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SEPARATION AND FENCHEL-TYPE DUALITY THEOREMS FOR FUNCTIONS WITH INTEGER AND REAL VARIABLES

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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ABSTRACT

The concepts of convex and concave functions are of critical importance in optimization theory. These concepts arise for both functions of real (that is, continuous) variables and functions of integer variables (that is, discrete) variables, and have been studied extensively by many researchers. In particular, in both real and integer convexity theory many results on the separation of convex sets and necessary conditions for an extremum (so-called Fenchel duality results) have been obtained. However, to the best of our knowledge little or no attention has been given to the study of convexity (and concavity) notions for functions that depend simultaneously on both real and integer variables, a class of functions that we will call "mixed convex/concave functions." In this dissertation we introduce a variety of notions of mixed convexity (and mixed concavity) for both functions and sets. After introducing these various notions, we derive some of their elementary properties and then prove separation and Fenchel duality results for each type of mixed convexity that is introduced.

CHAPTER 1

INTRODUCTION

1.1 CONVEXITY THEORY

In this section, basic information about three types of convexity, discrete, real and mixed convexity, is provided along with a discussion of the organization of the dissertation.

1.1.1 REAL (CONTINUOUS) CONVEX ANALYSIS

In real convex analysis, separation theorems for convex-concave functions and convex sets are well known (see Rockafellar (1970), Stoer-Witzgall (1970), Rockafellar (1974), Rockafellar-Wets (1998), Borwein and Lewis (2000), and Hiriart-Urruty and Lemaréchal (2000)). The Legendre-Fenchel duality theorem is another well known result in real convex analysis (see Stoer-Witzgall (1970) for Fenchel-type duality theorem and related examples). Separation and Legendre-Fenchel duality theorems have important applications for optimization problems with real variables. In this work, we will have a brief review of the separation and Legendre-Fenchel duality theorems in real convex analysis.

In certain optimization problems the independent variable (or certain coordinates of it) may naturally be restricted to integer values. This has inspired many researchers to define new notions of discrete and mixed convex-concave functions. Some of the well known results from real convex analysis, such as the separation and Legendre-Fenchel duality theorems for real convex-concave functions and convex sets, have counterparts in discrete convex analysis for various definitions of discrete convex functions.

1.1.2 DISCRETE CONVEX ANALYSIS

The theory of the convexity of real valued functions of an integer variable is initiated from real convexity theory, but also relies heavily on combinatorial arguments. Along with the functions which are real extensible, several definitions of discrete convex function have been introduced. The classical definition states that a discrete function of a single variable is convex if its first forward difference is increasing or at least non-decreasing, as defined by Denardo (1982), Fox (1966) and many others in the literature. Some of the discrete convex function definitions and their introducers are; "discretely convex functions" by Miller (1971), "integrally-convex functions" by Favati and Tardella (1990), "$M^{\natural}$-convex functions" by Murota and Shioura (1999), "$L^{\natural}$-convex functions" by Fujishige and Murota (2000), "L-convex functions" and "M-convex functions" by Fujishige and Murota (2000), "strongly discrete convex functions" by Yüceer (2002), and "D-convex and semistrictly quasi D-convex functions" by Ui (2006). One comprehensive discrete convex function definition, the notion of D-convex function, was introduced by Ui (2006), which has a unified form that includes discretely convex, integrally convex, M convex, $M^{\natural}$ convex, L convex and $L^{\natural}$ convex functions in local settings. An important application of discrete convexity is to solve optimization problems in applied mathematics such as network flow optimization problems. The three discrete convexity concepts, discrete condense convexity, discrete L-convexity and discrete M-convexity, will play the primary role for the results of this work. Some of the results in discrete convex analysis, such as separation and Fenchel-type duality theorems for condense discrete, discrete L and discrete M convex-concave functions, will be used to derive the main results of this work.

The separation and Fenchel-type duality theorems for discrete convex functions were stated and proven in the early 1980's. These include the discrete separation theorem by Frank (1982), the Fenchel-type duality theorem for discrete convex functions by Fujishige (1984) , and the separation and Fenchel-type duality theorems for L/M-convex functions by Murota (1996) .

1.1.3 MIXED CONVEX ANALYSIS

The optimization of discrete and real convex functions' is an important part of applied mathematics. The mixed convexity of a mixed function $\alpha : \mathbb{Z}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$, a function that is real extensible with integer and real variables was introduced by Tokgöz, Maalouf and Kumin (2009). Tokgöz et al. also introduced a mixed Hessian matrix of a mixed convex function with similar properties to the Hessian matrix of a C^2 real convex function. In the literature, there are well known problems that include both real and integer variables, such as a two parameter design problem associated with the $M/E_k/1$ Queueing system suggested by Kumin (1973) and a two parameter design problem associated with the $M/M/s$ Queueing system. Tokgöz, Maalouf and Kumin (2009) proposed a solution to the convexity and optimization problem of the two parameter design problem suggested by Kumin (1973) after introducing a mixed Hessian matrix and showing how it could be used to obtain optimization results.

Tokgöz (2009) defined T, T^*, E and E^* convex-concave functions by using the definitions of L, $L^\natural$, M and $M^\natural$ convex (concave) functions, respectively. In addition, the mixed Hessian matrices corresponding to T, T^*, E and E^* convex-concave functions and optimization results are given.

1.2 ORGANIZATION OF THE DISSERTATION

In this work, we examine closely the definition of a real convex-concave function and important results for real variable functions such as separation and Fenchel-type duality theorems. The definitions of condense discrete, discrete L and discrete M convex-concave functions are given along with versions of the separation and Fenchel-type duality theorems. Tokgöz, Maalouf and Kumin (2009) defined mixed convexity of a mixed function, introduced a mixed Hessian matrix and proved optimization results regarding to the mixed convex functions specific to each definition. The results of Tokgöz, Maalouf and Kumin (2009) are related to the convexity and optimization of mixed functions, and do not include separation or Fenchel duality results. Considering the condense mixed convex (resp. concave), T, T^*, E, and E^* (resp. concave) functions, the separation and Fenchel-type duality theorems have not been proposed by any researcher. Therefore, in this work, Fenchel-type duality and separation theorems are stated and proven by considering the mixed *condense*, T, T^*, E and E^* convex-concave functions. In addition, T_1, T_1^*, E_1 and E_1^* convex-concave functions are defined and related Fenchel-type duality and separation theorems are stated and proven.

Chapter 2

REAL CONVEX-CONCAVE FUNCTIONS

In this section we recall the definitions relevant to the separation and Fenchel-type duality theorems in real convex analysis. The separation and Fenchel-type duality theorems are stated but not proven.

2.1 REAL CONVEX-CONCAVE FUNCTIONS

The convexity-concavity of real variable functions have been studied extensively by many researchers. Sufficient conditions for optimality of a real variable function are strengthened in the presence of convexity-concavity conditions of the function, which is important in both pure and applied mathematics. In particular, two important types of results in convexity-concavity of functions with real variables are separation and Fenchel type duality theorems. In this section we will review the definitions of real convex and real concave functions and their properties (see Murota (2003) or Rockafellar (1970) for details).

A set $D \subseteq \mathbb{R}^n$ is called convex if it satisfies the condition

$$x, y \in D,\ 0 \leq a \leq 1 \Rightarrow ax + (1-a)\,y \in D.$$

Definition 2.1 (Convex function): A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called a convex function on a convex set $D \subseteq \mathbb{R}^n$ if and only if for $\forall\ x, y \in D$ and $0 \leq a \leq 1$, the inequality

$$f(ax + (1-a)y) \leq af(x) + (1-a)f(y) \tag{2.1}$$

holds. f is called strictly convex if the inequality in (2.1) is a strict inequality when $0 < a < 1$.

Definition 2.2 (Concave function): A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a concave function (resp., strictly concave) if and only if $-g$ is convex (resp., strictly convex).

If $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, its Hessian matrix

$$H = \left[\frac{\partial^2 f}{\partial x_i^2}\right]_{n\times n}$$

which is the matrix of second partial derivatives of the C^2 function f, can determine the convexity-concavity conditions of f. It is well known in the convexity of real variable functions that a C^2 function is convex (resp. concave) if and only if its corresponding Hessian matrix is positive (resp. negative) semidefinite. Furthermore f is strictly convex (resp. concave) if and only if its Hessian matrix is strictly positive (resp. negative) definite.

Definition 2.3 (Convex and concave conjugate): For a (not necessarily convex) function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ its convex conjugate $f^{\blacksquare} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$f^{\blacksquare}(y) = \sup_{x\in\mathbb{R}^n} \{\langle y, x\rangle - f(x)\}$$

where $y \in \mathbb{R}^n$ and the inner product $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product.

Similarly the concave conjugate of a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$g^{\square}(y) = \inf_{x\in\mathbb{R}^n} \{\langle y, x\rangle - g(x)\}.$$

Note that convex conjugate function is also known as the *convex Legendre-Fenchel transform* of the given function, and the transformation $\phi : f \mapsto f^{\blacksquare}$ is called the *convex Legendre-Fenchel transformation.*

Throughout this work $f^{\blacksquare\blacksquare} = \left(f^{\blacksquare}\right)^{\blacksquare}$ will mean the convex conjugate of the convex conjugate of f, which will be called *convex biconjugate* of f. Similarly $g^{\square\square} = \left(g^{\square}\right)^{\square}$

will mean the concave conjugate of the concave conjugate of g, which will be called *concave biconjugate* of g

Definition 2.4 (Effective domain): For an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, the effective domain of f is defined by

$$Dom\,(f) = \{x \in \mathbb{R}^n : -\infty < f(x) < \infty\}.$$

Definition 2.5 (Proper convex function): A convex function is said to be a proper convex function if it has non-empty effective domain.

Definition 2.6 (Epigraph): The epigraph of a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, denoted by $EP(f)$, is the set of points in $\mathbb{R}^n \times \mathbb{R}$ lying above the graph of $Y = f(x)$. That is,

$$EP(f) = \left\{(x, Y) \in \mathbb{R}^{n+1} : Y \geq f(x)\right\}.$$

Remark 2.1: A well known result in real convex analysis is that a function is convex if and only if its epigraph is a convex set.

Definition 2.7 (Closed convex function-CCF): A function f is called closed convex if $EP(f)$ is a closed convex set in $\mathbb{R}^{n+1}$.

Definition 2.8 (Closed proper convex function): A function f is called closed proper convex if $EP(f)$ is a closed convex set in $\mathbb{R}^{n+1}$ and f has non-empty effective domain.

Let $V \subseteq \mathbb{R}^n$ be a non-empty set. Then we set

$$B_\epsilon^V(x) = \{y \in \mathbb{R}^n : \|y - x\| < \epsilon\} \cap AH\,(V),$$

where $\epsilon > 0$ and $AH(V)$ is the *affine hull* of a set V, which is the smallest affine set (a translation of a linear space) that contains V.

Definition 2.9 (Relative interior): The relative interior of a set V, denoted by $RI\,(V)$, is the set

$$RI\,(V) = \left\{x \in V : B_\epsilon^V(x) \subset V\right\}$$

for some $\epsilon > 0$.i.e. That is, $RI(V)$ is the set of interior points of V with respect to the topology induced from $AH(V)$.

Definition 2.10 (Cone, Convex cone): A set S is a cone if it satisfies

$$x \in S,\ \alpha > 0 \Rightarrow \alpha x \in S.$$

A set S is a convex cone if it is a cone that is convex. *i.e.* S is a convex cone if and only if it satisfies the condition

$$x, y \in S,\ a, b > 0 \Rightarrow ax + by \in S.$$

Definition 2.11 (Convex polyhedron): A convex polyhedron is a convex set S that can be described by a finite number of linear inequalities as

$$S = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} c_{ij} x(j) \leq d_i \text{ for } i = 1, ..., m \right\}$$

where $c_{ij} \in \mathbb{R}$, $d_i \in \mathbb{R}$, $(1 \leq i \leq m$ and $1 \leq j \leq n)$ and $x = (x(1), ..., x(n))$. If $d_i = 0$ for all i, then S is a convex cone.

Definition 2.12 (Polyhedral convex function): If the epigraph of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex polyhedron in $\mathbb{R}^{n+1}$ then f is called a polyhedral convex function.

Following Murota (2003), we define the subdifferential as follows:

Definition 2.13 (Subdifferential): The subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ at a point $x \in dom f$ is defined to be the set

$$\partial_{\mathbb{R}} f(x) = \{p \in \mathbb{R}^n : f(y) - f(x) \geq \langle p, y - x \rangle \ \forall y \in \mathbb{R}^n\},$$

and the set of minimizers of f, denoted by

$$\arg\min f = \{x \in \mathbb{R}^n : f(y) \geq f(x) \ \forall y \in \mathbb{R}^n\}.$$

Note that $\arg\min f$ is a convex set for a convex function f. Also for a given convex function f and x in the relative interior of domain of f we have $\partial_{\mathbb{R}} f(x) \neq \emptyset$.

For a function f and a vector p, we denote by $f[-p]$ the function defined by

$$f[-p](x) = f(x) - \langle p, x \rangle, \quad (x \in \mathbb{R}^n)$$

2.1.1 SEPARATION AND FENCHEL-TYPE DUALITY THEOREMS FOR REAL VARIABLE FUNCTIONS

In the real variable convex function case, the known conjugacy theorem is as follows (Murota (2003), theorem 3.2, pg. 82):

Theorem 2.1 (Convex conjugacy of real variable functions): If f is a closed proper convex function then $f^{\blacksquare}$ is also a closed proper convex function and $f^{\blacksquare\blacksquare} = f$. That is, in the class of all closed proper convex functions, the Legendre-Fenchel transformation $\phi : f \mapsto f^{\blacksquare}$ gives a symmetric one-to-one correspondence.

As a consequence of theorem 2.1 and the definition of subdifferential of a function f, the relationship

$$\begin{aligned} p &\in \partial_{\mathbb{R}} f(x) \Leftrightarrow x \in \arg\min f[-p] \\ &\Leftrightarrow f^{\blacksquare}(p) = \langle p, x \rangle - f(x) \\ &\Leftrightarrow p \in \arg\min f^{\blacksquare}[-p] \\ &\Leftrightarrow x \in \partial_{\mathbb{R}} f^{\blacksquare}(p) \end{aligned}$$

can be obtained for a closed proper convex function f and vectors x, $p \in \mathbb{R}^n$.

The following separation theorem for disjoint real convex sets is due to Rockafellar (1970) (see theorem 11.3 pg. 97).

Theorem 2.2 (Separation for convex sets): If A_1 and A_2 are disjoint convex sets in $\mathbb{R}^n$ then there exists a nonzero vector p^* such that

$$\inf_{x \in A_1} \{\langle p^*, x \rangle\} \geq \sup_{x \in A_2} \{\langle p^*, x \rangle\}.$$

The following separation theorem for real variable functions is due to Stoer-Witzgall (1970) (see corollary 5.1.6, pg. 180) where epigraphs of functions are considered.

Theorem 2.3 (Separation of real variable functions): Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper convex function and $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be a proper concave function such that one of the following regularity conditions hold.

$$RI\,(Dom\,(f)) \cap RI\,(Dom\,(g)) \neq \emptyset. \tag{A-1}$$

$$f \text{ and } g \text{ are polyhedral, and } Dom\,(f) \cap Dom\,(g) \neq \emptyset. \tag{A-2}$$

If $f(x) \geq g(x)$ for $\forall x \in \mathbb{R}^n$ then $\exists\ l_1 \in \mathbb{R}$ and $l_2 \in \mathbb{R}^n$ such that the following inequality holds

$$f(x) \geq l_1 + \langle l_2, x \rangle \geq g(x), \ \ \forall x \in \mathbb{R}^n. \tag{2.2}$$

Example 2.1: A simple example of a separating hyperplane is $h(x,y) = \frac{x}{2} + \frac{y}{2} - 1$ for $f(x,y) = (x-1)^2 + y^2 + 1$ and $g(x,y) = -(x-1)^2 - y^2 - 1$ as it can be seen in figure 2.1.

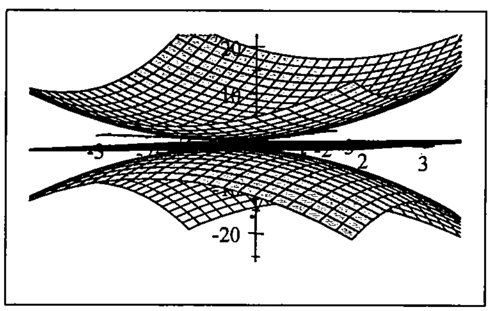

Figure 2.1. The separating plane in example 2.1

Fenchel duality is the infimum-supremum relationship between a convex function f and a concave function g and their conjugate functions $f^{\blacksquare}$ and $g^{\square}$, respectively. A standard version of the Fenchel Duality theorem is presented in Murota (2003) (see theorem 3.6 pg. 85) and goes as follows:

Theorem 2.4 (Fenchel-type duality of real variable functions): Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper convex function and $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be a proper concave function such that either $(A-1)$, $(A-2)$ or one of the following regularity conditions hold.

$$f \textit{ and } -g \textit{ are CCF, and } RI\left(Dom\left(f^{\blacksquare}\right)\right) \cap RI\left(Dom\left(g^{\square}\right)\right) \neq \emptyset. \quad \text{(A-3)}$$

$$f \text{ and } g \text{ are polyhedral, and } Dom\left(f^{\blacksquare}\right) \cap Dom\left(g^{\square}\right) \neq \emptyset. \quad \text{(A-4)}$$

Then

$$\inf_{x \in \mathbb{R}^n} \{f(x) - g(x)\} = \sup_{y \in \mathbb{R}^n} \left\{g^{\square}(y) - f^{\blacksquare}(y)\right\} \tag{2.3}$$

holds. Moreover, if the obtained common value is finite then the supremum is attained for some $y \in Dom\left(f^{\blacksquare}\right) \cap Dom\left(g^{\square}\right)$ under the assumption of $(A-1)$ or $(A-2)$ and the infimum is attained by some $x \in Dom(f) \cap Dom(g)$ under the assumption of $(A-3)$ or $(A-4)$.

Example 2.2: An example of duality is when $f(x) = e^x$ for $x \in \mathbb{R}$,

$$\begin{aligned} f^{\blacksquare}(x_1) &= \sup_{x \in \mathbb{R}} \{x x_1 - e^x\}, \\ &\Rightarrow f^{\blacksquare}(x_1) = \begin{cases} x_1 \log(x_1) - x_1 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 = 0 \\ +\infty & \text{if } x_1 < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} f^{\blacksquare\blacksquare}(x) &= \sup_{x_1 \in \mathbb{R}} \left\{x x_1 - f^{\blacksquare}(x_1)\right\} \\ &= \sup_{x_1 \in \mathbb{R}} \{x x_1 - x_1 \log(x_1) + x_1 | x_1 > 0\} \\ &= \sup_{x_1 \in \mathbb{R}} \{x x_1 - x_1 x + e^x | x_1 > 0\} \\ &= e^x. \end{aligned}$$

The Fenchel duality theorem is useful for confirming a minimum or a maximum that is already given for the difference of the concave and convex functions as above.

The existence of a dual vector that corresponds to the optimal point in the dual space can be obtained from the Fenchel duality theorem.

CHAPTER 3

CONDENSE DISCRETE CONVEX-CONCAVE FUNCTIONS

In this chapter we introduce the concept of a discrete condense convex-concave function and then state the separation and Fenchel-type duality theorems for condense discrete functions defined on $\mathbb{Z}^n$, which are induced from the well known separation and Fenchel-type duality theorems in real convex analysis. A subclass of condense mixed convex-concave function class is introduced following the mixed convexity definitions of Tokgöz, Maalouf and Kumin (2009).

3.1 CONDENSE DISCRETE CONVEX-CONCAVE FUNCTIONS

Similar to the convexity of the real variable real valued functions, real valued functions with integer variables have important applications in applied mathematics. Real valued integer variable function optimization results can be obtained by knowing the corresponding function convexity properties. The classical definition states that a discrete function of a single variable is discrete convex if its first forward differences are increasing or at least non-decreasing, as defined by Fox (1966) and Denardo (1982). The discrete convexity definition introduced by Denardo (1982) and Fox (1966) is used by many others in the literature such as Favati and Tardella (1990), Dyer and Proll (1976), and Weber (1980). The condense discrete convexity introduced in this section is similar to the discrete convexity defined by Denardo (1982) and Fox (1966). The definition of condense discrete convex (resp. concave) function $\gamma_1 : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ requires the assumption of the existence of a real convex (resp. concave) function $\overline{\gamma_1} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which coincides with γ_1 on the integer lattice.

Definition 3.1 (Condense discrete convex ($CDCx$) set): A condense discrete convex ($CDCx$) set B in $\mathbb{Z}^n$ if there exists a convex set $\overline{B} \subseteq \mathbb{R}^n$ such that $B = \overline{B} \cap \mathbb{Z}^n$.

Definition 3.2 (Condense discrete convex ($CDCx$) function): A function $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is a (strict) condense discrete convex ($CDCx$) function if there exists a (strict) real convex function $\overline{\gamma_1} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ where γ_1 coincides with $\overline{\gamma_1}$ on the integer lattice.

Definition 3.3 (Condense discrete concave function ($CDCc$)): A function $\gamma_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is condense discrete concave ($CDCc$) if $-\gamma_2$ is a $CDCx$ function.

Definition 3.4 (Discrete effective domain): The discrete effective domain of a function $\gamma : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$Dom_{\mathbb{Z}^n}(\gamma) = \{x \in \mathbb{Z}^n : -\infty < \gamma(x) < +\infty\}.$$

Definition 3.5 (Proper CDCx function): A $CDCx$ function $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if it has non-empty discrete effective domain.

Definition 3.6 (Discrete epigraph): The discrete epigraph of a $CDCx$ function $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, denoted by $EP_{\mathbb{Z}}(\gamma_1)$, is the set of points in $\mathbb{Z}^n \times \mathbb{R}$ lying above the graph of $Y = \gamma_1(x)$. That is,

$$EP_{\mathbb{Z}}(\gamma_1) = \{(x, Y) \in \mathbb{Z}^n \times \mathbb{R} : Y \geq \gamma_1(x)\}.$$

Note that a $CDCx$ function γ_1 is closed $CDCx$ since its epigraph $EP_{\mathbb{Z}}(\gamma_1)$ is a closed convex set in $\mathbb{Z}^n \times \mathbb{R}$.

Definition 3.7 (Closed proper $CDCx$ function): A $CDCx$ function γ_1 is said to be closed proper $CDCx$ if its convex extension $\overline{\gamma_1} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed and proper.

Let $AH_{\mathbb{Z}}(V)$ denote the discrete *affine hull* of a set $V \subseteq \mathbb{Z}^n$, which is the smallest discrete affine set (a translation of a discrete linear space) that contains V. For $x \in \mathbb{Z}^n$ set

$$U(x) = \{y \in \mathbb{Z}^n : \|y - x\|_\infty \leq 1\}.$$

Definition 3.8 (Relative interior of a discrete set): The relative interior of a discrete set V, $RI(co(V))$, is the relative interior of the convex hull of V.

Definition 3.9 (Discrete cone): A $CDCx$ set $S \subseteq \mathbb{Z}^n$ is a discrete cone if it satisfies

$$x \in S,\ \alpha \in \mathbb{Z}^+ \Rightarrow \alpha x \in S.$$

Definition 3.10 (CDCx cone): A $CDCx$ set $S \subseteq \mathbb{Z}^n$ is a $CDCx$ cone if and only if it satisfies the condition

$$x, y \in S,\ a, b \in \mathbb{Z}^+ \Rightarrow ax + by \in S.$$

Definition 3.11 (CDCx polyhedron): A $CDCx$ polyhedron is a $CDCx$ set S described by a finite number of linear inequalities as

$$S = \left\{ x \in \mathbb{Z}^n : \sum_{j=1}^{n} c_{ij} x(j) \leq d_i \text{ for } i = 1, ..., n \right\}$$

where $c_{ij} \in \mathbb{R}$ and $d_i \in \mathbb{R}$. If $d_i = 0$ for all i, then S is a $CDCx$ cone.

Definition 3.12 (Polyhedral CDCx function): If the discrete epigraph of a $CDCx$ function $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is a discrete convex polyhedron in $\mathbb{Z}^n \times \mathbb{R}$ then it is called polyhedral $CDCx$ function.

The inner product defined on $\mathbb{Z}^n$ is the inner product induced from $\mathbb{R}^n$ for integer vectors:

$$\langle k, k_0 \rangle = \sum_{i=1}^{n} k_i k_{0_i}$$

For $CDCx$ function $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and $CDCc$ function $\gamma_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, the condense discrete versions of the Legendre-Fenchel transformations are defined by

$$\begin{aligned}\gamma_1^{\bullet}(p_1) &= \sup_{k\in\mathbb{Z}^n}\{\langle p_1,k\rangle-\gamma_1(k)\} \quad (p_1\in\mathbb{R}^n),\\ \gamma_2^{\circ}(p_2) &= \inf_{k\in\mathbb{Z}^n}\{\langle p_2,k\rangle-\gamma_2(k)\} \quad (p_2\in\mathbb{R}^n).\end{aligned}$$

We denote the discrete convex conjugate of the discrete convex conjugate of a $CDCx$ function γ_1 by $(\gamma_1^{\bullet})^{\bullet}=\gamma_1^{\bullet\bullet}$. The integer subdifferential of γ_1 at $k\in dom_{\mathbb{Z}}\gamma_1$ is defined by

$$\partial_{\mathbb{Z}}\gamma_1(k)=\partial_{\mathbb{R}}\gamma_1(k)\cap\mathbb{Z}^V$$

The following result for discrete variable real valued functions characterizes the conjugacy conditions of $CDCx$ functions (see for example Murota (2003), proposition 8.11).

Theorem 3.1: For a $CDCx$ function γ_1 and a point $k\in Dom\gamma_1$ we have $\gamma_1^{\bullet\bullet}(k)=\gamma_1(k)$ if $\partial_{\mathbb{Z}}\gamma_1(k)\neq\emptyset$.

Proof: For $k\in Dom\gamma_1$ and $p\in\partial_{\mathbb{Z}}\gamma_1(k)$ we have $\gamma_1^{\bullet}(p)=\langle p,k\rangle-\gamma_1(k)$. Therefore

$$\gamma_1^{\bullet\bullet}(k)=\sup_{q\in\mathbb{R}^n}\{\langle q,k\rangle-\gamma_1^{\bullet}(q)\}\geq\langle p,k\rangle-\gamma_1^{\bullet}(p)=\gamma_1(k).$$

On the other hand $\gamma_1^{\bullet\bullet}(k)\leq\gamma_1(k)$ for any γ_1 and k.

The relationship between the discrete $CDCx$ and $CDCc$ Legendre-Fenchel transformations is $\gamma_1^{\circ}(p_1)=-(-\gamma_1)^{\bullet}(-p_1)$, which follows from the corresponding result for the real variable Legendre-Fenchel transformations.

Next we introduce the separation and Fenchel type duality theorems for condense discrete convex-concave functions, as well as the separation theorem for condense discrete convex sets.

3.1.1 SEPARATION AND FENCHEL - TYPE DUALITY THEOREMS FOR CONDENSE DISCRETE FUNCTIONS

The following results follow from separation and Fenchel-type duality theorems in real convex analysis (see Rockafellar (1970)). They will be used to state and prove the separation and Fenchel-type duality theorems for condense mixed convex functions in the next section.

Let $D_1, D_2 \subset \mathbb{Z}^n$ be two non-empty $CDCx$ sets in $\mathbb{R}^n$. A real hyperplane H is said to separate D_1 and D_2 in $\mathbb{Z}^n$ if D_1 is contained in one of the discrete half-spaces associated with H and D_2 lies in the opposite discrete half-space. A hyperplane H is said to be a regular separation of D_1 and D_2 in $\mathbb{Z}^n$ if at most one of D_1 and D_2 has non-empty intersection with H. H is a strong separating hyperplane of D_1 and D_2 in $\mathbb{Z}^n$ if both D_1 and D_2 are disjoint from H.

The following result is due to Rockafellar (1970) (see corollary 11.4.2 and theorem 11.3) when we restrict $\mathbb{R}^n$ into $\mathbb{Z}^n$.

Corollary 3.1: Let D_1 and D_2 be non-empty disjoint $CDCx$ sets in $\mathbb{Z}^n$. If at least one set is bounded and $RI\,(co\,(D_1)) \cap RI\,(co\,(D_2)) = \emptyset$ holds, then there exists a hyperplane H that separates D_1 and D_2 regularly.

The following result is due to Rockafellar (1970) (see corollary 11.4.2 and theorem 11.1) when we restrict $\mathbb{R}^n$ into $\mathbb{Z}^n$.

Corollary 3.2: Let D_1 and D_2 be two non-empty $CDCx$ sets such that at least one set is bounded in $\mathbb{Z}^n$. If there exists a hyperplane H separating D_1 and D_2 regularly then there exists a vector $d \in \mathbb{R}^n$ and a $c > 0$ such that

$$\inf_{x_1 \in D_1} \{\langle x_1, d\rangle\} - \sup_{x_2 \in D_2} \{\langle x_2, d\rangle\} \geq c$$

holds.

Proof: Assume there exists a hyperplane H separating D_1 and D_2 regularly where at least one of the sets is bounded. Without loss of generality assume $D_1 \cap H =$

$\emptyset$ where D_1 is bounded. We can choose $\beta \in \mathbb{R}$ and $c > 0$ such that

$$\begin{aligned}\langle x_1, d\rangle &\geq \beta + c, \forall x_1 \in D_1,\\ \langle x_2, d\rangle &\leq \beta, \forall x_2 \in D_2.\end{aligned}$$

Therefore

$$\begin{aligned}\inf_{x_1 \in D_1} \{\langle x_1, d\rangle\} &\geq \beta + c,\\ \sup_{x_2 \in D_2} \langle x_2, d\rangle &\leq \beta,\end{aligned}$$

which indicates that

$$\inf_{x_1 \in D_1} \{\langle x_1, d\rangle\} - \sup_{x_2 \in D_2} \langle x_2, d\rangle \geq c$$

holds for some $c > 0$.

Under stronger assumptions for the hyperplane considered in corollary 3.2, we can specify c for a certain class of $CDCx$ sets.

An integer hyperplane is a hyperplane in $\mathbb{Z}^n$ of the following form for some fixed $d \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}$,

$$H = \{x \in \mathbb{Z}^n : \langle x, d\rangle = \beta\}.$$

An integer hyperplane H is said to separate D_1 and D_2 if D_1 is contained in one of the discrete half-spaces associated with H and D_2 lies in the opposite discrete half space. An integer hyperplane H is said to be a regular separation of D_1 and D_2 if at most one of D_1 and D_2 has non-empty intersection with H. H is a strong separating integer hyperplane of D_1 and D_2 if both D_1 and D_2 are disjoint from H.

Corollary 3.3: Let D_1 and D_2 be two non-empty $CDCx$ sets in $\mathbb{Z}^n$ that have a regular separating integer hyperplane H. Then there exists a vector $d \in \mathbb{Z}^n$ such that

$$\inf_{x_1 \in D_1} \{\langle x_1, d\rangle\} - \sup_{x_2 \in D_2} \{\langle x_2, d\rangle\} \geq 1. \tag{3.1}$$

Proof: Let D_1 and D_2 be non-empty $CDCx$ sets in $\mathbb{Z}^n$ that are separated by an integer hyperplane H and associated discrete half spaces containing D_1 and D_2 can be expressed by

$$\begin{aligned} H &= \{x \in \mathbb{Z}^n : \langle x, d \rangle = \beta\}, \\ D_1 &\subset \{x_1 : \langle x_1, d \rangle \geq \beta\}, \end{aligned}$$

and

$$D_2 \subset \{x_2 : \langle x_2, d \rangle \leq \beta\}$$

for some $d \neq 0$ and β. Since H is a regular separating hyperplane, D_1 and D_2 are not both contained in H. Therefore, $\langle x_1, d \rangle \geq \beta$ for every $x_1 \in D_1$, and $\langle x_2, d \rangle \leq \beta$ for every $x_2 \in D_2$ with strict inequality for either every $x_1 \in D_1$ or every $x_2 \in D_2$. Suppose without loss of generality the strict inequality holds for all $x_1 \in D_1$. Note that

$$|\langle x_1, d \rangle - \beta| \geq 1$$

since $\langle x_1, d \rangle - \beta$ is a positive integer. Therefore we have

$$\begin{aligned} \inf_{x_1 \in D_1} \{\langle x_1, d \rangle\} &\geq \beta + 1, \\ \sup_{x_2 \in D_2} \{\langle x_2, d \rangle\} &\leq \beta, \\ \inf_{x_1 \in D_1} \{\langle x_1, d \rangle\} - \sup_{x_2 \in D_2} \{\langle x_2, d \rangle\} &\geq 1, \end{aligned}$$

which completes the proof.

The following separation theorem follow from corollary 3.2 and the corresponding theorems in real convex analysis (Rockafellar (1970)) where discrete epigraphs are considered in the proof.

Theorem 3.2 (Separation of condense discrete functions): Let $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper $CDCx$ function, and $\gamma_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a proper $CDCc$ function such that one of the regularity conditions holds:

$$RI\left(co\left(Dom_{\mathbb{Z}^n}(\gamma_1)\right)\right) \cap RI\left(co\left(Dom_{\mathbb{Z}^n}(\gamma_2)\right)\right) \neq \emptyset. \tag{3.2}$$

$$\gamma_1 \text{ and } -\gamma_2 \text{ are polyhedral } CDCx\text{, and } Dom_{\mathbb{Z}^n}(\gamma_1) \cap Dom_{\mathbb{Z}^n}(\gamma_2) \neq \emptyset. \qquad (3.3)$$

If $\overline{\gamma_1}(k) \geq \overline{\gamma_2}(k)$ for all $k \in \mathbb{R}^n$ then $\exists\ \eta_1 \in \mathbb{R}$ and $\eta_2 \in \mathbb{R}^n$ such that the following inequality holds for $\forall k \in \mathbb{Z}^n$:

$$\gamma_1(k) \geq \eta_1 + \langle \eta_2, k \rangle \geq \gamma_2(k) \qquad (3.4)$$

The following Fenchel-type duality holds under the conditions of theorem 3.2 for $CDCx$ and $CDCc$ functions:

Theorem 3.3 (Fenchel-type duality for condense discrete functions): Let $\gamma_1 : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ be a proper $CDCx$ function and $\gamma_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a proper $CDCc$ function such that one of the regularity conditions (3.2), (3.3),

$$\gamma_1 \textit{ and } -\gamma_2 \textit{ are closed } CDCx\textit{, and } RI\left(co\left(Dom_{\mathbb{Z}^n}(\gamma_1^\bullet)\right)\right) \cap RI\left(co\left(Dom_{\mathbb{Z}^n}(\gamma_2^\circ)\right)\right) \neq \emptyset, \qquad (3.5)$$

or

$$\gamma_1 \textit{ and } -\gamma_2 \textit{ are polyhedral } CDCx\textit{, and } Dom_{\mathbb{R}^n}(\gamma_1^\bullet) \cap Dom_{\mathbb{R}^n}(\gamma_2^\circ) \neq \emptyset \qquad (3.6)$$

holds. Then

$$\inf_{k \in \mathbb{Z}^n} \{\gamma_1(k) - \gamma_2(k)\} = \sup_{p_1 \in \mathbb{R}^n} \{\gamma_2^\circ(p_1) - \gamma_1^\bullet(p_1)\}. \qquad (3.7)$$

3.2 CONDENSE MIXED CONVEX SETS

In this section, we apply condense discrete convex and real convex set definitions to define condense mixed convex sets.

We use the notation $z = (k, \mu)$, $p = (p_1, p_2)$, $p_0 = (p_0^1, p_0^2)$, $p^* = (p_1^*, p_2^*)$, $p^{**} = (p_1^{**}, p_2^{**})$, $z_0 = (k_0, \mu_0)$, $z_1 = (k_1, \mu_1)$, $\Im_V = \mathbb{R}^n \times \mathbb{R}^m$, $\Im_{\mathbb{Z}\times\mathbb{R}} = \mathbb{Z}^n \times \mathbb{R}^m$ and $\Im_{\mathbb{Z}} = \mathbb{Z}^n \times \mathbb{Z}^m$ throughout this section.

A mixed condense set $A = B \times C \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ is a set where B is a $CDCx$ set in $\mathbb{Z}^n$ and C is a real convex set in $\mathbb{R}^m$.

The mixed inner product is an inner product of the form

$$\begin{aligned}\langle z_0, z\rangle_{\Im_{\mathbb{Z}\times\mathbb{R}}} &= \langle (k_0, \mu_0), (k, \mu)\rangle_{\Im_{\mathbb{Z}\times\mathbb{R}}} \\ &= \langle k_0, k\rangle_{\mathbb{Z}^n} + \langle \mu_0, \mu\rangle_{\mathbb{R}^m} \\ &= \sum_{i=1}^{n} k_{0_i} k_i + \sum_{j=1}^{m} \mu_{0_j}\mu_j.\end{aligned}$$

where the inner product on the mixed space $\Im_{\mathbb{Z}\times\mathbb{R}}$ is induced from the inner product of $(n+m)$-dimensional Euclidean space.

Definition 3.14 (Mixed effective domain): The mixed effective domain of a function $\Gamma : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$Dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}(\Gamma) = \{z \in \Im_{\mathbb{Z}\times\mathbb{R}} : -\infty < \Gamma(z) < +\infty\}.$$

Let $\Gamma_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{\infty\}$. Associated to Γ_1, we define Γ_1^{μ} to be $\Gamma_1^{\mu}(k) = \Gamma_1(k,\mu)$ for each fixed real vector $\mu \in \mathbb{R}^m$ and Γ_1^{k} to be $\Gamma_1^{k}(\mu) = \Gamma_1(k,\mu)$ for each fixed integer vector $k \in \mathbb{Z}^n$.

Definition 3.15 (Mixed epigraph): The mixed epigraph of a $CMCx$ function $\Gamma_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$, denoted by $EP_{\mathbb{Z}\times\mathbb{R}}(\Gamma_1)$, is the set of points

$$EP_{\mathbb{Z}\times\mathbb{R}}(\Gamma_1) = \{(z, Y) : Y \geq \Gamma_1(z)\} \subseteq \Im_{\mathbb{Z}\times\mathbb{R}} \times \mathbb{R}.$$

Note that for $z = (k,\mu) \in \Im_{\mathbb{Z}\times\mathbb{R}}$ we have $(k, Y) \in EP_{\mathbb{Z}}(\Gamma_1^{\mu})$ and $(\mu, Y) \in EP_{\mathbb{R}}(\Gamma_1^{k})$.

Definition 3.16 (Mixed relative interior): The mixed relative interior of $V = V_1 \times V_2$, $RI(V)$, is the set of points $z = (k,\mu) \in V$ such that k lies in the discrete relative interior of the convex hull of V_1 and $\mu \in RI(V_2)$.

Definition 3.17 (Mixed cone, Mixed convex cone): A set $S = S_1 \times S_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ is a mixed cone if it satisfies

$$z = (k,\mu) \in S \text{ and } \alpha_1 \in \mathbb{Z}^+, \alpha_2 \in \mathbb{R}^+ \Rightarrow (\alpha_1 k, \alpha_2 \mu) \in S.$$

Let $z_1 = (k_1, \mu_1)$ and $z_2 = (k_2, \mu_2)$. A set $S \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ is a mixed convex cone if and only if it satisfies the condition

$$z_1, z_2 \in S \text{ and } a_1, b_1 \in \mathbb{Z}^+ a_2, b_2 \in \mathbb{R}^+ \Rightarrow (a_1k_1 + a_2k_2, b_1\mu_1 + b_2\mu_2) \in S.$$

Definition 3.18 (Mixed convex polyhedron): A mixed convex polyhedron is a mixed convex set $S = S_1 \times S_2$ described by a finite number of linear inequalities as

$$S = \left\{ z \in \Im_{\mathbb{Z}\times\mathbb{R}} : \sum_{j=1}^{n} c_{tj}^{(1)} k(j) \leq d_t^{(1)} \text{ and } \sum_{j=1}^{m} c_{ij}^{(2)} \mu(j) \leq d_i^{(2)} \text{ for } 1 \leq t \leq n,\ 1 \leq i \leq m \right\}$$

where $c_{tj}^{(1)}, c_{ij}^{(2)} \in \mathbb{R}$ and $d_i^{(1)}, d_i^{(2)} \in \mathbb{R}$. If $\left(d_i^{(1)}, d_i^{(2)}\right) = (0,0)$ for all i, then S is a mixed convex cone.

3.2.1 SEPARATION OF CONDENSE MIXED CONVEX SETS

Let B_1 and B_2 be $CDCx$ sets in $\mathbb{Z}^n$, and C_1 and C_2 be real convex sets in $\mathbb{R}^m$. Two mixed convex sets $A_1 = B_1 \times C_1$ and $A_2 = B_2 \times C_2$ are called $\tilde{}$ disjoint, *i.e.* $A_1 \tilde{\cap} A_2 = \emptyset$, if both B_1, B_2 and C_1, C_2 have no elements in common; that is, $A_1 \tilde{\cap} A_2 \neq \emptyset$ if both B_1, B_2 and C_1, C_2 have elements in common. The conditions where

$$B_1 \cap B_2 = \emptyset \text{ and } C_1 \cap C_2 \neq \emptyset,$$

and

$$B_1 \cap B_2 \neq \emptyset \text{ and } C_1 \cap C_2 = \emptyset$$

are ruled out when we consider $\tilde{}$ intersection of $CMCx$ sets. The following theorem characterizes the separation property for $CMCx$ sets.

Theorem 3.4 (Separation of condense mixed convex sets-1): Let $A_1 = B_1 \times C_1$ and $A_2 = B_2 \times C_2$ be two mixed convex sets with either B_1 or B_2 bounded sets in $\mathbb{Z}^n$ such that $RI\,(A_1)\,\widetilde{\cap}RI\,(A_2) = \emptyset$. Then there exists a $p^* \in \Im_V$ and a $c > 0$ such that

$$\inf_{z \in A_1} \{\langle p^*, z\rangle\} - \sup_{z \in A_2} \{\langle p^*, z\rangle\} \geq c. \tag{3.8}$$

Proof: Applying corollary 3.1 and 3.3 to the disjoint $CDCx$ sets B_1 and B_2, there exists a $p_1 \in \mathbb{R}^n$ such that

$$\inf_{k \in B_1} \{\langle p_1, k\rangle\} - \sup_{k \in B_2} \{\langle p_1, k\rangle\} \geq c \tag{3.9}$$

holds where $c > 0$. Applying the real convex set separation theorem to the disjoint sets C_1 and C_2, there exists a $p_2 \in \mathbb{R}^m$ such that

$$\inf_{\mu \in C_1} \{\langle p_2, \mu\rangle\} \geq \sup_{\mu \in C_2} \{\langle p_2, \mu\rangle\}. \tag{3.10}$$

By adding (3.9) and (3.10), we have

$$\begin{aligned}
\inf_{k \in B_1} \{\langle p_1, k\rangle\} + \inf_{\mu \in C_1} \{\langle p_2, \mu\rangle\} &\geq c + \sup_{k \in B_2} \{\langle p_1, k\rangle\} + \sup_{\mu \in C_2} \{\langle p_2, \mu\rangle\}, \\
\inf_{k \in B_1, \mu \in C_1} \{\langle p_1, k\rangle + \langle p_2, \mu\rangle\} &\geq c + \sup_{k \in B_2, \mu \in C_2} \{\langle p_1, k\rangle + \langle p_2, \mu\rangle\}, \\
\inf_{(k,\mu) \in A_1} \{\langle (p_1, p_2), (k, \mu)\rangle\} &\geq c + \sup_{(k,\mu) \in A_2} \{\langle (p_1, p_2), (k, \mu)\rangle\}, \\
\inf_{z \in A_1} \{\langle p, z\rangle\} &\geq c + \sup_{z \in A_2} \{\langle p, z\rangle\}, \\
\inf_{z \in A_1} \{\langle p, z\rangle\} - \sup_{z \in A_2} \{\langle p, z\rangle\} &\geq c. \qquad (3.11)
\end{aligned}$$

which completes the proof.

Under the assumptions of corollary 3.3 where we considered integer hyperplanes for separation of $CDCx$ sets, we have the following separation theorem.

Theorem 3.5 (Separation of condense mixed convex sets-2): Let $A_1 = B_1 \times C_1$ and $A_2 = B_2 \times C_2$ be two mixed convex sets such that $RI\,(A_1)\,\widetilde{\cap}RI\,(A_2) = \emptyset$ where B_1 and B_2 are separated by an integer hyperplane. Then there exists a $p^* \in \Im_V$ such that

$$\inf_{z \in A_1} \{\langle p^*, z\rangle\} - \sup_{z \in A_2} \{\langle p^*, z\rangle\} \geq 1. \tag{3.12}$$

Proof: Applying corollary 3.3 to the disjoint $CDCx$ sets B_1 and B_2, there exists a $p_1 \in \mathbb{R}^n$ such that

$$\inf_{k\in B_1}\{\langle p_1,k\rangle\}-\sup_{k\in B_2}\{\langle p_1,k\rangle\}\geq 1 \tag{3.13}$$

holds. Applying the real convex set separation theorem to the disjoint sets C_1 and C_2, there exists a $p_2 \in \mathbb{R}^m$ such that

$$\inf_{\mu\in C_1}\{\langle p_2,\mu\rangle\}\geq \sup_{\mu\in C_2}\{\langle p_2,\mu\rangle\}. \tag{3.14}$$

By adding (3.13) and (3.14), we have

$$\begin{aligned}
\inf_{k\in B_1}\{\langle p_1,k\rangle\}+\inf_{\mu\in C_1}\{\langle p_2,\mu\rangle\} &\geq 1+\sup_{k\in B_2}\{\langle p_1,k\rangle\}+\sup_{\mu\in C_2}\{\langle p_2,\mu\rangle\},\\
\inf_{k\in B_1,\mu\in C_1}\{\langle p_1,k\rangle+\langle p_2,\mu\rangle\} &\geq 1+\sup_{k\in B_2,\mu\in C_2}\{\langle p_1,k\rangle+\langle p_2,\mu\rangle\},\\
\inf_{(k,\mu)\in A_1}\{\langle (p_1,p_2),(k,\mu)\rangle\} &\geq 1+\sup_{(k,\mu)\in A_2}\{\langle (p_1,p_2),(k,\mu)\rangle\},\\
\inf_{z\in A_1}\{\langle p,z\rangle\} &\geq 1+\sup_{z\in A_2}\{\langle p,z\rangle\},\\
\inf_{z\in A_1}\{\langle p,z\rangle\}-\sup_{z\in A_2}\{\langle p,z\rangle\} &\geq 1
\end{aligned} \tag{3.15}$$

which completes the proof.

3.3 CONVEX ENVELOPE AND MIXED SEPARATION THEOREMS

Independent from the mixed convexity discussion above, assume that we are given the mixed functions $h_1:\Im_{\mathbb{Z}\times\mathbb{R}}\to\mathbb{R}\cup\{\infty\}$ and $h_2:\Im_{\mathbb{Z}\times\mathbb{R}}\to\mathbb{R}\cup\{-\infty\}$. We define the convex envelope of a function h by following Witsenhausen (1968).

Let $z=(k,\mu)$ and $\overline{z}=\left(\overline{k},\overline{\mu}\right)$.

Definition 3.19: The convex envelope of a mixed function $h_1:\Im_{\mathbb{Z}\times\mathbb{R}}\to\mathbb{R}\cup\{\infty\}$ with $Domh_1\neq\emptyset$ is the function $\widehat{h}_1:\Im_V\to\mathbb{R}\cup\{\infty\}$ defined by

$$\widehat{h}_1(\overline{z})=\sup_{\substack{\alpha\in\mathbb{R}\\ \xi\in\mathbb{R}^n,\eta\in\mathbb{R}^m}}\left\{\alpha+\left\langle\xi,\overline{k}\right\rangle+\langle\eta,\overline{\mu}\rangle:\alpha+\langle\xi,k\rangle+\langle\eta,\mu\rangle\leq h_1(z)\ \text{ for }\forall z\in\Im_{\mathbb{Z}\times\mathbb{R}}\right\},$$

Definition 3.20: The concave envelope of a mixed function $h_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ is the function $\widehat{h}_2 : \Im_V \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\widehat{h}_2(\overline{z}) = \inf_{\substack{\alpha\in\mathbb{R} \\ \xi\in\mathbb{R}^n, \eta\in\mathbb{R}^m}} \left\{\alpha + \langle \xi, \overline{k}\rangle + \langle \eta, \overline{\mu}\rangle : \alpha + \langle \xi, k\rangle + \langle \eta, \mu\rangle \geq h_2(z) \text{ for } \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}\right\}.$$

The correspondence between the mixed function hyperplane separation and the mixed convex/concave envelope function hyperplane separation is stated in the following theorem.

Theorem 3.6: Let $h_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$ and $h_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be two mixed functions with the corresponding envelopes $\widehat{h}_1$ and $\widehat{h}_2$ satisfying the following conditions:

$$\begin{aligned} Domh_1 \cap Domh_2 &\neq \emptyset \\ Dom\widehat{h}_1 \cap Dom\widehat{h}_2 &\neq \emptyset \end{aligned}$$

Then there exist $z_0 \in \Im_{\mathbb{Z}\times\mathbb{R}}$ and $\eta \in \mathbb{R}$ such that

$$h_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq h_2(k,\mu), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$$

if and only if there exist $z_0 \in \Im_{\mathbb{Z}\times\mathbb{R}}$ and $\eta \in \mathbb{R}$ such that

$$\widehat{h}_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq \widehat{h}_2(k,\mu), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}.$$

Proof: Suppose

$$\begin{aligned} Domh_1 \cap Domh_2 &\neq \emptyset \\ Dom\widehat{h}_1 \cap Dom\widehat{h}_2 &\neq \emptyset \end{aligned}$$

hold, and let $z_0 \in \Im_{\mathbb{Z}\times\mathbb{R}}$, $\eta \in \mathbb{R}$ be such that

$$h_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq h_2(k,\mu), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$$

By definition of the convex envelope

$$h_1(k,\mu) \geq \widehat{h}_1(k,\mu) \geq \eta + \langle z_0, z\rangle,$$

and by the definition of the concave envelope

$$\eta + \langle z_0, z\rangle \geq \widehat{h}_2(k,\mu) \geq h_2(k,\mu)$$

Therefore

$$\widehat{h}_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq \widehat{h}_2(k,\mu)$$

For the reverse implication let $z_0 \in \Im_{\mathbb{Z}\times\mathbb{R}}$, $\eta \in \mathbb{R}$ be such that

$$\widehat{h}_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq \widehat{h}_2(k,\mu), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}.$$

Clearly by the inequalities above we also have

$$h_1(k,\mu) \geq \widehat{h}_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq \widehat{h}_2(k,\mu) \geq h_2(k,\mu)$$

which indicates

$$h_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq h_2(k,\mu)$$

and completes the proof.

Corollary 3.4: Let $h_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{\infty\}$ be a mixed function with the corresponding convex envelope $\widehat{h}_1$ and $h_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ be a mixed function with the corresponding concave envelope $\widehat{h}_2$. Then h_1 and h_2 have a mixed separating hyperplane if $\widehat{h}_1$ and $\widehat{h}_2$ satisfy the conditions of theorem 2.3.

Proof: By theorem 3.8 we have

$$h_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq h_2(k,\mu), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$$

if and only if

$$\widehat{h}_1(k,\mu) \geq \eta + \langle z_0, z\rangle \geq \widehat{h}_2(k,\mu), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}.$$

Therefore if the convex envelope $\widehat{h}_1$ and the concave envelope $\widehat{h}_2$ satisfy the conditions of theorem 2.3 then proof follows from theorem 3.8.

CHAPTER 4

DISCRETE $M/M^\natural$, MIXED E/E^*, AND MIXED E_1/E_1^* CONVEX-CONCAVE FUNCTIONS

In this chapter, we first review some basic facts about discrete $M/M^\natural$ convexity concepts and related results, after which mixed E/E^* and mixed E_1/E_1^* convexity concepts and related results are stated and proven.

4.1 DISCRETE M AND $M^\natural$ CONVEX-CONCAVE FUNCTIONS

The concept of discrete M-convex functions was introduced by Murota (1996) and that of discrete $M^\natural$ convex functions by Murota-Shioura (1999). Researchers such as Murota ((1996), (1998), (1999) and (2000)) and Shioura ((1999) and (2000)) worked on the theory of discrete variable M and $M^\natural$ convex functions. Important examples of discrete M-convex functions arise in network flow problems (Murota (2003)). In this section, following Murota (2003), we start with the basic definitions and state the necessary results to prove the separation and Fenchel-type duality theorems for discrete variable $M/M^\natural$ functions.

Fix a positive integer n and let $V = \{1, 2, ..., n\}$. When we deal with M and L convexity, we will use the notations $\mathbb{Z}^V$ and $\mathbb{Z}^n$ interchangeably.

Definition 4.1 (Effective domain): The effective domains of $f : \mathbb{Z}^V \to \mathbb{R} \cup \{\pm\infty\}$ and $h : \mathbb{R}^V \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$dom_{\mathbb{Z}}f = \left\{x \in \mathbb{Z}^V : -\infty < f(x) < \infty\right\},$$
$$dom_{\mathbb{R}}h = \left\{x \in \mathbb{R}^V : -\infty < h(x) < \infty\right\},$$

respectively.

Definition 4.2 (Positive and negative supports): The positive and negative supports of a vector $x = (x(v)\,|v \in V) \in \mathbb{Z}^V$ are defined by

$$supp^{+}(x) = \{v \in V|x(v) > 0\}, \ supp^{-}(x) = \{v \in V|x(v) < 0\},$$

respectively.

Definition 4.3 (Characteristic vector): The characteristic vector of a subset $X \subset V$ is defined by

$$\chi_X(v) = \begin{cases} 1 & \text{if} \quad v \in X \\ 0 & \text{if} \quad v \in V \backslash X \end{cases}$$

The motivation of the M-convex function comes from the inequality

$$\Theta_1(x) + \Theta_1(y) \geq \Theta_1(x - \alpha(x-y)) + \Theta_1(y + \alpha(x-y)) \tag{4.1}$$

which follows from the definition of the real convex function $\Theta_1 : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ for every α with $0 \leq \alpha \leq 1$. This inequality follows from adding the inequalities (2.1) with $a = \alpha$ and (2.1) with $a = 1 - \alpha$.

The inequality (4.1) shows that the sum of the two function values evaluated at x and y does not increase if the two points approach each other by the same distance on the line segment connecting them. For a function defined on discrete points of $\mathbb{Z}^V$, this property is simulated by moving two points along the coordinate axes rather than on the connecting line segment.

Definition 4.4 (M-convex (-concave) function): A function $\Theta_1 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is called M-convex if for $x, y \in dom_{\mathbb{Z}}\Theta_1$ and $u \in supp^{+}\{x - y\}$, there exists $v \in supp^{-}\{x-y\}$ such that

$$\Theta_1(x) + \Theta_1(y) \geq \Theta_1(x - \chi_u + \chi_v) + \Theta_1(y + \chi_u - \chi_v).$$

A function $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$ is called M-concave if $-\Theta_2$ is M-convex.

Let 0 denote a new element not in V and let $\widetilde{V} = V \cup \{0\}$.

Definition 4.5 (M$^\natural$-convex (-concave) function): A function $\Theta_1 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ is called $M^\natural$ convex if the function $\widetilde{\Theta}_1 : \mathbb{Z}^{\widetilde{V}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\widetilde{\Theta}_1(x_0, x) = \begin{cases} \Theta_1(x) & \text{if} \quad x_0 = -x(V), \ i.e.\ x(0) = -\sum\limits_{v \in V} x(v) \\ +\infty & \quad otherwise \end{cases}.$$

is an M-convex function where $x_0 \in \mathbb{Z}$ and $x \in \mathbb{Z}^V$. A function $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$ is called $M^\natural$ concave if $-\Theta_2$ is $M^\natural$ convex.

Definition 4.6 (Integer interval, restriction of a function): For two integer vectors a, $b \in (\mathbb{Z} \cup \{\pm\infty\})^n$, the integer interval $[a,b] = [a,b]_{\mathbb{Z}}$ is defined by

$$[a,b] = [a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z}^n : a(i) \leq x(i) \leq b(i), \ i = 1, ..., n\}.$$

The restriction of a function $\Theta : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ to an interval is defined to be the function $\Theta_{[a,b]} : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Theta_{[a,b]}(x) = \begin{cases} \Theta(x) & \text{if} \quad x \in [a,b] \\ +\infty & \text{if} \quad x \notin [a,b] \end{cases}.$$

Note that the following convex closure definition is similar to the convex envelope notion introduced in chapter 3 but the use of the term convex closure is customary in M/L convex functions.

Definition 4.7 (Convex closure, convex extensible function): Let $\Theta : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ be a function defined on the integer lattice with non-empty effective domain $dom_{\mathbb{Z}}\Theta$. The convex closure of Θ is defined to be a function $\overline{\Theta} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\overline{\Theta}(x) = \sup_{p \in \mathbb{R}^n, \alpha \in \mathbb{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq \Theta(y) \ \text{ for } \forall y \in \mathbb{Z}^n\}, (x \in \mathbb{R}^n).$$

In the case where $\overline{\Theta}$ coincides with Θ on integer points, i.e., if

$$\overline{\Theta}(x) = \Theta(x), \forall x \in \mathbb{Z}^n$$

then we say Θ is convex extensible and call $\overline{\Theta}$ the convex extension of Θ.

Definition 4.8 (Integral neighborhood): The integral neighborhood of $x \in \mathbb{R}^n$ is defined by

$$\begin{aligned} N(x) &= \{y \in \mathbb{Z}^n : \lfloor x(i) \rfloor \leq y(i) \leq \lceil x(i) \rceil, \ 1 \leq i \leq n\}, (x \in \mathbb{R}^n) \\ &= \{y \in \mathbb{Z}^n : \|x - y\|_\infty < 1 \text{ for } \forall x \in \mathbb{R}^n\} \end{aligned}$$

where $\lceil a \rceil$ is the ceiling of a, $\lfloor b \rfloor$ is the floor of b, and we used the l_∞ norm $\|z\|_\infty = \max_{1 \leq i \leq n} |z(i)|$ for all $z \in \mathbb{R}^n$.

Definition 4.9 (Local convex extension): Considering the local integral neighborhood $N(x)$, we define the local convex extension of Θ by

$$\widetilde{\Theta}(x) = \sup_{p \in \mathbb{R}^n, \alpha \in \mathbb{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq \Theta(y) \text{ for } \forall y \in N(x)\}.$$

Note that

$$\begin{aligned} \widetilde{\Theta}(x) &\geq \overline{\Theta}(x), \ x \in \mathbb{R}^n, \\ \widetilde{\Theta}(x) &= \Theta(x), \ x \in \mathbb{Z}^n. \end{aligned}$$

Letting $\mathbf{1} = (1, 1, , ..., 1)$, the local convex extension $\widetilde{\Theta}$ of Θ is convex on every unit interval

$$[z, z + \mathbf{1}]_\mathbb{R} = \{x \in \mathbb{R}^n : z(i) \leq x(i) \leq z(i) + 1 \text{ for all } i, 1 \leq i \leq n\}$$

with an integral point $z \in \mathbb{Z}^n$, but is not necessarily convex in the entire space $\mathbb{R}^n$.

Definition 4.10 (Integrally convex (-concave) function): If $\widetilde{\Theta}$ is convex on $\mathbb{R}^n$, the function Θ is said to be integrally convex. Alternatively, we can define

$$\Theta \text{ is integrally convex} \iff \widetilde{\Theta}(x) = \overline{\Theta}(x), \ x \in \mathbb{R}^n.$$

A function $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$ is integrally concave if $-\Theta_2$ is integrally convex. In particular, an integrally convex function is convex extensible.

Recall that a function defined on $\mathbb{R}^n$ is said to be polyhedral convex if its epigraph is a convex polyhedron in $\mathbb{R}^{n+1}$. A polyhedral convex function $\Theta : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ with $dom_{\mathbb{R}}\Theta \neq \emptyset$ is said to be M-convex if it satisfies the following exchange property:

$(EXH-1)$ For $x, y \in dom_{\mathbb{R}}\Theta$ and $u \in supp^+(x-y)$, there exists $v \in supp^-(x-y)$ and a number $\alpha_0 \in \mathbb{R}^+$ such that

$$\Theta(x) + \Theta(y) \geq \Theta(x - \alpha(\chi_u - \chi_v)) + \Theta(y + \alpha(\chi_u - \chi_v))$$

holds for all $\alpha \in [0, \alpha_0]$.

The following result is due to Murota-Shioura (2000).

Theorem 4.1: The convex extension of $\overline{\Theta}$ of an M-convex function $\Theta : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ on the integer lattice is a polyhedral $M-$convex function provided that $\overline{\Theta}$ is polyhedral.

The following result is due to Murota (2003).

Proposition 4.1: For a function $\Theta : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$,

$$\Theta \text{ is integrally convex} \iff \Theta_{[a,b]} \text{ is integrally convex for any } a, b \in \mathbb{Z}^n.$$

The well known directional derivative of a function Θ at a point $x \in dom_{\mathbb{R}}\Theta$ in a direction $d \in \mathbb{R}^n$ is defined by

$$\Theta'(x; d) = \lim_{t \downarrow 0} \frac{\Theta(x + dt) - \Theta(x)}{t}$$

when the limit exists, where $t \downarrow 0$ means t tends to 0 from the positive side. For convex Θ, the limit exists for all d, and $\Theta'(x; d)$ is a convex function in d. For polyhedral M-convex Θ, $\exists\ \epsilon > 0$, independent from $x \in dom_{\mathbb{R}}\Theta$ such that

$$\Theta'(x; d) = \Theta(x + d) - \Theta(x), (\|d\|_1 \leq \epsilon)$$

where $\|d\|_1 = \sum_{i \in V} |d(i)|$ represents the $l_1 - norm$ of the vector $d \in \mathbb{R}^V$ and $\Theta_1'(x; \cdot)$ is the directional derivative of Θ_1 at x. Therefore, the directional derivative of a polyhedral convex function Θ_1 is defined as follows: For each $x \in dom_{\mathbb{R}}\Theta_1$, there exists $\epsilon > 0$ such that

$$\Theta_1'(x; d) = \Theta_1(x + d) - \Theta_1(x), \ \ \|d\|_1 \leq \epsilon.$$

Definition 4.11 (Convex subdifferential, subgradient): Given a function $\Theta_1 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in dom_{\mathbb{Z}}\Theta_1$, we define

$$\partial_{\mathbb{R}}\Theta_1(x) = \left\{p \in \mathbb{R}^V : \Theta_1(y) - \Theta_1(x) \geq \langle p, y - x\rangle \text{ for } \forall y \in \mathbb{Z}^V\right\}$$

and call it the subdifferential of the function Θ_1 at x. An element of $\partial_{\mathbb{R}}\Theta_1$ is called a subgradient of Θ_1 at x.

Note that, being the intersection of infinitely many half spaces indexed by y, $\partial_{\mathbb{R}}\Theta_1(x)$ is convex (possibly empty) for any Θ_1 and any x. The set $\partial_{\mathbb{R}}\Theta_1(x)$ is non-empty for Θ_1 convex and x in the relative interior of $dom\Theta_1$.

If Θ_1 is convex extensible then we have

$$\partial_{\mathbb{R}}\Theta_1(x) = \partial_{\mathbb{R}}\overline{\Theta_1}(x), \ \forall x \in dom_{\mathbb{Z}}\Theta_1 \tag{4.2}$$

where $\overline{\Theta_1}$ is the convex extension of Θ_1 (Murota (2003), pg. 166).

Definition 4.12 (Concave subdifferential): The concave subdifferential $\partial_{\mathbb{R}}'$ of a concave function $\Theta_2 : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is defined by $\partial_{\mathbb{R}}'\Theta_2(x) = -\partial_{\mathbb{R}}(-\Theta_2)(x)$.

Definition 4.13 (Integer subdifferential): The set of integer valued subgradients

$$\partial_{\mathbb{Z}}\Theta_1(x) = \partial_{\mathbb{R}}\Theta_1(x) \cap \mathbb{Z}^V \tag{4.3}$$

is called the integer subdifferential of Θ_1 at $x \in dom_{\mathbb{Z}}\Theta_1$. The notation $\partial_{\mathbb{R}}'\Theta_2$ stands for the subdifferential of a concave function Θ_2 at x.

The directional derivatives and subdifferential of M-convex functions are characterized by theorem 6.61 (Murota (2003)).

Definition 4.14 (Integer concave subdifferential): The integer concave subdifferential $\partial'_{\mathbb{Z}}$ of $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$ is defined by $\partial'_{\mathbb{Z}}\Theta_2(x) = -\partial_{\mathbb{Z}}(-\Theta_2)(x)$.

Definition 4.15 (M-convex set): Given a non-empty set $B \subseteq \mathbb{Z}^V$, we say that B is M-convex if for $x, y \in B$ and $u \in supp^+(x-y)$, there exists $v \in supp^-(x-y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Due to Murota and Shioura (1999) (see theorem 6.3, Murota (2003)), an M-convex function is also an $M^\natural$-convex function. Conversely, an $M^\natural$-convex function is M-convex if and only if the effective domain is contained in $\{x \in \mathbb{Z}^V : x(V) = r\}$ for some $r \in \mathbb{Z}$.

The following result is due to Murota (1998).

Theorem 4.2 (Separation of M-convex sets): Let B_1 $(\subseteq \mathbb{Z}^V)$ and B_2 $(\subseteq \mathbb{Z}^V)$ be M-convex sets. If they are disjoint $(B_1 \cap B_2 = \emptyset)$, there exists $p^* \in \{0,1\}^V \cup \{0,-1\}^V$ such that

$$\inf \{\langle p^*, x\rangle : x \in B_1\} - \sup \{\langle p^*, x\rangle : x \in B_2\} \geq 1. \tag{4.4}$$

The following result follows from Murota (1996) (see theorem 6.42, pg.159).

Theorem 4.3: An $M^\natural$-convex function is integrally convex. In particular, an $M^\natural$-convex function is convex extensible.

The following result follows from Murota (1996) (see proposition 8.14, pg. 217).

Proposition 4.2: If $\Theta_1, -\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ are $M^\natural$-convex functions then

$$\Theta_1(x) \geq \Theta_2(x) \text{ for } \forall x \in \mathbb{Z}^V \Rightarrow \overline{\Theta_1}(x) \geq \overline{\Theta_2}(x) \text{ for } \forall x \in \mathbb{R}^V.$$

For M-convex $\Theta_1 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ and M-concave $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$, discrete versions of the Legendre-Fenchel transformations are defined by

$$\Theta_1^{\bullet}(p) = \sup\left\{\langle p, x\rangle - \Theta_1(x) : x \in \mathbb{Z}^V\right\} \quad \left(p \in \mathbb{R}^V\right),$$
$$\Theta_2^{\circ}(p) = \inf\left\{\langle p, x\rangle - \Theta_2(x) : x \in \mathbb{Z}^V\right\} \quad \left(p \in \mathbb{R}^V\right).$$

By using the results and definitions stated in this section we are ready to state and prove the separation and Fenchel-type duality theorems for M and $M^{\natural}$ convex-concave functions.

Definition 4.16 (Global minimizer, set of minimizers): A global minimizer of $\Theta_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a point x such that $\Theta_1(x) \leq \Theta_1(y)$ for all y. The set of the minimizers of Θ_1, denoted by

$$\arg\min \Theta_1 = \{x \in \mathbb{R}^n | \Theta_1(x) \leq \Theta_1(y) \text{ for } \forall y \in \mathbb{R}^n\}$$

is a convex set for a convex function Θ_1.

For a given function $\Theta_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a vector p, we denote by $\Theta_1[-p]$ the function

$$\Theta_1[-p](x) = \Theta_1(x) - \langle p, x\rangle, \; x \in \mathbb{R}^n$$

which is a convex function for convex Θ_1.

Definition 4.17 (Integral polyhedral M-convex function): A polyhedral convex function $\Theta_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is integral polyhedral convex if

$$\arg\min \Theta_1[-p] \text{ is an integral polyhedron for every } x \in dom_{\mathbb{R}}\Theta_1.$$

Polyhedral M-convex functions with the integrality condition stated above are called integrally polyhedral M-convex functions.

4.1.1 Separation and Fenchel Type Duality Theorems for Discrete M Functions

Some of the M-convex function properties can be extended to real convex function properties: The local minimization yields global minimization, and moreover Fenchel type duality and separation theorems hold. The separation theorem for discrete M functions was proved by Murota (1996) (see also (1998) and (1999)) and has important applications. In this section we state and review Murota's proofs for the separation and Fenchel type duality theorems for discrete M functions.

Let $\overline{S}$ denote the convex hull of the set S.

Theorem 4.4 (Separation of discrete M functions): Let $\Theta_1 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be an $M^\natural$-convex function and $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$ be an $M^\natural$-concave function such that either

$$dom_{\mathbb{Z}}\Theta_1 \cap dom_{\mathbb{Z}}\Theta_2 \neq \emptyset, \tag{4.5}$$

or

$$dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ} \neq \emptyset \tag{4.6}$$

holds. If $\Theta_1(x) \geq \Theta_2(x)$ $\left(\forall x \in \mathbb{Z}^V\right)$, then there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^V$ such that

$$\Theta_1(x) \geq \alpha^* + \langle p^*, x\rangle \geq \Theta_2(x) \tag{4.7}$$

for all $x \in \mathbb{Z}^V$. Moreover, if Θ_1 and Θ_2 are integer valued then there exist $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}^V$ such that (4.7) holds.

Proof: Assume that Θ_1 and $-\Theta_2$ are two $M^\natural$-convex functions.

Case 1: Suppose $dom_{\mathbb{Z}}\Theta_1 \cap dom_{\mathbb{Z}}\Theta_2 \neq \emptyset$. For the convex closure $\overline{\Theta_1}$ of Θ_1 and concave closure $\overline{\Theta_2}$ of Θ_2, we have $\overline{\Theta_1}(x) \geq \overline{\Theta_2}(x)$ for $\forall x \in \mathbb{R}^V$ by proposition 4.2. Since $dom_{\mathbb{R}}\overline{\Theta_1} \cap dom_{\mathbb{R}}\overline{\Theta_2} \neq \emptyset$ holds, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^V$ such that for every $x \in \mathbb{R}^V$, $\overline{\Theta_1}(x) \geq \alpha^* + \langle p^*, x\rangle \geq \overline{\Theta_2}(x)$ holds by the separation theorem for real variable convex functions (theorem 2.3). This implies for every $x \in \mathbb{Z}^V$, $\Theta_1(x) \geq \alpha^* + \langle p^*, x\rangle \geq \Theta_2(x)$ holds since $\overline{\Theta_1}(x) = \Theta_1(x)$ and $\overline{\Theta_2}(x) = \Theta_2(x)$ for all $x \in \mathbb{Z}^V$ by theorem 4.3.

The integrality assertion is proved from the facts that the integer subdifferential of an integer valued M-convex function is an L-convex set and that

L-convex sets have the property of convexity in intersection. We may assume that $\inf \{\Theta_1(x) - \Theta_2(x) | x \in \mathbb{Z}^V\} = 0$. Then there exists $x_0 \in \mathbb{Z}^V$ with $\Theta_1(x_0) - \Theta_2(x_0) = 0$ by the integrality of the function value. By (4.2) and a theorem of Murota ((2003), theorem 6.61 – (2), pg. 166-167) we have

$$\partial_{\mathbb{R}}\overline{\Theta_1}(x_0) \cap \partial'_{\mathbb{R}}\overline{\Theta_2}(x_0) = \partial_{\mathbb{R}}\Theta_1(x_0) \cap \partial'_{\mathbb{R}}\Theta_2(x_0) = \overline{\partial_{\mathbb{Z}}\Theta_1(x_0)} \cap \overline{\partial'_{\mathbb{Z}}\Theta_2(x_0)}$$

which is not empty since $p^* \in \partial_{\mathbb{R}}\overline{\Theta_1}(x_0) \cap \partial'_{\mathbb{R}}\overline{\Theta_2}(x_0)$. $\partial_{\mathbb{Z}}\Theta_1(x_0)$ and $\partial'_{\mathbb{Z}}\Theta_2(x_0)$ are L-convex sets. A result of Murota (see Murota (2003), theorem 5.7) indicates convexity in intersection of L-convex sets, i.e.

$$\overline{\partial_{\mathbb{Z}}\Theta_1(x_0)} \cap \overline{\partial'_{\mathbb{Z}}\Theta_2(x_0)} \neq \emptyset \Rightarrow \partial_{\mathbb{Z}}\Theta_1(x_0) \cap \partial'_{\mathbb{Z}}\Theta_2(x_0) \neq \emptyset$$

which guarantees the existence of an integer vector $p^{**} \in \partial_{\mathbb{Z}}\Theta_1(x_0) \cap \partial'_{\mathbb{Z}}\Theta_2(x_0)$. With this p^{**} and $\alpha^{**} = \Theta_2(x_0) - \langle p^{**}, x_0\rangle \in \mathbb{R}$ the inequality (4.7) is satisfied.

Case 2: Next we suppose that $dom_{\mathbb{Z}}\Theta_1 \cap dom_{\mathbb{Z}}\Theta_2 = \emptyset$ and $dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ} \neq \emptyset$. For a fixed $p_0 \in dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ}$, and for any $p \in \mathbb{R}^V$, we have

$$\begin{aligned}
\Theta_1^{\bullet}(p) &= \sup_{x \in dom_{\mathbb{Z}}\Theta_1} \{\langle p - p_0, x\rangle + [\langle p_0, x\rangle - \Theta_1(x)]\}, \\
&\leq \sup_{x \in dom_{\mathbb{Z}}\Theta_1} \langle p - p_0, x\rangle + \Theta_1^{\bullet}(p_0), \\
\Theta_2^{\circ}(p) &= \inf_{x \in dom_{\mathbb{Z}}\Theta_2} \{\langle p - p_0, x\rangle + [\langle p_0, x\rangle - \Theta_2(x)]\}, \\
&\geq \inf_{x \in dom_{\mathbb{Z}}\Theta_2} \langle p - p_0, x\rangle + \Theta_2^{\circ}(p_0)
\end{aligned}$$

from which follows

$$\Theta_2^{\circ}(p) - \Theta_1^{\bullet}(p) \geq \inf_{x \in dom_{\mathbb{Z}}\Theta_2} \langle p - p_0, x\rangle - \sup_{x \in dom_{\mathbb{Z}}\Theta_1} \langle p - p_0, x\rangle + \Theta_2^{\circ}(p_0) - \Theta_1^{\bullet}(p_0). \quad (4.8)$$

Since $dom_{\mathbb{Z}}\Theta_1$ and $dom_{\mathbb{Z}}\Theta_2$ are disjoint M-convex sets, the separation theorem for M-convex sets, theorem 4.2, gives $p^* \in \mathbb{R}^V$ such that the right hand side of (4.8) with $p = p^*$ is non-negative. Therefore we have $dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ} \neq \emptyset$ and

$\Theta_2^{\circ}(p) \geq \Theta_1^{\bullet}(p)$, hence by the separation theorem for real variable functions there exist $\alpha \in \mathbb{R}$ and $p_0 \in \mathbb{R}^V$ such that

$$\begin{aligned}
\Theta_2^{\circ}(p) &\geq \alpha + \langle p_0, x\rangle \geq \Theta_1^{\bullet}(p), \\
\inf_{x \in dom_{\mathbb{Z}}\Theta_2} \{\langle p_1, x\rangle - \Theta_2(x)\} &\geq \alpha + \langle p_0, x\rangle \geq \sup_{x \in dom_{\mathbb{Z}}\Theta_1} \{\langle p_1, x\rangle - \Theta_1(x)\}, \\
\inf_{x \in dom_{\mathbb{Z}}\Theta_2} \{\langle p_1, x\rangle - \Theta_2(x)\} &\geq \alpha + \langle p_0, x\rangle \geq \sup_{x \in dom_{\mathbb{Z}}\Theta_1} \{\langle p_1, x\rangle - \Theta_1(x)\}.
\end{aligned}$$

Using this inequality and noting that

$$\begin{aligned}
\langle p_1, x\rangle - \Theta_2(x) &\geq \inf_{x \in dom_{\mathbb{Z}}\Theta_2} \{\langle p_1, x\rangle - \Theta_2(x)\} = \Theta_2^{\circ}(p), \ \forall x \in \mathbb{Z}^V, \\
\Theta_1^{\bullet}(p) &= \sup_{x \in dom_{\mathbb{Z}}\Theta_1} \{\langle p_1, x\rangle - \Theta_1(x)\} \geq \langle p_1, x\rangle - \Theta_1(x), \ \forall x \in \mathbb{Z}^V,
\end{aligned}$$

we have

$$\begin{aligned}
\langle p_1, x\rangle - \Theta_2(x) &\geq \alpha + \langle p_0, x\rangle \geq \langle p_1, x\rangle - \Theta_1(x), \\
-\Theta_2(x) &\geq \alpha + \langle p_0 - p_1, x\rangle \geq -\Theta_1(x), \\
\Theta_2(x) &\leq -\alpha + \langle p^{**}, x\rangle \leq \Theta_1(z).
\end{aligned}$$

With this $p^{**} = p_0 - p_1$ and $\alpha^* = -\alpha \in \mathbb{R}$ the inequality (4.7) is satisfied.

For integer valued Θ_1 and $-\Theta_2$, we have $\Theta_1^{\bullet}$ and $-\Theta_2^{\circ}$ to be integral polyhedral L-convex functions, and hence $dom_{\mathbb{R}}\Theta_1^{\bullet}$ and $dom_{\mathbb{R}}\Theta_2^{\circ}$ are integral L-convex polyhedra. We may assume $p_0 \in \mathbb{Z}^V$ by the convexity in intersection of L-convex sets and $p^* \in \mathbb{Z}^V$ by theorem 4.2. Then $\Theta_1^{\bullet}(p^*)$ and $\Theta_2^{\circ}(p^*)$ are integers therefore we can take an integer $\alpha^* \in \mathbb{Z}^V$.

The following Fenchel-type duality theorem for discrete variable M-convex functions is also due to Murota (1996) (see also Murota (1998)).

Theorem 4.5 (Fenchel type duality of discrete M functions): Let $\Theta_1 : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be an $M^{\natural}$-convex function and $\Theta_2 : \mathbb{Z}^V \to \mathbb{R} \cup \{-\infty\}$ be an $M^{\natural}$-concave function such that either

$$dom_{\mathbb{Z}}\Theta_1 \cap dom_{\mathbb{Z}}\Theta_2 \neq \emptyset, \tag{4.9}$$

or

$$dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ} \neq \emptyset \tag{4.10}$$

holds. Then we have

$$\inf_{x\in\mathbb{Z}^V} \{\Theta_1(x) - \Theta_2(x)\} = \sup_{p\in\mathbb{R}^V} \{\Theta_2^{\circ}(p) - \Theta_1^{\bullet}(p)\}. \tag{4.11}$$

If this common value is finite, the supremum is attained by some $p \in dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ}$.

Proof: Suppose that $dom_{\mathbb{Z}}\Theta_1 \cap dom_{\mathbb{Z}}\Theta_2 \neq \emptyset$. By the definitions of discrete versions of the Legendre-Fenchel transformations,

$$\begin{aligned} \inf_{x\in\mathbb{Z}^V} \{\Theta_1(x) - \Theta_2(x)\} &\geq \inf_{x\in\mathbb{R}^V} \{\overline{\Theta_1}(x) - \overline{\Theta_2}(x)\} \geq & (4.12) \\ \sup_{p\in\mathbb{R}^V} \{\Theta_2^{\circ}(p) - \Theta_1^{\bullet}(p)\} &\geq \sup_{p\in\mathbb{Z}^V} \{\Theta_2^{\circ}(p) - \Theta_1^{\bullet}(p)\} & (4.13) \end{aligned}$$

hold. By using inequalities (4.12) and (4.13), we can assume that

$$\epsilon = \inf_{x\in\mathbb{Z}^V} \{\Theta_1(x) - \Theta_2(x)\}$$

is finite. By the separation theorem for discrete M functions, for $(\Theta_1 - \epsilon, \Theta_2)$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^V$ such that

$$\Theta_1(x) - \epsilon \geq \alpha^* + \langle p^*, x\rangle \geq \Theta_2(x) \tag{4.14}$$

for all $x \in \mathbb{Z}^V$ which implies $\Theta_2^{\circ}(p^*) - \Theta_1^{\bullet}(p^*) \geq \epsilon$. (4.12) and (4.13) combined with (4.14) gives (4.11) with the supremum at p^*.

Next we suppose that $dom_{\mathbb{Z}}\Theta_1 \cap dom_{\mathbb{Z}}\Theta_2 = \emptyset$ and $dom_{\mathbb{R}}\Theta_1^{\bullet} \cap dom_{\mathbb{R}}\Theta_2^{\circ} \neq \emptyset$. The separation theorem for M-convex sets applied to $dom_{\mathbb{Z}}\Theta_1$ and $dom_{\mathbb{Z}}\Theta_2$ gives $p^* \in \{0, \pm 1\}^V$ such that (4.4) holds. Plugging in $p = p_0 + cp^*$ in the proof of (4.8) and letting $c \to \infty$, we obtain

$$\sup_{p\in\mathbb{R}^V} \{\Theta_2^{\circ}(p) - \Theta_1^{\bullet}(p)\} = +\infty,$$

whereas

$$\inf_{x\in\mathbb{Z}^V}\{\Theta_1(x)-\Theta_2(x)\}=+\infty$$

by $dom_{\mathbb{Z}}\Theta_1\cap dom_{\mathbb{Z}}\Theta_2=\emptyset$.

The statements and proofs of the separation and Fenchel-type duality theorems for M convex-concave functions will play an important role in the statements and proofs of the separation and Fenchel-type duality theorems for mixed E convex-concave functions.

4.2 MIXED E AND E^* CONVEX-CONCAVE FUNCTIONS

In this section we define mixed E and E^* convex-concave functions by using the definitions of M, $M^\natural$ and proper real convex (concave) functions. In addition, the necessary definitions are stated and related results are proven to state and prove the separation and Fenchel-type duality theorems of mixed E convex-concave functions.

Let $V_1=\{1,2,...,n\}$ and $V_2=\{1,2,...,m\}$ be finite sets. We will use the notation $z=(x,y)$, $p=(p_1,p_2)$, $p_0=(p_0^1,p_0^2)$, $p^*=(p_1^*,p_2^*)$, $p^{**}=(p_1^{**},p_2^{**})$, $z_0=(x_0,y_0)$, $\Im_V=\mathbb{R}^{V_1}\times\mathbb{R}^{V_2}$, $\Im_{\mathbb{Z}\times\mathbb{R}}=\mathbb{Z}^{V_1}\times\mathbb{R}^{V_2}$, $\Im_{\mathbb{Z}}=\mathbb{Z}^{V_1}\times\mathbb{Z}^{V_2}$, $\Im_{\mathbb{Z}_i}=\mathbb{Z}^{V_i}$ and $\Im_{\mathbb{R}_i}=\mathbb{R}^{V_i}$ (for $i=1,2$) throughout this work.

Let $\Theta_1:\Im_{\mathbb{Z}\times\mathbb{R}}\to\mathbb{R}\cup\{\pm\infty\}$. Associated to Θ_1 we have two classes of functions; $\Theta_1^{\Im_{\mathbb{Z}_1}}$, the set of integer variable functions for each fixed real vector in $\Im_{\mathbb{R}_2}$, and $\Theta_1^{\Im_{\mathbb{R}_2}}$, the set of real variable functions for each fixed integer vector in $\Im_{\mathbb{Z}_1}$. *i.e.*

$$\Theta_1^{\Im_{\mathbb{Z}_1}} = \{\Theta_1^y:\Im_{\mathbb{Z}_1}\to\mathbb{R}\cup\{\pm\infty\}\,|y\in\Im_{\mathbb{R}_2}\ \&\ \Theta_1^y(x)=\Theta_1(x,y),\forall(x,y)\in\Im_{\mathbb{Z}\times\mathbb{R}}\}$$
$$\Theta_1^{\Im_{\mathbb{R}_2}} = \{\Theta_1^x:\Im_{\mathbb{R}_2}\to\mathbb{R}\cup\{\pm\infty\}\,|x\in\Im_{\mathbb{Z}_1}\ \&\ \Theta_1^x(y)=\Theta_1(x,y),\forall(x,y)\in\Im_{\mathbb{Z}\times\mathbb{R}}\}$$

Definition 4.18 (Mixed convex extension): A function $\Theta_1:\Im_{\mathbb{Z}\times\mathbb{R}}\to\mathbb{R}\cup\{+\infty\}$ is said to be mixed convex extensible if there exists $\overline{\Theta_1}:\Im_V\to\mathbb{R}\cup\{+\infty\}$, a proper real convex function, such that $\overline{\Theta_1}(x,y)=\Theta_1(x,y)$ where the real extension of the integer variables is done by using definition 4.7 for $\forall(x,y)\in\Im_{\mathbb{Z}\times\mathbb{R}}$.

Definition 4.19 (Mixed E convex-concave function): A mixed function $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}$ is called mixed E convex (concave) if it is mixed convex extensible, M-convex (-concave) with respect to its integer variables and proper convex (concave) with respect to its real variables.

Definition 4.20 (Mixed E*-convex (concave) function): A mixed function $\Theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}$ is called mixed E^* convex (concave) if it is mixed convex extensible, $M^\natural$ convex (concave) with respect to its integer variables and proper convex (concave) with respect to its real variables.

Definition 4.21 (E-convex set): A set $C_1 = A_1 \times B_1 \subset \Im_{\mathbb{Z}\times\mathbb{R}}$ is called a mixed E-convex set if A_1 is an M-convex set and B_1 is a real convex set.

Definition 4.22 $\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1\right)$**:** The domain of a mixed E-convex function Θ_1 $\left(i.e.\ dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1\right)$ is the set of points in $\Im_{\mathbb{Z}\times\mathbb{R}}$ where Θ_1 is finite. Henceforth we assume domain is a mixed E-convex set of the form

$$dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 = dom_{\Im_{\mathbf{Z}_1}}\Theta_1 \times dom_{\Im_{\mathbf{R}_2}}\Theta_1$$

That is, the domain is a product set in $\Im_{\mathbb{Z}\times\mathbb{R}}$.

Definition 4.23 $\left(RI\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1\right)\right)$**:** The relative interior of the mixed E-convex set $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1$ $\left(i.e.\ RI\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1\right)\right)$ is the set

$$RI\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1\right) = RI\left(dom_{\Im_{\mathbf{Z}_1}}\Theta_1\right) \times RI\left(dom_{\Im_{\mathbf{R}_2}}\Theta_1\right).$$

Note that $RI\left(dom_{\Im_{\mathbf{Z}_1}}\overline{\Theta_1}\right)$ is the set of integer points in the relative interior of the domain of $\overline{\Theta_1}$.

The integer convex conjugate of a mixed E-function Θ_1 has the form

$$\Theta_1^{\bullet}(p_1, y) = \sup_{x\in\Im_{\mathbf{Z}_1}} \left\{\langle p_1, x\rangle - \Theta_1(x, y)\right\},$$

and the real convex conjugate of Θ_1 has the form

$$\Theta_1^{\blacksquare}(x, p_2) = \sup_{y\in\Im_{\mathbf{R}_2}} \left\{\langle p_2, y\rangle - \Theta_1(x, y)\right\}.$$

Definition 4.24 (Conjugate of a mixed E-convex function): The convex conjugate of a mixed E function $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \neq \emptyset$ is the function

$$\Theta_1^{\blacklozenge}(p) := \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\}.$$

The following two lemmas characterize the correspondence between the conjugate of a mixed E function and the real and discrete conjugates of a mixed E function.

Lemma 4.1: The convex conjugate of a mixed E function $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \neq \emptyset$ satisfies $\Theta_1^{\blacklozenge} \equiv (-\Theta_1^{\bullet})^{\blacksquare} \equiv \left(-\Theta_1^{\blacksquare}\right)^{\bullet}$. If $\Theta_1(z) = \phi_1(x) + \varphi_1(y)$ in particular then $\Theta_1^{\blacklozenge} \equiv \phi_1^{\bullet}(x) + \varphi_1^{\blacksquare}(y)$.

Proof: By definition 4.24,

$$\begin{aligned}
\Theta_1^{\blacklozenge}(p) &= \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\} \\
&= \sup_{x\in\Im_{\mathbf{Z}_1}} \sup_{y\in\Im_{\mathbf{R}_2}} \{\langle p_1, x\rangle + \langle p_2, y\rangle - \Theta_1(z)\} \\
&= \sup_{x\in\Im_{\mathbf{Z}_1}} \left\{ \langle p_1, x\rangle + \sup_{y\in\Im_{\mathbf{R}_2}} \{\langle p_2, y\rangle - \Theta_1(z)\} \right\} \\
&= \sup_{x\in\Im_{\mathbf{Z}_1}} \left\{\langle p_1, x\rangle + \Theta_1^{\blacksquare}(x, p_2)\right\} \\
&= \sup_{x\in\Im_{\mathbf{Z}_1}} \left\{\langle p_1, x\rangle - \left(-\Theta_1^{\blacksquare}\right)(x, p_2)\right\} \\
&= \left(-\Theta_1^{\blacksquare}\right)^{\bullet}(p_1, p_2) \\
&= \left(-\Theta_1^{\blacksquare}\right)^{\bullet}(p), \forall p \in \Im_V.
\end{aligned}$$

Therefore $\Theta_1^{\blacklozenge} \equiv \left(-\Theta_1^{\blacksquare}\right)^{\bullet}$. Similarly $\Theta_1^{\blacklozenge} \equiv (-\Theta_1^{\bullet})^{\blacksquare}$ follows.

Suppose in particular $\Theta_1(z) = \phi_1(x) + \varphi_1(y)$ then

$$\begin{aligned}
\Theta_1^{\blacklozenge}(p) &= \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\} \\
&= \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \phi_1(x) - \varphi_1(y)\} \\
&= \sup_{y\in\Im_{\mathbf{R}_2}} \sup_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle + \langle p_2, y\rangle - \phi_1(x) - \varphi_1(y)\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle - \phi_1(x)\} + \sup_{y\in\Im_{\mathbf{R}_2}} \{\langle p_2, y\rangle - \varphi_1(y)\} \\
&= \phi_1^{\bullet}(x) + \varphi_1^{\blacksquare}(y).
\end{aligned}
$$

Definition 4.25 (Conjugate of a mixed E-concave function): The concave conjugate of a mixed E function $\Theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 \neq \emptyset$ is the function

$$\Theta_2^{\Diamond}(p) := \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\}.$$

The integer concave conjugate of a mixed E-function Θ_2 has the form

$$\Theta_2^{\circ}(p_1, y) = \inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle - \Theta_2(x, y)\},$$

and the real convex conjugate of Θ_2 has the form

$$\Theta_2^{\square}(x, p_2) = \inf_{y\in\Im_{\mathbf{R}_2}} \{\langle p_2, y\rangle - \Theta_2(x, y)\}.$$

Lemma 4.2: The concave conjugate of a mixed E-concave function $\Theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 \neq \emptyset$ satisfies $\Theta_2^{\Diamond} \equiv (-\Theta_2^{\circ})^{\square} \equiv \left(-\Theta_2^{\square}\right)^{\circ}$. If $\Theta_2(z) = \phi_2(x) + \varphi_2(y)$ in particular then $\Theta_2^{\Diamond} \equiv \phi_2^{\circ}(x) + \varphi_2^{\square}(y)$.

Proof: By definition 4.25,

$$
\begin{aligned}
\Theta_2^{\Diamond}(p) &= \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} \\
&= \inf_{y\in\Im_{\mathbf{R}_2}} \inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle + \langle p_2, y\rangle - \Theta_2(z)\} \\
&= \inf_{y\in\Im_{\mathbf{R}_2}} \left\{\langle p_2, y\rangle + \inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle - \Theta_2(z)\}\right\} \\
&= \inf_{y\in\Im_{\mathbf{R}_2}} \left\{\langle p_2, y\rangle - \left(-\inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle - \Theta_2(z)\}\right)\right\} \\
&= \inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_2, y\rangle - (-\Theta_2^{\circ})(p_1, y)\} \\
&= (-\Theta_2^{\circ})^{\square}(p), \forall p \in \Im_V.
\end{aligned}
$$

Therefore $\Theta_2^{\Diamond} \equiv (-\Theta_2^{\circ})^{\square}$. Similarly $\Theta_2^{\Diamond} \equiv \left(-\Theta_2^{\square}\right)^{\circ}$ follows.

Suppose in particular $\Theta_2(z) = \phi_2(x) + \varphi_2(y)$ then

$$\begin{aligned}
\Theta_2^{\lozenge}(p) &= \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} \\
&= \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \phi_2(x) - \varphi_2(y)\} \\
&= \inf_{y\in\Im_{\mathbf{R}_2}} \inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle + \langle p_2, y\rangle - \phi_2(x) - \varphi_2(y)\} \\
&= \inf_{x\in\Im_{\mathbf{Z}_1}} \{\langle p_1, x\rangle - \phi_2(x)\} + \inf_{y\in\Im_{\mathbf{R}_2}} \{\langle p_2, y\rangle - \varphi_2(y)\} \\
&= \phi_2^{\circ}(x) + \varphi_2^{\square}(y).
\end{aligned}$$

The following lemma characterizes the conjugate correspondence between the mixed E-convex and mixed E-concave functions.

Lemma 4.3: $\Theta_1^{\lozenge}(p) = -(-\Theta_1)^{\blacklozenge}(-p)$ for $p \in \Im_V$.

Proof:

$$\begin{aligned}
\Theta_1^{\lozenge}(p) &= \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\} \\
&= -\sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle -p, z\rangle - (-\Theta_1(z))\} \\
&= -(-\Theta_1)^{\blacklozenge}(-p), \ \forall p \in \Im_V.
\end{aligned}$$

Definition 4.26 (Convex (Concave) extension of a set): Given a mixed function $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{\pm\infty\}$ the convex (concave) extension of the family of functions $\Theta_1^{\Im_{\mathbf{Z}_1}}$ introduced just prior to definition 4.18 is the family of extended functions

$$\overline{\Theta_1}^{\Im_{\mathbf{Z}_1}} = \left\{\overline{\Theta_1^y} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\} \,|y \in \mathbb{R}^m\right\}.$$

where the extension of Θ_1^y from $\mathbb{Z}^n$ to $\mathbb{R}^n$ is found by definition 4.7. If all elements of $\Theta_1^{\Im_{\mathbf{Z}_1}}$ are real convex (concave) extensible (in definition 4.7 sense) then the set $\overline{\Theta_1}^{\Im_{\mathbf{Z}_1}}$ is said to be the convex (concave) extension of $\Theta_1^{\Im_{\mathbf{Z}_1}}$.

Next, we state and prove the separation and Fenchel-type duality theorems for mixed E convex-concave functions and the separation theorem for mixed E convex sets.

4.2.1 SEPARATION AND FENCHEL-TYPE DUALITY THEOREMS FOR MIXED E FUNCTIONS

For mixed E convex sets $C_1 = A_1 \times B_1 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ and $C_2 = A_2 \times B_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$, the notion $C_1 \widetilde{\cap} C_2 = \emptyset$ means $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. $C_1 \widetilde{\cap} C_2 \neq \emptyset$ means that $A_1 \cap A_2 \neq \emptyset$ and $B_1 \cap B_2 \neq \emptyset$.

Definition 4.27 (Polyhedral real extensible mixed E/E*-convex function): A mixed E (resp. E^*) convex function $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ is polyhedral real extensible if its extension $\overline{\Theta_1}$ is a polyhedral real convex function.

Theorem 4.6 (Separation theorem for mixed E-convex sets): Let $C_1 = A_1 \times B_1 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ and $C_2 = A_2 \times B_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ be mixed E convex sets. If C_1 and C_2 are $\widetilde{\cap}$ disjoint ($A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$) then there exists $p_1^* \in \{0,1\}^{V_1} \cup \{0,-1\}^{V_1}$ and a nonzero vector $p_2^* \in \Im_{\mathbb{R}_2}$ giving $p^* = (p_1^*, p_2^*)$ such that

$$\inf_{z\in C_1} \{\langle p^*, z\rangle\} - \sup_{z\in C_2} \{\langle p^*, z\rangle\} \geq 1$$

holds.

Proof: By the separation theorem for disjoint M-convex sets A_1 and A_2 there exists a $p_1^* \in \{0,1\}^{V_1} \cup \{0,-1\}^{V_1}$ such that

$$\inf_{x\in A_1} \{\langle p_1^*, x\rangle\} \geq 1 + \sup_{x\in A_2} \{\langle p_1^*, x\rangle\}. \tag{4.15}$$

Note that B_1 and B_2 are two disjoint convex sets. Therefore, by the separation theorem for real convex functions, there exists $p_2^* \in \Im_{\mathbb{R}_2}$ such that

$$\inf_{y\in B_1} \{\langle p_2^*, y\rangle\} \geq \sup_{y\in B_2} \{\langle p_2^*, y\rangle\} \tag{4.16}$$

By adding (4.15) and (4.16), we have

$$\begin{aligned} \inf_{x\in A_1} \{\langle p_1^*, x\rangle\} + \inf_{y\in B_1} \{\langle p_2^*, y\rangle\} &\geq 1 + \sup_{x\in A_2} \{\langle p_1^*, x\rangle\} + \sup_{y\in B_2} \{\langle p_2^*, y\rangle\} \\ \inf_{x\in A_1, y\in B_1} \{\langle p_1^*, x\rangle + \langle p_2^*, y\rangle\} &\geq 1 + \sup_{x\in A_2, y\in B_2} \{\langle p_1^*, x\rangle + \langle p_2^*, y\rangle\} \end{aligned}$$

$$\begin{aligned}\inf_{(x,y)\in C_1}\left\{\langle (p_1^*,p_2^*),(x,y)\rangle\right\} &\geq 1+\sup_{(x,y)\in C_2}\left\{\langle (p_1^*,p_2^*),(x,y)\rangle\right\}\\ \inf_{z\in C_1}\left\{\langle p,z\rangle\right\} &\geq 1+\sup_{z\in C_2}\left\{\langle p,z\rangle\right\}\\ \inf_{z\in C_1}\left\{\langle p,z\rangle\right\}-\sup_{z\in C_2}\left\{\langle p,z\rangle\right\} &\geq 1 \end{aligned} \tag{4.17}$$

which completes the proof.

By $\{\Theta_i^x(y)\}$ we denote the family of functions with respect to y indexed by x for each $x \in \mathbb{Z}^n$ when $i = 1, 2$.

Theorem 4.7 (Separation for mixed E functions): Let $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed E^*-convex function and $\Theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed E^*-concave function such that one of the following holds:

$$\Theta_1 \text{ and } -\Theta_2 \text{ are polyhedral real extensible and } dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \cap dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 \neq \emptyset \tag{4.18}$$

$$RI\left(dom_{\Im_V}\overline{\Theta_1}\right) \cap RI\left(dom_{\Im_V}\overline{\Theta_2}\right) \neq \emptyset \tag{4.19}$$

$$dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 = \emptyset \text{ and } dom_{\Im_V}\Theta_1^{\blacklozenge} \cap dom_{\Im_V}\Theta_2^{\lozenge} \neq \emptyset \tag{4.20}$$

for all $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$. If $\Theta_1(z) \geq \Theta_2(z)$ for $\forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$, then there exists $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that

$$\Theta_1(z) \geq \alpha^* + \langle p^*, z\rangle \geq \Theta_2(z) \tag{4.21}$$

for all $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Proof: Suppose Θ_1 and $-\Theta_2$ are mixed E^*-convex functions such that $\Theta_1(z) \geq \Theta_2(z)$ for $\forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Case 1: Let Θ_1 and $-\Theta_2$ be polyhedral real extensible convex functions. Therefore the convex extensions $\overline{\Theta_1}(z)$ and $\overline{\Theta_2}(z)$ are polyhedral and satisfy $\overline{\Theta_1}(z) = \Theta_1(z)$ and $\overline{\Theta_2}(z) = \Theta_2(z)$ for all $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$. By the assumption we have $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 \neq \emptyset$. Hence we have $dom_{\Im_{\mathbf{Z}_1}}\Theta_1 \cap dom_{\Im_{\mathbf{Z}_1}}\Theta_2 \neq \emptyset$ and $dom_{\Im_{\mathbf{R}_2}}\Theta_1 \cap dom_{\Im_{\mathbf{R}_2}}\Theta_2 \neq \emptyset$.

By the definition of a mixed E^*-convex function, $\Theta_1^y(x)$ and $-\Theta_2^y(x)$ are $M^\natural$-convex functions for all $y \in \Im_{\mathbb{R}_2}$ which are real extensible. Hence the convex extension $\overline{\Theta_1}^y(x)$ of $\Theta_1^y(x)$ and the concave extension $\overline{\Theta_2}^y(x)$ of $\Theta_2^y(x)$ satisfy $\overline{\Theta_1}^y(x) \geq \overline{\Theta_2}^y(x)$ for $\forall y \in \Im_{\mathbb{R}_2}$ by proposition 4.2. This indicates that $\overline{\Theta_1}(x,y) \geq \overline{\Theta_2}(x,y)$ for $\forall (x,y) \in \Im_V$.

In addition, noting that $\overline{\Theta_1}(x,y)$ and $\overline{\Theta_2}(x,y)$ are polyhedral real convex functions, the separation theorem for real variable convex functions indicates the existence of $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that the inequality

$$\overline{\Theta_1}(z) \geq \alpha^* + \langle p^*, z\rangle \geq \overline{\Theta_2}(z), \ \forall z \in \Im_V$$

holds. Therefore, by restriction to $\Im_{\mathbb{Z}\times\mathbb{R}}$ we have

$$\Theta_1(z) \geq \alpha^* + \langle p^*, z\rangle \geq \Theta_2(z), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$$

since $\overline{\Theta_1}(z) = \Theta_1(z)$ and $\overline{\Theta_2}(z) = \Theta_2(z)$ for all $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Case 2: Suppose (4.18) does not hold and (4.19) holds. This indicates that

$$RI\left(dom_{\Im_{\mathbb{Z}_1}}\overline{\Theta_1}\right) \cap RI\left(dom_{\Im_{\mathbb{Z}_1}}\overline{\Theta_2}\right) \neq \emptyset$$

and

$$RI\left(dom_{\Im_{\mathbb{R}_2}}\overline{\Theta_1}\right) \cap RI\left(dom_{\Im_{\mathbb{R}_2}}\overline{\Theta_2}\right) \neq \emptyset$$

The convex extension $\overline{\Theta_1}^y(x)$ of $\Theta_1^y(x)$ and the concave extension $\overline{\Theta_2}^y(x)$ of $\Theta_2^y(x)$ satisfy $\overline{\Theta_1}^y(x) \geq \overline{\Theta_2}^y(x)$ for $\forall y \in \Im_{\mathbb{R}_2}$ by proposition 4.2. This indicates that $\overline{\Theta_1}(x,y) \geq \overline{\Theta_2}(x,y)$ for $\forall (x,y) \in \Im_{\mathbb{Z}\times\mathbb{R}}$. By the separation theorem for real convex functions there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that

$$\overline{\Theta_1}(z) \geq \alpha^* + \langle p^*, z\rangle \geq \overline{\Theta_2}(z), \ \forall z \in \Im_V$$

which implies by restriction to $\Im_{\mathbb{Z}\times\mathbb{R}}$

$$\Theta_1(z) \geq \alpha^* + \langle p^*, z\rangle \geq \Theta_2(z), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$$

Case 3: Next suppose (4.18) and (4.19) do not hold and (4.20) holds. Therefore we have $dom_{\Im_V}\Theta_1^{\blacklozenge} \cap dom_{\Im_V}^{\lozenge}\Theta_2 \neq \emptyset$. For a fixed $p_0 \in dom_{\Im_V}\Theta_1^{\blacklozenge} \cap dom_{\Im_V}\Theta_2^{\lozenge}$, and for any $p \in \Im_V$, we have

$$\begin{aligned}
\Theta_1^{\blacklozenge}(p) &= \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1} \{\langle p-p_0, z\rangle + [\langle p_0, z\rangle - \Theta_1(z)]\} \\
&\leq \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1} [\langle p-p_0, z\rangle] + \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1} [\langle p_0, z\rangle - \Theta_1(z)] \\
&= \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1} [\langle p-p_0, z\rangle] + \Theta_1^{\blacklozenge}(p_0)
\end{aligned}$$

and

$$\begin{aligned}
\Theta_2^{\lozenge}(p) &= \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2} \{\langle p-p_0, z\rangle + [\langle p_0, z\rangle - \Theta_2(z)]\} \\
&\geq \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2} [\langle p-p_0, z\rangle] + \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2} [\langle p_0, z\rangle - \Theta_2(z)] \\
&= \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2} \langle p-p_0, z\rangle + \Theta_2^{\lozenge}(p_0)
\end{aligned}$$

from which follows

$$\Theta_2^{\lozenge}(p) - \Theta_1^{\blacklozenge}(p) \geq \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2} \langle p-p_0, z\rangle - \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1} \langle p-p_0, z\rangle + \Theta_2^{\lozenge}(p_0) - \Theta_1^{\blacklozenge}(p_0) \tag{4.22}$$

Noting that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1$ and $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2$ are disjoint mixed E-convex sets, by the separation theorem for mixed E-convex sets (theorem 4.6) there exists $p^* \in \Im_V$ such that with $p = p^*$ the right hand side of (4.22) is non-negative;

$$\Theta_2^{\lozenge}(p^*) - \Theta_1^{\blacklozenge}(p^*) \geq 1 + \Theta_2^{\lozenge}(p_0) - \Theta_1^{\blacklozenge}(p_0) \geq 0$$

Hence we have $\Theta_2^{\lozenge}(p^*) \geq \Theta_1^{\blacklozenge}(p^*)$ and by the assumption $dom_{\Im_V}\Theta_1^{\blacklozenge} \cap dom_{\Im_V}\Theta_2^{\lozenge} \neq \emptyset$, we can apply the real separation theorem to the real variable functions $\Theta_2^{\lozenge}$ and $\Theta_1^{\blacklozenge}$ to obtain $\alpha^* \in \mathbb{R}$ and $p^{**} \in \Im_V$ such that for all p

$$\begin{aligned}
\Theta_2^{\lozenge}(p) &\geq \alpha^* + \langle p^{**}, z\rangle \geq \Theta_1^{\blacklozenge}(p) \\
\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} &\geq \alpha^* + \langle p^{**}, z\rangle \geq \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\} \\
\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} &\geq \alpha^* + \langle p^{**}, z\rangle \geq \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\}
\end{aligned}$$

Using this inequality and noting that

$$\begin{aligned} \langle p, z\rangle - \Theta_2(z) &\geq \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} \\ \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\} &\geq \langle p, z\rangle - \Theta_1(z) \end{aligned}$$

we have

$$\begin{aligned} \langle p, z\rangle - \Theta_2(z) &\geq \alpha^* + \langle p^{**}, z\rangle \geq \langle p, z\rangle - \Theta_1(z) \\ -\Theta_2(z) &\geq \alpha^* + \langle p^{**} - p, z\rangle \geq -\Theta_1(z) \\ \Theta_2(z) &\leq \alpha^{**} + \langle p_1^{**}, z\rangle \leq \Theta_1(z) \end{aligned}$$

$\forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$, where $\alpha^{**} = -\alpha^*$ and $p_1^{**} = p - p^{**}$. This completes the proof.

The following Fenchel type duality theorem for mixed E functions follows the conditions of the separation theorem for mixed E convex-concave functions.

Theorem 4.8 (Fenchel type duality for E-functions): Let $\Theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed E^*-convex function and $\Theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed E^*-concave function such that one of the following conditions holds for every $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$

$$\Theta_1 \text{ and } -\Theta_2 \text{ are polyhedral real extensible, } dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \cap dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 \neq \emptyset \quad (4.23)$$

$$RI\left(dom_{\Im_V}\overline{\Theta_1}\right) \cap RI\left(dom_{\Im_V}\overline{\Theta_2}\right) \neq \emptyset \quad (4.24)$$

$$dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 = \emptyset \text{ and } dom_{\Im_V}\Theta_1^{\blacklozenge} \cap dom_{\Im_V}\Theta_2^{\lozenge} \neq \emptyset \quad (4.25)$$

Then we have

$$\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\Theta_1(z) - \Theta_2(z)\} = \sup_{p\in\Im_V} \left\{\Theta_2^{\lozenge}(p) - \Theta_1^{\blacklozenge}(p)\right\} \quad (4.26)$$

If this common value is finite, the supremum is attained by some $p \in dom_{\Im_V}\Theta_1^{\blacklozenge} \cap dom_{\Im_V}\Theta_2^{\lozenge}$.

Proof: Suppose Θ_1 and $-\Theta_2$ are mixed E^*-convex functions.

Case 1: Suppose that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Theta_2 \neq \emptyset$ holds. By the definitions of discrete and real versions of the Legendre-Fenchel transformations, the inequalities

$$\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\Theta_1(z) - \Theta_2(z)\} \geq \inf_{z\in\Im_V} \left\{\overline{\Theta_1}(z) - \overline{\Theta_2}(z)\right\} \geq \tag{4.27}$$

$$\sup_{p\in\Im_V} \left\{\Theta_2^{\lozenge}(p) - \Theta_1^{\blacklozenge}(p)\right\} \geq \sup_{p\in\Im_{\mathbf{Z}\times\mathbf{R}}} \left\{\Theta_2^{\lozenge}(p) - \Theta_1^{\blacklozenge}(p)\right\} \tag{4.28}$$

hold. By using the inequalities (4.27) and (4.28), we can assume that

$$\epsilon = \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\Theta_1(z) - \Theta_2(z)\}$$

is finite. By the separation theorem for mixed E-functions, for $(\Theta_1 - \epsilon, \Theta_2)$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that

$$\Theta_1(z) - \epsilon \geq \alpha^* + \langle p^*, z\rangle \geq \Theta_2(z), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}} \tag{4.29}$$

which implies

$$\begin{aligned}
\Theta_1(z) - \Theta_2(z) &\geq \epsilon \\
-\langle p, z\rangle + \Theta_1(z) + \langle p, z\rangle - \Theta_2(z) &\geq \epsilon \\
\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{-\langle p, z\rangle + \Theta_1(z)\} + \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} &\geq \epsilon \\
-\sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_1(z)\} + \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \Theta_2(z)\} &\geq \epsilon \\
\Theta_2^{\lozenge}(p) - \Theta_1^{\blacklozenge}(p) &\geq \epsilon
\end{aligned} \tag{4.30}$$

and taking the supremum of both sides of (4.30) with respect to $p \in \Im_V$ we have

$$\sup_{p\in\Im_V} \left\{\Theta_1^{\lozenge}(p) - \Theta_2^{\blacklozenge}(p)\right\} \geq \epsilon. \tag{4.31}$$

(4.27) and (4.28) combined with (4.31) give (4.26) with the supremum attained at p^*.

Case 2: Suppose (4.23) does not hold and (4.24) holds. This indicates that

$$RI\left(dom_{\Im_V}\overline{\Theta_1}\right) \cap RI\left(dom_{\Im_V}\overline{\Theta_2}\right) \neq \emptyset$$

By the Fenchel-type duality for real variable convex functions

$$\inf_{(x,y)\in\Im_V}\left\{\overline{\Theta_1}(x,y)-\overline{\Theta_2}(x,y)\right\}=\sup_{(p_1,p_2)\in\Im_V}\left\{\overline{\Theta_2^{\Diamond}}(p_1,p_2)-\overline{\Theta_1^{\blacklozenge}}(p_1,p_2)\right\} \tag{4.32}$$

holds. Since $\Im_{\mathbb{Z}\times\mathbb{R}}\subseteq\Im_V$ and $\overline{\Theta_i}|_{\Im_V}=\Theta_i$ we have

$$\begin{aligned}\inf_{(x,y)\in\Im_V}\left\{\overline{\Theta_1}(x,y)-\overline{\Theta_2}(x,y)\right\} &\leq \inf_{(x,y)\in\Im_{\mathbb{Z}\times\mathbb{R}}}\left\{\Theta_1(x,y)-\Theta_2(x,y)\right\}\\ &\leq \sup_{(x,y)\in\Im_{\mathbb{Z}\times\mathbb{R}}}\left\{\Theta_1(x,y)-\Theta_2(x,y)\right\}\\ &\leq \sup_{(x,y)\in\Im_V}\left\{\overline{\Theta_1}(x,y)-\overline{\Theta_2}(x,y)\right\}\end{aligned}$$

But (4.32) then implies that the inequalities in the previous expression are actually equalities and we then obtain the desired formula.

Case 3: Next we suppose that (4.23) and (4.24) does not hold but (4.25) holds. The separation theorem for mixed E-convex sets applied to $dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Theta_1$ and $dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Theta_2$ gives a p^* such that (4.22) holds. Plugging in $p=p_0+cp^*$ in (4.22) and letting $c\longrightarrow\infty$, we obtain

$$\sup\left\{\Theta_2^{\Diamond}(p)-\Theta_1^{\blacklozenge}(p):p\in\Im_V\right\}=+\infty$$

whereas

$$\inf\left\{\Theta_1(z)-\Theta_2(z):z\in\Im_{\mathbb{Z}\times\mathbb{R}}\right\}=+\infty$$

which completes the proof.

Next, we introduce mixed E_1 and E_1^* convex-concave function concepts. Separation and Fenchel-type duality theorems for mixed E_1 convex-concave functions will be stated and proven in addition to the statement and proof of the separation theorem for mixed E_1 convex sets.

4.3 MIXED E_1 AND E_1^* CONVEX-CONCAVE FUNCTIONS

In the previous section we introduced real extensible mixed functions called E-functions where real convexity and discrete M-convexity definitions and results are used. In this section we introduce mixed E_1 and E_1^* convex-concave functions, whose definitions originate from the definitions of discrete $M/M^\natural$ convex-concave functions and convex-concave envelopes. Note that an E-convex function is real convex extensible function whereas an E_1-convex function has its restriction function an M-convex function. Recall we let $\Im_V = \mathbb{R}^n \times \mathbb{R}^m$, $\Im_{\mathbb{Z}\times\mathbb{R}} = \mathbb{Z}^n \times \mathbb{R}^m$ and $\Im_{\mathbb{Z}} = \mathbb{Z}^n \times \mathbb{Z}^m$.

Definition 4.28 (Restriction function): A restriction function of a mixed function $\theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}$ is the discrete variable function $_*\theta_1 : \Im_{\mathbb{Z}} \to \mathbb{R}$ which satisfies $\theta_1(z) = {_*\theta_1}(z)$ for all $z \in \Im_{\mathbb{Z}}$.

Definition 4.29 (Discretized set): A set $_*S \subseteq \Im_{\mathbb{Z}}$ is the discrete set of $S \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ if $_*S \cap S = {_*S}$ holds.

Definition 4.30 (Mixed E$_1$-convex (concave) function): A mixed function $\theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{+\infty\}$ is called E_1-mixed convex if $\theta_1^x : \mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$ is the convex envelope of $_*\theta_1^x : \mathbb{Z}^m \to \mathbb{R}\cup\{\infty\}$, its restriction function $_*\theta_1$ is discrete $M-$convex and it is $M-$convex with respect to its integer variables. $\theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ is E_1-concave if $-\theta_2$ is E_1-convex.

Definition 4.31 (Mixed E$_1^*$-convex (concave) function): A mixed function $\theta_3 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{+\infty\}$ is called mixed E_1^* convex if $\theta_3^x : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ is the convex envelope of $_*\theta_3^x : \mathbb{Z}^n \to \mathbb{R}\cup\{\infty\}$, its restriction function $_*\theta_3$ is discrete $M^\natural$-convex and it is $M^\natural$-convex with respect to its integer variables. $\theta_4 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ is mixed E_1^*-concave if $-\theta_4$ is mixed E_1^*-convex.

Note that a mixed E_1-convex set is the mixed set $K = H \times W \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ where discretized set $_*K$ of K is an M-convex set in $\Im_{\mathbb{Z}}$, H is an M-convex set in $\Im_{\mathbb{Z}_1}$ and W is a real convex set in $\Im_{\mathbb{R}_2}$.

4.3.1 SEPARATION AND FENCHEL-TYPE DUALITY THEOREMS FOR MIXED E_1 FUNCTIONS

We first state and prove the separation theorem for mixed E_1-convex sets. Second, by using the definitions of mixed E_1 and E_1^* convex-concave functions, we will state and prove the separation and Fenchel-type duality theorems for mixed E_1 convex-concave functions.

Theorem 4.9 (Separation theorem of mixed $\mathbf{E_1}$-convex sets): Let $K_1 = H_1 \times W_1$ and $K_2 = H_2 \times W_2$ be two $\widetilde{\cap}$ disjoint mixed E_1-convex sets in $\Im_{\mathbb{Z}\times\mathbb{R}}$. There exists a $p_1^* \in \{0,1\}^{V_1} \cup \{0,-1\}^{V_1}$ and $p_2^* \in \Im_{\mathbb{R}_2}$ with $p^* = (p_1^*, p_2^*)$ such that

$$\inf_{z\in K_1} \{\langle p^*, z\rangle\} - \sup_{z\in K_2} \{\langle p^*, z\rangle\} \geq 1$$

holds.

Proof: Suppose that K_1 and K_2 are $\widetilde{\cap}$ disjoint mixed E_1-convex sets. Therefore, $H_1 \cap H_2 = \emptyset$ and $W_1 \cap W_2 = \emptyset$. The separation theorem for M-convex sets applied to the pair of convex sets H_1 and H_2 indicates the existence of $p_1^* \in \{0,1\}^{V_1} \cup \{0,-1\}^{V_1}$ such that

$$\inf_{x\in H_1} \{\langle p_1^*, x\rangle\} \geq 1 + \sup_{x\in H_2} \{\langle p_1^*, x\rangle\} \tag{4.33}$$

hold. Applying the separation theorem for real convex sets to the disjoint convex sets W_1 and W_2, there exists a $p_2^* \in \Im_{\mathbb{R}_2}$ such that

$$\inf_{y\in W_1} \{\langle p_2^*, y\rangle\} \geq \sup_{y\in W_2} \{\langle p_2^*, y\rangle\} \tag{4.34}$$

holds. By adding (4.34) and (4.33) we obtain

$$\begin{aligned}
\inf_{x\in H_1} \{\langle p_1^*, x\rangle\} + \inf_{y\in W_1} \{\langle p_2^*, y\rangle\} &\geq 1 + \sup_{x\in H_2} \{\langle p_1^*, x\rangle\} + \sup_{y\in W_2} \{\langle p_2^*, y\rangle\} \\
\inf_{z\in K_1} \{\langle p^*, z\rangle\} &\geq 1 + \sup_{z\in K_2} \{\langle p^*, z\rangle\} \\
\inf_{z\in K_1} \{\langle p^*, z\rangle\} - \sup_{z\in K_2} \{\langle p^*, z\rangle\} &\geq 1
\end{aligned}$$

which completes the proof.

By $\{\theta_i^x(y)\}$ we denote the family of functions with respect to y indexed by x for each $x \in \mathbb{Z}^n$ when $i = 1, 2$. Let θ_1 and $-\theta_2$ be two mixed E_1^* convex functions.

Theorem 4.10 (Separation of mixed $\mathbf{E}_1$ functions): Let $\theta_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed E_1^*-convex function and $\theta_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed E_1^*-concave function such that either

$$ {}_*\theta_1 \text{ and } -({}_*\theta_2) \text{ are polyhedral real extensible, } dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\theta_1 \tilde{\cap} dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\theta_2 \neq \emptyset \quad (4.35) $$

or

$$ dom_{\Im_V}(\theta_1)^{\blacklozenge} \cap dom_{\Im_V}(\theta_2)^{\lozenge} \neq \emptyset \tag{4.36} $$

holds $\forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$. If $\theta_1(z) \geq \theta_2(z)$ for $\forall z \in \Im_{\mathbb{Z}\times\mathbb{R}}$, then there exists $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that

$$ \theta_1(z) \geq \alpha^* + \langle p^*, z\rangle \geq \theta_2(z), \ \forall z \in \Im_{\mathbb{Z}\times\mathbb{R}} \tag{4.37} $$

for all $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Proof: Assume that θ_1 and $-\theta_2$ are two mixed E_1^*-convex functions.

Case 1: Suppose (4.35) holds. This indicates that $dom_{\Im_{\mathbb{Z}_1}}\theta_1 \cap dom_{\Im_{\mathbb{Z}_1}}\theta_2 \neq \emptyset$ and $dom_{\Im_{\mathbb{R}_2}}\theta_1 \cap dom_{\Im_{\mathbb{R}_2}}\theta_2 \neq \emptyset$. For the convex extension $\overline{\theta}_1^{\Im_{\mathbb{Z}_1}}$ of $\theta_1^{\Im_{\mathbb{Z}_1}}$ and concave extension $\overline{\theta}_2^{\Im_{\mathbb{Z}_1}}$ of $\theta_2^{\Im_{\mathbb{Z}_1}}$, we have $\overline{\theta}_1^{\Im_{\mathbb{Z}_1}}(z) \widetilde{\geq} \overline{\theta}_2^{\Im_{\mathbb{Z}_1}}(z)$ for $\forall y \in \mathbb{R}^V$ by proposition 4.2. Since $dom_{\Im_{\mathbb{R}_1}}\overline{\theta}_1 \cap dom_{\Im_{\mathbb{R}_1}}\overline{\theta}_2 \neq \emptyset$ and $dom_{\Im_{\mathbb{R}_2}}\theta_1 \cap dom_{\Im_{\mathbb{R}_2}}\theta_2 \neq \emptyset$, $dom_{\Im_V}\overline{\theta}_1 \cap dom_{\Im_V}\overline{\theta}_2 \neq \emptyset$. By definition, ${}_*\theta_1$ and ${}_*\theta_2$ are $M^\natural$-convex functions. This and theorem 4.3 shows that $\overline{\theta}_1$ and $-\overline{\theta}_2$ are integrally convex functions. By definitions 3.19 and 4.7, the convex envelope structure $\widehat{\theta}_1$ of θ_1 with respect to its real variables in $\mathbb{R}^m$ coincides with the convex extension $\overline{\theta}_1$ of the $M^\#$ discrete convex function ${}_*\theta_1$ with respect to its variables in $\mathbb{R}^m$; therefore, $\overline{\theta}_1(x, y) = \widehat{\theta}_1(x, y)$ for all $y \in \mathbb{R}^m$ when $x \in \mathbb{Z}^n$ is fixed where

$$ \overline{\theta}_1(z) = \sup_{p \in \Im_V, \alpha \in \mathbb{R}} \{\langle p, z\rangle + \alpha \mid \langle p, z_1\rangle + \alpha \leq \theta_1(z_1) \text{ for } \forall z_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\}, (z \in \Im_V). $$

In addition, $\theta_1(z) \geq \theta_2(z)$ for every $z \in \Im_{\mathbb{Z}\times\mathbb{R}}$ by the assumption and $\overline{\theta}_1(z) \geq \overline{\theta}_2(z)$ by proposition 4.2 for every $z \in \Im_V$. Since ${}_*\theta_1$ and $-({}_*\theta_2)$ are polyhedral real extensible convex functions, by theorem 4.1 it follows that $\overline{\theta}_1$ and $\overline{\theta}_2$ are polyhedral M-convex functions. The separation theorem for the real variable functions (theorem 2.3) indicates the existence of $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that

$$\overline{\theta}_1(z) \geq \alpha^* + \langle p^*, z\rangle \geq \overline{\theta}_2(z), \forall z \in \Im_V.$$

In addition,

$$\overline{\theta}_1(z) = \sup_{p\in\Im_V,\alpha\in\mathbb{R}} \{\langle p, z\rangle + \alpha \mid \langle p, z_1\rangle + \alpha \leq \theta_1(z_1) \text{ for all } z_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\}$$

and

$$\overline{\theta}_2(z) = \inf_{p\in\Im_V,\alpha\in\mathbb{R}} \{\langle p, z\rangle + \alpha \mid \langle p, z_1\rangle + \alpha \geq \theta_2(z_1) \text{ for all } z_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\}$$

Therefore,

$$\sup_{p\in\Im_V,\alpha\in\mathbb{R}} \{\langle p, z\rangle + \alpha\} = \overline{\theta}_1(z) \geq \alpha^* + \langle p^*, z\rangle \geq \overline{\theta}_2(z) = \inf_{p\in\Im_V,\alpha\in\mathbb{R}} \{\langle p, z\rangle + \alpha\},$$

and noting that $\langle p, z_1\rangle + \alpha \leq \theta_1(z_1)$ and $\langle p, z_1\rangle + \alpha \geq \theta_2(z_1)$ for every $z_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}$ we have

$$\begin{aligned}
\theta_1(z_1) &\geq \sup_{p\in\Im_V,\alpha\in\mathbb{R}} \{\langle p, z\rangle + \alpha \mid \langle p, z_1\rangle + \alpha \leq \theta_1(z_1) \text{ for all } z_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\} \\
&\geq \alpha^* + \langle p^*, z_1\rangle \\
&\geq \inf_{p\in\Im_V,\alpha\in\mathbb{R}} \{\langle p, z\rangle + \alpha \mid \langle p, z_1\rangle + \alpha \geq \theta_2(z_1) \text{ for all } z_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\} \\
&\geq \theta_2(z_1)
\end{aligned}$$

which completes the proof.

Case 2: Next we suppose that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2 = \emptyset$ and $dom_{\Im_V}\theta_1^{\blacklozenge} \cap dom_{\Im_V}\theta_2^{\lozenge} \neq \emptyset$. That is $dom_{\Im_{\mathbf{Z}_1}}\theta_1 \cap dom_{\Im_{\mathbf{Z}_1}}\theta_2 = \emptyset$, $dom_{\Im_{\mathbf{R}_2}}\theta_1 \cap dom_{\Im_{\mathbf{R}_2}}\theta_2 = \emptyset$ and

$dom_{\Im_V}\theta_1^{\blacklozenge} \cap dom_{\Im_V}\theta_2^{\lozenge} \neq \emptyset$. For a fixed $p_0 \in dom_{\Im_V}\theta_1^{\blacklozenge} \cap dom_{\Im_V}\theta_2^{\lozenge}$, and for any $p \in \Im_V$, we have

$$\begin{aligned}
\theta_1^{\blacklozenge}(p) &= \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1} \{\langle p-p_0, z\rangle + [\langle p_0, z\rangle - \theta_1(z)]\} \\
&\leq \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1} \langle p-p_0, z\rangle + \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1} [\langle p_0, z\rangle - \theta_1(z)] \\
&= \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1} \langle p-p_0, z\rangle + \theta_1^{\blacklozenge}(p_0) \\
\theta_2^{\lozenge}(p) &= \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2} \{\langle p-p_0, z\rangle + [\langle p_0, z\rangle - \theta_2(z)]\} \\
&\geq \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2} \langle p-p_0, z\rangle + \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2} [\langle p_0, z\rangle - \theta_2(z)] \\
&= \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2} \langle p-p_0, z\rangle + \theta_2^{\lozenge}(p_0)
\end{aligned}$$

from which it follows that

$$\theta_2^{\lozenge}(p) - \theta_1^{\blacklozenge}(p) \geq \inf_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2} \langle p-p_0, z\rangle - \sup_{z\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1} \langle p-p_0, z\rangle + \theta_2^{\lozenge}(p_0) - \theta_1^{\blacklozenge}(p_0) \tag{4.38}$$

Since $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1$ and $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2$ are disjoint mixed E_1-convex sets, by the separation theorem for mixed E_1-convex sets there exists a $p^* \in \Im_V$ such that the right hand side of (4.38) with $p = p^*$ is non-negative. Hence we have $\theta_2^{\lozenge}(p^*) \geq \theta_1^{\blacklozenge}(p^*)$ and by the assumption $dom_{\Im_V}\theta_1^{\blacklozenge} \cap dom_{\Im_V}\theta_2^{\lozenge} \neq \emptyset$, therefore there exist $\alpha^* \in \mathbb{R}$ in addition to $p^{**} \in \Im_V$ by the separation theorem applied to the real variable functions $\theta_2^{\lozenge}$ and $\theta_1^{\blacklozenge}$ such that

$$\begin{aligned}
\theta_2^{\lozenge}(p) &\geq \alpha^* + \langle p^{**}, z\rangle \geq \theta_1^{\blacklozenge}(p) \\
\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \theta_2(z)\} &\geq \alpha^* + \langle p^{**}, z\rangle \geq \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \theta_1(z)\} \\
\inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \theta_2(z)\} &\geq \alpha^* + \langle p^{**}, z\rangle \geq \sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \theta_1(z)\}
\end{aligned}$$

Using this inequality and noting that

$$\begin{aligned}
\langle p, z\rangle - \theta_2(z) &\geq \inf_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \theta_2(z)\} \\
\sup_{z\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle p, z\rangle - \theta_1(z)\} &\geq \langle p, z\rangle - \theta_1(z)
\end{aligned}$$

we have

$$\begin{aligned}
\langle p,z\rangle-\theta_2(z) &\geq \alpha^*+\langle p^{**},z\rangle\geq\langle p,z\rangle-\theta_1(z)\\
-\theta_2(z) &\geq \alpha^*+\langle p^{**}-p,z\rangle\geq-\theta_1(z)\\
\theta_2(z) &\leq \alpha^{**}+\langle p_1^{**},z\rangle\leq\theta_1(z)
\end{aligned}$$

holding $\forall z\in\Im_{\mathbb{Z}\times\mathbb{R}}$, where $\alpha^{**}=-\alpha^*$ and $p_1^{**}=p-p^{**}$. This completes the proof.

Note : Theorem 4.10 holds good in the case where the assumption "${}_*\theta_1$ and $-({}_*\theta_2)$ are polyhedral real extensible" stated in (4.35) is replaced with "θ_1 and $-\theta_2$ are polyhedral real extensible."

Next, we state and prove the Fenchel-type duality theorem for mixed E_1 convex-concave functions.

Theorem 4.11 (Fenchel type duality for mixed E$_1$-functions): Let $\theta_1:\Im_{\mathbb{Z}\times\mathbb{R}}\rightarrow\mathbb{R}\cup\{+\infty\}$ be a mixed E_1^*-convex function and $\theta_2:\Im_{\mathbb{Z}\times\mathbb{R}}\rightarrow\mathbb{R}\cup\{-\infty\}$ be a mixed E_1^*-concave function such that either

$${}_*\theta_1 \text{ and } -({}_*\theta_2) \text{ are polyhedral real extensible, } dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\theta_1\widetilde{\cap}dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\theta_2\neq\emptyset \quad (4.39)$$

or

$$dom_{\Im_V}\theta_1^{\blacklozenge}\cap dom_{\Im_V}\theta_2^{\lozenge}\neq\emptyset \quad (4.40)$$

holds. Then we have

$$\inf_{x\in\Im_{\mathbb{Z}\times\mathbb{R}}}\{\theta_1(x)-\theta_2(x)\}=\sup_{p\in\Im_V}\{\theta_2^{\lozenge}(p)-\theta_1^{\blacklozenge}(p)\} \quad (4.41)$$

If this common value is finite, the supremum is attained by some $p\in dom_{\Im_V}\theta_1^{\blacklozenge}\cap dom_{\Im_V}\theta_2^{\lozenge}$.

Proof: Suppose that (4.39) holds. By the definitions of the discrete and real versions of the Legendre-Fenchel transformations,

$$\begin{aligned}
\inf_{x\in\Im_{\mathbb{Z}\times\mathbb{R}}}\{\theta_1(x)-\theta_2(x)\} &\geq \inf_{x\in\Im_V}\{\overline{\theta_1}(x)-\overline{\theta_2}(x)\}\geq &(4.42)\\
\sup_{p\in\Im_V}\{\theta_2^{\lozenge}(p)-\theta_1^{\blacklozenge}(p)\} &\geq \sup_{p\in\Im_{\mathbb{Z}\times\mathbb{R}}}\{\theta_2^{\lozenge}(p)-\theta_1^{\blacklozenge}(p)\} &(4.43)
\end{aligned}$$

hold. By using inequalities (4.42) and (4.43) , we can assume that

$$\epsilon = \inf_{x \in \Im_{\mathbf{Z}\times\mathbf{R}}} \{\theta_1(x) - \theta_2(x)\}$$

is finite. Applying the separation theorem of mixed E_1-functions to $\theta_1 - \epsilon$ and θ_2, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \Im_V$ such that

$$\theta_1(x) - \epsilon \geq \alpha^* + \langle p^*, x\rangle \geq \theta_2(x) \tag{4.44}$$

for all $x \in \Im_{\mathbb{Z}\times\mathbb{R}}$. By (4.44)

$$\theta_1(z) - \theta_2(z) \geq \epsilon$$

and taking the infimum of both sides with respect to $z \in \Im_V$,

$$\inf_{z \in \Im_V} \{\theta_1(z) - \theta_2(z)\} \geq \epsilon \tag{4.45}$$

Applying the Fenchel-type duality theorem for real variable convex functions to the left side of the inequality (4.45) we have

$$\sup_{z \in \Im_V} \left\{\theta_2^{\lozenge}(p^*) - \theta_1^{\blacklozenge}(p^*)\right\} \geq \epsilon. \tag{4.46}$$

(4.42) and (4.43) combined with (4.46) give (4.41) where the supremum is attained at p^*.

Next suppose that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2 = \emptyset$ and $dom_{\Im_V}\theta_1^{\blacklozenge} \cap dom_{\Im_V}\theta_2^{\lozenge} \neq \emptyset$. The separation theorem for mixed E_1-convex sets applied to $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_1$ and $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\theta_2$ gives a p^* such that (4.38) holds. Plugging in $p = p_0 + cp^*$ in inequality (4.38) and letting $c \to \infty$, we obtain

$$\sup_{p \in \Im_V} \left\{\theta_2^{\lozenge}(p) - \theta_1^{\blacklozenge}(p)\right\} = +\infty$$

whereas

$$\inf_{x \in \Im_{\mathbf{Z}\times\mathbf{R}}} \{\theta_1(x) - \theta_2(x)\} = +\infty$$

CHAPTER 5

DISCRETE $L/L^\natural$, MIXED T/T^*, AND MIXED T_1/T_1^* CONVEX-CONCAVE FUNCTIONS

In this chapter, after providing brief information about $L/L^\natural$ convexity concepts and related results, mixed T/T^* and mixed T_1/T_1^* separation and Fenchel duality theorems will be stated and proven.

5.1 DISCRETE L AND $L^\natural$ CONVEX-CONCAVE FUNCTIONS

The motivation behind the L-convex function concept introduced by Murota (1998) arose from the generalization of the Lovász extension of submodular set functions. $L^\natural$ convex functions are defined by Fujishige and Murota (2000). In this section, submodular set functions, their Lovász extension and $L/L^\natural$ convex-concave functions will be introduced. In addition, separation and Fenchel-type duality theorems for L and $L^\natural$ convex-concave functions will be stated and proven.

5.1.1 SUBMODULAR FUNCTIONS

The concept of submodular functions underlies the notion of L-convex function idea; therefore, we will have an introductory look at the basic definitions and results related to submodular set functions following Murota (2003). Let 2^U denote the power set of U.

Definition 5.1 (Submodular-Supermodular functions): A set function $\psi : 2^U \to \mathbb{R} \cup \{+\infty\}$, which assigns a real number (or $+\infty$) to each subset of a given finite set, is said to be submodular if for $\forall X, Y \subseteq U$ the submodularity inequality,

$$\psi(X) + \psi(Y) \geq \psi(X \cup Y) + \psi(X \cap Y),$$

holds, where the inequality holds by convention when $\psi(X)$ or $\psi(Y)$ is equal to $+\infty$. It will be assumed that $\psi(\emptyset) = 0$ and $\psi(U) < +\infty$. A function $\mu : 2^U \to \mathbb{R} \cup \{-\infty\}$ is supermodular when $-\mu$ is submodular.

The relationship between the submodularity and convexity can be formulated in terms of the Lovász extension which is also called *Choquet integral* or *the linear extension.*

If $U = \{u_i\}_{i=1}^n$ is a finite set then for each $q \in \mathbb{R}^U$, we index the elements of $U = \{u_i\}_{i=1}^n$ in a non-increasing order in the components of U; *i.e.*, $U = \{u_1, u_2, ..., u_n\}$ and

$$q(u_1) \geq q(u_2) \geq ... \geq q(u_n)$$

where $|U| = n$. Let $q_i = q(u_i)$, $U_i = \{u_1, u_2, ..., u_i\}$ for $i = 1, 2, ..., n$. We have

$$q = \sum_{i=1}^{n-1} (q_i - q_{i+1}) \chi_{U_i} + q_n \chi_{U_n}$$

which is an expression of q as a linear combination of the characteristic vectors of the subsets U_i.

The linear interpolation of ψ according to this expression yields to the definition of the Lovász extension as follows:

Definition 5.2 (Lovász extension): For any set function $\psi : 2^U \to \mathbb{R} \cup \{\pm\infty\}$, the Lovász extension $\widehat{\psi} : \mathbb{R}^U \to \mathbb{R} \cup \{\pm\infty\}$ of ψ is defined by

$$\widehat{\psi}(q) = \sum_{i=1}^{n-1} (q_i - q_{i+1}) \psi(U_i) + q_n \psi(U_n).$$

Note that here we consider $0 \times \{\pm\infty\} = 0$ in the definition of Lovász extension $\widehat{\psi}$ of ψ. The Lovász extension $\widehat{\psi}$ is indeed an extension of ψ in that $\widehat{\psi}(\chi_X) = \psi(X)$ for $X \subseteq U$. The following theorem is due to Lovász (1983).

Theorem 5.1 (Lovász): A set function ψ is submodular if and only if its Lovász extension function $\widehat{\psi}$ is convex.

Duality of a pair of submodular-supermodular functions is formulated in the following discrete separation theorem where we use the notation

$$x(X) = \sum_{u \in X} x(u)$$

for a vector $x = \{x(u) \mid u \in U\} \in \mathbb{R}^U$ and a subset $X \subseteq U$.

The following separation theorem for submodular set functions is due to Frank (1982).

Theorem 5.2 (Frank's discrete separation): Let $\psi : 2^U \to \mathbb{R} \cup \{+\infty\}$ be a submodular function such that $\psi(\emptyset) = 0$ and $\psi(U) < +\infty$ and $\mu : 2^U \to \mathbb{R} \cup \{-\infty\}$ be a supermodular function such that $\mu(\emptyset) = 0$ and $\mu(U) > -\infty$. If $\psi(X) \geq \mu(X)$ for $\forall\ X \subseteq U$ then $\exists\ x^* \in \mathbb{R}^U$ such that

$$\psi(X) \geq x^*(X) \geq \mu(X), \quad \forall X \subseteq U.$$

Moreover, if ψ and μ are integer valued, the vector x^* can be chosen to be integer valued.

Now we are ready to introduce L and $L^\natural$ convex-concave function concepts.

5.1.2 Discrete L and $L^\natural$ Convex-Concave Functions

Let $\psi : 2^U \to \mathbb{R} \cup \{+\infty\}$ be a submodular set function and $\widehat{\psi}$ be its Lovász extension, which is indeed an extension of ψ in the sense that $\widehat{\psi}(\chi_X) = \psi(X)$ for $X \subseteq U$.

Definition 5.3 (Componentwise maxima-minima): The vectors of componentwise maxima and minima of $p, q \in \mathbb{R}^U$ are defined by

$$p \vee q = \max(p(u), q(u)) \text{ and } p \wedge q = \min(p(u), q(u)) \quad (u \in U),$$

respectively.

The submodularity of ψ on 2^U, or that $\widehat{\psi}$ on $\{0,1\}^U$, extends to the entire space. In fact, it can be shown that $\omega = \widehat{\psi}$ satisfies

$$\omega(p) + \omega(q) \geq \omega(p \vee q) + \omega(p \wedge q), \ \forall p, q \in \mathbb{R}^U. \tag{5.1}$$

Note that the submodularity inequality for ψ is a special case of (5.1) with $p = \chi_X$ and $q = \chi_Y$ because of the identities

$$\chi_{X \cup Y} = \chi_X \vee \chi_Y \text{ and } \chi_{X \cap Y} = \chi_X \wedge \chi_Y.$$

The definition of Lovász extension immediately indicates that

$$\omega(p + \alpha \mathbf{1}) = \omega(p) + \alpha r, \quad \left(\forall p \in \mathbb{R}^U, \forall \alpha \in \mathbb{R}\right) \tag{5.2}$$

for $r = \psi(U)$, where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^U$. This shows the linearity of ω with respect to the translation of p in the direction of $\mathbf{1}$.

The properties (5.1) and (5.2) of the Lovász extension of a submodular set function are discretized to the following definition of L-convex functions.

Definition 5.4 (Discrete L-Convex function): A function $\omega : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\}$ with $dom_{\mathbb{Z}}\omega \neq \emptyset$ is discrete L-convex if it satisfies

$$\begin{aligned} \omega(p) + \omega(q) &\geq \omega(p \vee q) + \omega(p \wedge q), \ \forall p, q \in \mathbb{Z}^U, \\ \text{and } \exists r &\in \mathbb{R} \text{ such that } \omega(p + \mathbf{1}) = \omega(p) + r, \ \forall p \in \mathbb{Z}^U. \end{aligned} \tag{5.3}$$

Let 0 denote a new element not in U and put $\widetilde{U} = U \cup \{0\}$.

Definition 5.5 (Discrete L♮-Convex function): A function $\omega : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\}$ is called discrete $L^\natural$ convex if $\widetilde{\omega} : \mathbb{Z}^{\widetilde{U}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\widetilde{\omega}(p_0, p) = \omega(p - p_0 \mathbf{1}) \quad \left(p_0 \in \mathbb{Z}, p \in \mathbb{Z}^U\right)$$

is an L-convex function.

Definition 5.6 (L-Convex set): A non-empty set of integer points $D \subseteq \mathbb{Z}^U$ is said to be an L-convex set if it satisfies the following two conditions:

$$\begin{aligned} p, q &\in D \Rightarrow p \vee q,\ p \wedge q \in D, \\ p &\in D \Rightarrow p \pm \mathbf{1} \in D. \end{aligned}$$

Recall that a polyhedral convex function $f_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is integral polyhedral convex if

$$\arg\min f_1[-p] \text{ is an integral polyhedron for every } x \in dom_{\mathbb{R}} f_1.$$

Polyhedral L-convex functions with the integrality condition stated above are called integrally polyhedral L-convex functions.

An example of an $\mathbf{L}-$ convex function: Moriguchi and Murota (2005) introduced a Hessian matrix that characterizes the $L^\natural$-convexity of functions and showed that the function $\omega : \mathbb{Z}_+^n \to \mathbb{R}$ defined by Miller (1971)

$$\omega(x) = \sum_{k=0}^{\infty} \left(1 - \prod_{j=1}^{n} \gamma_j(x_j + k)\right) + \lambda \sum_{j=1}^{n} c_j x_j$$

is an $L^\natural$-convex function where $\lambda > 0$, $c_j > 0$, and $\gamma_j(\cdot)$ is a cumulative distribution function of a discrete nonnegative Poisson random variable represented as

$$\gamma_j(k) = \sum_{m=0}^{k} e^{-\lambda_j} \frac{\lambda_j^m}{m!},\ k, m \in \mathbb{Z}_+$$

with $1 \leq j \leq n$.

For L-convex $\omega_1 : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\}$ and L-concave $\omega_2 : \mathbb{Z}^U \to \mathbb{R} \cup \{-\infty\}$, discrete versions of the Legendre-Fenchel transformations are defined by

$$\begin{aligned} \omega_1^\bullet(p) &= \sup_{x \in \mathbb{Z}^U} \{\langle p, x \rangle - \omega_1(x)\} \quad \left(p \in \mathbb{R}^U\right), \\ \omega_2^\circ(p) &= \inf_{x \in \mathbb{Z}^U} \{\langle p, x \rangle - \omega_2(x)\} \quad \left(p \in \mathbb{R}^U\right). \end{aligned}$$

The following results will be important when we prove the separation and Fenchel type duality theorems for both L and T convex-concave functions.

The real extension of discrete L convex functions is defined in the same manner as for the real convex extension of discrete M convex functions. Therefore we continue to use definition 4.7 for real extension $\overline{\omega_1}$ of a given L convex function ω_1.

The following proposition is due to Murota (2003).

Proposition 5.1: If ω_1 , $-\omega_2 : \mathbb{Z}^U \rightarrow \mathbb{R} \cup \{+\infty\}$ are $L^\natural-$ convex functions then

$$\omega_1(x) \geq \omega_2(x) \text{ for } \forall x \in \mathbb{Z}^U \Rightarrow \overline{\omega_1}(x) \geq \overline{\omega_2}(x) \text{ for } \forall x \in \mathbb{R}^U.$$

Theorem 5.3 is due to Murota (1998).

Theorem 5.3: An $L^\natural$ convex function is integrally convex. In particular, an $L^\natural$-convex function is convex extensible.

The following theorem is due to Murota-Shioura (2000).

Theorem 5.4: The convex extension $\overline{\omega_1}$ of an L-convex function $\omega_1 : \mathbb{Z}^U \rightarrow \mathbb{R} \cup \{+\infty\}$ on the integer lattice is a polyhedral L-convex function provided that $\overline{\omega_1}$ is polyhedral.

5.1.3 Separation and Fenchel Type Duality for L Convex-Concave Functions

Some of the L-convex function properties can be extended to the real convex function properties: For example local minimization yields global minimization, and Fenchel-type duality and separation theorems hold. In this section we state and prove the separation and Fenchel-type duality theorems for discrete variable L-convex/concave functions, as well as the separation theorem for L convex sets. The following two theorems are due to Murota (1998).

Theorem 5.5 (Separation of L-convex sets): Let D_1 $(\subseteq \mathbb{Z}^U)$ and D_2 $(\subseteq \mathbb{Z}^U)$ be L-convex sets. If they are disjoint $(D_1 \cap D_2 = \emptyset)$ then there exists $x^* \in \{-1, 0, 1\}^U$ such that

$$\inf_{p \in D_1} \{\langle p, x^* \rangle\} - \sup_{p \in D_2} \{\langle p, x^* \rangle\} \geq 1. \tag{5.4}$$

Theorem 5.6 (Separation theorem for discrete L functions): Let $\omega_1 : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\}$ be an $L^\natural$-convex function and $\omega_2 : \mathbb{Z}^U \to \mathbb{R} \cup \{-\infty\}$ be an $L^\natural$-concave function such that

$$dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 \neq \emptyset, \tag{5.5}$$

or

$$dom_{\mathbb{R}}\left(\omega_1^\bullet\right) \cap dom_{\mathbb{R}}\left(\omega_2^\circ\right) \neq \emptyset. \tag{5.6}$$

If $\omega_1(p) \geq \omega_2(p)$ for $\forall p \in \mathbb{Z}^U$ then there exist $\beta^* \in \mathbb{R}$ and $x^* \in \mathbb{R}^U$ such that

$$\omega_1(p) \geq \beta^* + \langle p, x^* \rangle \geq \omega_2(p) \tag{5.7}$$

for all $p \in \mathbb{Z}^U$. Moreover, if ω_1 and ω_2 are integer valued, there exist $\beta^* \in \mathbb{Z}$ and $x^* \in \mathbb{Z}^U$ such that (5.7) holds.

Proof: We may assume that ω_1 and $-\omega_2$ are two L-convex functions.

Case 1: Suppose $dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 \neq \emptyset$. For the convex closure $\overline{\omega_1}$ of ω_1 and concave closure $\overline{\omega_2}$ of ω_2, we have $\overline{\omega_1}(p) \geq \overline{\omega_2}(p)$ for $\forall p \in \mathbb{R}^U$ by proposition 5.1. Since $dom_{\mathbb{R}}\overline{\omega_1} \cap dom_{\mathbb{R}}\overline{\omega_2} \neq \emptyset$ holds, there exist $\beta^* \in \mathbb{R}$ and $x^* \in \mathbb{R}^U$ such that for every $p \in \mathbb{R}^U$, $\overline{\omega_1}(p) \geq \beta^* + \langle p, x^* \rangle \geq \overline{\omega_2}(p)$ holds by the separation theorem for real variable convex functions (theorem 2.3). This implies that for every $p \in \mathbb{Z}^U$, $\omega_1(p) \geq \beta^* + \langle p, x^* \rangle \geq \omega_2(p)$ holds since $\overline{\omega_1}(p) = \omega_1(p)$ and $\overline{\omega_2}(p) = \omega_2(p)$ for all $p \in \mathbb{Z}^U$ by theorem 5.3.

The integrality assertion is proved from the facts that the integer subdifferential of an integer valued L-convex function is an M-convex set and that M-convex sets have the property of convexity in intersection. We may assume that $\inf\left\{\omega_1(p) - \omega_2(p) \,|\, p \in \mathbb{Z}^U\right\} = 0$. Then there exists $p_0 \in \mathbb{Z}^U$ with $\omega_1(p_0) - \omega_2(p_0) = 0$ by the integrality of the function value. By ω_1 being convex extensible and a theorem of Murota ((2003) , theorem 7.43 – (2) , pg. 196) we have

$$\partial_{\mathbb{R}}\overline{\omega_1}(p_0) \cap \partial'_{\mathbb{R}}\overline{\omega_2}(p_0) = \partial_{\mathbb{R}}\omega_1(p_0) \cap \partial'_{\mathbb{R}}\omega_2(p_0) = \overline{\partial_{\mathbb{Z}}\omega_1(p_0)} \cap \overline{\partial'_{\mathbb{Z}}\omega_2(p_0)}$$

which is not empty since $p^* \in \partial_{\mathbb{R}}\overline{\omega_1}(p_0) \cap \partial'_{\mathbb{R}}\overline{\omega_2}(p_0)$. Since $\partial_{\mathbb{Z}}\omega_1(p_0)$ and $\partial'_{\mathbb{Z}}\omega_2(p_0)$ are M-convex sets, the separation theorem for M-convex sets indicates the convexity in intersection of M-convex sets, i.e.

$$\overline{\partial_{\mathbb{Z}}\omega_1(x_0)} \cap \overline{\partial'_{\mathbb{Z}}\omega_2(x_0)} \neq \emptyset \Rightarrow \partial_{\mathbb{Z}}\omega_1(x_0) \cap \partial'_{\mathbb{Z}}\omega_2(x_0) \neq \emptyset$$

which guarantees the existence of an integer vector $x^{**} \in \partial_{\mathbb{Z}}\omega_1(p_0) \cap \partial'_{\mathbb{Z}}\omega_2(p_0)$. With this x^{**} and $\beta^{**} = \omega_2(p_0) - \langle p_0, x^{**}\rangle \in \mathbb{R}$ the inequality (5.7) is satisfied.

Case 2: Next suppose that $dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 = \emptyset$ and $dom_{\mathbb{R}}\omega_1^{\bullet} \cap dom_{\mathbb{R}}\omega_2^{\circ} \neq \emptyset$. For a fixed $x_0 \in dom_{\mathbb{R}}\omega_1^{\bullet} \cap dom_{\mathbb{R}}\omega_2^{\circ}$ and for any $x \in \mathbb{R}^U$, we have

$$\begin{aligned}
\omega_1^{\bullet}(x) &= \sup_{p \in dom_{\mathbb{Z}}\omega_1} \{\langle p, x - x_0\rangle + [\langle p, x_0\rangle - \omega_1(p)]\}, \\
&\leq \sup_{p \in dom_{\mathbb{Z}}\omega_1} \langle p, x - x_0\rangle + \omega_1^{\bullet}(x_0), \\
\omega_2^{\circ}(x) &= \inf_{p \in dom_{\mathbb{Z}}\omega_2} \{\langle p, x - x_0\rangle + [\langle p, x_0\rangle - \omega_2(p)]\}, \\
&\geq \inf_{p \in dom_{\mathbb{Z}}\omega_2} \langle p, x - x_0\rangle + \omega_2^{\circ}(x_0),
\end{aligned}$$

from which follows

$$\omega_2^{\circ}(x) - \omega_1^{\bullet}(x) \geq \inf_{p \in dom_{\mathbb{Z}}\omega_2} \langle p, x - x_0\rangle - \sup_{p \in dom_{\mathbb{Z}}\omega_2} \langle p, x - x_0\rangle + \omega_2^{\circ}(x_0) - \omega_1^{\bullet}(x_0). \quad (5.8)$$

Since $dom_{\mathbb{Z}}\omega_1$ and $dom_{\mathbb{Z}}\omega_1$ are disjoint L-convex sets, the separation theorem for L-convex sets, theorem 5.5, gives $x^* \in \mathbb{R}^U$ such that the right hand side of (5.8) with $x = x^*$ is non-negative. With this x^* and $\beta^* \in \mathbb{R}$ such that $\omega_1^{\bullet}(x^*) \leq -\beta^* \leq \omega_2^{\circ}(x^*)$, the inequality (5.7) is satisfied.

For integer valued ω_1 and $-\omega_2$, we have $\omega_1^{\bullet}$ and $-\omega_2^{\circ}$ to be integral polyhedral M-convex functions, and hence $dom_{\mathbb{R}}\omega_1^{\bullet}$ and $dom_{\mathbb{R}}\omega_2^{\circ}$ are integral L-convex polyhedra. We may assume $x_0 \in \mathbb{Z}^U$ by the convexity in intersection of M-convex sets and $x^* \in \mathbb{Z}^U$ by the separation theorem for L convex sets. Then $\omega_1^{\bullet}(x^*)$ and $\omega_2^{\circ}(x^*)$ are integers therefore we can take an integer $\beta^* \in \mathbb{Z}$.

The following Fenchel-type duality result is due to Murota (2001).

Theorem 5.7 (Fenchel-type duality for discrete L functions): Let $\omega_1 : \mathbb{Z}^U \to \mathbb{R} \cup \{+\infty\}$ be an $L^\natural$-convex function and $\omega_2 : \mathbb{Z}^U \to \mathbb{R} \cup \{-\infty\}$ be an $L^\natural$-concave function such that

$$dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 \neq \emptyset, \tag{5.9}$$

or

$$dom_{\mathbb{R}}\omega_1^\bullet \cap dom_{\mathbb{R}}\omega_2^\circ \neq \emptyset. \tag{5.10}$$

holds. Then

$$\inf_{p\in\mathbb{Z}^U} \{\omega_1(p) - \omega_2(p)\} = \sup_{x\in\mathbb{R}^U} \{\omega_2^\circ(x) - \omega_1^\bullet(x)\}. \tag{5.11}$$

If this common value is finite, the supremum is attained by some $x \in dom_{\mathbb{R}}\omega_1^\bullet \cap dom_{\mathbb{R}}\omega_2^\circ$.

Proof: Suppose that $dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 \neq \emptyset$. By the definitions of discrete versions of the Legendre-Fenchel transformations, we have the inequalities

$$\inf_{p\in\mathbb{Z}^U} \{\omega_1(p) - \omega_2(p)\} \geq \inf_{p\in\mathbb{R}^U} \{\overline{\omega_1}(p) - \overline{\omega_2}(p)\} \geq \tag{5.12}$$

$$\sup_{x\in\mathbb{R}^U} \{\omega_2^\circ(x) - \omega_1^\bullet(x)\} \geq \sup_{x\in\mathbb{Z}^U} \{\omega_2^\circ(x) - \omega_1^\bullet(x)\}. \tag{5.13}$$

By using inequalities (5.12) and (5.13), we can assume that

$$\delta = \inf_{p\in\mathbb{Z}^U} \{\omega_1(p) - \omega_2(p)\}$$

is finite. By the separation theorem for L-functions applied to the pair $(\omega_1 - \delta, \omega_2)$, there exist $\alpha^* \in \mathbb{R}$ and $x^* \in \mathbb{R}^U$ such that

$$\omega_1(p) - \delta \geq \alpha^* + \langle p, x^* \rangle \geq \omega_2(p) \tag{5.14}$$

for all $p \in \mathbb{Z}^U$ which implies $\omega_2^\circ(p^*) - \omega_1^\bullet(p^*) \geq \delta$. (5.12) and (5.13) combined with (5.14) give (5.11) with the supremum at x^*.

Next we suppose that $dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 = \emptyset$ and $dom_{\mathbb{R}}\omega_1^\bullet \cap dom_{\mathbb{R}}\omega_2^\circ \neq \emptyset$. The separation theorem for L-convex sets applied to $dom_{\mathbb{Z}}\omega_1$ and $dom_{\mathbb{Z}}\omega_2$ gives $p^* \in$

$\{0, \pm 1\}^U$ such that (5.4) holds. Plugging in $x = x_0 + cx^*$ in inequality (5.8) and letting $c \to \infty$, we obtain

$$\sup_{x \in \mathbb{Z}^U} \{\omega_2^\circ(x) - \omega_1^\bullet(x)\} = +\infty,$$

whereas

$$\inf_{p \in \mathbb{Z}^U} \{\omega_1(p) - \omega_2(p)\} = +\infty.$$

by $dom_{\mathbb{Z}}\omega_1 \cap dom_{\mathbb{Z}}\omega_2 = \emptyset$.

The statements and proofs of the separation and Fenchel-type duality theorems for L convex-concave functions will play an important role in that of the separation and Fenchel-type duality theorems for mixed T convex-concave functions. These results are presented in the next section.

5.2 MIXED T AND T^* CONVEX-CONCAVE FUNCTIONS

In this section we define mixed T and T^* convex-concave functions by using the definitions of L, $L^\natural$ and proper real convex (concave) functions. In addition, the necessary definitions are stated and related results are formulated to state and prove the separation and Fenchel-type duality theorems of mixed T convex-concave functions.

Let $n, m \in \mathbb{Z}^+$ such that $U_1 = \{1, 2, ..., n\}$ and $U_2 = \{1, 2, ..., m\}$. We will use the notation $s = (k, h)$, $q = (q_1, q_2)$, $q_0 = (q_0^1, q_0^2)$, $q^* = (q_1^*, q_2^*)$, $q^{**} = (q_1^{**}, q_2^{**})$, $s_0 = (k_0, h_0)$, $\Im_U = \mathbb{R}^{U_1} \times \mathbb{R}^{U_2}$, $\Im_{\mathbb{Z}\times\mathbb{R}} = \mathbb{Z}^{U_1} \times \mathbb{R}^{U_2}$, $\Im_{\mathbb{Z}} = \mathbb{Z}^{U_1} \times \mathbb{Z}^{U_2}$, $\Im_{\mathbb{Z}_i} = \mathbb{Z}^{U_i}$ and $\Im_{\mathbb{R}_i} = \mathbb{R}^{U_i}$ (for $i = 1, 2$) throughout this work.

Let $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{\pm\infty\}$. Associated to Ω_1 we have two classes of functions; $\Omega_1^{\Im_{\mathbb{Z}_1}}$, the set of integer variable functions for each fixed vector in $\Im_{\mathbb{R}_2}$ and $\Omega_1^{\Im_{\mathbb{R}_2}}$, the set of real variable functions for each fixed integer vector in $\Im_{\mathbb{Z}_1}$. *i.e.*

$$\begin{aligned}
\Omega_1^{\Im_{\mathbb{Z}_1}} &= \left\{\Omega_1^h : \Im_{\mathbb{Z}_1} \to \mathbb{R} \cup \{\pm\infty\} \,|h \in \Im_{\mathbb{R}_2} \;\&\; \Omega_1^h(k) = \Omega_1(h, k), \forall (h, k) \in \Im_{\mathbb{Z}\times\mathbb{R}}\right\}, \\
\Omega_1^{\Im_{\mathbb{R}_2}} &= \left\{\Omega_1^k : \Im_{\mathbb{R}_2} \to \mathbb{R} \cup \{\pm\infty\} \,|k \in \Im_{\mathbb{Z}_1} \;\&\; \Omega_1^k(h) = \Omega_1(k, h), \forall (k, h) \in \Im_{\mathbb{Z}\times\mathbb{R}}\right\}.
\end{aligned}$$

Definition 5.7 (Mixed convex extension): A function $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be mixed convex extensible if there exists $\overline{\Omega_1} : \Im_U \rightarrow \mathbb{R} \cup \{+\infty\}$, a proper real convex function, such that $\overline{\Omega_1}(k, h) = \Omega_1(k, h)$ for $\forall (k, h) \in \Im_{\mathbb{Z}\times\mathbb{R}}$ where the real extension of the integer variables is done by using definition 4.7 for $\forall (k, h) \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Definition 5.8 (Mixed T convex (concave) function): A mixed function $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}$ is called mixed T convex (concave) if it is mixed convex extensible, L convex (concave) with respect to its integer variables and proper convex (concave) with respect to its real variables.

Definition 5.9 (Mixed T^* convex (concave) function): A mixed function $\Omega_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}$ is called mixed T^* convex (concave) if it is mixed convex extensible, $L^\natural$ convex (concave) with respect to its integer variables and proper convex (concave) with respect to its real variables.

Definition 5.10 (Mixed T-convex set): A set $D = D_1 \times D_2 \subset \Im_{\mathbb{Z}\times\mathbb{R}}$ is a mixed T-convex set if D_1 is an L-convex set and D_2 is a real convex set.

Definition 5.11 $(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1)$**:** The domain of a mixed T-convex function Ω_1 $(i.e.\ dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1)$ is the set of points in $\Im_{\mathbb{Z}\times\mathbb{R}}$ where Ω_1 is finite. Henceforth we assume the domain is a mixed T-convex set of the form

$$dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1 = dom_{\Im_{\mathbf{Z}_1}}\Omega_1 \times dom_{\Im_{\mathbf{R}_2}}\Omega_1.$$

That is, the domain is a product set in $\Im_{\mathbb{Z}\times\mathbb{R}}$.

Definition 5.12 $\left(RI\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1\right)\right)$**:** The relative interior of the mixed T-convex set $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1$ $\left(i.e.\ RI\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1\right)\right)$ is the set

$$RI\left(dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1\right) = RI\left(dom_{\Im_{\mathbf{Z}_1}}\Omega_1\right) \times RI\left(dom_{\Im_{\mathbf{R}_2}}\Omega_1\right).$$

Note that $RI\left(dom_{\Im_{\mathbf{Z}_1}}\overline{\Omega_1}\right)$ is the set of integer points in the relative interior of the domain of $\overline{\Omega_1}$.

The integer convex conjugate of a mixed T-function Ω_1 has the form

$$\Omega_1^{\bullet}(q_1, y) = \sup_{x \in \Im_{\mathbf{Z}_1}} \{\langle q_1, x\rangle - \Omega_1(x, y)\},$$

and the real convex conjugate of Ω_1 has the form

$$\Omega_1^{\blacksquare}(k, q_2) = \sup_{y \in \Im_{\mathbf{R}_2}} \{\langle q_2, y\rangle - \Omega_1(k, y)\}.$$

Definition 5.13 (Conjugate of a mixed T convex function): The convex conjugate of a mixed T function $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R} \cup \{+\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1 \neq \emptyset$ is the function

$$\Omega_1^{\blacklozenge}(q) := \sup_{s \in \Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_1(s)\}.$$

The following two lemmas characterize the correspondence between the conjugate of a mixed T function and the real and discrete conjugates of a mixed T function.

Lemma 5.1: The convex conjugate of a mixed T function $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R} \cup \{+\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1 \neq \emptyset$ satisfies $\Omega_1^{\blacklozenge} \equiv (-\Omega_1^{\bullet})^{\blacksquare} \equiv \left(-\Omega_1^{\blacksquare}\right)^{\bullet}$. If $\Omega_1(s) = f_1(k) + g_1(h)$ in particular then $\Omega_1^{\blacklozenge} \equiv f_1^{\bullet}(k) + g_1^{\blacksquare}(h)$.

Proof: By definition 5.13,

$$\begin{aligned}
\Omega_1^{\blacklozenge}(q) &= \sup_{s \in \Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_1(s)\} \\
&= \sup_{k \in \Im_{\mathbf{Z}_1}} \sup_{h \in \Im_{\mathbf{R}_2}} \{\langle q_1, k\rangle + \langle q_2, h\rangle - \Omega_1(s)\} \\
&= \sup_{k \in \Im_{\mathbf{Z}_1}} \left\{ \langle q_1, k\rangle + \sup_{h \in \Im_{\mathbf{R}_2}} \{\langle q_2, h\rangle - \Omega_1(s)\} \right\} \\
&= \sup_{k \in \Im_{\mathbf{Z}_1}} \left\{\langle q_1, k\rangle + \Omega_1^{\blacksquare}(k, q_2)\right\} \\
&= \sup_{k \in \Im_{\mathbf{Z}_1}} \left\{\langle q_1, k\rangle - \left(-\Omega_1^{\blacksquare}\right)(k, q_2)\right\} \\
&= \left(-\Omega_1^{\blacksquare}\right)^{\bullet}(q_1, q_2) \\
&= \left(-\Omega_1^{\blacksquare}\right)^{\bullet}(q), \forall q \in \Im_U.
\end{aligned}$$

Therefore $\Omega_1^{\blacklozenge} \equiv \left(-\Omega_1^{\blacksquare}\right)^{\bullet}$. Similarly $\Omega_1^{\blacklozenge} \equiv (-\Omega_1^{\bullet})^{\blacksquare}$ follows.

Suppose in particular $\Omega_1(s) = f_1(k) + g_1(h)$ then

$$\begin{aligned}
\Omega_1^{\blacklozenge}(q) &= \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q,s\rangle - \Omega_1(s)\} \\
&= \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q,s\rangle - f_1(k) - g_1(h)\} \\
&= \sup_{h\in\Im_{\mathbf{R}_2}} \sup_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1,k\rangle + \langle q_2,h\rangle - f_1(k) - g_1(h)\} \\
&= \sup_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1,k\rangle - f_1(k)\} + \sup_{h\in\Im_{\mathbf{R}_2}} \{\langle q_2,h\rangle - g_1(h)\} \\
&= f_1^{\bullet}(k) + g_1^{\blacksquare}(h).
\end{aligned}$$

Definition 5.14 (Conjugate of a mixed T concave function): The concave conjugate of a mixed T function $\Omega_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2 \neq \emptyset$ is the function

$$\Omega_2^{\lozenge}(q) := \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q,s\rangle - \Omega_2(s)\}.$$

The integer concave conjugate of a mixed T-function Ω_2 has the form

$$\Omega_2^{\circ}(q_1,h) = \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1,k\rangle - \Omega_2(k,h)\},$$

and the real convex conjugate of Ω_2 has the form

$$\Omega_2^{\square}(k,q_2) = \inf_{h\in\Im_{\mathbf{R}_2}} \{\langle q_2,h\rangle - \Omega_2(k,h)\}.$$

Lemma 5.2: The concave conjugate of a mixed T-concave function $\Omega_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R}\cup\{-\infty\}$ with $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2 \neq \emptyset$ satisfies $\Omega_2^{\lozenge} \equiv (-\Omega_2^{\circ})^{\square} \equiv \left(-\Omega_2^{\square}\right)^{\circ}$. If $\Omega_2(s) = f_2(k) + g_2(h)$ in particular then $\Omega_2^{\lozenge} \equiv f_2^{\circ}(k) + g_2^{\square}(h)$.

Proof: By definition 5.14,

$$\begin{aligned}
\Omega_2^{\lozenge}(q) &= \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q,s\rangle - \Omega_2(s)\}, \\
&= \inf_{h\in\Im_{\mathbf{R}_2}} \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1,k\rangle + \langle q_2,h\rangle - \Omega_2(s)\}, \\
&= \inf_{h\in\Im_{\mathbf{R}_2}} \left\{\langle q_2,h\rangle + \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1,k\rangle - \Omega_2(s)\}\right\},
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{h\in\Im_{\mathbf{R}_2}} \left\{ \langle q_2, h\rangle - \left(- \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1, k\rangle - \Omega_2(s)\} \right) \right\}, \\
&= \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_2, h\rangle - (-\Omega_2^{\circ})(q_1, h)\}, \\
&= (-\Omega_2^{\circ})^{\square}(q), \forall q \in \Im_U.
\end{aligned}
$$

Therefore $\Omega_2^{\Diamond} \equiv (-\Omega_2^{\circ})^{\square}$. Similarly $\Omega_2^{\Diamond} \equiv \left(-\Omega_2^{\square}\right)^{\circ}$ follows.

Suppose in particular $\Omega_2(s) = f_2(k) + g_2(h)$ then

$$
\begin{aligned}
\Omega_2^{\Diamond}(q) &= \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_2(s)\}, \\
&= \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - f_2(k) - g_2(h)\}, \\
&= \inf_{h\in\Im_{\mathbf{R}_2}} \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1, k\rangle + \langle q_2, h\rangle - f_2(k) - g_2(h)\}, \\
&= \inf_{k\in\Im_{\mathbf{Z}_1}} \{\langle q_1, k\rangle - f_2(k)\} + \inf_{h\in\Im_{\mathbf{R}_2}} \{\langle q_2, h\rangle - g_2(h)\}, \\
&= f_2^{\circ}(k) + g_2^{\square}(h).
\end{aligned}
$$

The following lemma characterizes the conjugate correspondence between the mixed T-convex and mixed T-concave functions.

Lemma 5.3: If $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ is a mixed T-convex function then $\Omega_1^{\Diamond}(q) = -(-\Omega_1)^{\blacklozenge}(-q)$ for every $q \in \Im_U$.

Proof:

$$
\begin{aligned}
\Omega_1^{\Diamond}(q) &= \inf_{(k,h)\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle (q_1, q_2), (k, h)\rangle - \Omega_1(k, h)\}, \\
&= \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_1(s)\}, \\
&= - \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle -q, s\rangle - (-\Omega_1(s))\}, \\
&= -(-\Omega_1)^{\blacklozenge}(-q), \ \forall q \in \Im_U.
\end{aligned}
$$

Definition 5.15 (Convex (Concave) extension of a set): Given a mixed function $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{\pm\infty\}$ the convex (concave) extension of the family of functions $\Omega_1^{\Im_{\mathbf{Z}_1}}$ introduced just prior to definition 5.7 is the family of extended functions

$$
\overline{\Omega_1}^{\Im_{\mathbf{Z}_1}} = \left\{ \overline{\Omega_1^h} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\} \,|\, h \in \mathbb{R}^m \right\}
$$

where the extension of Ω_1^h from $\mathbb{Z}^n$ to $\mathbb{R}^n$ is found by definition 5.7. If all elements of $\Omega_1^{\Im_{\mathbf{Z}_1}}$ are real convex (concave) extensible (in definition 5.7 sense) then the set $\overline{\Omega_1}^{\Im_{\mathbf{Z}_1}}$ is said to be the convex (concave) extension of $\Omega_1^{\Im_{\mathbf{Z}_1}}$.

Next, we state and prove the separation and Fenchel-type duality theorems for mixed T convex-concave functions and the separation theorem for mixed T convex sets.

5.2.1 SEPARATION AND FENCHEL - TYPE DUALITY THEOREMS FOR MIXED T FUNCTIONS

For mixed T convex sets $D = D_1 \times D_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ and $G = G_1 \times G_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$, the notation $D \widetilde{\cap} G = \emptyset$ means that $D_1 \cap G_1 = \emptyset$ and $D_2 \cap G_2 = \emptyset$. The notation $D \widetilde{\cap} G \neq \emptyset$, $\widetilde{\cap}$ means that $D_1 \cap G_1 \neq \emptyset$ and $D_2 \cap G_2 \neq \emptyset$.

Definition 5.16 (Polyhedral real extensible mixed T/T^*-convex function): A mixed T (resp. T^*) convex function $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ is polyhedral real extensible if its extension $\overline{\Omega_1}$ is a polyhedral real convex function.

Theorem 5.8 (Separation theorem for mixed T-convex sets): Let $D = D_1 \times D_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ and $G = G_1 \times G_2 \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ be mixed T convex sets. If D and G are $\widetilde{\cap}$ disjoint ($D_1 \cap G_1 = \emptyset$ and $D_2 \cap G_2 = \emptyset$) then there exists $q_1^* \in \{-1, 0, 1\}^{U_1}$ and a nonzero vector $q_2^* \in \Im_{\mathbb{R}_2}$ giving $q^* = (q_1^*, q_2^*)$ such that

$$\inf_{s\in D} \{\langle q^*, s\rangle\} - \sup_{s\in G} \{\langle q^*, s\rangle\} \geq 1$$

holds.

Proof: By the separation theorem for disjoint L-convex sets D_1 and G_1 there exists a $q_1^* \in \{-1, 0, 1\}^{U_1}$ such that

$$\inf_{k\in D_1} \{\langle q_1^*, k\rangle\} \geq 1 + \sup_{k\in G_1} \{\langle q_1^*, k\rangle\} . \tag{5.15}$$

Note that D_2 and G_2 are two disjoint convex sets. Therefore, by the separation theorem for real convex functions, there exists $q_2^* \in \Im_{\mathbb{R}_2}$ such that

$$\inf_{h\in D_2}\{\langle q_2^*, h\rangle\} \geq \sup_{h\in G_2}\{\langle q_2^*, h\rangle\}. \tag{5.16}$$

By adding (5.15) and (5.16), we have

$$\begin{aligned}
\inf_{k\in D_1}\{\langle q_1^*, k\rangle\} + \inf_{h\in D_2}\{\langle q_2^*, h\rangle\} &\geq 1 + \sup_{k\in G_1}\{\langle q_1^*, k\rangle\} + \sup_{h\in G_2}\{\langle q_2^*, h\rangle\}, \\
\inf_{k\in D_1, h\in D_2}\{\langle q_1^*, k\rangle + \langle q_2^*, h\rangle\} &\geq 1 + \sup_{k\in G_1, h\in G_2}\{\langle q_1^*, k\rangle + \langle q_2^*, h\rangle\}, \\
\inf_{(k,h)\in D}\{\langle (q_1^*, q_2^*), (k,h)\rangle\} &\geq 1 + \sup_{(k,h)\in G}\{\langle (q_1^*, q_2^*), (k,h)\rangle\}, \\
\inf_{s\in D}\{\langle q^*, s\rangle\} &\geq 1 + \sup_{s\in G}\{\langle q^*, s\rangle\}, \\
\inf_{s\in D}\{\langle q^*, s\rangle\} - \sup_{s\in G}\{\langle q^*, s\rangle\} &\geq 1, \qquad (5.17)
\end{aligned}$$

which completes the proof.

By $\{\Omega_i^k(h)\}$ we denote the family of functions with respect to h indexed by k for each $k \in \mathbb{Z}^n$ when $i = 1, 2$.

Theorem 5.9 (Separation for mixed T functions): Let $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed T^*-convex function and $\Omega_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed T^*-concave function such that one of the following holds:

$$\Omega_1 \text{ and } -\Omega_2 \text{ are polyhedral real extensible, and } dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_1 \cap dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_2 \neq \emptyset, \tag{5.18}$$

$$RI\left(dom_{\Im_V}\overline{\Omega_1}\right) \cap RI\left(dom_{\Im_V}\overline{\Omega_2}\right) \neq \emptyset, \tag{5.19}$$

$$dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_1 \widetilde{\cap} dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_2 = \emptyset \text{ and } dom_{\Im_U}\Omega_1^{\blacklozenge} \cap dom_{\Im_U}\Omega_2^{\lozenge} \neq \emptyset. \tag{5.20}$$

If $\Omega_1(s) \geq \Omega_2(s)$ for $\forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}$, then there exists $\beta^* \in \mathbb{R}$ and $q^* \in \Im_U$ such that

$$\Omega_1(s) \geq \beta^* + \langle q^*, s\rangle \geq \Omega_2(s) \tag{5.21}$$

for all $s \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Proof: Suppose Ω_1 and $-\Omega_2$ are mixed T^*-convex functions such that $\Omega_1(s) \geq \Omega_2(s)$ for $\forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Case 1: Let Ω_1 and $-\Omega_2$ be polyhedral real extensible convex functions. Therefore the convex extensions $\overline{\Omega_1}$ and $\overline{\Omega_2}$ are polyhedral and satisfy $\overline{\Omega_1}(s) = \Omega_1(s)$ and $\overline{\Omega_2}(s) = \Omega_2(s)$ for all $s \in \Im_{\mathbb{Z}\times\mathbb{R}}$. By the assumption we have $dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_1 \cap dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_2 \neq \emptyset$. So it follows that $dom_{\Im_V}\overline{\Omega_1} \cap dom_{\Im_V}\overline{\Omega_2} \neq \emptyset$.

By the definition of a mixed T^*-convex function, $\Omega_1^h(k)$ and $-\Omega_2^h(k)$ are $L^\natural$-convex functions for all $h \in \Im_{\mathbb{R}_2}$ which are real extensible. Hence the convex extension $\overline{\Omega_1}^h(k)$ of $\Omega_1^h(k)$ and the concave extension $\overline{\Omega_2}^h(k)$ of $\Omega_2^h(k)$ satisfy $\overline{\Omega_1}^h(k) \geq \overline{\Omega_2}^h(k)$ for $\forall h \in \Im_{\mathbb{R}_2}$ by proposition 5.1. This indicates that $\overline{\Omega_1}(k,h) \geq \overline{\Omega_2}(k,h)$ for $\forall (k,h) \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

In addition, noting that $\overline{\Omega_1}(k,h)$ and $\overline{\Omega_2}(k,h)$ are polyhedral real convex functions whose domains have nonempty intersection, the separation theorem for real variable convex functions indicate the existence of $\alpha^* \in \mathbb{R}$ and $q^* \in \Im_U$ such that the inequality

$$\overline{\Omega_1}(s) \geq \alpha^* + \langle q^*, s\rangle \geq \overline{\Omega_2}(s), \ \forall s \in \Im_U$$

holds. Therefore, by restriction to $\Im_{\mathbb{Z}\times\mathbb{R}}$ we have

$$\Omega_1(s) \geq \alpha^* + \langle q^*, s\rangle \geq \Omega_2(s), \ \forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}$$

since $\overline{\Omega_1}(s) = \Omega_1(s)$ and $\overline{\Omega_2}(s) = \Omega_2(s)$ for all $s \in \Im_{\mathbb{Z}\times\mathbb{R}}$.

Case 2: Suppose (5.18) does not hold and (5.19) holds. This indicates that

$$RI\left(dom_{\Im_{\mathbb{Z}_1}}\overline{\Omega_1}\right) \cap RI\left(dom_{\Im_{\mathbb{Z}_1}}\overline{\Omega_2}\right) \neq \emptyset,$$

and

$$RI\left(dom_{\Im_{\mathbb{R}_2}}\overline{\Omega_1}\right) \cap RI\left(dom_{\Im_{\mathbb{R}_2}}\overline{\Omega_2}\right) \neq \emptyset.$$

The convex extension $\overline{\Omega_1}^h(k)$ of $\Omega_1^h(k)$ and the concave extension $\overline{\Omega_2}^h(k)$ of $\Omega_2^h(k)$ satisfy $\overline{\Omega_1}^h(k) \geq \overline{\Omega_2}^h(k)$ for $\forall h \in \Im_{\mathbb{R}_2}$ by proposition 5.1. This indicates that

$\overline{\Omega_1}(k,h) \geq \overline{\Omega_2}(k,h)$ for $\forall (k,h) \in \Im_{\mathbb{Z}\times\mathbb{R}}$. By the separation theorem for real convex functions there exist $\alpha^* \in \mathbb{R}$ and $q^* \in \Im_U$ such that

$$\overline{\Omega_1}(s) \geq \alpha^* + \langle q^*, s\rangle \geq \overline{\Omega_2}(s), \ \forall s \in \Im_U,$$

which implies by restriction the existence of $\alpha^* \in \mathbb{R}$ and $q^* \in \Im_U$ such that

$$\Omega_1(s) \geq \alpha^* + \langle q^*, s\rangle \geq \Omega_2(s), \ \forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}.$$

Case 3: Next suppose (5.18) and (5.19) do not hold and (5.20) holds. Therefore we have $dom_{\Im_U}\Omega_1^{\blacklozenge} \cap dom_{\Im_U}\Omega_2^{\lozenge} \neq \emptyset$. For a fixed $q_0 \in dom_{\Im_U}\Omega_1^{\blacklozenge} \cap dom_{\Im_U}\Omega_2^{\lozenge}$, and for any $q \in \Im_U$, we have

$$\begin{aligned}
\Omega_1^{\blacklozenge}(q) &= \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1} \{\langle q-q_0, s\rangle + [\langle q_0, s\rangle - \Omega_1(s)]\}, \\
&\leq \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1} \{\langle q-q_0, s\rangle\} + \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1} [\langle q_0, s\rangle - \Omega_1(s)], \\
&= \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1} \langle q-q_0, s\rangle + \Omega_1^{\blacklozenge}(q_0),
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2^{\lozenge}(q) &= \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2} \{\langle q-q_0, s\rangle + [\langle q_0, s\rangle - \Omega_2(s)]\}, \\
&\geq \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2} (\langle q-q_0, s\rangle) + \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2} [\langle q_0, s\rangle - \Omega_2(s)], \\
&= \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2} (\langle q-q_0, s\rangle) + \Omega_2^{\lozenge}(q_0),
\end{aligned}$$

from which follows

$$\Omega_2^{\lozenge}(q) - \Omega_1^{\blacklozenge}(q) \geq \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2} \langle q-q_0, s\rangle - \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1} \langle q-q_0, s\rangle + \Omega_2^{\lozenge}(q_0) - \Omega_1^{\blacklozenge}(q_0). \tag{5.22}$$

Noting that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1$ and $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2$ are disjoint mixed T-convex sets, by the separation theorem for mixed T-convex sets (theorem 5.8) there exists $q^* \in \Im_U$ such that with $q = q^*$ the right hand side of (5.22) is non-negative;

$$\Omega_2^{\lozenge}(q^*) - \Omega_1^{\blacklozenge}(q^*) \geq 1 + \Omega_2^{\lozenge}(q_0) - \Omega_1^{\blacklozenge}(q_0) \geq 0$$

Hence we have $\Omega_2^{\lozenge}(q^*) \geq \Omega_1^{\blacklozenge}(q^*)$ and by the assumption $dom_{\Im_U}\Omega_1^{\blacklozenge} \cap dom_{\Im_U}\Omega_2^{\lozenge} \neq \emptyset$, we can apply the separation theorem to the real variable functions $\Omega_2^{\lozenge}$ and $\Omega_1^{\blacklozenge}$ to obtain $\alpha^* \in \mathbb{R}$, $q^{**} \in \Im_U$ such that $\forall q$

$$\begin{aligned}
\Omega_2^{\lozenge}(q) &\geq \alpha^* + \langle q^{**}, s\rangle \geq \Omega_1^{\blacklozenge}(q), \\
\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_2(s)\} &\geq \alpha^* + \langle q^{**}, s\rangle \geq \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_1(s)\}, \\
\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_2(s)\} &\geq \alpha^* + \langle q^{**}, s\rangle \geq \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_1(s)\},
\end{aligned}$$

Using this inequality and noting that

$$\begin{aligned}
\langle q, s\rangle - \Omega_2(s) &\geq \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_2(s)\}, \\
\sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \Omega_1(s)\} &\geq \langle q, s\rangle - \Omega_1(s),
\end{aligned}$$

we have

$$\begin{aligned}
\langle q, s\rangle - \Omega_2(s) &\geq \alpha^* + \langle q^{**}, s\rangle \geq \langle q, s\rangle - \Omega_1(s), \\
-\Omega_2(s) &\geq \alpha^* + \langle q^{**} - q, s\rangle \geq -\Omega_1(s), \\
\Omega_2(s) &\leq \alpha^{**} + \langle q_1^{**}, s\rangle \leq \Omega_1(s),
\end{aligned}$$

$\forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}$, where $\alpha^{**} = -\alpha^*$ and $q_1^{**} = q - q^{**}$. This completes the proof.

The following Fenchel type duality theorem for mixed T functions follows the conditions of the separation theorem for mixed T convex-concave functions.

Theorem 5.10 (Fenchel type duality for mixed T-functions): Let $\Omega_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed T^*-convex function and $\Omega_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed T^*-concave function such that one of the following conditions holds:

$$\Omega_1 \text{ and } -\Omega_2 \text{ are polyhedral real extensible, } dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1 \cap dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2 \neq \emptyset. \quad (5.23)$$

$$RI\left(dom_{\Im_V}\overline{\Omega_1}\right) \cap RI\left(dom_{\Im_V}\overline{\Omega_2}\right) \neq \emptyset, \quad (5.24)$$

$$dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2 = \emptyset \text{ and } dom_{\Im_U}\Omega_1^{\blacklozenge} \cap dom_{\Im_U}\Omega_2^{\lozenge} \neq \emptyset. \quad (5.25)$$

Then we have

$$\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{\Omega_1(s)-\Omega_2(s)\}=\sup_{q\in\Im_U}\left\{\Omega_2^{\Diamond}(q)-\Omega_1^{\blacklozenge}(q)\right\}. \tag{5.26}$$

If this common value is finite, the supremum is attained by some $q\in dom_{\Im_U}\Omega_1^{\blacklozenge}\cap dom_{\Im_U}\Omega_2^{\Diamond}$.

Proof: Suppose Ω_1 and $-\Omega_2$ are mixed T^*-convex functions.

Case 1: Suppose that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_1\widetilde{\cap}dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\Omega_2\neq\emptyset$ holds. By the definitions of discrete and real versions of the Legendre-Fenchel transformations, the inequalities

$$\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{\Omega_1(s)-\Omega_2(s)\} \;\geq\; \inf_{s\in\Im_U}\left\{\overline{\Omega_1}(s)-\overline{\Omega_2}(s)\right\}\geq \tag{5.27}$$

$$\sup_{q\in\Im_U}\left\{\Omega_2^{\Diamond}(q)-\Omega_1^{\blacklozenge}(q)\right\} \;\geq\; \sup_{q\in\Im_{\mathbf{Z}\times\mathbf{R}}}\left\{\Omega_2^{\Diamond}(q)-\Omega_1^{\blacklozenge}(q)\right\} \tag{5.28}$$

hold. By using the inequalities (5.27) and (5.28), we can assume that

$$\delta=\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{\Omega_1(s)-\Omega_2(s)\}$$

is finite. By the separation theorem for mixed T-functions applied to the pair $(\Omega_1-\delta,\Omega_2)$, there exist $\beta^*\in\mathbb{R}$ and $q^*\in\Im_U$ such that

$$\Omega_1(s)-\delta\geq\beta^*+\langle q^*,s\rangle\geq\Omega_2(s),\ \forall s\in\Im_{\mathbb{Z}\times\mathbb{R}} \tag{5.29}$$

which implies

$$\begin{aligned}
\Omega_1(s)-\Omega_2(s) &\geq \delta\\
-\langle q,s\rangle+\Omega_1(s)+\langle q,s\rangle-\Omega_2(s) &\geq \delta\\
\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{-\langle q,s\rangle+\Omega_1(s)\}+\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{\langle q,s\rangle-\Omega_2(s)\} &\geq \delta\\
-\sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{\langle q,s\rangle-\Omega_1(s)\}+\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}}\{\langle q,s\rangle-\Omega_2(s)\} &\geq \delta\\
\Omega_2^{\Diamond}(q)-\Omega_1^{\blacklozenge}(q) &\geq \delta
\end{aligned} \tag{5.30}$$

Taking the supremum of both sides of (4.30) with respect to $q\in\Im_U$ we have

$$\sup_{q\in\Im_U}\left\{\Omega_1^{\Diamond}(q)-\Omega_2^{\blacklozenge}(q)\right\}\geq\delta. \tag{5.31}$$

(5.27) and (5.28) combined with (5.31) give (5.26) with the supremum attained at q^*.

Case 2: Suppose (5.23) does not hold and (5.24) holds. This indicates that

$$RI\left(dom_{\Im_U}\overline{\Omega_1}\right) \cap RI\left(dom_{\Im_U}\overline{\Omega_2}\right) \neq \emptyset$$

By the Fenchel-type duality for real variable convex functions

$$\inf_{(k,h)\in\Im_U}\left\{\overline{\Omega_1}(k,h) - \overline{\Omega_2}(k,h)\right\} = \sup_{(p_1,p_2)\in\Im_U}\left\{\overline{\Omega_2^{\Diamond}}(q_1,q_2) - \overline{\Omega_1^{\blacklozenge}}(q_1,q_2)\right\} \tag{5.32}$$

holds. Since $\Im_{\mathbb{Z}\times\mathbb{R}} \subseteq \Im_U$ and $\overline{\Omega_i}|_{\Im_U} = \Omega_i$ we have

$$\begin{aligned}
\inf_{(k,h)\in\Im_U}\left\{\overline{\Omega_1}(k,h) - \overline{\Omega_2}(k,h)\right\} &\leq \inf_{(k,h)\in\Im_{\mathbb{Z}\times\mathbb{R}}}\left\{\Omega_1(k,h) - \Omega_2(k,h)\right\} \\
&\leq \sup_{(k,h)\in\Im_{\mathbb{Z}\times\mathbb{R}}}\left\{\Omega_1(k,h) - \Omega_2(k,h)\right\} \\
&\leq \sup_{(k,h)\in\Im_U}\left\{\overline{\Omega_1}(k,h) - \overline{\Omega_2}(k,h)\right\}
\end{aligned}$$

But (5.32) then implies that the inequalities in the previous expression are actually equalities and we then obtain the desired formula.

Case 3: Next we suppose that (5.23) and (5.24) do not hold, but (5.25) holds. The separation theorem for mixed T-convex sets applied to $dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_1$ and $dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\Omega_2$ gives a q^* such that (5.22) holds. Plugging in $q = q_0 + cq^*$ in (5.22) and letting $c \to \infty$, we obtain

$$\sup\left\{\Omega_2^{\Diamond}(k) - \Omega_1^{\blacklozenge}(k) : k \in \Im_U\right\} = +\infty,$$

whereas

$$\inf\left\{\Omega_1(k) - \Omega_2(k) : k \in \Im_{\mathbb{Z}\times\mathbb{R}}\right\} = +\infty.$$

5.3 MIXED T_1 AND T_1^* CONVEX-CONCAVE FUNCTIONS

In this section we introduce mixed T_1 and T_1^* convex-concave functions, whose definitions originate from the definitions of discrete L and $L^\natural$ convex-concave functions.

Separation and Fenchel-type duality theorems for mixed T_1 convex-concave functions will be stated and proven in addition to the statement and proof of the separation theorem for mixed T_1 convex sets.

Definition 5.17 (Restriction function): A restriction function of a mixed function $\phi_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}$ is the discrete variable function ${}_*\phi_1 : \Im_{\mathbb{Z}} \rightarrow \mathbb{R}$ which satisfies $\phi_1(s) =_* \phi_1(s)$ for all $s \in \Im_{\mathbb{Z}}$.

Definition 5.18 (Discretized set): A set ${}_*S \subseteq \Im_{\mathbb{Z}}$ is the discrete set of $S \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ if ${}_*S \cap S =_* S$ holds.

Definition 5.19 (Mixed T_1 convex (concave) function): A mixed function $\phi_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}\cup\{+\infty\}$ is called mixed T_1 convex if $\phi_1^x : \mathbb{R}^m \rightarrow \mathbb{R}\cup\{\infty\}$ is the convex envelope of ${}_*\phi_1^x : \mathbb{Z}^m \rightarrow \mathbb{R}\cup\{\infty\}$, its restriction function ${}_*\phi_1$ is discrete $L-$convex and it is $L-$convex with respect to its integer variables. $\phi_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}\cup\{-\infty\}$ is mixed T_1-concave if $-\phi_2$ is mixed T_1-convex.

Definition 5.20 (Mixed T_1^* convex (concave) function): A mixed function $\phi_3 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}\cup\{+\infty\}$ is called mixed T_1^* convex if $\phi_3^x : \mathbb{R}^m \rightarrow \mathbb{R}\cup\{\infty\}$ is the convex envelope of ${}_*\phi_3^x : \mathbb{Z}^m \rightarrow \mathbb{R}\cup\{\infty\}$, its restriction function ${}_*\phi_3$ is discrete $L^\natural$-convex and it is $L^\natural$-convex with respect to its integer variables. $\phi_4 : \Im_{\mathbb{Z}\times\mathbb{R}} \rightarrow \mathbb{R}\cup\{-\infty\}$ is mixed T_1^*-concave if $-\phi_4$ is mixed T_1^*-convex.

Note that a mixed T_1-convex set is the mixed set $K = H \times W \subseteq \Im_{\mathbb{Z}\times\mathbb{R}}$ where discretized set ${}_*K$ of K is an L-convex set in $\Im_{\mathbb{Z}}$, H is an L-convex set in $\Im_{\mathbb{Z}_1}$ and W is a real convex set in $\Im_{\mathbb{R}_2}$.

5.3.1 SEPARATION AND FENCHEL -TYPE DUALITY THEOREMS FOR MIXED T_1 FUNCTIONS

We first state and prove the separation theorem for mixed T_1-convex sets. Second, by using the definitions of mixed T_1 and T_1^* convex-concave functions, we will state and prove the separation and Fenchel-type duality theorems for mixed T_1 convex-concave functions.

Theorem 5.11 (Separation theorem for mixed T_1-convex sets): Let $K_1 = H_1 \times W_1$ and $K_2 = H_2 \times W_2$ be two $\widetilde{\cap}$ disjoint mixed T_1-convex sets in $\Im_{\mathbb{Z}\times\mathbb{R}}$. Then there exists a $q_1^* \in \{-1,0,1\}^{U_1}$ and $q_2^* \in \Im_{\mathbb{R}_2}$ with $q^* = (q_1^*, q_2^*)$ such that

$$\inf_{s\in K_1} \{\langle q^*, s\rangle\} - \sup_{s\in K_2} \{\langle q^*, s\rangle\} \geq 1.$$

holds.

Proof: Suppose K_1 and K_2 are $\widetilde{\cap}$ disjoint mixed T_1-convex sets. Therefore, $H_1 \cap H_2 = \emptyset$ and $W_1 \cap W_2 = \emptyset$. The separation theorem for L-convex sets applied to the pair of convex sets H_1 and H_2 indicates the existence of $q_1^* \in \{-1,0,1\}^{U_1}$ such that

$$\inf_{k\in H_1} \{\langle q_1^*, k\rangle\} \geq 1 + \sup_{k\in H_2} \{\langle q_1^*, k\rangle\} \tag{5.33}$$

hold. Applying the separation theorem for real convex sets to the disjoint convex sets W_1 and W_2, there exists a $q_2^* \in \Im_{\mathbb{R}_2}$ such that

$$\inf_{h\in W_1} \{\langle q_2^*, h\rangle\} \geq \sup_{h\in W_2} \{\langle q_2^*, h\rangle\} \tag{5.34}$$

holds. By adding (5.34) and (5.33)

$$\begin{aligned} \inf_{k\in H_1} \{\langle q_1^*, k\rangle\} + \inf_{h\in W_1} \{\langle q_2^*, h\rangle\} &\geq 1 + \sup_{k\in H_2} \{\langle q_1^*, k\rangle\} + \sup_{h\in W_2} \{\langle q_2^*, h\rangle\}, \\ \inf_{s\in K_1} \{\langle q^*, s\rangle\} &\geq 1 + \sup_{s\in K_2} \{\langle q^*, s\rangle\}, \end{aligned}$$

which completes the proof.

By $\{\phi_i^k(h)\}$ we denote the family of functions with respect to h indexed by k for each $k \in \mathbb{Z}^n$ when $i = 1, 2$. Let ϕ_1 and $-\phi_2$ be two mixed T_1^* convex functions.

Theorem 5.12 (Separation of mixed T_1 functions): Let $\phi_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed T_1^*-convex function and $\phi_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed T_1^*-concave function such that either

$${}_*\phi_1 \text{ and } -({}_*\phi_2) \text{ are polyhedral real extensible, } dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\phi_1 \widetilde{\cap} dom_{\Im_{\mathbb{Z}\times\mathbb{R}}}\phi_2 \neq \emptyset, \tag{5.35}$$

or

$$dom_{\Im_U} \phi_1^{\blacklozenge} \cap dom_{\Im_U} \phi_2^{\lozenge} \neq \emptyset \tag{5.36}$$

holds. If $\phi_1(s) \geq \phi_2(s)$ for $\forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}$, then there exists $\beta^* \in \mathbb{R}$ and $q^* \in \Im_U$ such that

$$\phi_1(s) \geq \beta^* + \langle q^*, s \rangle \geq \phi_2(s), \ \forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}. \tag{5.37}$$

Proof: Assume that ϕ_1 and $-\phi_2$ are two mixed T_1^*-convex functions.

Case 1: Suppose (5.35) holds. This indicates that $dom_{\Im_{\mathbb{Z}_1}} \phi_1 \cap dom_{\Im_{\mathbb{Z}_1}} \phi_2 \neq \emptyset$ and $dom_{\Im_{\mathbb{R}_2}} \phi_1 \cap dom_{\Im_{\mathbb{R}_2}} \phi_2 \neq \emptyset$. For the convex extension $\overline{\phi}_1^{\Im_{\mathbb{Z}_1}}$ of $\phi_1^{\Im_{\mathbb{Z}_1}}$ and concave extension $\overline{\phi}_2^{\Im_{\mathbb{Z}_1}}$ of $\phi_2^{\Im_{\mathbb{Z}_1}}$, we have $\overline{\phi}_1^{\Im_{\mathbb{Z}_1}}(s) \widetilde{\geq} \overline{\phi}_2^{\Im_{\mathbb{Z}_1}}(s)$ for $\forall h \in \mathbb{R}^U$ by proposition 5.1. Since $dom_{\Im_{\mathbb{R}_1}} \overline{\phi}_1 \cap dom_{\Im_{\mathbb{R}_1}} \overline{\phi}_2 \neq \emptyset$ and $dom_{\Im_{\mathbb{R}_2}} \phi_1 \cap dom_{\Im_{\mathbb{R}_2}} \phi_2 \neq \emptyset$, $dom_{\Im_U} \overline{\phi}_1 \cap dom_{\Im_U} \overline{\phi}_2 \neq \emptyset$. By definition, ${}_*\phi_1$ and ${}_*\phi_2$ are $L^\natural$-convex functions. This and theorem 5.4 show that $\overline{\phi}_1$ and $-\overline{\phi}_2$ are integrally convex functions. By definitions 3.19 and 4.7, the convex envelope structure $\widehat{\phi}_1$ of ϕ_1 with respect to its real variables in $\mathbb{R}^m$ coincides with the convex extension $\overline{\phi}_1$ of the $L^\#$ discrete convex function ${}_*\phi_1$ with respect to its variables in $\mathbb{R}^m$; therefore, $\overline{\phi}_1(k,h) = \widehat{\phi}_1(k,h)$ for all $h \in \mathbb{R}^m$ when $k \in \mathbb{Z}^n$ is fixed where

$$\overline{\phi}_1(s) = \sup_{q \in \Im_V, \beta \in \mathbb{R}} \{\langle q, s \rangle + \beta \mid \langle q, s_1 \rangle + \beta \leq \phi_1(s_1) \text{ for } \forall s_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\}, (s \in \Im_V).$$

In addition, $\phi_1(s) \geq \phi_2(s)$ for every $s \in \Im_{\mathbb{Z}\times\mathbb{R}}$ by the assumption and $\overline{\phi}_1(s) \geq \overline{\phi}_2(s)$ by proposition 4.2 for every $s \in \Im_V$. Since ${}_*\phi_1$ and $-({}_*\phi_2)$ are polyhedral real extensible convex functions, by theorem 5.4 it follows that $\overline{\phi}_1$ and $\overline{\phi}_2$ are polyhedral L-convex functions. The separation theorem for the real variable functions (theorem 2.3) indicates the existence of $\beta^* \in \mathbb{R}$ and $q^* \in \Im_V$ such that

$$\overline{\phi}_1(s) \geq \beta^* + \langle q^*, s \rangle \geq \overline{\phi}_2(s), \forall s \in \Im_V.$$

In addition,

$$\overline{\phi}_1(s) = \sup_{q \in \Im_V, \beta \in \mathbb{R}} \{\langle q, s \rangle + \beta \mid \langle q, s_1 \rangle + \beta \leq \phi_1(s_1) \text{ for all } s_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\}$$

and

$$\overline{\phi}_2(s) = \inf_{q\in\Im_V,\beta\in\mathbb{R}} \{\langle q, s\rangle + \beta | \langle q, s_1\rangle + \beta \geq \phi_2(s_1) \text{ for all } s_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\}$$

Therefore,

$$\sup_{q\in\Im_V,\beta\in\mathbb{R}} \{\langle q, s\rangle + \beta\} = \overline{\phi}_1(s) \geq \beta^* + \langle q^*, s\rangle \geq \overline{\phi}_2(s) = \inf_{q\in\Im_V,\beta\in\mathbb{R}} \{\langle q, s\rangle + \beta\},$$

and noting that $\langle q, s_1\rangle + \beta \leq \phi_1(s_1)$ and $\langle q, s_1\rangle + \beta \geq \phi_2(s_1)$ for every $s_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}$

$$\begin{aligned}
\phi_1(s_1) &\geq \sup_{q\in\Im_V,\beta\in\mathbb{R}} \{\langle q, s\rangle + \beta \mid \langle q, s_1\rangle + \beta \leq \phi_1(s_1) \text{ for all } s_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\} \\
&\geq \beta^* + \langle q^*, s_1\rangle \\
&\geq \inf_{q\in\Im_V,\beta\in\mathbb{R}} \{\langle q, s\rangle + \beta | \langle q, s_1\rangle + \beta \geq \phi_2(s_1) \text{ for all } s_1 \in \Im_{\mathbb{Z}\times\mathbb{R}}\} \\
&\geq \phi_2(s_1)
\end{aligned}$$

which completes the proof.

Case 2: Next we suppose that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2 = \emptyset$ and $dom_{\Im_U}\phi_1^{\blacklozenge} \cap dom_{\Im_U}\phi_2^{\lozenge} \neq \emptyset$. That is $dom_{\Im_{\mathbf{Z}_1}}\phi_1 \cap dom_{\Im_{\mathbf{Z}_1}}\phi_2 = \emptyset$, $dom_{\Im_{\mathbf{R}_2}}\phi_1 \cap dom_{\Im_{\mathbf{R}_2}}\phi_2 = \emptyset$ and $dom_{\Im_U}\phi_1^{\blacklozenge} \cap dom_{\Im_U}\phi_2^{\lozenge} \neq \emptyset$. For a fixed $q_0 \in dom_{\Im_U}\phi_1^{\blacklozenge} \cap dom_{\Im_U}\phi_2^{\lozenge}$, and for any $q \in \Im_U$, we have

$$\begin{aligned}
\phi_1^{\blacklozenge}(q) &= \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1} \{\langle q - q_0, s\rangle + [\langle q_0, s\rangle - \phi_1(s)]\}, \\
&\leq \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1} \langle q - q_0, s\rangle + \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1} [\langle q_0, s\rangle - \phi_1(s)], \\
&= \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1} \langle q - q_0, s\rangle + \phi_1^{\blacklozenge}(q_0), \\
\phi_2^{\lozenge}(q) &= \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2} \{\langle q - q_0, s\rangle + [\langle q_0, s\rangle - \phi_2(s)]\}, \\
&\geq \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2} \langle q - q_0, s\rangle + \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2} [\langle q_0, s\rangle - \phi_2(s)], \\
&= \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2} \langle q - q_0, s\rangle + \phi_2^{\lozenge}(q_0),
\end{aligned}$$

from which follows

$$\phi_2^{\lozenge}(q) - \phi_1^{\blacklozenge}(q) \geq \inf_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2} \langle q - q_0, s\rangle - \sup_{s\in dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1} \langle q - q_0, s\rangle + \phi_2^{\lozenge}(q_0) - \phi_1^{\blacklozenge}(q_0). \tag{5.38}$$

Since $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1$ and $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2$ are disjoint mixed T_1-convex sets, by the separation theorem for mixed T_1-convex sets there exists a $q^* \in \Im_U$ such that the right hand side of (5.38) with $q = q^*$ is non-negative.

Hence we have $\phi_2^{\lozenge}(q^*) \geq \phi_1^{\blacklozenge}(q^*)$ and by the assumption $dom_{\Im_V}\phi_1^{\blacklozenge} \cap dom_{\Im_V}\phi_2^{\lozenge} \neq \emptyset$. Therefore by the separation theorem applied to the real variable functions $\phi_2^{\lozenge}$ and $\phi_1^{\blacklozenge}$ there exist $\beta^* \in \mathbb{R}$ and $q^{**} \in \Im_U$ such that

$$\begin{aligned}
\phi_2^{\lozenge}(q) &\geq \beta^* + \langle q^{**}, s\rangle \geq \phi_1^{\blacklozenge}(q) \\
\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \phi_2(s)\} &\geq \beta^* + \langle q^{**}, s\rangle \geq \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \phi_1(s)\} \\
\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \phi_2(s)\} &\geq \beta^* + \langle q^{**}, s\rangle \geq \sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \phi_1(s)\}
\end{aligned}$$

Using this inequality and noting that

$$\begin{aligned}
\langle q, s\rangle - \phi_2(s) &\geq \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \phi_2(s)\} \\
\sup_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\langle q, s\rangle - \phi_1(s)\} &\geq \langle q, s\rangle - \phi_1(s)
\end{aligned}$$

we have

$$\begin{aligned}
\langle q, s\rangle - \phi_2(s) &\geq \beta^* + \langle q^{**}, s\rangle \geq \langle q, s\rangle - \phi_1(s) \\
-\phi_2(s) &\geq \beta^* + \langle q^{**} - q, s\rangle \geq -\phi_1(s) \\
\phi_2(s) &\leq \beta^{**} + \langle q_1^{**}, s\rangle \leq \phi_1(s)
\end{aligned}$$

holding $\forall s \in \Im_{\mathbb{Z}\times\mathbb{R}}$, where $\beta^{**} = -\beta^*$ and $q_1^{**} = q - q^{**}$. This completes the proof.

Note: Theorem 5.12 holds good in the case where the assumption "$_*\phi_1$ and $-({}_*\phi_2)$ are polyhedral real extensible" stated in (5.35) is replaced with "ϕ_1 and $-\phi_2$ are polyhedral real extensible."

Next, we state and prove the Fenchel-type duality theorem for mixed T_1 convex-concave functions.

Theorem 5.13 (Fenchel type duality for mixed T_1-functions): Let $\phi_1 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$ be a mixed T_1^*-convex function and $\phi_2 : \Im_{\mathbb{Z}\times\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ be a mixed T_1^*-concave function such that either

$$_*\phi_1 \text{ and } -(_*\phi_2) \text{ are polyhedral real extensible, } dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1 \widetilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2 \neq \emptyset, \quad (5.39)$$

or

$$dom_{\Im_U}\phi_1^{\blacklozenge} \cap dom_{\Im_U}\phi_2^{\lozenge} \neq \emptyset \quad (5.40)$$

holds. Then we have

$$\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\phi_1(s) - \phi_2(s)\} = \sup_{q\in\Im_U} \{\phi_2^{\lozenge}(q) - \phi_1^{\blacklozenge}(q)\}. \quad (5.41)$$

If this common value is finite, the supremum is attained by some $q \in dom_{\Im_U}\phi_1^{\blacklozenge} \cap dom_{\Im_U}\phi_2^{\lozenge}$.

Proof: Suppose that (5.39) holds. By the definitions of the discrete and real versions of the Legendre-Fenchel transformations, the inequalities

$$\inf_{k\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\phi_1(s) - \phi_2(s)\} \geq \inf_{s\in\Im_U} \{\overline{\phi_1}(s) - \overline{\phi_2}(s)\} \geq \quad (5.42)$$

$$\sup_{q\in\Im_U} \{\phi_2^{\lozenge}(q) - \phi_1^{\blacklozenge}(q)\} \geq \sup_{q\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\phi_2^{\lozenge}(q) - \phi_1^{\blacklozenge}(q)\} \quad (5.43)$$

hold. By using inequalities (5.42) and (5.43), we can assume that

$$\delta = \inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\phi_1(s) - \phi_2(s)\}$$

is finite. Applying the separation theorem of mixed T_1-functions to $\phi_1 - \delta$ and ϕ_2, there exist $\beta^* \in \mathbb{R}$ and $q^* \in \Im_U$ such that

$$\phi_1(s) - \delta \geq \beta^* + \langle q^*, s\rangle \geq \phi_2(s) \quad (5.44)$$

for all $s \in \Im_{\mathbb{Z}\times\mathbb{R}}$. By (5.44)

$$\phi_1(s) - \phi_2(s) \geq \delta,$$

and taking the infimum of both sides with respect to $s \in \Im_U$, we obtain

$$\inf_{s\in\Im_U} \{\phi_1(s) - \phi_2(s)\} \geq \delta. \quad (5.45)$$

Applying the Fenchel-type duality theorem for real variable convex functions to the left side of the inequality (5.45) we have

$$\sup_{s\in\Im_U} \left\{\phi_2^{\lozenge}(q^*) - \phi_1^{\blacklozenge}(q^*)\right\} \geq \delta. \tag{5.46}$$

(5.42) and (5.43) combined with (5.46) give (5.41) where the supremum is attained at q^*.

Next suppose that $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1 \tilde{\cap} dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2 = \emptyset$ and $dom_{\Im_U}\phi_1^{\blacklozenge} \cap dom_{\Im_U}\phi_2^{\lozenge} \neq \emptyset$. The separation theorem for mixed T_1-convex sets applied to $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_1$ and $dom_{\Im_{\mathbf{Z}\times\mathbf{R}}}\phi_2$ gives a q^* such that (5.38) holds. Plugging in $q = q_0 + cq^*$ in inequality (5.38) and letting $c \longrightarrow \infty$, we obtain

$$\sup_{q\in\Im_U} \left\{\phi_2^{\lozenge}(q) - \phi_1^{\blacklozenge}(q)\right\} = +\infty,$$

whereas

$$\inf_{s\in\Im_{\mathbf{Z}\times\mathbf{R}}} \{\phi_1(s) - \phi_2(s)\} = +\infty.$$

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 CONCLUSION

For (continuous) Euclidean space $\mathbb{R}^n$ there is a single, universally agreed upon notion of convex set and single, universally agreed upon notions convex/concave functions. These notions lead to various forms of separation and Fenchel duality theorems that have significant importance in optimization theory. On the other hand, for functions defined on (or for subsets of) the (discrete) integer space $\mathbb{Z}^n$, there are multiple notions of convexity that are useful in applications, each of which has its corresponding set of separation and Fenchel duality results, which are also of importance in discrete optimization problems. The principle goal of this work has been to introduce various notions of "mixed convexity" (that is, problems with both discrete and continuous variables) for convex sets and functions, with the goal of achieving a synthesis of the continuous and discrete separation and Fenchel duality results. This goal is motivated by the fact that many mixed problems (that is, involving both discrete and continuous variables) arise quite naturally in applications. One such example is the two parameter design problem associated with the $M/E_k/1$ Queueing system stated by Kumin in 1973. Considering mixed convex and concave functions, separation and Fenchel-type duality theorems have not been proposed previously. In this work, by using the similarities in the statements and proofs of separation and Fenchel-type duality theorems of discrete and real convex-concave functions, we have stated and proven the separation and Fenchel-type duality theorems for certain classes of mixed convex-concave functions.

6.2 FUTURE WORK

Convexity definitions are stated in the introduction, and we used some of those discrete convexity definitions to obtain the main results of this work. Improvement in the mixed convexity theory can be achieved by using similar results from both real and integer convexity theories. If separation and Fenchel-type duality theorems exist for other classes of discrete functions, separation and Fenchel-type duality theorems might also be shown for other classes of mixed functions. By using the mixed convexity definition, the convexity of mixed functions that exist in queueing theory might also be shown.

Another future goal is to find refinements of the results presented in this work with weaker and more easily verifiable assumptions.

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