times nk)$ matrix:

$$\mathbb{M} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{B}_1 \\ -\mathbf{B}_2 & \mathbf{A}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_3 & \mathbf{A}_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{B}_{n-1} & \mathbf{A}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{B}_n & \mathbf{A}_n \end{pmatrix}.$$

Since each $\mathbf{A}_i$ is in canonical form, the first $(n-1)k$ columns of $\mathbb{M}$ are linearly independent. Therefore, when $\mathbb{M}$ is converted to normal form through row and column operations, there will be at least $(n-1)k$ non-zero vectors. Thus the free abelian rank of $H_1(X_\Gamma)$ is at most $k+1$.

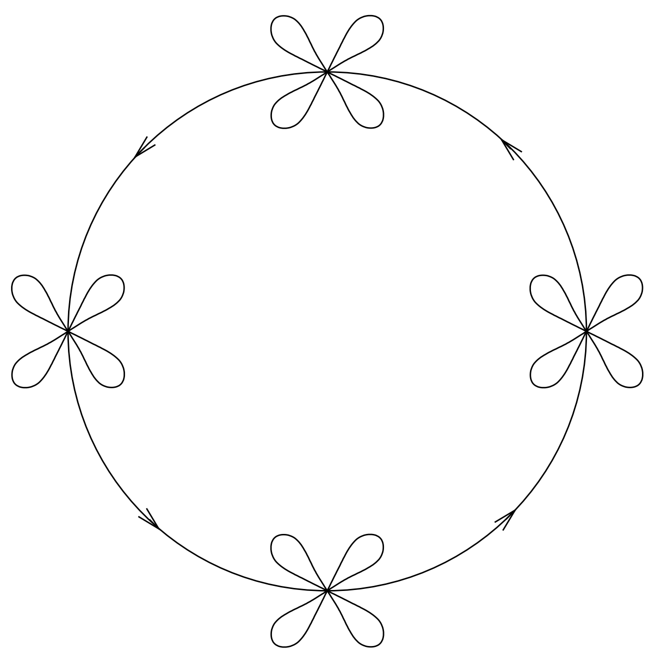


Figure 4.4: The 1-skeleton of the total space X_Γ as in Lemma 4.3 for $n = 4$ and $k = 4$.

□

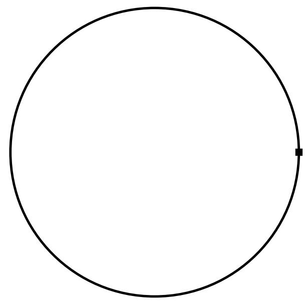


Figure 4.5: A graph of groups as described in Theorem 4.4.

Theorem 4.4. *Let $k \geq 1$, and let G be the fundamental group of a rank 1 graph of groups $\mathcal{G}$ consisting of a single edge and a single vertex such that the vertex group and edge group are both $\mathbb{Z}^k$. Let $\mathbf{A}$ and $\mathbf{B}$ be the half-edges of the graph and let $A, B \leq \mathbb{Z}^k$ be the subgroups they represent. If $(|\mathbf{A}|, |\mathbf{B}|) = 1$, then G is not large.*

Proof. The subgroups $A, B \leq \mathbb{Z}^k$ will have canonical matrix representatives

$$\mathbf{A} \sim_{col} \begin{pmatrix} a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & a_{k-1} & 0 \\ * & \cdots & * & a_k \end{pmatrix}$$

and

$$\mathbf{B} \sim_{col} \begin{pmatrix} b_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & b_{k-1} & 0 \\ * & \cdots & * & b_k \end{pmatrix}.$$

Let $a = a_1 a_2 \cdots a_k$ and $b = b_1 b_2 \cdots b_k$, then $|\mathbf{A}| = a$ and $|\mathbf{B}| = b$. Since $(|\mathbf{A}|, |\mathbf{B}|) = 1$, we have $(a, b) = 1$ and $(a_i, b_j) = 1$ for all $i, j \in \{1, \ldots, k\}$.

Let G' be a normal finite index subgroup of G which is the fundamental group of a regular covering $\mathcal{G}'$ (with underlying graph Γ') of $\mathcal{G}$. Then each vertex in Γ' represents the same finite index subgroup $L \leq \mathbb{Z}^k$ with canonical matrix representative:

$$\mathbf{L} \sim_{col} \begin{pmatrix} l_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & l_{k-1} & 0 \\ * & \cdots & * & l_k \end{pmatrix}.$$

Since G' is normal in G, and since $\mathbf{A}$ and $\mathbf{B}$ are half-edges of the same edge, the number of half-edges covering $\mathbf{A}$ must be the same as the number of half-edges covering $\mathbf{B}$; therefore,

$$\frac{[\mathbb{Z}^k : L]}{[A : A \cap L]} = \frac{[\mathbb{Z}^k : L]}{[B : B \cap L]}$$

and

$$[A : A \cap L] = [B : B \cap L]. \tag{4.1}$$

We will show that there is only 1 branch for each half-edge at any vertex in Γ', and therefore Γ' is a single cycle.

First we will investigate the canonical matrix representatives of $A \cap L$ and $B \cap L$. Following the algorithm given in [21], we get matrix representatives

$$\mathbf{A} \wedge \mathbf{L} \sim_{col} \begin{pmatrix} s_1[a_1, l_1] & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & s_{k-1}[a_{k-1}, l_{k-1}] & 0 \\ * & \cdots & * & [a_k, l_k] \end{pmatrix}$$

and

$$\mathbf{B} \wedge \mathbf{L} \sim_{col} \begin{pmatrix} t_1[b_1, l_1] & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & t_{k-1}[b_{k-1}, l_{k-1}] & 0 \\ * & \cdots & * & [b_k, l_k] \end{pmatrix}$$

for these intersections. Rewriting these matrices in terms of the basis elements of A and B respectively we get

$$(\mathbf{A} \wedge \mathbf{L})_{\mathbf{A}} = \begin{bmatrix} \frac{s_1[a_1,l_1]}{a_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & \frac{s_2[a_{k-1},l_{k-1}]}{a_{k-1}} & 0 \\ * & \cdots & * & \frac{[a_k,l_k]}{a_k} \end{bmatrix}_{\mathbf{A}}$$

and

$$(\mathbf{B} \wedge \mathbf{L})_{\mathbf{B}} = \begin{bmatrix} \frac{t_1[b_1,l_1]}{b_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & \frac{t_{k-1}[b_{k-1},l_{k-1}]}{b_{k-1}} & 0 \\ * & \cdots & * & \frac{[b_k,l_k]}{b_k} \end{bmatrix}_{\mathbf{B}}.$$

The determinant of these matrices will give us the index of each intersection in A and B respectively. Let $s = s_1 s_2 \cdots s_{k-1}$ and $t = t_1 t_2 \cdots t_{k-1}$. Equation (4.1)implies

$$\frac{s[a_1, l_1] \cdots [a_k, l_k]}{a} = \frac{t[b_1, l_1] \cdots [b_k, l_k]}{b}.$$

Using the fact that $[m, n] = \frac{mn}{(m,n)}$, we can simplify each side of the equation to get

$$\frac{sl}{(a_1, l_1) \cdots (a_k, l_k)} = \frac{tl}{(b_1, l_1) \cdots (b_k, l_k)}.$$

This gives us the relationship:

$$\frac{s}{t} = \frac{(a_1, l_1) \cdots (a_k, l_k)}{(b_1, l_1) \cdots (b_k, l_k)}.$$

Since $(a_i, b_j) = 1$ for all $i, j \in \{1, \cdots, k\}$, the right-hand side of this equation is a reduced fraction which means that

$$(a_1, l_1) \cdots (a_k, l_k) | s. \tag{4.2}$$

Now we will consider the number of branches which is:

$$\frac{[\mathbb{Z}^k : L]}{[A : A \cap L]} = \frac{l}{\frac{sl}{(a_1,l_1)\cdots(a_k,l_k)}} = \frac{(a_1, l_1) \cdots (a_k, l_k)}{s}.$$

This number must be a positive integer; therefore,

$$s | (a_1, l_1) \cdots (a_k, l_k). \tag{4.3}$$

From (4.2) and (4.3) we get $s = (a_1, l_1) \cdots (a_k, l_k)$, so the number of branches is 1. Hence all regular coverings $\mathcal{G}'$ of $\mathcal{G}$ are single cycles as in Figure 4.6. If G is large, then by Lemma 4.2, some normal finite index subgroup of G will map onto a non-abelian free group and will have arbitrarily high first virtual Betti-number. Since all our normal subgroups will come from a graph of groups with this structure, Lemma 4.3 tells us that the homology of any normal subgroup is bounded by $k + 1$, therefore G is not large.

$\square$

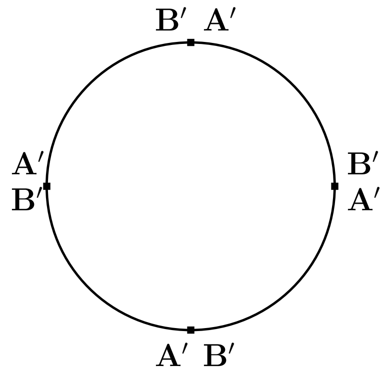


Figure 4.6: Any finite regular cover of the graph of groups describe in Lemma 4.3 will have this form where $\mathbf{A}'$ and $\mathbf{B}'$ are matrix representatives of the subgroups $A \cap L$ and $B \cap L$ respectively.

4.3 Results for Graphs of $\mathbb{Z}^2$ Groups

We can improve upon the result of Theorem 4.4, and give some positive results, if we further restrict to graphs of $\mathbb{Z}^2$ groups.

Theorem 4.5. *Suppose G is the fundamental group of a graph of groups $\mathcal{G}$ such that the underlying graph Γ consists of a single vertex and a single edge, and both the vertex and edge groups are $\mathbb{Z}^2$. Let the half-edges of $\mathcal{G}$ be labeled by the matrices $\mathbf{H}_1$ and $\mathbf{H}_2$, and let $H_1, H_2 \leq \mathbb{Z}^2$ be the subgroups represented by the two half-edges. Assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ are give in canonical form and*

$$\mathbf{H}_i = \begin{pmatrix} 1 & 0 \\ w_i & n \end{pmatrix}$$

for $n \geq 2$. Then G is large.

Proof. If $H_1 H_2 = L \neq \mathbb{Z}^2$ then, by Theorem 3.6, G is large; however not all of the groups described in the statement of the theorem will satisfy this condition.

For example, if

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

then $H_1 H_2 = \mathbb{Z}^k$.

Suppose

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 0 \\ w_1 & n \end{pmatrix}$$

and

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 0 \\ w_2 & n \end{pmatrix}$$

for $n \geq 2$ and $0 \leq w_1 < w_2 < n$. If $w_1 = w_2$ we can take $L = H_1 = H_2$ and apply Theorem 3.6 to show that G is large. Let

$$\mathbf{L} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix},$$

and let L be the subgroup which $\mathbf{L}$ represents. L is an index n^2 subgroup of $\mathbb{Z}^2$ and L is contained in both H_1 and H_2. Let $\widehat{X_v}$ be the n^2-fold cover of X_v such that $\pi_1(\widehat{X_v}) \cong L$. Taking this cover of the vertex space, in the covering graph of groups $\widehat{\mathcal{G}}$ we will have

$$\frac{[\mathbb{Z}^2 : L]}{[H_i : H_i \cap L]} = \frac{n^2}{n} = n$$

branches covering each half-edge. Each of these branches will be an n-fold covering of the corresponding half-edge.

If the constructed graph of groups $\mathcal{G}'$ is a legitimate covering of $\mathcal{G}$, we will have $\sigma_{\mathbf{H}_1}^{-1}(L) = \sigma_{\mathbf{H}_2}^{-1}(L)$ in the edge group $\mathbb{Z}^2$. Let $\{e_1, e_2\}$ be the basis vectors for the edge $\mathbb{Z}^2$ and let $\{f_1, f_2\}$ be the basis vectors for the vertex $\mathbb{Z}^2$. The map $\sigma_{\mathbf{H}_1}$ takes the basis elements $e_1 \mapsto f_1 + w_1 f_2$ and $e_2 \mapsto n f_2$. Solving for $\{nf_1, nf_2\}$ gives the matrix

$$\begin{pmatrix} n & 0 \\ -w_1 & 1 \end{pmatrix} \sim_{col} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

which represents the subgroup $\sigma_{\mathbf{H}_1}^{-1}(L)$ of the edge $\mathbb{Z}^2$. The map $\sigma_{\mathbf{H}_2}$ takes the basis elements $e_1 \mapsto f_1 + w_2 f_2$ and $e_2 \mapsto n f_2$. Solving for $\{nf_1, nf_2\}$ gives the matrix

$$\begin{pmatrix} n & 0 \\ -w_2 & 1 \end{pmatrix} \sim_{col} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

which represents the subgroup $\sigma_{\mathbf{H}_2}^{-1}(L)$ of the edge $\mathbb{Z}^2$. Since $\sigma_{\mathbf{H}_1}^{-1}(L) = \sigma_{\mathbf{H}_2}^{-1}(L)$ as subgroups, $\mathcal{G}'$ is a covering of $\mathcal{G}$. The underlying graph of $\mathcal{G}'$ has rank n; therefore, G is large.

□

The next Theorem is an improvement of the negative result given in Theorem 4.4 for graphs of $\mathbb{Z}^2$ groups.

Theorem 4.6. *Suppose G is the fundamental group of a graph of groups $\mathcal{G}$ such that the underlying graph Γ consists of a single vertex and a single edge, and both the vertex and edge groups are $\mathbb{Z}^2$. Let the half-edges of $\mathcal{G}$ be labeled by the matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ in canonical form, and let $H_1, H_2 \leq \mathbb{Z}^2$ be the subgroups* *represented by the two half-edges. Suppose*

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 0 \\ w & q \end{pmatrix}$$

and

$$\mathbf{H}_2 = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

for q prime. Then G is not large.

Proof. Suppose that we have a regular finite covering $\mathcal{G}'$ of $\mathcal{G}$. Any vertex of the underlying graph Γ' will then represent the same covering space of the vertex space and the same subgroup $L \cong \mathbb{Z}^2$ of the $\mathbb{Z}^2$ vertex group in Γ. Let

$$\mathbf{L} \sim_{col} \begin{pmatrix} m & 0 \\ x & n \end{pmatrix}$$

for some $m, n > 0$ and $0 \leq x < n$, be the canonical matrix representative of L.

Since there is only a single edge in Γ, any vertex $v' \in \Gamma'$ the number of branches covering $\mathbf{H}_1$ must be the same as the number of branches covering $\mathbf{H}_2$. In order to calculate the branching, we must find $[H_i : H_i \cap L]$ for $i = 1, 2$. Following the algorithm in [21] we get the following canonical matrix representatives for the groups $H_1 \cap L$ and $H_2 \cap L$ respectively:

$$\mathbf{H}_1 \wedge \mathbf{L} \sim_{col} \begin{pmatrix} p_1 m & 0 \\ z_1 & [q, n] \end{pmatrix}$$

where $1 \leq p_1 \leq min\{[q, mn], (q, n)\}$, $0 \leq z_1 < [q, n]$, and satisfying the following equations

$$\begin{bmatrix} p_1 m \\ z_1 \end{bmatrix} = p_1 m \begin{bmatrix} 1 \\ v \end{bmatrix} + s_1 \begin{bmatrix} 0 \\ q \end{bmatrix} \tag{4.4}$$

$$\begin{bmatrix} p_1 m \\ z_1 \end{bmatrix} = p_1 \begin{bmatrix} m \\ x \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ n \end{bmatrix} \tag{4.5}$$

for $s_1, t_1 \in \mathbb{Z}$, and

$$\mathbf{H}_2 \wedge \mathbf{L} \sim_{col} \begin{pmatrix} p_2[q,m] & 0 \\ z_2 & n \end{pmatrix}$$

where $1 \leq p_2 \leq min\{[q, mn], (1, n)\}$, $0 \leq z_2 < n$, and satisfying the following equations:

$$\begin{bmatrix} p_2[q,m] \\ z_2 \end{bmatrix} = \frac{p_2[q,m]}{q} \begin{bmatrix} q \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.6}$$

$$\begin{bmatrix} p_2[q,m] \\ z_2 \end{bmatrix} = \frac{p_2[q,m]}{m} \begin{bmatrix} m \\ x \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ n \end{bmatrix} \tag{4.7}$$

for $s_2, t_2 \in \mathbb{Z}$.

A necessary condition for $\mathcal{G}'$ to be a covering of $\mathcal{G}$ we need $[H_1 : H_1 \cap L] = [H_2 : H_2 \cap L]$. To find these values we look at the subgroup intersections in terms of the basis elements of H_1 and H_2 respectively, giving

$$\begin{bmatrix} p_1 m & 0 \\ * & \frac{[q,n]}{q} \end{bmatrix}_{H_1}$$

and

$$\begin{bmatrix} \frac{p_2[q,m]}{q} & 0 \\ * & n \end{bmatrix}_{H_2} .$$

The number of branches at each vertex in Γ' is given by:

$$\frac{[\mathbb{Z}^2 : L]}{[H_1 : H_1 \cap L]} = \frac{mn(q,n)}{p_1 mn} = \frac{(q,n)}{p_1}.$$

The number of branches is also given by:

$$\frac{[\mathbb{Z}^2 : L]}{[H_2 : H_2 \cap L]} = \frac{mn(q,m)}{p_2 mn} = \frac{(q,m)}{p_2}.$$

The number of branches must be a positive integer, so if there is more than one branch then q divides both m and n, and $p_1 = p_2 = 1$.

From Equation (4.5), we have

$$\begin{bmatrix} m \\ z_1 \end{bmatrix} = \begin{bmatrix} m \\ x \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ n \end{bmatrix} \longrightarrow z_1 = x + t_1 n, \tag{4.8}$$

so the inequalities $0 \leq z_1 < n$ and $0 \leq x < n$ imply that $z_1 = x$. Similarly Equation (4.7) implies that $z_2 = x$, therefore $\mathbf{H}_1 \wedge \mathbf{L} \sim_{col} \mathbf{H}_2 \wedge \mathbf{L} \sim_{col} \mathbf{L}$ and $L \subset H_1 \cap H_2$.

Since the matrix representative of $H_1 \cap H_2$ is

$$\mathbf{H}_1 \wedge \mathbf{H}_2 \sim_{col} \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix},$$

we have

$$\mathbf{L} \sim_{col} \begin{pmatrix} \alpha_1 q & 0 \\ \alpha_2 q & \alpha_3 q \end{pmatrix}$$

for some integers $\alpha_1, \alpha_3 > 0$ and $0 \leq \alpha_2 < \alpha_3$.

If the constructed graph of groups $\mathcal{G}'$ is a legitimate covering of $\mathcal{G}$, we will have $\sigma_{\mathbf{H}_1}^{-1}(L) = \sigma_{\mathbf{H}_2}^{-1}(L)$ in the edge group $\mathbb{Z}^2$. Let $\{e_1, e_2\}$ be the basis vectors for the edge $\mathbb{Z}^2$ and let $\{f_1, f_2\}$ be the basis vectors for the vertex $\mathbb{Z}^2$. The map $\sigma_{\mathbf{H}_1}$ takes the basis elements $e_1 \mapsto f_1 + wf_2$ and $e_2 \mapsto qf_2$. This gives the matrix

$$\begin{pmatrix} \alpha_1 q & 0 \\ \alpha_2 - \alpha_1 w & \alpha_3 \end{pmatrix}$$

which represents the subgroup $\sigma_{\mathbf{H}_1}^{-1}(L)$ of the edge $\mathbb{Z}^2$. The map $\sigma_{\mathbf{H}_2}$ takes the basis elements $e_1 \mapsto qf_1$ and $e_2 \mapsto f_2$. This gives the matrix

$$\begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 q & \alpha_3 q \end{pmatrix}$$

which represents the subgroup $\sigma_{\mathbf{H}_2}^{-1}(L)$ of the edge $\mathbb{Z}^2$. Since $\alpha_1 \neq \alpha_1 q$, these two matrices cannot be equivalent. This contradiction was obtained by assuming that the number of branches was greater than one; therefore, any covering of $\mathcal{G}$ will have an underlying graph which is a single cycle, and therefore Lemma 4.2 and Lemma 4.3 imply that G is not large.

$\square$

Chapter 5

Largeness of Graphs of Cyclic Groups

As we saw in Chapter 4, we can determine the largeness of several different families of graphs of infinite abelian groups, but a complete classification is difficult to achieve. The problem is that for each integer $n > 1$, $\mathbb{Z}^k$ has "too many" subgroups of index n when $k \geq 2$. For example, there are 2,015 index 24 subgroups of $\mathbb{Z}^3$ [21]. Because cyclic groups will have at most 1 subgroup of any given index we can get complete classifications for graphs of these groups.

5.1 Classification of GBS Groups

Before we begin proving our results we need to define a *modular homomorphism* for rank 1 GBS-graphs. The modular homomorphism is usually defined by a cycle of edges [24]. The following formulation is given in terms of half-edges.

Definition 5.1. Let $h : H_1(\mathcal{G}) \to \mathbb{Q}^{\times}_{>0}$ be defined by a cycle of half-edges $(q_1, p_1, \ldots, q_k, p_k) \mapsto \prod_{j=1}^{k} |q_j|/|p_j|$ We call h the *modular homomorphism* for the cycle $(q_1, p_1, \ldots, q_k, p_k)$. See Figure 5.1.

Theorem 5.2. *Let G be a generalized Baumslag-Solitar group.*

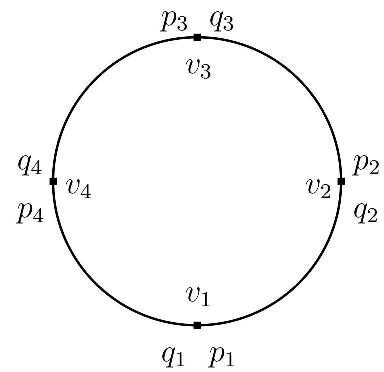


Figure 5.1: The value of the modular homomorphism for this cycle is $\prod_{j=1}^{4} |q_j|/|p_j|$.

1. *If G admits a graph of groups representation of rank greater than 1, then G is large.*

2. *If G admits a graph of groups with rank 1, then G is large if and only if the numerator and the denominator of the modular homomorphism of the cycle of the graph of groups are not relatively prime or there is a terminal half-edge m.*

3. *If G admits a graph of groups with rank 0, then G is large if and only if it is not isomorphic to $\mathbb{Z}$ or the Klein-bottle group.*

Before we begin the proof of Theorem 5.2, let's explore the relationship between this theorem and the proofs in Chapter 4. For the rank 1 case, if our GBS-graph has a terminal half-edge, then we know that the GBS group is large from Theorem 3.4, so we need only consider graphs with no terminal vertices. For the rank 0 case, Theorem 3.3, Theorem 3.5, and Theorem 4.1 show that all GBS groups with rank 0 graphs are large unless, after collapse moves, the graph is only a single vertex or a single edge with both half-edges labeled by 2. In the first case the group is $\mathbb{Z}$ and in the second case the groups is the Klein-bottle groups.

Proof. Γ can be described as a single cycle with k vertices and k edges.

Case 1. Assume that the numerator and the denominator of the modular homomorphism of the cycle of the graph of groups are not relatively prime. This means that there are half-edges m and n such that $\mathcal{H}_{m,n} = \mathcal{H}_{n,m}$ and $(|m|, |n|) > 1$. We can also assume, using expansion moves that $|m| = |n|$ and that they are a maximal pair in the sense that if p is half-edge such that $(|p|, |n|) > 1$ then $p \subset \mathcal{H}_{m,n}$. See Figure 5.2.

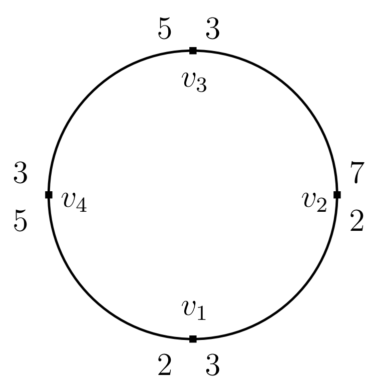


Figure 5.2: A graph of groups as described in Theorem 5.2 *Case 1.* The half-edges labeled by 3 at the vertices v_1 and v_4 form a maximal pair.

We will begin building a finite cover of X_Γ with the $|m|$-fold covers of X_{v_m} and X_{v_n} respectively. If $v_m = v_n$, there will only be one cover. We will add to the cover $|m|$ copies of $\mathcal{H}_{m,n}$ which will be attached to the covers $\widehat{X_{v_m}}$ and $\widehat{X_{v_n}}$ by lifts of the continuous maps f_m and f_n respectively. If $v_m = v_n$ then $\widehat{X_\Gamma}$ is complete, if not, we will attach one copy of the $|m|$-fold cover of the total space of $\mathcal{H}^c_{m,n}$ to the covers $\widehat{X_{v_m}}$ and $\widehat{X_{v_n}}$. See Figure 5.3. There is no branching required so $\widehat{X_\Gamma}$ is an $|m|$-fold cover of X_Γ. $\pi_1(\widehat{X_\Gamma})$ maps onto a non-abelian free group, so G is large.

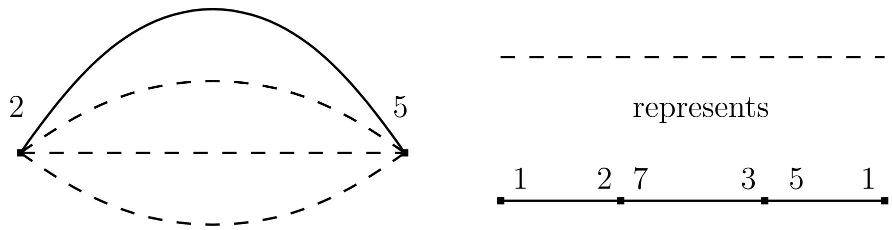


Figure 5.3: The graph of groups on the left is a 3-fold covering of the graph of groups in Figure 5.2.

Case 2. Now assume that the numerator and the denominator of the modular homomorphism of the cycle of the graph of groups are relatively prime. We will label the half-edges cyclicly so that $v_{p_i} = v_{q_i} = v_i$ for $i = 1, .., k$ and, using modulo-k arithmetic, the half-edges p_i and q_{i+1} are contained in the same edge. See Figure 5.4. By assumption we have $(|p_i|, |q_j|) = 1$ for all $i, j \in \{1, ..., k\}$. Let $X_{\Gamma'}$ be an M-fold, finite, regular cover of X_Γ. This cover will represent an index M subgroup G' of G.

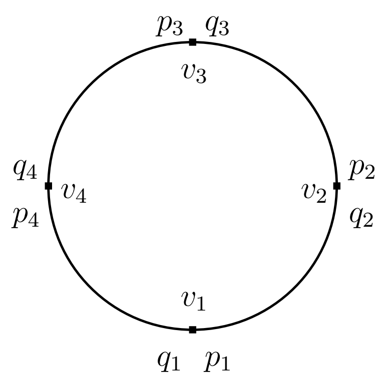


Figure 5.4: A GBS group with this graph of groups structure where $(|p_i|, |q_j|) = 1$ for all $i, j \in \{1, \ldots, k\}$, is not large.

For each vertex $v_i \in \Gamma$ there will be vertices, $v_i' \in \Gamma'$ which represent M_i-fold covers of v_i. Since the cover is regular there will be $\frac{M}{M_i}$ such vertices in Γ' all with the same valence. Thus Γ' will contain $\frac{M}{M_i}(M_i, |p_i|)$ half-edges covering p_i and

$\frac{M}{M_i}(M_i, |q_i|)$ half-edges covering q_i. Since p_i and q_{i+1} are half-edges of the same edge, we have that

$$\frac{M}{M_i}(M_i, |p_i|) = \frac{M}{M_{i+1}}(M_{i+1}, |q_{i+1}|) \tag{5.1}$$

which gives

$$M_i = M_{i+1}\frac{(M_i, |p_i|)}{(M_{i+1}, |q_{i+1}|)}.$$

By continuing to substitute using the equality in Equation 5.1 we get,

$$M_i = M_i\frac{(M_1, |p_1|)\dots(M_k, |p_k|)}{(M_1, |q_1|)\dots(M_k, |q_k|)}$$

which implies

$$\frac{(M_1, |p_1|)\dots(M_k, |p_k|)}{(M_1, |q_1|)\dots(M_k, |q_k|)} = 1.$$

Since $(|p_i|, |q_j|) = 1$, we now know that $(M_i, |p_i|) = (M_i, |q_i|) = 1$ for all $i = 1, ..., k$. This implies that Γ' is a 2-valent graph, and consequently a single closed path with $k\frac{M}{M_i}$ vertices. Lemma 4.2 and Lemma 4.3 imply that G is not large.

□

5.2 Classification of Graphs of Finite Cyclic Groups

Suppose we have a graph of finite cyclic groups with an edge group $\mathbb{Z}_j$ and a vertex group $\mathbb{Z}_k$. The half-edge will be labeled by a positive integer m such that $m|k$, representing the multiplication by m map $\mathbb{Z}_j \to \mathbb{Z}_k$. This relationship between the edge groups, vertex groups and half-edges, makes it possible to prove the following theorem.

Theorem 5.3. *Let G be the fundamental group of a graph of groups $\mathcal{G}$ of finite* *cyclic groups.*

1. *If $\mathcal{G}$ has rank greater than* 1*, then G is large.*

2. *If $\mathcal{G}$ has rank* 1*, then G is large if and only if it is not isomorphic to $\mathbb{Z}_k \rtimes \mathbb{Z}$ for some $k \geq 1$.*

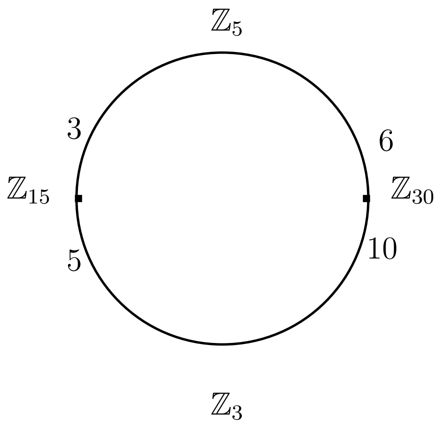
3. *If $\mathcal{G}$ has rank* 0*, then G is large if and only if it is not isomorphic to $\mathbb{Z}_k$ or virtually $\mathbb{Z}_k \rtimes \mathbb{Z}$ for some $k \geq 1$.*

Rank 1 *Case:* If there is a separating half-edge m such that $|m| \geq 2$ and $\mathcal{H}_m$ is a rank one graph, we know from Theorem 3.4 that G is large. We may therefore assume that Γ is homeomorphic to a circle.

Now suppose that there are half-edges m and n such that $|m|, |n| \geq 2$, $\mathcal{H}_{m,n} = \mathcal{H}_{n,m}$, and $(|m|, |n|) = d > 1$. If such a pair exists we can assume that it is maximal in the sense that any other half-edge p with $(d, |p|) > 1$ then $p \subset \mathcal{H}_{m,n}$. We will build $\widehat{X_\Gamma}$ starting with vertex space $\widehat{X_{v_m}}$ and $\widehat{X_{v_n}}$ that will represent d-fold covers of the spaces X_{v_m} and X_{v_n} respectively. In the case that $v_m = v_n$, there will be only one cover $\widehat{X_{v_m}}$.

We will attach d copies of the total space of $\mathcal{H}_{m.n}$ to $\widehat{X_{v_m}}$ and $\widehat{X_{v_n}}$ by the lifts of f_m and f_n respectively. If $v_n = v_m$ we are done. If not, we notice that for all vertices and edges v and e in $\mathcal{H}^c_{m,n}$ that $d|k$ and $d|j$ where $G_v \cong \mathbb{Z}_k$ and $G_e \cong \mathbb{Z}_j$ so there is a d-fold cover for this portion of the total space. See Figure 5.5.

Now suppose there exists no such pair of half-edges. We will label the half-edges cyclicly so that $v_{p_i} = v_{q_i} = v_i$ for $i = 1, .., l$ and, using modulo-l arithmetic, the half-edges p_i and q_{i+1} are contained in the same edge j_i. See Figure 5.6. By assumption we have $(|p_i|, |q_{i'}|) = 1$ for all $i, i' \in \{1, ..., l\}$. Let $\mathbb{Z}_{j_i}$ be the edge group for the edge connecting v_i to v_{i+1} and let $\mathbb{Z}_{k_i}$ be the vertex group



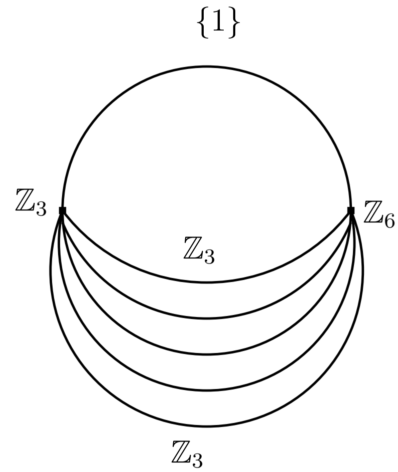


Figure 5.5: The graph of groups on the right is a 5-fold covering of the graph of groups on the left.

represented by v_i. Since $p_i|k_i$ and $q_i|k_i$ for all i we have $k_i = k_i' p_i q_i$ for all i. We also have $j_i = k_i' q_i = k_{i+1}' p_{i+1}$.

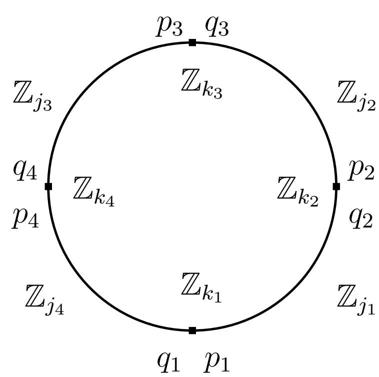


Figure 5.6:

This gives us $k_i' = k_{i+1}' \frac{p_{i+1}}{q_i} = k_i' \frac{p_i \ldots p_l}{q_i \ldots q_l}$. This implies that $p_i = q_i = 1$ for all i, thus $k_i = k = j_i$ for all i. Thus after collapse moves our graph is a single vertex representing $\mathbb{Z}_k$ and a single edge also representing $\mathbb{Z}_k$ and $G \cong \mathbb{Z}_k \rtimes \mathbb{Z}$.

□

Rank 0 *Case:* Theorem 3.3, Theorem 3.5, and Theorem 4.1 show all rank 0 graphs of finite cyclic groups are large except for the graphs with a single vertex which will have fundamental group $\mathbb{Z}_k$ for some k, and graphs with two vertices and a single edge where the vertex groups are the same and the edge group is an index 2 subgroup of these groups. In this case we have $G_{v_1} = G_{v_2} = \mathbb{Z}_{2k}$ and $G_e = \mathbb{Z}_k$ for $k \geq 1$. We can build the 2-fold cover of this graph as in Theorem 3.5. The fundamental group of this graph of groups will be $\mathbb{Z}_k \rtimes \mathbb{Z}$, so G is virtually $\mathbb{Z}_k \rtimes \mathbb{Z}$.

□

Remark 5.4. Theorem 5.3 can be restated as follows: *Let G be the fundamental group of a graph of finite cyclic groups. Then G is large if and only if it is not isomorphic to $\mathbb{Z}_k$ or virtually $\mathbb{Z}_k \rtimes \mathbb{Z}$.*

Chapter 6

Graphs of Abelian Groups with Infinite Index Edge Groups

In this chapter we give results for graphs of groups with all vertex groups isomorphic to $\mathbb{Z}^n$ and each edge groups isomorphic to $\mathbb{Z}^k$ for some $k < n$. Tubular groups are examples of these types of groups where $n = 2$. Tubular groups were introduced in [9] and have been further investigated in [10], [19], and [40].

Theorem 6.1. *Let G be the fundamental group of a graph of groups $\mathcal{G}$, with at least one edge, such that all vertex groups are $\mathbb{Z}^n$ and each edge group is isomorphic $\mathbb{Z}^{k_i}$ for some $k_i < n$. Then G is large.*

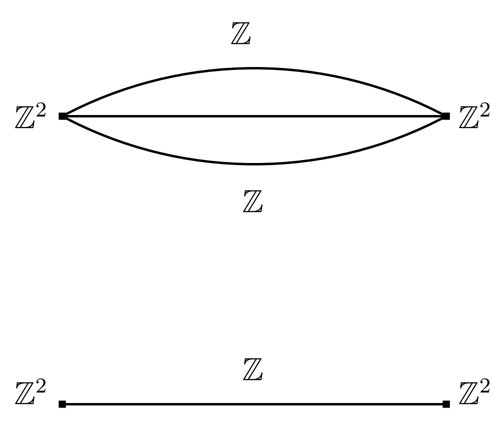


Figure 6.1: A graph of groups and cover as in Theorem 6.1 for $p = 3$.

Proof. The graph of groups $\mathcal{G}$ will have $m > 0$ edges. Let $G_{e_i} = \mathbb{Z}^{k_i}$ for each edge, let $G_v = \mathbb{Z}^n$ for all vertices of $\mathcal{G}$ and let $\mathbf{H}_i$ and $\mathbf{K}_i$ be the half-edges contained in the edge e_i. The half-edges $\mathbf{H}_i$ and $\mathbf{K}_i$ will be $k_i \times n$ matrices that determine the image of each generator of $\mathbb{Z}^{k_i}$. Since the underlying graph of $\mathcal{G}$ is finite, there are only a finite number of half-edges, and hence a finite number of matrix entries. Let p be a prime greater than 2 such that p does not divide any entry of any half-edge matrix of $\mathcal{G}$. Consider the p^n-fold cover of a vertex space corresponding to the subgroup P of $\mathbb{Z}^n$ with diagonal matrix representative where all diagonal entries are p. The number of branches in the cover corresponding to the half-edges $\mathbf{H}_i$ and $\mathbf{K}_i$ will be

$$\frac{[\mathbb{Z}^n : P]}{[H_i : H_i \cap P]} = \frac{p^n}{p^{k_i}} = \frac{[\mathbb{Z}^n : P]}{[K_i : K_i \cap P]},$$

so we can build a cover of the total space starting with this cover at each vertex space. See Figure 6.1. The total space $\widehat{X_\Gamma}$ will retract onto a graph of rank greater than or equal to 2, so G is large. □

Chapter 7

Future Directions

Building on the results of this thesis, there are several avenues of interest for further research. The first and most obvious direction is to attempt a complete classification for graphs of $\mathbb{Z}^2$ groups which could be extended to graphs of $\mathbb{Z}^k$ groups. We can determine the largeness of all rank 0 graphs of free infinite abelian groups, but there are still many rank 1 graphs for which no results have been given. The next step would be to look at graphs of general finitely generated infinite abelian groups.

Largeness of graphs of other types of groups, such as nilpotent or polycyclic groups, is another direction in which this research could be taken. Eventually, I would like to understand the largeness of graphs of free groups. This seems to be a very difficult problem. For example, the case where the graph has a single vertex and a single edge labeled F_k with injective homomorphisms of index 1 is still far from being completely understood, and the largeness of hyperbolic free-by-cyclic groups is completely open.

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