UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

GENERATING SIMULATION RELATIONS FOR CERTAIN NONLINEAR CONTROL SYSTEMS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma
2011

GENERATING SIMULATION RELATIONS
FOR CERTAIN NONLINEAR CONTROL SYSTEMS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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ACKNOWLEDGEMENTS

To my husband, Dr. Marius Munteanu, with love and utmost respect and consideration,

to my Ph.D. advisor, Dr. Kevin Grasse, for having taught me and helped me so much throughout the years, and for having always been so kind with me,

to my Ph.D. committee members: Dr. Tomasz Przebinda, for having encouraged me to start doing research in the first place, Dr. Murad Ozaydin, for having always been such an example of a professor and researcher for me, Dr. Walter Wei, Dr. John Fagan, and to all of my Romanian and American professors - I thank you all!

To the memories of my mother in law, Ioana Munteanu, and my grandfather, Iordache Talpalaru, for their sacrifices,

to my sister, Dr. Aurelia Balan, for having always been there for me,

and to my parents, Dr. Aurel and Maria Florescu - I love you both very much!

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ABSTRACT

In this dissertation, we introduce a new algorithm for generating simulation relations between nonlinear control systems that are affine in inputs and disturbances and provide precise mathematical conditions ensuring that the algorithm works as intended. Moreover, we prove that under appropriate conditions, making the "right choices" in the algorithm leads to a maximal simulation relation of the first system by the second. We also construct several illustrative examples showing in detail how the algorithm works in specific instances and also indicate some of the limitations of the algorithm.

0. INTRODUCTION

In broad terms, control theory is the study of ordinary differential equations or dynamical systems containing a parameter called a control. Different values of the control parameter induce different solutions or system trajectories, thus allowing one to influence or "control" the evolution of the system. Consequently, starting at a specified initial value of the system, one can reach a multitude of states depending on the choice of the control parameter. In this context, there are several questions that can be raised. For example, in investigating the reachability problem, one of the questions of interest is whether or not one can control the system in order to reach a desired final state by starting from a given initial state. A thorough discussion of this question, as well as other fundamental foundational issues, can be found in [**7**] and [**17**].

One of the important steps in the development of the theory of control systems was the realization that the state space, i.e., the space in which the trajectories of the system evolve, is in many cases not an Euclidean space or even a space diffeomorphic to an Euclidean space but rather a differentiable manifold. For example, when studying the dynamics of a rigid body, the state space naturally associated to the system is $SO(3)\times\mathbb{R}^3$, where $SO(3)$ denotes the group of all three-dimensional rotations about the origin. For other interesting examples and applications involving mechanical control systems, the reader is referred to [**2**] or [**17**]. Since, in general, the state space of a control system is a differentiable manifold, this led to the development of a new branch of control theory called geometric control theory in which differential geometric concepts such as vector fields, distributions, or vector bundles are employed in the study of control systems.

An overarching problem in geometric control theory is represented by the classification of control systems. In one of the first attempts in this direction, control systems have been classified based on state space equivalence. When regarded as families of vector fields indexed by a control parameter, control systems are said to be state space equivalent if there exists a diffeomorphism between the state spaces of the two systems whose differential "conjugates"the two families of vector fields (see [**2**]). So, if we assume for simplicity that for both systems the state space is $\mathbb{R}^m$ and the control space is $\mathbb{R}^d$, then the systems

$$\dot{x} = f(x,u) \text{ and } \dot{y} = g(y,u), \quad f,g : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m, \quad x(t), y(t) \in \mathbb{R}^m, u(t) \in \mathbb{R}^d,$$

are said to be state space equivalent if there exists a diffeomorphism $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\text{for any } x \in \mathbb{R}^m, \omega \in \mathbb{R}^d \text{ we have } \Phi_{*x}(f(x,\omega)) = g(\Phi(x),\omega).$$

For systems that are analytic and transitive, Krener ([**10**]) obtained a complete characterization of local state space equivalence in terms of (infinitely many) iterated Lie brackets involving the vector fields associated to the two systems. A similar characterization for global equivalence was derived by Sussman in [**18**] and [**19**] under additional assumptions. The problem for smooth systems has been further analyzed in [**8**]. However, as it has been noted, despite its natural definition, state space equivalence induces too many equivalence classes. So one needs to look for more general equivalence relations.

One step in this direction was the introduction of feedback equivalence (see [**1**] and [**9**]). Two systems as above are called feedback equivalent if, in addition to the diffeomorphism Φ considered above, we also have a map $\Psi : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ such that the map $(x, \bar{x}) \to (\Phi(x), \Psi(x, \omega))$ is a diffeomorphism and

$$\text{for any } x \in \mathbb{R}^m, \omega \in \mathbb{R}^d \text{ we have } \Phi_{*x}(f(x, \omega)) = g(\Phi(x), \Psi(x, \omega)).$$

The latter equivalence notion generalizes the former in that while the state space equivalence requires that both systems have the same input, for feedback equivalence the corresponding input for the second system is now a function of the state parameter x and the input ω. However, one common drawback is that in both cases the state spaces must be diffeomorphic and hence have the same dimension.

A different approach to generalizing the state space equivalence concept has been taken in [**11**] and [**12**], where the authors employ model reduction techniques to reduce the dimension of the state space of the second system while still requiring the same input function for both systems. More recently, Pappas introduced the concept of $\Phi-$related control systems (see [**14**]), which further generalizes both state space and feedback equivalence. The main idea in the aforementioned reference is that we start with a "complex"system (where complexity could refer to a large dimension of the state space or a high degree of nonlinearity) and we want to "relate"or "abstract"it with a "simpler"system. So, two systems given as

$$\dot{x} = f(x, u), \quad f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m, \quad x(t) \in \mathbb{R}^m, u(t) \in \mathbb{R}^d,$$

$$\dot{y} = g(y, v), \quad g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, \quad y(t) \in \mathbb{R}^n, v(t) \in \mathbb{R}^p$$

are said to be $\Phi-$related if there exists a mapping $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ whose fundamental property is that it sends trajectories of the first system onto trajectories of the second. We also say that the second system is an abstraction of the first. As noted in [**14**], this condition is equivalent to

$$\Phi_{*x}\{f(x, \omega) | \omega \in \mathbb{R}^d\} \subseteq \{g(\Phi(x, \theta)) | \theta \in \mathbb{R}^p\}$$

for all $x \in \mathbb{R}^m$, which can be seen to be a generalization of both state space and feedback equivalence.

It is interesting to observe that, regardless of whether we have state space equivalence or $\Phi-$related systems, the trajectories of the first system can be "mapped" to trajectories of the second in a nice geometric manner, that is, by means of Graph(Φ) $\subset \mathbb{R}^m \times \mathbb{R}^n$, where Φ is the function "relating" the two systems. Indeed, any trajectory of the first system can be "lifted" to a curve in Graph(Φ), which then projects onto a trajectory of the second system. Based on this idea, one can actually define a new way of relating two systems by replacing Graph(Φ) $\subset \mathbb{R}^m \times \mathbb{R}^n$ by a subset $\mathcal{R}$ of $\mathbb{R}^m \times \mathbb{R}^n$ with the property that certain trajectories of the first systems can be "paired up" with trajectories of the second through $\mathcal{R}$. We call $\mathcal{R}$ a simulation relation of the first system by the second and, if the simulation relation persists when the roles of the two systems can be reversed, we then say that $\mathcal{R}$ is a bisimulation relation. The reason that we are not requiring all trajectories of the first system to paired up with trajectories of the second system is that we are allowing relations $\mathcal{R}$ which may not project onto the entire state space of the first system. However, if every trajectory of the first system can be paired up with some trajectory of the second system, we say that the second system is a simulation of the first. If the same holds when replacing the first system by the second, then the two systems are called bisimilar.

The (bi)simulation concept is a relatively new development in geometric control theory, which was inspired by analogous concepts in automata theory. In [**6**], (bi)simulation relations were first discussed in the context of mathematical control systems in the relatively abstract setting of morphisms and categories. In the case of linear systems, specific results characterizing bisimulation relations induced by linear surjections have been established in [15]. For nonlinear systems that are affine in the control, in [20], Tabuada and Pappas discussed bisimulation relations induced by nonlinear submersions and provided an algebraic characterization for local bisimulation relations. Further results focusing on "admissible" controls and disturbances for both simulation and bisimulation relations have been obtained by Grasse (see [3],[4],[5]).

An obvious problem in the study of (bi)simulation relations is to determine whether or not there exists a (bi)simulation relation between two given control systems. A partial answer to this question was provided in [16] by van der Schaft who introduced an algorithm for computing maximal (bi)simulation relations between systems that are affine in inputs and disturbances. However, it should be noted that, while innovative, the algorithm (especially in the nonlinear case) is only presented in a broad, heuristic manner, with few mathematical details and no examples.

In this dissertation, inspired by [16], we introduce a new algorithm for generating simulation relations between nonlinear control systems that are affine in inputs and disturbances and provide precise mathematical conditions ensuring that the algorithm works as intended. Relative to the maximality issue, we prove that under appropriate conditions, making the "right choices" in the algorithm leads to a maximal simulation relation of the first system by the second. In addition, we construct several illustrative examples showing in detail how the algorithm works in specific instances and also indicate some of the limitations of the algorithm.

In Chapter 1, we review some basic differential geometric concepts and discuss the Constant Rank Theorem and its proof in the context of differentiable manifolds. We pay close attention to connectivity, which is an aspect that is often overlooked in the discussion of this theorem, yet it plays a significant role in our investigations. In addition, we present a mild generalization of the Constant Rank Theorem showing that, under certain conditions, the preimage of a submanifold is a submanifold as well. We note that the case we are addressing is in some sense dual to the classical case when the function is transverse to the submanifold.

In Chapter 2, we discuss some fundamental constructions involving vector bundles such as the Whitney sum of two vector bundles and the bundle of homomorphisms associated to a vector bundle. We also show that, given a vector subbundle of a vector bundle, we can construct a smooth section of the homomorphism bundle of the larger vector bundle having a special property with respect to sections of the smaller bundle. This section allows us to construct a function which is instrumental in obtaining some of our main results and whose regularity properties, when satisfied, ensure that some of the sets we consider in Chapter 4 are submanifolds.

In Chapter 3, we define the concept of input-disturbance-output (IDO) system as a refinement of the control system concept by taking into account the different types of controls that could arise (inputs and disturbances) and the fact that, in many applications, it is the external behavior rather than the behavior of the system itself that is of interest. In section 3.2., we introduce pointwise and admissible simulation relations between IDO systems and note that, as shown in [**4**], under appropriate conditions, the two definitions are equivalent (see Theorem 3.2.8). The introduction of both of these definitions is motivated by the fact that, in general, the algorithm we introduce in Chapter 4 generates pointwise simulation relations. However, from the point of view of trajectories and controls, the proper concept is that of admissible simulation relations.

Chapter 4 contains our main results. First, we construct an algorithm for generating (pointwise) simulation relations between two systems and prove that the algorithm does indeed generate a simulation relation as claimed (provided such a relation exist) and that termination occurs in a finite number of steps (see Theorem 4.1.). In addressing the problem of generating a maximal simulation relation, we introduce the notion of regular pre-simulation relations up to some specified order and show that if this pre-simulation condition is satisfied up to a certain order depending only on the dimensions of the state spaces, then, by making the proper choices in the algorithm, the set we obtain at termination does become a maximal (pointwise) simulation relation. We also obtain that this set is maximal among all admissible simulations satisfying a certain "disturbance constant rank" condition.

In Chapter 5, we present several examples illustrating how the algorithm and our results from Chapter 4 can be applied in specific cases. In particular, our examples show that the various potential outcomes resulting from the application of the algorithm are not only theoretical possibilities but can in fact occur for specific control systems.

1. GENERALITIES ON DIFFERENTIAL GEOMETRY

1.1. Basic Definitions

In all of our investigations, we will use the definition of a differentiable manifold as given in [**21**]. So, M^m is an $m-$dimensional differentiable manifold of class C^k ($k \in \mathbb{N}$ or $k = \infty$) if M is an $m-$dimensional, second countable, locally Euclidean space endowed with a differentiable structure of class C^k. We will refer to differentiable manifolds of class C^∞ as smooth manifolds, or simply as manifolds. The tangent space to M at a point $p \in M$ will be denoted by T_pM.

Definition 1.1.1. Given two smooth manifolds M^m and N^n, a function $f : M^m \to N^n$ is said to be *differentiable at* $p \in M$ if for any local charts $(U, \phi)(\phi : U \to \phi(U) \subseteq \mathbb{R}^m)$ around p and $(V, \psi)(\psi : V \to \psi(V) \subseteq \mathbb{R}^n)$ around $f(p)$, the function

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \mathbb{R}^n$$

is differentiable at $\phi(p)$. We use the terms "map" and "function" interchangeably.

If $f : M \to N$ is a smooth function, then its differential at $p \in M$ will be denoted by $f_{*p} : T_pM \to T_{f(p)}N$.

Definition 1.1.2. Let $f : M \to N$ be a smooth map.

a) f is called an *immersion* if $f_{*p} : T_pM \to T_{f(p)}N$ is injective for all $p \in M$.

b) The pair (M, f) is called a *submanifold* of N if f is a one-to-one immersion.

c) The function f is called an *imbedding* if f is a one-to-one immersion and a homeomorphism onto $f(M)$ with the subspace topology.

d) If f is one-to-one and onto and if f^{-1} is smooth then f is called a *diffeomorphism.*

Remark 1.1.3. As it was noted in [**21**], there exists an equivalence relation on the set of submanifolds of a given manifold defined as follows. Two submanifolds (M_1, f_1) and (M_2, f_2) of the manifold N are equivalent if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $f_1 = f_2 \circ \phi$. In this manner, for each equivalence class there exists a representative of the form (A, i), where A is a subset of N and $i : A \to N$ is the inclusion. Indeed, if (M, f) is any other representative in the given class, then we can let $A = f(M)$. The manifold structure on A is the one induced from M by requiring f to be a diffeomorphism.

While immersions are fundamental in the definition of submanifolds, regular points and submersions are essential tools in the construction and better understanding of submanifolds. We define these concepts below.

Definition 1.1.4. If $f : M^m \to N^n$ is a smooth map then a point $p \in M$ is called a *regular point of f* if the rank of f_{*p} is maximal. Otherwise, p is called a *critical point.* A point $q \in N$ is called a *regular value of f* if $f^{-1}(q)$ consists only of regular points.

Definition 1.1.5 Given a smooth map $f : M^m \to N^n$, f is called a *submersion* if it is surjective and f_{*p} is onto for every point $p \in M$.

As we will see in the next section, submersions and, in general, constant rank mappings allow one to construct submanifolds as preimages of single points.

1.2. The Constant Rank Theorem

Under appropriate circumstances, the following theorem provides the means of recognizing a subset of a manifold as a submanifold. The theorem is well known, but we chose to provide a proof for completeness.

Theorem 1.2.1 (The Constant Rank Theorem). Let M^m and N^n be two smooth manifolds and let $f : M^m \to N^n$ be a smooth mapping of constant rank $k \le \min(m, n)$. For every $q \in N$, the set $\{p \in M | f(p) = q\}$ is a union of at most countably many disjoint, connected, closed, imbedded submanifolds of M of codimension k.

For the proof we will first need the Rank Theorem.

Theorem 1.2.2. (The Rank Theorem - Euclidean Case). Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map having constant rank $k \le \min(m, n)$. For each $p \in \mathbb{R}^m$, there are local charts (U, ϕ) at p and (V, ψ) at $f(p)$ such that

$$\psi \circ f \circ \phi^{-1}(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0).$$

Proof. Without loss of generality, we may assume that $p = 0 \in \mathbb{R}^m$, $f(p) = 0 \in \mathbb{R}^n$, and the $k \times k$ matrix in the upper left corner of the Jacobian matrix associated to f_{*0} is nonsingular. Consider the function $\phi : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\phi(x) = (f^1(x), f^2(x), \ldots, f^k(x), x_{k+1}, \ldots, x_m)$$

and observe that the Jacobian associated to ϕ at 0 is

$$J_\phi(0) = \begin{bmatrix} \frac{\partial(f^1, f^2, \ldots, f^k)}{\partial(x_1, x_2, \ldots, x_k)}(0) & * \\ 0 & I_{m-k} \end{bmatrix},$$

where I_{m-k} denotes the identity matrix of order $(m - k)$.

Since the determinant of $J_\phi(0)$ is nonzero, by the Inverse Function Theorem (see [**21**]), there exists an open neighborhood of $0 \in \mathbb{R}^m$ so that the restriction of ϕ to this neighborhood is a local diffeomorphism onto its image. If we denote the image of this neighborhood under ϕ by U, then ϕ^{-1} is well defined and smooth on U.

If we let $\pi_i : \mathbb{R}^m \to \mathbb{R}$ the projection onto the $i-$th coordinate, $1 \le i \le n$, then, from the definition of ϕ, it follows that $\pi_i \circ \phi = f^i, 1 \le i \le k$, thus $\pi_i = f^i \circ \phi^{-1}$ on U.

Moreover, if we let $g_i := f^i \circ \phi^{-1}, k+1 \leq i \leq n$ then, for all $x \in U$, we have

$$f \circ \phi^{-1}(x) = (x_1, x_2, \ldots, x_k, g_{k+1}(x), \ldots, g_n(x)).$$

This way, the Jacobian of $f \circ \phi^{-1}$ at any $x \in U$ is

$$J_{f\circ\phi^{-1}}(x) = \begin{bmatrix} I_k & 0 \\ * & \frac{\partial(g_{k+1}, g_{k+2}, \ldots, g_n)}{\partial(x_{k+1}, x_{k+2}, \ldots, x_m)}(x) \end{bmatrix}.$$

As $J_{f\circ\phi^{-1}}(x) = J_f(\phi^{-1}(x)) \circ J_{\phi^{-1}}(x)$ and since ϕ^{-1} is a local diffeomorphism, we have $\mathrm{rank}(J_{f\circ\phi^{-1}}(x)) = \mathrm{rank}(J_f(\phi^{-1}(x))) = \text{constant} = k$, since f has constant rank k. On the other hand, by the form of $J_{f\circ\phi^{-1}}(x)$, its rank is at least k with equality if and only if $\frac{\partial g_i}{\partial x_j}(x) = 0$ for all i, j with $k+1 \leq i \leq n, k+1 \leq j \leq m$, and all $x \in U$. It follows that $g_{k+1}, \ldots, g_n$ are independent of $x_{k+1}, \ldots, x_m$ on U.

By shrinking U if necessary, we may assume that for all $x \in U \subseteq \mathbb{R}^m$ we have $(x_1, x_2, \ldots, x_k, 0, \ldots, 0) \in U$. Thus, we can define $\psi : U \to \mathbb{R}^n$ as follows. For any $x = (x_1, x_2, \ldots, x_m) \in U$, let

$$\psi(x) = (x_1, \ldots, x_k, x_{k+1} - g_{k+1}(x_1, \ldots, x_k, 0, \ldots, 0), \ldots, x_n - g_n(x_1, \ldots, x_k, 0, \ldots, 0)).$$

Now we have that for any $x \in U$

$$\psi \circ f \circ \phi^{-1}(x) = \psi(x_1, \ldots, x_k, g_{k+1}(x), \ldots, g_n(x)) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

To finish the proof, note that ψ is a local diffeomorphism. Indeed, its Jacobian at 0 is

$$J_\psi(0) = \begin{bmatrix} I_k & 0 \\ * & I_{n-k} \end{bmatrix}.$$

So, by the Implicit Function Theorem, ψ is a diffeomorphism on a neighborhood V of 0. $\square$

Theorem 1.2.3. (The Rank Theorem - General Case) If M^m and N^n are smooth manifolds and if $f : M^m \to N^n$ is a smooth map having constant rank $k \leq \min(m, n)$, then for each $p \in M$, there exist local charts (U, ϕ) at p and (V, ψ) at $f(p)$ such that

$$\psi \circ f \circ \phi^{-1}(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0).$$

Proof. This can be easily seen to follow from the Euclidean case. Indeed, by considering any local charts (U_1, ϕ_1) at p and (V_1, ψ_1) at $f(p)$, the function $\psi_1 \circ f \circ \phi_1^{-1}$ satisfies the hypotheses of the Euclidean case and the local charts (U, ϕ) and (V, ψ) as in the conclusion of the general case can be constructed by appropriately changing the original charts. $\square$

The following proposition shows that being an imbedded submanifold is a local property and also plays an important role in the proof of the Constant Rank Theorem.

Proposition 1.2.4. Let M be a smooth manifold and let $S \subseteq M$. If for each $x \in M$ there exists an open set $U \subseteq M$ containing x such that $U \cap S$ is an imbedded submanifold of U of dimension s, then there exists a differentiable structure on S such that S is an imbedded submanifold of M of dimension s.

Proof. Let $\mathcal{U}$ be the collection of all open subsets U as above. For each $U \in \mathcal{U}$, let $i_{U \cap S, U} : U \cap S \to U$ denote the inclusion. By the definition of an imbedded submanifold, it follows that there exists a differentiable structure on $U \cap S$ with respect to which i is a one-to-one immersion and i is a homeomorphism into. This implies that the topological structure on $U \cap S$ is the restriction of the topological structure on U (and hence M) to S. First, let us construct a topological structure on S as follows: a subset $V \subset S$ is open if it is a union of open subsets of $U \cap S$, with U ranging over a subset of $\mathcal{U}$. Observe that the topology on S defined this way coincides with the subspace topology induced from M. Clearly, every open set in S is the intersection of an open set in M with S. To show that for every open subset $W \subseteq M$ we have $W \cap S$ is open in S with the topology constructed above, note that

$$W \cap S = \bigcup_{x \in W \cap S} (U_x \cap W) \cap S,$$

where U_x denotes any open subset of M for which $U_x \cap S$ is a submanifold of U. But $(U_x \cap W) \cap S$ is an imbedded submanifold of $U_x \cap W$ since the intersection of an imbedded submanifold with an open set from the ambient manifold is an imbedded submanifold in the chosen open set, with the role of the ambient manifold being played by $U_x \cap S$ and that of the open set by $U_x \cap W$. In summary, we obtain that $U_x \cap W \in \mathcal{U}$ for all $x \in W \cap S$, which, by the relation above, implies that $W \cap S$ is open in the constructed topology of S.

Since the topology on S is the restriction of the topology on M and since M is Hausdorff and second countable, it follows that S has the same properties. In particular, as a consequence of the fact that S is second countable, we obtain that S has countably many components. This observation plays an important role in our investigations.

To define the differentiable structure on S, consider $U \in \mathcal{U}$ and a local chart (V, ϕ) in $U \cap S$. Since V is open in $U \cap S$, it is also open in S and we can declare (V, ϕ) to be a local chart in S. To establish the compatibility of the charts, consider two charts (V_1, ϕ_1) and (V_2, ϕ_2) around some $x \in S$. If both of them are charts in $U \cap S$ for some $U \in \mathcal{U}$, then the charts are clearly compatible. Otherwise, let us assume that (V_1, ϕ_1) is a chart in $U_1 \cap S$ and (V_2, ϕ_2) is a chart in $U_2 \cap S$ with $U_1 \cap U_2 \cap S \neq \emptyset$. If $i_{U \cap S, U}$ denotes the inclusion of $U \cap S$ into U, then by the Rank Theorem, there exist charts $(\bar{V}_1, \bar{\phi}_1)$ and $(\bar{V}_2, \bar{\phi}_2)$ around x on $U_1 \cap S$ and $U_2 \cap S$, respectively, and $(\bar{U}_1, \psi_1)$ and

$(\bar{U}_2, \psi_2)$ on U_1 and U_2, respectively, such that

$$\psi_1 \circ i_{\bar{U}_1 \cap S, \bar{U}_1} \circ \bar{\phi}_1^{-1}(x_1, x_2, \ldots, x_s) = \psi_1 \circ \bar{\phi}_1^{-1}(x_1, x_2, \ldots, x_s) = (x_1, x_2, \ldots, x_s, 0, \ldots, 0)$$

and

$$\psi_2 \circ i_{\bar{U}_2 \cap S, \bar{U}_2} \circ \bar{\phi}_2^{-1}(x_1, x_2, \ldots, x_s) = \psi_2 \circ \bar{\phi}_2^{-1}(x_1, x_2, \ldots, x_s) = (x_1, x_2, \ldots, x_s, 0, \ldots, 0).$$

Note that it is enough to establish the compatibility of $(\bar{V}_1, \bar{\phi}_1)$ and $(\bar{V}_2, \bar{\phi}_2)$ since (V_1, ϕ_1) and (V_2, ϕ_2) are compatible with $(\bar{V}_1, \bar{\phi}_1)$ and $(\bar{V}_2, \bar{\phi}_2)$, respectively, and $V_1 \cap \bar{V}_1 \neq \emptyset$, $V_2 \cap \bar{V}_2 \neq \emptyset$. So, without loss of generality we may assume that $(V_1, \phi_1) = (\bar{V}_1, \bar{\phi}_1)$ and $(V_2, \phi_2) = (\bar{V}_2, \bar{\phi}_2)$. But then, since $\psi_2 \circ \phi_2^{-1} = \psi_1 \circ \phi_1^{-1}$ on $\phi_1(V_1 \cap V_2) \cap \phi_2(V_1 \cap V_2)$, it follows that

$$\phi_1 \circ \phi_2^{-1} = \psi_1 \circ (\psi_1^{-1} \circ \phi_1) \circ \psi_2^{-1} \circ i,$$

where $i : \mathbb{R}^s \to \mathbb{R}^n$ denotes the inclusion of the first s factors and s denotes the common dimension of $V_1 \cap S$ and $V_2 \cap S$, Clearly, all maps in the composition above are smooth, thus showing compatibility.

Next, we need to prove that the inclusion $i_{S,M} : S \to M$ is an imbedding. To this end, note that for any $x \in S$, $i_{S,M*x}$ is one-to-one since $i_{S,M}$ coincides with $i_{U,M} \circ i_{U \cap S, U}$ on $U \cap S$ for some $U \in \mathcal{U}$ with $x \in U$ and both $i_{U,M}$ and $i_{U \cap S, U}$ are immersions. To show that $i_{S,M}$ is a homeomorphism it is enough to recall that the manifold topology on S coincides with the subspace topology coming from M. $\square$

Proof of the Constant Rank Theorem. Let $p \in f^{-1}(q)$ and consider charts (U, ϕ) around p and (V, ψ) around q as in the conclusion of Rank Theorem. Note that

$$x \in f^{-1}(q) \cap U \Leftrightarrow \psi(f(x)) = 0 \Leftrightarrow \phi(x) \in \{(y_1, y_2, \ldots, y_m) \in \phi(U) | y_1 = \cdots = y_k = 0\}.$$

Also note that the set appearing on the right side of the line above is a submanifold of $\phi(U)$. Indeed, the set $P = \{(y_1, y_2, \dots, y_m) \in \mathbb{R}^m | y_1 = \cdots = y_k = 0\}$ is an $(m-k)$-plane, hence a connected (second countable) manifold. Its intersection with the open set $\phi(U)$ is open in P, which implies that any one of its components is an open subset of P. Since the components are open and mutually disjoint subsets of P and since P is second countable, it follows that $P \cap \phi(U)$ has countably many components. But each component is clearly an imbedded submanifold of $\phi(U)$, which, based on the previous comments and the fact that P is closed in $\mathbb{R}^m$, implies that $P \cap \phi(U)$ is a closed, imbedded submanifold of $\phi(U)$.

As ϕ is a diffeomorphism, it follows that

$$f^{-1}(q) \cap U = \phi^{-1}(\{(y_1, y_2, \dots, y_m) \in \phi(U) | y_1 = \cdots = y_k = 0\})$$

is an imbedded submanifold of U. Thus, by Proposition 1.2.4, it follows that $f^{-1}(q)$ is an imbedded submanifold of M having countably many components. Since $f^{-1}(q)$ is closed in M and since the connected components of any set are closed in that set, it follows that each one of the connected components of $f^{-1}(q)$ is a closed subset of M. $\square$

Observation 1.2.5. In many instances we will be required to ascertain whether or not the preimage of an imbedded submanifold $S \subset N$ under a function $f : M \to N$ is itself an imbedded submanifold. The following section provides one solution to this problem. However, if S is the preimage of a point $p \in P$ under some function $g : N \to P$ and if the rank of $g \circ f$ is constant on M then, by the Constant Rank Theorem, $f^{-1}(S) = (g \circ f)^{-1}(p)$ will be a union of mutually disjoint, closed, connected, imbedded submanifolds of M.

1.3. A Mild Generalization of the Constant Rank Theorem

The Constant Rank Theorem shows that the preimage of a point under a constant rank mapping is an imbedded submanifold. If we consider preimages of submanifolds of positive dimension, the same conclusion may no longer be satisfied. In particular, the preimage of an imbedded submanifold is, in general, not an imbedded submanifold, even under constant rank assumptions. Indeed, if we consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x_1, x_2) = (x_1, 0)$, then the rank of f is constant and equal to one. If we also consider a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 0$ iff $-1 \leq x \leq 1$, then the preimage under f of the submanifold $\{(x, g(x)) | x \in \mathbb{R}\} \subset \mathbb{R}^2$ under f is the set $[-1, 1] \times \mathbb{R}$, which is not an imbedded (or even an immersed) submanifold of $\mathbb{R}^2$.

Despite the negative result above, one does have some positive results in this direction given certain transversality assumptions (see Theorem 1.39 in [**21**]). However, there are many instances in which the transversality condition is not satisfied and we would still like to be able to recognize the preimage of an imbedded submanifold as an imbedded submanifold. In this section we investigate a generalization of the Constant Rank Theorem in which the constant rank assumption is substituted by the following:

Definition 1.3.1. Let M and N be smooth manifolds, let $f : M \to N$ be a smooth map, and let $S \subseteq N$ be an imbedded submanifold of N. We say that S is *neat with respect to* f if the following condition is satisfied:

($*$) For every $x \in f^{-1}(S)$ there exists an open set $V \subseteq N$ such that $f(x) \in V$, a function $g : V \to \mathbb{R}^k$ such that $S \cap V = g^{-1}(0)$, and an open set $U \subseteq M$ such that $x \in U, f(U) \subseteq V$, and $\dim[(\ker g_{*f(y)} + \mathrm{im} f_{*y})/\ker g_{*f(y)}]$ is constant for $y \in U$.

Observation 1.3.2. A point $p \in N$ (regarded as a submanifold) is neat with respect to any constant rank map $f : M \to N$. Indeed, for any point $x \in f^{-1}(p)$, consider a chart (V, g) around p such that $g(p) = 0$ and let $U = f^{-1}(V)$. Clearly,

$$\dim[(\ker g_{*f(y)} + \mathrm{im} f_{*y})/\ker g_{*f(y)}] = \dim[(\{0\} + \mathrm{im} f_{*y})/\{0\}] = \mathrm{rank} f_{*y}$$

is independent on $y \in U$ since f has constant rank. Moreover, the remaining conditions in $(*)$ are easily checked.

Observation 1.3.3. Any imbedded submanifold $S \subseteq N^n$ is neat with respect to the inclusion map $i_S : S \to N$. To see this, let $x \in S$ and consider a manifold chart (V, ϕ) of N around x and a neighborhood U of x in S with the property that U is a slice of (V, ϕ), i.e.

$$U = \{z \in V | (\pi_i \circ \phi)(z) = 0, s + 1 \leq i \leq n\},$$

where $1 \leq s \leq n$ represents the dimension of S and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ represents the projection onto the i-th factor (see also [**21**]). Consider now the map $g : V \to \mathbb{R}^{n-s}$ given as $g = (\pi_1 \circ \phi, \ldots, \pi_{n-s} \circ \phi)$. Since S is imbedded, it follows that $g^{-1}(0) = U = S \cap V$. In addition, for any $y \in U$

$$\dim[(\ker g_{*i_S(y)} + \mathrm{im} i_{S*y})/\ker g_{*i_S(y)}] = \dim[(T_yS + T_yS)/T_yS] = 0$$

is constant on U, thus showing that condition $(*)$ is satisfied.

The following lemma clarifies the meaning of the constant term in the definition above.

Lemma 1.3.4. Let X, Y, Z be finite dimensional linear spaces and consider two linear mappings $F : X \to Y$ and $G : Y \to Z$. Then, we have:

$$\mathrm{rank}(G \circ F) = \dim[(\ker(G) + \mathrm{im}(F))/\ker(G)].$$

Proof. Indeed, first note that $\operatorname{rank}(G \circ F) = \dim(G(F(X))) = \dim(\operatorname{im}(G|_{F(X)}))$. If we consider the linear mapping $G|_{F(X)} : F(X) \to Z$ and apply the first isomorphism theorem, we obtain that $\operatorname{im}(G|_{F(X)})$ is isomorphic to $\operatorname{im}(F)/\ker(G|_{F(X)})$. Moreover, since $\ker(G|_{F(X)}) = \ker(G) \cap \operatorname{im}(F)$, we have

$$\operatorname{im}(G|_{F(X)}) \cong \operatorname{im}(F)/(\ker(G) \cap \operatorname{im}(F)) \cong (\ker(G) + \operatorname{im}(F))/\ker(G),$$

where the second isomorphism above follows from the second isomorphism theorem. $\square$

Returning to Definition 1.3.1, we see that the expression appearing in condition $(*)$ equals $\operatorname{rank}(g \circ f)_{*y}$. Based on this observation, we can prove the following:

Theorem 1.3.5. Let M and N be smooth manifolds and let $f : M \to N$ be a smooth map. If $S \subseteq N$ is a closed submanifold of N and if S is also neat submanifold with respect to f, then $f^{-1}(S)$ is a union of at most countably many disjoint, connected, closed, imbedded submanifolds of M.

Proof. We will show that for every $x \in f^{-1}(S)$ there exists an open set $U \subseteq M$ such that $f^{-1}(S) \cap U$ is an imbedded submanifold of U. Since S is neat with respect to f, there exists an open subset $U \subseteq M$ containing x, an open subset V containing $f(x)$, a function $g : V \to \mathbb{R}^k$ such that $f(U) \subseteq V, S \cap V = g^{-1}(0)$ and $\dim[(\ker g_{*f(y)} + \operatorname{im} f_{*y})/\ker g_{*f(y)}]$ is constant for $y \in U$. Note that, by Lemma 1.3.4, we obtain that $\operatorname{rank}(g \circ f)$ is constant on U. So, by the Constant Rank Theorem applied to the function $(g \circ f)|_U : U \to \mathbb{R}^k$ we obtain that $((g \circ f)|_U)^{-1}(0)$ is an imbedded submanifold of U of codimension $\operatorname{rank}(g \circ f)$. But

$$((g \circ f)|_U)^{-1}(0) = (f|_U)^{-1}((g|_{f(U)})^{-1}(0)) = (f|_U)^{-1}(g^{-1}(0) \cap f(U)) =$$

$$= (f|_U)^{-1}(S \cap V \cap f(U)) = (f|_U)^{-1}(S \cap f(U)) = f^{-1}(S) \cap U.$$

So, by Proposition 1.2.4, $f^{-1}(S)$ is an imbedded submanifold of M, hence so are its countably many connected components. Since f is continuous and S is closed in N, it follows that $f^{-1}(S)$ is closed in M. Moreover, the connected components of $f^{-1}(S)$ are closed in $f^{-1}(S)$ and hence, by the previous observation, they are also closed in M. To summarize, $f^{-1}(S)$ is a union of at most countably many disjoint, connected, closed, imbedded submanifolds. $\square$

Observation 1.3.6. The theorem above generalizes the Constant Rank Theorem. Indeed, for any function $f : M \to N$ having constant rank, by Observation 1.3.2, any point $p \in N$ is neat with respect to f. But then, by Theorem 1.3.5, the conclusion of the Constant Rank Theorem follows.

2. VECTOR BUNDLES AND DISTRIBUTIONS

2.1. Introduction

Definition 2.1.1. A triple (E, π, M) is called a $k-$*dimensional (real) vector bundle* if the following three conditions are satisfied:

(1) E, M are smooth manifolds and $\pi : E \to M$ is a surjective, smooth map.

(2) For every $p \in M, \pi^{-1}(p)$ is a $k-$dimensional (real) vector space.

(3) For every $p \in M$, there exists an open neighborhood U of p and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ with the following properties

(i) $pr_1 \circ \psi = \pi|_{\pi^{-1}(U)}$

(ii) For any $q \in U, \psi_q := pr_2 \circ \psi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \mathbb{R}^k$ is a linear isomorphism, where $pr_1 : U \times \mathbb{R}^k \to U$ and $pr_2 : U \times \mathbb{R}^k \to \mathbb{R}^k$ denote the canonical projections.

We call (U, ψ) a chart of the vector bundle. E, M, and π are called the total space, base space, and projection of the bundle, respectively. For each $p \in M, \pi^{-1}(p)$ is called the fiber over p.

Definition 2.1.2. Given two vector bundles $\pi : E \to M$ and $\tau : F \to M$, we say that τ is a *subbundle of* π if F is a submanifold of $E, \pi|_F = \tau$, and $\tau^{-1}(p)$ is a vector subspace of $\pi^{-1}(p)$ for all $p \in M$.

Definition 2.1.3. Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be two vector bundles. A smooth map $h : E_1 \to E_2$ is called a *bundle map* if there exists a smooth map

$\bar{h} : M_1 \to M_2$ such that $\pi_2 \circ h = \bar{h} \circ \pi_1$ and for any $p_1 \in M_1$ the function $h|_{\pi_1^{-1}(p_1)}$: $\pi_1^{-1}(p_1) \to \pi_2^{-1}(\bar{h}(p_1))$ is linear.

In particular, if $M_1 = M_2$ and there exists a bundle map h that is a diffeomorphism and such that $\bar{h} = id_{M_1}$, then the two vector bundles are said to be *isomorphic.*

Definition 2.1.4. A smooth map $s : M \to E$ is called a *section* of the vector bundle $\pi : E \to M$ if $\pi \circ s = id_M$.

2.2. Examples of vector bundles

Among the many vector bundle constructions, we mention the product (trivial) bundle $\pi_1 : M \times \mathbb{R}^k \to M$, the tangent bundle of a manifold $\pi : TM \to M$, the product of two vector bundles, the pull-back of a vector bundle, the Whitney sum of two vector bundles, and the homomorphism bundle associated to a vector bundle. For our purposes, we will briefly review some of these classical constructions below.

- The product of two vector bundles

Given two vector bundles $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2, \pi_1 \times \pi_2 : E_1 \times E_2 \to M_1 \times M_2$ is a vector bundle whose bundle charts are given by $(U_1 \times U_2, \psi_1 \times \psi_2)$, where (U_1, ψ_1) and (U_2, ψ_2) are bundle charts for π_1 and π_2, respectively.

- The pull-back of a bundle

Let $\pi : E \to M$ be a $k-$dimensional vector bundle and let $f : N \to M$ be a smooth map. Consider the set

$$f^*E = \{(q, v) \in N \times E | f(q) = \pi(v)\}.$$

Also, define $\pi_1 : f^*E \to N$ by $\pi_1(q, v) = q$. It follows that $\pi_1 : f^*E \to N$ (which we will also denoted by $f^*\pi$) has the structure of a $k-$dimensional vector bundle. To see this, choose a bundle chart (U, ψ) for π and a coordinate neighborhood V of $q \in N$ such that $f(V) \subset U$. Then we have $f^*E \cap (V \times \pi^{-1}(U)) = \{(r, \psi_{f(r)}^{-1}(x)); r \in V, x \in \mathbb{R}^k\}$. f^*E has a differentiable structure that makes it a submanifold of $N \times E$. Moreover, if we define $\psi_1 : \pi_1^{-1}(V) \to V \times \mathbb{R}^k$ by $\psi_1(r, v) = (r, \psi_{f(r)}(v))$ then $\pi_1 : f^*E \to M$ becomes a vector bundle with charts of the form (V, ψ_1).

Observation 2.2.1. If $f : N \to M$ is a smooth map and if (E, π, M) is a vector bundle then any smooth section s of E induces a smooth section f^*s of f^*E as follows:

$$f^*s : N \to f^*E, \quad (f^*s)(n) = (n, s(f(n))), n \in N.$$

Note that f^*s is indeed a section of f^*E since $\pi_1(f^*s)(n) = \pi_1(n, s(f(n))) = n$. In particular, if $E = TM$ is the tangent bundle of a manifold and if X is a smooth section of this vector bundle, i.e., a smooth vector field, then f^*X is a smooth section of f^*TM.

Observation 2.2.2. Let $i : N \to M$ be a submanifold of M, let X be a smooth vector field on M, and consider the restriction $X \circ i$ of X to N. Notice that, by the previous observation, i^*X is a section of the vector bundle $(i^*(TM), \pi_1, N)$. Let us also consider the bundle map $\hat{i} : i^*(TM) \to TM$ given by $\hat{i}(p, v) = v, p \in N, v \in T_pM$. We claim that for all $p \in N$ we have $\hat{i}(i^*X(p)) = (X \circ i)(p)$. Indeed, $\hat{i}(i^*X(p)) = \hat{i}(p, X(p)) = X(p)$. This relation allows us to identify restrictions of vector fields on M along a submanifold N with sections of the vector bundle $(i^*(TM), \pi_1, N)$ with the identification map given by $\hat{i}$. Similarly, we can identify the restriction to N of a subbundle D of TM with i^*D.

Observation 2.2.3. If $i : N \to M$ is a submanifold of M, by using the injection $TN \to i^*TM$ given by $v \to (n, i_{*n}v), v \in T_nN, n \in N$, we can also identify (any vector subbundle of) TN with a vector subbundle of i^*TM.

- The Whitney sum

Consider two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ over the same base. If $\Delta : M \to M \times M$ is the diagonal map, then $\pi_1 \oplus \pi_2 := \Delta^*(\pi_1 \times \pi_2)$ is called the Whitney sum of the two vector bundles. By the two constructions above, it follows that the fiber of the Whitney sum at p is $E_{1,p} \oplus E_{2,p}$ and if (U_i, ψ_i) is a bundle chart for $\pi_i, i = 1, 2$, then $(U_1 \cap U_2, \psi)$ is a bundle chart for $\pi_1 \oplus \pi_2$, where $\psi(p, r_1, r_2) = (p, \psi_{1p}(r_1), \psi_{2p}(r_2)), p \in U_1 \cap U_2, r_i \in \pi_i^{-1}(p), i = 1, 2$.

- The bundle of homomorphisms associated to a vector bundle

Let $\pi_i : E_i \to M$ be two vector bundles of dimension $k_i, i = 1, 2$. For each $p \in M$, consider $\mathrm{Hom}(E_{1,p}, E_{2,p})$, the set of all homomorphisms from the fiber $E_{1,p}$ to the fiber $E_{2,p}$. Define $\mathrm{Hom}(E_1, E_2) := \bigcup_{p \in M} \mathrm{Hom}(E_{1,p}, E_{2,p})$ and $\pi_{\mathrm{Hom}} : \mathrm{Hom}(E_1, E_2) \to M$ by sending $L \in \mathrm{Hom}(E_{1,p}, E_{2,p})$ to p. In this manner, $\pi_{\mathrm{Hom}} : \mathrm{Hom}(E_1, E_2) \to M$ becomes a vector bundle of dimension k_1k_2. If $(U_i, \psi_i) : \psi_1^{-1}(U_i) \to U_i \times \mathbb{R}^{k_i}$ are local charts for the two bundles, then a local chart for $\mathrm{Hom}(E_1, E_2)$ is given as $(U_1 \cap U_2, \phi)$, where $\phi : \pi_{\mathrm{Hom}}^{-1}(U_1 \cap U_2) \to (U_1 \cap U_2) \times \mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})$ is defined as follows: for each $L_p \in \pi_{\mathrm{Hom}}^{-1}(p), \phi(L_p) = (x, \tilde{L}_p)$, where $\tilde{L}_p \in \mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})$ is given by $\tilde{L}_p = \psi_{2,p} \circ L_p \circ \psi_{1,p}^{-1}$.

If $E_1 = E_2 = E$, then, if (U, ψ) is a bundle chart for E with $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, we have that $(U, \phi), \phi : \pi_{\mathrm{Hom}}^{-1}(U) \to U \times \mathrm{Hom}(\mathbb{R}^k)$ is a bundle chart for $\mathrm{Hom}(E)$, where ϕ is defined as follows: for each $p \in U$ and $L_p \in \mathrm{Hom}(E_p)$, $\phi(L_p) = (p, \tilde{L}_p)$, with $\tilde{L}_p = \psi_p \circ L_p \circ \psi_p^{-1}$. In particular, by the discussion above, the bundle charts for $\mathrm{Hom}(E_1 \oplus E_2)$ can be constructed in the following manner: consider bundle charts (U_i, ψ_i) for $E_i, i = 1, 2$ and let $U = U_1 \cap U_2$. For any $p \in U$ and any $L_p \in \mathrm{Hom}(E_{1,p} \oplus E_{2,p})$, define $\phi(L_p) = (p, \bar{L}_p)$, where $\bar{L}_p \in \mathrm{Hom}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$ is defined as $\bar{L}_p = (\psi_{1,p}, \psi_{2,p}) \circ L_p \circ (\psi_{1,p}^{-1}, \psi_{2,p}^{-1})$.

2.3. A special section of Hom$(E_1 \oplus E_2)$

In our investigations of distributions associated to control systems on a manifold, we encounter a special section of $\mathrm{Hom}(E_1 \oplus E_2)$. This section is defined as

$$S : M \to \mathrm{Hom}(E_1 \oplus E_2), \quad S(p)(v_{1,p}, v_{2,p}) = (v_{1,p}, 0), \quad (v_{1,p}, v_{2,p}) \in E_{1,p} \oplus E_{2,p},$$

and is instrumental in proving a key technical result needed in the development of our simulation algorithm. We want to prove that this section is smooth. To do this, let $(U, \xi), \xi : U \to \mathbb{R}^m$, be a manifold chart on M such that (U, ϕ) is a bundle chart for $\mathrm{Hom}(E_1 \oplus E_2)$, as above. Notice that ϕ can be used to define a manifold chart on $\mathrm{Hom}(E_1 \oplus E_2)$. Indeed, if we define $\Phi : \pi_{\mathrm{Hom}}^{-1}(U) \to \mathbb{R}^m \times \mathrm{Hom}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}) \simeq \mathbb{R}^m \times \mathbb{R}^{(k_1+k_2)^2}$ by $\Phi(L) = ((\xi \circ \pi_{\mathrm{Hom}})(L), pr_2 \circ \phi(L))$, then $(\pi_{\mathrm{Hom}}^{-1}(U), \Phi)$ is a manifold chart on $\mathrm{Hom}(E_1 \oplus E_2)$. But then, the local representation of the section S in the charts (U, ξ) and $(\pi_{\mathrm{Hom}}^{-1}(U), \Phi)$ is

$$\Phi \circ S \circ \xi^{-1}(p) = \Phi\left(S_{\xi^{-1}(p)}\right) = \left((\xi \circ \pi_{Hom})\left(S_{\xi^{-1}(p)}\right), (pr_2 \circ \phi)\left(S_{\xi^{-1}(p)}\right)\right) = (p, \bar{S}_p),$$

for all $p \in \xi(U)$, where $\bar{S}_p \in \mathrm{Hom}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$ is given by

$$\bar{S}_p(u_1, u_2) = \Big((\psi_{1,\xi^{-1}(p)}, \psi_{2,\xi^{-1}(p)}) \circ S_{\xi^{-1}(p)} \circ (\psi_{1,\xi^{-1}(p)}^{-1}, \psi_{2,\xi^{-1}(p)}^{-1})\Big)(u_1, u_2).$$

Let $\psi_{i,\xi^{-1}(p)}^{-1}(u_i) = v_{i,\xi^{-1}(p)}, i = 1, 2$. With this notation,

$$\bar{S}_p(u_1, u_2) = ((\psi_{1,\xi^{-1}(p)}, \psi_{2,\xi^{-1}(p)})\left(S_{\xi^{-1}(p)}(v_{1,\xi^{-1}(p)}, v_{2,\xi^{-1}(p)})\right) =$$

$$= (\psi_{1,\xi^{-1}(p)}, \psi_{2,\xi^{-1}(p)})(v_{1,\xi^{-1}(p)}, 0) = (u_1, 0).$$

So, $\bar{S}_p$ is independent of p and, hence, differentiable. But then, $\Phi \circ S \circ \xi^{-1}$ is differentiable as well.

2.4. Generalities on Distributions

Definition 2.4.1. a) A *distribution* D on a manifold M is an assignment of a vector subspace $D_x \subseteq T_xM$, for each $x \in M$.

Let $vf^{\infty}[D] = \{Y \in \mathcal{X}^{\infty}(M) | Y(x) \in D_x, \forall x \in M\}$, where $\mathcal{X}^{\infty}(M)$ denotes the set of all smooth vector fields on M.

b) A distribution D is called *involutive* if $vf^{\infty}[D]$ is a Lie subalgebra of $\mathcal{X}^{\infty}(M)$.

c) A distribution D is *smooth* if $vf^{\infty}[D](x) = D_x$.

d) A distribution D is *regular* if the function $x \to \dim D_x$ is constant on M. We call this constant the rank of D.

e) If $\pi : E \to M$ is a vector bundle then a *generalized distribution* D on M is an assignment of a vector subspace $D_x \subseteq E_x$, for each $x \in M$. By replacing vector fields on M by sections of E we can introduce terminology similar to a),c), and d) above for generalized distributions.

Observation 2.4.2. While every vector subbundle of TM induces a distribution on M, not every distribution on M comes from a vector subbundle. But if D is a smooth and regular distribution on a manifold M, then (D, π_D, M) is a vector subbundle of (TM, π, M), where π_D is the restriction of π to D. Indeed, since D is smooth, there exist smooth vector fields $X_1, \ldots, X_k$, where k is the rank of D, such that, for all $p \in M, X_i(p) \in D_p, 1 \leq i \leq k$, and $\{X_1(x), \ldots, X_k(x)\}$ form a basis of D_x. By continuity, there exists a neighborhood U of x such that for each $y \in U$, the vectors $X_1(y), \ldots, X_k(y)$ are linearly independent. But then we can define $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ as follows: for any $v_y = a_1X_1(y) + \ldots a_kX_k(y) \in D_y, y \in U, a_i \in \mathbb{R}$ let $\psi(v_y) = (y, a_1, \ldots, a_k)$. It follows that (U, ψ) can be considered as a local chart and

(D, π_D, M) is a vector subbundle. Note that by using a similar argument, it follows that a smooth and regular generalized distribution is a vector subbundle.

Observation 2.4.3. If S is smooth manifold and if $i : S \to M$ is the inclusion mapping then the restriction of TM to S can be identified with i^*TM. Finally, if D is a smooth distribution on M and $\bar{D}$ is a smooth distribution on S, then the restriction of $D+\bar{D}$ to S can be interpreted as a vector subbundle of the pullback bundle $i^*(TM)$, as long as the mapping $s \to \dim(D_{i(s)} + \bar{D}_s), s \in S$, is constant on S. This follows from the comments above and the second part of Observation 2.4.2.

Proposition 2.4.4. Let M be a manifold, let $\pi_E : E \to M$ be a vector bundle over M, and let $\pi_D : D \to M$ be a vector subbundle of E. Then there is a smooth section $S : M \to \text{Hom}(E)$ of the vector bundle $\text{Hom}(E)$ such that a section $\sigma : M \to E$ satisfies

$$\sigma(x) \in D_x \,\forall x \in M \quad \Leftrightarrow \quad S(x)\sigma(x) = \sigma(x) \,\forall x \in M.$$

Proof. Since M is paracompact, there exists an inner product on E, i.e., a smooth map $\langle \cdot, \cdot \rangle : E \oplus E \to \mathbb{R}$ which restricts to a positive definite, symmetric bilinear form on each fiber ([13]). For each $x \in M$ let us consider $D_x^\perp$, the orthogonal complement of D_x in E_x. If $D^\perp := \bigcup_{x \in M} D_x^\perp$ and $\pi_{D^\perp}$ is the restriction of π_E to $D^\perp$, then $\phi : D \oplus D^\perp \to E$ given by $\phi_x(v_x, w_x) = v_x + w_x$, for all $x \in M, v_x \in \mathcal{D}_x$, and $w_x \in D_x^\perp$, is a vector bundle isomorphism (see [13]).

Let us now define a smooth section $\bar{S}$ of $\text{Hom}(D \oplus D^\perp)$ as follows: for any $x \in M, v_x \in D_x$, and $w_x \in D_x^\perp$, let $\bar{S}(x)(v_x, w_x) = (v_x, 0)$. Note that by the result from the previous section, $\bar{S}$ is smooth. Now we can construct a smooth section S of $\text{Hom}(E)$. Indeed, for any $x \in M$, and $e \in E_x$, let $S(x)(e) = \phi_x(\bar{S}(x)(\phi_x^{-1}(e)))$. Now let's check that S satisfies the required condition. If $\sigma : M \to E$ is a section of E then

$$S(x)(\sigma(x)){=}\sigma(x) \;\Leftrightarrow\; \phi_x(\bar{S}(x)(\phi_x^{-1}(\sigma(x)))){=}\sigma(x) \;\Leftrightarrow\; \bar{S}(x)(\phi_x^{-1}(\sigma(x))){=}\phi_x^{-1}(\sigma(x)).$$

But $\bar{S}(x)(v_x, w_x) = (v_x, w_x)$ iff $w_x = 0$. Hence, the relation above can be reformulated as $\phi_x^{-1}(\sigma(x)) \in \mathcal{D}_x \times \{0\}$, which is equivalent to $\sigma(x) \in \mathcal{D}_x$. $\square$

Proposition 2.4.5. a) Let $\pi_E : E \to M$ be a smooth vector bundle of rank k over a manifold M, let D be a vector subbundle of E, and let $\sigma_1, \ldots, \sigma_s : M \to E$ be smooth sections of E. Let (by Proposition 2.4.4.) S be a smooth section of $\mathrm{Hom}(E)$ such that for every section $\sigma : M \to E$ we have $\sigma(x) \in D_x$ if and only if $S(x)\sigma(x) = \sigma(x)$ for every $x \in M$. Then

$$\mathrm{span}\{\sigma_1(x), \ldots, \sigma_s(x)\} \subseteq D_x \quad\Leftrightarrow\quad \Phi(x) = (0_x, \ldots, 0_x) \in \overbrace{E_x \times \cdots \times E_x}^{s \text{ times}},$$

where $\Phi : M \to \oplus_{i=1}^s E$ (s-fold Whitney sum of E with itself) is the smooth map

$$\Phi(x) = ((S(x) - I)\sigma_1(x), \ldots, (S(x) - I)\sigma_s(x)).$$

b) If the image of zero-section of $(\oplus_{i=1}^s E, \oplus_{i=1}^s \pi_E, M)$ is neat with respect to the function Φ above then the set

$$\{x \in M \,|\, \mathrm{span}\{\sigma_1(x), \ldots, \sigma_s(x)\} \subseteq D_x\}$$

is a union of disjoint, connected, closed, imbedded submanifold of M.

Proof. To establish a), let us observe that $\mathrm{span}\{\sigma_1(x), \ldots, \sigma_s(x)\} \subseteq D_x$ if and only if $\sigma_i(x) \in D_x$ for all $i, 1 \le i \le s$. By Proposition 2.4.4, this is equivalent to $S(x)\sigma_i(x) - \sigma_i(x) = 0_x \in E_x$ for all $i, 1 \le i \le s$, which, in turn, is equivalent to $\Phi(x) = (0_x, \ldots, 0_x)$.

To prove b), it is enough to note that by a), the set $\{x \in M | \, \mathrm{span}\{\sigma_1(x), \ldots, \sigma_s(x)\} \subseteq D_x\}$ is the preimage of the image zero-section of $(\oplus_{i=1}^{s} E, \oplus_{i=1}^{s} \pi_E, M)$ under the smooth function Φ. As the zero-section of any vector bundle is an imbedded submanifold, the conclusion follows based on Theorem 1.3.5. $\square$

Notation. From now on we will denote the function Φ defined in part a) of Proposition 2.4.5. by $\Phi_{M,E,D}$. It should also be noted that, in fact, $\Phi_{M,E,D}$ also depends on the sections $\sigma_i, 1 \leq i \leq s$. However, we suppress the dependence on these sections to simplify the notation.

Consequence 2.4.6. a) Let $f : N \to M$ be a smooth function, let $\pi : E \to M$ be a vector bundle of rank k, let D be a vector subbundle of f^*E, and let $\sigma_1, \ldots, \sigma_s : M \to E$ be smooth sections of E. Then we have

$$\mathrm{span}\{f^*\sigma_1(x), \ldots, f^*\sigma_s(x)\} \subseteq D_x \Leftrightarrow$$

$$\Leftrightarrow (\Phi_{N,f^*E,D})(x) = (0_x, \ldots, 0_x) \in \overbrace{(f^*E)_x \times \ldots \times (f^*E)_x}^{s \text{ times}}.$$

b) If the image of the zero section of $(\oplus_{i=1}^{s} f^*E, \oplus_{i=1}^{s} f^*\pi, N)$ is neat with respect to the function $\Phi_{N,f^*E,D} : N \to \oplus_{i=1}^{s} f^*E$ then the set

$$\{x \in N | \, \mathrm{span}\{f^*\sigma_1(x), \ldots, f^*\sigma_s(x)\} \subseteq D_x\}$$

is a disjoint union of connected, closed, imbedded submanifold of N.

Proof. It is enough to apply Proposition 2.4.5. to the vector bundle f^*E, the subbundle D, and the sections $f^*\sigma_1, \ldots, f^*\sigma_s$. $\square$

Observation 2.4.7. If (E, π, M) (and hence $(\oplus_{i=1}^{s} f^*E, \oplus_{i=1}^{s} f^*\pi, N)$) is a trivial vector bundle then, in order to obtain the same conclusion as above, we may replace the assumption in part b) of Consequence 2.4.6 as follows: If $\psi : \oplus_{i=1}^{s} f^*E \to N \times \mathbb{R}^{ks}$

is a trivialization function, if $\pi_2 : N \times \mathbb{R}^{ks} \to \mathbb{R}^{ks}$ represents the projection onto the second factor, and if

$$\Psi_{N,f^*E,D} \stackrel{\text{def}}{=} \pi_2 \circ \psi \circ \Phi : N \to \mathbb{R}^{ks}$$

has constant rank then, since the image zero section of $(\oplus_{i=1}^{s} f^*E, \oplus_{i=1}^{s} f^*\pi, N)$ is the preimage of $0 \in \mathbb{R}^{ks}$ under the function $\pi_2 \circ \psi$, it follows by Observation 1.2.5 that the conclusion of part b) of Consequence 2.4.6 remains valid.

3. CONTROL SYSTEMS AND SIMULATION RELATIONS

3.1. Introduction

We begin our discussion of control systems by establishing some notations and assumptions and by introducing the necessary terminology. Let M be a smooth manifold (assumed, as in Chapter 1, to be second countable and Hausdorff) and let Λ be a separable metric space. The following definition, as well as many of the considerations below, are adopted from [**3**] and [**4**].

Definition 3.1.1. A C^1 *control system* with state space M and control space Λ is a function $F : M \times \Lambda \to TM$ satisfying the following properties:

a) For each $\lambda \in \Lambda$, the function $x \to F(x, \lambda)$ is C^1 and satisfies $(\pi_M \circ F(x, \lambda)) = x$, and

b) For every coordinate chart $\phi : U \to \mathbb{R}^m$ of M, the local representation of F in this chart, that is, the function $F_\phi : \phi(U) \times \Lambda \to \mathbb{R}^m$ given by

$$F_\phi(y, \lambda) = \phi_{*\phi^{-1}(y)} F(\phi^{-1}(y), \lambda)$$

is C^1 in its first component.

Typically, a control system is given as $\dot{x} = F(x, u(t))$, with the understanding that once an input or control function $t \to u(t) \in \Lambda$ is specified, the result is $\dot{x} = F(x, u(t))$, which can be interpreted as an ordinary differential equation on M. To address the existence and uniqueness of solutions, we introduce the following definitions.

Definition 3.1.2. A measurable function $u : \mathbb{R} \to \Lambda$ is called a *potential control* if u is Lebesgue measurable. We denote the set of all Lebesgue measurable functions into Λ by $\mathcal{U}^{\Lambda}_{\text{meas}}$.

Definition 3.1.3. A potential control $u : \mathbb{R} \to \Lambda$ is called an *admissible control* for the C^1 control system F if the function $F_u : M \times \mathbb{R} \to TM$ defined as $F_u(x,t) = F(x, u(t))$ is such that it satisfies the C^1 Caratheodory conditions (see [**3**]) in any local chart of M. We denote the set of all admissible controls for F by $\mathcal{U}^{\Lambda}_{\text{meas}}(F)$.

The introduction of the technical conditions in the definition above is motivated by standard results in the theory of ordinary differential equations. More precisely, given a C^1 control system F, an initial point $x_0 \in M$, and an element $u \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, there exists an open interval $J \subseteq \mathbb{R}$ containing 0 and a unique function $\psi : J \to M$ such that $\psi(0) = x_0, \psi$ is absolutely continuous on every compact subinterval of J and

$$\dot{\psi}(t) = F(\psi(t), u(t)) \quad \text{ for almost all } t \in J.$$

Definition 3.1.4. Given a control system $F : M \times \Lambda \to TM$ as above, an admissible control $u \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, and an initial state $x_0 \in M$, the function $\psi : \mathbb{R} \to M$ defined above is called a *trajectory of the system* F corresponding to the initial condition x_0 and the control u. In some instances we will use the notation $\psi(t) = \psi(t, x_0, u)$ to emphasize the dependance of the trajectory on the initial condition and control.

In many applications involving control systems, not all of the variables in the state space are readily available or can be measured. A somewhat similar situation occurs in computer-aided design where one is only interested in the "external bahavior"of the system, i.e., the system's parameters that can be measured by an external device interacting with system. To capture situations of this type, we define a continuous function $h : M \times \Lambda \to O$ from $M \times \Lambda$ to a topological space O. We call h the *output* *mapping* and note that, in some sense, h summarizes the information about the control system that is either needed or available.

While in the definition of a control system we think of the controls u as representing external factors affecting the system, as noted in [**4**], we can refine our understanding of the controls given that the external factors may or may not be under our full control. This way we classify our controls as deterministic (also called inputs) and non-deterministic (also called disturbances).

Based on the two observations above, we introduce the following definitions (as in [**4**]):

Definition 3.1.5. a) A C^1 *input-disturbance (ID) system* is a C^1 control system $F : M \times \Lambda \to TM$ whose control space can be written as $\Lambda = \Omega \times \Delta$. We call Ω the *input space* and Δ the *disturbance space.*

b) A C^1 *input-disturbance-output (IDO) system* is a pair (F, h), where F is a C^1 ID system (as above) and $h : M \times \Omega \to O$ is a continuous mapping into a topological space O. We call h the *output mapping* and O the *output space.*

c) If $\mathcal{U} \subseteq \mathcal{U}^{\Lambda}_{\text{meas}}$ and $\mathcal{D} \subseteq \mathcal{U}^{\Omega}_{\text{meas}}$ are such that $\mathcal{U} \times \mathcal{D} \subseteq \mathcal{U}^{\Omega\times\Delta}_{\text{meas}}(F)$, then we call the four-tuple $(F, h, \mathcal{U}, \mathcal{D})$ an *(IDO) system with admissible inputs* $\mathcal{U}$ *and admissible disturbances* $\mathcal{D}$.

3.2. Simulation relations for IDO systems

In this section we introduce the notions of pointwise and admissible simulation relations as given in [**4**] and observe that, as noted in the aforementioned reference, for IDO systems that are affine in both inputs and disturbances the two notions are equivalent provided that a certain "disturbance constant rank" condition is satisfied. As noted in the Introduction, the motivation for having two types of simulation relations comes from the fact that the pointwise simulation concept is based on a condition involving tangent vectors and tangent spaces at a point and, being an algebraic condition, can be manipulated with relative ease. However, when one is interested in trajectories and admissible controls, the proper concept is that of an admissible simulation relation.

Let M and $\bar{M}$ be differentiable manifolds, let O be a topological space, let Ω, Δ, and $\bar{\Delta}$ be separable metric spaces, and suppose we have two C^1 IDO systems

$$F : M \times \Omega \times \Delta \to TM, \quad h : M \times \Omega \to O, \quad u \in \mathcal{U}, \quad d \in \mathcal{D}$$

and

$$\bar{F} : \bar{M} \times \Omega \times \bar{\Delta} \to T\bar{M}, \quad \bar{h} : \bar{M} \times \Omega \to O, \quad u \in \mathcal{U}, \quad \bar{d} \in \bar{\mathcal{D}}$$

having common input space Ω and common output space O. In addition, we also assume that the two IDO systems have a common family of admissible inputs $\mathcal{U}$ and admissible disturbances $\mathcal{D}$ and $\bar{\mathcal{D}}$, respectively, where

$$\mathcal{U} \subseteq \mathcal{U}^{\Omega}_{\text{meas}}, \mathcal{D} \subseteq \mathcal{U}^{\Delta}_{\text{meas}}, \bar{\mathcal{D}} \subseteq \mathcal{U}^{\bar{\Delta}}_{\text{meas}}$$

satisfy

$$\mathcal{U} \times \mathcal{D} \subseteq \mathcal{U}^{\Omega\times\Delta}_{\text{meas}}(F) \quad \text{and} \quad \mathcal{U} \times \bar{\mathcal{D}} \subseteq \mathcal{U}^{\Omega\times\bar{\Delta}}_{\text{meas}}(\bar{F}).$$

Definition 3.2.1. A nonempty subset $\mathcal{R} \subseteq M \times \bar{M}$ is called an *admissible simulation relation* of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$ if for every $(x_0, \bar{x}_0) \in \mathcal{R}$, for every $u \in \mathcal{U}$, and for every $d \in \mathcal{D}$ there exists $\bar{d} \in \bar{\mathcal{D}}$ and a compact interval I containing 0 in its interior such that for every $t \in I$ both $\psi(t, x_0, u, d)$ and $\bar{\psi}(t, \bar{x}_0, u, \bar{d})$ are defined (ψ and $\bar{\psi}$ stand for the trajectory mappings of F and $\bar{F}$, respectively), and

$$\text{for all}\, t \in I \,\text{we have}\, (\psi(t, x_0, u, d), \bar{\psi}(t, \bar{x}_0, u, \bar{d})) \in \mathcal{R}$$

and

$$\text{for all}\, t \in I \,\text{we have}\, h(\psi(t, x_0, u, d), u(t)) = \bar{h}(\bar{\psi}(t, \bar{x}_0, u, \bar{d}), u(t)).$$

Definition 3.2.2. A subset $\mathcal{R} \subseteq M \times \bar{M}$ is called a *pointwise simulation relation* of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$ if $\mathcal{R}$ is a union of connected, disjoint submanifolds of $M \times \bar{M}$ (possibly with different dimensions) and if the following "simulation condition" is satisfied:

(SC) For every $(x, \bar{x}) \in \mathcal{R}$, for every $\omega \in \Omega$, and for every $\delta \in \Delta$, there exists $\bar{\delta} \in \bar{\Delta}$

such that

$$\begin{bmatrix} F(x, \omega, \delta) \\ \bar{F}(\bar{x}, \omega, \bar{\delta}) \end{bmatrix} \in T_{(x,\bar{x})}\mathcal{R} \quad \text{and} \quad h(x, \omega) = \bar{h}(\bar{x}, \omega).$$

Definition 3.2.3. A pointwise (admissible) simulation relation $\mathcal{R}$ of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$ is called *maximal* if for any pointwise (admissible) simulation relation $\mathcal{R}'$ of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$ we have $\mathcal{R}' \subseteq \mathcal{R}$.

As mentioned before, the first two simulation concepts are closely related. In particular, if $\bar{F}$ is affine in its disturbances, i.e., if

$$\bar{F}(\bar{x}, \omega, \bar{\delta}) = \bar{f}(\bar{x}, \omega) + \bar{g}(\bar{x})\bar{\delta} \quad \text{for all} \quad (\bar{x}, \omega, \bar{\delta}) \in \bar{M} \times \Omega \times \mathbb{R}^q,$$

where $\bar{g}(\bar{x}) = [\bar{g}_1(\bar{x}), \ldots, \bar{g}_q(\bar{x})]$ and $\bar{g}_1, \ldots, \bar{g}_q$ are vector fields on $\bar{M}$, then, in order to clarify the connection between the two concepts, we introduce the following:

Definition 3.2.4. Given a smooth, immersed submanifold $\mathcal{R}$ of $M \times \bar{M}$ and a control system $\bar{F}$ as above we say that $\bar{F}$ *has disturbance constant rank along* $\mathcal{R}$ if the dimension of the vector space

$$\bar{\mathcal{V}}_{(x,\bar{x})} := T_{(x,\bar{x})}\mathcal{R} + \operatorname{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}$$

is constant as $(x, \bar{x})$ varies over each connected component of $\mathcal{R}$.

The following result appears in [**4**] and will play an important role in our investigations as it shows that under certain conditions, the concepts of admissible and pointwise simulation relations coincide.

Theorem 3.2.5. Let (F, h) and $(\bar{F}, \bar{h})$ be two IDO systems with common input space Ω and common output space O such that $\bar{F}$ is affine in disturbances. Let $\mathcal{R}$ be a smooth submanifold (or a union of disjoint submanifolds with possibly different dimensionss) of $M \times \bar{M}$ and assume that $\bar{F}$ has disturbance constant rank along $\mathcal{R}$ (along each submanifold in the union). Furthermore, consider $\mathcal{U} \subseteq \mathcal{U}^{\Omega}_{\text{meas}}(\bar{f}), \mathcal{D} \subseteq \mathcal{U}^{\Delta}_{\text{meas}}$ such that $\mathcal{U} \times \mathcal{D} \subseteq \mathcal{U}^{\Omega\times\Delta}_{\text{meas}}(F)$ and let $\bar{\mathcal{D}} \subseteq L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ (where $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ denotes the space of equivalence classes of all Lebesgue measurable functions from $\mathbb{R}$ into $\mathbb{R}^q$ that are integrable on compact subintervals of $\mathbb{R}$). Then the following statements hold:

a) If $\mathcal{R}$ is an admissible simulation relation of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$ and if $\mathcal{U}$ and $\mathcal{D}$ contain all constant mappings into their respective images, then $\mathcal{R}$ is a pointwise simulation relation of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$.

b) If $\mathcal{R}$ is a pointwise simulation relation of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, \bar{\mathcal{D}})$, then $\mathcal{R}$ is an admissible simulation relation of $(F, h, \mathcal{U}, \mathcal{D})$ by $(\bar{F}, \bar{h}, \mathcal{U}, L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q))$.

Remark 3.2.6. Consider a control system $F : M \times \Omega \times \Delta \to TM$ that is affine in both inputs and disturbances, i.e.,

$$F(x, \omega, \delta) = a(x) + b(x)\omega + g(x)\delta, \quad x \in M, \omega \in \Omega, \delta \in \Delta,$$

where $\Omega = \mathbb{R}^c, \Delta = \mathbb{R}^p, a(x)$ is a vector field on M, and the columns of the matrices $b(x)$ and $g(x)$ are vector fields on M. For such systems, a natural class of inputs and disturbances to be considered is that of Lebesgue measurable functions from $\mathbb{R}$ into their respective images that are integrable on compact intervals. As noted in [**4**] (Remark 3.2) and [**5**] (Example 2.8), these classes of inputs and disturbances are admissible for any control system that is affine in both inputs and disturbances. So, since we are only interested in control systems that are affine in inputs and disturbances, from now on we will let $\mathcal{U} = L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^c), \mathcal{D} = L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^p)$, and $\bar{\mathcal{D}} = L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$. In addition, whenever there is no danger of confusion, we will also refer to $(F, h, L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^c), L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^p))$ and $(\bar{F}, \bar{h}, L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^c), L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q))$ simply as (F, h) and $(\bar{F}, \bar{h})$.

Remark 3.2.7. Under the assumption that both F and $\bar{F}$ are affine in inputs and disturbances, condition (SC) becomes equivalent to the following:

For every $(x, \bar{x}) \in \mathcal{R}$ we have $h(x, \omega) = \bar{h}(\bar{x}, \omega)$. In addition, for every $(x, \bar{x}) \in \mathcal{R}, \omega \in \mathbb{R}^c$, we have

$$\begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im} \begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im} \begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})}\mathcal{R} + \mathrm{im} \begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}.$$

In light of the remarks above and the fact that $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^c)$ and $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^p)$ contain the constant mappings into their respective images, we obtain the following consequence of Theorem 3.2.5 for control systems that are affine in inputs as well as disturbances.

Theorem 3.2.8. Let (F, h) and $(\bar{F}, \bar{h})$ be two (IDO) systems with $F : M \times \mathbb{R}^c \times \mathbb{R}^p \to TM, \bar{F} : \bar{M} \times \mathbb{R}^c \times \mathbb{R}^q \to T\bar{M}$ given by

$$F(x, \omega, \delta) = a(x) + b(x)\omega + g(x)\delta \quad x \in M, \omega \in \mathbb{R}^c, \delta \in \mathbb{R}^p,$$

$$\bar{F}(\bar{x}, \omega, \bar{\delta}) = \bar{a}(\bar{x}) + \bar{b}(\bar{x})\omega + \bar{g}(\bar{x})\bar{\delta} \quad x \in \bar{M}, \omega \in \mathbb{R}^c, \bar{\delta} \in \mathbb{R}^q,$$

and $h : M \to \mathbb{R}^r, \bar{h} : \bar{M} \to \mathbb{R}^r$. Observe that $a, \bar{a}$, and the columns of the matrices $b, \bar{b}, g$, and $\bar{g}$ are vector fields on M. Let $\mathcal{R} \subseteq M \times \bar{M}$ be a union of disjoint, connected submanifolds of $M \times \bar{M}$ and consider the following affine simulation conditions

$$\text{(ASC)}_1 \qquad (x, \bar{x}) \in \mathcal{R} \quad \Rightarrow \quad \begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \operatorname{im} \begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \operatorname{im} \begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})}\mathcal{R} + \operatorname{im} \begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}$$

$$\text{(ASC)}_2 \qquad (x, \bar{x}) \in \mathcal{R} \quad \Rightarrow \quad h(x) = \bar{h}(\bar{x}).$$

Then the following statements hold:

a) $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\bar{F}, \bar{h})$ if and only if conditions $(ASC)_1$ and $(ASC)_2$ are satisfied.

b) If $\bar{F}$ has disturbance constant rank along each connected submanifold of $\mathcal{R}$, then $\mathcal{R}$ is an admissible simulation relation of (F, h) by $(\bar{F}, \bar{h})$ if and only if conditions $(ASC)_1$ and $(ASC)_2$ are satisfied.

4. THE MAIN RESULTS

In this chapter we present our main contributions to the problem of determining simulation relations for a given pair of control systems. First, we introduce our simulation algorithm and then show in Theorem 4.1 that the manifold at termination, if nonempty, is a simulation relation of the first system by the second. The same theorem also shows that the algorithm terminates (successfully or not) in a finite number of steps. At this point we should mention that, in contrast to the algorithm in [**16**], our algorithm is always well defined. In addition, we also paid close attention to the connectedness of the sets $\tilde{\mathcal{R}}^k$ (as defined in our algorithm) and to the possibility that these sets may not be submanifolds; these issues were not discussed in [**16**]. In doing so, we allow for greater flexibility in the algorithm in that the choice of a certain submanifold $\mathcal{R}^k$ needs to be made in going from one step to the next. While for specific IDO systems this choice can be made by "inspection", in Observation 4.2 we show how to make such a choice for two general IDO systems in a systematic fashion.

As mentioned above, running the algorithm involves making certain choices. So, different choices result in different submanifolds at termination. In our quest for a maximal simulation relation, we need to make the "proper"choices. To investigate how these choices should be made, we start by introducing the notion of regular pre-simulations up to order l (see Definition 4.3). It is interesting to note that, as shown in Lemma 4.5, under certain conditions, regular pre-simulation up to order l implies regular pre-simulation up to order $l+1$, for any $l \geq 1$.

Proposition 4.6 is one of our key results and shows that under the hypothesis that two IDO systems are regularly pre-simulated up to a specified order (depending on the dimensions of the state spaces), the set we obtain at termination is a nonempty pointwise simulation relation. Moreover, under the disturbance constant rank assumption, the set at termination is also an admissible simulation relation. It is worth noting that, when running the algorithm, some of the sets involved may be unions of closed component submanifolds of possibly different dimensions. Due to the way the algorithm is defined, if we insist on getting the maximal simulation relation, we are faced with a "branching"of the algorithm depending on the various dimensions of these component submanifolds. So, the set at (overall) termination may be a union of submanifolds of different dimensions.

The main reason for imposing the regular pre-simulation conditions in order to obtain a maximal simulation relation is to ensure that at each step of the algorithm (or any of its "branches") the sets $\mathcal{R}^k$ are unions of mutually disjoint, closed, connected, imbedded submanifolds. However, this condition may be difficult to check. Proposition 4.8 offers an alternate way of ensuring the aforementioned property by replacing the regular pre-simulation condition with the hypothesis that certain functions $\Psi_{k,\alpha}$ have constant rank. The construction of these functions is based on Consequence 2.4.6 and Observation 2.4.7.

In Theorem 4.9, we prove that, under the same assumptions as in Proposition 4.6, the set we obtain at termination is, in fact, a maximal pointwise simulation relation. Actually, the same set is also maximal among all admissible simulation relations satisfying the disturbance constant rank condition.

Finally, we end the chapter by considering the somewhat unlikely case of a $0-$dimensional simulation relation and analyze in Lemma 4.10 how this could happen for systems that are fully nonlinear. In the case of systems that are affine in inputs and disturbances we obtain a simpler characterization in Lemma 4.11. We should also note that, in spite of the fact that this is a rare occurrence, in Chapter 5 we provide an example of two systems for which the maximal simulation relation is in fact a point.

Let us now introduce an algorithm for computing simulation relations between two IDO systems, similar to the one introduced by van der Schaft in [**16**] for computing the maximal (pointwise) bisimulation relations.

The Algorithm.

Consider two IDO systems (F, h) and $(\bar{F}, \bar{h})$ such that both F and $\bar{F}$ are affine in disturbances as well as inputs, i.e.,

$$F(x, \omega, \delta) = a(x) + b(x)\omega + g(x)\delta \quad \text{and} \quad \bar{F}(\bar{x}, \omega, \bar{\delta}) = \bar{a}(\bar{x}) + \bar{b}(\bar{x})\omega + \bar{g}(\bar{x})\bar{\delta},$$

and $h : M \to \mathbb{R}^r, \bar{h} : \bar{M} \to \mathbb{R}^r$.

Step 0. Let $\mathcal{R}^0 = M \times \bar{M}$.

Step 1. Let $\tilde{\mathcal{R}}^1 = \{(x, \bar{x}) \in M \times \bar{M} | h(x) = \bar{h}(\bar{x})\}$. Is $\tilde{\mathcal{R}}^1 = \emptyset$?

If true, then the algorithm ends unsuccessfully.

If false, then let $\mathcal{R}^1$ be a (possibly $0-$ dimensional) submanifold of $\mathcal{R}^0$ contained in $\tilde{\mathcal{R}}^1$.

Step 2. Let $k := 1$

Step 3. Consider the set $\tilde{\mathcal{R}}^{k+1}$ defined as follows:

$$\tilde{\mathcal{R}}^{k+1} = \left\{ (x, \bar{x}) \in \mathcal{R}^k | \begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \operatorname{im} \begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \operatorname{im} \begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})}\mathcal{R}^k + \operatorname{im} \begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix} \right\}.$$

Step 4. Is $\tilde{\mathcal{R}}^{k+1} = \emptyset$?

If true, then the algorithm ends unsuccessfully.

If false, then select a (possibly 0−dimensional) submanifold $\mathcal{R}^{k+1}$ of $\mathcal{R}^k$ contained in $\tilde{\mathcal{R}}^{k+1}$.

Step 5. Is $\dim\mathcal{R}^k = \dim\mathcal{R}^{k+1}$?

If true, then the algorithm ends successfully with $\mathcal{R}^{k+1}$.

If false, then let $k := k+1$ and return to Step 3.

Theorem 4.1. a) If for some integer $k, k \geq 1$, we have $\dim\mathcal{R}^k = \dim\mathcal{R}^{k+1}$, with $\mathcal{R}^k$ as in the algorithm, then $\mathcal{R}^{k+1}$ is a pointwise simulation relation of (F, h) by $(\bar{F}, \bar{h})$. Hence, if the algorithm terminates successfully, the submanifold at termination, i.e. $\mathcal{R}^{k+1}$, is a pointwise simulation relation of (F, h) by $(\bar{F}, \bar{h})$. If, in addition, $\bar{F}$ has disturbance constant rank along $\mathcal{R}^{k+1}$, then $\mathcal{R}^{k+1}$ is an admissible simulation relation of (F, h) by $(\bar{F}, \bar{h})$.

b) The algorithm terminates in a finite number of steps.

Proof. To establish a), let us first observe that $\dim\mathcal{R}^k = \dim\mathcal{R}^{k+1}$ implies that $\mathcal{R}^{k+1}$ is an open subset of $\mathcal{R}^k$. But then, for all $(x, \bar{x}) \in \mathcal{R}^{k+1}, T_{(x,\bar{x})}\mathcal{R}^k = T_{(x,\bar{x})}\mathcal{R}^{k+1}$, which implies

$$\begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})}\mathcal{R}^{k+1} + \mathrm{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}$$

for all $(x, \bar{x}) \in \mathcal{R}^{k+1}$. This way, we obtain that conditions $(ASC)_1$ and $(ASC)_2$ in the hypothesis of Theorem 3.2.8, part a), are satisfied, and thus $\mathcal{R}^{k+1}$ is a pointwise simulation relation. Based on part b) of the same theorem, we obtain the rest of the claim.

To prove b), let's note that the algorithm actually terminates in at most $3(\dim M + \dim\bar{M} + 1) + 2$ steps. Indeed, since the dimension of $\mathcal{R}^1$ is at most $\dim M + \dim\bar{M}$, if the algorithm doesn't end successfully with $\mathcal{R}^2$ then, after the first five steps, the dimension of $\mathcal{R}^2$ is at most $\dim M + \dim \bar{M} - 1$. By repeating the same argument, if the algorithm doesn't terminate after $3k+2$ steps for some positive integer k, then the dimension of $\mathcal{R}^{k+1}$ is at most $\dim M + \dim \bar{M} - k$. In particular, if $k = \dim M + \dim \bar{M}$, the dimension of $\mathcal{R}^{\dim M + \dim \bar{M} + 1}$ must be zero. By running the algorithm three more steps, assuming it doesn't end earlier, the dimension of $\mathcal{R}^{\dim M + \dim \bar{M} + 2}$ must be zero. So $\mathcal{R}^{\dim M + \dim \bar{M} + 2}$ and $\mathcal{R}^{\dim M + \dim \bar{M} + 1}$ have the same dimensions and the algorithm terminates. $\square$

Observation 4.2. Since $\tilde{\mathcal{R}}^{k+1}$ is, in general, not a submanifold of $\mathcal{R}^k$, in order for the algorithm to be well-defined, in step 4 we need to consider a submanifold $\mathcal{R}^{k+1}$ of $\mathcal{R}^k$ contained in $\tilde{\mathcal{R}}^{k+1}$ so that we can continue the process. For now, let us indicate how to construct a specific submanifold that, in certain instances, can be taken as $\mathcal{R}^{k+1}$. Later on, we will come back to this step and investigate sufficient conditions ensuring that $\tilde{\mathcal{R}}^{k+1} = \mathcal{R}^{k+1}$, i.e., $\tilde{\mathcal{R}}^{k+1}$ is a submanifold of $\mathcal{R}^k$.

Let us assume that $\bar{F}$ has disturbance constant rank along $\mathcal{R}^k$ for some $k \geq 1$, let $i_k : \mathcal{R}^k \to \mathcal{R}^0$ be the inclusion map, and let $\bar{\mathcal{V}}^k$ be the vector bundle over $\mathcal{R}^k$ defined by

$$(x, \bar{x}) \to T_{(x,\bar{x})}\mathcal{R}^k + \operatorname{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}.$$

The restrictions to $\mathcal{R}^k$ of the vector field $\begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix}$, the c column vector fields of $\begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix}$, and the p column vector fields of $\begin{bmatrix} g(x) \\ 0 \end{bmatrix}$ represent $1 + c + p$ sections of $i_k^* T\mathcal{R}^0$. By using these sections, we can define the mapping

$$\Phi_k = \Phi_{\mathcal{R}^k, i_k^* T\mathcal{R}^0, \bar{\mathcal{V}}^k} : \mathcal{R}^k \to \bigoplus_{j=1}^{1+c+p} i_k^* T\tilde{\mathcal{R}}^0,$$

as outlined in Consequence 2.4.6. (We note that the vector fields mentioned above correspond to the sections denoted by $\sigma_1, \sigma_2, \ldots, \sigma_{1+c+p}$ in Consequence 2.4.6.) If we consider the set

$$\mathcal{M}^k := \{(x,\bar{x}) \in \mathcal{R}^k | \Phi_k \text{ has maximal rank at } (x,\bar{x})\},$$

then, since the rank of a function is locally non-decreasing, $\mathcal{M}^k$ is open in $\mathcal{R}^k$. But then, $\tilde{\Phi}_k$, the restriction of Φ_k to $\mathcal{M}^k$, has constant rank on $\mathcal{M}^k$ and, consequently, by the constant rank theorem, $(\tilde{\Phi}_k)^{-1}(0) = \mathcal{M}^k \cap (\Phi_k)^{-1}(0)$ is a submanifold of $\mathcal{M}^k$ and, hence, of $\mathcal{R}^k$. To summarize, if $\bar{F}$ has constant disturbance rank on $\mathcal{R}^k$ (so that Φ_k is well defined) and $\mathcal{M}_k \cap \Phi_k^{-1}(0) \neq \emptyset$ then then we can choose $\mathcal{R}^{k+1}$ in Step 4 of the algorithm to be $\mathcal{M}^k \cap (\Phi_k)^{-1}(0)$.

Let us now revisit step 4 of the algorithm in order to determine sufficient conditions ensuring that in certain instances we can choose $\mathcal{R}^{k+1}$ to be $\tilde{\mathcal{R}}^{k+1}$. First, we need a definition.

Definition 4.3. Let (F,h) and $(\bar{F},\bar{h})$ be two smooth IDO systems as in the algorithm. Let $\tilde{\mathcal{R}}_0 = M \times \bar{M}$, let $H : \tilde{\mathcal{R}}_0 \to \mathbb{R}^r$ be defined by $H(x,\bar{x}) = h(x) - \bar{h}(\bar{x})$, and let l be a positive integer. We say that (F,h) is *regularly pre-simulated* by $(\bar{F},\bar{h})$ up to order l if there exist nonempty sets $\tilde{\mathcal{R}}^1, \ldots, \tilde{\mathcal{R}}^l$ with the following properties:

(i) $\tilde{\mathcal{R}}^0 \supset \tilde{\mathcal{R}}^1 \supseteq \cdots \supseteq \tilde{\mathcal{R}}^l$;

(ii) $\tilde{\mathcal{R}}^1 = \{(x,\bar{x}) \in \tilde{\mathcal{R}}^0 = M \times \bar{M} | H(x,\bar{x}) = 0\}$;

(iii) For $k = 1, \ldots, l$ the set $\tilde{\mathcal{R}}^k$ is a countable union of pairwise disjoint, closed, connected, imbedded submanifolds $\mathcal{R}^k_\alpha$ of $\tilde{\mathcal{R}}^0 = M \times \bar{M}$; that is,

$$\tilde{\mathcal{R}}^k = \bigcup_{\alpha \in \mathcal{I}_k} \mathcal{R}^k_\alpha,$$

where $\mathcal{I}_k$ is a countable index set (we call the submanifolds $\mathcal{R}^k_\alpha$ the *component submanifolds* of $\tilde{\mathcal{R}}^k$; for a given k it is not required that these component submanifolds all be of the same dimension); we assume that the index sets $\mathcal{I}_k, k = 1, \dots, l$ are pairwise disjoint and for notational convenience we set $\tilde{\mathcal{R}}^0 = \mathcal{R}^0_{\alpha_0}$, where $\alpha_0 \notin \cup_{k=1}^l \mathcal{I}_k$, and $\mathcal{I}_0 = \{\alpha_0\}$;

(iv) For each $1 \le k \le l-1, \tilde{\mathcal{R}}^{k+1}$ is related to $\tilde{\mathcal{R}}^k = \cup_{\alpha \in \mathcal{I}_k} \mathcal{R}^k_\alpha$ in the following manner: For each $\alpha \in \mathcal{I}_k$,

$$\mathcal{S}^k_\alpha = \left\{ (x, \bar{x}) \in \mathcal{R}^k_\alpha \,\middle|\, \begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im} \begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im} \begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})} \mathcal{R}^k_\alpha + \mathrm{im} \begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix} \right\}$$

is either empty or is a union of a sub-family of the family of component submanifolds $\{\mathcal{R}^{k+1}_\beta | \beta \in \mathcal{I}_{k+1}\}$ of $\tilde{\mathcal{R}}^{k+1}$, and $\mathcal{S}^k_\alpha$ is nonempty for at least one $\alpha \in \mathcal{I}_k$. Moreover, every component submanifold $\mathcal{R}^{k+1}_\beta$ of $\tilde{\mathcal{R}}^{k+1}$ is contained in at least one of the sets $\mathcal{S}^k_\alpha$. Thus, if for $\alpha \in \mathcal{I}_k$ we let $\mathcal{J}_{k+1,\alpha} = \{\beta \in \mathcal{I}_{k+1} | \mathcal{R}^{k+1}_\beta \subseteq \mathcal{S}^k_\alpha\}$, then

$$\mathcal{S}^k_\alpha = \bigcup_{\beta \in \mathcal{J}_{k+1,\alpha}} \mathcal{R}^{k+1}_\beta,$$

$\mathcal{I}_{k+1} = \cup_{\alpha \in \mathcal{I}_k} \mathcal{J}_{k+1,\alpha}$ (a pairwise disjoint union), and at least one of the sets $\mathcal{J}_{k+1,\alpha}$ is nonempty. It follows that $\tilde{\mathcal{R}}^{k+1}$ admits the various representations

$$\tilde{\mathcal{R}}^{k+1} = \bigcup_{\beta \in \mathcal{I}_{k+1}} \mathcal{R}^{k+1}_\beta = \bigcup_{\alpha \in \mathcal{I}_k} \bigcup_{\beta \in \mathcal{J}_{k+1,\alpha}} \mathcal{R}^{k+1}_\beta = \bigcup_{\alpha \in \mathcal{I}_k} \mathcal{S}^k_\alpha.$$

Remark 4.4. Condition (iv) of the previous definition is interpreted as being vacuous in the case when $l = 1$. In a typical situation one constructs the sets $\tilde{\mathcal{R}}^k$ by starting with $\tilde{\mathcal{R}}^1 = H^{-1}(0)$, checking that $\tilde{\mathcal{R}}^1$ satisfies condition (iii) when $k = 1$, and then proceeding sequentially to the sets $\tilde{\mathcal{R}}^2, \tilde{\mathcal{R}}^3, \dots,$ etc., by determining whether or not the sets $\mathcal{S}^k_\alpha$ defined in (iv) are indeed disjoint unions of closed, connected, imbedded submanifolds of $\tilde{\mathcal{R}}^0 = M \times \bar{M}$. The verification of this last property may be ascertained by inspection in simple examples, but can also be guaranteed by the conditions set forth in the following lemma.

Lemma 4.5. Let (F, h) and $(\bar{F}, \bar{h})$ be two smooth IDO systems as in the algorithm, let $\tilde{\mathcal{R}}^0 = M \times \bar{M}$, and let $H : \tilde{\mathcal{R}}^0 \to \mathbb{R}^r$ be defined by $H(x, \bar{x}) = h(x) - \bar{h}(\bar{x})$. Then the following hold:

a) If H has constant rank on $\tilde{\mathcal{R}}^0$, then $\tilde{\mathcal{R}}^1 = H^{-1}(0)$ is a countable union of pairwise disjoint, closed, connected, imbedded submanifolds of $\tilde{\mathcal{R}}^0$ (in other words, (F, h) is regularly pre-simulated by $(\bar{F}, \bar{h})$ up to order 1).

b) Let l be a positive integer and suppose (F, h) is regularly pre-simulated by $(\bar{F}, \bar{h})$ up to order l. Further suppose that the following two conditions hold:

i) For each $\alpha \in \mathcal{I}_l, \bar{F}$ has disturbance constant rank along the component submanifolds $\mathcal{R}^l_\alpha$, which results in the mapping

$$\Phi_{l,\alpha} \stackrel{\text{def}}{=} \Phi_{\mathcal{R}^l_\alpha, i^*_{l,\alpha} T\tilde{\mathcal{R}}^0, \tilde{\mathcal{V}}^l_\alpha} : \mathcal{R}^l_\alpha \to \bigoplus_{j=1}^{1+c+p} i^*_{l,\alpha} T\tilde{\mathcal{R}}^0$$

(as constructed in Consequence 2.4.6) being well defined, where $\tilde{\mathcal{V}}^l_\alpha$ is the vector bundle over $\mathcal{R}^l_\alpha$ defined by

$$(x, \bar{x}) \to T_{(x,\bar{x})}\mathcal{R}^l_\alpha + \operatorname{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}$$

and the sections characterizing the construction of $\Phi_{\mathcal{R}^l_\alpha, i^*_{l,\alpha} T\tilde{\mathcal{R}}^0, \tilde{\mathcal{V}}^l_\alpha}$ from Consequence 2.4.6 are given by the vector field $\begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix}$, the c column vector fields of $\begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix}$, and the p column vector fields of $\begin{bmatrix} g(x) \\ 0 \end{bmatrix}$.

ii) the zero section of the bundle $\oplus_{j=1}^{1+c+p} i^*_{l,\alpha} T\tilde{\mathcal{R}}^0$ is neat with respect to the mapping $\Phi_{l,\alpha}$.

Then (F, h) is regularly pre-simulated by $(\bar{F}, \bar{h})$ up to order $l+1$ or $\mathcal{S}^l_\alpha = \emptyset$ for all $\alpha \in \mathcal{I}_l$, with the two possibilities being mutually exclusive.

Proof. By the constant rank theorem applied to the function $x \to H(x, \bar{x}) := h(x) - \bar{h}(\bar{x})$ on $\tilde{\mathcal{R}}^0$, it follows that $\tilde{\mathcal{R}}^1 = H^{-1}(0)$ is a union of at most countably many connected, disjoint, closed, imbedded submanifolds of $M \times \bar{M}$.

To prove b), we will need some properties we developed in Chapter 2. Based on Observations 2.2.2 and 2.2.3, for each $\alpha \in \mathcal{I}_l$, if we denote by $i_{l,\alpha} : \mathcal{R}^l_\alpha \to \tilde{\mathcal{R}}^0$ the inclusion map, then we can interpret $T\mathcal{R}^l_\alpha$ as a subbundle of $i^*_{l,\alpha}T\tilde{\mathcal{R}}^0$ and the column vector fields of $\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}$ as sections $i^*_{l,\alpha}T\mathcal{R}^0$. This way, if we assume that the generalized distribution $\bar{\mathcal{V}}^l_\alpha = T\mathcal{R}^l_\alpha + \mathrm{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix}$ is regular, i.e., $\bar{F}$ has constant disturbance rank along $\mathcal{R}^l_\alpha$, then, by Observation 2.4.2, $\bar{\mathcal{V}}^l_\alpha$ becomes a subbundle of $i^*_{l,\alpha}T\tilde{\mathcal{R}}^0$.

Let us also note that the generalized distribution

$$(x, \bar{x}) \to \mathrm{span}\begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq i^*_{l,\alpha}T_{(x,\bar{x})}\tilde{\mathcal{R}}^0$$

is spanned by smooth sections $i^*_{l,\alpha}\sigma_1, \ldots, i^*_{l,\alpha}\sigma_{1+c+p}$, where $\sigma_1, \ldots, \sigma_{1+c+p}$ are vector fields on $\tilde{\mathcal{R}}^0$ determined by column vector fields of the matrices $\begin{bmatrix} a \\ \bar{a} \end{bmatrix}$, $\begin{bmatrix} b \\ \bar{b} \end{bmatrix}$, and $\begin{bmatrix} g \\ 0 \end{bmatrix}$. Hence, based on the assumption that the zero section of the bundle $\oplus_{j=1}^{1+c+p} i^*_{l,\alpha}T\tilde{\mathcal{R}}^0$ is neat with respect to $\Phi_{l,\alpha}$ and Consequence 2.4.6, we obtain that the set

$$\mathcal{S}^l_\alpha = \left\{ (x, \bar{x}) \in \mathcal{R}^l_\alpha | \begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})}\mathcal{R}^l_\alpha + \mathrm{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix} \right\}$$

is the preimage of the zero section of the vector bundle $i^*_{l,\alpha}T\tilde{\mathcal{R}}^0$ under the mapping $\Phi_{l,\alpha}$, and hence a union of at most countably many mutually disjoint, closed, connected, imbedded submanifolds of $\mathcal{R}^l_\alpha$. Since $\mathcal{I}_l$ is countable, the collection of all these imbedded submanifolds taken over all $\alpha \in \mathcal{I}_k$ is also countable and let us denote the indexing set by $\mathcal{I}_{l+1}$. Let us also denote an arbitrary element of this collection by $\mathcal{R}^{l+1}_\beta$ for some $\beta \in \mathcal{I}_{l+1}$. If we let

$$\tilde{\mathcal{R}}^{l+1} = \bigcup_{\beta \in \mathcal{I}_{l+1}} \mathcal{R}^{l+1}_\beta,$$

then, since being a closed set, a connected set, and an imbedded submanifold are all hereditary properties, by condition (iii) of the regular pre-simulation condition up to order l, it follows that $\tilde{\mathcal{R}}^{l+1}$ is a union of at most countably many mutually disjoint disjoint, closed, connected, imbedded submanifolds of $\tilde{\mathcal{R}}^0$. Thus, condition (iii) of the regular pre-simulation condition up to order $l+1$ is satisfied.

Conditions (i) and (iv) of the regular pre-simulation condition up to order $l+1$ can now be easily checked based on the work above as long as $\mathcal{S}^l_\alpha \neq \emptyset$ for some $\alpha \in \mathcal{I}_l$. Note that this relation is what ensures that the nonempty stipulation in conditions (iv) and (i) is satisfied. $\square$

Notation: Given two IDO systems that are regularly pre-simulated up to order l, the definition above introduces the sets $\mathcal{I}_k, \tilde{\mathcal{R}}^k, \mathcal{R}^k_\alpha$, all defined for $1 \leq k \leq l, \alpha \in \mathcal{I}_k$, and the sets $\mathcal{S}^k_\alpha$ defined for $1 \leq k \leq l-1, \alpha \in \mathcal{I}_k$. For the purpose of the following proposition we extend the definitions of these sets inductively as follows: For any $k \geq l$ and $\alpha \in \mathcal{I}_k$ consider the set

$$\mathcal{T}^k_\alpha = \left\{ (x,\bar{x}) \in \mathcal{R}^k_\alpha | \begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im} \begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im} \begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subseteq T_{(x,\bar{x})}\mathcal{R}^k_\alpha + \mathrm{im} \begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix} \right\}$$

As a subset of $\tilde{\mathcal{R}}^0, \mathcal{T}^k_\alpha$ is either empty or a union of its connected components. For each $\alpha \in \mathcal{I}_k$ with $\mathcal{T}^k_\alpha \neq \emptyset$, let $\mathcal{J}_{k+1,\alpha}$ denote the set indexing all connected components of $\mathcal{T}^k_\alpha$. If $\mathcal{T}^k_\alpha = \emptyset$, we let $\mathcal{J}_{k,\alpha}$ consist of a singleton (indexing the empty set). Now we define $\mathcal{I}_{k+1} = \bigcup_{\alpha \in \mathcal{I}_k} J_{k+1,\alpha}$ and for $\beta \in \mathcal{J}_{k+1,\alpha} \subseteq \mathcal{I}_{k+1}$ we let $\mathcal{R}^{k+1}_\beta$ be the an arbitrary, closed, connected, nonempty (but possibly zero-dimensional) submanifold of $\tilde{\mathcal{R}}^0$ of largest possible dimension contained in the β-component of $\mathcal{T}^k_\alpha$ if such a submanifold exists or the empty set otherwise. Finally, we introduce the set $\mathcal{S}^k_\alpha = \cup_{\beta \in \mathcal{J}_{k+1,\alpha}} \mathcal{R}^{k+1}_\beta \subseteq \mathcal{T}^k_\alpha$ and let $\tilde{\mathcal{R}}^{k+1} = \cup_{\beta \in \mathcal{I}_{k+1}} \mathcal{R}^{k+1}_\beta$. Clearly, $\mathcal{R}^{k+1}_\beta, \mathcal{S}^k_\alpha$, and $\tilde{\mathcal{R}}^{k+1}$ are not unique. Note that, except for the uniqueness, the introduction of the sets above for $k \geq l$ generally agrees with the definition of the corresponding sets for $k < l$. In particular, for any $k \geq 1, \mathcal{R}_{\beta}^{k+1}$ is always a connected component of some $\mathcal{S}_{\alpha}^{k}$.

While the purpose of the following proposition is to illustrate several important properties of IDO systems that are pre-simulated up to order $\dim M + \dim \bar{M} + 2$, the proposition also shows that if $l = \dim M + \dim \bar{M} + 2$ then for any $k \geq l+1$, the sets $\tilde{\mathcal{R}}^k, \mathcal{R}_{\alpha}^{k}$, and $\mathcal{S}_{\alpha}^{k-1}$ introduced above are actually forced to be unique, just like the corresponding sets for $k < l$ are unique due to their definition.

Proposition 4.6. Consider two IDO systems (F, h) and $(\bar{F}, \bar{h})$ as in the algorithm and suppose that (F, h) is regularly pre-simulated by $(\bar{F}, \bar{h})$ up to order $\dim M + \dim \bar{M} + 2$. Then we have the following:

a) If $\mathcal{R}_{\alpha_1}^{1}, \alpha_1 \in \mathcal{I}_1$, is a nonempty component submanifold of $\tilde{\mathcal{R}}^1$ and if for all $k \geq 1, \mathcal{R}_{\alpha_{k+1}}^{k+1}$ is a nonempty component submanifold of $\mathcal{S}_{\alpha_k}^{k}$ for some $\alpha_{k+1} \in \mathcal{J}_{k+1,\alpha_k} \subseteq \mathcal{I}_{k+1}$, then

$$\mathcal{R}_{\alpha_1}^{1} \supseteq \mathcal{R}_{\alpha_2}^{2} \supseteq \cdots \supseteq \mathcal{R}_{\alpha_k}^{k} \supseteq \mathcal{R}_{\alpha_{k+1}}^{k+1} \supseteq \cdots \quad \text{and}$$

$$\tilde{\mathcal{R}}^0 \supseteq \tilde{\mathcal{R}}^1 \supseteq \cdots \tilde{\mathcal{R}}^k \supseteq \tilde{\mathcal{R}}^{k+1} \supseteq \cdots$$

b) There exists a sequence $\{\mathcal{R}_{\alpha_k}^{k}\}_{k \geq 1}, \alpha_k \in \mathcal{I}_k$, as in a) such that $\mathcal{R}_{\alpha_k}^{k} \neq \emptyset$ for all $k \geq 1$.

c) If we let Λ be the set of all sequences $\alpha := \{\alpha_k\}_{k \geq 1}$ satisfying the property from b), then for each such sequence there exist a nonnegative integer $l_\alpha \leq \dim M + \dim \bar{M} + 1$ such that

$$\mathcal{R}_{\alpha_{l_\alpha}}^{l_\alpha} = \mathcal{R}_{\alpha_{l_\alpha+1}}^{l_\alpha+1} = \cdots =: \mathcal{R}_{\alpha}^{*}.$$

In addition, there exists some nonnegative integer $l \leq \dim M + \dim \bar{M} + 1$ such that

$$\tilde{\mathcal{R}}^l = \tilde{\mathcal{R}}^{l+1} = \cdots = \mathcal{R}^*.$$

d) For each sequence $\alpha \in \Lambda$, $\mathcal{R}^*_\alpha$ and $\mathcal{R}^*$ are pointwise simulation relations of (F, h) by $(\bar{F}, \bar{h})$. Moreover, if $(\bar{F}, \bar{h})$ has disturbance constant rank on $\mathcal{R}^*_\alpha$ then $\mathcal{R}^*_\alpha$ is an admissible simulation relation as well. If the constant disturbance rank condition is satisfied on $\mathcal{R}^*_\alpha$ for all $\alpha \in \Lambda$, then $\mathcal{R}^*$ is also an admissible simulation relation.

Proof. Clearly, a) holds by the definition/construction of the sets $\mathcal{R}^k_{\alpha_k}$ and $\tilde{\mathcal{R}}^k$.

For b), let us first observe that it is possible for a sequence as in a) not to be continuable after a finite number of steps, i.e. for the algorithm to terminate with the empty set, because for some $k \leq \dim M + \dim \bar{M} + 1$ and some component submanifold $\mathcal{R}^k_{\alpha_k}$, the defining condition for $\tilde{\mathcal{R}}^{k+1}$ from the algorithm/pre-simulation condition may not be satisfied at any point in $\mathcal{R}^k_{\alpha_k}$. However, due to the "nonempty" stipulation in part (iv) of the definition of the pre-simulation relation, we do have at least one sequence satisfying a) such that $\mathcal{R}^k_{\alpha_k} \neq \emptyset$ for all integers k with $1 \leq k \leq \dim M + \dim \bar{M} + 2$.

Next, let us observe that for all $k \geq 1$, if $\dim \mathcal{R}^{k+1}_{\alpha_{k+1}} = \dim \mathcal{R}^k_{\alpha_k}$, then $\mathcal{R}^{k+1}_{\alpha_{k+1}}$ is an open subset of $\mathcal{R}^k_{\alpha_k}$. By condition (iii) of the pre-simulation condition and the construction of the sets $\mathcal{R}^{k+1}_{\alpha_{k+1}}$ for $k \geq \dim M + \dim \bar{M} + 2$ outlined in the comments preceding this proposition, it follows that $\mathcal{R}^{k+1}_{\alpha_{k+1}}$ is also a closed subset of $\mathcal{R}^k_{\alpha_k}$. As $\mathcal{R}^k_{\alpha_k}$ is connected, it follows that $\mathcal{R}^k_{\alpha_k} = \mathcal{R}^{k+1}_{\alpha_{k+1}}$.

As mentioned earlier, there exists at least one sequence satisfying a) such that $\mathcal{R}^k_{\alpha_k} \neq \emptyset$ for all integers k with $1 \leq k \leq \dim M + \dim \bar{M} + 2$. Now note that for such a sequence of sets, if we let $N := \dim M + \dim \bar{M} + 1$, then we can consider the finite sequence $\{\dim \mathcal{R}^k_{\alpha_k}\}_{1 \leq k \leq N+1}$. This is a nonincresing sequence consisting of $N + 1$ nonegative integers, with the largest term $(\dim \mathcal{R}^1_{\alpha_1})$ being less than or equal to $N - 1$. This implies that there exists some (smallest) integer l_α with $1 \leq l_\alpha \leq N$ associated to the sequence $\{\alpha_k\}_{k \geq 1}$ (and in fact only to the first l_α terms) such that

$\dim\mathcal{R}^{l_\alpha+1}_{\alpha_{l_\alpha+1}} = \dim\mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}}$. By the observation from the previous paragraph, we must have $\mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} = \mathcal{R}^{l_\alpha+1}_{\alpha_{l_\alpha+1}} = \cdots = \mathcal{R}^{N+1}_{\alpha_{N+1}}$. Indeed, to see this, it is enough to look at the definition of the sets $\mathcal{R}^k_{\alpha_k}$ as described in part (iv) of the pre-simulation condition. But then, the only connected, closed submanifold of $\mathcal{R}^{N+1}_{\alpha_{N+1}}$ of largest dimension is $\mathcal{R}^{N+1}_{\alpha_{N+1}}$, which, by the comments preceding this proposition, forces $\mathcal{R}^{N+2}_{\alpha_{N+2}}$ and, by a similar argument all $\mathcal{R}^k_{\alpha_k}$, to be equal to $\mathcal{R}^{N+1}_{\alpha_{N+1}}$ for all $k \geq N+1$. In summary, since $\mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} \neq \emptyset$, it follows that all of the terms of the sequence $\{\mathcal{R}^k_{\alpha_k}\}_{k\geq 1}$ are nonempty.

Let us note that the argument above also proves the first set of equalities in c). For the second part of the claim, let us consider the set $A := \{l_\alpha | \alpha \in \Lambda\}$. Since A is a set of nonnegative integers bounded above by $\dim M + \dim\bar{M} + 1$, if we consider the largest element l of A then, for each $\alpha \in \Lambda$, we have $l_\alpha \leq l \leq \dim M + \dim\bar{M} + 1$. This implies $\mathcal{R}^l_{\alpha_l} = \mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} = \mathcal{R}^*_\alpha$. But then, for all $k \geq l$,

$$\tilde{\mathcal{R}}^k = \bigcup_{\alpha\in\Lambda} \mathcal{R}^k_{\alpha_k} = \bigcup_{\alpha\in\Lambda} \mathcal{R}^l_{\alpha_l} = \tilde{\mathcal{R}}^l \left(= \bigcup_{\alpha\in\Lambda} \mathcal{R}^*_\alpha \right).$$

To prove the last part of the proposition, let us first show that for each $\alpha \in \Lambda, \mathcal{R}^*_\alpha$ satisfies $(ASC)_1$ and $(ASC)_2$. Clearly, $\mathcal{R}^*_\alpha$ satisfies $(ASC)_2$ since $\mathcal{R}^*_\alpha = \mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} \subseteq \mathcal{R}^1_{\alpha_1}$. Moreover, since $\mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} = \mathcal{R}^{l_\alpha+1}_{\alpha_{l_\alpha}+1}$, by the definition of $\mathcal{R}^{l_\alpha+1}_{\alpha_{l_\alpha+1}}$ we obtain

$$(x,\bar{x}) \in \mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} \quad \Rightarrow \quad \begin{bmatrix} a(x) \\ \bar{a}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} b(x) \\ \bar{b}(\bar{x}) \end{bmatrix} + \mathrm{im}\begin{bmatrix} g(x) \\ 0 \end{bmatrix} \subset T_{(x,\bar{x})}\mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} + \mathrm{im}\begin{bmatrix} 0 \\ \bar{g}(\bar{x}) \end{bmatrix},$$

which is exactly condition $(ASC)_1$ for $\mathcal{R}^{l_\alpha}_{\alpha_{l_\alpha}} = \mathcal{R}^*_\alpha$. Obviously, conditions $(ASC)_1$ and $(ASC)_2$ are satisfied on $\mathcal{R}^*$ since they are satisfied on each one of its connected components. To finish the proof, note that each one of the connected components of $\mathcal{R}^*$ are imbedded submanifolds of $M \times \bar{M}$ by the pre-simulation condition. In addition, by the observation above, conditions $(ASC)_1$ and $(ASC)_2$ are satisfied. So, the conclusion follows by Theorem 3.2.8. $\square$

Observation 4.7. In general, $\mathcal{R}^*$ may not be a submanifold since it may consist of connected components with different dimensions (see Example 5 from Chapter 5).

In Lemma 4.5 we provided conditions under which the regular pre-simulation condition in the hypothesis of Proposition 4.6 can be ascertained based on the neatness of the zero section of $\oplus_{j=1}^{1+c+p} i_{l,\alpha}^* T\tilde{\mathcal{R}}^0$ with respect to the mapping $\Phi_{l,\alpha}$. While, in general, it may be difficult to check the neatness condition, the main reason why neatness is needed is to conclude that the preimage of a certain submanifold is a union of at most countably many mutually disjoint, connected, closed, imbedded submanifolds. In certain instances, the same conclusion can be reached by ensuring that certain specifically constructed functions have constant rank, which is an easier condition to verify.

Proposition 4.8. Let $\tilde{\mathcal{R}}^k = \cup_{\alpha \in \mathcal{I}_k} \mathcal{R}_\alpha^k$ be a union of mutually disjoint, closed, connected, imbedded submanifolds of $\tilde{\mathcal{R}}^0$ and assume that each $\alpha \in \mathcal{I}_k$, $\bar{F}$ has disturbance constant rank along the component submanifolds $\mathcal{R}_\alpha^k$, which results in the mapping

$$\Phi_{k,\alpha} = \Phi_{\mathcal{R}_\alpha^k, i_{k,\alpha}^* T\tilde{\mathcal{R}}^0, \tilde{\mathcal{V}}_\alpha^k} : \mathcal{R}_\alpha^k \to \bigoplus_{j=1}^{1+c+p} i_{k,\alpha}^* T\tilde{\mathcal{R}}^0$$

being well defined. If, in addition, the vector bundle $\bigoplus_{j=1}^{1+c+p} i_{k,\alpha}^* T\tilde{\mathcal{R}}^0$ is trivial, if

$$\psi : \bigoplus_{j=1}^{1+c+p} i_{k,\alpha}^* T\tilde{\mathcal{R}}^0 \to \mathcal{R}_\alpha^k \times \mathbb{R}^{(1+c+p)(\dim M + \dim \bar{M})}$$

is a trivialization function, if $\pi_2 : \mathcal{R}_\alpha^k \times \mathbb{R}^{(1+c+p)(\dim M + \dim \bar{M})} \to \mathbb{R}^{(1+c+p)(\dim M + \dim \bar{M})}$ represents the projection onto the second factor, and if

$$\Psi_{k,\alpha} = \pi_2 \circ \psi \circ \Phi_{k,\alpha} : \mathcal{R}_\alpha^k \to \mathbb{R}^{(1+c+p)(\dim M + \dim \bar{M})}$$

has constant rank on $\mathcal{R}_\alpha^k$ (or an open subset of $\mathcal{R}_\alpha^k$ containing $(\Psi_{k,\alpha})^{-1}(0)$), then the preimage of the zero section of the vector bundle $\bigoplus_{j=1}^{1+c+p} i_{k,\alpha}^* T\tilde{\mathcal{R}}^0$ is the preimage of $0 \in \mathbb{R}^{(1+c+p)(\dim M + \dim \bar{M})}$ under $\Psi_{k,\alpha}$ and, hence, it is a union of at most countably many mutually disjoint, closed, connected, imbedded submanifolds of $\mathcal{R}^k_\alpha$.

Proof. The proof follows by Observation 1.2.5. $\square$

Theorem 4.9. Consider two IDO systems (F, h) and $(\bar{F}, \bar{h})$ such that F and $\bar{F}$ are affine in both inputs and disturbances and assume that (F, h) is regularly pre-simulated by $(\bar{F}, \bar{h})$ up to order $\dim M + \dim \bar{M} + 2$. Then the following hold:

a) If $\mathcal{R}'$ is any pointwise simulation relation of (F, h) by $(\bar{F}, \bar{h})$, then $\mathcal{R}' \subseteq \mathcal{R}^*$.

b) If $\mathcal{R}'$ is any submanifold of $M \times \bar{M}$ which is also an admissible simulation relation of (F, h) by $(\bar{F}, \bar{h})$ such that $\bar{F}$ has disturbance constant rank along $\mathcal{R}'$, then $\mathcal{R}' \subseteq \mathcal{R}^*$.

Proof. First, let us note that in both cases we can apply Theorem 3.2.8 to conclude that $\mathcal{R}'$ must satisfy conditions $(ASC)_1$ and $(ASC)_2$. If we consider a connected component of $\mathcal{R}'$ (denoted by $\mathcal{R}'_{\text{comp}}$), then this component must be a subset of one of the connected components of $\tilde{\mathcal{R}}^1$ since $\mathcal{R}'_{\text{comp}}$ must satisfy condition $(ASC)_2$. Similarly, using the same connectivity argument and the fact that $\mathcal{R}'_{\text{comp}}$ satisfies $(ASC)_1$, we can conclude that $\mathcal{R}'$ is a subset of one of the connected components of $\tilde{\mathcal{R}}^2$. In fact, by repeating the argument based on the $(ASC)_1$ condition, we can conclude that there exists a sequence $\alpha \in \Lambda$ as in Proposition 4.6 such that $\mathcal{R}'_{\text{comp}}$ is a subset of $\mathcal{R}^k_{\alpha_k}$, for all $k \geq 0$. By Proposition 4.6, part c), the sequence $\{\mathcal{R}^k_{\alpha_k}\}_{k \geq 0}$ stabilizes at $\mathcal{R}^*_\alpha$. So, we have $\mathcal{R}'_{\text{comp}} \subseteq \mathcal{R}^*_\alpha \subseteq \mathcal{R}^*$. As all connected components of $\mathcal{R}'$ are subsets of $\mathcal{R}^*$, we obtain that $\mathcal{R}' \subseteq \mathcal{R}^*$.

Let us conclude by noting that $\mathcal{R}^*$ is maximal among all pointwise simulation relations of (F, h) by $(\bar{F}, \bar{h})$. Moreover, if we consider all submanifolds of $M \times \bar{M}$ which are admissible simulation relations of the given systems and on which $\bar{F}$ has disturbance constant rank, then, by the theorem above, $\mathcal{R}^*$ is maximal among these simulation relations.

Returning to the algorithm in the general case, in step 3, we may have that $\mathcal{R}^k$ is a $0-$ dimensional set. Although it is somewhat unusual for a simulation relation to consist of a point or a $0-$dimensional submanifold, we discuss this possibility below.

Lemma 4.10 Let $\mathcal{R}$ be a $0-$dimensional subset of $M \times \bar{M}$ and consider two IDO systems (F, h) and $(\bar{F}, \bar{h})$, where $F : M \times \Omega \times \Delta \to TM$ and $\bar{F} : \bar{M} \times \Omega \times \bar{\Delta} \to T\bar{M}$. Then $\mathcal{R}$ is an admissible simulation relation of (F, h) by $(\bar{F}, \bar{h})$ if and only if for any $(x_0, \bar{x}_0) \in \mathcal{R}, h(x_0) = \bar{h}(\bar{x}_0)$ and for every $\omega \in \Omega$, and $\delta \in \Delta$, there exists $\bar{d} \in \bar{D}$ such that $F(x_0, \omega, \delta) = 0$ and $\bar{F}(\bar{x}_0, \omega, \bar{d}(t)) = 0$ for almost all t in some interval containing 0.

Proof. Let $\mathcal{R}$ be an admissible simulation relation of F by $\bar{F}$ and consider $(x_0, \bar{x}_0) \in \mathcal{R}, \omega \in \Omega$, and $\delta \in \Delta$. By Definition 3.2.1, if we choose the input and disturbance to be constant, i.e., $u(t) \equiv \omega$ and $d(t) \equiv \delta$, there exists $\bar{d} \in \bar{\mathcal{D}}$ such that $(\psi(t, x_0, \omega, \delta), \bar{\psi}(t, \bar{x}_0, \omega, \bar{d}(t))) \in \mathcal{R}$, where ψ and $\bar{\psi}$ denote the trajectories of F and $\bar{F}$. Moreover, $h(x_0) = \bar{h}(\bar{x}_0)$. Since ψ and $\bar{\psi}$ are continuous, it follows that the image of any interval containing 0 under $(\psi, \bar{\psi})$ is path-connected. But then, since $\mathcal{R}$ is $0-$dimensional, we must have $(\psi(t, x_0, \omega, \delta), \bar{\psi}(t, \bar{x}_0, \omega, d(t))) \equiv (x_0, \bar{x}_0)$. Differentiating this relation, we obtain the conclusion.

For the implication in the opposite direction, it is enough to note that, under the given hypothesis, the only trajectory ψ of F is $\psi(t, x_0, u, d) \equiv x_0$ and that there exists $\bar{d} \in \bar{\Delta}$ such that $\bar{\psi}(t, x_0, u, \bar{d}(t)) \equiv \bar{x}_0$ is a trajectory of $\bar{F}$. $\square$

In particular, if $F(x, \omega, \delta) = a(x) + b(x)\omega + g(x)\delta, x \in M, \omega \in \Omega, \delta \in \Delta$ and $\bar{F}(\bar{x}, \omega, \bar{\delta}) = \bar{a}(\bar{x}) + \bar{b}(\bar{x})\omega + \bar{g}(\bar{x})\bar{\delta}, \bar{x} \in M, \omega \in \Omega, \bar{\delta} \in \bar{\Delta}$, then the lemma above implies:

Lemma 4.11 Let $\mathcal{R}$ be a $0-$dimensional subset of $M \times \bar{M}$ and consider two IDO systems affine in inputs and disturbances as above. Then $\mathcal{R}$ is an admissible simulation relation of (F,h) by $(\bar{F},\bar{h})$ if and only if $h(x_0)=\bar{h}(\bar{x}_0), a(x_0)=b(x_0)=g(x_0)=0$ and $\bar{a}(x_0)+\mathrm{im}(\bar{b}(\bar{x}_0))\subseteq\mathrm{im}(\bar{g}(\bar{x}_0))$.

Proof. If $\mathcal{R}$ is an admissible simulation relation then, by Lemma 4.10, $F(x_0,\omega,\delta)=0$ for all $\omega\in\Omega,\delta\in\Delta$. This implies $a(x_0)=b(x_0)=g(x_0)$ by using $\omega=\delta\equiv 0$, followed by $\omega\equiv 0$ and δ arbitrary, and, lastly, $\delta\equiv 0$ and ω arbitrary. On the other hand, if we choose $\omega\equiv 0$, we obtain $\bar{a}(\bar{x}_0)=-\bar{g}(\bar{x}_0)\bar{d}(t)$ for almost all t in an open interval containing 0, which implies $\bar{a}(\bar{x}_0)\in\mathrm{im}(\bar{g}(\bar{x}_0))$. Since $\bar{F}(\bar{x}_0,\omega,\bar{d}(t))=0$, the previous relation implies that $\mathrm{im}(\bar{b}(\bar{x}_0))\subseteq\mathrm{im}(\bar{g}(\bar{x}_0))$, thus showing that $\bar{a}(\bar{x}_0)+\mathrm{im}(\bar{b}(\bar{x}_0))\subseteq\mathrm{im}(\bar{g}(\bar{x}_0))$.

The converse holds by choosing $\bar{d}(t)\equiv\bar{\delta}$, where $\bar{\delta}$ is the solution of the equation $\bar{F}(\bar{x}_0,\omega,\bar{\delta})=\bar{a}(\bar{x}_0)+\bar{b}(\bar{x}_0)\omega+\bar{g}(\bar{x}_0)\bar{\delta}=0$ guaranteed by $\bar{a}(\bar{x}_0)+\mathrm{im}(\bar{b}(\bar{x}_0))\subseteq\mathrm{im}(\bar{g}(\bar{x}_0))$. $\square$

5. EXAMPLES

Example 1. Given the nonlinear systems:

$$(1)\begin{cases} \dot{x}_1 = f_1(x_1) + b_1(x_1)u + g_1(x_1, x_2, x_3)d \\ \dot{x}_2 = f_2(x_1, x_2, x_3) + b_2(x_1, x_2, x_3)u + d \\ \dot{x}_3 = g_1(x_1, x_2, x_3)(f_3(x_1, x_2, x_3) + b_3(x_1, x_2, x_3)u + g_3(x_1, x_2, x_3)d) \\ h(x_1, x_2, x_3) = (x_1, x_2) \end{cases}$$

and

$$(2)\begin{cases} \dot{z}_1 = f_1(z_1) + b_1(z_1)u \\ \dot{z}_2 = \bar{f}_2(z_1, z_2) + \bar{b}_2(z_1, z_2)u + \bar{d} \\ \bar{h}(z_1, z_2) = (z_1, z_2), \end{cases}$$

we apply the algorithm in Chapter 4 to find a simulation relation of the first system by the second.

$\mathcal{R}^0 = \mathbb{R}^3 \times \mathbb{R}^2 = \mathbb{R}^5$

$$\tilde{\mathcal{R}}^1 = \{(x_1, x_2, x_3, z_1, z_2) | h(x_1, x_2, x_3) = \bar{h}(z_1, z_2)\} =$$

$$= \{(x_1, x_2, x_3, z_1, z_2) | (x_1, x_2) = (z_1, z_2)\} = \{(x_1, x_2, x_3, x_1, x_2) | (x_1, x_2, x_3) \in \mathbb{R}^3\}.$$

Since $\tilde{\mathcal{R}}^1$ is a submanifold of $\mathcal{R}^0$, we can choose $\mathcal{R}^1 = \tilde{\mathcal{R}}^1$ in step 2 of the algorithm and continue to the next step. In doing so, let's first note that for any $(x, z) \in \mathcal{R}^1$, $T_{(x,z)}\mathcal{R}^1 \cong \mathcal{R}^1$.

$$\tilde{\mathcal{R}}^2 = \left\{ (x_1,x_2,x_3,x_1,x_2) \in \mathcal{R}^1 \middle| \begin{bmatrix} f_1(x_1) \\ f_2(x_1,x_2,x_3) \\ g_1(x_1,x_2,x_3)f_3(x_1,x_2,x_3) \\ \bar{f}_1(x_1) \\ \bar{f}_2(x_1,x_2) \end{bmatrix} + \right.$$

$$+\mathrm{im} \begin{bmatrix} b_1(x_1) \\ b_2(x_1,x_2,x_3) \\ g_1(x_1,x_2,x_3)b_3(x_1,x_2,x_3) \\ \bar{b}_1(x_1) \\ \bar{b}_2(x_1,x_2) \end{bmatrix} + \mathrm{im} \begin{bmatrix} g_1(x_1,x_2,x_3) \\ 1 \\ g_1(x_1,x_2,x_3)g_3(x_1,x_2,x_3) \\ 0 \\ 0 \end{bmatrix} \subset$$

$$\left. \subset T_{(x,z)}\mathcal{R}^1 + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

So, we have $\tilde{\mathcal{R}}^2 = \big\{ (x_1,x_2,x_3,x_1,x_2) \in \mathbb{R}^5 | (\forall)\lambda,\mu \in \mathbb{R}, (\exists)\lambda_i \in \mathbb{R}, 1 \le i \le 4,$

$$\left. \begin{bmatrix} f_1(x_1) + \lambda b_1(x_1) + \mu g_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) + \lambda b_2(x_1,x_2,x_3) + \mu \\ g_1(x_1,x_2,x_3)(f_3(x_1,x_2,x_3) + \lambda b_3(x_1,x_2,x_3) + \mu g_3(x_1,x_2,x_3)) \\ \bar{f}_1(x_1) + \lambda \bar{b}_1(x_1) \\ \bar{f}_2(x_1,x_2) + \lambda \bar{b}_2(x_1,x_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_1 \\ \lambda_2 + \lambda_4 \end{bmatrix} \right\}.$$

Hence, $(x_1,x_2,x_3,x_1,x_2) \in \tilde{\mathcal{R}}^2$ iff for all $\lambda,\mu \in \mathbb{R}$, there exist $\lambda_i \in \mathbb{R}$ with $1 \le i \le 4$ such that

$$f_1(x_1) + \lambda b_1(x_1) + \mu g_1(x_1,x_2,x_3) = \lambda_1, \qquad f_2(x_1,x_2,x_3) + \lambda b_2(x_1,x_2,x_3) + \mu = \lambda_2,$$

$$g_1(x_1,x_2,x_3)(f_3(x_1,x_2,x_3) + \lambda b_3(x_1,x_2,x_3) + \mu g_3(x_1,x_2,x_3)) = \lambda_3,$$

$$\bar{f}_1(x_1) + \lambda \bar{b}_1(x_1) = \lambda_1, \qquad \bar{f}_2(x_1,x_2) + \lambda \bar{b}_2(x_1,x_2) = \lambda_2 + \lambda_4.$$

Comparing the first and fourth equations and keeping in mind that μ is arbitrary, we obtain $g_1(x_1,x_2,x_3) = 0$. Note that the remaining equations give no additional restrictions; λ_i and can be selected as follows:

$$\lambda_1 = f_1(x_1) + \lambda b_1(x_1) \qquad \lambda_3 = 0$$

$$\lambda_2 = f_2(x_1,x_2,x_3) + \lambda b_2(x_1,x_2,x_3) + \mu \qquad \lambda_4 = \mu + \bar{f}_2(x_1,x_2) + \lambda \bar{b}_2(x_1,x_2) - \lambda_2.$$

Hence $\tilde{\mathcal{R}}^2 = \{(x_1,x_2,x_3,x_1,x_2) | g_1(x_1,x_2,x_3) = 0\}$. Let us further investigate this set. Consider the function $G : \mathbb{R}^5 \to \mathbb{R}^3$ given by $G(x_1,x_2,x_3,x_4,x_5) =$

$(g_1(x_1,x_2,x_3), x_1-x_4, x_2-x_5)$ and note that $\tilde{\mathcal{R}}^2 = G^{-1}(0,0,0)$. The Jacobian matrix of G is

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

Note that the rank of the Jacobian at (x_1,x_2,x_3,x_4,x_5) is three iff any one of the partial derivatives of g_1 at the (x_1,x_2,x_3) is nonzero. If this is the case, then $(0,0,0)$ is a regular value for G and $\tilde{\mathcal{R}}^2$ is a submanifold of $\mathbb{R}^5$. In fact, a similar type of argument can be used to show that $\tilde{\mathcal{R}}^2$ is a submanifold of $\mathcal{R}^1$ and we can proceed to the next step by considering $\mathcal{R}^2 = \tilde{\mathcal{R}}^2$. On the other hand, if there exist $(x_1,x_2,x_3) \in \mathbb{R}^3$ such that all partial derivatives of g_1 at (x_1,x_2,x_3) are zero, then $\tilde{\mathcal{R}}^2$ may not be a submanifold. For example, this is the case when $g_1(x_1,x_2,x_3) = x_1^2+x_2^2-x_3^2$. At $(0,0,0)$, all partial derivatives of g_1 vanish. In fact $\tilde{\mathcal{R}}^2 = \{(x_1,x_2,x_3,x_1,x_2)|x_1^2+x_2^2 = x_3^2\}$ is a cone in $\mathbb{R}^5$, and hence it cannot be a submanifold of $\mathcal{R}^1$. However, note that $(0,0,0,0,0)$ is the only critical point, so we can remove it from $\tilde{\mathcal{R}}^2$ and get a submanifold by using the constant rank theorem on $\mathbb{R}^5-\{(0,0,0,0,0)\}$. This way we can choose $\mathcal{R}^2 = \tilde{\mathcal{R}}^2-\{(0,0,0,0,0)\}$. While this is an interesting special case, in what follows we will still consider a general g_1 and work under the assumption that we can choose $\mathcal{R}^2$ to be $\tilde{\mathcal{R}}^2$ minus the set of critical points.

In the next run of the algorithm, we need to construct

$$\tilde{\mathcal{R}}^3 = \left\{ x \in \mathcal{R}^2 | \begin{bmatrix} f_1(x_1) \\ f_2(x_1,x_2,x_3) \\ 0 \\ \bar{f}_1(x_1) \\ \bar{f}_2(x_1,x_2) \end{bmatrix} + \operatorname{im} \begin{bmatrix} b_1(x_1) \\ b_2(x_1,x_2,x_3) \\ 0 \\ \bar{b}_1(x_1) \\ \bar{b}_2(x_1,x_2) \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset T_x\mathcal{R}^2 + \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Let us now consider $T_x\mathcal{R}^2$. Since $\mathcal{R}^2$ is obtained as the preimage of a regular value in both cases above, its tangent space is given by the kernel of the Jacobian of G, i.e., the set

$$\left\{[\lambda_1, \lambda_2, \lambda_3, \lambda_1, \lambda_2]^T | \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_1}{\partial x_2} + \lambda_3 \frac{\partial g_1}{\partial x_3} = 0\right\}$$

So, we have $(x_1, x_2, x_3, x_1, x_2) \in \tilde{\mathcal{R}}^3$ iff $x \in \mathcal{R}^2$ and for all $\lambda, \mu \in \mathbb{R}$, there exist $\lambda_i \in \mathbb{R}, 1 \le i \le 4$ such that

$$f_1(x_1) + \lambda b_1(x_1) = \lambda_1, \quad f_2(x_1, x_2, x_3) + \lambda b_2(x_1, x_2, x_3) + \mu = \lambda_2, \quad 0 = \lambda_3$$

$$\bar{f}_2(x_1, x_2) + \lambda \bar{b}_2(x_1, x_2) = \lambda_2 + \lambda_4, \quad \text{and} \quad \lambda_1 \tfrac{\partial g_1}{\partial x_1} + \lambda_2 \tfrac{\partial g_1}{\partial x_2} + \lambda_3 \tfrac{\partial g_1}{\partial x_3} = 0.$$

We can find $\lambda_i, 1 \le i \le 4$, satisfying the equations iff

$$(f_1(x_1) + \lambda b_1(x_1)) \frac{\partial g_1}{\partial x_1} + (f_2(x_1, x_2, x_3) + \lambda b_2(x_1, x_2, x_3) + \mu) \frac{\partial g_1}{\partial x_2} = 0.$$

Since λ and μ are arbitrary, the relation above holds iff

$$f_1(x_1) \frac{\partial g_1}{\partial x_1} = 0, \quad \frac{\partial g_1}{\partial x_2} = 0, \quad \text{and} \quad b_1(x_1) \frac{\partial g_1}{\partial x_1} = 0.$$

Hence

$$\tilde{\mathcal{R}}^3 = \left\{x \in \mathcal{R}^2 | f_1(x_1) \frac{\partial g_1}{\partial x_1} = 0, \frac{\partial g_1}{\partial x_2} = 0 \,\text{and}\, b_1(x_1) \frac{\partial g_1}{\partial x_1} = 0\right\}.$$

Case 1. If $f_1(x_1) \frac{\partial g_1}{\partial x_1} \equiv 0, b_1(x_1) \frac{\partial g_1}{\partial x_1} \equiv 0$ and $\frac{\partial g_1}{\partial x_2} \equiv 0$ on $\mathcal{R}^2$ then $\tilde{\mathcal{R}}^3 = \mathcal{R}^2$ and the algorithm terminates successfully since we can choose $\mathcal{R}^3 = \tilde{\mathcal{R}}^3$. For example,

a) if $f_1(x_1) = 0, b_1(x_1) = 0$ for all $x_1 \in \mathbb{R}$ and $g_1(x_1, x_2, x_3) = x_1^2 + x_3^2 - 1$ then

$$\tilde{\mathcal{R}}^2 = \mathcal{R}^2 = \tilde{\mathcal{R}}^3 = \mathcal{R}^3 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | x_1^2 + x_3^2 = 1\}$$

is the image in $\mathbb{R}^5$ of the two-dimensional cylinder $x_1^2 + x_3^2 = 1$ in $\mathbb{R}^3$ under the imbedding $(x_1, x_2, x_3) \to (x_1, x_2, x_3, x_1, x_2)$. The algorithm terminates successfully with $\mathcal{R}^3$. Moreover, $\bar{F}$ has disturbance constant rank on $\mathcal{R}^3$ since the disturbance vector field $[0, 0, 0, 0, 1]^T$ is never tangent to $\mathcal{R}^3$. Hence $\mathcal{R}^3$ is a (pointwise and admissible) simulation relation.

b) if $f_1(x_1) = 0, b_1(x_1) = 0$ for all $x_1 \in \mathbb{R}$ and $g_1(x_1, x_2, x_3) = x_1^2 - x_3^2$ then

$$\tilde{\mathcal{R}}^2 = \{(x_1, x_2, x_3, x_1, x_2) | x_1 = \pm x_3\}.$$

If we choose $\mathcal{R}^2 = \tilde{\mathcal{R}}^2 - \{(0, x_2, 0, 0, x_2)\}$ then the algorithm terminates successfully with $\mathcal{R}^2$, the union of four half-planes. Again, $\mathcal{R}^2$ does satisfy the disturbance constant rank condition for the same reason as above and $\mathcal{R}^2$ is a (pointwise and admissible) simulation relation.

It should be noted that while $\mathcal{R}^2$ in case 1b) above has been chosen in a "maximal manner", $\mathcal{R}^2$ is not a "maximal"simulation relation (see Theorem 4.9). Indeed, if we consider

$$\mathcal{R} = \{(x_1, x_2, x_1, x_1, x_2) | (x_1, x_2) \in \mathbb{R}^2\},$$

then $\mathcal{R}$ is a (pointwise and admissible) simulation relation of the first system by the second since conditions $(ASC)_1$ and $(ASC)_2$ are satisfied on $\mathcal{R}$ and $\bar{F}$ has disturbance constant rank along $\mathcal{R}$. Yet $\mathcal{R} \subsetneq \mathcal{R}^2$. The reason why we cannot apply Theorem 4.9 to conclude maximality is that the regular pre-simulation condition up to order seven from the hypothesis of this theorem is not satisfied. More specifically, while $\bar{F}$ does have disturbance constant rank along $\mathcal{R}^1$, $\tilde{\mathcal{R}}^2$ is not a union of disjoint and *closed* submanifolds of $\tilde{\mathcal{R}}^0$.

If we disregard the observation above about $\tilde{\mathcal{R}}^2$ and pretend for a moment that $\tilde{\mathcal{R}}^2$ has not yet been computed, we can re-compute $\tilde{\mathcal{R}}^2$ based on Proposition 4.8 in the hope that it will be a union of closed and *disjoint* submanifolds. Let us first make the observation that since $i_{1,1}^* T\mathcal{R}^0, i_{1,1} : \mathcal{R}^1 \to \mathcal{R}^0$, is a trivial bundle over $\mathcal{R}^1$, we only need to investigate the rank of a single function $\Psi_{1,1} : \mathbb{R}^3 \to \mathbb{R}^{15}$ (as defined in Proposition 4.8), where $\mathcal{R}^1$ has been identified with $\mathbb{R}^3$.

Let us now recall that in order to find $\Psi_{1,1}$ we first need to determine S, the section of $\mathrm{Hom}(i_{1,1}^* T\mathcal{R}^0)$ defined by the property that, for any $x \in \mathcal{R}^1$, $S(x)$ sends any vector in $i_{1,1}^* T_x\mathcal{R}^0$ (identified with $\mathbb{R}^5$) to its projection onto $T_x\mathcal{R}^1 + \mathrm{im}([0,0,0,0,1]^T)$ (see Proposition 4.8 and the proof of Proposition 2.4.4). Note that for any $x \in \mathcal{R}^1$, the vectors

$$\begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \text{ and } \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

span the vector space $T_x\mathcal{R}^1 + \mathrm{im}([0,0,0,0,1]^T)$. This is a four dimensional vector space whose orthogonal complement is spanned by $[1,0,0,-1,0]^T$. Since $(S-I)(x)$ takes any vector $v \in \mathbb{R}^5$ to the negative of its orthogonal projection onto this orthogonal complement, we obtain that for any $x \in \mathbb{R}^3$,

$$(S-I)(x)([v_1,v_2,v_3,v_4,v_5]^T) = \left[\frac{1}{2}(v_4 - v_1), 0, 0, \frac{1}{2}(v_1 - v_4), 0\right].$$

Thus, $\Psi_{1,1}$ sends (x_1,x_2,x_3) to the 15$-$dimensional vector consisting of zeros on all coordinates except for the eleventh and fourteenth place where the entries are $-\frac{1}{2}g_1(x_1,x_2,x_3)$ and $\frac{1}{2}g_1(x_1,x_2,x_3)$, respectively. Clearly, the rank of $\Psi_{1,1}$ at (x_1,x_2,x_3) is one or zero depending on whether or not the gradient of g_1 at (x_1,x_2,x_3) is nonzero. In particular, for $g_1(x_1,x_2,x_3) = x_1^2 - x_3^2$, the rank of g_1 is zero at $(0,x_2,0)$ for any $x_2 \in \mathbb{R}$ and one everywhere else. Hence, the rank of $\Psi_{1,1}$ cannot be constant (on any neighborhood of $(0,0,0,0,0)$) and, since $(0,0,0,0,0)$ is contained in $\tilde{\mathcal{R}}^2$, the rank of $\Psi_{1,1}$ will vary (on any neighborhood of $\tilde{\mathcal{R}}^2$).

Case 2. If $f_1(x_1)\frac{\partial g_1}{\partial x_1} \equiv 0, b_1(x_1)\frac{\partial g_1}{\partial x_1} \equiv 0$ on $\mathcal{R}^2$ but $\frac{\partial g_1}{\partial x_2}$ is not identically 0 then let us consider the following cases:

a) $f_1(x_1) = b_1(x_1) = 0$ for all $x_1 \in \mathbb{R}$ and $g_1(x_1,x_2,x_3) = x_2 + \phi(x_1,x_3)$. Here,

$\frac{\partial g_1}{\partial x_2}$ is never zero, hence $\tilde{\mathcal{R}}^3 = \emptyset$ and the algorithm terminates unsuccessfully.

b) $f_1(x_1) = b_1(x_1) = 0$ for all $x_1 \in \mathbb{R}$ and $g_1(x_1, x_2, x_3) = e^{x_2}(1 - \phi(x_1, x_3))$, where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function whose Jacobian matrix is never zero (so that $\tilde{\mathcal{R}}^2$ is a submanifold). In this case

$$\tilde{\mathcal{R}}^3 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | \phi(x_1, x_3) = 1\} = \mathcal{R}^2 = \tilde{\mathcal{R}}^2.$$

and the algorithm terminates successfully.

Case 3. $f_1(x_1) = 0, b_1(x_1) = x_1$, and $g_1(x_1, x_2, x_3) = x_1^2 + x_3^2 - 1$. In this case

$$\mathcal{R}^2 = \tilde{\mathcal{R}}^2 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | x_1^2 + x_3^2 = 1\} \quad \text{and}$$

$$\tilde{R}^3 = \left\{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | x_1^2 + x_3^2 = 1, x_1 = 0\right\} = \left\{(0, x_2, \pm 1, 0, x_2) \in \mathbb{R}^5 | x_2 \in \mathbb{R}\right\},$$

which is the union of two parallel lines on the cylinder. Hence we can choose $\mathcal{R}^3 = \tilde{\mathcal{R}}^3$ (or any one of the two component lines). Note that a tangent vector to $\mathcal{R}^3$ is $[0, 1, 0, 0, 1]^T$. But then

$$\tilde{\mathcal{R}}^4 = \left\{ x \in \mathcal{R}^3 \middle| \begin{bmatrix} 0 \\ f_2(0, x_2, \pm 1) \\ 0 \\ 0 \\ \bar{f}_2(0, x_2) \end{bmatrix} + \mathrm{im} \begin{bmatrix} 0 \\ b_2(0, x_2, \pm 1) \\ 0 \\ 0 \\ \bar{b}_2(0, x_2) \end{bmatrix} + \mathrm{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \mathrm{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Equivalently, $(0, x_2, \pm 1, 0, x_2) \in \tilde{\mathcal{R}}^4$ iff $(\forall)\lambda, \mu \in \mathbb{R}$, there exist $\lambda_i \in \mathbb{R}, 1 \leq i \leq 1$ such that

$$f_2(0, x_2, \pm 1) + \lambda b_2(0_1, x_2, \pm 1) + \mu = \lambda_1 \quad \text{and} \quad \bar{f}_2(0, x_2) + \lambda \bar{b}_2(0, x_2) = \lambda_1 + \lambda_2$$

Clearly, one can always choose λ_1 and λ_2 with the properties above. So $\mathcal{R}^4 = \tilde{\mathcal{R}}^4 = \mathcal{R}^3$ and, since $\bar{F}$ has disturbance constant rank on $\mathcal{R}^4$, the algorithm terminates successfully with $\mathcal{R}^4$, the union of two disjoint submanifolds with the same dimension.

Case 4. $f_1(x_1) \equiv 0, b_1(x_1) = x_1$, and $g_1(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3$. Here

$$\tilde{\mathcal{R}}^2 = \mathcal{R}^2 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | x_1^2 + x_2^2 = x_3\}$$

is the image in $\mathbb{R}^5$ of the paraboloid of revolution $x_1^2 + x_2^2 = x_3$ in $\mathbb{R}^3$ under the imbedding $(x_1, x_2, x_3) \to (x_1, x_2, x_3, x_1, x_2)$. Moreover,

$$\tilde{\mathcal{R}}^3 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | x_1^2 + x_2^2 = x_3, x_2 = 0, x_1 = 0\} = \{(0,0,0,0,0)\}$$

So, we obtain that $\tilde{\mathcal{R}}^3$ is a point. But then, based on Lemma 4.11, the algorithm terminates unsuccessfully since the second component of the disturbance vector field $[g_1(x_1, x_2, x_2), 1, g_1(x_1, x_2, x_3)g_3(x_1, x_2, x_3)]^T$ is never zero. Alternatively, if we choose $\mathcal{R}^3 = \{(0,0,0,0,0)\}$ and continue the algorithm, then we obtain $\mathcal{R}^4 = \emptyset$ since it is impossible to have

$$\begin{bmatrix} 0 \\ f_2(0,0,0) \\ 0 \\ 0 \\ \bar{f}_2(0, x_2) \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 \\ b_2(0,0,0) \\ 0 \\ 0 \\ \bar{b}_2(0, x_2) \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Case 5. Let us make a minor change in the disturbance vector field of the original systems in order to get the following new systems:

$$(1) \begin{cases} \dot{x}_1 = f_1(x_1) + b_1(x_1)u + g_1(x_1, x_2, x_3)d \\ \dot{x}_2 = f_2(x_1, x_2, x_3) + b_2(x_1, x_2, x_3)u + x_2 d \\ \dot{x}_3 = g_1(x_1, x_2, x_3)(f_3(x_1, x_2, x_3) + b_3(x_1, x_2, x_3)u + g_3(x_1, x_2, x_3)d) \\ h(x_1, x_2, x_3) = (x_1, x_2) \end{cases}$$

and

$$(2) \begin{cases} \dot{z}_1 = f_1(z_1) + b_1(z_1)u \\ \dot{z}_2 = \bar{f}_2(z_1, z_2) + \bar{b}_2(z_1, z_2)u + \bar{d} \\ \bar{h}(z_1, z_2) = (z_1, z_2). \end{cases}$$

This way we obtain the same general $\tilde{\mathcal{R}}^2 = \mathcal{R}^2$ as before, but this time $(x_1, x_2, x_3, x_1, x_2) \in \tilde{\mathcal{R}}^3$ iff $g_1(x_1, x_2, x_3) = 0$ and for all $\lambda, \mu \in \mathbb{R}$, we can find $\lambda_i, 1 \leq i \leq 4$, such that

$$f_1(x_1) + \lambda b_1(x_1) = \lambda_1, \quad f_2(x_1, x_2, x_3) + \lambda b_2(x_1, x_2, x_3) + \mu x_2 = \lambda_2, \quad 0 = \lambda_3$$

$$\bar{f}_2(x_1,x_2)+\lambda\bar{b}_2(x_1,x_2)=\lambda_2+\lambda_4, \quad \text{and} \quad \lambda_1\tfrac{\partial g_1}{\partial x_1}+\lambda_2\tfrac{\partial g_1}{\partial x_2}+\lambda_3\tfrac{\partial g_1}{\partial x_3}=0.$$

We can do so as long as

$$(f_1(x_1)+\lambda b_1(x_1))\frac{\partial g_1}{\partial x_1}+(f_2(x_1,x_2,x_3)+\lambda b_2(x_1,x_2,x_3)+\mu x_2)\frac{\partial g_1}{\partial x_2}=0,$$

which is equivalent to

$$f_1(x_1)\frac{\partial g_1}{\partial x_1}+f_2(x_1,x_2,x_3)\frac{\partial g_1}{\partial x_2}=0,\ x_2\frac{\partial g_1}{\partial x_2}=0, \text{ and } b_1(x_1)\frac{\partial g_1}{\partial x_1}+b_2(x_1,x_2,x_3)\frac{\partial g_1}{\partial x_2}=0.$$

Let us now look at a specific case of the above in which $g_1(x_1,x_2,x_3)=x_1^2+x_2^2-x_3$, $b_1(x_1)=x_1$, and $b_2(x_1,x_2,x_3)=f_2(x_1,x_2,x_3)\equiv 0$ (or, more generaly, any "multiple"of g_1). We obtain

$$\tilde{\mathcal{R}}^3=\{(x_1,x_2,x_3,x_1,x_2)\in\mathbb{R}^5|x_1^2+x_2^2=x_3,x_1=0,x_2=0\}=\{(0,0,0,0,0)\}.$$

This time, the relation in Lemma 4.11 is satisfied iff $f_1(0)=0$ and

$$\begin{bmatrix}0\\ \bar{f}_2(0,0)\end{bmatrix}+\operatorname{im}\begin{bmatrix}0\\ \bar{b}_2(0,0)\end{bmatrix}\subset\operatorname{im}\begin{bmatrix}0\\1\end{bmatrix}.$$

So, it is enough to have $f_1(0)=0$. Thus, we obtain a $0-$dimensional simulation relation provided that $g_1(x_1,x_2,x_3)=x_1^2+x_2^2-x_3, b_1(x_1)=x_1, b_2(x_1,x_2,x_3)=f_2(x_1,x_2,x_3)\equiv 0$, $f_1(0)=0$, and all other functions are arbitrary.

Example 2. The following example (without the algorithm) can be found in [**4**] and shows that, in general, without assuming that $\bar{F}$ has disturbance constant rank, the submanifold at termination may not be an admissible simulation relation.

Consider the following systems

$$1)\begin{cases}\dot{x}_1=1+x_1^3d\\ \dot{x}_2=d\\ h(x_1,x_2)=x_1\end{cases}\quad \text{and}\quad (2)\begin{cases}\dot{z}_1=1+z_1^2\bar{d}\\ h(z_1)=z_1\end{cases}$$

Clearly, $\tilde{\mathcal{R}}^1=\mathcal{R}^1=\{(x_1,x_2,x_1)\in\mathbb{R}^3|x_1,x_2\in\mathbb{R}\}$. Continuing the algorithm, we have

$$\tilde{\mathcal{R}}^2 = \left\{ (x_1, x_2, x_1) \in \mathcal{R}^1 \middle| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \mathrm{im} \begin{bmatrix} x_1^3 \\ 1 \\ 0 \end{bmatrix} \subseteq T_{(x_1,x_2,x_1)}\mathcal{R}^1 + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ x_1^2 \end{bmatrix} \right\}.$$

Since $T_{(x_1,x_2,x_1)}\mathcal{R}^1$ is spanned by $[1,0,1]^T$ and $[0,1,0]^T$, $(x_1,x_2,x_1) \in \tilde{\mathcal{R}}^2$ iff for any $\lambda \in \mathbb{R}$ there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$1 + \lambda x_1^3 = \lambda_1, \lambda = \lambda_2, \text{ and } 1 = \lambda_1 + \lambda_3 x_1^2.$$

Since we can always find $\lambda_i, 1 \leq i \leq 3$, satisfying the three relations above, we have $\tilde{\mathcal{R}}^2 = \mathcal{R}^1$ and the algorithm terminates. However, as it was pointed out in the reference mentioned above, $\bar{F}$ does not have disturbance constant rank and $\mathcal{R}^2$ is not an admissible simulation relation (but is a pointwise simulation relation, as expected).

Example 3. In contrast to the previous example, in this example, despite the disturbance constant rank condition not being satisfied on the submanifold we obtain at termination, the submanifold at termination is actually an admissible simulation relation. This shows that the disturbance constant rank condition is sufficient for admissibility, but not necessary.

Consider the systems:

$$(1) \begin{cases} \dot{x}_1 = x_1 + x_1 u + \dfrac{2}{3} x_1 d \\ \dot{x}_2 = \dfrac{3}{2} x_2 + \dfrac{3}{2} x_2 u + x_2 d \\ h(x_1, x_2) = (x_1, x_2) \end{cases} \quad \text{and} \quad (2) \begin{cases} \dot{z}_1 = z_1 + z_1 u \\ \dot{z}_2 = z_2 \bar{d} \\ \bar{h}(z_1, z_2) = (z_2^2, z_2^3). \end{cases}$$

$\tilde{\mathcal{R}}^1 = \{(x_1, x_2, z_1, z_2) \in \mathbb{R}^4 | x_1 = z_2^2, x_2 = z_2^3\} = \{(z_2^2, z_2^3, z_1, z_2) | z_1, z_2 \in \mathbb{R}\}$. Note that $\tilde{\mathcal{R}}^1$ is the preimage of $0 \in \mathbb{R}^2$ under the function $G : \mathbb{R}^4 \to \mathbb{R}^2$ given by $G(x_1, x_2, x_3, x_4) = (x_1 - x_4^2, x_2 - x_4^3)$. The Jacobian of this function at (x_1, x_2, x_3, x_4) is

$$\begin{bmatrix} 1 & 0 & 0 & -2x_4 \\ 0 & 1 & 0 & -3x_4^2 \end{bmatrix}.$$

Clearly, the rank of the Jacobian is constant. Hence, $\tilde{\mathcal{R}}^1$ is a (two-dimensional) submanifold of $\mathbb{R}^4$ and we can choose $\mathcal{R}^1 = \tilde{\mathcal{R}}^1$. The tangent space of $\mathcal{R}^1$ at (z_2^2, z_2^3, z_1, z_2) is given by the kernel of the Jacobian at this point. So, a vector $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$ is in the tangent space iff $\lambda_1 = 2z_2\lambda_4, \lambda_2 = 3z_2^2\lambda_4$. To summarize,

$$T_{(x,z)}\tilde{\mathcal{R}}^1 = \operatorname{span}\left\{ \begin{bmatrix} 2z_2 \\ 3z_2^2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Let us now compute $\tilde{\mathcal{R}}^2$.

$$\tilde{\mathcal{R}}^2 = \left\{ (z_2^2, z_2^3, z_1, z_2) \in \mathcal{R}^1 | \begin{bmatrix} z_2^2 \\ \frac{3}{2}z_2^3 \\ z_1 \\ 0 \end{bmatrix} + \operatorname{im} \begin{bmatrix} z_2^2 \\ \frac{3}{2}z_2^3 \\ z_1 \\ 0 \end{bmatrix} + \operatorname{im} \begin{bmatrix} \frac{2}{3}z_2^2 \\ z_2^3 \\ 0 \\ 0 \end{bmatrix} \subset T_{(x,z)}\mathcal{R}^1 + \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ z_2 \end{bmatrix} \right\} =$$

$$= \left\{ (z_2^2, z_2^3, z_1, z_2) \in \mathbb{R}^4 | (\forall)\lambda, \mu \in \mathbb{R}, (\exists)\lambda_i, 1 \le i \le 3, \right.$$

$$\left. \begin{bmatrix} z_2^2 + \lambda z_2^2 + \frac{2}{3}\mu z_2^2 \\ \frac{3}{2}z_2^3 + \frac{3}{2}z_2^3\lambda + z_2^3\mu \\ z_1 + \lambda z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2z_2\lambda_1 \\ 3z_2^2\lambda_1 \\ \lambda_2 \\ \lambda_1 + \lambda_3 z_2 \end{bmatrix} \right\}.$$

So, $(z_2^2, z_2^3, z_1, z_2) \in \tilde{\mathcal{R}}^2$ iff for all $\lambda, \mu \in \mathbb{R}$, there exists $\lambda_i, 1 \le i \le 3$ such that

$$z_2^2 + \lambda z_2^2 + \frac{2}{3}\mu z_2^2 = 2z_2\lambda_1,\ \frac{3}{2}z_2^3 + \frac{3}{2}z_2^3\lambda + z_2^3\mu = 3z_2^2\lambda_1,\ z_1 + \lambda z_1 = \lambda_2 \text{ and } 0 = \lambda_1 + \lambda_3 z_2.$$

Note that the second equation can be obtained by multiplying the first equation by $\frac{3}{2}z_2$. If $z_2 = 0$, with an arbitrary choice of λ_1, the first, and hence second equation, will be trivially satisfied. The value of λ_2 is given by the third equation and λ_3 can be chosen arbitrarily if we select $\lambda_1 = 0$. If $z_2 \neq 0$, then we can choose $\lambda_1 = z_2(\frac{1}{2} + \frac{1}{2}\lambda + \frac{1}{3}\mu), \lambda_2 = z_1(1+\lambda)$, and $\lambda_3 = -\frac{\lambda_1}{z_2}$. In summary, we have $\tilde{\mathcal{R}}^2 = \mathcal{R}^1$. Since $\tilde{\mathcal{R}}^2$ is a submanifold, we have $\mathcal{R}^2 = \tilde{\mathcal{R}}^2 = \mathcal{R}^1$.

We claim that the constant disturbance rank condition is not satisfied along $\mathcal{R}^2$. Indeed, along the line $\{(0, 0, z_1, 0) | z_1 \in \mathbb{R}\}$, the tangent space to $\mathcal{R}^2$ is spanned by the vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. So,

$$T_{(0,0,z_1,0)}\mathcal{R}^2 + \text{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

On the other hand, along the complement of the line above in $\mathcal{R}^2$,

$$T_{(z_2^2,z_2^2,z_1,z_2)}\mathcal{R}^2 + \text{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ z_2 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2z_2 \\ 3z_2^2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ z_2 \end{bmatrix} \right\}.$$

Since the three vectors on the right side of the equality are linearly independent, we obtain different dimensions depending on whether $z_2 = 0$ or $z_2 \neq 0$. So, the disturbance constant rank condition is not satisfied.

Now, we claim that $\mathcal{R}^2$ is actually an admissible simulation relation of the first system by the second. Let $(z_{20}^2, z_{20}^3, z_{10}, z_{20}) \in \mathcal{R}^2$. To prove that $\mathcal{R}^2$ is a simulation relation, we need to show that for any locally integrable functions u and d, there exists a locally integrable function $\bar{d}(t)$ such that

$$(\phi_1(t), \phi_2(t), \psi_1(t), \psi_2(t)) \in \mathcal{R}^2 \; a.e.,$$

where $(\phi_1(t), \phi_2(t))$ is the trajectory of the first system with initial conditions (z_{20}^2, z_{20}^3) and $(\psi_1(t), \psi_2(t))$ is the trajectory of the second system with initial conditions (z_{10}, z_{20}). If we select $\bar{d} = \frac{1}{2}(1+u+\frac{2}{3}d)$ then $\bar{d}$ is clearly locally integrable. Moreover, if we consider $\alpha(t) = \int_0^t [1+u(s)+\frac{2}{3}d(s)]\, ds$, then $t \to (z_{20}^2 e^{\alpha(t)}, z_{20}^3 e^{\frac{3}{2}\alpha(t)})$ is a trajectory for the first system starting at (z_{20}^2, z_{20}^3). If we also let $\beta(t) = \int_0^t [1 + u(s)]\, ds$ then $t \to (z_{10} e^{\beta(t)}, z_{20} e^{\frac{1}{2}\alpha(t)})$ is a trajectory of the second system starting at (z_{10}, z_{20}). As we can easily check,

$$t \to (z_{20}^2 e^{\alpha(t)}, z_{20}^3 e^{\frac{3}{2}\alpha(t)}, z_{10} e^{\beta(t)}, z_{20} e^{\frac{1}{2}\alpha(t)}) \in \mathcal{R}^2.$$

Example 4. The example below illustrates that for two systems there exist descending sequences as in Proposition 4.6 which do terminate with the empty set while others don't. In addition, the example also shows that it is possible to have $\alpha_1, \alpha_2 \in \mathcal{I}_k$ such that $\mathcal{R}^k_{\alpha_1}$ and $\mathcal{R}^k_{\alpha_2}$ have the same dimensions while at the next level, for some for $\bar{\alpha}_1, \bar{\alpha}_2$ the submanifolds $\mathcal{R}^{k+1}_{\bar{\alpha}_1}$ and $\mathcal{R}^{k+1}_{\bar{\alpha}_2}$ will have different dimensions, with $\mathcal{R}^{k+1}_{\bar{\alpha}_1}$ being obtained from $\mathcal{R}^k_{\alpha_1}$ and $\mathcal{R}^{k+1}_{\bar{\alpha}_2}$ being obtained from $\mathcal{R}^k_{\alpha_2}$. So, let us consider the nonlinear systems:

$$1) \begin{cases} \dot{x}_1 = (x_1 - 2)(x_1 x_2 + x_1 x_2 u + (x_1^2 + x_2^2 - 1)d) \\ \dot{x}_2 = -(x_1 - 2)(x_1^2 + x_1^2 u + d) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) + b_3(x_1, x_2, x_3)u + g_3(x_1, x_2, x_3)d \\ h(x_1, x_2, x_3) = (x_1, x_2) \end{cases}$$

and

$$(2) \begin{cases} \dot{z}_1 = (z_1 - 2)(z_1 z_2 + z_1 z_2 u) \\ \dot{z}_2 = -(z_1 - 2)(z_1^2 + z_2^2 u) + \bar{d} \\ \bar{h}(z_1, z_2) = (z_1, z_2). \end{cases}$$

By using a similar argument as for the systems in Example 1, we have

$$\tilde{\mathcal{R}}^2 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | (x_1 - 2)(x_1^2 + x_2^2 - 1) = 0\}.$$

So $\tilde{\mathcal{R}}^2$ is the union of two disjoint submanifolds of $\mathbb{R}^5$ of the same dimension:

$$\mathcal{R}_1^2 = \{(2, x_2, x_3, 2, x_2) | x_2, x_3 \in \mathbb{R}\} \text{ and } \mathcal{R}_2^2 = \{(x_1, x_2, x_3, x_1, x_2) \in \mathbb{R}^5 | x_1^2 + x_2^2 = 1\}.$$

Next we will construct $\tilde{\mathcal{R}}^3$ by using Proposition 4.8. First, let us show that the hypotheses of the proposition are satisfied. To check the disturbance constant rank condition, it is enough to note that the vector $[0, 0, 0, 0, 1]^T$ is never tangent to either $\mathcal{R}_1^2$ or $\mathcal{R}_2^2$.

Let us now compute $\Psi_{2,1}$ and $\Psi_{2,2}$. For any $x \in \mathcal{R}_1^2$, we have that $T_x\mathcal{R}_1^2 + \operatorname{im}\left([0,0,0,0,1]^T\right)$ is spanned by the vectors $[0,1,0,0,1]^T$, $[0,0,1,0,0]^T$, and $[0,0,0,0,1]^T$. Its orthogonal complement is then spanned by $[1,0,0,0,0]^T$ and $[0,0,0,1,0]^T$. As $(S-I)(x)$ is the negative of the projection onto this orthogonal complement, as one can easily check, $\Psi_{2,1}$ sends $(2,x_2,x_3,2,x_2) \in \mathcal{R}_1^2$ to the zero vector in $\mathbb{R}^5 \times \mathbb{R}^5 \times \mathbb{R}^5$. Clearly, the rank of $\Psi_{2,1}$ is constant and equal to zero. Thus

$$\left\{ x \in \mathcal{R}_1^2 \middle| \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ f_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ b_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} (x_1-2)(x_1^2+x_2^2-1) \\ -(x_1-2) \\ g_3(x_1,x_2,x_3) \\ 0 \\ 0 \end{bmatrix} \subseteq \right.$$

$$\left. \subseteq T_x\mathcal{R}_1^2 + \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathcal{R}_1^2.$$

For any $x \in \mathcal{R}_2^2$, $T_x\mathcal{R}_2^2 + \operatorname{im}\left([0,0,0,0,1]^T\right)$ is spanned by the vectors $[x_2,-x_1,0,x_2,-x_1]^T$, $[0,0,1,0,0]^T$, $[0,0,0,0,1]^T$. Its orthogonal complement in $\mathbb{R}^5$ is then spanned by the vectors $[x_1,x_2,0,0,0]^T$ and $[1,0,0,-1,0]^T$.

Hence, for any $x \in \mathcal{R}_2^2$ and any $v = [v_1,v_2,v_3,v_4,v_5]^T \in \mathbb{R}^5$, we have

$$((S-I)(x))(v) = \begin{bmatrix} \frac{x_1[(x_1-1)(v_1-v_4)+(v_1x_1+v_2x_2)(x_1-2)]}{x_1^2-2} \\ \frac{x_2[x_1(v_1-v_4)-2(v_1x_1+v_2x_2)]}{x_1^2-2} \\ 0 \\ \frac{(v_1-v_4)-x_1(v_1x_1+v_2x_2)}{x_1^2-2} \\ 0 \end{bmatrix}.$$

This way, by using the three vector fields $\sigma_1, \sigma_2, \sigma_3$ given by

$$\begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ f_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix}, \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ b_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix}, \text{ and } \begin{bmatrix} (x_1-2)(x_1^2+x_2^2-1) \\ -(x_1-2) \\ g_3(x_1,x_2,x_3) \\ 0 \\ 0 \end{bmatrix},$$

it follows that $\Psi_{2,2} : \mathcal{R}_2^2 \to \mathbb{R}^5 \times \mathbb{R}^5 \times \mathbb{R}^5$ sends any (x_1,x_2,x_3,x_1,x_2) with $x_1^2+x_2^2=1$ to the vector consisting of zeros on all components except for the eleventh, twelfth, and fourteenth component where the entries are

$$(*) \qquad -\frac{2x_1x_2(x_1-2)}{x_1^2-2}, \frac{2x_2^2(x_1-2)}{x_1^2-2}, \text{ and } \frac{x_1x_2(x_1-2)}{x_1^2-2},$$

respectively. Clearly, the preimage of $0 \in \mathbb{R}^{15}$ under $\Psi_{2,2}$ is the set

$$\{(1,0,x_3,1,0)|x_3 \in \mathbb{R}\} \cup \{(-1,0,x_3,-1,0)|x_3 \in \mathbb{R}\},$$

which can be directly seen to be the union of two one-dimensional disjoint, closed, imbedded submanifolds of $\mathcal{R}_2^2$. On the other hand, if we insist on applying Proposition 4.8, we can first parameterize a neighborhood of (say) the first set in the union above by

$$\lambda : (-\epsilon, \epsilon) \times \mathbb{R},\ \lambda(\theta, x_3) = (\cos\theta, \sin\theta, x_3).$$

The rank of $\Psi_{2,2}$ on $\lambda((-\epsilon,\epsilon)\times\mathbb{R})$ is then the same as the rank of $\tilde{\Psi}_{2,2} = \Psi_{2,2} \circ \lambda$ on $((-\epsilon,\epsilon)\times\mathbb{R})$. As one can check by inspection, for any $x_3 \in \mathbb{R}$, the rank of $\tilde{\Psi}_{2,2}$ at $(0, x_3)$ is constant, indeed maximal, and equal to one. Moreover, for any point $(\theta, x_3), \theta \neq 0$, where the rank of $\tilde{\Psi}_{2,2}$ might be zero, we must have that the derivative of the first and second components with respect to θ in $(*)$ above (after replacing x_1 and x_2 by $\cos\theta$ and $\sin\theta$) is zero. But then, the derivative of the quotient between the first and second component must be zero. Note that the quotient is well defined as long as $\epsilon < \pi$. As it turns out, this quotient is $\cot\theta$ and its derivative is clearly nonzero. This shows that there exists an open neighborhood of $\{(1,0,x_3,1,0)|x_3 \in \mathbb{R}\}$ on which the rank of $\Psi_{2,2}$ is constant and we can now apply Proposition 4.8. The same argument can be used for $\{(-1,0,x_3,-1,0)|x_3 \in \mathbb{R}\}$.

Thus, $\mathcal{R}_1^3 = \mathcal{R}_1^2 = \{(2,x_2,x_3,2,x_2)|x_2,x_3 \in \mathbb{R}\}$ is a two-dimensional submanifold while $\mathcal{R}_2^3 = \{(1,0,x_3,1,0)|x_3 \in \mathbb{R}\}$ and $\mathcal{R}_3^3 = \{(-1,0,x_3,-1,0)|x_3 \in \mathbb{R}\}$ are one-dimensional submanifolds.

As one can easily check,

$$\left\{ x \in \mathcal{R}_2^3 \middle| \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ f_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix} + \mathrm{im} \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ b_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix} + \mathrm{im} \begin{bmatrix} (x_1-2)(x_1^2+x_2^2-1) \\ -(x_1-2) \\ g_3(x_1,x_2,x_3) \\ 0 \\ 0 \end{bmatrix} \subseteq \right.$$

$$\left. \subseteq T_x\mathcal{R}_2^3 + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \emptyset \quad \text{and}$$

$$\left\{ x \in \mathcal{R}_3^3 \middle| \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ f_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix} + \mathrm{im} \begin{bmatrix} (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \\ b_3(x_1,x_2,x_3) \\ (x_1-2)x_1x_2 \\ -(x_1-2)x_1^2 \end{bmatrix} + \mathrm{im} \begin{bmatrix} (x_1-2)(x_1^2+x_2^2-1) \\ -(x_1-2) \\ g_3(x_1,x_2,x_3) \\ 0 \\ 0 \end{bmatrix} \subseteq \right.$$

$$\left. \subseteq T_x\mathcal{R}_3^3 + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \emptyset,$$

thus resulting in $\tilde{\mathcal{R}}^4 = \{(2, x_2, x_3, 2, x_2) | x_2, x_3 \in \mathbb{R}\}$ and the termination of the algorithm.

It should be noted that while the computations for $\tilde{\mathcal{R}}^3$ have been achieved by using the function $\Psi_{2,2}$ and Proposition 4.8, the same answer for $\tilde{\mathcal{R}}^3$ can be obtained by inspection, i.e., by applying the algorithm directly.

Example 5. Regardless of whether or not the rank conditions in Proposition 4.8 or the regular pre-simulation condition in Theorem 4.6 are satisfied, one may still have that the conclusions of Proposition 4.6 are satisfied. More precisely, when applying the original algorithm, one only chooses a submanifold $\mathcal{R}^{k+1}$ of $\mathcal{R}^k$ contained in $\tilde{\mathcal{R}}^{k+1}$ and continues the algorithm. But if $\tilde{\mathcal{R}}^{k+1}$ is a union of (say) two connected, disjoint submanifolds of different dimensions, then $\mathcal{R}^{k+1}$ can be chosen in two "maximal"ways, thus getting a "branching"of the algorithm. Hence, one may still obtain an "overall"termination set which, only for the purpose of this example, will also be denoted by $\mathcal{R}^*$. (We recall that the definition of $\mathcal{R}^*$ has been introduced in Proposition 4.6 and assumes the regular pre-simulation conditions in that proposition). As noted in Observation 4.7, this set may not be a submanifold. In this example, we show that $\mathcal{R}^*$ may consist of component submanifolds with different dimensions.

Consider the nonlinear systems:

$$1)\ \begin{cases} \dot{x}_1 = f_1(x_1,x_2) + b_1(x_1,x_2)u + g_1(x_1,x_2,x_3)d \\ \dot{x}_2 = f_2(x_1,x_2,x_3) + b_2(x_1,x_2,x_3)u + g_2(x_1,x_2,x_3)d \\ \dot{x}_3 = f_3(x_1,x_2,x_3) + b_3(x_1,x_2,x_3)u + g_3(x_1,x_2,x_3)d \\ h(x_1,x_2,x_3) = (x_1,x_2) \end{cases} \quad \text{and}$$

$$(2)\ \begin{cases} \dot{z}_1 = f_1(z_1,z_2) + b_1(z_1,z_2)u \\ \dot{z}_2 = \bar{f}_2(z_1,z_2) + \bar{b}_2(z_1,z_2)u + \bar{g}_2(z_1,z_2)\bar{d} \\ \bar{h}(z_1,z_2) = (z_1,z_2). \end{cases}$$

such that

i) $b_2(x_1,x_2,x_3) = \psi_1(x_1)k_1(x_1,x_2,x_3) + \bar{b}_2(x_1,x_2)$, and $\psi_1(\pm 1) = 0$ (so $b_2(\pm 1,x_2,x_3) = \bar{b}_2(\pm 1,x_2)$),

ii) $f_2(x_1,x_2,x_3) = \psi_2(x_1)k_2(x_1,x_2,x_3) + \bar{f}_2(x_1,x_2)$ and $\psi_2(\pm 1) = 0$ (so $f_2(\pm 1,x_2,x_3) = \bar{f}_2(\pm 1,x_2)$)

iii) $g_1(x_1,x_2,x_3) = 0$ iff $x_1 = \pm 1$,

iv) $g_2(x_1,x_2,x_3) = 0$ iff $x_1 = 1$ or $x_2 = 0$,

v) $\bar{g}_2(\pm 1,x_2) = 0$,

vi) $f_1(1,x_2) = b_1(1,x_2) = 0$,

vii) $f_1(-1,0) = b_1(-1,0) = 0$, and

viii) $\bar{f}_2(-1,0) = \bar{b}_2(-1,0) = 0$.

(In particular, we can make the following selection:

$f_1(x_1,x_2)=b_1(x_1,x_2)=(x_1-1)x_2, g_1(x_1,x_2,x_3)=x_1^2-1,$

$f_2(x_1,x_2,x_3)=b_2(x_1,x_2,x_3)=(x_1+1)x_2, g_2(x_1,x_2,x_3)=(x_1-1)x_2,$

f_3, b_3, g_3 arbitrary,

$\bar{f}_2(x_1,x_2)=\bar{b}_2(x_1,x_2)=(x_1+1)x_2, \bar{g}_2(x_1,x_2)=x_1^2-1.)$

This way, we have $\mathcal{R}^0=\mathbb{R}^3\times\mathbb{R}^2=\mathbb{R}^5$,

$$\tilde{\mathcal{R}}^1=\{(x_1,x_2,x_3,z_1,z_2)|h(x_1,x_2,x_3)=\bar{h}(z_1,z_2)\}$$

$$=\{(x_1,x_2,x_3,z_1,z_2)|(x_1,x_2)=(z_1,z_2)\}=\{(x_1,x_2,x_3,x_1,x_2)|(x_1,x_2,x_3)\in\mathbb{R}^3\}.$$

Since $\tilde{\mathcal{R}}^1$ is a submanifold of $\mathcal{R}^0$, we can choose $\mathcal{R}^1=\tilde{\mathcal{R}}^1$ in step 2 of the algorithm and continue to the next step. In doing so, let us first observe that for any $(x,z)\in\mathcal{R}^1$, we have $T_{(x,z)}\mathcal{R}^1\cong\mathcal{R}^1$.

Let us compute $\tilde{\mathcal{R}}^2$ next.

$$\tilde{\mathcal{R}}^2=\left\{(x_1,x_2,x_3,x_1,x_2)\in\mathcal{R}^1|\begin{bmatrix}f_1(x_1,x_2)\\f_2(x_1,x_2,x_3)\\f_3(x_1,x_2,x_3)\\\bar{f}_1(x_1,x_2)\\\bar{f}_2(x_1,x_2)\end{bmatrix}+\mathrm{im}\begin{bmatrix}b_1(x_1,x_2)\\b_2(x_1,x_2,x_3)\\b_3(x_1,x_2,x_3)\\\bar{b}_1(x_1,x_2)\\\bar{b}_2(x_1,x_2)\end{bmatrix}+\right.$$
$$\left.+\mathrm{im}\begin{bmatrix}g_1(x_1,x_2,x_3)\\g_2(x_1,x_2,x_3)\\g_3(x_1,x_2,x_3)\\0\\0\end{bmatrix}\subset T_{(x,z)}\mathcal{R}^1+\mathrm{im}\begin{bmatrix}0\\0\\0\\0\\\bar{g}_2(x_1,x_2)\end{bmatrix}\right\}=$$
$$=\Bigg\{(x_1,x_2,x_3,x_1,x_2)\in\mathbb{R}^5|(\forall)\lambda,\mu\in\mathbb{R},(\exists)\lambda_i\in\mathbb{R},1\le i\le 4,$$
$$\left.\begin{bmatrix}f_1(x_1,x_2)+\lambda b_1(x_1,x_2)+\mu g_1(x_1,x_2,x_3)\\f_2(x_1,x_2,x_3)+\lambda b_2(x_1,x_2,x_3)+\mu g_2(x_1,x_2,x_3)\\f_3(x_1,x_2,x_3)+\lambda b_3(x_1,x_2,x_3)+\mu g_3(x_1,x_2,x_3)\\\bar{f}_1(x_1,x_2)+\lambda\bar{b}_1(x_1,x_2)\\\bar{f}_2(x_1,x_2)+\lambda\bar{b}_2(x_1,x_2)\end{bmatrix}=\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_1\\\lambda_2+\lambda_4\bar{g}_2(x_1,x_2)\end{bmatrix}\right\}$$

So, $(x_1,x_2,x_3,x_1,x_2)\in\tilde{\mathcal{R}}^2$ iff for all $\lambda,\mu\in\mathbb{R}$, there exist $\lambda_i\in\mathbb{R},1\le i\le 4$ such that

$f_1(x_1,x_2)+\lambda b_1(x_1,x_2)+\mu g_1(x_1,x_2,x_3)=\lambda_1$

$f_2(x_1,x_2,x_3)+\lambda b_2(x_1,x_2,x_3)+\mu g_2(x_1,x_2,x_3)=\lambda_2$

$f_3(x_1,x_2,x_3)+\lambda b_3(x_1,x_2,x_3+\mu g_3(x_1,x_2,x_3))=\lambda_3$

$\bar{f}_1(x_1,x_2)+\lambda\bar{b}_1(x_1,x_2)=\lambda_1$

$\bar{f}_2(x_1,x_2) + \lambda\bar{b}_2(x_1,x_2) = \lambda_2 + \lambda_4\bar{g}_2(x_1,x_2)$

Comparing the first and fourth equations and keeping in mind that μ is arbitrary, we obtain $g_1(x_1,x_2,x_3) = 0$, which is equivalent to $x_1 = \pm 1$, by iii). Since λ_1 and λ_3 can be easily found, we need to investigate if and when λ_2 and λ_4 can be found in terms of λ and μ.

If $x_1 = 1$, then, since $g_2(1,x_2,x_3) = 0$ by condition (iv), the second equation becomes $\lambda_2 = f_2(1,x_2,x_3) + \lambda b_2(1,x_2,x_3)$. Since $b_2(1,x_2,x_3) = \bar{b}_2(1,x_2)$ and $f_2(1,x_2,x_3) = \bar{f}_2(1,x_2)$ by conditions (i) and (ii), and since $\bar{g}_2(1,x_2,x_3) = 0$ by condition (v), the second and fifth equations both reduce to

$$\lambda_2 = f_2(1,x_2,x_3) + \lambda b_2(1,x_2,x_3) = \bar{f}_2(1,x_2) + \lambda\bar{b}_2(1,x_2),$$

which provide a consistent definition for λ_2. Clearly, λ_4 can be chosen in an arbitrary manner since $\bar{g}_2(1,x_2) = 0$.

On the other hand, if $x_1 = -1$, then, replacing λ_2 from the second equation in the last equation, we have

$$\bar{f}_2(-1,x_2) + \lambda\bar{b}_2(-1,x_2) = f_2(-1,x_2,x_3) + \lambda b_2(-1,x_2,x_3) + \\ + \mu g_2(-1,x_2,x_3) + \lambda_4\bar{g}_2(-1,x_2).$$

Again, by i) and ii), the first two terms on either side of the equation are equal. So, since $\bar{g}_2(-1,x_2) = 0$ by (v), the relation above becomes $0 = \mu g_2(-1,x_2,x_3)$. The only way this can be true for all μ is if $x_2 = 0$. So, if $x_2 = -1$, then λ_2 and λ_4 can be found (with λ_4 arbitrary) iff $x_2 = 0$.

To summarize, $\tilde{\mathcal{R}}^2 = \{(1,x_2,x_3,1,x_2) \in \mathbb{R}^5 | x_2,x_3 \in \mathbb{R}\} \cup \{(-1,0,x_3,-1,0) \in \mathbb{R}^5 | x_3 \in \mathbb{R}\}$. Thus, $\mathcal{R}^2$ is a union of disjoint, closed, connected submanifolds with different dimensions.

With $\mathcal{R}_1^2 := \{(1, x_2, x_3, 1, x_2) \in \mathbb{R}^5 | x_2, x_3 \in \mathbb{R}\}$, we want to use the algorithm to compute

$$\tilde{\mathcal{R}}_1^3 = \left\{ (1,x_2,x_3,1,x_2) \middle| \begin{bmatrix} f_1(1,x_2) \\ f_2(1,x_2,x_3) \\ f_3(1,x_2,x_3) \\ f_1(1,x_2) \\ \bar{f}_2(1,x_2) \end{bmatrix} + \mathrm{im} \begin{bmatrix} b_1(1,x_2) \\ b_2(1,x_2,x_3) \\ b_3(1,x_2,x_3) \\ b_1(1,x_2) \\ \bar{b}_2(1,x_2) \end{bmatrix} + \right.$$

$$\left. \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ g_3(1,x_2,x_3) \\ 0 \\ 0 \end{bmatrix} \subset T_{(1,x_2,x_3,1,x_2)} \mathcal{R}_1^2 \right\}.$$

(It should be noted that the term $\mathrm{im}[0, \bar{g}(\bar{x})]^T$ does not appear on the right side of the inclusion above since it is zero). So, since $T_{1,x_2,x_3,1,x_2} \mathcal{R}_1^2$ is spanned by $[0, 1, 0, 0, 1]^T$ and $[0, 0, 1, 0, 0]^T$, $(1, x_2, x_3, 1, x_2) \in \tilde{\mathcal{R}}_1^3$ iff for any $\lambda, \mu \in \mathbb{R}$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$f_1(1, x_2) + \lambda b_1(1, x_2) = 0$$

$$f_2(1, x_2, x_3) + \lambda b_2(1, x_2, x_3) = \lambda_1$$

$$f_3(1, x_2, x_3) + \lambda b_3(1, x_2, x_3) + \mu g_3(1, x_2, x_3) = \lambda_2$$

$$f_1(1, x_2) + \lambda b_1(1, x_2) = 0$$

$$\bar{f}_2(1, x_2) + \lambda \bar{b}_2(1, x_2) = \lambda_1$$

By i) and ii), the second relation coincides with the last. Moreover, we can find λ_1 and λ_2 such that all five relations are satisfied iff $f_1(1, x_2) = b_1(1, x_2) = 0$, which holds by vi). Thus, $\tilde{\mathcal{R}}_1^3 = \mathcal{R}_1^2$ and the algorithm terminates.

With $\mathcal{R}_2^2 := \{(-1, 0, x_3, -1, 0) \in \mathbb{R}^5 | x_3 \in \mathbb{R}\}$, in order to compute $\tilde{\mathcal{R}}_2^3$, let us first note that with $x_1 = -1$ and $x_2 = 0$, we have

$$f_2(-1, 0, x_3) = \bar{f}_2(-1, 0) = b_2(-1, 0, x_3) = \bar{b}_2(-1, 0) = 0$$

by the assumed properties of the functions above. Thus $\tilde{\mathcal{R}}_2^3$ is defined as follows:

$$\tilde{\mathcal{R}}_2^3 = \left\{ (-1,0,x_3,-1,0) \middle| \begin{bmatrix} 0 \\ 0 \\ f_3(-1,0,x_3) \\ 0 \\ 0 \end{bmatrix} + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ b_3(-1,0,x_3) \\ 0 \\ 0 \end{bmatrix} + \right.$$

$$\left. \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ g_3(-1,0,x_3) \\ 0 \\ 0 \end{bmatrix} \subset T_{(-1,0,x_3,-1,0)} \mathcal{R}_2^2 \right\},$$

and this is satisfied for every $x_3 \in \mathbb{R}$ (that is, for every $(-1,0,x_3,-1,0) \in \mathcal{R}_2^2$) because the tangent space to $\mathcal{R}_2^2$ at $(-1,0,x_3,-1,0)$ is spanned by $[0,0,1,0,0]^T$ and the inclusion above holds if for every $\lambda, \mu \in \mathbb{R}$ there exists $\lambda_1 \in \mathbb{R}$ such that

$$f_3(-1,0,x_3) + \lambda b_3(-1,0,x_3) + \mu g_3(-1,0,x_3) = \lambda_1,$$

which, of course, holds trivially since we can just define λ_1 by the formula above. Thus

$$\mathcal{R}^* = \{(1,x_2,x_3,1,x_2) \in \mathbb{R}^5 | x_2,x_3 \in \mathbb{R}\} \cup \{(-1,0,x_3,-1,0) \in \mathbb{R}^5 | x_3 \in \mathbb{R}\}$$

is the union between a two-dimensional and a one-dimensional submanifold. It should also be noted that $\bar{F}$ has disturbance constant rank on each of the two components since the disturbance vector field $[0,0,0,0,\bar{g}_2(x_1,x_2)]^T$ is zero in either of the cases $x_1 = 1$ or $x_1 = -1$, thus making $\mathcal{R}^*$ a pointwise as well as an admissible simulation relation.

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