UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

GEODESIC FIBRATIONS OF
ELLIPTIC 3-MANIFOLDS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma
2013

GEODESIC FIBRATIONS OF
ELLIPTIC 3-MANIFOLDS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Acknowledgements

Let this work be to the glory of God. Thank you LORD for creating such a beautifully ordered universe and allowing me the opportunity to study it.

Thanks to my advisor, Dr. Andy Miller, for your seemingly endless patience and time. It was a pleasure to work with you on this project. I will miss our weekly meetings, which have been a highlight of my week for the past three years. You have taught me so much and have been a great source of encouragement. I cannot thank you enough.

Thank you Dr. Turner and Dr. Wantz. Without your advice, prompting, and encouragement I would not have made it this far. Thank you to my committee members for your willingness to serve. Thanks to my classmates, especially Matt and Rachel, for being in the thick of it with me.

Thanks to my family and friends for encouraging me on this journey. Thanks Mom and Dad for supporting me and being proud of me. Thanks Ashley, Blake, Macie, Brax, Granny, Debbie, and Carlous for adding fun and love to my life, and for being interested and excited about my progress. Thanks to my best friends Bekah and Rebekah and to my church family, including Craig, Anita, Rhonda, Dave, Ben, Danielle, Nate, Amber, Catie, Arielle, and Jennie for loving me just the way that I am, math nerd and all.

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Abstract

The well-known Hopf fibration of S^3 is interesting in part because its fibers are geodesics, or great circles, of S^3. However, this is not the only great circle fibration of S^3. In 1983, Herman Gluck and Frank Warner used the fact that the space of all oriented geodesics of the 3-sphere is homeomorphic to $S^2 \times S^2$ to establish that there are many other great circle fibrations of S^3. They showed that a submanifold of $S^2 \times S^2$ corresponds to a fibration of S^3 by oriented great circles if and only if it is the graph of a distance decreasing map from either S^2 factor to the other. Since S^3 is the universal cover of all elliptic 3-manifolds, we use this result to investigate geodesic Seifert fibrations of elliptic 3-manifolds. We also develop a different perspective on the space of oriented geodesics in S^3 than that used by Gluck and Warner, and we examine its role in studying the geometry of the 3-sphere.

Chapter 1

Introduction

In 1982, William Thurston classified the geometries of all 3-manifolds, as explained in [11]. He concluded that the geometry of any geometric 3-manifold can be modeled on either E^3, H^3, S^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL_2\mathbb{R}}$, Nil, or Sol. For six of these geometries (all but H^3 and Sol), all the closed manifolds are Seifert fiber spaces. We will specifically look at Seifert fibrations in which the fibers are geodesics.

We only consider elliptic (or spherical) 3-manifolds in this paper, which are those that have the geometry of S^3, the 3-sphere. In 1925, Hopf classified all elliptic 3-manifolds in [6]. Through the classification he identified four distinct types of elliptic 3-manifolds: cyclic type (also known as lens spaces), product type, tetrahedral type, and dihedral type. By looking at geodesic fibrations of the 3-sphere, we can draw conclusions about the geodesic Seifert fibrations of all types of elliptic 3-manifolds.

In order to better understand the geodesic fibrations of the 3-sphere, we first consider properties of great circles, which are the geodesics of the 3-sphere. The 3-sphere can be thought of as the set of unit quaternions. We use this perspective and make use of properties of quaternions in our analysis. The space of all oriented great circles is homeomorphic to $S^2 \times S^2$. So each oriented great circle of S^3 corresponds to a point of $S^2 \times S^2$. This structure allows us to examine various properties of the oriented great circles of S^3. We consider the distance between two great circles, Clifford parallelism, the intersection of great circles, and the action of isometries of S^3 on the space of oriented great circles.

The Hopf fibration is a well-known fibration of the 3-sphere by great circles. We say that two fibrations are equivalent if there is an isometry of S^3 that carries the fibers of one to the fibers of the other. So there are many fibrations of S^3 that are equivalent to the Hopf fibration, which we also call Hopf fibrations. These are not the only great circle fibrations of S^3 however. In 1983, Gluck and Warner classified all oriented great circle fibrations of S^3 in [4]. They showed that the set of oriented great circle fibers of a fibration corresponds to a subset of $S^2 \times S^2$ that is the graph of a distance decreasing function from either S^2 factor to the other. That is, for $f : S^2 \to S^2$ distance decreasing, the set $\{(x, f(x)) \mid x \in S^2\} \subset S^2 \times S^2$ corresponds to the fibers of a great circle fibration, as does the set $\{(f(x), x) \mid x \in S^2\} \subset S^2 \times S^2$. For example, a Hopf fibration corresponds to the graph of a constant function.

We use the result by Gluck and Warner to examine geodesic Seifert fibrations of the other elliptic 3-manifolds. Elliptic 3-manifolds are of the form $M = S^3/\Gamma$ where Γ is a subgroup of isometries of S^3 that act freely. We show that there is a one-to-one correspondence between oriented great circle fibrations of S^3 in which the fibers are preserved by Γ and oriented geodesic Seifert fibrations of M. The Hopf classification theorem includes the fact that all elliptic 3-manifolds admit a unitary structure. This implies that every elliptic 3-manifold inherits the Hopf fibration from S^3 and the fibers of this fibration are geodesics. We make use of the result by Gluck and Warner to classify all geodesic Seifert fibrations of any elliptic 3-manifold (Theorem 4.9).

We begin our exploration of different types of elliptic 3-manifolds by considering the cyclic type elliptic 3-manifolds, or lens spaces. These are denoted $L(m, n)$ for m, n relatively prime, and we note that $L(m, n)$ is isometric to $L(m', n')$ if and only if $m = m'$ and $n' = \pm n^{\pm 1} \mod m$. We conclude in Theorem 5.5 that the lens space $L(m, n)$ admits a geodesic Seifert fibration that is not a Hopf fibration if and only if one of $\gcd(m, n+1)$ and $\gcd(m, n-1)$ divides the other. In this case the lens space admits uncountably many non-equivalent geodesic Seifert fibrations that are not Hopf fibrations.

For example, consider the lens spaces of the form $L(12, n)$. There are only two distinct lens spaces of this form, $L(12, 1) \cong L(12, 11)$ and $L(12, 5) \cong L(12, 7)$. The first admits non-Hopf geodesic Seifert fibrations and the second only admits Hopf fibrations.

An amphichiral lens space is one that admits an orientation reversing isometry. All other lens spaces are called chiral. A lens space is amphichiral if and only if $n^2 = -1 \mod m$. In our previous example of lens spaces of the form $L(12, n)$, neither of the lens spaces are amphichiral. We conclude in Theorem 6.2 that chiral lens spaces admit exactly two non-equivalent Hopf fibrations, and amphichiral lens spaces and non-cyclic elliptic 3-manifolds admit a unique Hopf fibration up to equivalence.

A symplectic elliptic 3-manifold is of the form $M = S^3/\Gamma$, where Γ preserves the standard Hermitian form on $\mathbb{H}$. In Theorems 6.3 and 6.4, we conclude that a non-symplectic product type elliptic 3-manifold admits a unique geodesic Seifert fibration up to equivalence, which is a Hopf fibration, whereas a symplectic product type elliptic 3-manifold admits uncountably many geodesic Seifert fibrations. We then classify these geodesic Seifert fibrations using the Gluck and Warner result.

For the tetrahedral and dihedral type elliptic 3-manifolds, we show that there are only a few cases that admit non-Hopf geodesic Seifert fibrations. We conclude in Theorem 6.7 that all but one tetrahedral type elliptic 3-manifold admits a unique geodesic Seifert fibration up to equivalence, which is a Hopf fibration. The remaining tetrahedral type manifold in fact has uncountably many non-equivalent geodesic Seifert fibrations which are not Hopf fibrations (Theorem 6.8). Similarly for the dihedral type elliptic 3-manifolds there is a class that admits uncountably many non-equivalent geodesic Seifert fibrations which are not Hopf fibrations (Theorem 6.12), but the majority of dihedral type elliptic 3-manifolds admit a unique geodesic Seifert fibration up to equivalence, which is a Hopf fibration (Theorem 6.11).

The following is an outline of the remaining paper. In Chapter 2 we lay out basic properties of the 3-sphere and its geodesics. We introduce and explore basic properties of quaternions, which we make extensive use of in all of our analysis. We consider the space of all geodesics of the 3-sphere, which is called the Grassmannian manifold and is homeomorphic to $S^2 \times S^2$. The isometries of both the 2-sphere and the 3-sphere are explored, along with how the isometries of S^3 act on the space of great circles $S^2 \times S^2$. We use this structure to analyze properties of great circles of S^3. We look at how we can tell the distance between two great circles in S^3 by looking at their corresponding points in $S^2 \times S^2$, and also how we can tell if two great circles are Clifford parallel. There is a discussion of the stereographic projection as a means of picturing the 3-sphere. We conclude the chapter by exploring great spheres contained in the 3-sphere and their relationship to a specific decomposition of the 3-sphere into the great circles corresponding to $C \times C \subset S^2 \times S^2$ where C is a great circle of S^2.

Chapter 3 is concerned with great circle fibrations of the 3-sphere. We specifically consider the well-known Hopf fibration. The great circle fibers of the Hopf fibration correspond to $S^2 \times \{point\}$ or $\{point\} \times S^2$. We then introduce a result by Gluck and Warner from [4] that characterizes all great circle fibrations of the 3-sphere. The fibers of any great circle fibration correspond to the graph of a distance decreasing function from one S^2 factor to the other.

In Chapter 4 we lay out basic definitions and properties of elliptic 3-manifolds, including unitary and symplectic structures of elliptic 3-manifolds. We also define a Seifert fibration of a 3-manifold and make use of the Gluck and Warner result to classify all geodesic Seifert fibrations of the elliptic 3-manifolds. In Chapters 5 and 6 we consider each of the four types of elliptic 3-manifolds separately, drawing conclusions about the geodesic Seifert fibrations of each.

Chapter 2

Geodesics of S^3

The three-sphere, denoted by S^3, can be thought of as a subset of $\mathbb{R}^4$, $\mathbb{C}^2$, or $\mathbb{H}$. In $\mathbb{R}^4$, we think of S^3 as the unit sphere, i.e., the set of all points a distance 1 from the origin, $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$. If we let $u = x_1 + x_2 i \in \mathbb{C}$ and $v = x_3 + x_4 i \in \mathbb{C}$, then we can see S^3 as the set $\{(u, v) \in \mathbb{C} \times \mathbb{C} \mid |u|^2 + |v|^2 = 1\}$. This can be extended further to view S^3 as the group of unit quaternions, $\{q \in \mathbb{H} \mid |q| = 1\}$, by letting $q = u + vj$ for $u, v \in \mathbb{C}$. We will primarily think of the three-sphere as the set of unit quaternions.

2.1 Properties of unit quaternions

The set of quaternions, denoted by $\mathbb{H}$, is defined as $\mathbb{H} = \{x_1 + x_2 i + x_3 j + x_4 k \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$. Quaternion multiplication is then defined so that $i^2 = j^2 = k^2 = ijk = -1$. Note that quaternion multiplication is not commutative. The quaternion conjugate of $q = x_1 + x_2 i + x_3 j + x_4 k$ is $\overline{q} = x_1 - x_2 i - x_3 j - x_4 k$. Then the norm of a quaternion is $|q| = \sqrt{q\overline{q}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$, which is the Euclidean norm of the vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. For a quaternion $q = x_1 + x_2 i + x_3 j + x_4 k$ we define the *real part of* q to be $Re(q) = x_1$ and the *complex part of* q to be $Cx(q) = x_1 + x_2 i$.

The set of unit quaternions forms a group under quaternion multiplication. The inverse of a unit quaternion $q = u + vj$, where $u, v \in \mathbb{C}$, is given by $q^{-1} = \overline{q} = \overline{u} - vj$. Using this structure S^3 forms a compact Lie group with center $\{\pm 1\}$ and maximal torus $S^1 = \{e^{i\theta} \in \mathbb{C} \mid 0 \leq \theta < 2\pi\}$.

The notion of a Euclidean dot product extends to the group of unit quaternions in the following way. Let q, q' be unit quaternions such that $q = x_1 + x_2 i + x_3 j + x_4 k$ and $q' = x_1' + x_2' i + x_3' j + x_4' k$. The Euclidean dot product gives that $(x_1, x_2, x_3, x_4) \cdot (x_1', x_2', x_3', x_4') = x_1 x_1' + x_2 x_2' + x_3 x_3' + x_4 x_4'$. So we define the dot product on the set of unit quaternions by $q \cdot q' = Re(qq'^{-1}) = x_1 x_1' + x_2 x_2' + x_3 x_3' + x_4 x_4'$, which is consistent with the Euclidean dot product.

Lemma 2.1. *Two unit quaternions q_1 and q_2 are conjugate if and only if $Re(q_1) = Re(q_2)$.*

Proof. ($\Longrightarrow$) Suppose that $q_1 = pq_2p^{-1}$ for some unit quaternion $p = w + zj$

where $q_2 = u + vj$ for $u, v, w, z \in \mathbb{C}$. Then

$$\begin{aligned}
q_1 &= (w + zj)(u + vj)(\overline{w} - zj) \\
&= \Big(wu - z\overline{v} + (wv + z\overline{u})j\Big)(\overline{w} - zj) \\
&= \overline{w}(wu - z\overline{v}) + \overline{z}(wv + z\overline{u}) + \Big(z(z\overline{v} - wu) + w(wv + z\overline{u})\Big)j \\
&= |w|^2 u - z\overline{wv} + \overline{z}wv + |z|^2\overline{u} + \Big(z(z\overline{v} - wu) + w(wv + z\overline{u})\Big)j \\
&= (|w|^2 + |z|^2)Re(u) + (|w|^2 - |z|^2)Im(u)i + 2iIm(\overline{z}wv) + \\
&\qquad \Big(z(z\overline{v} - wu) + w(wv + z\overline{u})\Big)j \\
&= Re(u) + \Big((|w|^2 - |z|^2)Im(u) + 2Im(\overline{z}wv)\Big)i + \\
&\qquad \Big(z(z\overline{v} - wu) + w(wv + z\overline{u})\Big)j.
\end{aligned}$$

Thus $Re(q_1) = Re(q_2)$.

($\Longleftarrow$) Suppose that $Re(q_1) = Re(q_2)$. Every element of a Lie group is conjugate to an element of a maximal torus. Thus every element of S^3 is conjugate to an element of S^1. So $q_1 = re^{i\theta}r^{-1}$ and $q_2 = se^{i\phi}s^{-1}$ for some $r, s \in S^3$ and $\theta, \phi \in [-\pi, \pi]$. Therefore $\cos\theta = Re(q_1) = Re(q_2) = \cos\phi$. So either $\theta = \phi$ or $\theta = -\phi$. If $\theta = \phi$, then $r^{-1}q_1 r = e^{i\theta} = e^{i\phi} = s^{-1}q_2 s$. Thus q_1 and q_2 are conjugate. If $\theta = -\phi$, then $r^{-1}q_1 r = e^{i\theta} = e^{-i\phi} = je^{i\phi}j^{-1} = js^{-1}q_2 sj^{-1}$, showing that q_1 and q_2 are conjugate. $\square$

For q a quaternion, if $Re(q) = 0$ then we say that q is a *pure quaternion*. Note that for q a pure unit quaternion, $q^2 = -1$. Also, by Lemma 2.1, every pure unit quaternion is conjugate to i. We can identify $\mathbb{R}^3$ with the set of all pure quaternions in the following way: $ai + bj + ck \leftrightarrow (a, b, c) \in \mathbb{R}^3$. The set of pure unit quaternions can be identified with S^2. From now on, when we refer to S^2 we will be thinking of the set of pure unit quaternions.

The structure of quaternion multiplication can be used to study geometric properties in $\mathbb{R}^3$. For example, given two orthogonal pure quaternions $q_1 = ai + bj + ck$ and $q_2 = xi + yj + zk$, we see that $q_1 q_2 = (bz - cy)i + (cx - az)j + (ay - bx)k = q_1 \times q_2$.

We can write a unit quaternion q as $u + vj$ for $u, v \in \mathbb{C}$, but we can also express it as $\cos\theta + w\sin\theta$ where $0 \leq \theta \leq \pi$ and w is a pure unit quaternion. Every unit quaternion can be written uniquely in this way. This structure is explained more thoroughly in [12]. In this notation, the inverse of $\cos\theta + w\sin\theta$ is $\cos\theta - w\sin\theta$.

One final lemma concerning quaternions will be used in this paper.

Lemma 2.2. *The element $q \in S^3$ commutes with i if and only if $q \in S^1$.*

Proof. ($\Longleftarrow$) Clearly if $q \in S^1$, then $qiq^{-1} = i$ so q commutes with i.

($\Longrightarrow$) Suppose that $q \in S^3$ commutes with i. Then for $q = u + vj$, we have that $(u + vj)i = i(u + vj)$. Since $(u + vj)i = ui + vji = iu - vij = iu - ivj = i(u - vj)$ we need $v = 0$. Therefore $q \in S^1$. $\square$

2.2 Great circles

The metric on S^3 is the induced Riemannian metric from the standard Euclidean metric on $\mathbb{R}^4$. Note that S^3 equipped with this metric is a complete Riemannian manifold.

A *great circle* in S^3 is the intersection of a 2-dimensional subspace of $\mathbb{R}^4$ with S^3. Assuming the local existence and uniqueness of geodesics on any complete Riemannian manifold, we have that any geodesic of S^3 is a great circle. The following argument, which can be found in [10], shows why this is the case. Let q_1 and q_2 be elements of S^3 close enough so that there exists a unique geodesic arc l from q_1 to q_2. Let π be the plane through q_1, q_2, and 0, and let Σ be any 3-space containing π. Reflection through Σ is an isometry which fixes q_1 and q_2, thus it must fix l. Since this is the case for all 3-spaces containing π, we have that l is contained in π. Therefore, the geodesics of S^3 are great circles, and the distance between any two points in S^3 is the measure of the angle between the corresponding vectors, i.e., for $q_1, q_2 \in S^3$, $d(q_1, q_2) = \text{angle between } q_1 \text{ and } q_2 = \cos^{-1}(q_1 \cdot q_2) = \cos^{-1}(Re(q_1 q_2^{-1}))$.

This gives a one-to-one correspondence between oriented geodesics of S^3 and oriented 2-dimensional subspaces of $\mathbb{R}^4$, which constitute the Grassmannian manifold, denoted $\widetilde{G_2}(\mathbb{R}^4)$.

2.3 Isometries of S^3

As the previous section shows, the distance between two unit quaternions q_1 and q_2 is $d(q_1, q_2) = \cos^{-1}(q_1 \cdot q_2) = \cos^{-1}(Re(q_1 q_2^{-1}))$. Right multiplication by a unit quaternion p preserves this distance, since $d(q_1 p, q_2 p) = \cos^{-1}(Re(q_1 p p^{-1} q_2^{-1})) = d(q_1, q_2)$. Left multiplication by a unit quaternion p also preserves distance, since $d(pq_1, pq_2) = \cos^{-1}(Re(pq_1 q_2^{-1} p^{-1})) = \cos^{-1}(Re(q_1 q_2^{-1})) = d(q_1, q_2)$, by Lemma 2.1.

The following argument found in [10] will help us further understand the group of orientation preserving isometries of S^3, $Isom_+(S^3) = SO(4)$. Since left and right multiplication by a unit quaternion preserve distance, these give isometries on S^3. Also, any orientation reversing isometry of S^3 has a fixed point, but left and right multiplication by a unit quaternion $q \neq 1$

does not have a fixed point, so these isometries are orientation preserving. Consider the homomorphism

$$\rho : S^3 \times S^3 \to Isom_+(S^3)$$

defined by

$$\rho(q_1, q_2)(q) = q_1 q q_2^{-1}. \tag{2.1}$$

If (q_1, q_2) lies in the kernel of ρ, then $p = q_1 p q_2^{-1}$ for all $p \in S^3$. Specifically taking $p = 1$, we see that $q_1 = q_2$. Thus $q_1 = q_2$ must be in the center of S^3, which consists only of ± 1. So the kernel of ρ is $\{(1,1), (-1,-1)\}$. Since the kernel has dimension 0, the image of ρ is 6-dimensional. But $SO(4) = Isom_+(S^3)$ is also 6-dimensional since every matrix of $SO(4)$ is determined by the 6 elements above the main diagonal. Also $SO(4)$ is compact and connected. Thus ρ is surjective. This gives the following isomorphism

$$\tilde{\rho} : (S^3 \times S^3)/\mathbb{Z}_2 \to Isom_+(S^3).$$

Lemma 2.3. *The element $\rho(q_1, q_2) \in Isom_+(S^3)$ has a fixed point if and only if $Re(q_1) = Re(q_2)$.*

Proof. The element $\rho(q_1, q_2) \in Isom_+(S^3)$ has a fixed point p precisely if $p = q_1 p q_2^{-1}$, which is equivalent to $q_2 = p^{-1} q_1 p$. By Lemma 2.1, q_1 and q_2 are conjugate if and only if $Re(q_1) = Re(q_2)$. □

Now consider orientation reversing isometries of S^3. To begin with consider the homomorphism $\gamma : S^3 \to S^3$ defined by $\gamma(q) = q^{-1}$. This is an isometry since $d(\gamma(q_1), \gamma(q_2)) = d(q_1^{-1}, q_2^{-1}) = \cos^{-1}(Re(q_1^{-1}q_2))$ $= \cos^{-1}(Re(q_1 q_2^{-1})) = d(q_1, q_2)$. Also, this is an orientation reversing isometry since it corresponds to a matrix of $O(4)$ with determinant -1. In fact, all orientation reversing isometries of S^3 can be expressed as $\gamma \circ \rho(q_1, q_2)$ for some $q_1, q_2 \in S^3$ since $[O(4) : SO(4)] = 2$. Note that if we conjugate an orientation preserving isometry by γ, we obtain another orientation preserving isometry, and in fact $\gamma \circ \rho(q_1, q_2) \circ \gamma^{-1} = \rho(q_2, q_1)$.

2.4 The Grassmannian manifold, $\widetilde{G_2}(\mathbb{R}^4)$

The Grassmannian manifold, denoted by $\widetilde{G_2}(\mathbb{R}^4)$, consists of all oriented 2-dimensional subspaces of $\mathbb{R}^4$. We can think of an element of this space as $(v_1, v_2) \in \mathbb{R}^4 \times \mathbb{R}^4$ where (v_1, v_2) is the ordered orthonormal basis of the 2-dimensional subspace. We can then consider the quotient space obtained by identifying basis elements that yield the same ordered 2-dimensional subspace. By applying the quotient topology, we obtain a natural topology on $\widetilde{G_2}(\mathbb{R}^4)$. In [4], Gluck and Warner show using exterior products that $\widetilde{G_2}(\mathbb{R}^4)$ is homeomorphic to $S^2 \times S^2$. In this section we present this result using a different perspective.

Let $\langle u, v\rangle$ denote the oriented 2-dimensional vector subspace of $\mathbb{R}^4$ spanned by the orthonormal basis $\{u, v\} \subset S^3$, with orientation from u to v. So the angle from u to v is $\frac{\pi}{2}$ in the oriented plane $\langle u, v\rangle$, but is $-\frac{\pi}{2}$ in the oriented plane $\langle v, u\rangle$. For example, consider the complex plane $\mathbb{C} \subset \mathbb{H}$ spanned by the vectors 1 and i. Depending on orientation, this plane can be denoted by $\langle e^{i\theta}, ie^{i\theta}\rangle$ or $\langle ie^{i\theta}, e^{i\theta}\rangle$ for any $\theta \in [0, 2\pi)$. Moreover, every ordered orthonormal basis for $\mathbb{C}$ has one of these forms. We say that $\mathbb{C} = \langle u, v\rangle$ is oriented *counterclockwise* if $uv^{-1} = -i$, and it is oriented *clockwise* if $uv^{-1} = i$.

Lemma 2.4. *$\rho(q_1, q_2)\langle 1, i\rangle = \langle 1, i\rangle$ if and only if $q_1, q_2 \in S^1$*

Proof. ($\Longleftarrow$) Suppose first that $q_1, q_2 \in S^1$. Then $q_1 q_2^{-1}, q_1 i q_2^{-1} \in S^1$. So the plane spanned by $\{q_1 q_2^{-1}, q_1 i q_2^{-1}\}$ is equal to the plane spanned by $\{1, i\}$. Also, $(q_1 q_2^{-1})(q_1 i q_2^{-1})^{-1} = -i$ which means that $\langle q_1 q_2^{-1}, q_1 i q_2^{-1}\rangle$ is oriented counterclockwise, and we have $\langle q_1 q_2^{-1}, q_1 i q_2^{-1}\rangle = \langle 1, i\rangle$.

($\Longrightarrow$) Now suppose that $\rho(q_1, q_2)\langle 1, i\rangle = \langle 1, i\rangle$. Then $\langle q_1 q_2^{-1}, q_1 i q_2^{-1}\rangle = \langle 1, i\rangle$. So $q_1 q_2^{-1} \in S^1$ and $q_1 i q_2^{-1} \in S^1$. Also, $\langle q_1 q_2^{-1}, q_1 i q_2^{-1}\rangle$ is oriented counterclockwise so $(q_1 q_2^{-1})(q_1 i q_2^{-1})^{-1} = -i$, which implies that $-q_1 i q_1^{-1} = -i$. Thus q_1 commutes with i. The only elements of S^3 that commute with i are in S^1 by Lemma 2.2. Therefore $q_1 \in S^1$. Since $q_1 \in S^1$ and $q_1 q_2^{-1} \in S^1$ we have that $q_2 \in S^1$ as well. $\square$

Lemma 2.5. *The left action of $S^3 \times S^3$ on $\widetilde{G_2}(\mathbb{R}^4)$ given by $\rho(q_1, q_2)\langle q, q'\rangle = \langle q_1 q q_2^{-1}, q_1 q' q_2^{-1}\rangle$ is transitive.*

Proof. Suppose we are given an arbitrary oriented plane $\langle q_1, q_2\rangle$. It is sufficient to find $r, s \in S^3$ such that $\rho(r, s)\langle 1, i\rangle = \langle q_1, q_2\rangle$. Indeed, we will show that there exists $r, s \in S^3$ with $rs^{-1} = q_1$ and $ris^{-1} = q_2$. Since q_1 and q_2 are orthogonal, $Re(q_1^{-1} q_2) = 0$ so $q_1^{-1} q_2$ is a pure unit quaternion. By Lemma 2.1, every pure unit quaternion is conjugate to i. Thus $q_1^{-1} q_2 = sis^{-1}$ for some $s \in S^3$. Now let $r = q_1 s$. Then $rs^{-1} = q_1$ and $ris^{-1} = q_2$. $\square$

Lemma 2.6. *$\rho(q_1, q_2)\langle 1, i\rangle = \rho(q_1', q_2')\langle 1, i\rangle$ if and only if $q_1 \in q_1' S^1$ and $q_2 \in q_2' S^1$*

Proof. We have that $\rho(q_1, q_2)\langle 1, i\rangle = \rho(q_1', q_2')\langle 1, i\rangle$ if and only if $\rho(q_1'^{-1}, q_2'^{-1})\rho(q_1, q_2)\langle 1, i\rangle = \langle 1, i\rangle$. This means that $\rho(q_1'^{-1} q_1, q_2'^{-1} q_2)\langle 1, i\rangle =$

$\langle 1, i\rangle$ and $q_1'^{-1}q_1 \in S^1, q_2'^{-1}q_2 \in S^1$ by Lemma 2.4. Therefore $q_1 \in q_1'S^1, q_2 \in q_2'S^1$. $\square$

Theorem 2.7. *The map*

$$H : \widetilde{G_2}(\mathbb{R}^4) \to S^2 \times S^2$$

given by $H(\rho(q_1, q_2)\langle 1, i\rangle) = (q_1 i q_1^{-1}, q_2 i q_2^{-1})$ *is a bijection.*

Proof. By Lemma 2.5, this map is defined on all elements of $\widetilde{G_2}(\mathbb{R}^4)$. To show that this map is well-defined, suppose that $\rho(q_1, q_2)\langle 1, i\rangle = \rho(q_1', q_2')\langle 1, i\rangle$. By Lemma 2.6, $q_1 = q_1'e^{i\theta}$ and $q_2 = q_2'e^{i\phi}$ for some θ and ϕ. So $q_1 i q_1^{-1} = q_1'e^{i\theta}i(q_1'e^{i\theta})^{-1} = q_1' i q_1'^{-1}$. Similarly $q_2 i q_2^{-1} = q_2' i q_2'^{-1}$.

Every pure unit quaternion is conjugate to i, so this map is surjective. To show that it is injective, suppose that $(q_1 i q_1^{-1}, q_2 i q_2^{-1}) = (q_1' i q_1'^{-1}, q_2' i q_2'^{-1})$. Since $q_1 i q_1^{-1} = q_1' i q_1'^{-1}$, we have that $q_1'^{-1}q_1$ commutes with i. The only elements of S^3 that commute with i are in S^1 by Lemma 2.2. Therefore $q_1 \in q_1'S^1$. Similarly, $q_2 \in q_2'S^1$. By Lemma 2.6, this means that $\rho(q_1, q_2)\langle 1, i\rangle = \rho(q_1', q_2')\langle 1, i\rangle$. $\square$

The map H is in fact a homeomorphism and $\widetilde{G_2}(\mathbb{R}^4) \cong S^2 \times S^2$, however we will not consider the topology of $\widetilde{G_2}(\mathbb{R}^4)$ in this paper. For the remainder of this paper we will say that the geodesic $\rho(q_1, q_2)\langle 1, i\rangle \cap S^3$ *corresponds to* the element $(q_1 i q_1^{-1}, q_2 i q_2^{-1}) \in S^2 \times S^2$. We will also at times find it convenient write to $\rho(q_1, q_2)\langle 1, i\rangle$ to denote the oriented geodesic $\rho(q_1, q_2)\langle 1, i\rangle \cap S^3$.

2.5 Isometries of S^2

Since $\widetilde{G_2}(\mathbb{R}^4) \cong S^2 \times S^2$, it will be useful to explore the isometries of S^2, $Isom(S^2) = O(3)$. First consider the orientation preserving isometries of S^2, $Isom_+(S^2) = SO(3)$. This group consists entirely of all rotations of the sphere.

Consider the homomorphism

$$\psi : S^3 \to SO(3) \tag{2.2}$$

given by $\psi(q)(r) = qrq^{-1}$ for $q \in S^3$ and $r \in S^2$. We can think of $\psi(q)$ as $\rho(q,q)|_{S^2}$. Since $\rho(q,q)$ is an isometry of S^3 which leaves S^2 invariant, it restricts to an isometry of S^2.

Proposition 2.8. *Let $q = \cos\theta + u\sin\theta$ where u is a pure unit quaternion and $0 \leq \theta \leq \pi$. Then the fixed points of $\psi(q)$ are u and $-u$, and the angle of rotation of $\psi(q)$ about u is 2θ.*

Proof. We have that

$$\begin{aligned}
quq^{-1} &= (\cos\theta + u\sin\theta)u(\cos\theta + u\sin\theta)^{-1} \\
&= (u\cos\theta + u^2\sin\theta)(\cos\theta - u\sin\theta) \\
&= u\cos^2\theta - u^2\cos\theta\sin\theta + u^2\cos\theta\sin\theta - u^3\sin^2\theta \\
&= u\cos^2\theta - u^3\sin^2\theta \\
&= u(\cos^2\theta - u^2\sin^2\theta) \\
&= u.
\end{aligned}$$

Thus u is fixed by $\psi(q)$, and $q(-u)q^{-1} = -quq^{-1} = -u$ so $-u$ is fixed by $\psi(q)$ as well.

Now note that for v a pure unit quaternion orthogonal to u, we have that $uv = u \times v$. Since $u \times v = -v \times u$, we also know that $uv = -vu$. So

$$\begin{aligned}
\psi(q)(v) &= qvq^{-1} \\
&= (\cos\theta + u\sin\theta)v(\cos\theta + u\sin\theta)^{-1} \\
&= (\cos\theta + u\sin\theta)(v\cos\theta - vu\sin\theta) \\
&= (\cos\theta + u\sin\theta)(v\cos\theta + uv\sin\theta) \\
&= (\cos\theta + u\sin\theta)(\cos\theta + u\sin\theta)v \\
&= (\cos 2\theta + u\sin 2\theta)v \\
&= v\cos 2\theta + (u \times v)\sin 2\theta.
\end{aligned}$$

The oriented plane $\langle v, u \times v\rangle$ is the 2-dimensional subspace in $\mathbb{R}^3$ orthogonal to u, which is left invariant by $\psi(q)$. So this plane is rotated by an angle of 2θ. Thus $\psi(q)$ is a rotation about u by an angle of 2θ. $\square$

Since every orientation preserving isometry of S^2 is a rotation, ψ is surjective. Also $\psi(q) = \psi(q')$ if and only if $q^{-1}q'$ is in the centralizer of S^2, which consists entirely of ± 1, so $q' = \pm q$. Therefore ψ is a two-to-one map.

Now consider the antipodal map $\phi : S^2 \to S^2$ defined by $\phi(r) = -r$. This is an orientation reversing isometry of S^2 since the corresponding matrix of $O(3)$ has determinant -1. All orientation reversing isometries of S^2 can be expressed as $\phi \circ \psi(q)$ for some $q \in S^3$ since $[O(3) : SO(3)] = 2$.

2.6 Action of $O(4)$ on $S^2 \times S^2$

Let $\rho(q_1', q_2')\langle 1, i\rangle \cap S^3$ be an arbitrary geodesic of S^3. This geodesic corresponds to the element $(q_1' i q_1'^{-1}, q_2' i q_2'^{-1}) \in S^2 \times S^2$. Consider how an isometry of S^3 acts on this geodesic. The orientation preserving isometry $\rho(q_1, q_2)$ acts on this geodesic in the following way: $\rho(q_1, q_2)\rho(q_1', q_2')\langle 1, i\rangle = \rho(q_1 q_1', q_2 q_2')\langle 1, i\rangle$, which corresponds to the element $(q_1 q_1' i q_1'^{-1} q_1^{-1}, q_2 q_2' i q_2'^{-1} q_2^{-1}) \in S^2 \times S^2$. We call the action of $\rho(q_1, q_2)$ on the space of oriented great circles $\hat{\rho}(q_1, q_2)$, and we can see that $\hat{\rho}(q_1, q_2)$ acts on $S^2 \times S^2$ by independent rotations in each of the S^2 factors. More specifically, we have the following lemma.

Lemma 2.9. *The isometry $\rho(q_1, q_2) \in Isom_+ S^3$ acts on the space of oriented great circles $S^2 \times S^2$ by $\hat{\rho}(q_1, q_2)(a, b) = (\psi(q_1)(a), \psi(q_2)(b))$.* □

Thus there are at least two fixed points in each of the S^2 factors. For $q_1 = \cos\theta_1 + u_1 \sin\theta_1 \neq \pm 1$ and $q_2 = \cos\theta_2 + u_2 \sin\theta_2 \neq \pm 1$, note that the two fixed points in the first S^2 factor are u_1 and $-u_1$, and the fixed points in the second S^2 factor are u_2 and $-u_2$, by Lemma 2.8. The angle of rotation in the first S^2 factor is $2\theta_1$, and the angle of rotation in the second S^2 factor is $2\theta_2$. Therefore the isometry $\rho(q_1, q_2)$ leaves the oriented great circles corresponding to $\pm(u_1, u_2) \in S^2 \times S^2$ and $\pm(u_1, -u_2) \in S^2 \times S^2$ fixed.

Now consider how an orientation reversing isometry $\gamma \circ \rho(q_1, q_2)$ acts on the space of geodesics $S^2 \times S^2$. This acts on the geodesic $\rho(q_1', q_2')\langle 1, i\rangle$ in the following way:

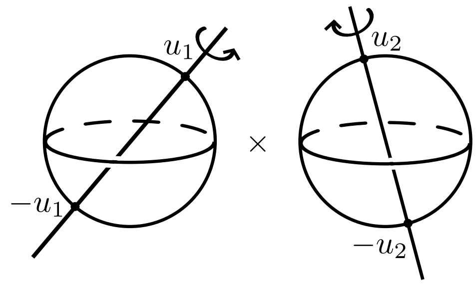

Figure 2.1: Rotations in the S^2 factors

$$\begin{aligned}\gamma\rho(q_1, q_2)\rho(q_1', q_2')\langle 1, i\rangle &= \gamma\langle q_1q_1'q_2'^{-1}q_2^{-1}, q_1q_1'iq_2'^{-1}q_2^{-1}\rangle \\ &= \langle q_2q_2'q_1'^{-1}q_1^{-1}, -q_2q_2'iq_1'^{-1}q_1^{-1}\rangle \\ &= \rho(q_2q_2'j, q_1q_1'j)\langle 1, i\rangle\end{aligned}$$

This corresponds to the element $(-q_2q_2'iq_2'^{-1}q_2^{-1}, -q_1q_1'iq_1'^{-1}q_1^{-1}) \in S^2 \times S^2$. Thus we have the following lemma.

Lemma 2.10. *The orientation reversing isometry $\gamma\rho(q_1, q_2) \in Isom_{-}S^3$ acts on the space of oriented great circles $S^2 \times S^2$ by $\widehat{\gamma\rho}(q_1, q_2)(a, b) = (-\psi(q_2)(b), -\psi(q_1)(a))$.* $\square$

We can see that an orientation reversing isometry of S^3 acts on the space of oriented geodesics $S^2 \times S^2$ by first performing an orientation reversing isometry on each of the S^2 factors and then switching the factors. Just as in the orientation preserving case, there will be some geodesics that are fixed by this action. Let $q_2q_1 = \cos\theta + r\sin\theta$. Note that r is fixed by the rotation $\psi(q_2q_1)$ by Lemma 2.8. Then the geodesics corresponding to $\pm(r, -q_1rq_1^{-1}) \in S^2 \times S^2$ are fixed by $\gamma\rho(q_1, q_2)$ since $\widehat{\gamma\rho}(q_1, q_2)(r, -q_1rq_1^{-1}) =$

$(q_2q_1rq_1^{-1}q_2^{-1}, -q_1rq_1^{-1}) = (r, -q_1rq_1^{-1})$. Also the geodesics which correspond to $\pm(r, q_1rq_1^{-1}) \in S^2 \times S^2$ will be sent to the geodesics corresponding to $\pm(-r, -q_1rq_1^{-1}) \in S^2 \times S^2$, which we will see in the next section are actually the same geodesics with opposite orientation.

2.7 Distances between geodesics in S^3

For $n \in S^3$ and C any great circle in S^3 define the distance from n to C to be $d(n, C) = \min\{d(n, m) \mid m \in C\}$. This distance is equal to the smallest angle in $\mathbb{R}^4$ that the line through n and the origin makes with the plane that determines C.

Let $P = \rho(q_1, q_2)\langle 1, i\rangle$ and $Q = \rho(q_1', q_2')\langle 1, i\rangle$ with $C = P \cap S^3$ and $C' = Q \cap S^3$. Define α_{min} to be the smallest angle that any line through the origin in P makes with Q, that is, $\alpha_{min} = \min\{d(n, C') \mid n \in C\}$. Define α_{max} to be the largest angle that any line through the origin in P makes with Q, that is, $\alpha_{max} = \max\{d(n, C') \mid n \in C\}$. Note that $\alpha_{min}, \alpha_{max} \in [0, \frac{\pi}{2}]$.

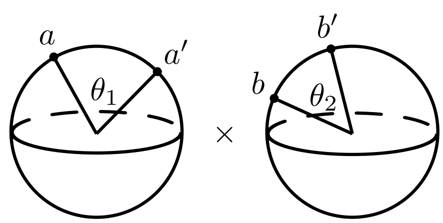

Figure 2.2: $S^2 \times S^2$ angles

Proposition 2.11. *Suppose P corresponds to $(a, b) \in S^2 \times S^2$, and Q cor-*

responds to $(a', b') \in S^2 \times S^2$. *Let* $\theta_1 = d_{S^2}(a, a')$ *and* $\theta_2 = d_{S^2}(b, b')$. *Then*

$$\alpha_{min} = \frac{1}{2}|\theta_1 - \theta_2| \tag{2.3}$$
$$\alpha_{max} = \frac{1}{2}\min\{\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)\}. \tag{2.4}$$

In the proof of Proposition 2.11 we will use the following lemma.

Lemma 2.12. *If* $f : \mathbb{C} \to \mathbb{C}$ *is defined by* $f(z) = Az + B\bar{z}$ *for some* $A, B \in \mathbb{C}$, *then* $f(S^1)$ *is a possibly degenerate ellipse* E *centered at the origin with* $major(E) = |A| + |B|$ *and* $minor(E) = \big||A| - |B|\big|$, *where* $major(E)$ *is the major radius of* E *and* $minor(E)$ *is the minor radius of* E.

Proof of Lemma 2.12. The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = Az + B\bar{z}$ for some $A, B \in \mathbb{C}$ gives a real linear transformation. Thus it maps a circle centered at the origin to an ellipse centered at the origin.

Let $e^{i\theta} \in S^1$. Then $f(e^{i\theta}) = Ae^{i\theta} + Be^{-i\theta}$. So

$$\begin{aligned}
|Ae^{i\theta} + Be^{-i\theta}|^2 &= (Ae^{i\theta} + Be^{-i\theta})(\overline{A}e^{-i\theta} + \overline{B}e^{i\theta}) \\
&= A\overline{A} + A\overline{B}e^{2i\theta} + B\overline{A}e^{-2i\theta} + B\overline{B} \\
&= |A|^2 + |B|^2 + 2Re(A\overline{B}e^{2i\theta})
\end{aligned}$$

The maximum that this value can be is $|A|^2+|B|^2+2|A||B| = (|A|+|B|)^2$, and the minimum that this value can be is $|A|^2+|B|^2-2|A||B| = (|A|-|B|)^2$. $\square$

Proof of Proposition 2.11. Let $P = \rho(q_1, q_2)\langle 1, i\rangle$ and $Q = \rho(q_1', q_2')\langle 1, i\rangle$. Then the isometry $\rho(q_1, q_2)^{-1}$ takes the plane P to $\langle 1, i\rangle$ and the plane Q to $\rho(q_1, q_2)^{-1}\rho(q_1', q_2')\langle 1, i\rangle = \rho(q_1^{-1}q_1', q_2^{-1}q_2')\langle 1, i\rangle$. This action preserves α_{min} and α_{max} since it preserves distances in S^3. It also preserves θ_1 and θ_2 since it acts on $S^2 \times S^2$ by a rotation in each of the factors. Therefore we may assume that $P = \langle 1, i \rangle$ and $Q = \rho(q_1, q_2)\langle 1, i \rangle$ with $C = P \cap S^3$, $C' = Q \cap S^3$, and $q_i = u_i + v_i j$. So the plane P corresponds to $(i, i) \in S^2 \times S^2$ and the plane Q corresponds to $(q_1 i q_1^{-1}, q_2 i q_2^{-1}) \in S^2 \times S^2$, by Theorem 2.7.

Consider the orthogonal projection $p : \mathbb{H} \to P$ given by $p(u + vj) = u$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = p \circ \rho(q_1, q_2)(z)$. Then $f(z) = p(q_1 z q_2^{-1}) = p((u_1 + v_1 j) z (\overline{u_2} - v_2 j)) = u_1 \overline{u_2} z + v_1 \overline{v_2} \bar{z}$. By Lemma 2.12, $f(S^1) = p(C')$ is an ellipse E in $\mathbb{C}$ centered at the origin with a major axis length of $major(E) = |u_1||u_2| + |v_1||v_2|$ and a minor axis length of $minor(E) = \big||u_1||u_2| - |v_1||v_2|\big|$.

Also, for $y \in C'$, we have $y \cdot p(y) = |y||p(y)| \cos\alpha = |p(y)| \cos\alpha$ for $\alpha = \angle(y, p(y)) = d(y, C)$, and $y \cdot p(y) = (y_1, y_2, y_3, y_4) \cdot (y_1, y_2, 0, 0) = y_1^2 + y_2^2 = |p(y)|^2$. Thus

$$|p(y)| = \cos\alpha \tag{2.5}$$

for $\alpha = d(y, C)$. Since $\alpha \in [0, \frac{\pi}{2}]$, we have that $major(E) = \cos\alpha_{min}$ and $minor E = \cos\alpha_{max}$ (refer to Figure 2.3).

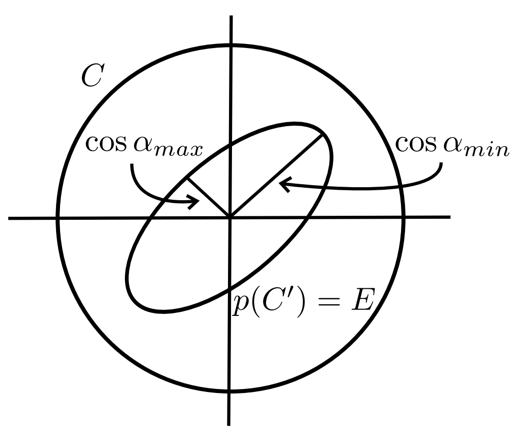

Figure 2.3: Projection to P

Combining the two expressions for the lengths of the major and minor axes, we get $\cos\alpha_{min} = |u_1||u_2| + |v_1||v_2|$ and $\cos\alpha_{max} = \big||u_1||u_2| - |v_1||v_2|\big|$.

Since $|u_1|^2+|v_1|^2 = 1$ and $|u_2|^2+|v_2|^2 = 1$, we can write $|u_1| = \cos\gamma, |v_1| = \sin\gamma$ for some $\gamma \in [0, \frac{\pi}{2}]$, and $|u_2| = \cos\beta, |v_2| = \sin\beta$ for some $\beta \in [0, \frac{\pi}{2}]$. This gives us the following

$$\cos\alpha_{min} = \cos\gamma\cos\beta + \sin\gamma\sin\beta = \cos(\gamma - \beta) \tag{2.6}$$

$$\cos\alpha_{max} = \big|\cos\gamma\cos\beta - \sin\gamma\sin\beta\big| = \big|\cos(\gamma+\beta)\big|. \tag{2.7}$$

Also, since P corresponds to $(i, i) \in S^2 \times S^2$ and Q corresponds to $(q_1 i q_1^{-1}, q_2 i q_2^{-1})$, we have that

$$\begin{aligned}
\cos\theta_1 &= q_1 i q_1^{-1} \cdot i \\
&= Re(q_1 i q_1^{-1} i^{-1}) \\
&= Re(-(u_1 + v_1 j) i (\overline{u_1} - v_1 j) i) \\
&= Re(|u_1|^2 - |v_1|^2 + 2u_1 v_1 j) \\
&= |u_1|^2 - |v_1|^2.
\end{aligned}$$

Similarly,

$$\cos\theta_2 = |u_2|^2 - |v_2|^2.$$

Since $|u_1| = \cos\gamma$ and $|v_1| = \sin\gamma$, we can see that $\cos\theta_1 = \cos^2\gamma - \sin^2\gamma$. Therefore, $2\gamma = \theta_1$ since $2\gamma, \theta_1 \in [0, \pi]$. Similarly $2\beta = \theta_2$.

From Equation 2.6, since $0 \leq \alpha_{min} \leq \frac{\pi}{2}$, we get that $\alpha_{min} = |\gamma - \beta| = \frac{1}{2}|\theta_1 - \theta_2|$.

From Equation 2.7, we get the following two cases:

Case 1: $0 \le \gamma + \beta \le \frac{\pi}{2}$. In this case, $\alpha_{max} = \gamma + \beta = \frac{1}{2}(\theta_1 + \theta_2)$.

Case 2: $\frac{\pi}{2} \le \gamma + \beta \le \pi$. In this case, $|\cos(\gamma+\beta)| = \cos(\pi - (\gamma+\beta))$. So $\alpha_{max} = \frac{1}{2}(2\pi - (\theta_1 + \theta_2))$.

Therefore, $\alpha_{max} = \frac{1}{2}\min\{\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)\}$. □

From Proposition 2.11 we see that given a geodesic $C \subset S^3$ corresponding to $(a, b) \in S^2 \times S^2$, the same geodesic with opposite orientation corresponds to $(-a, -b)$ since, in that case, $\alpha_{min} = \alpha_{max} = 0$. If $C = P \cap S^3$, then we define the *orthogonal geodesic of* C to be $C^\perp = P^\perp \cap S^3$, where $P^\perp$ is the unique 2-dimensional subspace of $\mathbb{R}^4$ that is orthogonal to P. Depending on orientation, the orthogonal geodesic corresponds to $(-a, b)$ or $(a, -b)$ since $\alpha_{min} = \alpha_{max} = \frac{\pi}{2}$. We also have the following results.

Corollary 2.13. *The geodesics C and C' of S^3 corresponding respectively to (a, b) and (a', b') in $S^2 \times S^2$ intersect if and only if $\theta_1 = \theta_2$, where $\theta_1 = d_{S^2}(a, a')$ and $\theta_2 = d_{S^2}(b, b')$. Moreover, their angle of intersection is* $\min\{\theta_1, \pi - \theta_1\}$.

Proof. C and C' intersect iff $\alpha_{min} = 0$ iff $\theta_1 = \theta_2$. The angle of intersection of C and C' is $\alpha_{max} = \frac{1}{2}\min\{\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)\}$. Then there are two cases.

Case 1: $\theta_1 = \theta_2 \le \frac{\pi}{2}$. In this case, $\alpha_{max} = \frac{1}{2}(\theta_1 + \theta_2) = \theta_1$.

Case 2: $\theta_1 = \theta_2 > \frac{\pi}{2}$. In this case, $\alpha_{max} = \frac{1}{2}(2\pi - (\theta_1 + \theta_2)) = \pi - \theta_1$. □

Two geodesics C and C' of S^3 are said to be *Clifford parallel* if $d(m, C') = d(n, C')$ for all $m, n \in C$. Clifford parallelism is explored extensively in [1].

Corollary 2.14. *The geodesics C and C' of S^3 corresponding respectively to (a, b) and (a', b') are Clifford parallel if and only if $a = \pm a'$ or $b = \pm b'$.*

Proof. The geodesics C and C' are Clifford parallel iff $\alpha_{min} = \alpha_{max}$. By Proposition 2.11 this happens precisely when $|\theta_1 - \theta_2| = \min\{\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)\}$.

There are four possibilities: (1) $\theta_1 - \theta_2 = \theta_1 + \theta_2$; (2) $\theta_1 - \theta_2 = -\theta_1 - \theta_2$; (3) $\theta_1 - \theta_2 = 2\pi - (\theta_1 + \theta_2)$; and (4) $\theta_1 - \theta_2 = -2\pi + (\theta_1 + \theta_2)$. These are respectively equivalent to (1) $\theta_2 = 0$; (2) $\theta_1 = 0$; (3) $\theta_1 = \pi$; and (4) $\theta_2 = \pi$. □

We say that the geodesics C and C' of S^3 corresponding respectively to (a, b) and (a', b') are *Clifford parallel of the first kind* if $a = \pm a'$. They are said to be *Clifford parallel of the second kind* if $b = \pm b'$.

Corollary 2.15. *The distance between Clifford parallel geodesics C and C' of the first kind is $\frac{1}{2}\theta_2$ if $\theta_1 = 0$, or $\frac{1}{2}(\pi - \theta_2)$ if $\theta_1 = \pi$. The distance between Clifford parallel geodesics C and C' of the second kind is $\frac{1}{2}\theta_1$ if $\theta_2 = 0$, or $\frac{1}{2}(\pi - \theta_1)$ if $\theta_2 = \pi$.*

Proof. If C and C' are Clifford parallel of the first kind, then $\theta_1 \in \{0, \pi\}$. If $\theta_1 = 0$, then $d(C, C') = \alpha_{min} = \alpha_{max} = \frac{1}{2}\theta_2$. If $\theta_1 = \pi$, then $d(C, C') = \alpha_{min} = \alpha_{max} = \frac{1}{2}(\pi - \theta_2)$. A similar argument holds for Clifford parallel geodesics of the second kind. □

Adopting notation from [1], for C a great circle of S^3, let $C_\alpha = \{n \in S^3 \mid d(n, C) = \alpha\}$ for $\alpha \in [0, \frac{\pi}{2}]$.

Corollary 2.16. *For $\alpha \in (0, \frac{\pi}{2})$, C_α is a torus in S^3.*

Proof. Let $C = P \cap S^3$ be a great circle in S^3. Suppose that $P = \rho(q_1, q_2)\langle 1, i\rangle$. The isometry $\rho(q_1, q_2)^{-1}$ takes the plane P to $\langle 1, i\rangle$. We may therefore assume that $P = \langle 1, i \rangle$. By Equation 2.5, we have

$$\begin{aligned}
C_\alpha &= \{y \in S^3 \mid d(y, C) = \alpha\} \\
&= \{y \in S^3 \mid \ |p(y)| = \cos\alpha\} \\
&= \{(y_1, y_2, y_3, y_4) \in S^3 \mid y_1^2 + y_2^2 = \cos^2\alpha\} \\
&= \{(y_1, y_2, y_3, y_4) \mid y_1^2 + y_2^2 = \cos^2\alpha, y_3^2 + y_4^2 = \sin^2\alpha\} \\
&= \{(y_1, y_2) \mid y_1^2 + y_2^2 = \cos^2\alpha\} \times \{(y_3, y_4) \mid y_3^2 + y_4^2 = \sin^2\alpha\} \\
&= \text{circle of radius } \cos\alpha \quad \times \quad \text{circle of radius } \sin\alpha.
\end{aligned}$$

$\square$

We call $C_{\frac{\pi}{4}}$ a *Clifford torus.* Note that $C_0 = C$ and $C_{\frac{\pi}{2}} = C^\perp$, and also if C' is Clifford parallel to C and $d(C', C) = \alpha$, then $C' \subset C_\alpha$.

Define a *cross-section of* S^2, $\kappa_{(\alpha,x)}$, to be the set of all points of S^2 a fixed distance α from a fixed point x of S^2. Consider the family of great circles of S^3 that correspond to the set $T = \{a\} \times \kappa_{(\alpha,x)} \subset S^2 \times S^2$ for fixed $a, x \in S^2$ and fixed $\alpha \in [0, \pi]$, as in Figure 2.4.

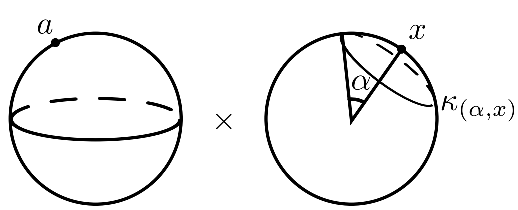

Figure 2.4: C_α in $S^2 \times S^2$

By Corollary 2.14, each great circle in T is Clifford parallel to the great circle C of S^3 that corresponds to the element (a, x) in $S^2 \times S^2$. Also, by Corollary 2.15, the distance between C and any great circle in T is $\alpha/2$.

Thus every great circle of T is contained in $C_{\alpha/2}$. By Corollary 2.13, none of the great circles of T intersect each other, and in fact, they are pairwise Clifford parallel by Corollary 2.14. As T is a circle of great circles, it is a torus. Therefore T corresponds to a great circle partition of $C_{\alpha/2}$.

Corollary 2.17. *Let $T = \{a\} \times \kappa_{(\alpha,x)} \subset S^2 \times S^2$ for fixed $a, x \in S^2$ and fixed $\alpha \in [0, \pi]$. Then T gives a great circle partition of $C_{\alpha/2}$, where C corresponds to $(a, x) \in S^2 \times S^2$.* □

2.8 The fixed geodesics of $\rho(q_1, q_2)$

Recall that $\rho(q_1, q_2)$ acts on $S^2 \times S^2$ by independent rotations in each of the S^2 factors. For $q_1 = \cos\theta + u_1 \sin\theta$ and $q_2 = \cos\theta + u_2 \sin\theta$, the fixed points in the first S^2 factor are u_1 and $-u_1$ and the fixed points in the second S^2 factor are u_2 and $-u_2$, by Lemma 2.8. The isometry $\rho(q_1, q_2)$ leaves at least one pair

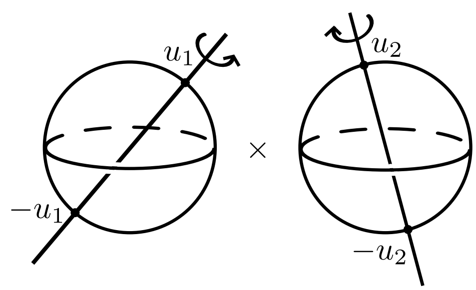

Figure 2.5: Rotations in the S^2 factors

of unoriented orthogonal geodesics fixed. One of these unoriented geodesics corresponds to the pair $\pm(u_1, u_2)$ in $S^2 \times S^2$ and the other corresponds to $\pm(u_1, -u_2)$. However these geodesics that are fixed by $\rho(q_1, q_2)$ do not necessarily remain pointwise fixed. If $q_1, q_2 \in \mathbb{C}$, then $q_1 = e^{i\theta}$ and $q_2 = e^{i\phi}$

for some $\theta, \phi \in [0, 2\pi]$. In this case, $\rho(q_1, q_2)$ fixes the planes $\mathbb{C} = \langle 1, i \rangle$ and $\mathbb{C}^\perp = \langle j, k \rangle$. Since $\rho(q_1, q_2)\langle 1, i \rangle = \langle e^{i\theta}e^{-i\phi}, e^{i\theta}ie^{-i\phi} \rangle = \langle e^{i(\theta-\phi)}, ie^{i(\theta-\phi)} \rangle$, we know that $\rho(q_1, q_2)$ rotates the plane $\mathbb{C} = \langle 1, i \rangle$ by an angle of $\theta - \phi$. Also, since $\rho(q_1, q_2)\langle j, k \rangle = \langle e^{i\theta}je^{-i\phi}, e^{i\theta}ke^{-i\phi} \rangle = \langle e^{i(\theta+\phi)}j, e^{i(\theta+\phi)}k \rangle$, we know that $\rho(q_1, q_2)$ rotates the plane $\mathbb{C}^\perp = \langle j, k \rangle$ by an angle of $\theta + \phi$.

Now suppose $q_1, q_2 \notin \mathbb{C}$ with $q_1 = \cos\theta + u_1 \sin\theta$ and $q_2 = \cos\phi + u_2 \sin\phi$ for u_1, u_2 pure unit quaternions. By Lemma 2.1, there exist unit quaternions σ_1 and σ_2 such that $q_1 = \sigma_1 e^{i\theta}\sigma_1^{-1}$ and $q_2 = \sigma_2 e^{i\phi}\sigma_2^{-1}$. Thus $\rho(q_1, q_2) = \rho(\sigma_1 e^{i\theta}\sigma_1^{-1}, \sigma_2 e^{i\phi}\sigma_2^{-1}) = \rho(\sigma_1, \sigma_2)\rho(e^{i\theta}, e^{i\phi})\rho(\sigma_1, \sigma_2)^{-1}$. Let $P := \rho(\sigma_1, \sigma_2)\langle 1, i \rangle$ and $P^\perp := \rho(\sigma_1, \sigma_2)\langle j, k \rangle$. Then P and $P^\perp$ are fixed by $\rho(q_1, q_2)$.

Let $r \in P \cap S^3$. Then $\rho(\sigma_1, \sigma_2)^{-1}(r) \in \langle 1, i \rangle$. Since $\rho(e^{i\theta}, e^{i\phi})$ rotates the plane $\langle 1, i \rangle$ by an angle of $\theta - \phi$, the angle between $\rho(\sigma_1, \sigma_2)^{-1}(r)$ and $\rho(e^{i\theta}, e^{i\phi})\rho(\sigma_1, \sigma_2)^{-1}(r)$ is $\theta - \phi$. Also, since $\rho(\sigma_1, \sigma_2)$ preserves angles, the angle between $\rho(\sigma_1, \sigma_2)\rho(\sigma_1, \sigma_2)^{-1}(r) = r$ and $\rho(\sigma_1, \sigma_2)\rho(e^{i\theta}, e^{i\phi})\rho(\sigma_1, \sigma_2)^{-1}(r) = \rho(q_1, q_2)(r)$ is $\theta - \phi$. So the angle of rotation of P is $\theta - \phi$. Similarly, the angle of rotation of $P^\perp$ is $\theta + \phi$.

Also note that $P = \rho(\sigma_1, \sigma_2)\langle 1, i \rangle$ corresponds to the element $(\sigma_1 i \sigma_1^{-1}, \sigma_2 i \sigma_2^{-1}) \in S^2 \times S^2$. Since $q_1 = \sigma_1 e^{i\theta}\sigma_1^{-1} = \cos\theta + \sigma_1 i \sigma_1^{-1} \sin\theta$, we know that $u_1 = \sigma_1 i \sigma_1^{-1}$. Therefore P corresponds to (u_1, u_2). We can similarly show that $P^\perp$ corresponds to $(u_1, -u_2)$.

Proposition 2.18. *The element $\rho(q_1, q_2) \in Isom_+(S^3)$, where $q_1 = \cos\theta + u_1 \sin\theta$ and $q_2 = \cos\phi + u_2 \sin\phi$ for u_1, u_2 pure unit quaternions, fixes at least one pair of orthogonal geodesics. More specifically, these geodesics correspond to (u_1, u_2) and $(u_1, -u_2)$ in $S^2 \times S^2$. The geodesic corresponding to (u_1, u_2)*

is rotated by an angle of $\theta - \phi$, and the geodesic corresponding to $(u_1, -u_2)$ is rotated by an angle of $\theta + \phi$. $\square$

2.9 Stereographic projection

Taking 1 as the "north pole" of S^3, we can use the stereographic projection π to map $S^3 - \{1\}$ onto the space of pure quaternions which is naturally identified with $\mathbb{R}^3$. This map, $\pi : S^3 - \{1\} \to \mathbb{R}^3$, is given by

$$\pi(x_1 + x_2 i + x_3 j + x_4 k) = \frac{x_2 i + x_3 j + x_4 k}{1 - x_1}. \tag{2.8}$$

If $p = \pi(x_1 + x_2 i + x_3 j + x_4 k) = xi + yj + zk$ then

$$\begin{aligned} |p|^2 &= \frac{x_2^2 + x_3^2 + x_4^2}{(1 - x_1)^2} \\ &= \frac{1 - x_1^2}{(1 - x_1)^2} \\ &= \frac{1 + x_1}{1 - x_1}, \end{aligned}$$

and this gives

$$x_1 = \frac{|p|^2 - 1}{|p|^2 + 1}. \tag{2.9}$$

Further computations yield

$$x_2 = \frac{2x}{|p|^2 + 1}, \quad x_3 = \frac{2y}{|p|^2 + 1}, \quad x_4 = \frac{2z}{|p|^2 + 1}. \tag{2.10}$$

Therefore the inverse of π is given by

$$\pi^{-1}(xi + yj + zk) = \frac{1}{|p|^2 + 1}\left((|p|^2 - 1) + 2xi + 2yj + 2zk\right). \tag{2.11}$$

Lemma 2.19. *The stereographic projection π maps spheres of S^3 to spheres or planes in $\mathbb{R}^3$. Moreover if S is a sphere then $\pi(S)$ is a plane if $1 \in S$ and $\pi(S)$ is a sphere if $1 \notin S$.*

Proof. A sphere S in S^3 is the intersection of a hyperplane of $\mathbb{R}^4$ with S^3. Thus for a fixed $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k \in S^3$ and $0 \leq \alpha_0 < 1$,

$$\begin{aligned} S &= \{q \in S^3 \mid q \cdot \alpha = \alpha_0\} \\ &= \{x_1 + x_2 i + x_3 j + x_4 k \in S^3 \mid \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_0\}. \end{aligned}$$

From Equations 2.9, 2.10, the points of the sphere S map to the points $p = xi + yj + zk \in \mathbb{R}^4$ satisfying the equation

$$\alpha_1(|p|^2 - 1) + 2\alpha_2 x + 2\alpha_3 y + 2\alpha_4 z = \alpha_0(|p|^2 + 1),$$

which is equivalent to

$$(\alpha_1 - \alpha_0)(x^2 + y^2 + z^2) + 2\alpha_2 x + 2\alpha_3 y + 2\alpha_4 z = \alpha_1 + \alpha_0.$$

If $1 \in S$, then $\alpha_1 = \alpha_0$ and the sphere S maps to the plane

$$2\alpha_2 x + 2\alpha_3 y + 2\alpha_4 z = \alpha_1 + \alpha_0. \tag{2.12}$$

If $1 \notin S$, then $\alpha_1 \neq \alpha_0$ and the sphere S maps to the sphere

$$\left(x + \frac{\alpha_2}{\alpha_1 - \alpha_0}\right)^2 + \left(y + \frac{\alpha_3}{\alpha_1 - \alpha_0}\right)^2 + \left(z + \frac{\alpha_4}{\alpha_1 - \alpha_0}\right)^2 = \frac{1 - \alpha_0^2}{(\alpha_1 - \alpha_0)^2}. \quad (2.13)$$

□

Lemma 2.20. *The stereographic projection π maps circles of S^3 to circles or lines in $\mathbb{R}^3$. Moreover if C is a circle then $\pi(C)$ is a line if $1 \in C$ and $\pi(C)$ is a circle if $1 \notin C$.*

Proof. A circle of S^3 is the intersection of a 2-dimensional plane in $\mathbb{R}^4$ with S^3. Any plane of $\mathbb{R}^4$ is the intersection of two 3-spaces of $\mathbb{R}^4$. Therefore any circle of S^3 is the intersection of two spheres in S^3.

Let C be a circle of S^3 that passes through the point 1. Then C is the intersection of two spheres S_1 and S_2 of S^3 that each pass through the point 1. By Lemma 2.19, $\pi(S_1)$ and $\pi(S_2)$ are distinct planes in $\mathbb{R}^3$. The planes $\pi(S_1)$ and $\pi(S_2)$ intersect in a line which is $\pi(C)$.

Let C be a circle of S^3 that does not pass through the point 1. Then C is the intersection of two spheres S_1 and S_2 of S^3, at least one of which does not pass through the point 1, say S_1. So $\pi(S_1)$ is a sphere in $\mathbb{R}^3$, and $\pi(S_2)$ is either a plane or a sphere in $\mathbb{R}^3$, by Lemma 2.19. Since S_1 and S_2 intersect in the 1-dimensional space C, $\pi(S_1)$ and $\pi(S_2)$ intersect in a 1-dimensional space. Therefore $\pi(C)$ is a circle. □

2.10 Great spheres

A *great sphere* is the intersection of a three-dimensional subspace of $\mathbb{R}^4$ with S^3. We have already been considering the great sphere consisting of the pure unit quaternions, which we have been denoting S^2. Any other great sphere can be denoted by $\rho(q_1, q_2)(S^2) = q_1 S^2 q_2^{-1}$. Note that any two distinct great spheres intersect in a great circle, since any two distinct three-dimensional subspaces of $\mathbb{R}^4$ intersect in a two-dimensional subspace.

Lemma 2.21. *Every great sphere S of S^3 can be expressed as $S = qS^2$ where $q \in S^3$ is orthogonal to S. More specifically, $q_1 S^2 q_2^{-1} = q_1 q_2^{-1} S^2$.*

Proof. The great sphere $q_1 S^2 q_2^{-1} = \langle q_1 i q_2^{-1}, q_1 j q_2^{-1}, q_1 k q_2^{-1} \rangle \cap S^3$ is the intersection of the subspace of $\mathbb{R}^4$ orthogonal to the element $q_1 q_2^{-1}$ with S^3. The subspace $\langle q_1 q_2^{-1} i, q_1 q_2^{-1} j, q_1 q_2^{-1} k \rangle$ is also orthogonal to the element $q_1 q_2^{-1}$ since $(q_1 q_2^{-1} i) \cdot (q_1 q_2^{-1}) = Re(q_1 q_2^{-1} i (q_1 q_2^{-1})^{-1}) = 0$ by Lemma 2.1. Similarly, $(q_1 q_2^{-1} j) \cdot (q_1 q_2^{-1}) = 0$ and $(q_1 q_2^{-1} k) \cdot (q_1 q_2^{-1}) = 0$. Therefore $q_1 q_2^{-1} S^2 = \langle q_1 q_2^{-1} i, q_1 q_2^{-1} j, q_1 q_2^{-1} k \rangle \cap S^3$ is also the intersection of the subspace of $\mathbb{R}^4$ orthogonal to the element $q_1 q_2^{-1}$ with S^3. $\square$

We will be considering great circles on these great spheres, so it will be useful to keep the following fact in mind. The space of all oriented great circles on a sphere is in one-to-one correspondence with the points on the sphere. Consider the following correspondence, illustrated in Figure 2.6.

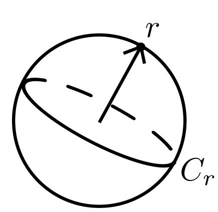

Figure 2.6: Correspondence between points of S^2 and great circles

Let r be a point on S^2. Then r is a pure unit quaternion thus is conjugate to k by Lemma 2.1, so $r = qkq^{-1}$ for some $q \in S^3$. Let C_r be the oriented great circle $\langle qiq^{-1}, qjq^{-1}\rangle \cap S^3 = \rho(q,q)\langle i,j\rangle \cap S^3$. Since $\rho(q,q)$ preserves the pure unit quaternions, $C_r \subset S^2$. Also note that C_r is contained in the plane orthogonal to r. Thus as r varies through S^2, C_r varies through all oriented great circles of S^2. Notice that C_{-r} is the same great circle as C_r with opposite orientation.

Lemma 2.22. *The oriented great circle $\rho(q_1,q_2)\langle i,j\rangle$ corresponds to $(q_1kq_1^{-1}, -q_2kq_2^{-1}) \in S^2 \times S^2$ via the correspondence given in Theorem 2.7.*

Proof. Let $r_1 = \frac{1}{2}(1+i-j+k)$ and $r_2 = \frac{1}{2}(1-i+j+k)$. Then $(q_1kq_1^{-1}, -q_2kq_2^{-1}) = (q_1r_1ir_1^{-1}q_1^{-1}, q_2r_2ir_2^{-1}q_2^{-1})$. By Theorem 2.7, this element of $S^2 \times S^2$ corresponds to $\rho(q_1r_1, q_2r_2)\langle 1,i\rangle \in \widetilde{G_2}(\mathbb{R}^4)$. But $\rho(q_1r_1, q_2r_2)\langle 1,i\rangle = \rho(q_1,q_2)\langle r_1r_2^{-1}, r_1ir_2^{-1}\rangle = \rho(q_1,q_2)\langle i,j\rangle$. □

Theorem 2.23. *The oriented geodesic in S^3 corresponding to $(a,b) \in S^2 \times S^2$ is contained in S^2 if and only if $b = -a$.*

Proof. Let $(a,-a) \in S^2 \times S^2$. Since a is a pure unit quaternion, $a = qkq^{-1}$ for some $q \in S^3$, by Lemma 2.1. Thus $(a,-a) = (qkq^{-1}, -qkq^{-1})$. This element in $S^2 \times S^2$ corresponds to the geodesic $\rho(q,q)\langle i,j\rangle$ by Lemma 2.22. Since $\rho(q,q)$ preserves the pure unit quaternions, $\rho(q,q)\langle i,j\rangle$ is a geodesic in the pure unit quaternions. Thus $(a,-a) \in S^2 \times S^2$ corresponds to a geodesic in the pure unit quaternions.

Now let C be an arbitrary oriented geodesic of S^3 contained in the pure unit quaternions. Then C is in the plane $\langle qiq^{-1}, qjq^{-1}\rangle = \rho(q,q)\langle i,j\rangle$ for some $q \in S^3$. This geodesic corresponds to the element $(qkq^{-1}, -qkq^{-1}) = (a,-a) \in S^2 \times S^2$ by Lemma 2.22. Thus all geodesics in the pure unit quaternions correspond to some $(a,-a) \in S^2 \times S^2$. □

Corollary 2.24. *The set of all oriented great circles contained in the great sphere qS^2 corresponds to the set $\{(qaq^{-1}, -a) \in S^2 \times S^2\}$. So the set of geodesics of any great sphere in S^3 corresponds to the graph of an orientation reversing isometry of S^2.*

Proof. By Lemma 2.21, every great sphere can be written as qS^2, where S^2 is the sphere consisting of pure unit quaternions. Since the set of geodesics of S^3 contained in S^2 corresponds to $\{(a, -a) \in S^2 \times S^2\}$, the set of geodesics of S^3 contained in the great sphere $qS^2 = \rho(q, 1)(S^2)$ corresponds to $\{(qaq^{-1}, -a) \in S^2 \times S^2\}$, by Theorem 2.9. This is the graph of the orientation reversing isometry ϕ of S^2 defined by $\phi(r) = -qrq^{-1}$. $\square$

Lemma 2.23 shows us that the family of geodesics in S^2 corresponds to the graph of the antipodal map on S^2.

Corollary 2.25. *Let C_r be a great circle in S^2. The set $\{(a, -a) \in S^2 \times S^2 \mid a \in C_r\}$ corresponds to the family of all great circles of S^3 contained in S^2 which pass through $\pm r$.*

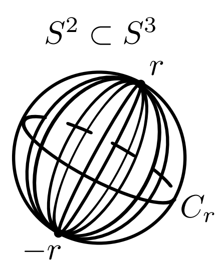

Figure 2.7: Corollary 2.25

Proof. Since a is a pure unit quaternion, $a = qkq^{-1}$ for some $q \in S^3$ by Lemma 2.1. So $(a, -a) = (qkq^{-1}, -qkq^{-1})$ which corresponds to the geodesic

$\rho(q,q)\langle i,j\rangle = \langle qiq^{-1}, qjq^{-1}\rangle$ by Lemma 2.22. This geodesic is contained in the pure unit quaternions and is orthogonal to $qkq^{-1} = a$. As a varies through C_r, we get all geodesics in S^2 through the points $\pm r$ (refer to Figure 2.8). $\square$

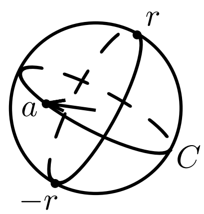

Figure 2.8: Great circles of S^2

Corollary 2.26. *Let C_r be a great circle in S^2. For $q \in S^3$, the set $\mathcal{S}_q = \{(qaq^{-1}, -a) \mid a \in C_r\}$ corresponds to the family of great circles in qS^2 which pass through $\pm qr$.*

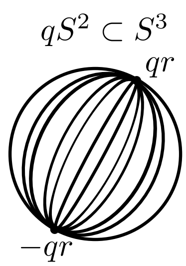

Figure 2.9: The set $\mathcal{S}_q$

Proof. Note that since a is a pure unit quaternion, $a = pkp^{-1}$ for some unit quaternion p. Thus $(qaq^{-1}, -a) = (qpkp^{-1}q^{-1}, -pkp^{-1})$, which corresponds to the geodesic $\rho(qp,p)\langle i,j\rangle = \rho(q,1)\rho(p,p)\langle i,j\rangle \in qS^2$. The geodesic $\rho(p,p)\langle i,j\rangle = \langle pip^{-1}, pjp^{-1}\rangle$ is orthogonal to $pkp^{-1} = a$. As a varies through

C_r, $\langle pip^{-1}, pjp^{-1}\rangle$ gives all great circles in S^2 passing through $\pm r$ by Corollary 2.25. Therefore, if we apply $\rho(q,1)$ to all of these geodesics, we obtain all great circles in qS^2 containing $\pm qr$. $\square$

2.11 A decomposition of S^3

As we saw in Corollary 2.25 and Corollary 2.26, we can decompose a sphere into all great circles which pass through two fixed antipodal points. We can similarly decompose S^3 into all great spheres which contain a fixed great circle.

Let C be a fixed oriented great circle of S^3 that corresponds to $(r', -r) \in S^2 \times S^2$. There exists a $q \in S^3$ such that $qrq^{-1} = r'$ by Lemma 2.1. Let $q_\theta = \cos\theta + r\sin\theta$ and consider the family of great spheres $\{qq_\theta S^2 \mid 0 \leq \theta < 2\pi\}$. This family is a decomposition of S^3 into all of the great spheres that contain C, as the following lemmas show.

Lemma 2.27. *Let C be an oriented great circle corresponding to $(r', -r) \in S^2 \times S^2$, let $q \in S^3$ satisfy $qrq^{-1} = r'$, and let $q_\theta = \cos\theta + r\sin\theta$. Then* $\displaystyle\bigcap_{\theta\in[0,2\pi)} qq_\theta S^2 = C$.

Proof. By Corollary 2.24, the family of all great circles of $qq_\theta S^2$ corresponds to the set $\{(qq_\theta a q_\theta^{-1} q^{-1}, -a) \mid a \in S^2\}$. Since $q_\theta r q_\theta^{-1} = r$ we know that $(qrq^{-1}, -r) = (r', -r)$ is in this set for all $\theta \in [0, 2\pi)$. Therefore C is in $qq_\theta S^2$ for all $\theta \in [0, 2\pi)$. $\square$

Lemma 2.28. *Let C be an oriented great circle corresponding to $(r', -r) \in S^2 \times S^2$, let $q \in S^3$ satisfy $qrq^{-1} = r'$, and let $q_\theta = \cos\theta + r\sin\theta$. If the great sphere pS^2 contains C, then $p = qq_\theta$ for some $\theta \in [0, 2\pi)$.*

Proof. Suppose pS^2 contains C, which corresponds to $(r', -r) \in S^2 \times S^2$. The family of all great circles of pS^2 is $\{(pap^{-1}, -a \mid a \in S^2\}$. So $(r', -r) = (pap^{-1}, -a)$ for some $a \in S^2$, specifically $a = r$ and $pap^{-1} = prp^{-1} = r'$. But $r' = qrq^{-1}$, so $prp^{-1} = qrq^{-1}$. This means that $q^{-1}prp^{-1}q = r$ and $q^{-1}p = q_\theta$ for some $\theta \in [0, 2\pi)$. Therefore $p = qq_\theta$ for some $\theta \in [0, 2\pi)$. □

Lemma 2.29. *Let C be an oriented great circle corresponding to $(r', -r) \in S^2 \times S^2$, let $q \in S^3$ satisfy $qrq^{-1} = r'$, and let $q_\theta = \cos\theta + r\sin\theta$. Then* $\bigcup_{\theta\in[0,2\pi)} qq_\theta S^2 = S^3$.

Proof. Let q' be an arbitrary element of S^3. If $q' \in C$ then by Lemma 2.27, $q' \in qq_\theta S^2$ for all $\theta \in [0, 2\pi)$. If $q' \notin C$ then $\langle q', C\rangle$ defines a 3 dimensional subspace of $\mathbb{R}^4$ and $\langle q', C\rangle \cap S^3$ is a great sphere in S^3. Thus q' is in a great sphere that contains C. So by Lemma 2.28, $q' \in qq_\theta S^2$ for some $\theta \in [0, 2\pi)$. □

There are a few observations we can make about this decomposition. First, consider the great circle $C^\perp$ which corresponds to $(r', r) \in S^2 \times S^2$. It can be shown that $C^\perp = \langle q, qr\rangle \cap S^3$. The antipodal points $\pm qq_\theta r$ are contained in $qq_\theta S^2$ and, since $qq_\theta r = qr\cos\theta - q\sin\theta$, the points $\pm qq_\theta r$ are also contained in $C^\perp$. Therefore each great sphere in the family $\{qq_\theta S^2 \mid 0 \leq \theta < 2\pi\}$ passes through the antipodal points $\pm qq_\theta r$ of the great circle $C^\perp$.

We can also picture this decomposition via the stereographic projection as Figure 2.10 shows. In Figure 2.10 we are assuming that $\pi(C^\perp)$ is a line.

We further consider this decomposition of S^3 into great spheres by also decomposing each great sphere into great circles that pass through the points of intersection with $C^\perp$. By Corollary 2.26 the family of all great circles of $qq_\theta S^2$

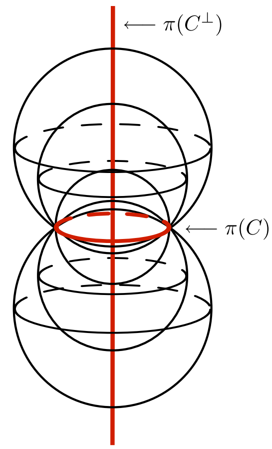

Figure 2.10: Decomposition of S^3 by great spheres

passing through the points $\pm qq_\theta r$ corresponds to the set $\{(qq_\theta a q_\theta^{-1} q^{-1}, -a) \mid a \in C_r\} \subset S^2 \times S^2$. Then

$$\begin{aligned} \bigcup_{\theta\in[0,2\pi)} \{(qq_\theta a q_\theta^{-1} q^{-1}, -a) \mid a \in C_r\} &= qC_r q^{-1} \times C_r \\ &= C_{qrq^{-1}} \times C_r \\ &= C_{r'} \times C_r \end{aligned}$$

This further decomposition into great circles can be pictured using the stereographic projection, as in Figure 2.11. Note that in Figure 2.11 it is assumed that $\pi(C^\perp)$ is a line. The following theorem summarizes the results in this section.

Theorem 2.30. *Let C be an oriented great circle corresponding to $(r', -r) \in S^2 \times S^2$, let $q \in S^3$ satisfy $qrq^{-1} = r'$, and let $q_\theta = \cos\theta + r\sin\theta$. The set $\{qq_\theta S^2 \mid 0 \le \theta < 2\pi\}$ decomposes S^3 into the family of all great spheres that contain the great circle C. Each of these great spheres can further be* *decomposed into the great circles passing through the points of intersection of the sphere and $C^{\perp}$. This family of great circles corresponds to the set $C_{r'} \times C_r \subset S^2 \times S^2$.* $\square$

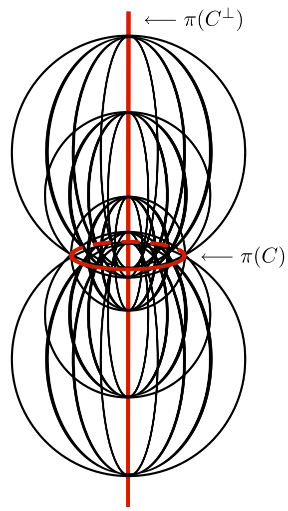

Figure 2.11: $C_{r'} \times C_r$ via the stereographic projection

We will now turn our attention to consider great circle fibrations of S^3. We use the structure of $S^2 \times S^2$ to classify such fibrations, and then extend this work to the other elliptic 3-manifolds.

Chapter 3

Great circle fibrations of S^3

A *fiber bundle* structure on a space E with fiber F consists of a *total space* E, a *base space* B and a continuous surjection $p : E \to B$, called a *fibration*, such that each point of B has a neighborhood U for which there is a homeomorphism $g_U : p^{-1}(U) \to U \times F$ satisfying $\pi \circ g_U = p$, where π is the projection onto the first coordinate. For each $b \in U$, $p^{-1}(b) = F_b$ is called a *fiber*. Notice that g_U maps each fiber F_b homeomorphically onto $\{b\} \times F$. Thus the fibers F_b are arranged locally as in the product $B \times F$, though this structure is not necessarily global. In short, we say that a space E is fibered by F if E is partitioned into fibers F_b, and locally the fibers are arranged as a product.

Given an orientable manifold E with a fixed orientation and an orientable base space B, we define an *oriented fibration* to be a fibration $p : E \to B$ together with a choice of orientation on B. The choice of orientation on B determines an orientation on the fibers F_b in the following way. Let $U \subset B$ be defined as above. Note that U is oriented based on the orientation chosen for B. Then choose the orientation on F so that the orientation on $p^{-1}(U)$

obtained by restricting the given orientation on E agrees via g_U with the product orientation on $U \times F$. Each unoriented fibration with orientable base and total spaces can be oriented in two ways, depending on the orientation chosen for B.

We say that two oriented fibrations $p : E \to B$ and $p' : E \to B$ are *(geometrically) equivalent* if there exists an isometry γ of E such that $p = p' \circ \gamma$. Two fibrations are *topologically equivalent* if there exists a homeomorphism h of E such that $p = p' \circ h$.

In this chapter we are concerned with oriented fibrations $f : S^3 \to B$ with great circle fibers. We subsequently refer to such fibrations as *oriented great circle fibrations.* Since great circles are 1-dimensional manifolds, the base space B is a 2-dimensional manifold. The space S^3 is compact and simply connected, thus since f is a continuous surjection, B is compact and simply connected as well. Therefore $B = S^2$.

By the correspondence given in Theorem 2.7, each oriented great circle fibration p determines a subset M_p of $S^2 \times S^2$ consisting of those oriented great circles which are fibers of p.

3.1 The Hopf fibration

The most well-known great circle fibration of S^3 is the *Hopf fibration* $h : S^3 \to S^2$ defined by

$$h(q) = qiq^{-1}. \tag{3.1}$$

By Lemma 2.1, since qiq^{-1} is conjugate to i, its real part is zero and $qiq^{-1} \in S^2$. Note also that, by Lemma 2.1, for $p \in S^2$, there exists $q \in S^3$ such that

$p = qiq^{-1}$, so h is surjective.

Lemma 3.1. *The fibers of the Hopf fibration are left cosets of S^1. More specifically, for $p = qiq^{-1} \in S^2$,*

$$h^{-1}(p) = h^{-1}(qiq^{-1}) = qS^1. \tag{3.2}$$

Proof. Lemma 2.2 shows that the centralizer of i in S^3 is S^1. Therefore

$$\begin{aligned}
h^{-1}(qiq^{-1}) &= \{r \in S^3 \mid rir^{-1} = qiq^{-1}\} \\
&= \{r \in S^3 \mid q^{-1}ri = iq^{-1}r\} \\
&= \{r \in S^3 \mid q^{-1}r \text{ commutes with i}\} \\
&= \{r \in S^3 \mid q^{-1}r \in S^1\} \\
&= \{r \in S^3 \mid r \in qS^1\} \\
&= qS^1.
\end{aligned}$$

□

Left cosets of S^1 are great circles, so h is a great circle fibration. We can orient this Hopf fibration in two ways, depending on the fixed orientation of S^3 and the chosen orientation of S^2. One orientation results in oriented fibers of the form $q\langle 1, i\rangle \cap S^3 = \rho(q, 1)\langle 1, i\rangle \cap S^3$. The other orientation results in fibers of the form $q\langle i, 1\rangle \cap S^3 = \rho(qij, j)\langle 1, i\rangle \cap S^3$.

Consider the subset M_h of $\widetilde{G_2}(\mathbb{R}^4) \cong S^2 \times S^2$ that corresponds to the Hopf fibration of S^3 via the correspondence given in Theorem 2.7. In the first case, these oriented planes correspond to $(qiq^{-1}, i) \in S^2 \times S^2$, so this Hopf fibration corresponds to the submanifold $S^2 \times \{i\} \subset \widetilde{G_2}(\mathbb{R}^4)$. In the second case, these oriented planes correspond to $(-qiq^{-1}, -i) \in S^2 \times S^2$, so this Hopf fibration corresponds to the submanifold $S^2 \times \{-i\} \subset \widetilde{G_2}(\mathbb{R}^4)$.

Now consider fibrations that are geometrically equivalent to the Hopf fibration h. Let $\hat{h} = h \circ \rho(q_1, q_2)$ for $\rho(q_1, q_2) \in Isom_+(S^3)$. This also gives a fibration of S^3 by oriented great circles. These fibrations are referred to as Hopf fibrations as well. The fibers of $\hat{h}$ are $\hat{h}^{-1}(qiq^{-1}) = \rho(q_1^{-1}, q_2^{-1}) \circ h^{-1}(qiq^{-1}) = \rho(q_1^{-1}, q_2^{-1})(qS^1) = q_1^{-1}qS^1q_2$. Since elements of $Isom_+(S^3)$ act on $S^2 \times S^2$ by independent rotations in the S^2 factors, these fibrations each correspond to the submanifold $S^2 \times \{w\} \subset \widetilde{G_2}(\mathbb{R}^4)$ for some $w \in S^2$.

Let $\gamma : S^3 \to S^3$ be defined by $\gamma(q) = q^{-1}$, which is an orientation reversing isometry on S^3. Consider the map $\tilde{h} = h \circ \gamma$. So

$$\tilde{h}(q) = h(q^{-1}) = q^{-1}iq.$$

The fibers of this fibration are

$$\tilde{h}^{-1}(qiq^{-1}) = \gamma^{-1} \circ h^{-1}(qiq^{-1}) = \gamma(qS^1) = S^1q^{-1},$$

which are right cosets of S^1. This also gives a fibration that is geometrically equivalent to the Hopf fibration. By Lemma 2.10, γ acts on $S^2 \times S^2$ by switching the S^2 factors and performing the antipodal map on each of the S^2 factors. Thus this Hopf fibration corresponds to the submanifold $\{i\} \times S^2 \subset \widetilde{G_2}(\mathbb{R}^4)$ or $\{-i\} \times S^2 \subset \widetilde{G_2}(\mathbb{R}^4)$, depending on the orientation chosen for the fibers. Again, we can compose this Hopf fibration with elements of $Isom_+(S^3)$ to obtain other Hopf fibrations, which correspond to the submanifold $\{w\} \times S^2 \subset \widetilde{G_2}(\mathbb{R}^4)$ for some $w \in S^2$.

These are all of the fibrations of S^3 by oriented great circles that are geometrically equivalent to the Hopf fibration, and we refer to them all as Hopf fibrations. We have shown that all Hopf fibrations correspond to either the submanifold $\{w\} \times S^2$ or the submanifold $S^2 \times \{w\}$ in $\widetilde{G_2}(\mathbb{R}^4)$ for $w \in S^2$.

By Corollary 2.14, each pair of great circles in a Hopf fibration are Clifford parallel. Recall that $\kappa_{(\alpha,x)}$ is the set of all points of S^2 a distance α from the point $x \in S^2$. Notice that

$$S^2 \times \{w\} = \left(\bigcup_{\alpha \in (0,\pi)} \left(\kappa_{(\alpha,x)} \times \{w\} \right) \right) \cup (x, w) \cup (-x, w).$$

By Corollary 2.17, we know that each $\kappa_{(\alpha,x)} \times \{w\}$ is a partition of the torus $C_{\alpha/2}$ into Clifford parallel great circles, where C is the great circle corresponding to $(x, w) \in S^2 \times S^2$. Therefore, a Hopf fibration partitions S^3 into the great circles corresponding to (x, w) and $(-x, w)$ along with a partition of each torus C_β into Clifford parallel great circles. Figure 3.1 [8] shows the stereographic projection of the Hopf fibration h.

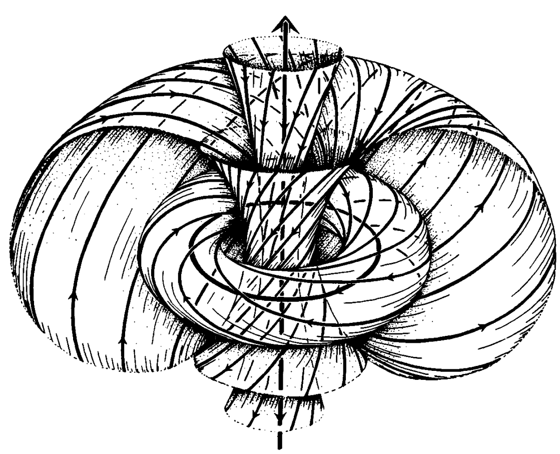

Figure 3.1: Decomposition of S^3 into tori by the Hopf fibration [8]

3.2 More general fibrations of S^3

The Hopf fibrations are very special fibrations of S^3 by oriented great circles. These are by no means the only such fibrations of S^3. In 1983, Gluck and Warner classified all possible fibrations of S^3 by oriented great circles. Consider the correspondence between the fibers of a great circle fibration of S^3 and points in $S^2 \times S^2$ as defined in Theorem 2.7.

Theorem 3.2 (Gluck and Warner, [4]). *A submanifold of $\widetilde{G_2}(\mathbb{R}^4) \cong S^2 \times S^2$ corresponds to a fibration of S^3 by oriented great circles if and only if it is the graph of a distance decreasing map from either S^2 factor to the other.*

For example, as discussed in the previous section, a Hopf fibration corresponds to the graph of a constant map from one S^2 factor to the other.

In Section 2.6 we considered the action of isometries of S^3 on $S^2 \times S^2$. With this in mind, we see that there are infinitely many geometrically different great circle fibrations of S^3. All great circle fibrations of S^3 are topologically equivalent, however.

Michael Gage extended the work of Gluck and Warner to investigate another special kind of fibration of S^3. Define a *skew-Hopf fibration* to be a fibration of S^3 obtained from a Hopf fibration by applying a linear transformation of $\mathbb{R}^4$ followed by a projection of the fibers back onto S^3.

Theorem 3.3 (Gage, [3]). *Each skew-Hopf fibration corresponds to a distance decreasing map from one S^2 factor to the other which can be decomposed as an orthogonal projection to a plane through the center of the sphere, followed by a distance decreasing linear map from one 2-plane to another, and finally the inverse projection back onto the sphere.*

By considering the great circle fibrations of S^3 we can learn about the geodesic Seifert fibrations of elliptic 3-manifolds, which have S^3 as their universal cover. We will use the result by Gluck and Warner to classify the geodesic Seifert fibrations of any elliptic 3-manifold. This is the focus of the remaining paper.

Chapter 4

Geodesic Seifert fibrations of elliptic 3-manifolds

An *elliptic 3-manifold* is of the form $M = S^3/\Gamma$, where Γ is a finite subgroup of $SO(4)$ which acts freely on S^3. We say that two elliptic 3-manifolds $M_1 = S^3/\Gamma_1$ and $M_2 = S^3/\Gamma_2$ are *isometric* if $\Gamma_2 = \alpha\Gamma_1\alpha^{-1}$ for some $\alpha \in O(4)$.

4.1 Unitary and symplectic structures

The standard Hermitian form on $\mathbb{H}$ is defined to be $\langle q_1, q_2\rangle = q_1\overline{q_2}$ where $\overline{q_2} = \overline{u_2 + v_2 j} = \overline{u_2} - vj$. Note that the complex part gives the standard Hermitian form on $\mathbb{C}^2$. That is, $Cx(\langle u_1 + v_1 j, u_2 + v_2 j\rangle) = u_1\overline{u_2} + v_1\overline{v_2}$. Also, the real part gives the Euclidean inner product on $\mathbb{R}^4$: $Re(\langle x_1 + y_1 i + z_1 j + w_1 k, x_2 + y_2 i + z_2 j + w_2 k\rangle) = x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2$. Recall from Section 2.3 that the group of orientation preserving isometries of S^3 is $SO(4) = \rho(S^3 \times S^3)$ where $\rho(q_1, q_2)$ is the isometry defined by $\rho(q_1, q_2)(q) = q_1 q q_2^{-1}$. All of these isometries preserve the Euclidean inner product. The group of unitary isometries of S^3, $U(2)$, is a subgroup of $SO(4)$ that preserves the standard Hermitian form on $\mathbb{C}^2$. The group of symplectic isometries of S^3, $Sp(1)$, is a subgroup of $SO(4)$ which preserves the standard Hermitian form on $\mathbb{H}$.

Lemma 4.1. *$U(2) = \rho(S^1 \times S^3)$ and $Sp(1) = \rho(\{\pm 1\} \times S^3)$*

Proof. The isometry $\rho(q_1, q_2)$ preserves the standard Hermitian form on $\mathbb{C}^2$ if and only if $Cx(\langle x, y\rangle) = Cx(\langle \rho(q_1, q_2)(x), \rho(q_1, q_2)(y)\rangle)$ for all $x, y \in S^3$. Thus $Cx(x\overline{y}) = Cx(q_1 x\overline{y}q_1^{-1})$ for all $x, y \in S^3$. For all $x\overline{y} \in S^1 j$, we have that $Cx(x\overline{y}) = 0$ and so $Cx(q_1 x\overline{y}q_1^{-1}) = 0$ iff $q_1 \in S^1$. And in fact, for any $q_1 \in S^1$ we have that $Cx(x\overline{y}) = Cx(q_1 x\overline{y}q_1^{-1})$ for all $x, y \in S^3$.

The isometry $\rho(q_1, q_2)$ preserves the standard Hermitian form on $\mathbb{H}$ if and only if $\langle x, y\rangle = \langle \rho(q_1, q_2)(x), \rho(q_1, q_2)(y)\rangle$ for all $x, y \in S^3$. This means that $x\overline{y} = q_1 x q_2^{-1}\overline{q_1 y q_2^{-1}} = q_1 x\overline{y}q_1^{-1}$ for all $x, y \in S^3$. Therefore $q_1 \in \{\pm 1\}$. $\square$

This gives the following hierarchy:

$$Sp(1) < U(2) < SO(4)$$

We say that an elliptic 3-manifold $M = S^3/\Gamma$ is *unitary* if $\Gamma < U(2)$. We say that an elliptic 3-manifold $M = S^3/\Gamma$ has *unitary type* if it is isometric to a unitary elliptic 3-manifold, that is, if $\alpha\Gamma\alpha^{-1} < U(2)$ for some $\alpha \in O(4)$. We say that an elliptic 3-manifold $M = S^3/\Gamma$ is *symplectic* if $\Gamma < Sp(1)$. An elliptic 3-manifold $M = S^3/\Gamma$ is of *symplectic type* if $\alpha\Gamma\alpha^{-1} < Sp(1)$ for some $\alpha \in O(4)$.

Lemma 4.2. *1. Every elliptic 3-manifold is of unitary type.*

2. *The elliptic 3-manifold $M = S^3/\Gamma$ is of symplectic type if and only if $\Gamma < \rho(\{\pm 1\} \times S^3)$ or $\Gamma < \rho(S^3 \times \{\pm 1\})$.*

Proof. 1. This is a result of the Hopf classification theorem.

2. We know that $M = S^3/\Gamma$ is of symplectic type if and only if $\alpha\Gamma\alpha^{-1} < Sp(1) = \rho(\{\pm 1\} \times S^3)$ for some $\alpha \in O(4)$. This means that $\Gamma < \alpha^{-1}\rho(\{\pm 1\} \times S^3)\alpha$ for some $\alpha \in O(4)$. If $\alpha = \rho(q_1, q_2) \in SO(4)$ this is equivalent to $\Gamma < \rho(\{\pm 1\} \times S^3)$. Note that for $\gamma(q) = q^{-1}$, we have that $\gamma\rho(r_1, r_2)\gamma = \rho(r_2, r_1)$ for all $\rho(r_1, r_2) \in SO(4)$. If $\alpha = \gamma\rho(q_1, q_2) \notin SO(4)$ the condition that $\Gamma < \alpha^{-1}\rho(\{\pm 1\} \times S^3)\alpha$ is equivalent to $\Gamma < \rho(S^3 \times \{\pm 1\})$.

$\square$

The classification of elliptic 3-manifolds up to isometry, developed by Hopf and discussed in detail in [11] and [10], tells us that there are four types of elliptic 3-manifolds: cyclic type, product type, tetrahedral type, and dihedral type. We will specifically define each of these types in the following chapters.

4.2 Geodesic Seifert fibrations

Define a *trivial fibered solid torus* to be $S^1 \times D^2$, with fibers $S^1 \times \{y\}$ for $y \in D^2$. Let $\sigma : S^1 \times D^2 \to S^1 \times D^2$ be defined as $\sigma(u, v) = (ue^{2\pi i/m}, ve^{2\pi in/m})$ for n, m relatively prime. There is a free $\mathbb{Z}_m$ action on the trivial fibered solid torus $S^1 \times D^2$ where $\mathbb{Z}_m = \langle\sigma\rangle$. The quotient space $(S^1 \times D^2)/\langle\sigma\rangle$ is called a *fibered solid torus*. So a trivial fibered solid torus is an m-fold cover of a fibered solid torus. We can also think of a fibered solid torus as being obtained by cutting a trivial fibered solid torus open along some $\{x\} \times D^2$ and gluing back together with a $2\pi n/m$ rotation for n, m relatively prime.

We say that two fibered solid tori are homeomorphic if there exists a fiber-preserving homeomorphism from one to the other. A non-trivial fibered solid torus is fibered by curves that wrap m times longitudinally and n times meridionally around the core circle. Since a Dehn twist can be performed along a meridian curve in this solid torus to obtain a homeomorphic solid torus, any solid fibered torus with $m = \pm 1$ is homeomorphic to a trivial solid fibered torus.

There is a continuous surjection f, called a fibration, from a nontrivial fibered solid torus to a disk orbifold with one cone point of order m, as the following commutative diagram shows:

$$\begin{array}{ccc} S^1 \times D^2 & \xrightarrow{/\langle\sigma\rangle} & (S^1 \times D^2)/\langle\sigma\rangle \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D^2 & \xrightarrow{/\langle\hat{\sigma}\rangle} & D^2/\langle\hat{\sigma}\rangle \end{array} \tag{4.1}$$

where π_2 is projection onto the second coordinate, $\sigma : S^1 \times D^2 \to S^1 \times D^2$ is defined as above, and $\hat{\sigma} : D^2 \to D^2$ is the restriction of σ to the second coordinate, that is $\hat{\sigma}(v) = ve^{2\pi in/m}$.

Let M be a 3-manifold. Define a *Seifert fibration of* M to be a continuous surjection $p : M \to X$ such that for all $x \in X$ there is a neighborhood U of x for which $p^{-1}(U)$ is homeomorphic to a fibered solid torus in E. Note that a trivial fibered solid torus will map to a disk and a nontrivial fibered solid torus will map to a disk with one cone point via the fibration f given in Diagram 4.1. Therefore the base space X is a closed 2-dimensional orbifold whose only singularities are cone points.

We say that two Seifert fibrations p and p' are *equivalent* if there exists an isometry of M, call it α, such that $p \circ \alpha = p'$. The *fibers of* p are the simple closed curves $p^{-1}(x)$ for $x \in X$. So two fibrations are equivalent if there is an isometry of M that carries the fibers of one fibration to the fibers of the other. A Seifert fibration of the 3-manifold M gives a decomposition of M into pairwise disjoint fibers, such that each fiber has a neighborhood that forms a fibered solid torus. As we can see, unlike geodesic fibrations that we have been dealing with in S^3, a Seifert fibration does not necessarily admit a local product structure. We are concerned with *geodesic Seifert fibrations* in which the fibers are geodesics of the space M. We say that a fiber is a *regular fiber* if it has a neighborhood that is homeomorphic to a trivial fibered solid torus. All other fibers we call *critical fibers*. Critical fibers get mapped to cone points in the base orbifold. A further introduction to Seifert fibrations can be found in [10].

For the remainder of this chapter suppose that $\Gamma < SO(4)$ acts freely on S^3. Thus $M = S^3/\Gamma$ is an elliptic 3-manifold.

Lemma 4.3. *Let $\tilde{F}$ be a great circle fibration of S^3 and suppose Γ preserves the fibers of this fibration. Then $\tilde{F}$ projects to a geodesic Seifert fibration of $M = S^3/\Gamma$.*

$$\begin{array}{ccc} S^3 & \xrightarrow{\tilde{F}} & S^2 \\ {\scriptstyle /\Gamma}\downarrow & & \downarrow{\scriptstyle /\Gamma'} \\ M & \xrightarrow{F} & X \end{array} \tag{4.2}$$

Proof. First, for $\gamma \in \Gamma$, define the map $\gamma' : S^2 \to S^2$ by $\gamma' = \tilde{F}\gamma\tilde{F}^{-1}$. Since γ maps fibers of $\tilde{F}$ to fibers, the map γ' is well-defined. Now let

$\Gamma' = \{\gamma' \mid \gamma \in \Gamma\}$, and let $X = S^2/\Gamma'$. Define the map F so that Diagram 4.2 commutes.

We will now show that F is a geodesic Seifert fibration. The fiber $F^{-1}(x)$ is a geodesic of M since $\tilde{F}$ has great circle fibers. To show that F is a Seifert fibration, we will show that every fiber of F has a neighborhood that is a fibered solid torus. Let x be a cone point of X. Then x is covered by a single point y of S^2 and a small neighborhood U of x lifts up to a small neighborhood $\tilde{U}$ of y. Since $\tilde{F}$ is a great circle fibration of S^3, $\tilde{F}^{-1}(\tilde{U})$ is a trivial fibered solid torus T. Note that Γ preserves the fibers of $\tilde{F}$ and $\gamma(T) = T$ for $\gamma \in \Gamma$ since $\gamma'(\tilde{U}) = \tilde{F}\gamma\tilde{F}^{-1}(\tilde{U})$. Thus Γ preserves the fibering of T. So $T/\Gamma = F^{-1}(U)$ is a solid fibered torus, and the trivial fibered torus T is an finite-sheeted covering space. Thus $F^{-1}(x)$ is a critical fiber of F.

Now suppose that $x \in X$ is not a cone point of X. Then x has a neighborhood U in X that does not contain a cone point. This neighborhood U is covered by several disjoint neighborhoods $\tilde{U}_1, ..., \tilde{U}_k$ in S^2. Then $\tilde{F}^{-1}(\tilde{U}_i)$ is a trivial fibered torus in S^3 for all $1 \leq i \leq k$. Since Γ permutes these tori, the quotient by Γ is a trivial fibered torus $F^{-1}(U)$ in M. Thus $F^{-1}(x)$ is a regular fiber. $\square$

We will find it useful for the next few lemmas to define the linking number of two closed curves in S^3. There are many different ways to define the linking number, all of which are equivalent, as explained thoroughly in [9]. We define the linking number using homology. Let γ_1 and γ_2 be disjoint oriented closed curves in S^3. Let $[\gamma_1]$ be the homology class of γ_1 in $H_1(S^3 - \gamma_2)$, and let α be a generator of $H_1(S^3 - \gamma_2) \cong \mathbb{Z}$. Then $[\gamma_1] = n\alpha$. Define the *linking number of* γ_1 *and* γ_2 to be $lk(\gamma_1, \gamma_2) = n$. For example, an (m, n)-curve γ_1 on the surface of a solid unknotted torus T in S^3 has linking number n with the core circle γ_2 of T. This is because $S^3 - \gamma_2$ deformation retracts to $S^3 - T$, which is a solid open torus T' with homology group generator α, which is a longitudinal curve of T'. Then α is a meridional curve of T. Therefore $[\gamma_1] = n\alpha$ and $lk(\gamma_1, \gamma_2) = n$.

Lemma 4.4. *Any two disjoint great circles of S^3 have linking number ± 1.*

Proof. Let $\rho(q_1, q_2)\langle 1, i\rangle$ and $\rho(q_1', q_2')\langle 1, i\rangle$ be disjoint great circles in S^3. We know that $\{q_1q_2, q_1iq_2, q_1'q_2', q_1'iq_2'\}$ forms a basis for $\mathbb{R}^4$. There is a linear transformation that takes $\rho(q_1, q_2)\langle 1, i\rangle$ to $\langle 1, i\rangle$ and $\rho(q_1', q_2')\langle 1, i\rangle$ to $\langle j, k\rangle$ or $\langle k, j\rangle$. These two great circles have linking number ± 1. Since this linear transformation preserves linking number, the linking number of $\rho(q_1, q_2)\langle 1, i\rangle$ and $\rho(q_1', q_2')\langle 1, i\rangle$ is ± 1. $\square$

Consider a Seifert fibration $\tilde{F} : M \to \tilde{X}$. As explained in [10], there is a short exact sequence

$$1 \to K \to \pi_1(M) \to \pi_1(\tilde{X}) \to 1, \tag{4.3}$$

where K is a cyclic subgroup of $\pi_1(M)$ generated by a regular fiber. For $M = S^3$, we have that $\pi_1(\tilde{X}) = 1$. Since the only S^2 orbifolds that are simply connected are those with at most two cone points having relatively prime order, $\tilde{X}$ is an orbifold with at most two cone points. If $\tilde{X}$ has no cone points then $\tilde{X} = S^2$ and $\tilde{F}$ is a fibration in the usual sense. If $\tilde{X}$ has a cone point then the inverse image of that cone point is a critical fiber in S^3.

Lemma 4.5. *A great circle Seifert fibration of S^3 is a great circle fibration of S^3.*

Proof. Let $\tilde{F}$ be a great circle Seifert fibration of S^3. Suppose that $\tilde{F}$ has a critical fiber C. Then C has a non-trivial fibered solid torus neighborhood T_1. Thus there is a great circle fiber C' that is an (m,n)-curve on the boundary of T_1, where $m \neq \pm 1$. The linking number of C and C' is n. If $n \neq \pm 1$ then this is a contradiction of Lemma 4.4.

Suppose that $n = \pm 1$. Since C is a great circle and is therefore contained in a plane and unknotted, there is a Heegaard decomposition of S^3 into the fibered solid tori T_1 and T_2. The great circle C' which is an (m,n)-curve on the boundary of T_1 is an (n,m)-curve on the boundary of T_2. So C' is an (n,m)-curve wrapping around the core circle of T_2. The linking number of these two great circles is m. Since $m \neq \pm 1$, this contradicts Lemma 4.4. □

Lemma 4.6. *Let F be a geodesic Seifert fibration of $M = S^3/\Gamma$. Then F lifts to a fibration of S^3 and Γ preserves the fibers of this fibration.*

Proof. Let $F : M \to X$ be a geodesic Seifert fibration of M. The space M has S^3 as its universal cover, and let $\tilde{X}$ be the universal cover of X. Then we can define a map $\tilde{F} : S^3 \to \tilde{X}$ so that Diagram 4.4 commutes. There is not necessarily a unique way to define $\tilde{F}$, but once a base point $\tilde{F}(1)$ is chosen in $\tilde{X}$ the map becomes well-defined. In this manner, it becomes clear that Γ preserves the fibers of $\tilde{F}$.

$$\begin{array}{ccc} S^3 & \xrightarrow{\tilde{F}} & \tilde{X} \\ {\scriptstyle /\Gamma}\big\downarrow & & \big\downarrow \\ M & \xrightarrow{F} & X \end{array} \tag{4.4}$$

We claim that $\tilde{F}$ is a great circle fibration of S^3. Since the fibers of F are geodesics, the fibers of $\tilde{F}$ are great circles. The map $\tilde{F}$ is a Seifert fibration since local properties are preserved by a covering space. By Lemma 4.5, $\tilde{F}$ is a great circle fibration of S^3 and the base space $\tilde{X}$ is S^2. $\square$

A *good orbifold* is an orbifold that has a manifold as its universal cover. The previous lemma immediately gives the following corollary.

Corollary 4.7. *Every geodesic Seifert fibration of an elliptic 3-manifold has a base space that is a good orbifold.* $\square$

From Lemmas 4.3 and 4.6, the set of all geodesic Seifert fibrations of $M = S^3/\Gamma$ is in one-to-one correspondence with the set of all great circle fibrations of S^3 in which Γ preserves the fibers. Consider for example the Hopf fibration $h : S^3 \to S^2$ defined by $h(q) = qiq^{-1}$. Recall from Section 3.1 that all Hopf fibrations are of the form $h \circ \rho(q_1, q_2)$ or $h \circ \gamma \circ \rho(q_1, q_2)$ for some $q_1, q_2 \in S^3$ and $\gamma(q) = q^{-1}$. The fibers of the fibration $h \circ \rho(q_1, q_2)$ are of the form $\rho(q_1, q_2)(qS^1)$ as q varies through S^3. The fibers of the fibration $h \circ \gamma \circ \rho(q_1, q_2)$ are of the form $\rho(q_1, q_2)(S^1 q)$ as q varies through S^3. We say that an elliptic 3-manifold $M = S^3/\Gamma$ admits a Hopf fibration if a Hopf fibration of S^3 projects to M. That is, if the fibers of a Hopf fibration are preserved by Γ.

Corollary 4.8. *Every elliptic 3-manifold M admits a Hopf fibration.*

Proof. If $M = S^3/\Gamma$ is unitary, then $\Gamma < U(2) = \rho(S^1 \times S^3)$. Consider the Hopf fibration $h \circ \gamma$ which has fibers $S^1 q$. Then for $\rho(q_1, q_2) \in \Gamma$, we have that $\rho(q_1, q_2)(S^1 q) = q_1 S^1 q q_2^{-1} = S^1 q q_2^{-1}$, which is a fiber of the fibration $h \circ \gamma$. Therefore the Hopf fibration $h \circ \gamma$ projects to a Hopf fibration of M.

By the Hopf classification theorem, every elliptic 3-manifold is of unitary type. Therefore every elliptic 3-manifold $M = S^3/\Gamma$ is isometric to a unitary elliptic 3-manifold $M' = S^3/\alpha\Gamma\alpha^{-1}$ where $\alpha\Gamma\alpha^{-1} < \rho(S^1 \times S^3)$ for some $\alpha \in O(4)$. Consider the Hopf fibration $h \circ \gamma \circ \alpha^{-1}$. This fibration has fibers $\alpha^{-1}(S^1 q)$. Let $\rho(q_1, q_2) \in \Gamma$, which means that $\rho(q_1, q_2)\alpha^{-1} \in \alpha^{-1}\rho(S^1 \times S^3)$. Then $\rho(q_1, q_2)(\alpha^{-1}(S^1 q)) = \alpha^{-1}\rho(e^{i\theta}, q')(S^1 q) = \alpha^{-1}(S^1 q q')$ for some $\theta \in [0, 2\pi]$ and $q' \in S^3$. So the fibers of this fibration are preserved by Γ and M admits a Hopf fibration. $\square$

Using Theorem 3.2 by Gluck and Warner we can classify all geodesic Seifert fibrations of any elliptic 3-manifold in terms of distance decreasing functions $f : S^2 \to S^2$. Define $\hat{\Gamma}$ to be the action of the group Γ on $S^2 \times S^2$. Recall that an oriented great circle fibration of S^3 has oriented fibers. By projecting a great circle fibration of S^3 to a Seifert fibration of M, we can project the orientation of the fibers as well. This gives an *oriented geodesic Seifert fibration of M*. Then we immediately obtain the following theorem.

Theorem 4.9. *The set of all oriented geodesic Seifert fibrations of the elliptic 3-manifold $M = S^3/\Gamma$ is in one-to-one correspondence with the set of all graphs of distance decreasing functions $f : S^2 \to S^2$ from either S^2 factor to the other preserved by $\hat{\Gamma}$.* $\square$

In order to classify the geodesic Seifert fibrations, we will consider graphs of functions that are preserved by $\hat{\Gamma}$. The following lemma will be useful in this process.

Lemma 4.10. 1. *The graph $\{(x, f(x)) \mid x \in S^2\}$ is preserved by $\hat{\Gamma}$ if and only if $f \circ \psi(q_1) = \psi(q_2) \circ f$ for all $\rho(q_1, q_2) \in \Gamma$.*

2. *The graph $\{(f(x), x) \mid x \in S^2\}$ is preserved by $\hat{\Gamma}$ if and only if $f \circ \psi(q_2) = \psi(q_1) \circ f$ for all $\rho(q_1, q_2) \in \Gamma$.*

Proof. The graph $G = \{(x, f(x)) \mid x \in S^2\}$ is preserved by $\hat{\Gamma}$ if and only if $\hat{\rho}(q_1, q_2)(x, f(x)) \in G$ for all $\rho(q_1, q_2) \in \Gamma$. Since $\hat{\rho}(q_1, q_2)(x, f(x)) = (\psi(q_1)(x), \psi(q_2)(f(x)))$, this means that $(\psi(q_1)(x), \psi(q_2)(f(x))) = (\psi(q_1)(x), f(\psi(q_1)(x)))$. Therefore, $f \circ \psi(q_1) = \psi(q_2) \circ f$ for all $\rho(q_1, q_2) \in \Gamma$. A similar argument can be applied to graphs of the form $\{(f(x), x) \mid x \in S^2\}$. $\square$

Note that two Seifert fibrations of $M = S^3/\Gamma$ are equivalent if and only if there is an isometry of M that carries the fibers of one fibration to the fibers of the other. Suppose that one fibration corresponds to the graph $G \subset S^2 \times S^2$ of a distance decreasing function $f : S^2 \to S^2$ and the other corresponds to a graph $G' \subset S^2 \times S^2$ of the distance decreasing function $f' : S^2 \to S^2$. These two fibrations are equivalent if and only if there exists $\alpha \in O(4)$ such that $\alpha\Gamma\alpha^{-1} = \Gamma$ and the action of α on $S^2 \times S^2$ carries G to G'.

We will now use these results to consider the geodesic Seifert fibrations of each type of elliptic 3-manifold as classified by Hopf, beginning with the cyclic type in the following chapter.

Chapter 5

Lens spaces, $L(m, n)$

The first type of elliptic 3-manifold that we will consider is the *cyclic type*, also known as a *lens space*, denoted $L(m, n)$. Let $\eta_{m,n} \in Isom_+(S^3)$ be defined by $\eta_{m,n}(u+vj) = e^{2\pi i/m}u + e^{2\pi in/m}vj$ for $u, v \in \mathbb{C}$ and m, n relatively prime, $m \neq 0$. Then $\eta_{m,n}$ acts freely on S^3 and $\langle \eta_{m,n} \rangle \cong \mathbb{Z}_m$. The lens space $L(m, n)$ is the space $S^3/\langle \eta_{m,n} \rangle$. In the 1930s Reidemeister showed that $L(m, n)$ is homeomorphic to $L(m', n')$ if and only if $m' = m$ and $n' = \pm n^{\pm 1}$ mod m. Note that $L(0, 1) = S^1 \times S^2$, which is not an elliptic 3-manifold, and $L(1, 0) = S^3$ and $L(2, 1) = \mathbb{RP}^3$.

5.1 Action of $\eta_{m,n}$ on $S^2 \times S^2$

Since $\eta_{m,n}$ is an orientation preserving isometry of S^3, we know how it acts on $S^2 \times S^2$. First note that $\eta_{m,n} = \rho(e^{i\pi(n+1)/m}, e^{i\pi(n-1)/m})$ since

$$
\begin{aligned}
\rho(e^{i\pi(n+1)/m}, e^{i\pi(n-1)/m})(u+vj) &= e^{i\pi(n+1)/m}(u+vj)e^{-i\pi(n-1)/m} \\
&= e^{i\pi(n+1)/m}ue^{-i\pi(n-1)/m} + e^{i\pi(n+1)/m}vje^{-i\pi(n-1)/m} \\
&= e^{2i\pi/m}u + e^{2i\pi n/m}vj \\
&= \eta_{m,n}(u+vj).
\end{aligned}
$$

Recall from Section 2.5 that $\psi : S^3 \to SO(3)$ is defined by $\psi(q)(r) = qrq^{-1}$. Proposition 2.8 tells us that for $q = \cos\theta + u\sin\theta$ where $u \in S^2$, $\psi(q)$ is a rotation of S^2 with fixed points $\pm u$ and angle of rotation 2θ. Let $(\hat{\eta}_1, \hat{\eta}_2)$ denote the action of $\eta_{m,n}$ on $S^2 \times S^2$. Referring to Lemma 2.9, since $\eta_{m,n} = \rho(e^{i\pi(n+1)/m}, e^{i\pi(n-1)/m})$ we know that $\eta_{m,n}$ acts on $S^2 \times S^2$ by $\hat{\eta}_1 = \psi(e^{i\pi(n+1)/m})$ in the first factor and $\hat{\eta}_2 = \psi(e^{i\pi(n-1)/m})$ in the second factor. Thus $\hat{\eta}_1$ is a rotation on the first S^2 factor by an angle of $2\pi(n+1)/m$ with fixed points $\pm i$, and $\hat{\eta}_2$ is a rotation on the second S^2 factor by an angle of $2\pi(n-1)/m$ with fixed points $\pm i$. Therefore the order of $\hat{\eta}_1$ is $\dfrac{m}{\gcd(m, n+1)}$ and the order of $\hat{\eta}_2$ is $\dfrac{m}{\gcd(m, n-1)}$.

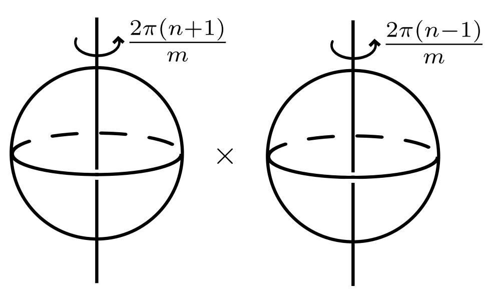

Figure 5.1: The action of $\eta_{m,n}$ on $S^2 \times S^2$

5.2 Geodesic Seifert fibrations of lens spaces

We have shown a correspondence between geodesic Seifert fibrations of $L(m, n)$ and great circle fibrations of S^3 satisfying certain conditions. Using Theorem 4.9, we are able to classify all geodesic Seifert fibrations of the lens space $L(m, n)$.

Theorem 5.1. *There is a one-to-one correspondence between oriented geodesic Seifert fibrations of $L(m, n)$ and graphs of distance decreasing functions $f : S^2 \to S^2$ satisfying either*

1. $\hat{\eta}_2 f = f \hat{\eta}_1$ *for the graph* $\{(x, f(x)) \mid x \in S^2\}$, *or*

2. $\hat{\eta}_1 f = f \hat{\eta}_2$ *for the graph* $\{(f(x), x) \mid x \in S^2\}$.

Proof. By Theorem 4.9 there is a one-to-one correspondence between geodesic Seifert fibrations of $L(m, n)$ and graphs of distance decreasing functions $f : S^2 \to S^2$ that are preserved by $\langle(\hat{\eta}_1, \hat{\eta}_2)\rangle$.

If the graph is of the form $\{(x, f(x)) \mid x \in S^2\}$ then this means that $\hat{\eta}_2 f = f \hat{\eta}_1$ by the first part of Lemma 4.10. If the graph is of the form $\{(f(x), x) \mid x \in S^2\}$ then this means that $\hat{\eta}_1 f = f \hat{\eta}_2$ by the second part of Lemma 4.10. □

Consider specifically the case of $L(2, 1) = \mathbb{RP}^3$. In this case, both $\hat{\eta}_1$ and $\hat{\eta}_2$ are the identity. All distance decreasing maps $f : S^2 \to S^2$ satisfy $\hat{\eta}_2 f = f \hat{\eta}_1$ and $\hat{\eta}_1 f = f \hat{\eta}_2$. Therefore $L(2, 1)$ inherits all oriented great circle fibrations of S^3, and there is a one-to-one correspondence between graphs of distance decreasing functions $f : S^2 \to S^2$ and oriented geodesic Seifert fibrations of $L(2, 1) = \mathbb{RP}^3$. This is a very special case, however. Most lens spaces admit far fewer geodesic Seifert fibrations than S^3, as we will see in the next sections.

5.3 The Hopf fibrations of $L(m, n)$

In Section 3.1, we talked specifically about Hopf fibrations of S^3. We say that a geodesic Seifert fibration of $L(m, n)$ is a Hopf fibration if it is a projection of a Hopf fibration of S^3. Define a *basic Hopf fibration* to be a fibration corresponding to the graph of the constant function $f(x) = i$ or $f(x) = -i$. There are four oriented basic Hopf fibrations. The graphs $\{(x, i) \mid x \in S^2\}$ and $\{(x, -i) \mid x \in S^2\}$ correspond to the same Hopf fibration with oppositely oriented fibers, and the graphs $\{(i, x) \mid x \in S^2\}$ and $\{(-i, x) \mid x \in S^2\}$ correspond to a different Hopf fibration.

Lemma 5.2. *Every lens space $L(m, n)$ admits the four basic oriented Hopf fibrations.*

Proof. Since $\hat{\eta}_1$ and $\hat{\eta}_2$ leave $\pm i$ fixed, by Theorem 5.1, the graphs $\{(x, i) \mid x \in S^2\}$, $\{(x, -i) \mid x \in S^2\}$, $\{(i, x) \mid x \in S^2\}$, and $\{(-i, x) \mid x \in S^2\}$ correspond to oriented geodesic Seifert fibrations of $L(m, n)$. $\square$

Most lens spaces admit only these Hopf fibrations, however symplectic type lens spaces admit more Hopf fibrations. Recall from Lemma 4.2 that a symplectic 3-manifold is of the form $M = S^3/\Gamma$ where $\Gamma < \rho(S^3 \times \{\pm 1\})$ or $\Gamma < \rho(\{\pm 1\} \times S^3$. The lens space $L(m, n) = S^3/\langle \rho(e^{i\pi(n+1)/m}, e^{i\pi(n-1)/m}) \rangle$ is therefore of symplectic type if either $\langle e^{i\pi(n+1)/m} \rangle < \{\pm 1\}$ or $\langle e^{i\pi(n-1)/m} \rangle < \{\pm 1\}$. This means that either $n = -1 \mod m$ in the first case or $n = 1 \mod m$ in the second case.

Lemma 5.3. *The lens space $L(m,n)$ admits non-basic Hopf fibrations if and only if it is of symplectic type.*

Proof. ($\Longrightarrow$) Suppose $L(m,n)$ admits a non-basic Hopf fibration. This fibration corresponds to the graph of a constant function $f(x) = y_0$ for $y_0 \neq \pm i$. If it corresponds to the graph $\{(x, y_0) \mid x \in S^2\}$, then from Theorem 5.1 we know that $\hat{\eta}_2 f = f\hat{\eta}_1$. But $f\hat{\eta}_1(x) = y_0$ for all $x \in S^2$. Thus $\hat{\eta}_2 f(x) = y_0$ and $\hat{\eta}_2$ must fix y_0. Therefore $\hat{\eta}_2$ must be the identity and $n = 1 \mod m$. Thus $L(m,n)$ is of symplectic type.

Similarly, if the fibration corresponds to the graph $\{(y_0, x) \mid x \in S^2\}$ then $n = -1 \mod m$ and $L(m,n)$ is of symplectic type.

($\Longleftarrow$) Suppose that $L(m,n)$ is of symplectic type. Then $n = \pm 1 \mod m$. If $n = 1 \mod m$, then $\hat{\eta}_2$ is the identity. Consider the graph $\{(x, y_0) \mid x \in S^2\}$ of the constant map $f(x) = y_0$ for $y_0 \neq \pm i$. This gives a Hopf fibration of S^3 and $\hat{\eta}_2 f = y_0 = f\hat{\eta}_1$. Therefore by Theorem 5.1, this projects to a geodesic Seifert fibration of $L(m,n)$. This is a non-basic Hopf fibration.

Similarly, if $n = -1 \mod m$ then the graph $\{(y_0, x) \mid x \in S^2\}$ corresponds to a non-basic Hopf fibration for any $y_0 \neq \pm i$. $\square$

If $\hat{\eta}_1$ is the identity on S^2 then all the oriented Hopf fibrations of S^3 corresponding to $\{w\} \times S^2 \subset S^2 \times S^2$ for any $w \in S^2$ descend to oriented Hopf fibrations of the lens space L. If $\hat{\eta}_1$ is not the identity on S^2 then the only oriented Hopf fibrations of S^3 of the form $\{w\} \times S^2 \subset S^2 \times S^2$ that descend to oriented Hopf fibrations of the lens space L are those for which $w \in \{\pm i\}$. If $\hat{\eta}_2$ is the identity then all the oriented Hopf fibrations of S^3 corresponding to $S^2 \times \{w\} \subset S^2 \times S^2$ for any $w \in S^2$ descend to oriented Hopf fibrations of the lens space L. If $\hat{\eta}_2$ is not the identity on S^2 then the only oriented Hopf fibrations of S^3 of the form $S^2 \times \{w\} \subset S^2 \times S^2$ that descend to oriented Hopf fibrations of the lens space L are those for which $w \in \{\pm i\}$. For S^3 and $\mathbb{RP}^3$ both $\hat{\eta}_1$ and $\hat{\eta}_2$ are the identity, so therefore the space of all oriented Hopf fibrations is isomorphic to $S^2 \sqcup S^2$. For all other symplectic lens spaces the space of oriented Hopf fibrations is isomorphic to $S^2 \sqcup \{\pm i\}$. For any non-symplectic lens space L the space of all oriented Hopf fibrations of L is isomorphic to $\{\pm i\} \sqcup \{\pm i\}$.

An isometry $\alpha \in O(4)$ of S^3 descends to an isometry of the lens space $L(m,n) = S^3/\langle \eta_{m,n} \rangle$ if and only if $\alpha \langle \eta_{m,n} \rangle \alpha^{-1} = \langle \eta_{m,n} \rangle$. For example, the isometry $\rho(j,j)$ descends to an isometry of any lens space. This is because

$$\begin{aligned}
\rho(j,j)\eta_{m,n}\rho(j,j)^{-1}(u+vj) &= \rho(j,j)\eta_{m,n}\rho(-j,-j)(u+vj) \\
&= \rho(j,j)\eta_{m,n}(\overline{u}+\overline{v}j) \\
&= \rho(j,j)(e^{2\pi i/m}\overline{u}+e^{2\pi in/m}\overline{v}j) \\
&= (e^{-2\pi i/m}u+e^{-2\pi in/m}vj) \\
&= \eta_{m,n}^{-1}(u+vj).
\end{aligned}$$

So conjugating by $\rho(j,j)$ takes a generator of $\langle \eta_{m,n} \rangle$ to a generator of $\langle \eta_{m,n} \rangle$. Therefore $\rho(j,j)\langle \eta_{m,n} \rangle \rho(j,j)^{-1} = \langle \eta_{m,n} \rangle$ for any m,n, and $\rho(j,j)$ is an isometry of any lens space.

Recall that two Hopf fibrations are *equivalent* if there exists an isometry that carries the fibers of one to the fibers of the other. Lens spaces with only orientation preserving isometries admit two non-equivalent Hopf fibrations, however those with orientation reversing isometries admit a unique Hopf fibration up to equivalence. We say that a lens space is *amphichiral* if it admits an orientation reversing isometry, and *chiral* if it does not. It is a well-known fact that a lens space $L(m,n)$ is amphichiral if and only if $n^2 = -1 \mod m$ (refer to [7] for more details).

Theorem 5.4. *Every amphichiral lens space admits a unique oriented Hopf fibration up to equivalence. Every chiral lens space admits two non-equivalent oriented Hopf fibrations.*

Proof. We know that every lens space admits the four basic oriented Hopf fibrations by Lemma 5.2. Consider the oriented basic Hopf fibration corresponding to the set $S^2 \times \{i\}$. Since $\rho(j,j)$ is an isometry of any lens space and $\hat{\rho}(j,j)(S^2 \times \{i\}) = S^2 \times \{-i\}$, the oriented basic Hopf fibrations corresponding to $S^2 \times \{i\}$ and $S^2 \times \{-i\}$ are equivalent in any lens space. Similarly the oriented basic Hopf fibrations corresponding to $\{i\} \times S^2$ and $\{-i\} \times S^2$ are equivalent in any lens space. The only lens spaces that admit other oriented Hopf fibrations are of symplectic type by Lemma 5.3.

Let $L = L(m,n)$ be an amphichiral lens space. Since $n^2 = -1 \mod m$, L is not of symplectic type unless L is S^3 or $\mathbb{RP}^3$. If $L = S^3$, then all Hopf fibrations are equivalent as explained in Section 3.1. If $L = \mathbb{RP}^3$, then since $\mathbb{RP}^3$ inherits every isometry from S^3 we know that all Hopf fibrations of L are equivalent. Otherwise L admits only the four basic oriented Hopf fibrations which constitute two non-equivalent oriented Hopf fibrations, one of which corresponds to the graph of a constant function from the first S^2 factor to the second and the other corresponds to the graph of a constant function from the second S^2 factor to the first. Recall that by Lemma 2.10 an orientation reversing isometry acts on $S^2 \times S^2$ by switching the two S^2 factors. Since L admits an orientation reversing isometry, there will be an isometry that carries one of these oriented basic Hopf fibrations to the other. Therefore these oriented Hopf fibrations are equivalent and L admits a unique oriented Hopf fibration.

Now suppose that $L = L(m, n)$ is a chiral lens space. If L is not of symplectic type then L admits only the two basic non-equivalent oriented Hopf fibrations, one corresponding to a constant function from the first S^2 factor to the second and the other corresponding to a constant function from the second factor S^2 factor to the first. Every orientation preserving isometry will preserve the S^2 factors. Thus an orientation preserving isometry does not carry one of these oriented Hopf fibrations to the other. Since L is chiral, it does not admit an orientation reversing isometry. Therefore there will not be an isometry that carries one of the oriented basic Hopf fibrations to the other. So L has two non-equivalent oriented Hopf fibrations.

If L is of symplectic type then $L = S^3/\langle\eta_{m,n}\rangle$ for $\langle\eta_{m,n}\rangle < \rho(\{\pm1\} \times S^3)$ or $\langle\eta_{m,n}\rangle < \rho(S^3 \times \{\pm1\})$ by Lemma 4.2. In the first case, the oriented Hopf fibrations of L correspond to $\{\omega\} \times S^2$ for $\omega \in S^2$ and the two equivalent oriented Hopf fibration corresponding to $S^2 \times \{i\}$ and $S^2 \times \{-i\}$. All fibrations corresponding to $\{\omega\} \times S^2$ are equivalent since $\rho(q, 1)$ is an isometry of the lens space for any $q \in S^3$. So there will be an isometry of the lens space that maps any oriented Hopf fibration of this form to any other oriented Hopf fibration of this form. Also, since L is chiral, there does not exist an orientation reversing isometry of L, therefore the fibration corresponding to $\{\omega\} \times S^2$ is not equivalent to the fibration corresponding to $S^2 \times \{i\}$. Thus there are two non-equivalent oriented Hopf fibrations. A similar argument shows that there are two non-equivalent oriented Hopf fibrations for the case in which $\langle \eta_{m,n} \rangle < \rho(S^3 \times \{\pm 1\})$. $\square$

5.4 The non-Hopf fibrations of $L(m, n)$

In many cases, the only geodesic Seifert fibrations of $L(m, n)$ are the Hopf fibrations. However there are lens spaces that admit non-Hopf fibrations as well.

Theorem 5.5. *The lens space $L(m, n)$ admits a geodesic Seifert fibration that is not a Hopf fibration if and only if one of $\gcd(m, n+1)$ and $\gcd(m, n-1)$ divides the other. In this case the lens space admits uncountably many non-equivalent geodesic Seifert fibrations that are not Hopf fibrations.*

Proof. ($\Longrightarrow$) Suppose $L(m, n)$ has a geodesic Seifert fibration that is not a Hopf fibration. By Theorem 5.1, this fibration corresponds to the graph of a distance decreasing function $\{(x, f(x)) \mid x \in S^2\}$ satisfying $\hat{\eta}_2 f = f \hat{\eta}_1$, or the graph $\{(f(x), x) \mid x \in S^2\}$ satisfying $\hat{\eta}_1 f = f \hat{\eta}_2$. Also, since this is not a Hopf fibration, f is not a constant map.

Consider the case in which this Seifert fibration corresponds to the graph $\{(x, f(x)) \mid x \in S^2\}$ satisfying $\hat{\eta}_2 f = f \hat{\eta}_1$. If $\hat{\eta}_2$ is the identity, then $n = 1$ and $\gcd(m, n-1) = m$ so $\gcd(m, n+1)$ divides $\gcd(m, n-1)$. If $\hat{\eta}_2$ is not the identity, recall that the order of $\hat{\eta}_1$ is $\dfrac{m}{\gcd(m, n+1)}$ and the order of $\hat{\eta}_2$ is $\dfrac{m}{\gcd(m, n-1)}$. Therefore since $\hat{\eta}_2 f = f \hat{\eta}_1$ we have that $\hat{\eta}_2 f \hat{\eta}_1^{\frac{m}{\gcd(m,n+1)}-1} = f$ and thus $\hat{\eta}_2^{\frac{m}{\gcd(m,n+1)}} f = f$. Since f is not a constant map and the only fixed points of $\hat{\eta}_2$ are $\pm i$, this means that $\dfrac{m}{\gcd(m, n-1)}$ divides $\dfrac{m}{\gcd(m, n+1)}$ and so $\gcd(m, n+1)$ divides $\gcd(m, n-1)$.

A similar argument shows that in the case in which the Seifert fibration corresponds to the graph $\{(f(x),x) \mid x \in S^2\}$ satisfying $\hat{\eta}_1 f = f\hat{\eta}_2$, we get that $\gcd(m,n-1)$ divides $\gcd(m,n+1)$.

($\Longleftarrow$) Now suppose that $\gcd(m,n+1)$ divides $\gcd(m,n-1)$. We will give an uncountable family of geodesic Seifert fibration of $L(m,n)$ that are not Hopf fibrations. Define l so that $l(n+1) = (n-1) \mod m$. Since $\gcd(m,n+1)$ divides $\gcd(m,n-1)$, such an l exists. If $l = 0$ then define $f : S^2 \to S^2$ so that $f = F_3 \circ F_2 \circ F_1$ where $F_1(\cos\phi\, i + \sin\phi\, e^{i\theta} j) = |\cos\phi|\, i + \sin\phi\, e^{i\theta} j$, $F_2(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = \cos\phi\ i + \sin\phi\ j$, and $F_3(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = \cos t\phi\, i + \sin t\phi\, e^{i\theta} j$ for $t \in (0,1)$. Since F_1 and F_2 do not increase distances and F_3 is a distance decreasing function, f is a distance decreasing function.

If $l \neq 0$ consider the function $f : S^2 \to S^2$ defined by $f = F_2^{-1} \circ F_3 \circ F_2 \circ F_1$ where $F_1 : S^2 \to S^2$ is defined by $F_1(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = |\cos\phi|\ i + \sin\phi\, e^{i\theta} j$, $F_2 : S^2 \to \hat{\mathbb{C}}$ is the stereographic projection that carries the positive i hemisphere to the unit disk in $\hat{\mathbb{C}}$ given by $F_2(\cos\phi\, i + \sin\phi\, e^{i\theta} j) = \dfrac{\sin\phi\ e^{i\theta}}{1+\cos\phi}$ with inverse $F_2^{-1}(z) = \dfrac{1-|z|^2}{1+|z|^2} i + \dfrac{2z}{1+|z|^2} j$, and $F_3 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined by $F_3(z) = tz^l$ where $t < \frac{1}{2l}$.

This function f is distance decreasing since F_1 does not increase distances and $F_2^{-1} \circ F_3 \circ F_2$ is a distance decreasing function on the positive i hemisphere of S^2. We can see this by considering the spherical metric on $\hat{\mathbb{C}}$ defined by $d(z,w) = \dfrac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$. Then $d(z,w) > d(tz^l, tw^l)$ if and only if

$$\frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}} > \frac{2t|z^l - w^l|}{\sqrt{(1+t^2|z^l|^2)(1+t^2|w^l|^2)}}$$

which is equivalent to

$$t < \frac{|z-w|}{|z^l - w^l|} \cdot \frac{\sqrt{(1+t^2|z^l|^2)(1+t^2|w^l|^2)}}{\sqrt{(1+|z|^2)(1+|w|^2)}} \tag{5.1}$$

Note that $\left|\frac{z-w}{z^l-w^l}\right| = \left|\frac{1}{z^{l-1}+wz^{l-2}+...+w^{l-2}z+w^{l-1}}\right| \geq \frac{1}{l}$ for $|z| \leq 1$ and $|w| \leq 1$. Also $\frac{\sqrt{(1+t^2|z^l|^2)(1+t^2|w^l|^2)}}{\sqrt{(1+|z|^2)(1+|w|^2)}} > \frac{1}{2}$. Therefore for $t < \frac{1}{2l}$ equation 5.1 is satisfied and $F_2^{-1} \circ F_3 \circ F_2$ is a distance decreasing function on the positive i hemisphere of S^2.

In either case ($l = 0$ or $l \neq 0$) we can directly check using the formulas for $f, \hat{\eta}_1$, and $\hat{\eta}_2$ that $\hat{\eta}_2 f = f\hat{\eta}_1$. Since f is distance decreasing and $\hat{\eta}_2 f = f\hat{\eta}_1$ in both cases, the function f corresponds to a geodesic Seifert fibration of $L(m,n)$ by Theorem 5.1. This is not a Hopf fibration since f is not a constant map. For different values of t we obtain non-equivalent geodesic Seifert fibrations since no isometry of $L(m,n)$ will carry the fibers of one to the fibers of another. We can see this because there is no isometry of S^2 that will carry the image of f with one value of t to the image of f with a different value of t.

If $\gcd(m, n-1)$ divides $\gcd(m, n+1)$ then let l be defined so that $l(n-1) = (n+1) \mod m$. Define f as above and note that $\hat{\eta}_1 f = f\hat{\eta}_2$ so by Theorem 5.1 the function f corresponds to a geodesic Seifert fibration of $L(m,n)$. $\square$

Theorem 5.5 can further be refined to give the following corollary.

Corollary 5.6. *The lens space $L(m,n)$ admits uncountably many geodesic Seifert fibration that are not Hopf fibrations if and only if either* $\gcd(m, n+1) = 1$ *or* 2*, or* $\gcd(m, n-1) = 1$ *or* 2.

Proof. ($\Longleftarrow$) Suppose that $\gcd(m, n+1) = 1$, then $\gcd(m, n+1)$ divides $\gcd(m, n-1)$. Suppose that $\gcd(m, n+1) = 2$. Then m and $n+1$ are even. So $n-1$ is also even. Thus $\gcd(m, n-1)$ is even and $\gcd(m, n+1)$ divides $\gcd(m, n-1)$. In either case, by Theorem 5.5 the lens space $L(m, n)$ admits a geodesic Seifert fibration that is not a Hopf fibration. A similar argument holds for $\gcd(m, n-1) = 1$ or 2.

($\Longrightarrow$) Suppose that the lens space $L(m, n)$ admits a geodesic Seifert fibration that is not a Hopf fibration. By Theorem 5.5 this means that either $\gcd(m, n+1)$ divides $\gcd(m, n-1)$ or $\gcd(m, n-1)$ divides $\gcd(m, n+1)$. Consider the case in which $\gcd(m, n+1)$ divides $\gcd(m, n-1)$. Note that if n is even then $\gcd(n-1, n+1) = 1$, and if n is odd then $\gcd(n-1, n+1) = 2$. Since $\gcd(m, n+1)$ divides $\gcd(m, n-1)$ and $\gcd(m, n-1)$ divides $(n-1)$, then $\gcd(m, n+1)$ divides $(n-1)$. Now since $\gcd(m, n+1)$ divides $(n-1)$ and $\gcd(m, n+1)$ divides $(n+1)$, we must have that either $\gcd(m, n+1) = 1$ or 2. Similarly, if $\gcd(m, n-1)$ divides $\gcd(m, n+1)$ then $\gcd(m, n-1) = 1$ or 2. $\square$

We have now classified all geodesic Seifert fibrations of any lens space and considered examples of such fibrations in the process. In the following chapter we turn our attention to the non-cyclic elliptic 3-manifolds.

Chapter 6

The non-cyclic elliptic 3-manifolds

There are three remaining types of elliptic 3-manifolds: product type, tetrahedral type, and dihedral type. In this chapter we will discuss these remaining types and draw conclusions about the geodesic Seifert fibrations of each type.

Let $\psi : S^3 \to SO(3)$ be defined as in Section 2.5 so that $\psi(q)(x) = qxq^{-1}$ for each $x \in S^2$. The kernel of ψ is $\{\pm 1\}$. If G is a finite subgroup of $SO(3)$, then let $\tilde{G}$ denote $\psi^{-1}(G) \subset S^3$. There are five types of finite subgroups of $SO(3)$ up to conjugation: $\mathbb{Z}_n$ (cyclic of order n), D_n (dihedral of order $2n$ where $n > 1$), $T \cong A_4$ (tetrahedral), $O \cong S_4$ (octahedral), and $I \cong A_5$ (icosahedral). As shown in [2], we specifically define the corresponding subgroups of S^3 in the following way:

$$\begin{aligned}
\tilde{\mathbb{Z}}_n &= \langle e^{\pi i/n} \rangle \\
\tilde{D}_n &= \langle e^{\pi i/n}, j \rangle \\
\tilde{T} &= \langle i, w \rangle \\
\tilde{O} &= \left\langle \frac{j+k}{\sqrt{2}}, w \right\rangle \\
\tilde{I} &= \left\langle \frac{i+\sigma j+\tau k}{2}, w \right\rangle
\end{aligned}$$

where $w = \frac{-1+i+j+k}{2}$, $\sigma = \frac{\sqrt{5}-1}{2}$, and $\tau = \frac{\sqrt{5}+1}{2}$.

6.1 Definition of non-cyclic types

We say that an elliptic 3-manifold M has *product type* if M is isometric to $M' = S^3/\rho(H)$, where

$$H = \tilde{\mathbb{Z}}_m \times \tilde{H}_2 \tag{6.1}$$

for H_2 one of D_n for $n > 1$, T, O, or I, and $\gcd(m, |H_2|) = 1$. Notice that $\hat{\rho}(H)$ acts on $S^2 \times S^2$ independently in each of the S^2 factors, unlike in the case of the lens space. More specifically, $\hat{\rho}(H) = \mathbb{Z}_m \times H_2$.

In order to define the *tetrahedral type*, let m be an odd integer, and define epimorphisms $f_1 : \tilde{\mathbb{Z}}_{3m} \to \mathbb{Z}_3$ and $f_2 : \tilde{T} \to \mathbb{Z}_3$. Note that $\ker f_1 = \langle e^{i\pi/m} \rangle$, $f_1^{-1}(e^{2i\pi/3}) = e^{i\pi/3m} \ker f_1$, and $f_1^{-1}(e^{4i\pi/3}) = e^{2i\pi/3m} \ker f_1$ (or vice versa). Also $\ker f_2 = \{\pm 1, \pm i, \pm j, \pm k\}$, $f_2^{-1}(e^{2i\pi/3}) = w \ker f_2$, and $f_2^{-1}(e^{4i\pi/3}) = w^2 \ker f_2$ (or vice versa).

There are two diagonal subgroups of $\tilde{\mathbb{Z}}_{3m} \times \tilde{T}$ with index three:

$$H = \{(h_1, h_2) \in \tilde{\mathbb{Z}}_{3m} \times \tilde{T} \mid f_1(h_1) = f_2(h_2)\} \tag{6.2}$$

and

$$H' = \{(h_1, h_2) \in \tilde{\mathbb{Z}}_{3m} \times \tilde{T} \mid f_1(h_1) = f_2(h_2^{-1})\}.$$

These subgroups are conjugate, however, so we will only consider H. We say that an elliptic 3-manifold M has *tetrahedral type* if it is isometric to $M' = S^3/\rho(H)$.

Now for the *dihedral type*, let $\gcd(n, 2m) = 1$, and consider the unique epimorphisms $g_1 : \tilde{\mathbb{Z}}_{2m} \to \mathbb{Z}_2$ and $g_2 : \tilde{D}_n \to \mathbb{Z}_2$. Note that $\ker g_1 = \langle e^{i\pi/m} \rangle$ and $g_1^{-1}(-1) = e^{i\pi/2m} \ker g_1$. Also $\ker g_2 = \langle e^{i\pi/n} \rangle$ and $g_2^{-1}(-1) = j \ker g_2$. Let

$$H = \{(h_1, h_2) \in \tilde{\mathbb{Z}}_{2m} \times \tilde{D}_n \mid g_1(h_1) = g_2(h_2)\} \tag{6.3}$$

which is the unique subgroup of $\tilde{\mathbb{Z}}_{2m} \times \tilde{D}_n$ of index 2. We say that an elliptic 3-manifold M has *dihedral type* if it is isometric to $M' = S^3/\rho(H)$.

By Theorem 4.9 there is a one-to-one correspondence between oriented geodesic Seifert fibrations of an elliptic 3-manifold $M = S^3/\Gamma$ and graphs of distance decreasing functions $f : S^2 \to S^2$ preserved by $\hat{\Gamma}$. Thus in order to classify all geodesic Seifert fibrations of M, we will consider graphs of distance decreasing functions $f : S^2 \to S^2$ preserved by $\hat{\Gamma}$. First we will show in the following lemma that, in fact, for any non-cyclic elliptic 3-manifold isometric to $M = S^3/\rho(H)$ for H defined as in equation 6.1, 6.2, or 6.3, there are no graphs of the form $\{(x, f(x)) \mid x \in S^2\}$ satisfying the condition given in part (1) of Lemma 4.10, and therefore there are no corresponding oriented geodesic Seifert fibrations of M.

Lemma 6.1. *For $M = S^3/\rho(H)$ where H is defined as in equation 6.1, 6.2, or 6.3, there is no oriented geodesic Seifert fibration of M corresponding to a graph of the form $\{(x, f(x)) \mid x \in S^2\}$ where $f : S^2 \to S^2$ is a distance decreasing function.*

Proof. Consider each type of manifold separately. First let $M = S^3/\rho(\tilde{\mathbb{Z}}_m \times \tilde{H}_2)$ be a product type manifold. In order for the graph $\{(x, f(x)) \mid x \in S^2\}$ to correspond to an oriented geodesic Seifert fibration of M, we must have $f \circ \psi(q_1) = \psi(q_2) \circ f$ for all $q_1 \in \tilde{\mathbb{Z}}_m$ and $q_2 \in \tilde{H}_2$ by Lemma 4.10. Let $q_1 = 1$, and q_2 vary throughout $\tilde{H}_2$. Then $\psi(q_2)(f(x)) = f(x)$ for all $q_2 \in \tilde{H}_2$. But $\psi(\tilde{H}_2) = H_2$ does not have a global fixed point. Therefore no graphs of the form $\{(x, f(x)) \mid x \in S^2\}$ correspond to oriented geodesic Seifert fibrations of M.

Now suppose that $M = S^3/\rho(H)$ has tetrahedral type. The graph $\{(x, f(x)) \mid x \in S^2\}$ will be preserved by $\hat{\rho}(H)$ if and only if $\psi(h_2) \circ f = f \circ \psi(h_1)$ for all $(h_1, h_2) \in H$ by Lemma 4.10. Let $h_1 = 1$ and let h_2 vary through $\ker f_2$. Then this condition gives that $\psi(h_2)(f(x)) = f(x)$ for all $h_2 \in \ker f_2$. However $\psi(\ker f_2)$ consists of all order 2 rotations of T and therefore has no global fixed points. Thus no graph of this form is left invariant by $\hat{\rho}(H)$.

Finally consider $M = S^3/\rho(H)$ of dihedral type. The graph $\{(x, f(x)) \mid x \in S^2\}$ will be preserved by $\hat{\rho}(H)$ if and only if $\psi(h_2) \circ f = f \circ \psi(h_1)$ for all $(h_1, h_2) \in H$ by Lemma 4.10. Let $h_1 = 1$ and let h_2 vary through $\ker g_2$. Then this condition gives that $\psi(h_2)(f(x)) = f(x)$ for all $h_2 \in \ker g_2$. Note that $\psi(\ker g_2)$ is cyclic of order n. The only global fixed points of $\psi(\ker g_2)$

are $\pm i \in S^2$. Thus we must have $f(x) = i$ or $f(x) = -i$. Now consider $h_1 \notin \ker g_1$ and $h_2 \notin \ker g_2$. If $f(x) = i$, then $\psi(h_2)(f(x)) = -i$, as elements not in $\psi(\ker g_2)$ send i to $-i$. Therefore, $\psi(h_2)(f(x)) \neq f(\psi(h_1)(x))$ for some $(h_1, h_2) \in H$. Thus this function does not correspond to a geodesic Seifert fibration of M. Similarly, we cannot have $f(x) = -i$. Therefore there are no oriented geodesic Seifert fibrations of M corresponding to graphs of the form $\{(x, f(x)) \mid x \in S^2\}$. $\square$

The following is a result that partially draws on Theorem 5.4 and will be further explored in the subsequent sections in Lemmas 6.5, 6.9, and 6.13.

Theorem 6.2. *Chiral lens spaces admit two non-equivalent oriented Hopf fibrations. Amphichiral lens spaces and non-cyclic elliptic 3-manifolds admit a unique oriented Hopf fibration up to equivalence.*

Let us now consider each type of non-cyclic elliptic 3-manifold separately.

6.2 Product type

Recall that a product type elliptic 3-manifold is defined to be isometric to some $M = S^3/\rho(H)$, for $H = \tilde{\mathbb{Z}}_m \times \tilde{H}_2$ where H_2 is one of D_n for $n > 1$, T, O, or I, and $\gcd(m, |H_2|) = 1$. In order to classify all geodesic Seifert fibrations of the product type manifold $M = S^3/\rho(H)$, we will consider graphs of the form $\{(f(x), x) \mid x \in S^2\}$, for $f : S^2 \to S^2$ distance decreasing, preserved by $\hat{\rho}(H) = \mathbb{Z}_m \times H_2$. In order for this graph to correspond to a geodesic Seifert fibration of M, we must have that $\psi(q_1) \circ f = f \circ \psi(q_2)$ for all $q_1 \in \tilde{\mathbb{Z}}_m, q_2 \in \tilde{H}_2$ by Lemma 4.10.

Theorem 6.3. *Let $M = S^3/\rho(H)$ be a non-symplectic product type elliptic 3-manifold. Then M admits a unique oriented geodesic Seifert fibration up to equivalence. This fibration is a Hopf fibration.*

Proof. Since M is not symplectic, this means that $m \neq 1$. Let q_1 vary throughout $\tilde{\mathbb{Z}}_m$ and $q_2 = 1$. So $q_1 = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then $\psi(q_1)(f(x)) = f(x)$ for all $q_1 \in \tilde{\mathbb{Z}}_m$ and $x \in S^2$. Since the only global fixed points of $\psi(\tilde{\mathbb{Z}}_m) = \mathbb{Z}_m$ are i and $-i$, we must have that $f(x) = i$ or $f(x) = -i$. For these constant functions, the condition $\psi(q_1) \circ f = f \circ \psi(q_2)$ for all $q_1 \in \tilde{\mathbb{Z}}_m, q_2 \in \tilde{H}_2$ is satisfied. Therefore the only oriented geodesic Seifert fibrations of M are the Hopf fibrations corresponding to $\{i\} \times S^2$ and $\{-i\} \times S^2$, which are actually the same fibration with oppositely oriented fibers. These oriented fibrations are in fact equivalent. The isometry $\rho(j, 1)$ is an isometry of M that sends the fibers of one of these fibrations to the fibers of the other. This is true because $\rho(j, 1)\rho(\tilde{\mathbb{Z}}_m \times \tilde{H}_2)\rho(j^{-1}, 1) = \rho(j\tilde{\mathbb{Z}}_m j^{-1} \times \tilde{H}_2) = \rho(\tilde{\mathbb{Z}}_m \times \tilde{H}_2)$. Also $\rho(j, 1)$ carries $\{i\} \times S^2$ to $\{-i\} \times S^2$, so these two oriented Hopf fibrations are equivalent. □

We will now consider the case in which M is a symplectic product type elliptic 3-manifold.

Theorem 6.4. *Let $M = S^3/\rho(H)$ be a symplectic product type elliptic 3-manifold. Then M admits uncountably many geodesic Seifert fibrations. There is a one-to-one correspondence between oriented geodesic Seifert fibrations of M and the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is a distance decreasing function that is constant on the orbits of H_2.*

Proof. Since M is symplectic, $M = S^3/\rho(\{\pm 1\} \times \tilde{H}_2)$, so $m = 1$. The group $\hat{\rho}(\{\pm 1\} \times \tilde{H}_2)$ acts on $S^2 \times S^2$ by $\{1\} \times H_2$. The graph $\{(f(x), x) \mid x \in S^2\}$

is left invariant by this action if and only if the function f is constant on the orbits of H_2. This gives many possibilities for geodesic Seifert fibrations of M. $\square$

Specifically this theorem tells us that the graph $\{(f(x), x) \mid x \in S^2\}$ of any constant map corresponds to an oriented geodesic Seifert fibration of M. The graph of a constant map corresponds to a Hopf fibration. For any $q \in S^3$, the isometry $\rho(q,1)$ is an isometry of the symplectic product type elliptic 3-manifold M since $\rho(q,1)\rho(\{\pm 1\} \times \tilde{H}_2)\rho(q^{-1},1) = \rho(\{\pm 1\} \times \tilde{H}_2)$. Therefore all of the Hopf fibrations of the form $\{w\} \times S^2$ are equivalent and there is a unique Hopf fibration up to equivalence of a symplectic product type elliptic 3-manifold.

Lemma 6.5. *Let $M = S^3/\rho(H)$ be a symplectic product type elliptic 3-manifold. Then M admits a unique oriented Hopf fibration up to equivalence.*

$\square$

The results in this section completely classify the geodesic Seifert fibrations of any product type elliptic 3-manifold. We have yet to look at the tetrahedral and dihedral type elliptic 3-manifolds, which are discussed in the following two sections.

6.3 Tetrahedral type

We have defined a tetrahedral type elliptic 3-manifold to be isometric to $M = S^3/\rho(H)$ where $H = \{(h_1, h_2) \in \tilde{\mathbb{Z}}_{3m} \times \tilde{T} \mid f_1(h_1) = f_2(h_2)\}$ for m odd and epimorphisms $f_1 : \tilde{\mathbb{Z}}_{3m} \to \mathbb{Z}_3$ and $f_2 : \tilde{T} \to \mathbb{Z}_3$. Recall that $\rho(H)$ acts on $S^2 \times S^2$ by a diagonal subgroup of $\mathbb{Z}_{3m} \times T$.

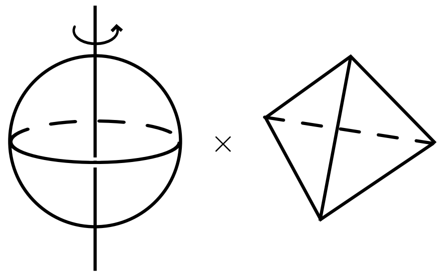

Figure 6.1: The action of $\mathbb{Z}_{3m} \times T$ on $S^2 \times S^2$

In order to classify all geodesic Seifert fibrations of the tetrahedral type manifold $M = S^3/\rho(H)$, we will consider graphs of distance decreasing functions $f : S^2 \to S^2$ of the form $\{(f(x), x) \mid x \in S^2\}$ preserved by $\hat{\rho}(H)$. In order for this graph to correspond to a geodesic Seifert fibration of M, we must have that $\psi(q_1) \circ f = f \circ \psi(q_2)$ for all $(h_1, h_2) \in H$ by Lemma 4.10.

Note that there does not exist a symplectic tetrahedral type elliptic 3-manifold. However, as the following lemma explores, there are tetrahedral type manifolds that admit special symplectic type coverings.

Lemma 6.6. *A tetrahedral type elliptic 3-manifold $M = S^3/\rho(H)$ has an index 3 symplectic type covering if and only if $m = 1$.*

Proof. ($\Longrightarrow$) Suppose that $m = 1$. Then consider the manifold $M' = S^3/\rho(\{\pm 1\} \times \ker f_2)$. Note that $\rho(\{\pm 1\} \times \ker f_2) < \rho(H)$ so M' is a cover of M. Also $|\rho(H)| = 24$ and $|\rho(\{\pm 1\} \times \ker f_2)| = 8$, so this is an index 3 covering.

($\Longrightarrow$) Suppose that a tetrahedral type elliptic 3-manifold $M = S^3/\rho(H)$ has an index 3 symplectic type covering M'. Since M' is of symplectic type, it must be that $M' = S^3/\Gamma'$ where $\Gamma' = \rho(H_1 \times \{\pm 1\})$ or $\Gamma' = \rho(\{\pm 1\} \times H_2)$ by Lemma 4.2. Also since it is an index 3 covering we must have that $\Gamma' < \rho(H)$

and $|\Gamma'| = 8m$ as $|\rho(H)| = 24m$. If $\Gamma' = \rho(H_1 \times \{\pm 1\})$ then this implies that $H_1 < \ker f_1 < \tilde{\mathbb{Z}}_{3m}$ and $|H_1| = 8m$, which is a contradiction since $|\tilde{\mathbb{Z}}_{3m}| = 6m$. Thus it must be that $\Gamma' = \rho(\{\pm 1\} \times H_2)$. Therefore $H_2 < \ker f_2$ and $|H_2| = 8m$. Since $|\ker f_2| = 8$, we have $m = 1$ and $\Gamma' = \rho(\{\pm 1\} \times \ker f_2)$. □

Theorem 6.7. *Let $M = S^3/\rho(H)$ be a tetrahedral type elliptic 3-manifold with $m \neq 1$. Then M admits a unique oriented geodesic Seifert fibration. This fibration is a Hopf fibration.*

Proof. Let $h_2 = 1$ and let h_1 vary through $\ker f_1$. We must have that $\psi(h_1)(f(x)) = f(x)$ for all $h_1 \in \ker f_1$. Since $m \neq 1$, the only fixed points of $\psi(\ker f_1)$ are $\pm i$. The only functions f that will satisfy this condition are $f(x) = i$ and $f(x) = -i$. In fact, these functions f satisfy the condition $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. Thus the only geodesic Seifert fibrations of M correspond to $\{i\} \times S^2$ and $\{-i\} \times S^2$. These give the same Hopf fibration of M with oppositely oriented fibers.

The isometry $\rho(j, e^{\pi i/4})$ is an isometry of M that sends the fibers of one of these oriented fibrations to the fibers of the other. Note that $\rho(j, e^{\pi i/4})\rho(H)\rho(j^{-1}, e^{-\pi i/4}) = \rho(\{(jh_1j^{-1}, e^{\pi i/4}h_2e^{-\pi i/4}) \mid (h_1, h_2) \in \tilde{\mathbb{Z}}_{3m} \times \tilde{T}, f_1(h_1) = f_2(h_2)\})$. We know that $e^{\pi i/4}$ is in the normalizer of $\tilde{T} < S^3$ since $\tilde{T} = \langle i, w \rangle$ for $w = \frac{-1+i+j+k}{2}$, and $e^{\pi i/4}ie^{-\pi i/4} = i$ and $e^{\pi i/4}we^{-\pi i/4} = -w^2 i$. Also $f_1(jh_1j^{-1}) = f_1(h_1^{-1}) = [f_1(h_1)]^{-1}$ and $f_2(e^{\pi i/4}h_2e^{-\pi i/4}) = [f_2(h_2)]^{-1}$. Therefore $\rho(j, e^{\pi i/4})\rho(H)\rho(j^{-1}, e^{-\pi i/4}) = \rho(H)$, so $\rho(j, e^{\pi i/4})$ is an isometry of M. Its action on $S^2 \times S^2$ sends $\{i\} \times S^2$ to $\{-i\} \times S^2$. So these two oriented fibrations are equivalent. □

Theorem 6.8. *Let $M = S^3/\rho(H)$ be a tetrahedral type elliptic 3-manifold with $m = 1$. Then M admits uncountably many non-equivalent geodesic*

Seifert fibrations. There is a one-to-one correspondence between oriented geodesic Seifert fibrations of M and the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is a distance decreasing function that satisfies $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$.

Proof. By Lemmas 4.10 and 6.1, the oriented geodesic Seifert fibrations of M are in one-to-one correspondence with the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is distance decreasing and $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$.

The following gives an infinite family of such functions f. Define $F_1 : S^2 \to S^2$ by $F_1(ai + bj + ck) = |a|i + |b|j + |c|k$. Let $F_2 : S^2 \to S^2$ be a rotation that carries $\frac{i+j+k}{\sqrt{3}}$ to i, and let $F_3 : S^2 \to S^2$ be defined by $F_3(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = \cos t\phi\ i + \sin t\phi\ e^{i\theta} j$ for $t \in (0, 1)$. Now let $f = F_3 \circ F_2 \circ F_1$. The function f is a distance decreasing function since F_1 does not increase distances, F_2 is an isometry of S^2, and F_3 is a distance decreasing function.

Now let us check that f satisfies the condition $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. For $(h_1, h_2) \in \ker f_1 \times \ker f_2$, we need $F_3 \circ F_2 \circ F_1 = F_3 \circ F_2 \circ F_1 \circ \psi(h_2)$ for all $h_2 \in \ker f_2 = \{\pm 1, \pm i, \pm j, \pm k\}$. Thus we must have that $F_1 = F_1 \circ \psi(h_2)$. Since $F_1 \circ \psi(h_2)(ai+bj+ck) = F_1(h_2(ai+bj+ck)h_2^{-1}) = F_1(\pm ai \pm bj \pm ck) = |a|i + |b|j + |c|k$, we know that $F_1 = F_1 \circ \psi(h_2)$ for all $h_2 \in \ker f_2$.

Now consider $(h_1, h_2) \in f_1^{-1}(e^{2\pi i/3}) \times f_2^{-1}(e^{2\pi i/3})$, where $f_1^{-1}(e^{2\pi i/3}) = \pm e^{\pi i/3}$ and $f_2^{-1}(e^{2\pi i/3}) = w^2 \ker f_2$ for $w = \frac{-1+i+j+k}{2}$. Note that $\psi(w^2) = F_2^{-1} \circ F_3^{-1} \circ \psi(h_1) \circ F_3 \circ F_2$ on the image of F_1 for $h_1 = \pm e^{\pi i/3}$. Thus we must have that $\psi(w^2) \circ F_1 = F_1 \circ \psi(w^2 u)$ for $u \in \ker f_2$. This is true since

$\psi(w^2) \circ F_1(ai + bj + ck) = w^2(|a|i + |b|j + |c|k)w = |a|j + |b|k + |c|i$, and $F_1 \circ \psi(w^2 u)(ai + bj + ck) = F_1(\pm aj \pm bk \pm ci) = |a|j + |b|k + |c|i$. A similar argument holds for $(h_1, h_2) \in f_1^{-1}(e^{4\pi i/3}) \times f_2^{-1}(e^{4\pi i/3})$.

Therefore f is distance decreasing and $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. So the graph $\{(f(x), x) \mid x \in S^2\}$ corresponds to an oriented geodesic Seifert fibration of M. For different values of $t \in (0, 1)$ and different rotations F_2, we get non-equivalent oriented geodesic Seifert fibrations of M since no isometry of M will carry the fibers of one to the fibers of another. We can see this because there is no isometry of S^2 that will carry the image of f with one value of t to the image of f with a different value of t. Therefore M admits uncountably many non-equivalent geodesic Seifert fibrations. $\square$

Note that there are only two constant maps satisfying this condition. They are $f(x) = i$ and $f(x) = -i$. These will correspond to the same Hopf fibration with oppositely oriented fibers. The oriented Hopf fibrations corresponding to $\{i\} \times S^2$ and $\{-i\} \times S^2$ are equivalent by a similar argument to that found at the end of the proof of Theorem 6.7.

Lemma 6.9. *Let $M = S^3/\rho(H)$ be a tetrahedral type elliptic 3-manifold with $m = 1$. Then M admits a unique oriented Hopf fibration up to equivalence.*

$\square$

This concludes our consideration of tetrahedral type elliptic 3-manifolds. We now turn our attention to the final type of elliptic 3-manifolds, the dihedral type.

6.4 Dihedral type

Recall that a dihedral type elliptic 3-manifold is defined to be isometric to some $M = S^3/\rho(H)$ where $H = \{(h_1, h_2) \in \tilde{\mathbb{Z}}_{2m} \times \tilde{D}_n \mid g_1(h_1) = g_2(h_2)\}$ for $\gcd(n, 2m) = 1$ and epimorphisms $g_1 : \tilde{\mathbb{Z}}_{2m} \to \mathbb{Z}_2$ and $g_2 : \tilde{D}_n \to \mathbb{Z}_2$. Recall that $\rho(H)$ acts on $S^2 \times S^2$ by a diagonal subgroup of $\mathbb{Z}_{2m} \times D_n$.

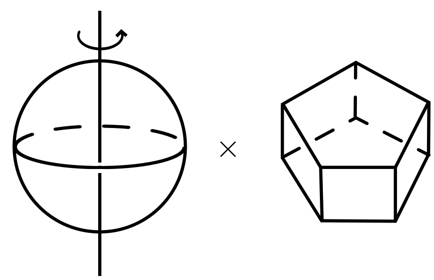

Figure 6.2: The action of $\mathbb{Z}_{2m} \times D_n$ on $S^2 \times S^2$

By Theorem 4.9, there is a one-to-one correspondence between geodesic Seifert fibrations of M and graphs of distance decreasing functions $f : S^2 \to S^2$ preserved by $\hat{\rho}(H)$.

As in the tetrahedral case, there does not exist a symplectic dihedral type elliptic 3-manifold. However, there are dihedral type manifolds that have a special relation to symplectic type manifolds.

Lemma 6.10. *A dihedral type elliptic 3-manifold $M = S^3/\rho(H)$ has an index 2 symplectic covering if and only if $m = 1$.*

Proof. ($\Longrightarrow$) Suppose that $m = 1$. Then consider the manifold $M' = S^3/\rho(\{\pm 1\} \times \ker g_2)$. Note that $\rho(\{\pm 1\} \times \ker g_2) < \rho(H)$ so M' is a cover of M. Also $|\rho(H)| = 4n$ and $|\rho(\{\pm 1\} \times \ker g_2)| = 2n$, so this is an index 2 covering.

($\Longrightarrow$) Suppose that a dihedral type elliptic 3-manifold $M = S^3/\rho(H)$ has an index 2 symplectic type covering M'. Since M' is of symplectic type, it must be that $M' = S^3/\Gamma'$ where $\Gamma' = \rho(H_1 \times \{\pm 1\})$ or $\Gamma' = \rho(\{\pm 1\} \times H_2)$ by Lemma 4.2. Also since it is an index 2 covering we must have that $\Gamma' < \rho(H)$ and $|\Gamma'| = 2nm$ as $|\rho(H)| = 4nm$. If $\Gamma' = \rho(H_1 \times \{\pm 1\})$ then this implies that $H_1 < \ker g_1 < \tilde{\mathbb{Z}}_{2m}$ and $|H_1| = 2nm$, which is a contradiction since $|\tilde{\mathbb{Z}}_{2m}| = 4m$ and $n \notin \{1, 2\}$. Thus it must be that $\Gamma' = \rho(\{\pm 1\} \times H_2)$. Therefore $H_2 < \ker g_2$ and $|H_2| = 2nm$. Since $|\ker g_2| = 2n$, we have $m = 1$ and $\Gamma' = \rho(\{\pm 1\} \times \ker g_2)$. $\square$

Theorem 6.11. *Let $M = S^3/\rho(H)$ be a dihedral type elliptic 3-manifold with $m \neq 1$. Then M admits a unique oriented geodesic Seifert fibration. This fibration is a Hopf fibration.*

Proof. Consider graphs of the form $\{(f(x), x) \mid x \in S^2\}$. This graph will be preserved by $\hat{\rho}(H)$ if and only if $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. Suppose that $m \neq 1$. Let $h_2 = 1$ and let h_1 vary through $\ker g_1$. We must have that $\psi(h_1)(f(x)) = f(x)$ for all $h_1 \in \ker f_1$. Since $m \neq 1$, the only fixed points of $\psi(\ker g_1)$ are $\pm i$. The only functions f that will satisfy this condition are $f(x) = i$ and $f(x) = -i$. In fact, these functions f satisfy the condition $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. Thus the only geodesic Seifert fibrations of M correspond to $\{i\} \times S^2$ and $\{-i\} \times S^2$. These give the same Hopf fibration of M with oppositely oriented fibers.

Note that $\rho(j, 1)$ is an isometry of M. This is true because $\rho(j,1)\rho(H)\rho(j^{-1},1) = \rho(\{(jh_1j^{-1}, h_2) \in \tilde{\mathbb{Z}}_{2m} \times \tilde{D}_n \mid g_1(h_1) = g_2(h_2)\}) = \rho(\{(h_1^{-1}, h_2) \in \tilde{\mathbb{Z}}_{2m} \times \tilde{D}_n \mid g_1(h_1) = g_2(h_2)\}) = \rho(H)$ since $g_1(h_1) = g_1(h_1^{-1})$. Therefore, since the action of $\rho(j, 1)$ on $S^2 \times S^2$ takes $\{i\} \times S^2$ to $\{-i\} \times S^2$, these two oriented Hopf fibrations are equivalent. $\square$

Theorem 6.12. *Let $M = S^3/\rho(H)$ be a dihedral type elliptic 3-manifold with $m = 1$. Then M admits uncountably many non-equivalent geodesic Seifert fibrations. There is a one-to-one correspondence between oriented geodesic Seifert fibrations of M and the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is a distance decreasing function that satisfies $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$.*

Proof. By Lemmas 4.10 and 6.1, the oriented geodesic Seifert fibrations of M are in one-to-one correspondence with the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is distance decreasing and $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$.

The following gives an infinite family of such functions f. Define the functions $F_1, F_2, F_3 : S^2 \to S^2$ by $F_1(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = \cos\phi\ i + \sin\phi\ j$, F_2 is a rotation that maps j to i, and $F_3(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = \cos t\phi\ i + \sin t\phi\ e^{i\theta} j$ for $t \in (0, 1)$. Let $f = F_3 \circ F_2 \circ F_1$. This is a distance decreasing function since F_1 does not increase distances, F_2 is an isometry, and F_3 is distance decreasing.

Now let us check that f satisfies the condition $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. For $(h_1, h_2) \in \ker g_1 \times \ker g_2$ this means $f = f \circ \psi(h_2)$ for all $h_2 \in \ker g_2 = \langle e^{i\pi/n} \rangle$. Since f is constant on the orbits of $\langle e^{i\pi/n} \rangle$, $f = f \circ \psi(h_2)$. Now for $(h_1, h_2) \in g_1^{-1}(-1) \times g_2^{-1}(-1) = \{\pm i\} \times j\langle e^{i\pi/n} \rangle$, note that $\psi(j) = F_2^{-1} \circ F_3^{-1} \circ \psi(i) \circ F_3 \circ F_2$ on the image of F_1. Thus we must have that $\psi(j) \circ F_1 = F_1 \circ \psi(je^{i\pi l/n})$ for $l \in \mathbb{Z}$. This is true since $\psi(j) \circ F_1(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = \psi(j)(\cos\phi\ i + \sin\phi\ j) = -\cos\phi\ i + \sin\phi\ j$, and $F_1 \circ \psi(je^{i\pi l/n})(\cos\phi\ i + \sin\phi\ e^{i\theta} j) = F_1(-\cos\phi\ i + \sin\phi\ e^{-i(\theta + 2\pi l/n)} j) =$

$-\cos\phi\, i + \sin\phi\, j$.

Therefore f is distance decreasing and $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$. So the graph $\{(f(x), x) \mid x \in S^2\}$ corresponds to an oriented geodesic Seifert fibration of M. For different values of $t \in (0, 1)$ and different rotations F_2, we get non-equivalent oriented geodesic Seifert fibrations of M since no isometry of M will carry the fibers of one to the fibers of another. We can see this because there is no isometry of S^2 that will carry the image of f with one value of t to the image of f with a different value of t. Therefore M admits uncountably many non-equivalent geodesic Seifert fibrations. $\square$

Note that there are only two constant maps satisfying this condition. They are $f(x) = i$ and $f(x) = -i$. So the only oriented Hopf fibrations of S^3 that project to $M = S^3/\rho(H)$ for $H = \{(h_1, h_2) \in \tilde{\mathbb{Z}}_2 \times \tilde{D}_n \mid g_1(h_1) = g_2(h_2)\}$ correspond to $\{i\} \times S^2$ and $\{-i\} \times S^2$. These are the same Hopf fibration with oppositely oriented fibers. By a similar argument to the one found at the end of the proof of Theorem 6.11, these two oriented Hopf fibrations are in fact equivalent. Therefore this type of manifold has a unique oriented Hopf fibration up to equivalence.

Lemma 6.13. *Let $M = S^3/\rho(H)$ be a dihedral type elliptic 3-manifold with $m = 1$. Then M admits a unique oriented Hopf fibration up to equivalence.*

$\square$

We have now classified all geodesic Seifert fibrations of the dihedral type elliptic 3-manifolds, and thus we have now considered each type of elliptic 3-manifold as classified by Hopf.

Chapter 7

Conclusion and further research

We have classified all geodesic Seifert fibrations of any type of elliptic 3-manifold up to isometry by considering the fibers of a Seifert fibration and their corresponding points in $S^2 \times S^2$. We have used the result by Gluck and Warner that every great circle fibration of S^3 corresponds to a submanifold of $S^2 \times S^2$ that is the graph of a distance decreasing function from one S^2 factor to the other, along with the fact that every geodesic Seifert fibration of an elliptic 3-manifold comes from a great circle fibration of S^3. By looking at which distance decreasing functions are preserved by $\hat{\Gamma}$, which is the action of Γ on $S^2 \times S^2$, we have been able to classify all geodesic fibrations of the elliptic 3-manifold $M = S^3/\Gamma$.

We have specifically looked at each of the four types of elliptic 3-manifolds as classified by Hopf: cyclic type (lens space), product type, tetrahedral type, and dihedral type. There is a one-to-one correspondence between oriented geodesic Seifert fibrations of the lens space $L(m, n)$ and graphs of distance decreasing functions $f : S^2 \to S^2$ satisfying either $\hat{\eta}_2 f = f \hat{\eta}_1$ for the graph $\{(x, f(x)) \mid x \in S^2\}$, or $\hat{\eta}_1 f = f \hat{\eta}_2$ for the graph $\{(f(x), x) \mid x \in S^2\}$. We have shown that chiral lens spaces admit two non-equivalent oriented Hopf fibrations, while amphichiral lens spaces and non-cyclic elliptic 3-manifolds admit a unique oriented Hopf fibration up to equivalence. The lens space $L(m, n)$ admits a geodesic Seifert fibration that is not a Hopf fibration if and only if one of $\gcd(m, n + 1)$ and $\gcd(m, n - 1)$ divides the other. In this case the lens space admits uncountably many non-equivalent geodesic Seifert fibrations that are not Hopf fibrations.

We have also shown that a non-symplectic product type elliptic 3-manifold admits a unique oriented geodesic Seifert fibration up to equivalence, and this is a Hopf fibration. A symplectic product type elliptic 3-manifold $M = S^3/\rho(H)$ (where H is defined as in equation 6.1) admits uncountably many geodesic Seifert fibrations. There is a one-to-one correspondence between oriented geodesic Seifert fibrations of M and the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is a distance decreasing function that is constant on the orbits of H_2. For tetrahedral and dihedral type elliptic 3-manifolds $M = S^3/\rho(H)$ (where H defined as in equations 6.2 or 6.3) if $m \neq 1$ then M admits a unique oriented geodesic Seifert fibration, and this fibration is a Hopf fibration. If $m = 1$ then M admits uncountably many geodesic Seifert fibrations. There is a one-to-one correspondence between oriented geodesic Seifert fibrations of M and the set of graphs $\{(f(x), x) \mid x \in S^2\}$ for which $f : S^2 \to S^2$ is a distance decreasing function that satisfies $\psi(h_1) \circ f = f \circ \psi(h_2)$ for all $(h_1, h_2) \in H$.

While we have classified all geodesic Seifert fibrations of the elliptic 3-manifolds, there are many interesting questions along these lines that remain open. In [3], Michael Gage classified the set of skew-Hopf fibrations of the

3-sphere. How can we describe the skew-Hopf fibrations of the other elliptic 3-manifolds? There are more properties of great circles of the 3-sphere that could be explored using the structure of $S^2 \times S^2$. In Section 2.11 we looked at the subset $C \times C$ of $S^2 \times S^2$ where C is a great circle of S^2. This corresponds to a decomposition of S^3 into great circles as pictured in Figure 2.11. For a general closed curve $\gamma \subset S^2$, what does $\gamma \times \gamma \subset S^2 \times S^2$ correspond to in S^3? In [4], Gluck and Warner proved that the space of all oriented great circle fibrations of the 3-sphere deformation retracts to the subspace of all Hopf fibrations and therefore has the homotopy type of a pair of disjoint spheres. Is there a similar result for the other elliptic 3-manifolds? Also what can we tell about the structure of a fibration by considering the image of its corresponding distance decreasing function from S^2 to S^2?

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