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BOCKSTEIN BASIS AND RESOLUTION THEOREMS IN EXTENSION THEORY

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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DEDICATION

to

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For

introducing me to topology

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ABSTRACT

Resolution refers to a map (a continuous function) between topological spaces, where the domain is in some way better than the range, and the fibers (point preimages) meet certain requirements. We will be interested in the relationship between covering dimension and cohomological dimension, so the resolution we obtain will be between a domain of finite covering dimension, and a range of finite cohomological dimension, with cell-like or G-acyclic fibers. Both domain and range will be compact metrizable spaces.

A useful tool in investigating dimension of spaces is extension of maps. An indispensable tool in cohomological dimension theory are results of M. F. Bockstein, usually referred to as Bockstein theory. Extending maps and Bockstein theory will be extensively used in this work, as well as the theory of inverse sequences and limits.

We will look at standard resolution theorems in extension theory by R. Edwards-J. Walsh, A. Dranishnikov and M. Levin. Also, we will mention how they generalize to the L. Rubin-P. Schapiro resolution theorem, and we will focus on the proof of the case that the Rubin-Schapiro proof did not cover, namely:

Theorem: *Let G be an abelian group with $P_G = \mathbb{P}$, where $P_G = \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in$ Bockstein Basis $\sigma(G)\}$. Let $n \in \mathbb{N}$, and let K be a connected* CW*-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leq k < n$. Then for every compact metrizable space X with $X\tau K$ (i.e., with K an absolute extensor for X), there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is cell-like, dim $Z \leq n$ and $Z\tau K$.*

Introduction

The main goal of this work will be to prove the following resolution theorem:

Theorem: *Let G be an abelian group with $P_G = \mathbb{P}$, where $P_G = \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in$ Bockstein Basis $\sigma(G)\}$. Let $n \in \mathbb{N}$ and let K be a connected* CW*-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leq k < n$. Then for every compact metrizable space X with $X\tau K$ (i.e., with K an absolute extensor for X), there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is cell-like, dim $Z \leq n$ and $Z\tau K$.*

Resolution refers to a map (a continuous function) between topological spaces where the domain is in some way better than the range, and the fibers (point preimages) meet certain requirements. The resolution we obtain will be between a domain of finite covering dimension and a range of finite cohomological dimension with cell-like fibers. Both the domain and range will be compact metrizable spaces.

The first two chapters of this work contain all the notions necessary to understand the statement of the main theorem. Chapter 1 includes definitions of covering dimension (dim) and cohomological dimension modulo an abelian group G ($\dim_G$). Both of these dimensions are characterized in terms of extending maps, so the notation for absolute extensors is also introduced. Furthermore, cell-like maps are defined, and since most resolution theorems require the maps to be G-acyclic, we define the notion of G-acyclicity as well. In addition, we define inverse sequences and inverse limits, the main tool in constructing the domains for the resolutions we build.

Chapter 2 is entirely dedicated to Bockstein Theory. During the 1950s, Meyer Feliksovich Bockstein developed an algorithm for the computation of cohomological dimension with respect to a given abelian group G by means of cohomological dimensions with coefficients taken from a countable family of abelian groups $\sigma(G)$. His definition of $\sigma(G)$ was also used by V. I. Kuz'minov ([Ku]), and later adapted by E. Dyer ([Dy]), and then by A. Dranishnikov ([Dr3]).

Thus there are three different definitions of a Bockstein basis $\sigma(G)$, which are not equivalent in general, but which are equivalent from the point of view of cohomological dimension. All three are listed in Chapter 2, together with the list of Bockstein inequalities.

In Chapter 3 we quote some standard resolution theorems in extension theory by R. Edwards-J. Walsh, A. Dranishnikov and M. Levin. Also, we mention how they generalize to the L. Rubin-P. Schapiro resolution theorem.

The Edwards-Walsh Resolution Theorem refers to integral cohomological dimension $\dim_{\mathbb{Z}}$ and cell-like maps:

Theorem 3.1 [Wa] *For every compact metrizable space X with $\dim_{\mathbb{Z}} X \leq n$, there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is cell-like, and $\dim Z \leq n$.*

The original motivation for this resolution theorem was the cell-like map dimension raising problem: can a surjective cell-like map of a finite dimensional space have range which is of higher dimension? According to the Edwards-Walsh Theorem, in order to show that a cell-like map can raise dimension, it is enough to find a compact metrizable space X with finite $\dim_{\mathbb{Z}} X$ and infinite $\dim X$. The solution to this problem is due to A. Dranishnikov [Dr1], 1988.

The Edwards-Walsh Theorem has been generalized to the class of arbitrary metrizable spaces by L. Rubin and P. Schapiro ([RS1]), and to the class of arbitrary compact Hausdorff spaces by S. Mardešić and L. Rubin ([MR]). A similar statement to the Edwards-Walsh Theorem was proved by A. Dranishnikov for the group $\mathbb{Z}/p$, where p is an arbitrary prime number and the map is $\mathbb{Z}/p$-acyclic (Theorem 3.2 in Chapter 3).

Later, A. Koyama and K. Yokoi ([KY1]) were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for the class of compact Hausdorff spaces. Dranishnikov proved a similar statement to the Edwards-Walsh theorem, for the group $\mathbb{Q}$ and $\mathbb{Q}$-acyclic maps ([Dr4]), but he could only obtain $\dim Z \leq n+1$, and if $n \geq 2$, then additionally $\dim_{\mathbb{Q}} Z \leq n$. This result was later improved by M. Levin (Theorem 3.3).

The obvious question was whether a theorem similar to Edwards-Walsh's could be stated for compact metrizable spaces and arbitrary abelian groups. In their work [KY2], Koyama and Yokoi made a substantial amount of progress in answering this question. Their method relied heavily on the existence of Edwards-Walsh complexes, which have been studied by J. Dydak and J. Walsh in [DW], and which had been applied originally, in a rudimentary form, in [Wa]. However, using a different approach from the one in [KY2], M. Levin has proved a very strong generalization (Theorem 3.4) for Theorems 3.1 and 3.2, concerning compact metrizable spaces and arbitrary abelian groups. This Theorem was generalized further by Leonard Rubin and Philip Schapiro [RS2] (Theorem 3.5), by replacing $\dim_G X \leq n$ by $X \tau K$, that is, replacing a $K(G,n)$ with a CW-complex K upon which the demands are less strict.

However, the proof of the Rubin-Schapiro Resolution Theorem does not cover all abelian groups; namely, the case when $P_G = \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in \sigma(G)\} = \mathbb{P}$ is not covered. In fact, the statement of this theorem will be true when $P_G = \mathbb{P}$, but in this case the statement can be improved. At the end of Chapter 3, we mention for the first time the statement of the Resolution Theorem covering the case $P_G = \mathbb{P}$, (Theorem 3.6).

Chapter 4 contains two important technical results needed for the proof of Theorem 3.6. A generalized version of Walsh's Lemma, Lemma 4.2, lists the properties needed in order to get a cell-like surjective map $\pi : Z \to X$ if we already know what Z is. The adapted version of Edwards' Theorem 4.4 tells us how to construct the bonding maps for the inverse sequences that we will need in Theorem 3.6.

Since the proof of the main result requires certain manipulations of inverse sequences of metric compacta, Chapter 5 will contain the needed results. Here we define, for a given compactum X, an inverse sequence (X_i, p_i^{i+1}) which is a representation of X, and which is stable and simplicially irreducible from index m, with associated sequence of stability (γ_i). We explain how to build $(\mathbf{K}_j, (\gamma_{(j),i}))_{j\in\mathbb{N}}$, i.e., a sequence of inverse sequences (with their stability sequences) that will participate in forming the inverse sequence $\mathbf{Z}$, whose limit Z will be the domain for our resolution map in Theorem 3.6.

Finally, Chapter 6 is entirely dedicated to the proof of Theorem 3.6.

Chapter 1

Preliminaries: $\dim$ and $\dim_G$, extension of maps, cell-likeness and G-acyclicity, K-modification, inverse sequences and limits

Let us start by introducing a notation for absolute extensors. Recall that a topological space Y is an *absolute extensor* for a topological space X if for any closed subset A of X and any map $f : A \to Y$, there is a continuous extension $F : X \to Y$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

The standard notation for this is $Y \in \mathrm{AE}(X)$, but we will be using the notation $X \tau Y$, which was introduced by Kuratowski in honor of Tietze. Note that if $X\tau Y$, then any closed subset $A \subset X$ inherits this property, i.e., $A\tau Y$.

Now we would like to define covering dimension for a topological space. First, we will define the notion of the order of a cover for a topological space, or, more generally, the notion of the order of a family of subsets of a set.

Let X be a set, and let $\mathcal{F}$ be a collection of subsets of X. For $x \in X$, we say that the *order of $\mathcal{F}$ at x* is the number of elements of $\mathcal{F}$ containing x, and we write it as $\operatorname{ord}_x \mathcal{F}$.

The order of the collection $\mathcal{F}$ is defined as $\operatorname{ord} \mathcal{F} := \sup\limits_{x\in X} \operatorname{ord}_x \mathcal{F}$.

Definition 1.1 *Let X be a topological space, and let $n \in \mathbb{Z}_{\geq -1}$. We write* $\dim X \leq -1$ *if $X = \emptyset$. If $X \neq \emptyset$, and $n \in \mathbb{Z}_{\geq 0}$, we write* $\dim X \leq n$ *if for each open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X which refines $\mathcal{U}$, and such that* $\operatorname{ord} \mathcal{V} \leq n+1$.

If there is no $n \in \mathbb{Z}_{\geq -1}$ *such that* $\dim X \leq n$*, we put* $\dim X = \infty$*. Otherwise, we define*

$$\dim X := \inf \{n \in \mathbb{Z}_{\geq -1} : \dim X \leq n\}.$$

We refer to $\dim X$ *as the* covering dimension of X*, or just* the dimension of X.

Proposition 1.2 $\dim X = -1$ *if and only if* $X = \emptyset$. $\Box$

Notice that if you form the *nerve* $\mathcal{N}(\mathcal{V})$ of the cover $\mathcal{V}$, that is, the simplicial complex whose vertices are all the nonempty elements of $\mathcal{V}$, and a finite subcollection $\mathcal{V}_0 \subset \mathcal{V}$ forms a simplex in $\mathcal{N}(\mathcal{V})$ if $\bigcap_{V \in \mathcal{V}_0} V \neq \emptyset$, then $\dim X \leq n \Leftrightarrow \operatorname{ord} \mathcal{V} \leq n+1$ means that the combinatorial dimension of $\mathcal{N}(\mathcal{V})$ is at most n.

We can characterize dim using extension of maps as follows:

Theorem 1.3 *For any nonempty paracompact Hausdorff space* X *and* $n \in \mathbb{Z}_{\geq 0}$*,* $\dim X \leq n \;\Leftrightarrow\; X \tau S^n$. $\Box$

Another important fact is:

Theorem 1.4 *For each metrizable space* X*, if* K *is a* CW*-complex such that* $X \tau K$ *and* $Y \subset X$*, then* $Y \tau K$. $\Box$

This subspace theorem is also true for the class of stratifiable spaces, which includes all metrizable spaces, and the proof can be found in [IR].

Therefore, if X is a metrizable space with $\dim X \leq n$, then for any $Y \subset X$ we have $\dim Y \leq n$.

Now we will define cohomological dimension modulo an abelian group.

Definition 1.5 *Let* G *be an abelian group,* X *a topological space, and* $n \in \mathbb{Z}_{\geq -1}$*. We define* the cohomological dimension of X modulo G*, (or* with respect to G*) to be* -1 *if and only if* $X = \emptyset$*. In that case we write* $\dim_G X = -1$.

If $X \neq \emptyset$, we consider two cases: if for each $n \in \mathbb{Z}_{\geq 0}$ there exists a closed subset $A \subset X$ such that the relative n-th Čech cohomology group $\check{H}^n(X, A; G) \neq 0$, then we define $\dim_G X = \infty$. Otherwise,

$$\dim_G X := \inf \{n \in \mathbb{Z}_{\geq 0} \ : \ \check{H}^k(X, A; G) = 0, \ \forall k \geq n + 1, \ \forall \ \textit{closed } A \subset X\}.$$

It can be shown that if $\check{H}^n(X, A; G) = 0$ for all closed $A \subset X$, then for all $i \in \mathbb{N}$, $\check{H}^{n+i}(X, A; G) = 0$ for all closed $A \subset X$. So we can say that $\dim_G X$ is the largest number n such that there is a closed subset $A \subset X$ with $\check{H}^n(X, A; G) \neq 0$.

In order to characterize $\dim_G$ using extension of maps, we need to introduce the notion of an Eilenberg-MacLane complex.

Definition 1.6 *Let G be an abelian group and $n \in \mathbb{N}$. An Eilenberg-MacLane complex of type (G, n), denoted by $K(G, n)$, is a connected* CW*-complex K having the property*

$$\pi_i(K) \cong \begin{cases} G & \textit{if } i = n \\ 0 & \textit{if } i \neq n. \end{cases}$$

Note that there is a $K(G, n)$ for any abelian group G and any $n \in \mathbb{N}$, and that any two $K(G, n)$'s are homotopy equivalent. An easy construction of a $K(\mathbb{Z}, n)$ can be found in [Ha], and we will be using the fact that $K(\mathbb{Z}, n)^{(n+1)} = K(\mathbb{Z}, n)^{(n)}$.

We also need Eilenberg-MacLane complexes because of the following:

Theorem 1.7 *For each nonempty paracompact Hausdorff space X, abelian group G and $n \in \mathbb{Z}_{\geq 0}$, $\dim_G X \leq n \ \Leftrightarrow \ X \ \tau \ K(G, n)$.* $\square$

As above, using Theorem 1.4, if X is a metrizable space with $\dim_G X \leq n$, then for any $Y \subset X$ we have $\dim_G Y \leq n$.

The general correspondence between dim and $\dim_G$ for a compact metrizable space X, is:

$$\dim_G X \leq \dim_{\mathbb{Z}} X \leq \dim X.$$

Theorem 1.8 (P. S. Aleksandrov) [Al] *If X is a compact metrizable space with $\dim X < \infty$, then $\dim_{\mathbb{Z}} X = \dim X$.* $\square$

The question of the existence of an example of a compact metrizable space with infinite dimension dim and finite cohomological dimension $\dim_{\mathbb{Z}}$ was known as Aleksandrov's problem, and remained open until 1988. The example from 1988 is due to A. Dranishnikov [Dr1] – he has shown how to construct a compact metrizable space X with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X \leq 3$.

Now, let us give a name to maps whose fibers have special properties.

Definition 1.9 *A map $\pi : Z \to X$ between topological spaces is called* cell-like *if it is proper and each of its fibers (point preimages) $\pi^{-1}(x)$ has the shape of a point, or, equivalently, for each $x \in X$ there is an inverse sequence of compact metrizable spaces (Z_i, p_i^{i+1}) whose inverse limit is $\pi^{-1}(x)$, and whose bonding maps p_i^{i+1} are nullhomotopic.*

The second property from the definition of a cell-like map is equivalent to saying that the fibers of the map π are cell-like sets, that is, for some $n \in \mathbb{N}$, $\pi^{-1}(x)$ can be embedded into $\mathbb{R}^n$ as an intersection of countably many nested n-cells. Yet another equivalent statement: for any CW-complex K and for any $x \in X$, every map $f : \pi^{-1}(x) \to K$ is nullhomotopic.

Definition 1.10 *A map $\pi : Z \to X$ between topological spaces is called G-*acyclic *if all its fibers $\pi^{-1}(x)$ have trivial reduced Čech cohomology with respect to the group G, or, equivalently, every map $f : \pi^{-1}(x) \to K(G, n)$ is nullhomotopic.*

The map $\pi : Z \to X$ being cell-like implies that π is also G-acyclic. The notion of G-acyclicity of a map may be generalized as follows:

Definition 1.11 *For a given* CW*-complex K, a map $\pi : Z \to X$ between topological spaces is called K-*acyclic *if every map $f : \pi^{-1}(x) \to K$ is nullhomotopic.*

In the following chapters we will need a way to measure closeness of maps that land in a simplicial complex.

Definition 1.12 *Let K be a simplicial complex, X a space, and $f : X \to |K|$ a map. A map $g : X \to |K|$ is called a K-*modification *of f if whenever $x \in X$ and $f(x) \in \sigma$, for some $\sigma \in K$, then $g(x) \in \sigma$. This is equivalent to the following: whenever $x \in X$ and $f(x) \in \mathring{\sigma}$, for some $\sigma \in K$, then $g(x) \in \sigma$.*

Therefore, if L is a simplicial complex, $X = |L|$, and $g : |L| \to |K|$ is a simplicial approximation to f, then g is a K-modification of f.

Other useful notions will be inverse sequences and inverse limits.

Definition 1.13 *An* inverse sequence $\mathbf{X} = (X_i, p_i^{i+1})$ *consists of countably many topological spaces X_i and maps $p_i^{i+1} : X_{i+1} \to X_i$, called* bonding maps*. The* inverse limit $\lim \mathbf{X}$ *is the subspace of the product space $\prod_{i=1}^{\infty} X_i$ defined by*

$$\lim\ \mathbf{X} := \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i \ :\ p_i^{i+1}(x_{i+1}) = x_i,\ \forall i \right\}.$$

The space $X = \lim\ (X_i, p_i^{i+1})$ inherits its topology from the product $\prod_{i=1}^{\infty} X_i$, and it can be shown that if every X_i is compact and metrizable, then so is X.

Theorem 1.14 *Every compact metrizable space can be represented as the inverse limit of an inverse sequence of compact polyhedra, with surjective and simplicial bonding maps.*

Chapter 2

Bockstein Basis, Bockstein Theorem, Bockstein Inequalities

As we have seen in Chapter 1, the cohomological dimension of a given compact metrizable space depends on the coefficient group. Any abelian group can be the coefficient group of a cohomology theory and there are uncountably many of them. It turns out that in the case of compact metrizable spaces, it suffices to consider only countably many groups. Solving Aleksandrov's problem ([Al]), M. F. Bockstein found an algorithm for computation of the cohomological dimension with respect to a given abelian group G by means of cohomological dimensions with coefficients taken from a countable family of abelian groups $\sigma(G)$. His definition of $\sigma(G)$ was also used by V. I. Kuz'minov ([Ku]), and later adapted by E. Dyer ([Dy]), and then by A. Dranishnikov ([Dr3]).

Thus there are three different definitions of a Bockstein basis $\sigma(G)$, which are not equivalent in general, but which are equivalent from the point of view of cohomological dimension.

Notation:

(1) $\mathbb{P}$ stands for the set of all prime numbers,

(2) $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} \ : \ n \text{ is not divisible by } p\}$ is called the *p-localization of the integers*, and

(3) $\mathbb{Z}/p^\infty = \{\frac{m}{n} \in \mathbb{Q}/\mathbb{Z} \ : \ n = p^k \text{ for some } k \geq 0\}$ is called *the quasi-cyclic p-group*.

Definition 2.1 *For an abelian group G, we say that* an element $g \in G$ is divisible by $n \in \mathbb{Z} \setminus \{0\}$ *if the equation $nx = g$ has a solution in G, G* is divisible by n *if* *all of its elements are divisible by* n*, and* G *is a* divisible group *if* G *is divisible by all* $n \in \mathbb{Z} \setminus \{0\}$.

Definition 2.2 *For an abelian group* G*,* $\operatorname{Tor} G$ *is the subgroup of all elements of* G *of finite order, and* p*–*$\operatorname{Tor} G$ *is the subgroup of all elements whose order is a power of* p*, that is,* $p\text{–}\operatorname{Tor} G = \{g \in G : p^k g = 0 \text{ for some } k \geq 1\}$.

Here are the three definitions of a Bockstein basis $\sigma(G)$:

(BI) Bockstein - Kuz'minov [Ku]: Let G be an abelian group, $G \neq 0$. Then $\sigma(G)$ is the subset of $\{\mathbb{Q}\} \cup \{\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}_{(p)} : p \in \mathbb{P}\}$ defined by:

$$
\begin{array}{llcl}
(\mathbf{1}) & \mathbb{Q} \in \sigma(G) & \Leftrightarrow & G \text{ contains an element of infinite order} \\
 & & \Leftrightarrow & G/\operatorname{Tor} G \neq 0 \\
(\mathbf{2}) & \mathbb{Z}_{(p)} \in \sigma(G) & \Leftrightarrow & G \text{ satisfies the following: } \exists g \in G \text{ such that } \forall k \in \mathbb{Z}_{\geq 0}, \\
 & & & p^k g \text{ is not divisible by } p^{k+1} \\
 & & \Leftrightarrow & G/\operatorname{Tor} G \text{ is not divisible by } p \\
(\mathbf{3}) & \mathbb{Z}/p \in \sigma(G) & \Leftrightarrow & G \text{ contains an element of order } p^k, \text{ for some } k \in \mathbb{N}, \\
 & & & \text{which is not divisible by } p \\
 & & \Leftrightarrow & p\text{–}\operatorname{Tor} G \text{ is not divisible by } p \\
(\mathbf{4}) & \mathbb{Z}/p^\infty \in \sigma(G) & \Leftrightarrow & G \text{ contains an element of order } p \\
 & & \Leftrightarrow & p\text{–}\operatorname{Tor} G \neq 0.
\end{array}
$$

The following definition was adapted from the original one by E. Dyer ([Dy]) by changing the property (4). It is also used in the papers by J. Dydak ([Dy1]) and A. Koyama and K. Yokoi ([KY1]).

(BII) Dyer: Let G be an abelian group, $G \neq 0$. Then $\sigma(G)$ is the subset of $\{\mathbb{Q}\} \cup \{\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}_{(p)} : p \in \mathbb{P}\}$ defined by:

(1)	$\mathbb{Q} \in \sigma(G)$	$\Leftrightarrow$	G contains an element of infinite order
		$\Leftrightarrow$	$G/\operatorname{Tor} G \neq 0$
(2)	$\mathbb{Z}_{(p)} \in \sigma(G)$	$\Leftrightarrow$	G satisfies the following: $\exists g \in G$ such that $\forall k \in \mathbb{Z}_{\geq 0}$, $p^k g$ is not divisible by p^{k+1}
		$\Leftrightarrow$	$G/\operatorname{Tor} G$ is not divisible by p
(3)	$\mathbb{Z}/p \in \sigma(G)$	$\Leftrightarrow$	G contains an element of order p^k, for some $k \in \mathbb{N}$, which is not divisible by p
		$\Leftrightarrow$	p–$\operatorname{Tor} G$ is not divisible by p
(4′)	$\mathbb{Z}/p^\infty \in \sigma(G)$	$\Leftrightarrow$	p–$\operatorname{Tor} G \neq 0$ and p–$\operatorname{Tor} G$ is divisible by p.

A. Dranishnikov ([Dr3]) introduced the third definition by changing property (1) in Dyer's definition. This definition is also used by M. Levin ([Le1]).

(BIII) Dranishnikov: Let G be an abelian group, $G \neq 0$. Then $\sigma(G)$ is the subset of $\{\mathbb{Q}\} \cup \{\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}_{(p)} : p \in \mathbb{P}\}$ defined by:

(1′)	$\mathbb{Q} \in \sigma(G)$	$\Leftrightarrow$	$G/\operatorname{Tor} G \neq 0$ and $G/\operatorname{Tor} G$ is divisible by all $p \in \mathbb{P}$
(2)	$\mathbb{Z}_{(p)} \in \sigma(G)$	$\Leftrightarrow$	G satisfies the following: $\exists g \in G$ such that $\forall k \in \mathbb{Z}_{\geq 0}$, $p^k g$ is not divisible by p^{k+1}
		$\Leftrightarrow$	$G/\operatorname{Tor} G$ is not divisible by p
(3)	$\mathbb{Z}/p \in \sigma(G)$	$\Leftrightarrow$	G contains an element of order p^k, for some $k \in \mathbb{N}$, which is not divisible by p
		$\Leftrightarrow$	p–$\operatorname{Tor} G$ is not divisible by p
(4′)	$\mathbb{Z}/p^\infty \in \sigma(G)$	$\Leftrightarrow$	p–$\operatorname{Tor} G \neq 0$ and p–$\operatorname{Tor} G$ is divisible by p.

Note that, according to Definitions BI and BII, $\sigma(\mathbb{Z}) = \{\mathbb{Q}\} \cup \{\mathbb{Z}_{(p)} : p \in \mathbb{P}\}$, while, according to Definition BIII, $\sigma(\mathbb{Z}) = \{\mathbb{Z}_{(p)} : p \in \mathbb{P}\}$. Also note that, according to Definition BI, $\sigma(\mathbb{Z}/p) = \{\mathbb{Z}/p, \mathbb{Z}/p^\infty\}$, while, according to BII and BIII, $\sigma(\mathbb{Z}/p) = \{\mathbb{Z}/p\}$.

But these definitions are going to be equivalent from the point of view of cohomological dimension because of the Bockstein Inequalities and the Bockstein Theorem.

Theorem 2.3 (Bockstein Inequalities) [Dr3]

For any compact metrizable space X the following inequalities hold:

(BI1) $\dim_{\mathbb{Z}/p^\infty} X \leq \dim_{\mathbb{Z}/p} X$,

(BI2) $\dim_{\mathbb{Z}/p} X \leq \dim_{\mathbb{Z}/p^\infty} X + 1$,

(BI3) $\dim_{\mathbb{Z}/p} X \leq \dim_{\mathbb{Z}_{(p)}} X$,

(BI4) $\dim_{\mathbb{Q}} X \leq \dim_{\mathbb{Z}_{(p)}} X$,

(BI5) $\dim_{\mathbb{Z}_{(p)}} X \leq \max\{\dim_{\mathbb{Q}} X, \dim_{\mathbb{Z}/p^\infty} X + 1\}$,

(BI6) $\dim_{\mathbb{Z}/p^\infty} X \leq \max\{\dim_{\mathbb{Q}} X, \dim_{\mathbb{Z}_{(p)}} X - 1\}$. $\square$

Theorem 2.4 (Bockstein Theorem) [Dy] *If G is an abelian group and X is a locally compact space, then*

$$\dim_G X = \sup_{H \in \sigma(G)} \dim_H X. \ \square$$

According to the Bockstein inequality (BI1), if $\mathbb{Z}/p$ and $\mathbb{Z}/p^\infty$ are in $\sigma(G)$ at the same time, the supremum $\sup\limits_{H \in \sigma(G)} \dim_H X$ will be affected by $\dim_{\mathbb{Z}/p} X$ only. So the definitions BI and BII are equivalent with respect to cohomological dimension.

Analogously, by the Bockstein inequality (BI4), if $\mathbb{Q}$ and $\mathbb{Z}_{(p)}$ are in $\sigma(G)$ at the same time, the supremum $\sup\limits_{H \in \sigma(G)} \dim_H X$ is affected by $\dim_{\mathbb{Z}_{(p)}} X$ only.

Thus the definitions BII and BIII are equivalent with respect to cohomological dimension.

Convention: We will be using BII as a definition for the Bockstein Basis $\sigma(G)$.

Now let $P_G := \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in \sigma(G)\}$.

Lemma 2.5 *If G is an abelian group such that $P_G = \mathbb{P}$, then for any compact metrizable space X, $\dim_G X = \dim_{\mathbb{Z}} X$.*

Proof: $P_G = \mathbb{P}$ means that for each $p \in \mathbb{P}$, $\mathbb{Z}_{(p)} \in \sigma(G)$. By the Bockstein Inequalities (BI4), (BI3) and (BI1), the supremum $\sup\limits_{H \in \sigma(G)} \dim_H X$ has to be achieved at $\sup\limits_{p \in \mathbb{P}} \dim_{\mathbb{Z}_{(p)}} X$. Since $\sigma(\mathbb{Z}) = \{\mathbb{Q}\} \cup \{\mathbb{Z}_{(p)} : p \in \mathbb{P}\}$, we get that

$$\sup_{H \in \sigma(G)} \dim_H X = \sup_{H \in \sigma(\mathbb{Z})} \dim_H X. \;\square$$

Chapter 3

Resolution Theorems: Edwards-Walsh, Dranishnikov, Levin, Rubin-Schapiro

The word resolution, as used here, refers to a map (a continuous function) between topological spaces, where the domain is in some way better than the range, and the fibers (point preimages) meet certain requirements. In particular, we are interested in the relationship between covering dimension and cohomological dimension, so the resolution we obtain will be between a domain of finite covering dimension, and a range of finite cohomological dimension, with cell-like or G-acyclic fibers. Both domain and range will be compact metrizable spaces.

Let us look at some examples of resolution theorems. Here is the cell-like resolution theorem, first stated by R. Edwards ([Ed]), and later proven by J. Walsh in [Wa]:

Theorem 3.1 (R. Edwards - J. Walsh, 1981) [Wa]*: For every compact metrizable space X with $\dim_{\mathbb{Z}} X \leq n$, there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is cell-like, and $\dim Z \leq n$.* $\square$

The original motivation for the Edwards-Walsh Resolution Theorem was the cell-like map dimension raising problem: can a surjective cell-like map of a finite dimensional space have range which is of higher dimension? This was important, for example, because by a theorem of R. Daverman ([Da]), if $f : \mathbb{R}^n \to X$ is cell-like and $\dim X < \infty$, then $X \times \mathbb{R}^2 \approx \mathbb{R}^{n+2}$. According to the Edwards-Walsh Theorem, in order to show that a cell-like map can raise dimension, it is enough to find a compact metrizable space X with finite $\dim_{\mathbb{Z}} X$ and infinite $\dim X$. As we have already mentioned, the solution to this problem, as well as to Aleksandrov's problem, is due to A. Dranishnikov [Dr1], 1988. An alternate proof of the Edwards-Walsh Theorem can be found in [ARS].

The Edwards-Walsh Theorem has been generalized to the class of arbitrary metrizable spaces by L. Rubin and P. Schapiro ([RS1]), and to the class of arbitrary compact Hausdorff spaces by S. Mardešić and L. Rubin ([MR]). A similar statement to the Edwards-Walsh Theorem was proved by A. Dranishnikov, for the group $\mathbb{Z}/p$, where p is an arbitrary prime number:

Theorem 3.2 (A. Dranishnikov, 1988) [Dr2]*: For every compact metrizable space X with* $\dim_{\mathbb{Z}/p} X \leq n$*, there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is $\mathbb{Z}/p$-acyclic, and* $\dim Z \leq n$*.* $\square$

Later, A. Koyama and K. Yokoi ([KY1]) were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for the class of compact Hausdorff spaces. Dranishnikov proved a similar statement to Theorem 3.2 for the group $\mathbb{Q}$ ([Dr4]), but he could only obtain $\dim Z \leq n+1$, and if $n \geq 2$, then additionally $\dim_{\mathbb{Q}} Z \leq n$. This result was later improved by M. Levin:

Theorem 3.3 (M. Levin, 2005) [Le2]*: Let $n \in \mathbb{N}_{\geq 2}$. Then for every compact metrizable space X with* $\dim_{\mathbb{Q}} X \leq n$*, there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is $\mathbb{Q}$-acyclic, and* $\dim Z \leq n$*.* $\square$

The obvious question was whether a theorem similar to Theorem 3.2 could be stated for compact metrizable spaces and arbitrary abelian groups. In their work [KY2], Koyama and Yokoi made a substantial amount of progress in answering this question. Their method relied heavily on the existence of Edwards-Walsh resolutions, which have been studied by J. Dydak and J. Walsh in [DW], and which had been applied originally, in a rudimentary form, in [Wa]. (Note that, here, the word "resolution" is not referring to a map – it refers to a CW-complex built upon the n-skeleton of a given polyhedron, so it would be more appropriate to call it an Edwards-Walsh "extension space"or "complex".) However, using a different approach from the one in [KY2], M. Levin has proved a very strong generalization for Theorems 3.1 and 3.2, concerning compact metrizable spaces and arbitrary abelian groups:

Theorem 3.4 (M. Levin, 2003) [Le1]*: Let G be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. Then for every compact metrizable space X with $\dim_G X \leq n$, there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that:*

(a) π *is G-acyclic,*

(b) $\dim Z \leq n+1$, *and*

(c) $\dim_G Z \leq n$. $\square$

The requirement of $n \in \mathbb{N}_{\geq 2}$ in Levin's Theorem cannot be improved because there is a counterexample for $n = 1$ ($G = \mathbb{Q}$, [Le1]). The requirement that $\dim Z \leq n+1$ cannot be improved either – there is a counterexample for $\dim Z \leq n$ ($G = \mathbb{Z}/p^\infty$, [KY2]). The part that may be improved is $\dim_G X \leq n \Leftrightarrow X\tau K(G,n)$, by replacing a $K(G,n)$ with a CW-complex upon which the demands will be less strict:

Theorem 3.5 (L. Rubin - P. Schapiro, 2005) [RS2]*: Let G be an abelian group with $P_G \neq \mathbb{P}$, where $P_G = \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in$ Bockstein basis $\sigma(G)\}$. Let $n \in \mathbb{N}_{\geq 2}$, and let K be a connected* CW*-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leq k < n$. Then for every compact metrizable space X with $X\tau K$, there exists a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that:*

(a) π *is* G*-acyclic,*

(b) $\dim Z \leq n+1$*, and*

(c) $Z\tau K$.

If in addition, $\pi_{n+1}(K) = 0$*, then we may replace* (a) *by the stronger statement:*

(aa) π *is* K*-acyclic.* $\square$

Note that Theorem 3.5 does not cover the case when $P_G = \mathbb{P}$. In fact, the statement of this theorem will be true when $P_G = \mathbb{P}$, but in this case the statement can be improved. The rest of this thesis will be dedicated to the proof of the following theorem:

Theorem 3.6 *Let* G *be an abelian group with* $P_G = \mathbb{P}$*, where* $P_G = \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in \sigma(G)\}$*. Let* $n \in \mathbb{N}$*, and let* K *be a connected* CW*-complex with* $\pi_n(K) \cong G$*,* $\pi_k(K) \cong 0$ *for* $0 \leq k < n$*. Then for every compact metrizable space* X *with* $X\tau K$*, there exists a compact metrizable space* Z *and a surjective map* $\pi : Z \to X$ *such that:*

(a) π *is cell-like,*

(b) $\dim Z \leq n$*, and*

(c) $Z\tau K$.

Note that because $P_G = \mathbb{P}$, this theorem works for $n = 1$, while the Rubin-Schapiro Resolution Theorem 3.5 works for $n \in \mathbb{N}_{\geq 2}$.

More importantly, note that Theorem 3.6 is a generalization of the Edwards-Walsh resolution Theorem 3.1: if $K = K(\mathbb{Z}, n)$, then $X\tau K \Leftrightarrow \dim_{\mathbb{Z}} X \leq n$, so we get the Edwards-Walsh Theorem as a corollary. In that case, statement (c) follows from (b).

Recall that $P_G = \mathbb{P}$ implies $\dim_G X = \dim_{\mathbb{Z}} X$ by Lemma 2.5, i.e., $X\tau K(G,n) \Leftrightarrow X\tau K(\mathbb{Z}, n)$. So if $K = K(G,n)$ in the statement of Theorem 3.6, then $X\tau K \Leftrightarrow$

$\dim_{\mathbb{Z}} X \leq n$, so we get the Edwards-Walsh Theorem 3.1 in another way. But K need not be a $K(G, n)$. So Theorem 3.6 is, indeed, a generalization of the Edwards-Walsh Theorem.

Chapter 4

Results needed for the proof of the Main Theorem: generalized Walsh and Edwards Theorems

This will be a statement needed to produce $\pi : Z \twoheadrightarrow X$, based on [Wa].

Notation: $B_r(x)$ stands for the *closed* ball with radius r, centered at x.

Lemma 4.1 (Generalized Walsh Lemma) *Let* $\mathbf{X} = (P_i, f_i^{i+1})$ *be an inverse sequence of compact metric polyhedra* (P_i, d_i) *of diameter less than* 1 *with surjective bonding maps,* $\mathbf{Z} = (M_i, g_i^{i+1})$ *an inverse sequence of Hausdorff compacta,* $X = \lim \mathbf{X}$ *and* $Z = \lim \mathbf{Z}$. *Assume also that we have maps* $\phi_i : M_i \to P_i$, *and, for each* $i \in \mathbb{N}$ *we have numbers* $0 < \varepsilon(i) < \frac{\delta(i)}{3} < 1$, *satisfying:*

(I) *for* $i \geq 2$, $\phi_{i-1} \circ g_{i-1}^i$ *and* $f_{i-1}^i \circ \phi_i$ *are* $\frac{\varepsilon(i-1)}{3}$ *- close,*

(II) *for* $i \geq 2$ *and for any* $y \in P_i$, $\mathrm{diam}\ (f_{i-1}^i(B_{\delta(i)}(y))) < \frac{\varepsilon(i-1)}{3}$, *and*

(III) *for* $i > j$ *and for any* $y \in P_i$, $\mathrm{diam}\ (f_j^i(B_{\varepsilon(i)}(y))) < \frac{\varepsilon(j)}{2^i}$.

Then there is a map $\pi : Z \to X$ *with fibers*

(IV) $\pi^{-1}(x) = \pi^{-1}((x_i)) = \lim\ (\phi_i^{-1}(B_{\delta(i)}(x_i)), g_i^{i+1}) = \lim\ (\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_i^{i+1})$

(here g_i^{i+1} *stands for the appropriate restriction).*

If, in addition, we have that:

(V) *for all* $x = (x_i) \in X$ *and for all* i, $\phi_i^{-1}(B_{\varepsilon(i)}(x_i)) \neq \emptyset$,

then $\pi^{-1}(x) \neq \emptyset$, *so the map* π *will be surjective.*

Proof: The following diagram will help in visualizing the steps of this proof.

$$\begin{tikzcd}
\cdots & M_i \arrow[l] \arrow[d, "\phi_i"'] & M_{i+1} \arrow[l, "g_i^{i+1}"'] \arrow[d, "\phi_{i+1}"] & \cdots \arrow[l] & Z \arrow[d, dashed, "\pi"] \\
\cdots & P_i \arrow[l] & P_{i+1} \arrow[l, "f_i^{i+1}"] & \cdots \arrow[l] & X
\end{tikzcd}$$

Let $z = (z_i)$ be an element of $Z \subset \prod_{i=1}^{\infty} M_i$; so $g_i^{i+1}(z_{i+1}) = z_i$ and $\phi_i(z_i) \in P_i$, for all $i \in \mathbb{N}$. Define a sequence in $\prod_{i=1}^{\infty} P_i$ as follows:

$$\begin{aligned}
x^1 &= (\phi_1(z_1), \phi_2(z_2), \phi_3(z_3), \phi_4(z_4), \ldots) \\
x^2 &= (f_1^2(\phi_2(z_2)), \phi_2(z_2), \phi_3(z_3), \phi_4(z_4,) \ldots) \\
x^3 &= (f_1^3(\phi_3(z_3)), f_2^3(\phi_3(z_3)), \phi_3(z_3), \phi_4(z_4), \ldots) \\
&\vdots \\
x^j &= (f_1^j(\phi_j(z_j)), f_2^j(\phi_j(z_j)), \ldots, f_{j-1}^j(\phi_j(z_j)), \phi_j(z_j), \phi_{j+1}(z_{j+1}), \ldots) \\
x^{j+1} &= (f_1^{j+1}(\phi_{j+1}(z_{j+1})), f_2^{j+1}(\phi_{j+1}(z_{j+1})), \ldots, f_j^{j+1}(\phi_{j+1}(z_{j+1})), \phi_{j+1}(z_{j+1}), \phi_{j+2}(z_{j+2}) \ldots) \\
&\vdots
\end{aligned}$$

Let $\pi_j : Z \to \prod_{i=1}^{\infty} P_i$ be defined by $\pi_j(z) := x^j$. We would like to show that $(\pi_j(z))_{j \in \mathbb{N}}$ is a Cauchy sequence in $\prod_{i=1}^{\infty} P_i$. Properties we will need are:

(1) for $j \geq 2$, $f_{j-1}^j(\phi_j(z_j))$ and $\phi_{j-1}(z_{j-1}) = \phi_{j-1}(g_{j-1}^j(z_j))$ are $\varepsilon(j-1)$-close, and

(2) for $i > j$, $f_j^{i+1}(\phi_{i+1}(z_{i+1}))$ and $f_j^i(\phi_i(z_i))$ are $\frac{\varepsilon(j)}{2^i}$-close.

Property (1) follows from (I). Property (2) is true because: by $(1)_{i+1}$, $f_i^{i+1}(\phi_{i+1}(z_{i+1}))$ and $\phi_i(z_i)$ are $\varepsilon(i)$-close, so $f_i^{i+1}(\phi_{i+1}(z_{i+1})) \in B_{\varepsilon(i)}(\phi_i(z_i))$. Therefore $f_j^{i+1}(\phi_{i+1}(z_{i+1})) = f_j^i(f_i^{i+1}(\phi_{i+1}(z_{i+1}))) \in f_j^i(B_{\varepsilon(i)}(\phi_i(z_i)))$, and diam $f_j^i(B_{\varepsilon(i)}(\phi_i(z_i))) < \frac{\varepsilon(j)}{2^i}$, by (III).

So $f_j^{i+1}(\phi_{i+1}(z_{i+1}))$ and $f_j^i(\phi_i(z_i))$ are $\frac{\varepsilon(j)}{2^i}$-close.

We shall employ the metric d on $\prod_{i=1}^{\infty} P_i$ given by

$$d((s_i),(r_i)) := \sum_{i=1}^{\infty} \frac{d_i(s_i,r_i)}{2^i}.$$

Note that by $(2)_{j>q}$ and $(1)_{j+1}$,

$$\begin{aligned} d(\pi_j(z),\pi_{j+1}(z)) &= \left(\sum_{q=1}^{j-1} \frac{d_q(f_q^j(\phi_j(z_j)), f_q^{j+1}(\phi_{j+1}(z_{j+1})))}{2^q}\right) + \frac{d_j(\phi_j(z_j), f_j^{j+1}(\phi_{j+1}(z_{j+1})))}{2^j} \\ &< \left(\sum_{q=1}^{j-1} \frac{\varepsilon(q)}{2^j}\frac{1}{2^q}\right) + \frac{\varepsilon(j)}{2^j} < \frac{1}{2^j}\left(\sum_{q=1}^{j-1}\frac{1}{2^q}\right) + \frac{1}{2^j} \\ &< \frac{1}{2^j}\left(\left(\sum_{q=1}^{\infty}\frac{1}{2^q}\right) + 1\right) = \frac{1}{2^{j-1}}. \end{aligned}$$

Therefore, for the indexes j and $j+k$ we get:

$$\begin{aligned} d(\pi_j(z),\pi_{j+k}(z)) &\le d(\pi_j(z),\pi_{j+1}(z)) + d(\pi_{j+1}(z),\pi_{j+2}(z)) + \ldots + d(\pi_{j+k-1}(z),\pi_{j+k}(z)) \\ &< \frac{1}{2^{j-1}} + \frac{1}{2^j} + \ldots + \frac{1}{2^{j+k-2}} < \frac{1}{2^{j-2}} \cdot \sum_{i=1}^{\infty}\frac{1}{2^i} = \frac{1}{2^{j-2}}. \end{aligned}$$

Thus $(\pi_j(z))_{j\in\mathbb{N}}$ is a Cauchy sequence in the compact metric space $\prod_{i=1}^{\infty} P_i$, and therefore it is convergent. Define $\pi(z) := \lim_{j\to\infty} \pi_j(z)$.

Notice that for any $k \in \mathbb{N}$, and for any $z \in Z$,

$$d(\pi_k(z),\pi(z)) \le \sum_{j=k}^{\infty} d(\pi_j(z),\pi_{j+1}(z)) < \sum_{j=k}^{\infty} \frac{1}{2^{j-1}} = \frac{1}{2^{k-2}}.$$

So the sequence $(\pi_j)_{j\in\mathbb{N}}$ converges uniformly to π. Therefore $\pi : Z \to \prod_{i=1}^{\infty} P_i$ is a continuous function.

We would like to see that $\pi(Z) \subset X$. If y_j is j-th coordinate of $\pi(z)$ for some

$z \in Z$, then $y_j = \lim_{i>j} f_j^i(\phi_i(z_i))$. Therefore if $j > 1$,

$$f_{j-1}^j(y_j) = f_{j-1}^j(\lim_{i>j} f_j^i(\phi_i(z_i))) = \lim_{i>j}(f_{j-1}^j(f_j^i(\phi_i(z_i)))) =$$
$$= \lim_{i>j}(f_{j-1}^i(\phi_i(z_i))) = \lim_{i>j-1}(f_{j-1}^i(\phi_i(z_i))) = y_{j-1}.$$

So $\pi(z) \in X$, i.e., $\pi(Z) \subset X$.

Now that we have a map $\pi : Z \to X$, we need to see what its fibers are. Take any $x = (x_i) \in X$. From $(\mathrm{II})_i$ and $(\mathrm{I})_i$, we will get that

(3) $g_{i-1}^i(\phi_i^{-1}(B_{\delta(i)}(x_i))) \ \subset \ \phi_{i-1}^{-1}(B_{\varepsilon(i-1)}(x_{i-1}))$.

Here is why: take any $y \in \phi_i^{-1}(B_{\delta(i)}(x_i))$, i.e., $\phi_i(y) \in B_{\delta(i)}(x_i)$. Note that $(\mathrm{II})_i$: diam $(f_{i-1}^i(B_{\delta(i)}(x_i))) < \frac{\varepsilon(i-1)}{3}$. Hence $d_{i-1}(f_{i-1}^i(\phi_i(y)), f_{i-1}^i(x_i)) < \frac{\varepsilon(i-1)}{3}$, i.e., $d_{i-1}(f_{i-1}^i(\phi_i(y)), x_{i-1}) < \frac{\varepsilon(i-1)}{3}$. By $(\mathrm{I})_i$: $d_{i-1}(\phi_{i-1}(g_{i-1}^i(y)), f_{i-1}^i(\phi_i(y))) < \frac{\varepsilon(i-1)}{3}$, and therefore

$$d_{i-1}(x_{i-1}, \phi_{i-1}(g_{i-1}^i(y))) \leq d_{i-1}(x_{i-1}, f_{i-1}^i(\phi_i(y))) + d_{i-1}(f_{i-1}^i(\phi_i(y)), \phi_{i-1}(g_{i-1}^i(y)))$$
$$< \frac{2\varepsilon(i-1)}{3} < \varepsilon(i-1).$$

So $\phi_{i-1}(g_{i-1}^i(y)) \in B_{\varepsilon(i-1)}(x_{i-1})$, and therefore $g_{i-1}^i(y) \in \phi_{i-1}^{-1}(B_{\varepsilon(i-1)}(x_{i-1}))$, so (3) is true.

As a consequence of (3) and the fact that $\varepsilon(i) < \delta(i)$, both $(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i|_{\phi_i^{-1}(B_{\delta(i)}(x_i))})$ and $(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i|_{\phi_i^{-1}(B_{\varepsilon(i)}(x_i))})$ are inverse sequences with the same limit. Now we would like to show that this limit is $\pi^{-1}(x)$.

Let us show that $\lim(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i) \subset \pi^{-1}(x)$, where g_{i-1}^i stands for the appropriate restriction. Take any $z = (z_i) \in \lim(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i)$. Note that

(4) the j-th coordinate of $\pi(z)$ is $\lim\limits_{i>j} f_j^i(\phi_i(z_i))$.

Since $z_i \in \phi_i^{-1}(B_{\varepsilon(i)}(x_i))$, we have that $\phi_i(z_i) \in B_{\varepsilon(i)}(x_i)$. Condition $(\mathrm{III})_i$: diam $(f_j^i(B_{\varepsilon(i)}(x_i))) < \frac{\varepsilon(j)}{2^i}$ implies that $f_j^i(\phi_i(z_i))$ and $x_j = f_j^i(x_i)$ are $\frac{\varepsilon(j)}{2^i}$-close. Therefore $\lim\limits_{i>j} f_j^i(\phi_i(z_i)) = x_j$, so $\pi(z) = x$, i.e., $z \in \pi^{-1}(x)$.

Let us demonstrate that $\pi^{-1}(x) \subset \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i)$. Suppose that $z = (z_i) \in Z$, and $z \notin \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i)$. We will show that $\pi(z) \neq x$.

Now $z \notin \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i)$ means that there is an index $j \in \mathbb{N}$ such that $z_j \notin \phi_j^{-1}(B_{\delta(j)}(x_j))$. So $d_j(\phi_j(z_j), x_j) > \delta(j)$. The inequality $\varepsilon(j) < \frac{\delta(j)}{3}$ assures that $B_{2\varepsilon(j)}(\phi_j(z_j)) \cap B_{\varepsilon(j)}(x_j) = \emptyset$. If we look at the distance between $\phi_j(z_j)$ and the j-th coordinate of $\pi(z)$ (see (4)), from $(1)_{j+1}$ and $(2)_{k>j}$ we get:

$$\begin{aligned} d_j(\phi_j(z_j), \lim_{i>j} f_j^i(\phi_i(z_i))) &\leq d_j(\phi_j(z_j), f_j^{j+1}(\phi_{j+1}(z_{j+1}))) \\ &\quad + \sum_{k=j+1}^{\infty} d_j(f_j^k(\phi_k(z_k)), f_j^{k+1}(\phi_{k+1}(z_{k+1}))) \\ &< \varepsilon(j) + \sum_{k=j+1}^{\infty} \frac{\varepsilon(j)}{2^k} = \varepsilon(j) + \frac{\varepsilon(j)}{2^j} \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} < 2\varepsilon(j). \end{aligned}$$

That is, the j-th coordinate of $\pi(z)$ is contained in $B_{2\varepsilon(j)}(\phi_j(z_j))$, implying $\pi(z) \neq x$, i.e., $z \notin \pi^{-1}(x)$.

So we get that

$$\lim(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i) \subset \pi^{-1}(x) \subset \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i),$$

and since the left and right side of this statement are equal, then (IV) is true.

If (V) is also true, i.e., $\pi^{-1}(x)$ is the inverse limit of an inverse sequence of compact nonempty spaces, then, according to Theorem 2.4 from Appendix II of [Du], $\pi^{-1}(x) \neq \emptyset$. Thus, the map $\pi : Z \to X$ is surjective. $\square$

Lemma 4.2 (Special version of Walsh Lemma) *Let* $\mathbf{X} = (P_i, f_i^{i+1})$ *be an inverse sequence of compact metric polyhedra* (P_i, d_i) *with diameter less than* 1 *and with surjective bonding maps, and let* L_i *be triangulations of* P_i. *Suppose that we have maps* $g_i^{i+1} : |L_{i+1}^{(n+1)}| \to |L_i^{(n+1)}|$ *such that* $g_i^{i+1}(|L_{i+1}^{(n)}|) \subset |L_i^{(n)}|$, *and let* $\mathbf{Z} = (|L_i^{(n)}|, g_i^{i+1})$ *be the inverse sequence of subpolyhedra* $|L_i^{(n)}| \subset P_i$, *where each* g_i^{i+1} *stands for the appropriate restriction. Let* $X = \lim \mathbf{X}$, $Z = \lim \mathbf{Z}$. *Assume that for each* $i \in \mathbb{N}$ *we have numbers* $0 < \varepsilon(i) < \frac{\delta(i)}{3} < 1$, *satisfying:*

(I) *for* $i \geq 2$, g_{i-1}^i *and* $f_{i-1}^i\big|_{|L_i^{(n)}|}$ *are* $\frac{\varepsilon(i-1)}{3}$ *- close,*

(II) *for* $i \geq 2$ *and for any* $y \in P_i$, $\operatorname{diam}\,(f_{i-1}^i(B_{\delta(i)}(y))) \;<\; \frac{\varepsilon(i-1)}{3}$, *and*

(III) *for* $i > j$ *and for any* $y \in P_i$, $\operatorname{diam}\,(f_j^i(B_{\varepsilon(i)}(y))) \;<\; \frac{\varepsilon(j)}{2^i}$.

Then there is a map $\pi : Z \to X$ *with fibers*

$$\pi^{-1}(x) \;=\; \pi^{-1}((x_i)) \;=\; \lim\,(B_{\delta(i)}(x_i) \cap |L_i^{(n)}|, g_i^{i+1}) \;=\; \lim\,(B_{\varepsilon(i)}(x_i) \cap |L_i^{(n)}|, g_i^{i+1})$$

(here g_i^{i+1} *stands for the appropriate restriction).*

If, in addition, we have that:

(IV) $\operatorname{mesh} L_i \;<\; \varepsilon(i)$, *for all* i,

then for all $x \in X$ *we have* $\pi^{-1}(x) \neq \emptyset$, *so the map* π *will be surjective.*
If we also have

(V) *for* $i \geq 1$ *and for any* $y \in P_i$, $B_{\varepsilon(i)}(y) \subset P_{y,i} \subset B_{\delta(i)}(y)$, *where* $P_{y,i}$ *is a contractible subpolyhedron of* $|L_i|$, *and*

(VI) *for* $i \geq 2$, $g_{i-1}^i(|L_i^{(n+1)}|) \subset |L_{i-1}^{(n)}|$,

then the map π *is cell-like.*

Proof: The following diagram will be useful.

$$\begin{tikzcd}
\cdots & {|L_i^{(n)}|} \arrow[l] \arrow[d, hook] & & {|L_{i+1}^{(n)}|} \arrow[ll, "g_i^{i+1}|_{|L_{i+1}^{(n)}|}"'] \arrow[d, hook] & \cdots \arrow[l] & Z \arrow[d, dashed, "\pi"] \\
\cdots & P_i = |L_i| \arrow[l] & & P_{i+1} = |L_{i+1}| \arrow[ll, "f_i^{i+1}"] & \cdots \arrow[l] & X
\end{tikzcd}$$

The existence of $\pi : Z \to X$ with the required properties of fibers follows from Lemma 4.1, when $P_i = |L_i|$, $M_i = |L_i^{(n)}|$ and $\phi_i = i : |L_i^{(n)}| \hookrightarrow |L_i|$ is the inclusion.

Note that $\phi_i^{-1}(B_{\delta(i)}(x_i)) = B_{\delta(i)}(x_i) \cap |L_i^{(n)}|$, so (IV) of Lemma 4.1 becomes:

$$\text{(IV}^*\text{)}\ \ \pi^{-1}(x) = \pi^{-1}((x_i)) = \lim(B_{\delta(i)}(x_i)\cap|L_i^{(n)}|, g_i^{i+1}) \ = \ \lim(B_{\varepsilon(i)}(x_i)\cap|L_i^{(n)}|, g_i^{i+1}).$$

Property (IV) will guarantee that, for any $x \in X$, $\pi^{-1}(x) \neq \emptyset$. This is true because, if we take any $x = (x_i) \in X$, $x_i \in P_i = |L_i|$ implies that there is a simplex $\sigma \in L_i$ such that $x_i \in \sigma$. Since $\operatorname{mesh} L_i < \varepsilon(i)$, we get that $\operatorname{diam}\sigma < \varepsilon(i)$, so $\sigma \subset B_{\varepsilon(i)}(x_i)$. Therefore $\sigma^{(n)} \subset B_{\varepsilon(i)}(x_i) \cap |L_i^{(n)}|$, so

$$\emptyset \neq B_{\varepsilon(i)}(x_i) \cap |L_i^{(n)}| \subset B_{\delta(i)}(x_i) \cap |L_i^{(n)}| = \phi_i^{-1}(B_{\delta(i)}(x_i)).$$

By (V) of Lemma 4.1, $\pi : Z \to X$ is surjective.

It remains to show that properties (V) and (VI) imply that π is cell-like. Note that from (V) and (IV*) we get that $\pi^{-1}(x) \ = \lim\ (P_{x_i,i} \cap |L_i^{(n)}|, g_i^{i+1})$, where g_i^{i+1} stands for the appropriate restriction. It will be sufficient to show that the maps $g_i^{i+1} : P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}| \to P_{x_i,i} \cap |L_i^{(n)}|$ are null-homotopic.

First note that $P_{x_{i+1},i+1}$ being contractible implies that the inclusion map $i : P_{x_{i+1},i+1}\cap|L_{i+1}^{(n)}| \hookrightarrow P_{x_{i+1},i+1}$ is null-homotopic. Since $\dim P_{x_{i+1},i+1}\cap|L_{i+1}^{(n)}| \leq n$,

i is null-homotopic as a map into $P_{x_{i+1},i+1} \cap |L_{i+1}^{(n+1)}|$, that is, this homotopy happens within the $(n+1)$-skeleton of L_{i+1}. Composing such a null-homotopy with $g_i^{i+1}|_{|L_{i+1}^{(n+1)}|} : |L_{i+1}^{(n+1)}| \to |L_i^{(n)}|$ yields the sought after null-homotopy for the restriction $g_i^{i+1}|_{P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}|}$. $\square$

The following Lemma will be useful in the proof of the new version of Edwards' Theorem.

Lemma 4.3 *For any finite simplicial complex C, there is a map $r : |C| \to |C|$ and an open cover $\mathcal{V} = \{V_\sigma : \sigma \in C\}$ of $|C|$ such that for all σ, $\tau \in C$:*

(i) $\mathring{\sigma} \subset V_\sigma$,

(ii) *if* $\sigma \neq \tau$ *and* $\dim \sigma = \dim \tau$, V_σ *and* V_τ *are disjoint,*

(iii) *if* $y \in \mathring{\tau}$, $\dim \sigma \geq \dim \tau$ *and* $\sigma \neq \tau$, *then* $y \notin V_\sigma$,

(iv) *if* $y \in \mathring{\tau} \cap V_\sigma$, *where* $\dim \sigma < \dim \tau$, *then* σ *is a face of* τ, *and*

(v) $r(V_\sigma) \subset \sigma$.

Proof: Since C is finite, let us suppose that $\dim C = q$. Note that the simplicial complex C has the property that for each k, there is an open neighborhood U_k of $|C^{(k)}|$ in $|C|$, and a surjective map $r_k : |C| \to |C|$ so that

(1) $r_k|_{|C^{(k)}|} = id_{|C^{(k)}|}$,

(2) r_k preserves simplexes, i.e., for any $\tau \in C$, $r_k(\tau) \subset \tau$, and

(3) $r_k(U_k) \subset |C^{(k)}|$.

Also note that for vertices $v \in C^{(0)}$ we have that $\mathring{v} = v$.

Here is how we will define the open cover $\mathcal{V} = \{V_\sigma : \sigma \in C\}$ for $|C|$:

(4) for each k-simplex σ of C, where $k = 0, \dots, q-1$, put

$V_\sigma := (r_k \circ r_{k+1} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})$ into $\mathcal{V}$, and

(5) for each q-simplex σ of C, put $V_\sigma := \mathring{\sigma}$ into $\mathcal{V}$.

Note that all elements of $\mathcal{V}$ are open sets: in (5) that is clear, and in (4): $(r_k \circ r_{k+1} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma}) = r_{q-1}^{-1}(\dots(r_{k+1}^{-1}(r_k^{-1}(\mathring{\sigma}))))$, and $r_k^{-1}(\mathring{\sigma})$ is open because $r_k|_{U_k} : U_k \to |C^{(k)}|$ is continuous, and $\mathring{\sigma}$ is open in $|C^{(k)}|$.

Let us check that (i) is true: $\mathring{\sigma} \subset V_\sigma$ is clear for case (5), and, for case (4), since $r_k, r_{k+1}, \dots, r_{q-1}$ are all the identity on $|C^{(k)}|$ and $\mathring{\sigma} \subset |C^{(k)}|$, then $\mathring{\sigma} \subset V_\sigma$. Hence $\mathcal{V}$ is a cover for $|C|$ because of (i).

If σ and τ are two different simplexes of the same dimension, then $\mathring{\sigma}$ and $\mathring{\tau}$ are disjoint. If $\dim \sigma = \dim \tau = q$, (ii) is clear. If $\dim \sigma = \dim \tau < q$, then (4) implies that V_σ and V_τ are disjoint, i.e., (ii) is true.

Let us prove property (iii). We know that $y \in \mathring{\tau} \subset V_\tau$. If τ and σ are of the same dimension, then (ii) implies $y \notin V_\sigma$. If $\dim \tau < \dim \sigma \leq q-1$, then $V_\sigma := (r_{\dim \sigma} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})$, so if y would be in V_σ, then $r_{\dim \sigma} \circ \dots \circ r_{q-1}(y) \in \mathring{\sigma}$. But $r_{\dim \sigma}, \dots, r_{q-1}$ are the identity on $|C^{(\dim \tau)}| \supset \tau$, so $r_{\dim \sigma} \circ \dots \circ r_{q-1}(y) = y \in \mathring{\sigma}$, which is in contradiction with $y \in \mathring{\tau}$. Thus $y \notin V_\sigma$. If $\dim \tau < \dim \sigma = q$, then $V_\sigma = \mathring{\sigma}$, so $y \in \mathring{\tau}$ and $\tau \neq \sigma$ imply that $y \notin V_\sigma$.

To prove (iv), suppose that $y \in V_\sigma$ for some $\sigma \in C$ with $\dim \sigma < \dim \tau$. Then $V_\sigma := (r_{\dim \sigma} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})$, so $r_{\dim \sigma} \circ \dots \circ r_{q-1}(y) \in \mathring{\sigma}$. Notice that $r_{\dim \tau}, r_{\dim \tau + 1}, \dots, r_{q-1}$ are the identity on τ, so $r_{\dim \sigma} \circ \dots \circ r_{q-1}(y) = r_{\dim \sigma} \circ \dots \circ r_{\dim \tau - 1}(y) \in \mathring{\sigma}$. The maps $r_{\dim \sigma}, \dots, r_{\dim \tau - 1}$ preserve simplexes, by (2), so $y \in \mathring{\tau}$ implies that $r_{\dim \sigma} \circ \dots \circ r_{\dim \tau - 1}(y) \in \tau$. Thus $\tau \cap \mathring{\sigma} \neq \emptyset$, so σ must be a face of τ.

It remains to define the map r and prove the property (v). Define $r :=$

$r_0 \circ r_1 \circ \ldots \circ r_{q-1} : |C| \to |C|$. For any k-simplex σ of C where $k = 1, \ldots, q-1$, by (4) we get that

$$r(V_\sigma) = r_0 \circ r_1 \circ \ldots \circ r_{q-1}((r_k \circ r_{k+1} \circ \ldots \circ r_{q-1})^{-1}(\mathring{\sigma})) = r_0 \circ r_1 \circ \ldots \circ r_{k-1}(\mathring{\sigma}),$$

since all r_i are surjective. Also, by (2), $r(V_\sigma) = r_0 \circ r_1 \circ \ldots \circ r_{k-1}(\mathring{\sigma}) \subset \sigma$.

Also, for any q-simplex σ of C, we get $r(V_\sigma) = r(\mathring{\sigma}) \subset \sigma$ for the same reason. For vertices $v \in C^{(0)}$, $r(V_v) = r \circ r^{-1}(v) = v$. So we conclude that (v) is true. $\square$

A version of Theorem 4.2 from [Wa], adapted for our situation follows:

Theorem 4.4 (New statement of Edwards Theorem) *Let $n \in \mathbb{N}$ and let Y be a compact metrizable space such that $Y = \lim\ (|L_i|, f_i^{i+1})$, where $|L_i|$ are compact polyhedra with $\dim L_i \leq n+1$, and f_i^{i+1} are surjections. Then $\dim_{\mathbb{Z}} Y \leq n$ implies that there exists an $s \in \mathbb{N}$, $s > 1$, and there exists a map $g_1^s : |L_s| \to |L_1^{(n)}|$ which is an L_1-modification of f_1^s .*

$$\begin{tikzcd}
|L_1^{(n)}| \arrow[d, hook] & & & \\
|L_1| & |L_s| \arrow[l, "f_1^s"'] \arrow[ul, dashed, "g_1^s"'] & \cdots \arrow[l] & Y
\end{tikzcd}$$

Proof: There will be two separate parts of this proof, for $n \geq 2$ and for $n = 1$.

Let us start with $n \geq 2$. We will build an Edwards-Walsh complex $\widehat{L}_1$ above $L_1^{(n)}$. Since $\dim L_1 \leq n+1$ and L_1 is finite, L_1 has to have finitely many $(n+1)$-simplexes, say, $\sigma_1, \ldots, \sigma_m$. Focus on $L_1^{(n)}$, and above each of $\sigma_i^{(n)} = \partial\sigma_i \approx S^n$, build a $K(\mathbb{Z}, n)$ by attaching cells of dimension $(n+2)$ and higher. Name the CW-complex that we get in this fashion $\widehat{L}_1$. Notice that we can write $\widehat{L}_1 = L_1^{(n)} \cup K(\sigma_1) \cup K(\sigma_2) \cup \ldots \cup K(\sigma_m)$, where each $K(\sigma_i)$ is a $K(\mathbb{Z}, n)$ attached to

$\partial\sigma_i$. Also notice that we can make the attaching maps piecewise linear, so that we will be able to triangulate $\widehat{L}_1$ keeping $L_1^{(n)}$ as a subcomplex.

Let $\theta : \widehat{L}_1 \to |L_1|$ be a map such that $\theta|_{|L_1^{(n)}|} = id_{|L_1^{(n)}|}$ and $\theta(K(\sigma_i)) \subset \sigma_i$. This θ can be constructed as follows: first, define $\theta|_{|L_1^{(n)}|} := id_{|L_1^{(n)}|}$. Since each σ_i is contractible, it is an absolute extensor for CW-complexes. Therefore the inclusion map $j : \sigma_i^{(n)} \to \sigma_i$ can be extended over $K(\sigma_i)$. Call this extension $\theta|_{K(\sigma_i)}$. Gluing together all of the extensions $\theta|_{K(\sigma_i)}$ for $i = 1, \ldots, m$ with $\theta|_{|L_1^{(n)}|}$ will produce the map θ.

Let $f_1 : Y \to |L_1|$ be the projection map from the inverse sequence. The map f_1 is surjective since all f_i^{i+1} are surjective. Extend $f_1|_{f_1^{-1}(|L_1^{(n)}|)} : f_1^{-1}(|L_1^{(n)}|) \to |L_1^{(n)}|$ to a map $h : Y \to \widehat{L}_1$ such that

(a) $h(f_1^{-1}(\sigma_i)) \subset \theta^{-1}(\sigma_i) = K(\sigma_i)$, for $i = 1, \ldots, m$.

This can be done using $\dim_{\mathbb{Z}} Y \leq n \iff Y \tau K(\mathbb{Z}, n)$: for any $(n+1)$-dimensional σ_i, take $f_1|_{f_1^{-1}(\sigma_i^{(n)})} : f_1^{-1}(\sigma_i^{(n)}) \to \sigma_i^{(n)}$ and compose it with the inclusion $i : \sigma_i^{(n)} \hookrightarrow K(\sigma_i) = K(\mathbb{Z}, n)$. Now $Y \tau K(\mathbb{Z}, n)$ implies $f_1^{-1}(\sigma_i)\, \tau K(\mathbb{Z}, n)$, so the map $i \circ f_1|_{f_1^{-1}(\sigma_i^{(n)})} : f_1^{-1}(\sigma_i^{(n)}) \to K(\sigma_i)$ can be extended over $f_1^{-1}(\sigma_i)$. Call this extension $h|_{f_1^{-1}(\sigma_i)}$. So we get the map h that we need by gluing together all of the extensions $h|_{f_1^{-1}(\sigma_i)}$, for $i = 1, \ldots, m$, with $h|_{f_1^{-1}(|L_1^{(n)}|)} = f_1|_{f_1^{-1}(|L_1^{(n)}|)}$.

Note that our inverse sequence $(|L_i|, f_i^{i+1})$ is a compact resolution for Y, so, in particular, it has the resolution property (R1): if we choose an open cover $\mathcal{V}$ for the minimum and hence finite subcomplex $\widehat{C}$ in $\widehat{L}_1$ such that $h(Y) \subset \widehat{C}$, then we can find an $s > 1$ and a map $h_1^s : |L_s| \to \widehat{C}$ such that h and $h_1^s \circ f_s$ are $\mathcal{V}$-close.

$$
\begin{tikzcd}
\widehat{L}_1 \arrow[d, "\theta"'] & \widehat{C} \arrow[l, hook'] & & \\
|L_1| & |L_s| \arrow[l, "f_1^s"] \arrow[u, dashed, "h_1^s"'] \arrow[u, dotted, bend left, "h_s"] & \cdots \arrow[l] & Y \arrow[ull, "h"'] \arrow[ll, bend left, "f_s"]
\end{tikzcd}
$$

Let us make a wise choice for $\mathcal{V}$. Start by triangulating $\widehat{C}$: let C denote a finite simplicial complex which is a triangulation of $\widehat{C}$ whose restriction to $|L_1^{(n)}|$ is a subcomplex. So $|C| = \widehat{C}$. Since C is finite, let us suppose that $\dim C = q$.

Define an open cover $\mathcal{V}$ for $|C|$, and a map $r : |C| \to |C|$ as in Lemma 4.3. For this cover $\mathcal{V}$ for $|C|$, we may apply resolution property (R1): we can find an $s > 1$ and a map $h_1^s : |L_s| \to |C|$ such that h and $h_1^s \circ f_s$ are $\mathcal{V}$-close. Define $h_s := r \circ h_1^s : |L_s| \to |C|$. Because of our choices, we get that

(b) whenever $h(y) \in \mathring{\tau}$ for some $\tau \in C$, then $h_s \circ f_s(y) \in \tau$.

This is true because, by (i), (iii) and (iv) of Lemma 4.3, $h(y) \in \mathring{\tau}$ implies that $h(y) \in V_\tau$, and possibly also $h(y) \in V_\sigma$ for some σ which is a face of τ, but $h(y)$ is in no other elements of $\mathcal{V}$. Since $h_1^s \circ f_s$ is $\mathcal{V}$-close to h, we have that either $h_1^s \circ f_s(y) \in V_\tau$, or $h_1^s \circ f_s(y) \in V_\sigma$, for some face σ of τ. But by (v) of Lemma 4.3, $r(V_\tau) \subset \tau$ and $r(V_\sigma) \subset \sigma \subset \tau$. Thus $h_s \circ f_s(y) = r \circ h_1^s \circ f_s(y) \in \tau$.

If $f_1(y) \in \sigma_i$ for some $(n+1)$-simplex σ_i of L_1, then, by (a), $h(y) \in K(\sigma_i)$, so $h(y) \in \mathring{\tau}$ for some $\tau \in C$ and $\tau \subset K(\sigma_i)$. By (b), $h_s(f_s(y)) \in \tau$. So we can conclude that

(c) if $f_1(y) \in \sigma_i$, for some $(n+1)$-simplex σ_i of L_1, then both $h(y)$ and $h_s \circ f_s(y)$ land in $K(\sigma_i)$.

Now we will construct a map $g_1^s : |L_s| \to |L_1^{(n)}|$ such that :

(d) $g_1^s|_{h_s^{-1}(|L_1^{(n)}|)} = h_s|_{h_s^{-1}(|L_1^{(n)}|)}$, and

(e) whenever $h_s(z) \in K(\sigma_i)$ for some $(n+1)$-simplex σ_i of L_1, then $g_1^s(z) \in \sigma_i$.

$$\begin{tikzcd}
\widehat{L}_1 \arrow[dd, "\theta"'] & \widehat{C} \arrow[l, hook'] & \\
& \widehat{C}^{(n+1)} \arrow[u, hook] & \\
|L_1| & & |L_s| \arrow[ll, "f_1^s"] \arrow[lu, dashed, "g_1^s"] \arrow[luu, "h_s"']
\end{tikzcd}$$

In fact, g_1^s will be the stability theory version of h_s. We know that $h_s : |L_s| \to |C| = \widehat{C}$, where C is a triangulation of the finite CW-subcomplex $\widehat{C}$ of $\widehat{L}_1$. Since $\widehat{C}$ is finite, we can pick a cell γ of maximal possible dimension $\dim \gamma = q$ (we have assumed that $\dim C = q$, so $\dim \widehat{C} = q$). It is safe to assume that $q \geq n+2$.

Pick a point w in $\mathring{\gamma}$ with an open neighborhood $W \subset \mathring{\gamma}$. Since $\dim |L_s| \leq n+1$ and $\dim \gamma > n+1$, the point w we picked is an unstable value for h_s. Therefore we can construct a new map $g_{1,\gamma}^s : |L_s| \to \widehat{C} \setminus \{w\}$ that agrees with h_s on $h_s^{-1}(\widehat{C} \setminus W)$, and $g_{1,\gamma}^s(h_s^{-1}(\gamma)) \subset \gamma \setminus \{w\}$. Retract $\gamma \setminus \{w\}$ to $\partial\gamma$ by a retraction $\tilde{r} : \widehat{C} \setminus \{w\} \to \widehat{C} \setminus \mathring{\gamma}$, such that $\tilde{r}|_{\widehat{C} \setminus \mathring{\gamma}} = id$. Replace h_s with $\tilde{r} \circ g_{1,\gamma}^s : |L_s| \to \widehat{C} \setminus \mathring{\gamma}$.

We will repeat this process, starting with $\widehat{C} \setminus \mathring{\gamma}$ and the map $\tilde{r} \circ g_{1,\gamma}^s$ instead of $\widehat{C}$ and h_s: pick a cell of maximal dimension in $\widehat{C} \setminus \mathring{\gamma}$, etc. This is done one cell at a time, until we get rid of all cells in $\widehat{C}$ with dimension $\geq n+2$. The map we end up with will be $g_1^s : |L_s| \to \widehat{C}^{(n+1)}$, where $\widehat{C}^{(n+1)}$ stands for the CW-skeleton of dimension $n+1$ for $\widehat{C}$. Notice that $\widehat{C}^{(n+1)} \subset \widehat{L}_1^{(n+1)}$, but the CW-skeleton of dimension $n+1$ for $\widehat{L}_1$ is equal to the CW-skeleton of dimension n for $\widehat{L}_1$, since we have built $\widehat{L}_1$ by attaching cells of dimension $n+2$ and higher to $L_1^{(n)}$. Thus $\widehat{L}_1^{(n+1)} = \widehat{L}_1^{(n)} = |L_1^{(n)}|$, where $L_1^{(n)}$ is the simplicial n-skeleton of L_1. So in fact,

$g_1^s : |L_s| \to |L_1^{(n)}|$.

By our construction, g_1^s agrees with h_s on $h_s^{-1}(|L_1^{(n)}|)$, so (d) is true. To prove property (e), let $h_s(z) \in K(\sigma_i)$. Then $h_s(z) \in \gamma$, for some cell γ of $K(\sigma_i)$. So $\tilde{r} \circ g_{1,\gamma}^s(z) \in \partial\gamma \subset K(\sigma_i)$. As we go on with our construction, we get $g_1^s(z) \in (K(\sigma_i))^{(n+1)} = \partial\sigma_i \subset \sigma_i$.

Finally, for any $z \in |L_s|$ we have that either $f_1^s(z) \in \overset{\circ}{\tau}$, for some $\tau \in L_1^{(n)}$, or $f_1^s(z) \in \overset{\circ}{\sigma}_i$, for some $(n+1)$-simplex σ_i of L_1. Since f_s is surjective, there is a $y \in Y$ such that $f_s(y) = z$.

So, if $f_1^s(z) \in \overset{\circ}{\tau}$ for some $\tau \in L_1^{(n)}$, then $f_1(y) = f_1^s(f_s(y)) = f_1^s(z) \in \overset{\circ}{\tau} \subset |L_1^{(n)}|$. Recall that on $f_1^{-1}(|L_1^{(n)}|)$, f_1 and h coincide. Thus $f_1(y) = h(y) \in \overset{\circ}{\tau}$. There is a simplex $\tau' \in C \cap |L_1^{(n)}|$ such that $\tau' \subset \tau$, and $f_1(y) = h(y) \in \overset{\circ}{\tau}{}'$. By (b) we get that $h_s \circ f_s(y) \in \tau' \subset \tau$, i.e., $h_s(z) \in \tau \in L_1^{(n)}$, so by (d), $g_1^s(z) = h_s(z) \in \tau$.

On the other hand, if $f_1^s(z) \in \overset{\circ}{\sigma}_i$, for some $(n+1)$-simplex σ_i of L_1, then $f_1(y) = f_1^s(f_s(y)) = f_1^s(z) \in \overset{\circ}{\sigma}_i$. By (c), $h_s \circ f_s(y) \in K(\sigma_i)$, i.e., $h_s(z) \in K(\sigma_i)$. Property (e) implies that $g_1^s(z) \in \sigma_i$.

So g_1^s is an L_1-modification of f_1^s.

It remains to prove this theorem for $n = 1$. First note that $\dim_{\mathbb{Z}} Y \leq 1$ implies that $\dim Y \leq 1$. We will not need to construct an Edwards-Walsh complex $\widehat{L}_1$ here. Instead, look at the map $f_1 : Y \to |L_1|$. Let $g_1 : Y \to |L_1^{(1)}|$ be a stability theory version of f_1. We construct g_1 as before: since we know that $\dim L_1 \leq 2$, pick any 2-simplex σ of L_1. We can pick a point $w \in \overset{\circ}{\sigma}$ with an open neighborhood $W \subset \overset{\circ}{\sigma}$, and since $\dim \sigma = 2$, the point w is an unstable value for f_1. So there exists a map $g_{1,\sigma} : Y \to |L_1| \setminus \{w\}$ which agrees with f_1 on $f_1^{-1}(|L_1| \setminus W)$, and such that $g_{1,\sigma}(f_1^{-1}(\sigma)) \subset \sigma \setminus \{w\}$. Now retract $\sigma \setminus \{w\}$ to $\partial\sigma$ by a retraction $\tilde{r}$ which is the identity on $|L_1| \setminus \overset{\circ}{\sigma}$. Finally, replace f_1 by $\tilde{r} \circ g_{1,\sigma} : Y \to |L_1| \setminus \overset{\circ}{\sigma}$.

Continue the process with one 2-simplex at a time. Since L_1 is finite, in finitely many steps we will reach the needed map $g_1 : Y \to |L_1^{(1)}|$. Note that from the construction of g_1, we get

(f) $g_1|_{f_1^{-1}(|L_1^{(1)}|)} = f_1|_{f_1^{-1}(|L_1^{(1)}|)}$, and for every 2-simplex σ of L_1, $\;g_1(f_1^{-1}(\sigma)) \subset \partial\sigma$.

$$\begin{tikzcd}
{|L_1^{(1)}|} \arrow[d, hook] & & & \\
{|L_1|} & {|L_s|} \arrow[l, "f_1^s"] \arrow[ul, dashed, "\widehat{g}_1^s"] \arrow[ul, dotted, "g_1^s"'] & \cdots \arrow[l] & Y \arrow[ll, bend left, "f_s"] \arrow[ulll, "g_1"']
\end{tikzcd}$$

Let us choose an open cover $\mathcal{V}$ of $L_1^{(1)}$ as before: apply Lemma 4.3 to $C = L_1^{(1)}$. Note that $q = 1$, so the map $r = r_0 : |L_1^{(1)}| \to |L_1^{(1)}|$.

Now use resolution property (R1): there is an index $s > 1$ and a map $\widehat{g}_1^s : |L_s| \to |L_1^{(1)}|$ such that $\widehat{g}_1^s \circ f_s$ and g_1 are $\mathcal{V}$-close. Define $g_1^s := r_0 \circ \widehat{g}_1^s : |L_s| \to |L_1^{(1)}|$.

Notice that for any $y \in Y$, if $g_1(y) \in \mathring{\tau}$ for some $\tau \in L_1^{(1)}$ (vertices included), then $g_1(y) \in V_\tau$, and possibly also $g_1(y) \in V_v$, where v is a vertex of τ. Then either $\widehat{g}_1^s \circ f_s(y) \in V_\tau$, or $\widehat{g}_1^s \circ f_s(y) \in V_v$. In any case, $r_0 \circ \widehat{g}_1^s \circ f_s(y) \in \tau$. Hence,

(g) for any $y \in Y$, $g_1(y) \in \mathring{\tau}$ for some $\tau \in L_1^{(1)}$, implies that $g_1^s(f_s(y)) \in \tau$.

Finally, for any $z \in |L_s|$, f_s is surjective implies that there is a $y \in Y$ such that $f_s(y) = z$. Then $f_1^s(z) = f_1^s(f_s(y)) = f_1(y)$. Now $f_1^s(z)$ is either in $\mathring{\sigma}$ for some 2-simplex σ in L_1, or in $\mathring{\tau}$ for some $\tau \in L_1^{(1)}$.

If $f_1^s(z) \in \mathring{\sigma}$, that is $f_1(y) \in \mathring{\sigma}$ for some 2-simplex σ, by (f) we get that $g_1(y) \in \partial\sigma$. Then by (g), $g_1^s(f_s(y)) \in \partial\sigma$, i.e., $g_1^s(z) \in \sigma$.

If $f_1^s(z) = f_1(y) \in \overset{\circ}{\tau}$ for some $\tau \in L_1^{(1)}$, then (f) implies that $g_1(y) = f_1(y) \in \overset{\circ}{\tau}$, so by (g), $g_1^s(f_s(y)) \in \tau$, i.e., $g_1^s(z) \in \tau$. Therefore, g_1^s is indeed an L_1-modification of f_1^s. $\square$

Lemma 4.5 *Let $n \in \mathbb{N}$, G be an abelian group and K be a connected* CW-*complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \le k < n$. If Y is a compact metrizable space with* $\dim Y \le n+1$, *then* $Y \tau K \;\Leftrightarrow\; \dim_G Y \le n$.

Proof: Build a $K(G,n)$ by attaching cells of dimension $n+2$ and higher to our CW-complex K.

First, assume that $Y \tau K$, and let us show $\dim_G Y \le n$. If we look at any closed set $A \subset Y$ and any map $f : A \to K(G,n)$, we have that $\dim A \le \dim Y \le n+1$, so we can homotope f into $K(G,n)^{(n+1)} = K^{(n+1)} \subset K$, i.e., there is a map $\overline{f} : A \to K$ which is homotopic to f. Now $Y \tau K$ implies the existence of a map $g : Y \to K$ which extends $\overline{f}$. Therefore, by the homotopy extension theorem, f can be extended continuously over Y, so we get that $Y \tau K \;\Rightarrow\; Y \tau K(G,n) \;\Rightarrow\; \dim_G Y \le n$.

Second, assume that $\dim_G Y \le n$, and let us show $Y \tau K$. Look at any closed set $A \subset Y$ and any map $f : A \to K$. Let $i : K \hookrightarrow K(G,n)$ be the inclusion map. Then $Y \tau K(G,n)$ implies that there is a map $\tilde{f} : Y \to K(G,n)$ extending $i \circ f : A \to K(G,n)$, i.e., $\tilde{f}|_A = i \circ f$.

Since Y is compact, $\tilde{f}(Y)$ is contained in a finite subcomplex $\widehat{C}$ of $K(G,n)$. There are finitely many cells in $\widehat{C} \setminus K$, and all of them have dimension $\ge n+2$. Pick a cell of maximal dimension $\gamma \in \widehat{C} \setminus K$, and a point $w \in \overset{\circ}{\gamma}$ with an open neighborhood $W \subset \overset{\circ}{\gamma}$. Since $\dim Y \le n+1$ and $\dim \gamma \ge n+2$, by stability theory the point w is an unstable value of the map $\tilde{f}$, so there is a map $g_\gamma : Y \to \widehat{C} \setminus \{w\}$ which agrees with $\tilde{f}$ on $\tilde{f}^{-1}(\widehat{C} \setminus W)$, and such that $g_\gamma(\tilde{f}^{-1}(\gamma)) \subset \gamma \setminus \{w\}$. Retract

$\gamma \setminus \{w\}$ to $\partial\gamma$ by a retraction $\tilde{r} : \widehat{C} \setminus \{w\} \to \widehat{C} \setminus \mathring{\gamma}$, such that $\tilde{r}|_{\widehat{C}\setminus\mathring{\gamma}} = id$. Replace $\tilde{f}$ with $\tilde{r} \circ g_\gamma : Y \to \widehat{C} \setminus \mathring{\gamma}$. Repeat this process one cell at a time until all cells of $\widehat{C} \setminus K$ are exhausted. The map we end up with will be $g : Y \to K$ such that $g|_{\tilde{f}^{-1}(K)} = \tilde{f}|_{\tilde{f}^{-1}(K)}$. Since $\tilde{f}(A) = f(A) \subset K$, that is, $A \subset \tilde{f}^{-1}(K)$, we get $g|_A = \tilde{f}|_A$. So $g : Y \to K$ is an extension of $f : A \to K$. Therefore $Y \tau K$. $\square$

Chapter 5

Lemmas for inverse sequences

The proof of the main result will require certain manipulations of inverse sequences of metric compacta. This Chapter will contain the needed results.

The next lemma follows from Corollary 1 of [MS2].

Lemma 5.1 *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of metric compacta* (X_i, d_i)*. Then there exists a sequence* (γ_i) *of positive numbers such that if* $\mathbf{Y} = (X_i, q_i^{i+1})$ *is an inverse sequence and* $d_i(p_i^{i+1}, q_i^{i+1}) < \gamma_i$ *for each* i*, then* $\lim \mathbf{Y} = \lim \mathbf{X}$*.* $\square$

We shall call such (γ_i) a *sequence of stability* for $\mathbf{X}$.

Let K be a simplicial complex, X a space, and $f : X \to |K|$ be a map. Recall Definition 1.12 for a K-modification. One calls f a K-*irreducible* map if each K-modification g of f is surjective. Note that, in this case, f is surjective and for any subdivision M of K, f is M-irreducible.

Lemma 5.2 *If* $f : X \to |K|$ *is a* K*-irreducible map, and* $g : X \to |K|$ *is a* K*-modification of* f*, then* g *is* K*-irreducible.*

Proof: We need to show that each K-modification of g is surjective. Let $h : X \to |K|$ be a K-modification of g. Since g is a K-modification of f, then h is also a K-modification of f: if $x \in X$, and $f(x) \in \mathring{\sigma}$ for some $\sigma \in K$, then $g(x) \in \sigma$, so $h(x) \in \sigma$, too. The K-irreducibility of f implies surjectivity of h. $\square$

From Theorem 3.11 of [JR] we may deduce the following.

Lemma 5.3 *Let X be a compact metrizable space. Then we may write X as the inverse limit of an inverse sequence $\mathbf{Q} = (|Q_i|, q_i^{i+1})$ of compact metric polyhedra, where each bonding map q_i^{i+1} is a Q_i-irreducible surjection.* $\square$

Lemma 5.4 *Let X be a compact metrizable space. Then there exists an inverse sequence $\mathbf{K} = (|K_i|, p_i^{i+1})$ of compact metric polyhedra $(|K_i|, d_i)$ along with a sequence of stability (γ_i) for $\mathbf{K}$ such that $\lim \mathbf{K} = X$, and for each $i \in \mathbb{N}$,* $\operatorname{mesh} K_i < \gamma_i$*. We may also specify that for some $m \in \mathbb{N}$, whenever $i \geq m$, then $p_i^{i+1} : |K_{i+1}| \to |K_i|$ is a K_i-irreducible simplicial map.*

Proof: Write $X = \lim \mathbf{Q}$, where $\mathbf{Q} = (|Q_i|, q_i^{i+1})$ is an inverse sequence of compact metric polyhedra $(|Q_i|, d_i)$ as in Lemma 5.3. By Lemma 5.1, we know that there is a sequence of stability (ρ_i) for $\mathbf{Q}$. For each i, put $\gamma_i = \rho_i/2$. Note that (γ_i) is also a sequence of stability for $\mathbf{Q}$.

Let K_1 be a subdivision of Q_1 with $\operatorname{mesh} K_1 < \gamma_1$. Suppose that $i \in \mathbb{N}$ and for each $1 \leq j \leq i$, we have chosen a subdivision K_j of Q_j with $\operatorname{mesh} K_j < \gamma_j$ and, when $1 < j$, a map $p_{j-1}^j : |K_j| \to |K_{j-1}|$ which is a simplicial approximation to q_{j-1}^j. Then select a subdivision K_{i+1} of Q_{i+1} with $\operatorname{mesh} K_{i+1} < \gamma_{i+1}$, and which supports a simplicial approximation $p_i^{i+1} : |K_{i+1}| \to |K_i|$ of q_i^{i+1}. Note that $d_i(q_i^{i+1}, p_i^{i+1}) < \gamma_i$, so q_i^{i+1} being Q_i-irreducible implies that each p_i^{i+1} is surjective.

Let us check that $\mathbf{K} := (|K_i|, p_i^{i+1})$ and $m = 1$ satisfy all of the requirements. Clearly $X = \lim \mathbf{K}$, since (γ_i) is a sequence of stability for $\mathbf{Q}$. It remains to show that the new bonding maps p_i^{i+1} are K_i-irreducible. First, note that q_i^{i+1} being Q_i-irreducible implies that q_i^{i+1} is also K_i-irreducible. Since p_i^{i+1} is a simplicial approximation of q_i^{i+1}, p_i^{i+1} is a K_i-modification of q_i^{i+1}. By Lemma 5.2, p_i^{i+1} is K_i-irreducible too. $\square$

Definition 5.5 Whenever X is a compact metrizable space, then we shall refer to an inverse sequence $\mathbf{K}$ of metric polyhedra $(|K_i|, d_i)$ which admits a sequence (γ_i) of positive numbers and $m \in \mathbb{N}$ so that the properties of Lemma 5.4 are satisfied as a representation of X which is *stable and simplicially irreducible from index* m with *associated sequence of stability* (γ_i).

Of course, Lemma 5.4 and its proof show that every compact metrizable space X has a representation $\mathbf{K}$ which is stable and simplicially irreducible from index $m = 1$.

Next, we want to define a certain type of move which when applied to such $\mathbf{K} = \mathbf{K}_0$ as in Definition 5.5 results in a $\mathbf{K}_1$ which is also a stable and simplicially irreducible (from some index m) representation of X. We will then show that if this procedure is repeated recursively in a controlled manner, resulting in a sequence $\mathbf{K}_1, \mathbf{K}_2, \ldots$, then there will be a limit $\mathbf{K}_\infty = \lim_{j\to\infty}(\mathbf{K}_j)$ which also will be a representation of X.

Lemma 5.6 *Let (ε_i) be a sequence of positive numbers. Let X be a compact metrizable space, let $\mathbf{K} = (|K_i|, p_i^{i+1})$ be a representation of X which is stable and simplicially irreducible from index m_1 with an associated sequence of stability (γ_i), and let $m \in \mathbb{N}_{\geq m_1}$. Define $\gamma'_i = \gamma_i$ if $1 \leq i < m$, $\gamma'_m = \frac{1}{2}[\gamma_m - \operatorname{mesh} K_m]$, and $\gamma'_i = \gamma_i/2$ if $i > m$. Let Σ be a subdivision of K_m with $\operatorname{mesh}\Sigma < \min\{\varepsilon_m, \gamma'_m\}$. Then there exists an inverse sequence $\mathbf{L} = (|L_i|, l_i^{i+1})$ as follows:*

(a) *in case $1 \leq i < m$, then $L_i = K_i$ and $l_i^{i+1} = p_i^{i+1}$,*

(b) $L_m = \Sigma$,

(c) *for each $i \geq m+1$, L_i is a subdivision of K_i with $\operatorname{mesh} L_i < \min\{\varepsilon_i, \gamma'_i\}$, and*

(d) *if* $i \geq m+1$, $l^i_{i-1} : |L_i| \to |L_{i-1}|$ *is a simplicial approximation to the map* p^i_{i-1}. $\square$

Definition 5.7 We shall call a pair $(\mathbf{L}, (\gamma'_i))$ as in Lemma 5.6 an *m-shift* of $(\mathbf{K}, (\gamma_i))$ from Σ.

Observe that $d_m(p^{m+1}_m, l^{m+1}_m) \leq \operatorname{mesh} \Sigma < \frac{1}{2}[\gamma_m - \operatorname{mesh} K_m] = \gamma'_m$. Hence if $g : |L_{m+1}| \to |L_m|$ is a map and $d_m(g, l^{m+1}_m) < \gamma'_m$, we may conclude that $d_m(g, p^{m+1}_m) < \gamma_m$. Indeed, the following is true:

(e) for each i, if $g : |L_{i+1}| \to |L_i|$ is a map and $d_i(g, l^{i+1}_i) < \gamma'_i$, then we have $d_i(g, p^{i+1}_i) < \gamma_i$.

Therefore we conclude:

Lemma 5.8 *Whenever* $(\mathbf{L}, (\gamma'_i))$ *is an* m-shift *of* $(\mathbf{K}, (\gamma_i))$ *from* Σ*, then* $\mathbf{L}$ *is a stable and simplicially irreducible representation of* X *from index* m *with associated sequence of stability* (γ'_i). $\square$

By exercising some additional care in the construction of $\mathbf{L}$, we may guarantee that for all i, $d_i(p^{i+1}_i, l^{i+1}_i) < \varepsilon_i$ (of course, $p^{i+1}_i = l^{i+1}_i$ if $i < m$).

It is routine to check that the next lemma holds true.

Lemma 5.9 *Let* X *be a compact metrizable space, and let* $\mathbf{K}_0$ *be a representation of* X *which is stable and simplicially irreducible from index* m_1*, with* $(\gamma_{(0),i})$ *a sequence of stability. For every* m_1*-shift* $(\mathbf{K}_1, (\gamma_{(1),i}))$ *of* $(\mathbf{K}_0, (\gamma_{(0),i}))$ *from* Σ_1 *(an appropriate subdivision of the triangulation of the* m_1*-term of* $\mathbf{K}_0$*),* $\mathbf{K}_1$ *is a representation of* X *which is stable and simplicially irreducible from index* m_1*, with* $(\gamma_{(1),i})$ *an associated sequence of stability. It satisfies property* (e) *with* $(\gamma'_i) =$

$(\gamma_{(1),i})$ *and* $(\gamma_i) = (\gamma_{(0),i})$. *The terms (as metric spaces) in* $\mathbf{K}_0$ *and* $\mathbf{K}_1$ *are equal. For* $i < m_1$, $\gamma_{(1),i} = \gamma_{(0),i}$, *the terms with index* i *have the same triangulations in* $\mathbf{K}_0$ *and* $\mathbf{K}_1$, *and the bonding maps in* $\mathbf{K}_0$ *and* $\mathbf{K}_1$ *with subscript* i *are equal. For* $i \geq m_1$, $\gamma_{(1),i}$ *need not equal* $\gamma_{(0),i}$, *the triangulation of the term in* $\mathbf{K}_1$ *with index* i *is a subdivision of that in* $\mathbf{K}_0$ *with the same index, and the bonding map with subscript* i *in* $\mathbf{K}_1$ *may differ from that in* $\mathbf{K}_0$ *with subscript* i.

If $i_0 \in \mathbb{N}$, $m_1 < \ldots < m_{i_0}$ *is a finite sequence in* $\mathbb{N}$, *and successively we have chosen* $(\mathbf{K}_j, (\gamma_{(j),i}))$ *an* m_j*-shift of* $(\mathbf{K}_{j-1}, (\gamma_{(j-1),i}))$ *from* Σ_j *(an appropriate subdivision of the* m_j*-term of* $\mathbf{K}_{j-1}$*),* $1 \leq j \leq i_0$, *then we may conclude that* $\mathbf{K}_{i_0}$ *is a representation of* X *which is stable and simplicially irreducible from index* m_{i_0}, *with* $(\gamma_{(i_0),i})$ *an associated sequence of stability; it satisfies property* (e) *with* $(\gamma'_i) = (\gamma_{(i_0),i})$ *and* $(\gamma_i) = (\gamma_{(i_0-1),i})$. *The terms (as metric spaces) in* $\mathbf{K}_0$ *and* $\mathbf{K}_{i_0}$ *are equal. For* $i < m_{i_0}$, $\gamma_{(i_0),i} = \gamma_{(i_0-1),i}$, *the terms with index* i *have the same triangulations in* $\mathbf{K}_{i_0-1}$ *and* $\mathbf{K}_{i_0}$, *and the bonding maps in* $\mathbf{K}_{i_0-1}$ *and* $\mathbf{K}_{i_0}$ *with subscript* i *are equal. For* $i \geq m_{i_0}$, $\gamma_{(i_0),i}$ *need not equal* $\gamma_{(i_0-1),i}$, *the triangulation of the term in* $\mathbf{K}_{i_0}$ *with index* i *is a subdivision of that in* $\mathbf{K}_{i_0-1}$ *with the same index, and the bonding map with subscript* i *in* $\mathbf{K}_{i_0}$ *may differ from that in* $\mathbf{K}_{i_0-1}$ *with subscript* i. $\square$

Henceforth we typically shall write $(|K_{(j),i}|, p^{i+1}_{(j),i})$ to denote such a representation $\mathbf{K}_j$, $0 \leq j \leq i_0$. One should note that, whenever $i_0 \geq j_0 \geq j \geq 1$, then $K_{(j),m_j} = K_{(j_0),m_j} = \Sigma_j$ when this occurs from the procedure in Lemma 5.9.

Definition 5.10 Let X be a compact metrizable space and let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function. Let $\mathbf{K}_0$ be a representation of X which is stable and simplicially irreducible from index $r(1)$, with $(\gamma_{(0),i})$ a sequence of stability. Suppose that $(\mathbf{K}_j, (\gamma_{(j),i}))$, $j \in \mathbb{N}$, is a sequence such that for each j, $(\mathbf{K}_j, (\gamma_{(j),i}))$

is an $r(j)$-shift of $(\mathbf{K}_{j-1}, (\gamma_{(j-1),i}))$ from Σ_j.

Then for each $k \in \mathbb{N}$, if m, l, and i are chosen so that $m \geq l \geq r(k) > i$, one sees that $p^{i+1}_{(l),i} = p^{i+1}_{(m),i}$ and $\gamma_{(l),i} = \gamma_{(m),i}$. So for each i, the sequences $(\gamma_{(j),i})_{j\in\mathbb{N}}$ and $(p^{i+1}_{(j),i})_{j\in\mathbb{N}}$ are eventually constant. Hence, in an obvious way, we may define an inverse sequence $\mathbf{K}_\infty = (|K_{(\infty),i}|, p^{i+1}_{(\infty),i}) = \lim\limits_{j\to\infty} \mathbf{K}_j$ and a sequence $(\gamma_{(\infty),i}) = \lim\limits_{j\to\infty}(\gamma_{(j),i})$ of positive numbers. Here, $K_{(\infty),i} = \lim\limits_{j\to\infty} K_{(j),i}$ and $p^{i+1}_{(\infty),i} = \lim\limits_{j\to\infty} p^{i+1}_{(j),i}$.

$$
\begin{tikzcd}
\mathbf{K}_0 : & {|K_{(0),1}|} & {|K_{(0),r(1)}|} \arrow[l, "p^{r(1)}_{(0),1}"'] \arrow[d, squiggly, "\Sigma_1"'] & \cdots \arrow[l, "p^{r(2)}_{(0),r(1)}"'] & & & & \\
\mathbf{K}_1 : & {|K_{(1),1}|} & {|K_{(1),r(1)}|} \arrow[l, "p^{r(1)}_{(\infty),1}"'] & {|K_{(1),r(2)}|} \arrow[l, "p^{r(2)}_{(1),r(1)}"'] \arrow[d, squiggly, "\Sigma_2"'] & \cdots \arrow[l, "p^{r(3)}_{(1),r(2)}"'] & & & \\
\mathbf{K}_2 : & {|K_{(2),1}|} & {|K_{(2),r(1)}|} \arrow[l, "p^{r(1)}_{(\infty),1}"'] & {|K_{(2),r(2)}|} \arrow[l, "p^{r(2)}_{(\infty),r(1)}"'] & \cdots \arrow[l, "p^{r(3)}_{(2),r(2)}"'] & & & \\
\vdots & & & & & & \vdots \arrow[d, squiggly, "\Sigma_{k-1}"'] & \\
\mathbf{K}_{k-1} : & & & & & {} & {|K_{(k-1),r(k-1)}|} \arrow[l, "p^{r(k-1)}_{(\infty),r(k-2)}"'] & {|K_{(k-1),r(k)}|} \arrow[l, "p^{r(k)}_{(k-1),r(k-1)}"'] \arrow[d, squiggly, "\Sigma_k"'] \\
\mathbf{K}_k : & & & & & & {} & {|K_{(k),r(k)}|} \arrow[l, "p^{r(k)}_{(\infty),r(k-1)}"'] \\
\vdots & & & & & & &
\end{tikzcd}
$$

From our construction and this definition, we can deduce the following:

Lemma 5.11 *Assume the notation of Definition* 5.10*. Then* $\mathbf{K}_\infty$ *is a representation of* X*. If* $i \in \mathbb{N}$*,* $g : |K_{(\infty),i+1}| \to |K_{(\infty),i}|$ *is a map, and* $d_i(g, p^{i+1}_{(\infty),i}) < \gamma_{(\infty),i}$*, then* $d_i(g, p^{i+1}_{(0),i}) < \gamma_{(0),i}$ *and hence* $(\gamma_{(\infty),i})$ *is a sequence of stability for* $\mathbf{K}_\infty$*.*

Proof: To show that $\mathbf{K}_\infty$ is a representation of X, it is enough to check that for all $i \in \mathbb{N}$, $d_i(p^{i+1}_{(\infty),i}, p^{i+1}_{(0),i}) < \gamma_{(0),i}$.

Take an $i \in \mathbb{N}$. If $i < r(1)$, then $p^{i+1}_{(\infty),i} = p^{i+1}_{(0),i}$ and $\gamma_{(\infty),i} = \gamma_{(0),i}$. Hence the statement $d_i(g, p^{i+1}_{(\infty),i}) < \gamma_{(\infty),i}$ implies that $d_i(g, p^{i+1}_{(0),i}) < \gamma_{(0),i}$.

If $i \geq r(1)$, then we know that $r(k-1) \leq i < r(k)$ for some $k \in \mathbb{N}_{\geq 2}$. The fact that $i < r(k)$ implies that $p^{i+1}_{(\infty),i} = p^{i+1}_{(k-1),i}$. On the other hand, $r(k-1) \leq i$ implies that $\gamma_{(j),i}$ has changed in every step of the construction from step 0 to $(k-1)$. That is, $\gamma_{(j),i} \leq \frac{1}{2}\gamma_{(j-1),i}$ for all $1 \leq j \leq k-1$, so $\gamma_{(j),i} \leq \frac{1}{2^j}\gamma_{(0),i}$. Therefore

$$\begin{aligned} d_i(p^{i+1}_{(\infty),i}, p^{i+1}_{(0),i}) = d_i(p^{i+1}_{(k-1),i}, p^{i+1}_{(0),i}) &\leq d_i(p^{i+1}_{(k-1),i}, p^{i+1}_{(k-2),i}) + \ldots + d_i(p^{i+1}_{(1),i}, p^{i+1}_{(0),i}) \\ &< \gamma_{(k-1),i} + \ldots + \gamma_{(1),i} \leq \frac{\gamma_{(0),i}}{2^{k-1}} + \ldots + \frac{\gamma_{(0),i}}{2} < \gamma_{(0),i} \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = \gamma_{(0),i}. \end{aligned}$$

By Lemma 5.1, $\lim \mathbf{K}_\infty = X$.

It remains to show that $d_i(g, p^{i+1}_{(\infty),i}) < \gamma_{(\infty),i}$ implies $d_i(g, p^{i+1}_{(0),i}) < \gamma_{(0),i}$. The fact that $i < r(k)$ implies that $\gamma_{(\infty),i} = \gamma_{(k-1),i}$.

So $d_i(g, p^{i+1}_{(\infty),i}) = d_i(g, p^{i+1}_{(k-1),i}) < \gamma_{(k-1),i}$. Therefore

$$\begin{aligned} d_i(p^{i+1}_{(0),i}, g) &\leq d_i(p^{i+1}_{(0),i}, p^{i+1}_{(1),i}) + d_i(p^{i+1}_{(1),i}, p^{i+1}_{(2),i}) + \ldots + d_i(p^{i+1}_{(k-2),i}, p^{i+1}_{(k-1),i}) + d_i(p^{i+1}_{(k-1),i}, g) \\ &< (\gamma_{(1),i} + \gamma_{(2),i} + \ldots + \gamma_{(k-1),i}) + \gamma_{(k-1),i} \\ &\leq \gamma_{(0),i} \cdot \left(\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{k-1}} \right) + \frac{1}{2^{k-1}} \right) = \gamma_{(0),i}. \ \square \end{aligned}$$

Chapter 6

Proof of the Main Theorem

Let us now prove Theorem 3.6.

Proof: We will construct, using induction:

- ⋄ an increasing function $r : \mathbb{N} \to \mathbb{N}$,
- ⋄ sequences of numbers $(\delta(i))_{i\in\mathbb{N}}$ and $(\varepsilon(i))_{i\in\mathbb{N}}$ such that $0 < \varepsilon(i) < \frac{\delta(i)}{3} < 1$, for all i,
- ⋄ a sequence of inverse sequences $\mathbf{K}_j = (|K_{(j),i}|, p^{i+1}_{(j),i})$, for $j \in \mathbb{Z}_{\geq 0}$, as described in Lemma 5.9, with terms that are compact polyhedra and with surjective bonding maps, and with $\lim \mathbf{K}_j = X$ (in fact, these sequences are representations for X that are stable and simplicially irreducible from index $r(j)$, with stability sequences $(\gamma_{(j),i})$, and $|K_{(j),i}| = |K_{(0),i}|$, for all i and j in $\mathbb{N}$),
- ⋄ a sequence of subdivisions Σ_i of $K_{(i-1),r(i)}$, for $i \in \mathbb{N}$, and
- ⋄ a sequence of maps $g^{r(i)}_{r(i-1)} : \ |\Sigma^{(n+1)}_i| \to |\Sigma^{(n)}_{i-1}|$, for $i \geq 2$,

such that for each i for which the statement makes sense, we have:

(I)$_i$ $g^{r(i)}_{r(i-1)}$ and $p^{r(i)}_{(i-1),r(i-1)}|_{|\Sigma^{(n+1)}_i|}$ are $\frac{\varepsilon(i-1)}{3}$ - close,

(II)$_i$ for any $y \in |K_{(i-1),r(i)}| = |\Sigma_i|$, diam $(p^{r(i)}_{(i-1),r(i-1)}(B_{\delta(i)}(y))) \ < \ \frac{\varepsilon(i-1)}{3}$,

(III)$_i$ for $i > j$ and for any $y \in |K_{(i-1),r(i)}| = |\Sigma_i|$, diam $(p^{r(i)}_{(j),r(j)}(B_{\varepsilon(i)}(y))) \ < \ \frac{\varepsilon(j)}{2^i}$,

(IV)$_i$ mesh $\Sigma_i < \min \{\frac{\varepsilon(i)}{3}, \gamma_{(i-1),r(i)}\}$, so mesh $\Sigma_i < \varepsilon(i)$, and

(V)$_i$ for any $y \in |K_{(i-1),r(i)}| = |\Sigma_i|$, $B_{\varepsilon(i)}(y) \subset P_{y,i} \subset B_{\delta(i)}(y)$, where $P_{y,i}$ is a contractible subpolyhedron of $|\Sigma_i|$.

In fact, this will prepare us to use Walsh's Lemma 4.2 with

$$\mathbf{X} = (|K_{(0),r(i)}|,\ p^{r(i+1)}_{(i),r(i)}),\quad \mathbf{Z} = (|\Sigma_i^{(n)}|,\ g^{r(i+1)}_{r(i)}|_{|\Sigma_i^{(n)}|}).$$

Let us start the construction by taking a representation for X which is stable and simplicially irreducible from index 1: $\mathbf{K}_0 = (|K_{(0),i}|, p^{i+1}_{(0),i})$, $\lim \mathbf{K}_0 = X$, with stability sequence $(\gamma_{(0),i})$.

Define $r(1) := 1$.

We will choose $0 < \delta(1) < 1$ any way we want. Next, we pick an intermediate subdivision $\widetilde{\Sigma}_1$ of $K_{(0),1}$ so that for any $y \in |K_{(0),1}|$, any closed $\widetilde{\Sigma}_1$-vertex star containing y is contained in the closed $\delta(1)$-ball $B_{\delta(1)}(y)$. (It is enough to make $\operatorname{mesh} \widetilde{\Sigma}_1 < \frac{\delta(1)}{2}$, so $\operatorname{diam}(\overline{\operatorname{st}}(w, \widetilde{\Sigma}_1)) \ \leq\ 2 \operatorname{mesh} \widetilde{\Sigma}_1 \ < \delta(1)$.)

Now choose an $\varepsilon(1)$ so that $0 < \varepsilon(1) < \frac{\delta(1)}{3}$, and for any $y \in |K_{(0),1}|$, the closed $\varepsilon(1)$-ball $B_{\varepsilon(1)}(y)$ sits inside an open vertex star with respect to $\widetilde{\Sigma}_1$. (This can be done as follows: form the open cover for $|K_{(0),1}|$ consisting of the open stars $\operatorname{st}(w, \widetilde{\Sigma}_1)$. There is a Lebesgue number λ for this cover, so make your $\varepsilon(1) < \frac{\lambda}{2}$. Then for any $y \in |K_{(0),1}|$, $\operatorname{diam} B_{\varepsilon(1)}(y) < \lambda \ \Rightarrow\ B_{\varepsilon(1)}(y) \subset \operatorname{st}(w_0, \widetilde{\Sigma}_1)$, for some $w_0 \in \widetilde{\Sigma}_1^{(0)}$. Fix such w_0 for each y.)

Note that for any $y \in |K_{(0),1}|$, $B_{\varepsilon(1)}(y) \subset \ |\overline{\operatorname{st}}(w_0, \widetilde{\Sigma}_1)| \ \subset B_{\delta(1)}(y)$. Define $P_{y,1} := |\overline{\operatorname{st}}(w_0, \widetilde{\Sigma}_1)|$, which is a contractible subpolyhedron of $|K_{(0),1}|$, so (V)$_1$ is satisfied.

Choose a subdivision Σ_1 of $\widetilde{\Sigma}_1$ with $\operatorname{mesh} \Sigma_1 < \min\{\frac{\varepsilon(1)}{3}, \gamma_{(0),1}\}$, which implies (IV)$_1$.

Let $(\mathbf{K}_1, (\gamma_{(1),i}))$ be a 1-shift of $(\mathbf{K}_0, (\gamma_{(0),i}))$ from Σ_1, i.e., $\mathbf{K}_1 = (|K_{(1),i}|, p_{(1),i}^{i+1})$ is an inverse sequence with $K_{(1),1} = \Sigma_1$, limit equal X, and stability sequence $(\gamma_{(1),i})$. Note that at this point, all bonding maps in $\mathbf{K}_1$ are simplicial because $\mathbf{K}_1$ is simplicially irreducible from index 1. Let

$$\mathbf{Y}_1 := (|K_{(1),i}^{(n+1)}|,\ p_{(1),i}^{i+1}|_{\ |K_{(1),i+1}^{(n+1)}|})$$

be the inverse sequence of the $(n+1)$-skeleta of the polyhedra in $\mathbf{K}_1$, where the bonding maps are the restrictions of the original bonding maps. Notice that every $p_{(1),i}^{i+1}|_{\ |K_{(1),i+1}^{(n+1)}|} : |K_{(1),i+1}^{(n+1)}| \to |K_{(1),i}^{(n+1)}|$ is still simplicial and surjective: since $p_{(1),i}^{i+1}$ is simplicial and surjective, for every simplex $\sigma \in K_{(1),i}^{(n+1)}$ with $\dim \sigma = k$, there exists a simplex $\tau \in K_{(1),i+1}$ such that $\dim \tau \geq k$ and $p_{(1),i}^{i+1}(\tau) = \sigma$. So there must be a k-face of τ which is mapped by $p_{(1),i}^{i+1}$ onto σ. In particular, for every $(n+1)$-dimensional $\sigma \in K_{(1),i}^{(n+1)}$, there exists an $(n+1)$-simplex in $K_{(1),i+1}$ that is mapped onto σ by $p_{(1),i}^{i+1}$.

Now let $Y_1 = \lim \mathbf{Y}_1$. Then $\dim Y_1 \leq n+1$, and $X \tau K$ implies $Y_1\ \tau K$. So by Lemma 4.5, we get $\dim_G Y_1 \leq n$.

Since $P_G = \mathbb{P}$, Lemma 2.5 implies that $\dim_{\mathbb{Z}} Y_1 = \dim_G Y_1 \leq n$, so we can apply Edwards' Theorem 4.4 to $\mathbf{Y}_1$: there exists an $s \in \mathbb{N}$, $s > 1$ and a map $\widehat{g}_1^s : |K_{(1),s}^{(n+1)}| \to |K_{(1),1}^{(n)}|$ so that if $z \in |K_{(1),s}^{(n+1)}|$, and $p_{(1),1}^s(z)$ lands in the combinatorial interior $\overset{\circ}{\sigma}$ of a simplex σ of $K_{(1),1}^{(n+1)}$, then $\widehat{g}_1^s(z)$ lands in σ. This will, eventually, lead to property $(\mathrm{I})_2$.

$$\begin{tikzcd}
|K_{(1),1}^{(n)}| \arrow[d, hook] & & & \\
|K_{(1),1}^{(n+1)}| & |K_{(1),r(2)}^{(n+1)}| \arrow[l, "p_{(1),1}^{r(2)}|"'] \arrow[ul, dashed, "\widehat{g}_1^{r(2)}"'] & \cdots \arrow[l] & Y_1
\end{tikzcd}$$

Define $r(2) := s$. Using uniform continuity of the map $p^{r(2)}_{(1),1}$, choose $0 < \delta(2) < 1$ so that $(\mathrm{II})_2$ is true: for any $y \in |K_{(1),r(2)}|$, diam $(p^{r(2)}_{(1),1}(B_{\delta(2)}(y))) < \frac{\varepsilon(1)}{3}$. Pick an intermediate subdivision $\widetilde{\Sigma}_2$ of $K_{(1),r(2)}$ so that for any $y \in |K_{(1),r(2)}| = |K_{(0),r(2)}|$, any closed $\widetilde{\Sigma}_2$-vertex star containing y is contained in $B_{\delta(2)}(y)$.

Now choose an $\varepsilon(2)$ so that $0 < \varepsilon(2) < \frac{\delta(2)}{3}$, and so that $(\mathrm{III})_2$ will be true: for any $y \in |K_{(1),r(2)}|$, diam $(p^{r(2)}_{(1),1}(B_{\varepsilon(2)}(y))) < \frac{\varepsilon(1)}{2^2}$. This follows from the uniform continuity of $p^{r(2)}_{(1),1}$. Also, make sure that $(\mathrm{V})_2$ is true: for any $y \in |K_{(1),r(2)}|$, the closed $\varepsilon(2)$-ball centered at y sits inside an open vertex star with respect to $\widetilde{\Sigma}_2$, i.e., $B_{\varepsilon(2)}(y) \subset \mathrm{st}(w_0, \widetilde{\Sigma}_2)$, for some $w_0 \in \widetilde{\Sigma}_2^{(0)}$. Therefore $B_{\varepsilon(2)}(y) \subset |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_2)| \subset B_{\delta(2)}(y)$. Define $P_{y,2} := |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_2)|$, which is a contractible subpolyhedron of $|K_{(1),r(2)}|$.

Choose a subdivision Σ_2 of $\widetilde{\Sigma}_2$ with mesh $\Sigma_2 < \gamma_{(1),r(2)}$, where $\gamma_{(1),r(2)}$ is from the stability sequence $(\gamma_{(1),i})$ for $\mathbf{K}_1$. Also make sure that mesh $\Sigma_2 < \frac{\varepsilon(2)}{3}$, which implies $(\mathrm{IV})_2$. Note that Σ_2 is a subdivision of $K_{(1),r(2)}$.

$$
\begin{tikzcd}
\mathbf{K}_1 : & & \cdots & |K_{(1),r(2)}| \arrow[l] & \cdots \arrow[l, "p^{r(2)+1}_{(1),r(2)}"'] & & \\
\mathbf{K}_2 : & \cdots & |\Sigma_2| = \arrow[l] & |K_{(2),r(2)}| \arrow[u, "id"] \arrow[u, dotted, shift right, "j"'] & |K_{(2),r(2)+1}| \arrow[l, "p^{r(2)+1}_{(2),r(2)}"'] & \cdots \arrow[l] & X \\
\mathbf{Y}_2 : & & |\Sigma_2^{(n+1)}| = & |K^{(n+1)}_{(2),r(2)}| \arrow[u, hook] & |K^{(n+1)}_{(2),r(2)+1}| \arrow[l, "p^{r(2)+1}_{(2),r(2)}|"'] \arrow[u, hook] & \cdots \arrow[l] & Y_2 \arrow[u, hook]
\end{tikzcd}
$$

Now we can build $\mathbf{K}_2 = (|K_{(2),i}|,\ p^{i+1}_{(2),i})$ as an $r(2)$-shift of $(\mathbf{K}_1, (\gamma_{(1),i}))$ from Σ_2, i.e., $\mathbf{K}_2 = (|K_{(2),i}|, p^{i+1}_{(2),i})$ is an inverse sequence with $K_{(2),r(2)} = \Sigma_2$ and limit X, and stability sequence $(\gamma_{(2),i})$. Notice that for $i \geq r(2)$, the maps $p^{i+1}_{(2),i}$ are simplicial.

Let $j : |\Sigma_2| \to |K_{(1),r(2)}|$ be a simplicial approximation to the identity map.

Since j is simplicial, $j(|\Sigma_2^{(n+1)}|) \subset |K^{(n+1)}_{(1),r(2)}|$, so treat $j|_{|\Sigma_2^{(n+1)}|} : |\Sigma_2^{(n+1)}| \to |K^{(n+1)}_{(1),r(2)}|$.

Define $g^{r(2)}_{r(1)} := \widehat{g}_1^{r(2)} \circ j|_{|\Sigma_2^{(n+1)}|} : |\Sigma_2^{(n+1)}| \to |K^{(n)}_{(1),1}| = |\Sigma_1^{(n)}|$. For any $y \in |\Sigma_2^{(n+1)}|$, y and $j(y)$ have to be contained in the same simplex of $K_{(1),r(2)}$. Since $p^{r(2)}_{(1),1} : |K_{(1),r(2)}| \to |K_{(1),1}|$ is simplicial, $p^{r(2)}_{(1),1}(y)$ and $p^{r(2)}_{(1),1}(j(y))$ land in the same simplex of $K_{(1),1} = \Sigma_1$. On the other hand, because of our choice of $\widehat{g}_1^{r(2)}$, if $p^{r(2)}_{(1),1}(j(y))$ lands in $\mathring{\sigma}$, for some simplex σ of $K^{(n+1)}_{(1),1}$, then $\widehat{g}_1^{r(2)}(j(y))$ lands in σ, too. Therefore

$$d_1(p^{r(2)}_{(1),1}(y), \widehat{g}_1^{r(2)}(j(y))) \ \leq \ \operatorname{mesh} K_{(1),1} = \operatorname{mesh} \Sigma_1 < \frac{\varepsilon(1)}{3}.$$

Thus $g^{r(2)}_{r(1)}$ and $p^{r(2)}_{(1),1}|_{|\Sigma_2^{(n+1)}|} = p^{r(2)}_{(1),r(1)}|_{|\Sigma_2^{(n+1)}|}$ are $\frac{\varepsilon(1)}{3}$-close, so $(\mathrm{I})_2$ is true. This concludes the basis of induction. The following diagram summarizes the preceding construction.

$$\begin{tikzcd}
|\Sigma_1^{(n)}| = |K^{(n)}_{(1),1}| \arrow[d, hook] & & \\
|\Sigma_1^{(n+1)}| = |K^{(n+1)}_{(1),1}| \arrow[d, hook] & |K^{(n+1)}_{(1),r(2)}| \arrow[l, "p^{r(2)}_{(1),1}|"'] \arrow[ul, "\widehat{g}_1^{r(2)}"'] \arrow[d, hook] & \\
|\Sigma_1| = |K_{(1),1}| & |K_{(1),r(2)}| \arrow[l, "p^{r(2)}_{(1),1}"'] & |K^{(n+1)}_{(2),r(2)}| = |\Sigma_2^{(n+1)}| \arrow[ul, "j|"'] \arrow[d, hook] \\
& & |K_{(2),r(2)}| \arrow[ul, "j"'] \arrow[ul, bend left, "id"]
\end{tikzcd}$$

Step of induction. Let $k \in \mathbb{N}_{\geq 3}$. Suppose that we have chosen, as required above,

◇ for $j = 1, \ldots, k-1$, the numbers $r(j)$, $\delta(j)$, $\varepsilon(j)$,

◇ for $j = 0, \ldots, k-1$, the inverse sequences $\mathbf{K}_j = (|K_{(j),i}|, p^{i+1}_{(j),i})$, which are stable and simplicially irreducible from index $r(j)$, with stability sequences $(\gamma_{(j),i})$,

◇ for $j = 1, \ldots, k-1$, subdivisions Σ_j of $K_{(j-1),r(j)}$, and

◇ for $j = 2, \ldots, k-1$, maps $g^{r(j)}_{r(j-1)} : |\Sigma^{(n+1)}_j| \to |\Sigma^{(n)}_{j-1}|$,

so that the properties $(\mathrm{I})_j$-$(\mathrm{V})_j$ are satisfied for each $j = 1, \ldots, k-1$ for which they make sense.

Focus on the inverse sequence $\mathbf{K}_{k-1} = (|K_{(k-1),i}|,\ p^{i+1}_{(k-1),i})$. For $i \geq r(k-1)$, the bonding maps $p^{i+1}_{(k-1),i}$ are simplicial. Recall that $\lim \mathbf{K}_{k-1} = X$, and notice that $K_{(k-1),r(k-1)} = \Sigma_{k-1}$. Let

$$\mathbf{Y}_{k-1} := (|K^{(n+1)}_{(k-1),i}|,\ p^{i+1}_{(k-1),i}|_{\ |K^{(n+1)}_{(k-1),i+1}|})_{i \geq r(k-1)}$$

be the inverse sequence of the $(n+1)$-skeleta of the polyhedra in $\mathbf{K}_{k-1}$, starting with the $(r(k-1))$-th polyhedron onward. As before, the bonding maps are restrictions of the original bonding maps, and these restrictions are simplicial and surjective.

Now let $Y_{k-1} = \lim \mathbf{Y}_{k-1}$. Then $\dim Y_{k-1} \leq n+1$, and $X \tau K$ implies $Y_{k-1}\, \tau K$. So by Lemma 4.5, we get $\dim_G Y_{k-1} \leq n$. As before, since $P_G = \mathbb{P}$, Lemma 2.5 implies $\dim_{\mathbb{Z}} Y_{k-1} = \dim_G Y_{k-1} \leq n$, so we can apply Edwards' Theorem 4.4 to $\mathbf{Y}_{k-1}$, noticing that the first entry in $\mathbf{Y}_{k-1}$ has index $r(k-1)$.

So there exists an $s \in \mathbb{N}$, $s > r(k-1)$ and a map $\widehat{g}^s_{r(k-1)} : |K^{(n+1)}_{(k-1),s}| \to |K^{(n)}_{(k-1),r(k-1)}|$ so that if $z \in |K^{(n+1)}_{(k-1),s}|$, and $p^s_{(k-1),r(k-1)}(z)$ lands in the combinatorial interior $\mathring{\sigma}$ of a simplex σ of $K^{(n+1)}_{(k-1),r(k-1)}$, then $\widehat{g}^s_{r(k-1)}(z)$ lands in σ. This will help us get the property $(\mathrm{I})_k$.

$$\begin{tikzcd}
{|K^{(n)}_{(k-1),r(k-1)}|} \arrow[d, hook] & & & \\
{|K^{(n+1)}_{(k-1),r(k-1)}|} & {|K^{(n+1)}_{(k-1),r(k)}|} \arrow[l, "p^{r(k)}_{(k-1),r(k-1)}|"] \arrow[ul, dashed, "\widehat{g}^{r(k)}_{r(k-1)}"'] & \cdots \arrow[l] & Y_{k-1}
\end{tikzcd}$$

Define $r(k) := s$. Using the uniform continuity of the map $p^{r(k)}_{(k-1),r(k-1)}$, choose $0 < \delta(k) < 1$ so that $(\mathrm{II})_k$ is true:

$$\forall y \in |K_{(k-1),r(k)}|, \quad \mathrm{diam}\ (p^{r(k)}_{(k-1),r(k-1)}(B_{\delta(k)}(y))) \ < \ \frac{\varepsilon(k-1)}{3}.$$

Pick an intermediate subdivision $\widetilde{\Sigma}_k$ of $K_{(k-1),r(k)}$ so that for any $y \in |K_{(k-1),r(k)}|$, any closed $\widetilde{\Sigma}_k$-vertex star containing y is contained in $B_{\delta(k)}(y)$.

Now choose an $\varepsilon(k)$ so that $0 < \varepsilon(k) < \frac{\delta(k)}{3}$, and so that $(\mathrm{III})_k$ and $(\mathrm{V})_k$ will be true. First make sure that for all $y \in |K_{(k-1),r(k)}|$, the closed $\varepsilon(k)$-ball centered at y sits inside an open $\widetilde{\Sigma}_k$-vertex star, i.e., $B_{\varepsilon(k)}(y) \subset \mathrm{st}(w_0, \widetilde{\Sigma}_k)$, for some $w_0 \in \widetilde{\Sigma}_k^{(0)}$. Therefore $B_{\varepsilon(k)}(y) \subset \ |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_k)| \ \subset B_{\delta(k)}(y)$. Define $P_{y,k} := |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_k)|$, which is a contractible subpolyhedron of $|K_{(k-1),r(k)}|$. So $(\mathrm{V})_k$ is satisfied. Next, we know that for all $j < k$, the maps $p^{r(k)}_{(j),r(j)}$ are uniformly continuous. We also know that, in our notation, $j < k$ implies that $p^{r(k)}_{(j),r(j)} = p^{r(k)}_{(k-1),r(j)}$. So we can make a choice of $\varepsilon(k)$ so that we have: for any $y \in |K_{(k-1),r(k)}|$,

$$\begin{aligned}
\operatorname{diam}\,(p^{r(k)}_{(1),r(1)}(B_{\varepsilon(k)}(y))) &< \frac{\varepsilon(1)}{2^k}, \\
\operatorname{diam}\,(p^{r(k)}_{(2),r(2)}(B_{\varepsilon(k)}(y))) &< \frac{\varepsilon(2)}{2^k}, \\
&\ \vdots \\
\operatorname{diam}\,(p^{r(k)}_{(k-1),r(k-1)}(B_{\varepsilon(k)}(y))) &< \frac{\varepsilon(k-1)}{2^k}.
\end{aligned}$$

So $(\mathrm{III})_k$ is true.

Choose a subdivision Σ_k of $\widetilde{\Sigma}_k$ with $\operatorname{mesh}\Sigma_k < \gamma_{(k-1),r(k)}$, where $\gamma_{(k-1),r(k)}$ is from the stability sequence $(\gamma_{(k-1),i})$ for $\mathbf{K}_{k-1}$. Also make sure that $\operatorname{mesh}\Sigma_k < \frac{\varepsilon(k)}{3}$, which implies $(\mathrm{IV})_k$. Note that Σ_k is a subdivision of $K_{(k-1),r(k)}$.

$$\begin{tikzcd}
\mathbf{K}_{k-1}: & & \cdots & {|K_{(k-1),r(k)}|} \arrow[l] & \cdots \arrow[l, "p^{r(k)+1}_{(k-1),r(k)}"'] & & \\
\mathbf{K}_k: & \cdots & {|\Sigma_k| =} \arrow[l] & {|K_{(k),r(k)}|} \arrow[u, "id"] \arrow[u, dotted, bend right, "j"'] & {|K_{(k),r(k)+1}|} \arrow[l, "p^{r(k)+1}_{(k),r(k)}"'] & \cdots \arrow[l] & X \\
\mathbf{Y}_k: & & {|\Sigma_k^{(n+1)}| =} & {|K^{(n+1)}_{(k),r(k)}|} \arrow[u, hook] & {|K^{(n+1)}_{(k),r(k)+1}|} \arrow[l, "p^{r(k)+1}_{(k),r(k)}|"'] \arrow[u, hook] & \cdots \arrow[l] & Y_k \arrow[u, hook]
\end{tikzcd}$$

Now we can build $\mathbf{K}_k = (|K_{(k),i}|,\ p^{i+1}_{(k),i})$ as an $r(k)$-shift of $(\mathbf{K}_{k-1}, (\gamma_{(k-1),i}))$ from Σ_k, i.e., $\mathbf{K}_k = (|K_{(k),i}|, p^{i+1}_{(k),i})$ is an inverse sequence with $K_{(k),r(k)} = \Sigma_k$ and limit X, and stability sequence $(\gamma_{(k),i})$. For index $i \geq r(k)$, the bonding maps $p^{i+1}_{(k),i}$ are simplicial.

Let $j : |\Sigma_k| \to |K_{(k-1),r(k)}|$ be a simplicial approximation to the identity map. Since j is simplicial, $j(|\Sigma_k^{(n+1)}|) \subset |K^{(n+1)}_{(k-1),r(k)}|$.

Define $g^{r(k)}_{r(k-1)} := \widehat{g}^{r(k)}_{r(k-1)} \circ j|_{|\Sigma_k^{(n+1)}|} : |\Sigma_k^{(n+1)}| \to |K^{(n)}_{(k-1),r(k-1)}| = |\Sigma^{(n)}_{k-1}|$. For any $y \in |\Sigma_k^{(n+1)}|$, y and $j(y)$ have to be contained in the same simplex of $K_{(k-1),r(k)}$.

Since $p^{r(k)}_{(k-1),r(k-1)} : |K_{(k-1),r(k)}| \to |K_{(k-1),r(k-1)}|$ is simplicial, $p^{r(k)}_{(k-1),r(k-1)}(y)$ and $p^{r(k)}_{(k-1),r(k-1)}(j(y))$ land in the same simplex τ of $K_{(k-1),r(k-1)} = \Sigma_{k-1}$. On the other hand, because of our choice of $\widehat{g}^{r(k)}_{r(k-1)}$, if $p^{r(k)}_{(k-1),r(k-1)}(j(y))$ lands in $\mathring{\sigma}$, for some simplex σ of $K^{(n+1)}_{(k-1),r(k-1)}$ which is a face of τ, then $\widehat{g}^{r(k)}_{r(k-1)}(j(y))$ lands in σ, too. Therefore

$$d_{k-1}(p^{r(k)}_{(k-1),r(k-1)}(y), \widehat{g}^{r(k)}_{r(k-1)}(j(y))) \;\leq\; \operatorname{mesh} K_{(k-1),r(k-1)} = \operatorname{mesh} \Sigma_{k-1} < \frac{\varepsilon(k-1)}{3}.$$

Hence $g^{r(k)}_{r(k-1)}$ and $p^{r(k)}_{(k-1),r(k-1)}|_{|\Sigma^{(n+1)}_k|}$ are $\frac{\varepsilon(k-1)}{3}$-close, so $(\mathrm{I})_k$ is true. This concludes the inductive step.

$$\begin{tikzcd}
{|\Sigma^{(n)}_{k-1}| = |K^{(n)}_{(k-1),r(k-1)}|} \arrow[d, hook] & & \\
{|\Sigma^{(n+1)}_{k-1}| = |K^{(n+1)}_{(k-1),r(k-1)}|} \arrow[d, hook] & {|K^{(n+1)}_{(k-1),r(k)}|} \arrow[l, "p^{r(k)}_{(k-1),r(k-1)}|"'] \arrow[ul, "\widehat{g}^{r(k)}_{r(k-1)}"'] \arrow[d, hook] & \\
{|\Sigma_{k-1}| = |K_{(k-1),r(k-1)}|} & {|K_{(k-1),r(k)}|} \arrow[l, "p^{r(k)}_{(k-1),r(k-1)}"'] & {|K^{(n+1)}_{(k),r(k)}| = |\Sigma^{(n+1)}_k|} \arrow[ul, "j|"'] \arrow[d, hook] \\
 & & {|K_{(k),r(k)}|} \arrow[ul, bend left, "j"'] \arrow[ul, bend left=30, "id"]
\end{tikzcd}$$

Notice that the inverse sequence

$$\mathbf{X} := (|K_{(0),r(i)}|,\ p^{r(i+1)}_{(i),r(i)}) \;=\; (|K_{(i),r(i)}|,\ p^{r(i+1)}_{(i),r(i)}) \;=\; (|\Sigma_i|,\ p^{r(i+1)}_{(i),r(i)})$$

is a subsequence of $\mathbf{K}_\infty = (|K_{(\infty),i}|,\ p^{i+1}_{(\infty),i}) = (|K_{(0),i}|,\ p^{i+1}_{(\infty),i})$. By Lemma 5.11, $\lim \mathbf{K}_\infty = X$, so $\lim \mathbf{X}$ is homeomorphic to X. Without loss of generality, assume that $\lim \mathbf{X} = X$.

Let $\mathbf{Z} := (|\Sigma_i^{(n)}|,\ g_{r(i)}^{r(i+1)}|_{|\Sigma_{i+1}^{(n)}|})$. Since $|\Sigma_i^{(n)}|$ are metrizable, compact and nonempty, $\lim \mathbf{Z} = Z$ is a nonempty compact metrizable space. Clearly, $\dim Z \leq n$, which also implies that $\dim_G Z \leq n$. Now $Z\tau K$ follows from Lemma 4.5.

Apply Walsh's Lemma 4.2 to these $\mathbf{X}$ and $\mathbf{Z}$: since the requirements (I)-(VI) of Lemma 4.2 are satisfied, there is a cell-like surjective map $\pi : Z \to X$. $\square$

Corollary 6.1 *Let G be an abelian group with $P_G = \mathbb{P}$. Let K be a connected* CW-*complex with $\pi_1(K) \cong G$. Then every compact metrizable space X with $X\tau K$ has to have* $\dim X \leq 1$.

Proof: Theorem 3.6 is true for $n = 1$, so for any compact metrizable space X with $X\tau K$, we can find a compact metrizable space Z with $\dim Z \leq 1$, $Z\tau K$ and a surjective cell-like map $\pi : Z \to X$. Cell-like maps are G-acyclic, so in particular, π is a $\mathbb{Z}$-acyclic map.

The Vietoris-Begle Theorem implies that a G-acyclic map cannot raise $\dim_G$-dimension. Since $\dim Z \leq 1$ implies that $\dim_{\mathbb{Z}} Z \leq 1$, and since π is a $\mathbb{Z}$-acyclic map, we have that $\dim_{\mathbb{Z}} X \leq 1$, too. Recall that $\dim_{\mathbb{Z}} X \leq 1 \;\Leftrightarrow\; \dim X \leq 1$. $\square$

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