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INVARIANT VECTORS AND LEVEL RAISING OPERATORS IN REPRESENTATIONS OF THE P-ADIC GROUP GL(3)

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Abstract

Each irreducible, admissible representation (π, V) of $\mathrm{GL}(n)$ over a non-archimedean local field F has associated fixed vector spaces $\mathrm{V}^{\mathrm{K}(m)}$, for $\mathrm{K}(m)$ a compact-open subgroup of $\mathrm{GL}(n)$. It is known that there exists some non-negative integer m such that $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 1$ and, if $m' < m$, then $\dim(\mathrm{V}^{\mathrm{K}(m')}) = 0$. Such an m is called the conductor of π which is denoted by $c(\pi)$. If the representation is also generic, an equation is given by Reeder for calculating the dimension of $\mathrm{V}^{\mathrm{K}(m)}$. In this paper, the dimension of $\mathrm{V}^{\mathrm{K}(m)}$ is determined when (π, V) is a non-generic representation of $\mathrm{GL}(3)$.

For $m \geq c(\pi)$, an element of $\mathrm{V}^{\mathrm{K}(m)}$ is considered to have level m. A non-zero element in $\mathrm{V}^{\mathrm{K}(c(\pi))}$ is a local newform, elements of higher level are known as oldforms. Level raising operators are maps from $\mathrm{V}^{\mathrm{K}(m)}$ to $\mathrm{V}^{\mathrm{K}(m+1)}$ that lift an element from one level to the next. In this paper, level raising operators are presented for $\mathrm{V}^{\mathrm{K}(m)}$ associated to representations of $\mathrm{GL}(3)$ and the main theorem proves that, when applied to a local newform, these level raising operators can be used to obtain a set of basis elements for each level.

In the generic case, the proof uses Whittaker functions, zeta integrals, Hecke operators and Satake parameters. For the non-generic case, it is shown that unramified characters of F play a role and the matrix of each level raising operator is used.

1 Introduction

A representation, denoted (π, V), of a group G consists of some complex vector space V and a homomorphism π from G to the group of linear operators on V. Of particular interest, are certain compact-open subgroups, $\mathrm{K}(m)$, of G and the vector spaces, $\mathrm{V}^{\mathrm{K}(m)} \subset \mathrm{V}$, whose elements are invariant under the action of $\mathrm{K}(m)$. Much is known about the dimension of $\mathrm{V}^{\mathrm{K}(m)}$ for certain types of representations and about the relationship between $\dim(\mathrm{V}^{\mathrm{K}(m)})$ and the conductor, $\mathfrak{p}^{c(\pi)}$, of π.

In this thesis, we shall consider both generic and non-generic representations of the group GL(3). We will present level raising operators $\beta_i : \mathrm{V}^{\mathrm{K}(m)} \to \mathrm{V}^{\mathrm{K}(m+1)}$, and will show that these level raising operators provide a surjective map from level m to level $m+1$. Part of the focus will be on representations that arise from induction on a particular parabolic subgroup, P, of GL(3). These are irreducible, admissible, non-generic representations, (π, V), for which we will determine the dimensions of $\mathrm{V}^{\mathrm{K}(m)}$. By combining this information with that which is already known, we can and do determine $\dim(\mathrm{V}^{\mathrm{K}(m)})$ for the remaining non-supercuspidal representations of GL(3).

The first two sections are basically a review of the definitions and concepts we will use to obtain our results. We begin with some basic definitions from general representation theory. We then review important concepts, such as characters, conductors, generic representations, and induction, that will be used throughout this paper. Various known coset decompositions for GL(n) are listed since some will play a role in what we do.

Following the general representation theory, the focus will be on known GL(2) theory that pertains to the new results for GL(3). Many facts about generic and non-generic representations are given, along with information about dimensions and conductors. The zeta integral is defined as are level raising operators. It is shown that all the oldforms can be obtained by applying the level raising operators to a newform and taking linear combinations. This is proven through the use of the zeta integral. Some information about Hecke operators and Satake parameters is given, but it is limited to what is useful for the main results of this thesis. The section on GL(2) concludes with a table of its representations which sums up some pertinent information.

The final section begins with a review of known GL(3) theory that is relevant to our work. It starts with a proof of the double coset decomposition of GL(3) that makes use of our parabolic subgroup P. This is followed by a description of parabolic induction specific to GL(3). A table of the representations of GL(3) which lists their irreducible constituents and specifies which are generic is given. The focus then moves onto the generic representations, specifically on the Whittaker models and zeta integrals.

The main results achieved in this thesis are then presented. Level raising operators are put forward and a proof is given showing that, in the generic case, one can obtain all oldforms from the local newform using said level raising operators. Finally, a non-generic representation, (π, V), is considered. The dimension of $\mathrm{V}^{\mathrm{K}(m)}$ is calculated and the necessary level raising operators are determined. Proof that all oldforms are obtained by applying the level raising operators to the newform and taking linear combinations is then given for any irreducible, admissible representation. This information is then used to present a table of the non-supercuspidal representations of GL(3) which now includes the dimensions of $\mathrm{V}^{\mathrm{K}(m)}$ for all of the representations.

1.1 Notations

Throughout the paper, F will be a non-archimedean, local field with ring of integers $\mathfrak{o}$, which has a maximal ideal $\mathfrak{p}$. Also, ϖ will be a fixed generator of $\mathfrak{p}$ with the property that $|\varpi| = q^{-1}$ for $q = |\mathfrak{o}/\mathfrak{p}|$. Furthermore, ν will be a valuation on F normalized so that $\nu(\varpi) = 1$. We will be working with $\mathrm{G} = \mathrm{GL}(n, \mathrm{F})$, the group of invertible $n \times n$ matrices with entries in F. The standard Borel subgroup of G consisting of upper triangular matrices will be denoted B. We set $\mathrm{K} := \mathrm{K}(0) := \mathrm{GL}(n, \mathfrak{o})$, a maximal compact-open subgroup of G. Finally, for integers $m > 0$, $\mathrm{K}(m)$ will denote the subgroup of K which consists of all matrices having the form

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \tag{1}$$

for $A \in \mathrm{GL}(n-1, \mathfrak{o})$, with c a $1 \times (n-1)$ row matrix with entries in $\mathfrak{p}^m$. For an element of F to be in $\mathfrak{p}^m$, simply means that said element is divisible by ϖ^m. Thus, b is an $(n-1) \times 1$ column matrix with entries in $\mathfrak{o}$, placing $d \in \mathfrak{o}^\times$.

2 Some basics

We will be interested in irreducible, admissible representations with trivial central character. Thus, some definitions are in order. There are concepts from representation theory that will be needed. Therefore, a review of these concepts is included in this section. Throughout this section, let (π, V) be a representation of G on the complex vector space V.

2.1 Definitions

Definition. The representation π is said to be *smooth* if

$$\mathrm{Stab}_{\mathrm{G}}(v) := \{g \in \mathrm{G} \mid \pi(g)v = v\}$$

is open for every $v \in \mathrm{V}$.

In fact, if we define

$$\mathrm{V}^{\tilde{\mathrm{K}}} := \{v \in \mathrm{V} \mid \pi(k)v = v,\ \forall\ k \in \tilde{\mathrm{K}}\}, \tag{2}$$

for $\tilde{\mathrm{K}}$ a compact-open subgroup of G, then π is smooth if and only if each $v \in \mathrm{V}$ is in some such $\mathrm{V}^{\tilde{\mathrm{K}}}$. In other words, we have

$$\mathrm{V} = \bigcup_{\tilde{\mathrm{K}}} \mathrm{V}^{\tilde{\mathrm{K}}},$$

where $\tilde{\mathrm{K}}$ ranges over all of the compact-open subgroups of G.

Definition. If a smooth representation has a subspace W such that $\pi(g)w \in \mathrm{W}$ for all $w \in \mathrm{W}$ and $g \in \mathrm{G}$, such a subspace is said to be *invariant*. A smooth representation is *irreducible* if its only invariant subspaces are $\{0\}$ and the space V itself.

Definition. A smooth representation is *admissible* if the space $\mathrm{V}^{\tilde{\mathrm{K}}}$ is finite dimensional for every compact-open subgroup $\tilde{\mathrm{K}}$ of G.

In fact, for such $\tilde{\mathrm{K}}$, a smooth representation of G is admissible if and only if every irreducible representation of $\tilde{\mathrm{K}}$ occurs a finite number of times in V.

Definition. Denote the center of G by Z and let $\mathbf{1}_{\mathrm{V}}$ be the identity on V. For a smooth, irreducible representation π of G, the smooth, one-dimensional representation ω_π of Z such that

$$\pi(z) = \omega_\pi(z)\mathbf{1}_{\mathrm{V}},$$

for $z \in \mathrm{Z}$, is the *central character* of π. For π to have *trivial central character*, means that $\omega_\pi(z) = z$ for every $z \in \mathrm{Z}$. Hence,

$$\pi(\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix})v = v, \tag{3}$$

for all $a \in \mathrm{F}^\times$ and all $v \in \mathrm{V}$.

2.2 Characters and conductors

A *character* of G is a continuous homomorphism $\mathrm{G} \to \mathbb{C}^\times$. Note that a character is a representation $(\chi, \mathbb{C})$ which is smooth. In fact, a one-dimensional representation is smooth if and only if it is equivalent to a representation that is defined by a character. In other words, a character χ of $\mathrm{F}^\times$ results in a character of G of the form $g \mapsto \chi(\det(g))$, and this gives all the characters of G. In fact, any finite dimensional, smooth, irreducible representation of G is one-dimensional and thus a character of G. If π is any representation of G, then π can be *twisted* by a character χ. The twist is denoted by $\pi \otimes \chi$ with $(\pi \otimes \chi)(g) = \chi(\det(g))\pi(g)$ and has the same representation space as π.

If χ is a nontrivial character of G, the *level* of χ is either zero, in which case $\chi|_{\mathfrak{o}^\times} = \{1\}$, or the smallest positive integer m such that $\chi|_{1+\mathfrak{p}^m} = \{1\}$. In general, for an irreducible, admissible, generic representation (π, V) of G, the level is associated with an ideal $\mathfrak{p}^{c(\pi)}$, where $c(\pi)$ is known as the *conductor* of π. One may also refer to the ideal itself as the conductor. In fact, the conductor for said representation is the smallest integer m such that $\mathrm{V}^{\mathrm{K}(m)} \neq \{0\}$ and for such m it is known that $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 1$, i.e, $\dim(\mathrm{V}^{\mathrm{K}(c(\pi))}) = 1$, [JPSS81]. We say the character is *unramified* if the conductor is zero, equivalently, if the conductor is $\mathfrak{o}^{\times}$. Otherwise, we say the character is *ramified*. If we are dealing with an additive character, the level is zero if the character is trivial on $\mathfrak{o}^{\times}$, otherwise the level is the integer $m > 0$ such that the character is trivial on $\mathfrak{p}^{m}$ but is not trivial on $\mathfrak{p}^{m-1}$, with the definition of conductor as above.

If χ is an unramified character of $\mathrm{F}^{\times}$, then χ is determined by its value on a prime element of $\mathrm{F}^{\times}$. For example, χ is determined by its value at ϖ. This value can be any non-zero complex number, i.e, $\chi(\varpi) = \alpha \in \mathbb{C}^{\times}$. Such an α is called a *Satake parameter* of χ.

There is a special character that occurs called the *modulus character*. It arises as a result of certain properties of the Haar measures of locally compact, topological groups. If P is such a group, then it has a left Haar measure, dx, which is unique up to scalars. Thus, for any $y \in \mathrm{P}$, one has $dx(xy^{-1})$ is also a left Haar measure. Hence, there exists a $\delta_{\mathrm{P}}(y) \in \mathbb{R}^{\times}_{>0}$ such that $dx(xy^{-1}) = \delta_{\mathrm{P}}(y)dx(x)$. The resulting map

$$\delta_{\mathrm{P}} : \mathrm{P} \to \mathbb{R}^{\times}_{>0}$$

is called the modulus character of P. For more details about the Haar measure, see [Mur].

2.3 Generic representations

We will consider both generic and non-generic representations. To define a *generic representation* of G, fix a non-trivial additive character $\psi : \mathrm{F} \to \mathbb{C}^{\times}$. Let N be the subgroup of unipotent, upper triangular matrices in G. Now define a one-dimensional representation θ_ψ of N by

$$\theta_\psi : \mathrm{N} \to \mathbb{C}^\times \; : \; ((u_{ij})) \mapsto \psi(u_{12} + \cdots + u_{n-1,n}).$$

For π any representation of G, consider $\mathrm{Hom}_{\mathrm{N}}(\pi|_{\mathrm{N}}, \theta_\psi)$. If π is smooth and irreducible with $\mathrm{Hom}_{\mathrm{N}}(\pi|_{\mathrm{N}}, \theta_\psi) \neq 0$, then we say π is generic. The property of π being generic does not depend on the choice of ψ.

For a generic representation (π, V) of G, which has conductor $c(\pi)$, from [Ree91] theorem 1, it is known that

$$\dim(\mathrm{V}^{\mathrm{K}(m+c(\pi))}) = \binom{m+n-1}{m}, \tag{4}$$

for $m \in \mathbb{Z}_{\geq 0}$. Equivalently, it can be written as

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = \binom{m - c(\pi) + n - 1}{m - c(\pi)}, \tag{5}$$

for all integers $m \geq c(\pi)$.

Whittaker model

Let $\lambda \in \mathrm{Hom}_{\mathrm{N}}(\pi|_{\mathrm{N}}, \theta_\psi)$ be given. In particular, this implies that λ commutes with the action of N. Thus, $\lambda(\pi(u)v) = \theta_\psi(u)\lambda(v)$ for all $u \in \mathrm{N}$ and all $v \in \mathrm{V}$. Now, for $v \in \mathrm{V}$, define a function $W_v : \mathrm{G} \to \mathbb{C}$ by $W_v(g) = \lambda(\pi(g)v)$. Thus, we have

$$\begin{aligned} W_v(ug) &= \lambda(\pi(ug)v) = \lambda(\pi(u)\pi(g)v) \\ &= \theta_\psi(u)\lambda(\pi(g)v) = \theta_\psi(u)W_v(g), \end{aligned}$$

for all $u \in \mathrm{N}$ and for all $g \in \mathrm{G}$. Such a λ is called a Whittaker functional.

Set

$$\mathcal{W}(\pi, \psi) := \{W_v \mid v \in \mathrm{V}\} \subset \{W : \mathrm{G} \to \mathbb{C} \mid W(ug) = \theta_\psi(u)W(g)\},$$

where the action of G is by right translation, i.e., $(\pi(h)W_v)(g) = W_v(gh)$ for all h and $g \in \mathrm{G}$. We call $\mathcal{W}(\pi, \psi)$ the Whittaker model of π with respect to ψ. Note that $\mathrm{V} \simeq \mathcal{W}$ via the map $v \mapsto W_v$.

Conversely, suppose that the Whittaker model $\mathcal{W}(\pi, \psi)$ exists. Define a map $\lambda : \mathcal{W}(\pi, \psi) \to \mathbb{C} : W \mapsto W(1)$. Thus, we have

$$\lambda(\pi(u)W) = (\pi(u)W)(1) = W(1u) = \theta_\psi(u)W(1) = \theta_\psi(u)\lambda(W),$$

implying that $\lambda \in \mathrm{Hom}_N(\pi|_N, \theta_\psi)$.

In conclusion, we see that $\mathcal{W}(\pi, \psi) \neq \{0\}$ if and only if $\mathrm{Hom}_{\mathrm{N}}(\pi|_{\mathrm{N}}, \theta_\psi) \neq \{0\}$. Therefore, a representation has a non-zero Whittaker model if and only if the representation is generic.

2.4 Coset decompositions of $\mathrm{GL}(n, \mathrm{F})$

There are many decompositions of G. These include the Bruhat decomposition which states

$$\mathrm{G} = \bigsqcup_{w \in \mathrm{W}} \mathrm{B}w\mathrm{B},$$

where W is the Weyl group of G; the Cartan decomposition which states

$$\mathrm{G} = \bigsqcup_{a \in \mathrm{A}} \mathrm{K}a\mathrm{K},$$

for $\mathrm{A} = \{\mathrm{diag}(\varpi^{m_1}, \ldots, \varpi^{m_n}) \mid m_i \in \mathbb{Z},\ m_1 \leq m_2 \leq \ldots \leq m_n\}$; and the Iwasawa decomposition which states that $\mathrm{G} = \mathrm{BK}$. Of these, we will make use of the Iwasawa decomposition in particular. To see proofs of these decompositions, see [PR08]. We will also make use of the decomposition

$$\mathrm{G} = \bigsqcup_{\substack{m \geq m_i \geq m_j \\ i > j}} \mathrm{B}\gamma_{m_1,\ldots,m_{n-1}}\mathrm{K}(m), \tag{6}$$

for $m, m_i,$ and $m_j \in \mathbb{Z}_{\geq 0}$, with i and $j \in \{1, \ldots, n\}$, found in [Ree91], where

$$\gamma_{m_1,\ldots,m_{n-1}} := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \varpi^{m_1} & \varpi^{m_2} & \ldots & \varpi^{m_{n-1}} & 1 \end{pmatrix}. \tag{7}$$

2.5 Induced representations

The focus of this thesis will be on *induced representations*. Let (σ, W) be a representation of H, a closed subgroup of G. Let X be the space of functions $f : \mathrm{G} \to \mathrm{W}$ such that

$$f(hg) = \sigma(h)f(g)$$

for all $h \in \mathrm{H}$ and all $g \in \mathrm{G}$, and for which there exists some compact-open subgroup $\tilde{K}$ of G on which f is right invariant. Thus, also

$$f(gk) = f(g)$$

for all $g \in \mathrm{G}$ and all $k \in \tilde{\mathrm{K}}$. Define a homomorphism $\Sigma : \mathrm{G} \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{X})$ by

$$\Sigma(g)f : x \mapsto f(xg),$$

for all g and x in G. Then (Σ, X) is a representation of G called the representation of G induced by σ. This representation is denoted by $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$.

For example, if χ is a character of B, then $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$ is the set of locally constant functions, also known as smooth functions, $f : \mathrm{G} \to \mathbb{C}$ with the property that

$$f(bg) = \chi(b)\delta(b)^{\frac{1}{2}} f(g),$$

for all $b \in \mathrm{B}$, all $g \in \mathrm{G}$, and for δ the modulus character of B. Let a_i, $i = 1, \ldots, n$, be the elements on the diagonal of b, then $\delta(b) = |\, a_1\,|^{n-1} |\, a_2\,|^{n-3} \ldots |\, a_n\,|^{-(n-1)}$ in this case.

The conductor of an induced representation is related to the conductors of the characters used in the induction. For example, in $\mathrm{GL}(2, \mathrm{F})$ if $\chi = \chi_1 \times \chi_2$ then $c(\chi) = c(\chi_1) + c(\chi_2)$, [Sch02].

Parabolic induction

We will make use of a special form of induction called *parabolic induction*. This type of induction is induction from a parabolic subgroup of G up to G. A *standard parabolic subgroup*, P, is made up of upper triangular block matrices, i.e, matrices of the form

$$\begin{pmatrix} A_{n_1} & * & * & * \\ & A_{n_2} & * & * \\ & & \ddots & \vdots \\ & & & A_{n_r} \end{pmatrix}$$

such that $\sum_{i=1}^{r} n_i = n$, where A_{n_i} is a $n_i \times n_i$ matrix in $\mathrm{GL}(n_i, \mathrm{F})$, and each $*$ represents an appropriate sized matrix with entries in F. A *parabolic subgroup* is a subgroup of G which is conjugate to one of the standard parabolic subgroups.

For parabolic induction, define M to be the block diagonal subgroup which consists of matrices of the form

$$\begin{pmatrix} A_{n_1} & & & \\ & A_{n_2} & & \\ & & \ddots & \\ & & & A_{n_r} \end{pmatrix},$$

for A_{n_i} as above. This is known as the *Levi subgroup* of P. Let (ρ, W) be a smooth representation of M. We can inflate ρ to a representation of P, also called ρ. The parabolically induced representation, $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\rho)$, is then the set of all locally constant functions $f : \mathrm{G} \to \mathbb{C}$ that have the following transformation property

$$f(pg) = \delta_{\mathrm{P}}(p)^{\frac{1}{2}} \rho(p) f(g), \tag{8}$$

for $p \in \mathrm{P}$, $g \in \mathrm{G}$, and where δ_{P} is the modulus character associated with P. The space of the parabolically induced representation $\pi = \pi(\chi_1, \ldots, \chi_r)$, where each χ_i, $i = 1, \ldots, r$, is a character of $\mathrm{GL}(n_i, \mathrm{F})$, is denoted $\mathrm{V}(\chi_1, \ldots, \chi_r)$.

Principal series representation

In the special case where $\mathrm{P} = \mathrm{B}$ and χ_i is a character of $\mathrm{F}^\times$, for $i = 1, 2, \ldots, n$, so that

$$\chi(\begin{pmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & \ddots & \vdots \\ & & & a_n \end{pmatrix}) = \chi_1(a_1)\chi_2(a_2)\ldots\chi_n(a_n), \tag{9}$$

the representation $\pi(\chi)$ obtained by parabolic induction is a *principal series representation* of G. Such a $\pi(\chi)$ is irreducible if and only if $\chi_i \neq \chi_j |\cdot|$, for every $i \neq j$, where $|\cdot|$ is the normalized absolute value on F, [Kud94].

Spherical representations

There are principal series representations called *spherical representations*, or *unramified representations*, which arise from parabolic induction with unramified characters and thus have conductor 0. Let (π, V) be a representation induced from B as above and consider a function $f \in \mathrm{V}^{\mathrm{K}}$. For $g \in \mathrm{G}$, use the Iwasawa decomposition to write $g = bk$ for some $b \in \mathrm{B}$ and some $k \in \mathrm{K}$. Then

$$f(g) = f(bk) = f(b) = \pi(b)\delta(b)^{\frac{1}{2}} f(1) \in \mathbb{C},$$

for δ the modulus character of B. If $f(1) = 0$, then $f(g) = 0$, for all $g \in \mathrm{G}$, implying that $f \equiv 0$. Therefore, the function $\mathrm{V}^{\mathrm{K}} \to \mathbb{C} : f \mapsto f(1)$ is an injective map. Thus, $\dim(\mathrm{V}^{\mathrm{K}}) \leq 1$. If $\dim(\mathrm{V}^{\mathrm{K}}) = 1$, we have a spherical representation. Note that, in this case, since each χ_i is a character of $\mathrm{F}^\times$, for $i = 1, \ldots, n$, in order to have a non-zero $f \in \mathrm{V}^{\mathrm{K}}$, each χ_i must be unramified. This is because the Iwasawa decomposition is NOT unique. But if $bk = b'k'$, then $b^{-1}b' \in \mathrm{B} \cap \mathrm{K}$, which means $b = b'\bar{b}$, for $\bar{b} \in \mathrm{B}(\mathfrak{o})$. In particular, all elements, b_{ii}, on the diagonal of $\bar{b}$ are in $\mathfrak{o}^\times$ and therefore, each b_{ii} is reduced to one by χ_i if and only if each χ_i is unramified. Conversely, if each χ_i is unramified, then there exists a unique, well-defined K-invariant vector f_o in V^{K} such that $f_o(1) = 1$. Such an f_o is called the *spherical vector*. Thus, we have a spherical representation and a corresponding spherical vector if and only if each χ_i is unramified.

Steinberg representation

The *Steinberg representation* is obtained as a result of parabolic induction on the Borel subgroup. Use the characters $\chi_1 = \nu^{\frac{1-n}{2}}, \chi_2 = \nu^{\frac{1-n}{2}+1}, \cdots, \chi_n = \nu^{\frac{n-1}{2}}$ of $\mathrm{F}^\times$ to obtain a character χ of B defined as per equation (9) and induce up to a representation of G. Note that this representation is reducible. The unique irreducible quotient of $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$ is called the Steinberg representation, which is denoted by $\mathrm{St}_{\mathrm{GL}(n)}$. It is square integrable and has trivial central character.

Oldforms and newforms

If an irreducible, admissible representation (π, V) arises from induction with some of the characters ramified, then $\dim(\mathrm{V}^{\mathrm{K}}) = 0$. But, as stated above, it is known that there exists some $\mathrm{V}^{\mathrm{K}(m)}$ such that $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 1$. In particular, the conductor of π is $\mathfrak{p}^m$, i.e., $c(\pi) = m$, and there exists a distinguished vector $v \in \mathrm{V}^{\mathrm{K}(c(\pi))}$, unique up to multiples, which spans this one-dimensional space. For any irreducible, admissible representation (π, V) that has invariant vectors, any non-zero vector in $\mathrm{V}^{\mathrm{K}(c(\pi))}$ is called a *local newform*. The spherical vector is a special case of a local newform. It is also known, for integers $m > c(\pi)$, that $\dim(V^{\mathrm{K}(m)}) > 1$. Elements of $\mathrm{V}^{\mathrm{K}(m)}$, $m > c(\pi)$, are known as *oldforms*.

2.6 Supercuspidal representations

An irreducible, admissible representation that does NOT occur as a subquotient of a representation arising from parabolic induction from a proper parabolic subgroup is called a *supercuspidal representation*. Such supercuspidal representations are the building blocks of all irreducible representations. In fact, if π is an irreducible, admissible representation of G, then there exists some parabolic subgroup P and a supercuspidal representation ρ of $\mathrm{M} \subset \mathrm{P}$, for M the Levi subgroup of P, so that π is a subrepresentation of $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\rho)$. All supercuspidal representations are generic.

3 $\mathrm{GL}(2,\mathrm{F})$ theory

It is known that an irreducible, admissible, representation of $\mathrm{G} := \mathrm{GL}(2,\mathrm{F})$ with trivial central character is either infinite-dimensional, and thus generic, or one-dimensional, and thus non-generic. Furthermore, we know the principal series representation $\pi(\chi_1, \chi_2)$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$. Note that $\pi(\chi_1, \chi_2) \cong \pi(\chi_2, \chi_1)$. The representation $\pi(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$ has two constituents; the trivial representation is the unique quotient and the Steinberg representation, $\mathrm{St}_{\mathrm{GL}(2)}$, is the unique subrepresentation. In general, for χ a character of $\mathrm{F}^{\times}$, the two constituents of the representation $\pi(\chi|\cdot|^{\frac{1}{2}}, \chi|\cdot|^{-\frac{1}{2}})$ are the one-dimensional representation $\chi \circ \det$, which is the quotient, and $\chi\mathrm{St}_{\mathrm{GL}(2)}$, the unique subrepresentation which is the twist of $\mathrm{St}_{\mathrm{GL}(2)}$. These twists of the Steinberg are called *special representations*.

3.1 Parabolic induction

There is only one form of parabolic induction in this case as B, the standard Borel subgroup of G, is the only proper standard parabolic subgroup of G. Start with a representation (π, V) of B which is induced to a representation of G referred to as the standard induced representation. Let χ_1 and χ_2 be characters of $\mathrm{F}^{\times}$, then

$$\chi := \chi_1 \times \chi_2 : \mathrm{B} \to \mathbb{C}^{\times} : \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$$

is a character of B, and $\mathrm{V}(\chi_1 \times \chi_2) := \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$ is the set of all locally constant functions $\phi : \mathrm{G} \to \mathbb{C}$ such that

$$\phi(\begin{pmatrix} a & b \\ & d \end{pmatrix} g) = \chi_1(a)\chi_2(d) \left|\frac{a}{d}\right|^{\frac{1}{2}} \phi(g),$$

for all

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} \in \mathrm{B}$$

and all $g \in \mathrm{G}$. In this case, the modulus character δ of B is $\left|\frac{a}{d}\right|$.

3.2 Spherical vector of the Whittaker model

Suppose (π, V) is a spherical representation with a Whittaker model $\mathcal{W}$. Let W_o be the spherical vector in $\mathcal{W}$. Because W_o is K-invariant, we need only consider $\mathrm{W}_o|_{\mathrm{B}}$. Note that here $\mathrm{K} := \mathrm{GL}(2, \mathfrak{o})$ and we are using the Iwasawa decomposition $\mathrm{G} = \mathrm{BK}$ to conclude $\mathrm{W}_o(\tilde{g}) = \mathrm{W}_o(\tilde{b}\tilde{k}) = \mathrm{W}_o(\tilde{b})$, for $\tilde{g} \in \mathrm{G}$, some $\tilde{b} \in \mathrm{B}$, and some $\tilde{k} \in \mathrm{K}$. But, for $\tilde{b} \in \mathrm{B}$,

$$\tilde{b} = \begin{pmatrix} a & b \\ & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & d \end{pmatrix}.$$

Thus, because of the transformation property of W_o, only diagonal matrices need to be considered. However, any diagonal matrix may be written

$$\begin{pmatrix} a & \\ & d \end{pmatrix} = \begin{pmatrix} ad^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} d & \\ & d \end{pmatrix},$$

and, as we are assuming that π has trivial central character, we have

$$\begin{aligned} \mathrm{W}_o(\begin{pmatrix} ad^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} d & \\ & d \end{pmatrix}) &= \pi(\begin{pmatrix} d & \\ & d \end{pmatrix})\mathrm{W}_o(\begin{pmatrix} ad^{-1} & \\ & 1 \end{pmatrix}) \\ &= \mathrm{W}_o(\begin{pmatrix} ad^{-1} & \\ & 1 \end{pmatrix}). \end{aligned}$$

Hence, the diagonal matrix may be ignored.

Now, replacing ad^{-1} with just a, our matrix may be written as

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix} = \begin{pmatrix} u\varpi^r & \\ & 1 \end{pmatrix} = \begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix} \begin{pmatrix} u & \\ & 1 \end{pmatrix},$$

for some $u \in \mathfrak{o}^\times$ and some $r \in \mathbb{Z}$. Again, using the fact that W_o is K-invariant and that the final matrix is an element of K, it can be ignored. Let's take a closer look at what we have left. Using K-invariance once more, let $r < 0$ and

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in \mathrm{K}.$$

This gives us

$$\begin{aligned} \mathrm{W}_o(\begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix}) &= \mathrm{W}_o(\begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \\ &= \mathrm{W}_o(\begin{pmatrix} 1 & x\varpi^r \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix}) \\ &= \psi(x\varpi^r)\mathrm{W}_o(\begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix}). \end{aligned}$$

Since x can be any element of $\mathfrak{o}$ and we are assuming $r < 0$, we know there exists an x such that $x\varpi^r \in \mathfrak{p}^{-1}$. Hence, we can find x such that $\psi(x\varpi^r) \neq 1$. Thus, we conclude that

$$\mathrm{W}_o(\begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix}) = 0$$

for all $r < 0$. Therefore, W_o is determined on

$$\{\begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix} \mid r \geq 0\}.$$

3.3 Zeta integrals

There exist zeta integrals for generic representations. In $\mathrm{GL}(2,\mathrm{F})$, for W in the Whittaker model, the zeta integral is defined as

$$\mathrm{Z}(s,\mathrm{W}) := \int\limits_{\mathrm{F}^\times} \mathrm{W}(\begin{pmatrix} a & \\ & 1 \end{pmatrix}) |\, a\,|^{s-\frac{1}{2}}\, d^\times a \tag{10}$$

for $s \in \mathbb{C}$. Here $d^\times a$ is a Haar measure on G. It is well known that the zeta integral converges for $\mathrm{Re}(s)$ large enough, and that it is not zero for $\mathrm{W} \not\equiv 0$.

The main use of the zeta integral for our purposes will be to aid in showing linear independence of certain elements that arise from repeated application of level raising operators to a local newform.

3.4 Level raising operators

Suppose (π, V) is an irreducible, admissible representation of G and that $m = c(\pi)$ is the smallest non-negative integer such that $\mathrm{V}^{\mathrm{K}(m)} \neq 0$. Then $\dim(\mathrm{V}^{\mathrm{K}(c(\pi))}) = 1$ and the conductor of π is $\mathfrak{p}^{c(\pi)}$, or just $c(\pi)$. Let v_o be the local newform. It is known that for all integers $m \geq c(\pi)$ we have $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 1 + m - c(\pi)$, [Cas73]. Since the dimension increases by one with each increase in level, two level raising operators are needed to obtain a basis of oldforms for $\mathrm{V}^{\mathrm{K}(m)}$ from the newform. They are

$$\beta_o = id, \text{ and } \beta_1 = \pi(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}).$$

First we need to show that the level raising operators are maps from $\mathrm{V}^{\mathrm{K}(m)}$ to $\mathrm{V}^{\mathrm{K}(m+1)}$. Then, since the basis is obtained through repeated applications of the level raising operators we need to show that, for $m > c(\pi)$, if

$$\sum_{\substack{i+j=m-c(\pi)\\ i,j\in\mathbb{Z}_{\geq 0}}} a_{i,j}(\beta_o^i\beta_1^j)(v_o) = 0, \tag{11}$$

then $a_{i,j} = 0$ for every i, j.

If we start with v_o, then

$$\beta_o v_o = v_o \in \mathrm{V}^{\mathrm{K}(c(\pi)+1)},$$

because $\mathrm{K}(c(\pi)+1) \subset \mathrm{K}(c(\pi))$. Note that this puts $v_o \in \mathrm{V}^{\mathrm{K}(m)}$ for every $m \geq c(\pi)$. Now we need to show that $\beta_1 v_o \in \mathrm{V}^{\mathrm{K}(c(\pi)+1)}$. Let

$$k = \begin{pmatrix} a & b \\ x & u \end{pmatrix} \in \mathrm{K}(c(\pi)+1).$$

Then

$$\begin{aligned}(\pi(k))(\beta_1 v_o) &= \pi(\begin{pmatrix} a & b \\ x & u \end{pmatrix}\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})v_o \\ &= \pi(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}\begin{pmatrix} a & b\varpi \\ x\varpi^{-1} & u \end{pmatrix})v_o \\ &= \pi(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})v_o = \beta_1 v_o,\end{aligned}$$

since

$$\begin{pmatrix} a & b\varpi \\ x\varpi^{-1} & u \end{pmatrix} \in \mathrm{K}(c(\pi)),$$

and v_o is $\mathrm{K}(c(\pi))$-invariant.

Performing the same calculations by replacing $c(\pi)$ with some $m > c(\pi)$ and v_o

with some $v_i \in \mathrm{V}^{\mathrm{K}(m)}$ shows that $\beta_o v_i$ and $\beta_1 v_i$ are indeed in $\mathrm{V}^{\mathrm{K}(m+1)}$.

To show that with repeated applications of the level raising operators we can obtain a basis for any $\mathrm{V}^{\mathrm{K}(m)}$ when $m > c(\pi)$, and in so doing proving that these two level raising operators are all that are needed to obtain all the oldforms from a newform, we will make use of the zeta integral. First, we need to look at how the zeta integral behaves with respect to the level raising operators. Clearly, for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough for convergence of the integral, and $W \in \mathcal{W}$, we have

$$\mathrm{Z}(s, \beta_o \mathrm{W}) = \mathrm{Z}(s, \mathrm{W}).$$

Applying the zeta integral to $\beta_1 \mathrm{W}$ gives us

$$\begin{aligned}
\mathrm{Z}(s, \beta_1 \mathrm{W}) &= \int\limits_{\mathrm{F}^\times} \pi(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})\mathrm{W}(\begin{pmatrix} a & \\ & 1 \end{pmatrix})| \, a \, |^{s-\frac{1}{2}} \, d^\times a \\
&= \int\limits_{\mathrm{F}^\times} \mathrm{W}(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})| \, a \, |^{s-\frac{1}{2}} \, d^\times a \\
&= \int\limits_{\mathrm{F}^\times} \mathrm{W}(\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} \begin{pmatrix} a\varpi^{-1} & \\ & 1 \end{pmatrix})| \, a \, |^{s-\frac{1}{2}} \, d^\times a.
\end{aligned}$$

Use the fact that π has trivial central character and make use of the Haar measure that allows us to send $a \rightarrow a\varpi$ to get

$$\begin{aligned}
\mathrm{Z}(s, \beta_1 \mathrm{W}) &= \int\limits_{\mathrm{F}^\times} \mathrm{W}(\begin{pmatrix} a & \\ & 1 \end{pmatrix})| \, a\varpi \, |^{s-\frac{1}{2}} \, d^\times a \\
&= q^{\frac{1}{2}-s}\mathrm{Z}(s, \mathrm{W}).
\end{aligned}$$

Now we use the isomorphism from V to its Whittaker model to map equation (11) to

$$\sum_{\substack{i+j=m-c(\pi)\\ i,j\in\mathbb{Z}_{\geq 0}}} a_{i,j}(\beta_o^i\beta_1^j)(W_o)=0. \tag{12}$$

In particular, note that W_o is a newform in the Whittaker model and is therefore non-zero. Next we use the relationship between the zeta integral and β_o and β_1, choosing s such that $\mathrm{Re}(s)$ is large enough to ensure that $\mathrm{Z}(s, \mathrm{W}_o)$ converges, to get

$$\mathrm{Z}(s, \sum_{\substack{i+j=m-c(\pi)\\ i,j\in\mathbb{Z}_{\geq 0}}} a_{i,j}(\beta_o^i\beta_1^j)(W_o)) = \sum_{\substack{i+j=m-c(\pi)\\ i,j\in\mathbb{Z}_{\geq 0}}} a_{i,j}q^{j(\frac{1}{2}-s)}(\mathrm{Z}(s, W_o)) = 0.$$

Because W_o is not zero, we know $\mathrm{Z}(s, \mathrm{W}_o) \neq 0$, thus,

$$\sum_{\substack{i+j=m-c(\pi)\\ i,j\in\mathbb{Z}_{\geq 0}}} a_{i,j}q^{j(\frac{1}{2}-s)} = 0,$$

which can be viewed as a polynomial in $q^{\frac{1}{2}-s}$. As such, we conclude that $a_{i,j} = 0$ for every i, j.

3.5 Hecke operators

Now let (π, V) be an irreducible, admissible representation of G and let θ be the characteristic function of

$$\mathrm{K}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathrm{K}.$$

Define

$$\mathrm{T}v := \int_{\mathrm{G}} \theta(g)\pi(g)v\,dg,$$

for all $v \in \mathrm{V}$, where dg is a Haar measure on G.

Note that

$$\mathrm{T}v = \int\limits_{\mathrm{K}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\mathrm{K}} \theta(g)\pi(g)v\,dg$$

$$= \int\limits_{\mathrm{K}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\mathrm{K}} \pi(g)v\,dg,$$

because

$$\theta(g) := \begin{cases} 1 & \text{if } g \in \mathrm{K}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathrm{K}; \\ 0 & \text{if } g \notin \mathrm{K}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathrm{K}. \end{cases}$$

Let $v \in \mathrm{V}^{\mathrm{K}}$ and $k \in \mathrm{K}$. Note that

$$k^{-1}g \in \mathrm{K}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathrm{K},$$

if and only if

$$g \in \mathrm{K}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathrm{K}.$$

Therefore, by virtue of the Haar measure,

$$\pi(k)\mathrm{T}v = \int\limits_{\mathrm{K}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\mathrm{K}} \pi(k)\pi(g)v\,dg$$

$$= \int\limits_{\mathrm{K}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\mathrm{K}} \pi(kg)v\,dg, \qquad \text{let } g \mapsto k^{-1}g$$

$$= \int\limits_{\mathrm{K}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\mathrm{K}} \pi(g)v\,dg = \mathrm{T}v.$$

Thus, Tv is K-invariant. Such a T is known as a *Hecke operator*. It acts on the space of K-invariant vectors, which is known to be one dimensional, via $v \mapsto \mathrm{T}v$. Therefore, we know $\mathrm{T}v = \lambda v$ for some eigenvalue λ. For more on the subject of Hecke operators, see [Bum97].

It will be helpful to have a more explicit formula for such a Hecke operator. In order to find this formula, we begin by making use of a known decomposition of K.

It is easy enough to show that

$$\mathrm{K} = \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix} \sqcup \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}. \tag{13}$$

This implies that

$$\mathrm{K}(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})\mathrm{K} = \bigcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} \mathrm{K}$$
$$\cup \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} \mathrm{K}.$$

In fact, this is a disjoint union. With a little work, one ends up with

$$\mathrm{K}(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})\mathrm{K} = \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} \mathrm{K} \sqcup \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} \mathrm{K}. \tag{14}$$

In particular, one can think of it as

$$\mathrm{K}(\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})\mathrm{K} = \bigsqcup_i \gamma_i \mathrm{K},$$

where the union is a finite one.

Normalizing the Haar measure so that

$$\int_{\mathrm{K}} dg = 1,$$

with v as above, gives

$$\begin{aligned}
\mathrm{T}v &= \int_{\bigsqcup_i \gamma_i \mathrm{K}} \pi(g)v\,dg = \sum_i \int_{\gamma_i \mathrm{K}} \pi(g)v\,dg \\
&= \sum_i \int_{\mathrm{K}} \pi(\gamma_i g)v\,dg = \sum_i \int_{\mathrm{K}} \pi(\gamma_i)v\,dg \\
&= \sum_i (\pi(\gamma_i)v) \int_{\mathrm{K}} dg = \sum_i \pi(\gamma_i)v \\
&= \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})v + \pi(\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})v.
\end{aligned}$$

In particular, this implies

$$\sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix})v + \pi(\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})v = \lambda v. \tag{15}$$

3.6 Satake parameters

Let χ_1 and χ_2 be unramified characters of $\mathrm{F}^\times$. The character $\chi_1 \times \chi_2$ of G, irreducible or not, has exactly one spherical representation, call it $\pi(\chi_1(\varpi), \chi_2(\varpi))$, where $\chi_i(\varpi)$ is a Satake parameter. The following well known theorem gives a connection between the Satake parameters and spherical representations.

Theorem 3.1. *Given any spherical representation π of* G*, there exist α_1 and α_2 in $\mathbb{C}^\times$ such that $\pi \simeq \pi(\alpha_1, \alpha_2)$.*

Table 1: Representations of $\mathrm{GL}(2,\mathrm{F})$

	constituent of	representation	generic	$c(\pi)$	
I	$\chi_1 \times \chi_2$ $\chi_1\chi_2^{-1} \neq \lvert\,\rvert^{\pm 1}$	$\chi_1 \times \chi_2$	•	$c(\chi_1) + c(\chi_2)$	
IIa	$\chi\nu^{\frac{1}{2}} \times \chi\nu^{-\frac{1}{2}}$	$\chi\mathrm{St}_{\mathrm{GL}(2)}$	•	1	$c(\chi) = 0$
				$2c(\chi)$	$c(\chi) > 0$
IIb		$\chi\mathbf{1}_{\mathrm{GL}(2)}$		0	$c(\chi) = 0$
				$2c(\chi)$	$c(\chi) > 0$
III	supercuspidals		•	≥ 2	

Note that in the case where the representation also has trivial central character, we would have $\chi_2 = \chi_1^{-1}$. Thus, the corresponding Satake parameters would be (α, α^{-1}).

In fact, there is an isomorphism between the set of Satake parameters of spherical representations which have trivial central character, $(\alpha, \alpha^{-1})/\sim$, for $\sim$ the equivalence relation $(\alpha, \alpha^{-1}) \sim (\alpha^{-1}, \alpha)$, and the set of Hecke eigenvalues, λ, via the map $(\alpha, \alpha^{-1}) \mapsto q^{\frac{1}{2}}(\alpha + \alpha^{-1}) = \lambda$. See [AS01] for a very nice description of the relationships between Satake parameters, spherical representations, and Hecke eigenvalues.

3.7 Table of representations of $\mathrm{GL}(2,\mathrm{F})$

Table (1) summarizes some of what we know about representations of $\mathrm{GL}(2,\mathrm{F})$. Note that, for a representation (π, V), it does not include the dimension of $\mathrm{V}^{\mathrm{K}(m)}$. However, for every $m \geq c(\pi)$, we know that $\dim(\mathrm{V}^{\mathrm{K}(m)}) = m + 1 - c(\pi)$ in group I and III, $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 1$ in group IIb, and $\dim(\mathrm{V}^{\mathrm{K}(m)}) = m - c(\pi)$ for group IIa. In all cases, we know $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 0$ for all $m < c(\pi)$.

4 $\mathrm{GL}(3, \mathrm{F})$ theory

For $\mathrm{G} := \mathrm{GL}(3, \mathrm{F})$, a one-dimensional representation is non-generic. However, unlike the case for $\mathrm{GL}(2, \mathrm{F})$, there are other non-generic representations. It is known that a representation parabolically induced from a generic representation of a Levi subgroup has a unique generic irreducible constituent, any other irreducible constituents being non-generic, [JPSS79]. In this section, set B to be the standard Borel subgroup of G, define $\mathrm{K} := \mathrm{K}(0) = \mathrm{GL}(3, \mathfrak{o})$, and, for an integer $m > 0$, let

$$\mathrm{K}(m) := \mathrm{K} \cap \{ \begin{pmatrix} * & * & * \\ * & * & * \\ \mathfrak{p}^m & \mathfrak{p}^m & * \end{pmatrix} \}.$$

4.1 Double coset decomposition of $\mathrm{GL}(3, \mathrm{F})$

Using what we know about the double coset decomposition of $\mathrm{GL}(n, \mathrm{F})$, see equations (6) and (7), we have

$$\mathrm{G} = \bigsqcup_{m \geq s \geq r} \mathrm{B}\gamma_{sr}\mathrm{K}(m), \tag{16}$$

for

$$\gamma_{sr} = \begin{pmatrix} 1 & & \\ & 1 & \\ \varpi^s & \varpi^r & 1 \end{pmatrix},$$

with $m, s,$ and $r \in \mathbb{Z}_{\geq 0}$.

Now define P to be the parabolic subgroup whose F-points are

$$\mathrm{P}(\mathrm{F}) := \{ \begin{pmatrix} A & * \\ & x \end{pmatrix} \in \mathrm{GL}(3, \mathrm{F}) \mid A \in \mathrm{GL}(2, \mathrm{F}) \}, \tag{17}$$

for $*$ a 2x1 column matrix with entries in F. This places $x \in \mathrm{F}^\times$. Define $\mathrm{P}(\mathfrak{o})$ by replacing F with $\mathfrak{o}$. We can then use the double coset decomposition $\mathrm{B}\backslash\mathrm{G}/\mathrm{K}(m)$ above to obtain the decomposition set forward in the following lemma.

Lemma 4.1. *Let m and r be non-negative integers. Define*

$$\gamma_r = \begin{pmatrix} 1 & & \\ & 1 & \\ & \varpi^r & 1 \end{pmatrix}, \tag{18}$$

then

$$\mathrm{G} = \bigsqcup_{r=0}^{m} \mathrm{P}(\mathrm{F})\gamma_r \mathrm{K}(m). \tag{19}$$

Proof. By the Iwasawa decomposition, it is enough to show that

$$\mathrm{GL}(3,\mathfrak{o}) = \bigsqcup_{r=0}^{m} \mathrm{P}(\mathfrak{o})\gamma_r \mathrm{K}(m).$$

Clearly

$$\bigcup_{r=0}^{m} \mathrm{P}(\mathfrak{o})\gamma_r \mathrm{K}(m) \subseteq \mathrm{GL}(3,\mathfrak{o}).$$

It remains to show that the other inclusion,

$$\mathrm{GL}(3,\mathfrak{o}) \subseteq \bigcup_{r=0}^{m} \mathrm{P}(\mathfrak{o})\gamma_r \mathrm{K}(m),$$

holds and that

$$\bigcup_{r=0}^{m} \mathrm{P}(\mathfrak{o})\gamma_r \mathrm{K}(m) = \bigsqcup_{r=0}^{m} \mathrm{P}(\mathfrak{o})\gamma_r \mathrm{K}(m).$$

For $A \in \mathrm{GL}(3,\mathfrak{o})$, we can write

$$A = \tilde{b} \begin{pmatrix} 1 & & \\ & 1 & \\ \varpi^s & \varpi^r & 1 \end{pmatrix} \tilde{k},$$

for some $\tilde{b} \in \mathrm{B}(\mathfrak{o})$, the standard Borel subgroup of $\mathrm{GL}(3,\mathfrak{o})$, some $\tilde{k} \in \mathrm{K}(m)$, and some non-negative integers s and r such that $m \geq s \geq r$. Therefore, we can write

$$A = \begin{pmatrix} a & d & e \\ & b & f \\ & & c \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ \varpi^s & \varpi^r & 1 \end{pmatrix} \begin{pmatrix} g & h & i \\ j & k & l \\ x & y & u \end{pmatrix}$$

$$= \begin{pmatrix} a - d\varpi^{s-r} & d & e \\ -b\varpi^{s-r} & b & f \\ & & c \end{pmatrix} \gamma_r \begin{pmatrix} g & h & i \\ g\varpi^{s-r} + j & h\varpi^{s-r} + k & i\varpi^{s-r} + l \\ x & y & u \end{pmatrix}.$$

Hence, A is in $\mathrm{P}(\mathfrak{o})\gamma_r \mathrm{K}(m)$ as desired.

Now suppose $p\gamma_r k = p'\gamma_s k'$ for some $p, p' \in \mathrm{P}(\mathfrak{o})$ and $k, k' \in \mathrm{K}(m)$. This means $\gamma_s^{-1}(p')^{-1}p\gamma_r \in \mathrm{K}(m)$. Say

$$(p')^{-1}p = \begin{pmatrix} a & b & e \\ c & d & f \\ & & w \end{pmatrix},$$

then

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & -\varpi^s & 1 \end{pmatrix} \begin{pmatrix} a & b & e \\ c & d & f \\ & & w \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & \varpi^r & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b + e\varpi^r & e \\ c & d + f\varpi^r & f \\ -c\varpi^s & w\varpi^r - d\varpi^s & w - f\varpi^s \end{pmatrix} \in \mathrm{K}(m).$$

Thus, both $c\varpi^s$ and $w\varpi^r - d\varpi^s \in \mathfrak{p}^m$. Furthermore, since $ad - bc \in \mathfrak{o}^\times$, either ad or $bc \in \mathfrak{o}^\times$. Hence, either $d \in \mathfrak{o}^\times$ or $c \in \mathfrak{o}^\times$. If $c \in \mathfrak{o}^\times$, then $s = m$ because $c\varpi^s \in \mathfrak{p}^m$. Thus, $d\varpi^s \in \mathfrak{p}^m$ putting $w\varpi^r \in \mathfrak{p}^m$. Since $w \in \mathfrak{o}^\times$, it follows that $r = m$. Therefore, $r = s$ as desired. If $d \in \mathfrak{o}^\times$, since $\nu(w\varpi^r) = r$ and $\nu(d\varpi^s) = s$ with $w\varpi^r - d\varpi^s \in \mathfrak{p}^m$, and both $r, s \leq m$, it must be that $r = s$. Else $r < s$, placing $w\varpi^r - d\varpi^s \in \mathfrak{p}^r \not\subset \mathfrak{p}^m$. $\square$

Thus, a set of double coset representatives for K, and hence G, is given by γ_r, with $0 \leq r \leq m$. Note that there are exactly $m+1$ cosets in this decomposition. The cosets for K are $\mathrm{P}(\mathfrak{o})\gamma_0\mathrm{K}(m), \mathrm{P}(\mathfrak{o})\gamma_1\mathrm{K}(m), \ldots, \mathrm{P}(\mathfrak{o})\gamma_m\mathrm{K}(m) = \mathrm{P}(\mathfrak{o})\mathrm{K}(m)$. In particular, for $m = 0$ there is just one coset, i.e., $\mathrm{P}(\mathfrak{o})\gamma_0\mathrm{K}(0) = \mathrm{GL}(3, \mathfrak{o})$.

4.2 Parabolic induction

For $n = 3$, there are two forms of parabolic induction. One is similar to that of the $n = 2$ case. Given characters χ_1, χ_2, and χ_3 of $\mathrm{F}^\times$, we have a character of B denoted

$$\chi : \mathrm{B} \to \mathbb{C}^\times : \begin{pmatrix} a & d & e \\ & b & f \\ & & c \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)\chi_3(c).$$

Thus, we have $\mathrm{V}(\chi) := \mathrm{V}(\chi_1, \chi_2, \chi_3) := \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$ is the set of all locally constant functions $\phi : \mathrm{G} \to \mathbb{C}$ such that

$$\phi(\begin{pmatrix} a & d & e \\ & b & f \\ & & c \end{pmatrix} g) = \chi_1(a)\chi_2(b)\chi_3(c) \left| \frac{a}{c} \right| \phi(g),$$

for all

$$\begin{pmatrix} a & d & e \\ & b & f \\ & & c \end{pmatrix} \in \mathrm{B}$$

and all $g \in \mathrm{G}$. Here we have δ, the modulus character of B, equals $\left| \frac{a}{c} \right|^2$. Recall that this representation is irreducible if and only if $\chi_i \neq \nu\chi_j$, for all $i \neq j$, where $i, j \in \{1, 2, 3\}$, [Kud94]. If the representation is irreducible, then it is generic and we know by equation (4) that

$$\dim(\mathrm{V}^{\mathrm{K}(m+c)}) = \binom{m+2}{m} = \binom{m+2}{2}. \tag{20}$$

Converting equation (20) into the $V^{K(m)}$ notation gives

$$\dim(V^{K(m)}) = \binom{m+2-c(\pi)}{2}. \tag{21}$$

The second form of parabolic induction involves the subgroup $P := P(F)$. What follows is a description of the explicit model for inducing a character of P. For this induction, start with two characters χ_1 and χ_2 of $F^\times$ to obtain the character of P which we define by

$$\chi : P \to \mathbb{C}^\times : \begin{pmatrix} A & * \\ & b \end{pmatrix} \mapsto \chi_1(\det(A))\chi_2(b). \tag{22}$$

We induce this up to G to get $V(\chi_1 \mathbf{1}_{GL(2)} \times \chi_2) := \mathrm{Ind}_P^G(\pi)$, the set of all locally constant functions $\phi : G \to \mathbb{C}$ that have the transformation property

$$\phi(\begin{pmatrix} A & * \\ & b \end{pmatrix} g) = \chi_1(\det(A))\chi_2(b)\frac{|\det(A)|^{\frac{1}{2}}}{|b|}\phi(g), \tag{23}$$

for all $g \in G$, all $A \in GL(2,F)$, and all $b \in F^\times$. Note that, in this case, the modulus character of P is

$$\delta_P = \frac{|\det(A)|}{|b|^2}.$$

Note: There is also the parabolic subgroup $P_{(1,2)} \simeq P$ whose F-points are

$$P_{(1,2)}(F) = \{\begin{pmatrix} x & * \\ & A \end{pmatrix} \in GL(3,F) \mid A \in GL(2,F)\}, \tag{24}$$

Table 2: Representations of $\mathrm{GL}(3,\mathrm{F})$

	constituent of	representation	generic
I	$\chi_1 \times \chi_2 \times \chi_3$	(irreducible)	•
IIa	$\nu^{\frac{1}{2}}\chi_1 \times \nu^{-\frac{1}{2}}\chi_1 \times \chi_2$	$\chi_1 \mathrm{St}_{\mathrm{GL}(2)} \times \chi_2$	•
IIb	$(\chi_2 \neq \nu^{\pm\frac{3}{2}}\chi_1)$	$\chi_1 \mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$	
IIIa		$\chi \mathrm{St}_{\mathrm{GL}(3)}$	•
IIIb	$\nu\chi \times \chi \times \nu^{-1}\chi$	$L(\nu\chi, \nu^{-\frac{1}{2}}\chi \mathrm{St}_{\mathrm{GL}(2)})$	
IIIc		$L(\nu^{\frac{1}{2}}\chi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1}\chi)$	
IIId		$\chi \mathbf{1}_{\mathrm{GL}(3)}$	

so $x \in \mathrm{F}^\times$, and $*$ is a 1×2 row matrix with entries in F. Use χ_1 and χ_2 as above to obtain the character of $\mathrm{P}_{(1,2)}$ defined by

$$\chi : \mathrm{P}_{(1,2)} \to \mathbb{C}^\times \; : \; \begin{pmatrix} b & * \\ & A \end{pmatrix} \mapsto \chi_1(b)\chi_2(\det(A)).$$

Upon induction, $\mathrm{Ind}_{\mathrm{P}_{(2,1)}}^{\mathrm{G}}(\pi)$ is the set of all locally constant functions $\phi : \mathrm{G} \to \mathbb{C}$ that have the transformation property

$$\phi(\begin{pmatrix} b & * \\ & A \end{pmatrix} g) = \chi_1(b)\chi_2(\det(A)) \frac{|\, b|}{|\det(A)|^{\frac{1}{2}}}\, \phi(g).$$

Although we work with the parabolic subgroup P in this thesis, it is clear that with some slight modification the work would be essentially the same if we had chosen to work with the parabolic subgroup $\mathrm{P}_{(1,2)}$.

4.3 Table of representations of $\mathrm{GL}(3,\mathrm{F})$

Table (2), which lists the non-supercuspidal representations of G along with their irreducible constituents, specifying which are generic, can easily be deduced from [Kud94].

Table 3: Constituents of $\nu\chi \times \chi \times \nu^{-1}\chi$

	$\nu^{\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)} \times \nu^{-1}\chi$	$\nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu^{-1}\chi$
$\nu\chi \times \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)}$	$\chi\mathrm{St}_{\mathrm{GL}(3)}$	$L(\nu\chi, \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)})$
$\nu\chi \times \nu^{-\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)}$	$L(\nu^{\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1}\chi)$	$\chi\mathbf{1}_{\mathrm{GL}(3)}$

Furthermore, it follows from induction in stages that

$$\begin{aligned}\nu\chi \times \chi \times \nu^{-1}\chi =&\nu^{\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)} \times \nu^{-1}\chi + \nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu^{-1}\chi\\ =&\nu\chi \times \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)} + \nu\chi \times \nu^{-\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)}.\end{aligned}$$

The first terms on the right are the subrepresentations, while the second terms are the quotients. All four of the representations on the right are reducible. Each has two irreducible components. Table (3) shows the relationships of these constituents. In the table, the quotients are on the bottom, respectively on the right.

4.4 Level raising operators

If (π, V) is a smooth representation of $\mathrm{GL}(3, \mathrm{F})$ and v is in $\mathrm{V}^{\mathrm{K}(m)}$, for $m \in \mathbb{Z}_{\geq 0}$, then the formulas

$$\beta_0(v) := v,$$

$$\beta_1(v) := \pi(\begin{pmatrix}1 & & \\ & 1 & \\ & & \varpi\end{pmatrix})v, \text{ and}$$

$$\beta_2(v) := \sum_{\mathrm{A} \in \mathrm{K}_2/\mathrm{K}_2(1)} \pi(\begin{pmatrix}\mathrm{A} & \\ & 1\end{pmatrix})\pi(\begin{pmatrix}1 & & \\ & \varpi & \\ & & \varpi\end{pmatrix})v,$$

take v to $\mathrm{V}^{\mathrm{K}(m+1)}$. In other words, these three equations define level raising operators. The notation $\mathrm{K}_2 = \mathrm{GL}(2, \mathrm{F})$, while $\mathrm{K}_2(1)$ is the subgroup of K_2 that has entries from $\mathfrak{p}$ in the $(2,1)$ position.

Proposition 4.2. *For any integer $m \geq 0$, the maps β_0, β_1, and β_2 define operators from $\mathrm{V}^{\mathrm{K}(m)}$ to $\mathrm{V}^{\mathrm{K}(m+1)}$ which pairwise commute.*

Proof. The fact that the maps are pairwise commutative, i.e., $(\beta_i\beta_j)(v) = (\beta_j\beta_i)(v)$, for every $v \in \mathrm{V}$ and for $i, j \in \{0, 1, 2\}$, is an immediate result from the defining formulas.

Since $\mathrm{V}^{\mathrm{K}(m)} \subset \mathrm{V}^{\mathrm{K}(m+1)}$, each $v \in \mathrm{V}^{\mathrm{K}(m)}$ is also in $\mathrm{V}^{\mathrm{K}(m+1)}$. Hence, each element in $\mathrm{V}^{\mathrm{K}(m)}$ will be raised to the same element in $\mathrm{V}^{\mathrm{K}(m+1)}$ by the level raising operator β_0.

For the second level raising operator, we want to show that

$$\beta_1(v) \in \mathrm{V}^{\mathrm{K}(m+1)},$$

for any $v \in \mathrm{V}^{\mathrm{K}(m)}$. Let

$$k_o = \begin{pmatrix} g & h & i \\ j & k & l \\ x & y & u \end{pmatrix} \in \mathrm{K}(m+1),$$

then

$$(\pi(k_o))(\beta_1(v)) = \pi(\begin{pmatrix} g & h & i \\ j & k & l \\ x & y & u \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix})v$$

$$= \pi(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix} \begin{pmatrix} g & h & i\varpi \\ j & k & l\varpi \\ x\varpi^{-1} & y\varpi^{-1} & u \end{pmatrix})v$$

$$= \pi(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix})v = \beta_1(v),$$

because x and $y \in \mathfrak{p}^{m+1}$ puts $x\varpi^{-1}$ and $y\varpi^{-1} \in \mathfrak{p}^m$, and v is invariant under the action of $\mathrm{K}(m)$ with

$$\begin{pmatrix} g & h & i\varpi \\ j & k & l\varpi \\ x\varpi^{-1} & y\varpi^{-1} & u \end{pmatrix} \in \mathrm{K}(m).$$

To show that $\beta_2(v) \in \mathrm{V}^{\mathrm{K}(m+1)}$ for all $v \in \mathrm{V}^{\mathrm{K}(m)}$, we need to show that $\beta_2(v)$ is invariant under the action of $\mathrm{K}(m+1)$. To do so, it suffices to show invariance on a set of generators of $\mathrm{K}(m+1)$.

Lemma 4.3. *Every $k \in \mathrm{K}(m+1)$ can be written as*

$$\begin{pmatrix} 1 & & \\ & 1 & \\ w & z & 1 \end{pmatrix} \begin{pmatrix} \tilde{A} & \\ & \tilde{u} \end{pmatrix} \begin{pmatrix} 1 & & e \\ & 1 & f \\ & & 1 \end{pmatrix},$$

for w and $z \in \mathfrak{p}^{m+1}$, $\tilde{A} \in \mathrm{GL}(2, \mathfrak{o})$, $\tilde{u} \in \mathfrak{o}^\times$, and e and $f \in \mathfrak{o}$.

Proof. Let

$$\begin{pmatrix} g & h & i \\ j & k & l \\ x & y & u \end{pmatrix} \in \mathrm{K}(m+1).$$

This places x and y in $\mathfrak{p}^{m+1}$, and u in $\mathfrak{o}^\times$. Thus, $gk - hj$ is in $\mathfrak{o}^\times$, and both $kx - jy$

and $gy - hx$ are in $\mathfrak{p}^{m+1}$. Therefore,

$$\begin{pmatrix} g & h & i \\ j & k & l \\ x & y & u \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ \frac{kx-jy}{gk-hj} & \frac{gy-hx}{gk-hj} & 1 \end{pmatrix} \begin{pmatrix} g & h & \\ j & k & \\ & & u - \frac{ikx-ijy-lhx+lgy}{gk-hj} \end{pmatrix} \begin{pmatrix} 1 & & \frac{ik-hl}{gk-hj} \\ & 1 & \frac{gl-ij}{gk-hj} \\ & & 1 \end{pmatrix}.$$

This is of the correct form since by our given we have

- $\frac{kx-jy}{gk-hj}$ and $\frac{gy-hx}{gk-hj} \in \mathfrak{p}^{m+1}$,
- $ikx - ijy - lhx + lgy \in \mathfrak{p}^{m+1}$, thus $u - \frac{ikx-ijy-lhx+lgy}{gk-hj} \in \mathfrak{o}^\times$,
- $\frac{ik-hl}{gk-hj}$ and $\frac{gl-ij}{gk-hj} \in \mathfrak{o}$.

□

Thus, the following 3 cases are enough to show the $\mathrm{K}(m+1)$-invariance of $\beta_2(v)$.

$i)$ Show

$$\pi(\begin{pmatrix} 1 & & \\ & 1 & \\ w & z & 1 \end{pmatrix})\beta_2(v) = \beta_2(v),$$

for all $v \in \mathrm{V}^{\mathrm{K}(m)}$.

We know that

$$\begin{pmatrix} 1 & & \\ & 1 & \\ w & z & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ x\varpi^{-1} & y & 1 \end{pmatrix},$$

where we set

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} w & z \end{pmatrix} A.$$

Thus, x and y must be in $\mathfrak{p}^{m+1}$ and therefore, $x\varpi^{-1} \in \mathfrak{p}^m$, placing the last matrix in $\mathrm{K}(m)$. Hence, for $v \in \mathrm{V}^{\mathrm{K}(m)}$, we have

$$\pi(\begin{pmatrix} 1 & & \\ & 1 & \\ w & z & 1 \end{pmatrix})\beta_2(v) = \beta_2(v).$$

ii) Show that

$$\pi(\begin{pmatrix} \tilde{A} & \\ & \tilde{u} \end{pmatrix})\beta_2(v) = \beta_2(v),$$

for all $v \in \mathrm{V}^{\mathrm{K}(m)}$.

We know that

$$\begin{pmatrix} \tilde{A} & \\ & \tilde{u} \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{A}A & \\ & \tilde{u} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} C & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \tilde{u} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} C & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \tilde{u} \end{pmatrix}.$$

Note we are setting $C = \tilde{A}A$, which just changes the coset representatives for

$K_2/K_2(1)$, and where

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \tilde{u} \end{pmatrix} \in \mathrm{K}(m).$$

Hence,

$$\pi(\begin{pmatrix} \tilde{A} & \\ & \tilde{u} \end{pmatrix})\beta_2(v) = \beta_2(v).$$

iii) Show that

$$\pi(\begin{pmatrix} 1 & & e \\ & 1 & f \\ & & 1 \end{pmatrix})\beta_2(v) = \beta_2(v),$$

for all $v \in \mathrm{V}^{\mathrm{K}(m)}$.

We know that

$$\begin{pmatrix} 1 & & e \\ & 1 & f \\ & & 1 \end{pmatrix} \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \multirow{2}{*}{$A^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}$$

$$= \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix} \begin{pmatrix} 1 & & a\varpi \\ & 1 & b \\ & & 1 \end{pmatrix}.$$

Note we are setting

$$\begin{pmatrix} a \\ b \end{pmatrix} = A^{-1} \begin{pmatrix} e \\ f \end{pmatrix}.$$

Therefore, a and $b \in \mathfrak{o}$. Thus, also $a\varpi \in \mathfrak{o}$. This places the last matrix in $\mathrm{K}(m)$.

Hence,

$$\pi(\begin{pmatrix}1 & & e\\ & 1 & f\\ & & 1\end{pmatrix})\beta_2(v)=\beta_2(v).$$

Thus, we see that the proposed three level raising operators do indeed take any $v \in \mathrm{V}^{\mathrm{K}(m)}$ to an element $\beta_i(v) \in \mathrm{V}^{\mathrm{K}(m+1)}$, for $i=0,1,2$. $\square$

It will be useful to have an explicit formula for β_2. Start with the decomposition of K_2 given by equation (13)

$$\mathrm{K}_2=\bigsqcup_{x\in\,\mathfrak{o}/\mathfrak{p}}\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix}\sqcup\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix}.$$

Furthermore, note that

$$\bigsqcup_{x\in\,\mathfrak{o}/\mathfrak{p}}\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix}=\bigsqcup_{x\in\,(\mathfrak{o}/\mathfrak{p})^\times}\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix}\sqcup\begin{pmatrix}1 & \\ & 1\end{pmatrix}\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix}. \tag{25}$$

This allows us to write

$$\begin{aligned}\beta_2(v)=&\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(\begin{pmatrix}1 & & \\ x & 1 & \\ & & 1\end{pmatrix})\pi(\begin{pmatrix}1 & & \\ & \varpi & \\ & & \varpi\end{pmatrix})v\\ &+\pi(\begin{pmatrix}1 & & \\ & \varpi & \\ & & \varpi\end{pmatrix})v+\pi(\begin{pmatrix} & 1 & \\ -1 & & \\ & & 1\end{pmatrix})\pi(\begin{pmatrix}1 & & \\ & \varpi & \\ & & \varpi\end{pmatrix})v.\end{aligned}$$

First note that

$$\begin{pmatrix} & 1 & \\ -1 & & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & \varpi & \\ & & \varpi\end{pmatrix}=\begin{pmatrix}\varpi & & \\ & 1 & \\ & & \varpi\end{pmatrix}\begin{pmatrix} & 1 & \\ -1 & & \\ & & 1\end{pmatrix}. \tag{26}$$

Also, for $x \in (\mathfrak{o}/\mathfrak{p})^\times$, the useful identity gives us

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & \\ & -x \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}. \tag{27}$$

In particular, this implies

$$\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & & \\ & -x & \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} & \begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix} \\ &= \begin{pmatrix} 1 & x^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & & \\ & -x & \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix} \\ &= \begin{pmatrix} 1 & x^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi \end{pmatrix} \begin{pmatrix} -x^{-1} & & \\ & -x & \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x^{-1}\varpi & \\ & 1 & \\ & & 1 \end{pmatrix}. \end{aligned}$$

Now note that the final three matrices are each in $\mathrm{K}(m)$ for any $m \geq 0$. Hence, since $v \in \mathrm{V}$ implies that v is invariant under the action of $\mathrm{K}(m)$ for some m, we end up with

$$\pi(\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix})\pi(\begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix})v = \pi(\begin{pmatrix} 1 & x^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix})\pi(\begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi \end{pmatrix})v \tag{28}$$

Using equation (26) and equation (28), we arrive at the explicit formula

$$\beta_2(v) = \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix})\pi(\begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi \end{pmatrix})v + \pi(\begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix})v. \tag{29}$$

Remark 4.4. *Because of the pairwise commutativity of the level raising operators, the maximum number of linearly independent vectors that can be obtained by applying the three level raising operators to one newform gives the progression* $1, 3, 6, 10, 15, \ldots$. *Note that, if* V *is an irreducible, admissible, generic representation, then formula (20) can be used to compute the dimension of* $\mathrm{V}^{\mathrm{K}(c(\pi)+m)}$. *Doing so results in exactly the same progression,* $1, 3, 6, 10, 15, \ldots$, *for the dimensions of* $\mathrm{V}^{\mathrm{K}(c(\pi))}, \mathrm{V}^{\mathrm{K}(c(\pi)+1)}, \ldots$.

In the generic case, to show that the vectors obtained via the level raising operators are linearly independent, zeta integrals will be employed. Therefore, we will explore more about generic representations in the following section. Afterward, non-generic representations, V, will be considered and the dimension of $\mathrm{V}^{\mathrm{K}(m)}$ in that case will be determined, too.

4.5 Generic representations

Much is known about the generic representations of $\mathrm{GL}(3, \mathrm{F})$. It is known that irreducible representations that arise from parabolic induction from the Borel with $\chi_i \neq \chi_j |\cdot|$, for $i \neq j$, where $i, j \in \{1, 2, 3\}$, are generic, as are all supercuspidal representations. We also know that reducible representations have a unique irreducible, generic constituent. There are methods to determine the conductors for the generic representations as well as a formula for $\dim(\mathrm{V}^{\mathrm{K}(m)})$ when (π, V) is an irreducible, generic representation.

Whittaker model

We know any generic representation has a Whittaker model. Fix an additive character ψ of F, with the conductor $c(\psi) = \mathfrak{o}$. Define a character θ of N, the unipotent, upper triangular matrices of G, by

$$\theta(\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}) = \psi(a)\psi(c).$$

The induced representation $\mathrm{I}(\theta) := \mathrm{Ind}_{\mathrm{N}}^{\mathrm{G}}(\theta)$ is then the set of all smooth functions $f : \mathrm{G} \to \mathbb{C}$ such that $f(ng) = \theta(n)f(g)$, for every $n \in \mathrm{N}$ and every $g \in \mathrm{G}$.

Since we are assuming that (π, V) is a generic representation, we then have

$$\mathrm{V} \subset \{\mathrm{W} : \mathrm{G} \to \mathbb{C} \mid \mathrm{W}(\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} g) = \psi(a)\psi(c)\mathrm{W}(g)\}$$

and $\mathrm{V} = \mathcal{W}(\pi, \psi)$, the Whittaker model of π with respect to ψ.

Spherical vector

Let $\mathcal{W}(\pi, \psi)$ be a spherical Whittaker model with spherical vector W_o, and assume π has trivial central character. Thus, by virtue of the properties of the spherical vector and by making use of the Iwasawa decomposition, for $\tilde{g} \in \mathrm{G}$, we have $\mathrm{W}_o(\tilde{g}) = \mathrm{W}_o(\tilde{b}\tilde{k}) = \mathrm{W}_o(\tilde{b})$ for some $\tilde{b} \in \mathrm{B}$ and some $\tilde{k} \in \mathrm{K}$. Therefore, W_o is determined on the Borel. Furthermore,

$$\mathrm{W}_o(\begin{pmatrix} a & d & e \\ & b & f \\ & & c \end{pmatrix}) = \mathrm{W}_o(\begin{pmatrix} 1 & db^{-1} & ec^{-1} \\ & 1 & fc^{-1} \\ & & 1 \end{pmatrix} \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}).$$

Hence, W_o is determined on A, the diagonal matrices in G. Also, because we are assuming that π has trivial central character,

$$\mathrm{W}_o(\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}) = \pi(\begin{pmatrix} c & & \\ & c & \\ & & c \end{pmatrix})\mathrm{W}_o(\begin{pmatrix} ac^{-1} & & \\ & bc^{-1} & \\ & & 1 \end{pmatrix})$$

$$=\mathrm{W}_o(\begin{pmatrix} ac^{-1} & & \\ & bc^{-1} & \\ & & 1 \end{pmatrix}).$$

Thus, W_o is determined on matrices in A that have 1 in the (3,3) position. Now $a = u\varpi^r$ and $b = v\varpi^t$ for some u and v in $\mathfrak{o}^\times$, and some r and t in $\mathbb{Z}$. Therefore,

$$\mathrm{W}_o(\begin{pmatrix} a & & \\ & b & \\ & & 1 \end{pmatrix}) = \mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}\begin{pmatrix} u & & \\ & v & \\ & & 1 \end{pmatrix}) = \mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}),$$

because W_o is right K-invariant.

In fact, W_o is determined on such matrices where $0 \leq t \leq r$. If $r < t$, then for any $x \in \mathfrak{o}$

$$\mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}) = \mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix})$$

$$= \mathrm{W}_o(\begin{pmatrix} 1 & x\varpi^{r-t} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix})$$

$$= \psi(x\varpi^{r-t})\mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}).$$

This implies that

$$\mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}) = 0,$$

for $r < t$, because there exists some $x \in \mathfrak{o}$ such that $\psi(x\varpi^{r-t}) \neq 1$. Similarly, if $t < 0$, then for any $x \in \mathfrak{o}$

$$\mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}) = \mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix})$$

$$= \mathrm{W}_o(\begin{pmatrix} 1 & & \\ & 1 & x\varpi^t \\ & & 1 \end{pmatrix}\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix})$$

$$= \psi(x\varpi^t)\mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}).$$

Hence,

$$\mathrm{W}_o(\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix}) = 0,$$

for $t < 0$, since there exists some $x \in \mathfrak{o}$ which gives $\psi(x\varpi^t) \neq 1$.

In conclusion, W_o is determined on the set

$$\{\begin{pmatrix} \varpi^r & & \\ & \varpi^t & \\ & & 1 \end{pmatrix} \mid 0 \leq t \leq r\}. \tag{30}$$

Zeta integrals

With π an irreducible, admissible, generic representation of G and τ an irreducible, admissible, generic representation of $\mathrm{GL}(2, \mathrm{F})$, the zeta integral for G is defined as

$$\mathrm{Z}(s, \mathrm{W}, \mathrm{W}') := \int_{\mathrm{N}_2\backslash\mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\mathrm{W}'(g)|\det(g)|^s \, dg, \tag{31}$$

where $\mathrm{G}_2 = \mathrm{GL}(2, \mathrm{F})$ and $\mathrm{N}_2 = \mathrm{N}(2, \mathrm{F})$, and for some $\mathrm{W} \in \mathcal{W}(\pi, \psi)$, some $\mathrm{W}' \in \mathcal{W}(\tau, \psi^{-1})$, and some $s \in \mathbb{C}$. The Haar measure, dg, is chosen so that the volume of K is one, and we will assume that s is such that $\mathrm{Re}(s)$ is large enough to ensure that the zeta integral converges.

For any $h \in \mathrm{GL}(2, \mathrm{F})$, the Haar measure gives us

$$\int_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\mathrm{W}'(g)|\det(g)|^s \, dg = \int_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} gh & \\ & 1 \end{pmatrix})\mathrm{W}'(gh)|\det(gh)|^s \, dg. \tag{32}$$

Now suppose $h \in \mathrm{K}_2 := \mathrm{GL}(2, \mathfrak{o})$, then we can use the $\mathrm{K}(m)$-invariance of W and the fact that $|\det(h)| = 1$ to show that, in this case, equation (32)

$$= \int_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\mathrm{W}'(gh)|\det(g)|^s \, dg.$$

Therefore, for dh the Haar measure on $\mathrm{GL}(2, \mathrm{F})$ chosen such that the volume of K_2 will be one,

$$\int_{\mathrm{K}_2} \int_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\mathrm{W}'(g)|\det(g)|^s \, dg \, dh$$

$$= \int_{\mathrm{K}_2} \int_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\mathrm{W}'(gh)|\det(g)|^s \, dg \, dh.$$

Thus,

$$\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1 \end{pmatrix})W'(g)|\det(g)|^s\,dg$$

$$=\int_{K_2}\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1 \end{pmatrix})W'(gh)|\det(g)|^s\,dg\,dh$$

$$=\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1 \end{pmatrix})(\int_{K_2} W'(gh)\,dh)|\det(g)|^s\,dg$$

Now set

$$W'' = \int_{K_2} W'(gh)\,dh. \tag{33}$$

This gives us $Z(s, W, W') = Z(s, W, W'')$, and by equation (33) we have W'' is right K_2-invariant. In other words, if $Z(s, W, W')$ is non-zero for some W', then it is non-zero for a W' which is right K_2-invariant. Therefore, we can and do assume τ is spherical with spherical vector W'. Furthermore, for $W \not\equiv 0$, there exists some spherical vector W' so that the zeta integral is non-zero, see [JPSS81] lemme (3.5) and the thèoréme on page 208.

The explicit formula given in part *(ii)* of the thèoréme on page 208, [JPSS81], is of particular interest for the results of this thesis. The formula implies that if W is a new vector in the Whittaker model, then $Z(s, W, W')$ depends holomorphically on the Satake parameters of W'.

Zeta integrals and level raising operators

Now we want to check how the zeta integral behaves under the level raising operators from section 4.4. We will assume both π and τ have trivial central character.

Since β_0 is the identity map, we have

$$\mathrm{Z}(s, \beta_0 \mathrm{W}, \mathrm{W}') = \mathrm{Z}(s, \mathrm{W}, \mathrm{W}'). \tag{34}$$

By virtue of the Haar measure and the trivial central character of both π and τ, we have

$$\mathrm{Z}(s, \beta_1 \mathrm{W}, \mathrm{W}') = \int\limits_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix})\mathrm{W}'(g)|\det(g)|^s\, dg$$

$$= \int\limits_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix})\mathrm{W}'(g\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix})|\det(g\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix})|^s\, dg$$

$$= q^{-2s} \int\limits_{\mathrm{N}_2\backslash \mathrm{G}_2} \pi(\begin{pmatrix} \varpi & & \\ & \varpi & \\ & & \varpi \end{pmatrix})\mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\tau(\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix})\mathrm{W}'(g)|\det(g)|^s\, dg$$

$$= q^{-2s} \int\limits_{\mathrm{N}_2\backslash \mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix})\mathrm{W}'(g)|\det(g)|^s\, dg.$$

Thus,

$$\mathrm{Z}(s, \beta_1 \mathrm{W}, \mathrm{W}') = q^{-2s}\mathrm{Z}(s, \mathrm{W}, \mathrm{W}'). \tag{35}$$

Using the explicit formula for β_2 gives

$$(\beta_2 \mathrm{W})(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) = \sum_{x \in \mathfrak{o}/\mathfrak{p}} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi \end{pmatrix})$$

$$+\mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix}).$$

In the zeta integral, the second term becomes

$$\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix})W'(g)|\det(g)|^s\, dg$$

$$= \int_{N_2\backslash G_2} W(\begin{pmatrix} g\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix})W'(g\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})|\det(g\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})|^s\, dg$$

$$=q^{-s} \int_{N_2\backslash G_2} \pi(\begin{pmatrix} \varpi & & \\ & \varpi & \\ & & \varpi \end{pmatrix})W(\begin{pmatrix} g & \\ & 1 \end{pmatrix})W'(g\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})|\det(g)|^s\, dg$$

$$= q^{-s} \int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1 \end{pmatrix})W'(g\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})|\det(g)|^s\, dg. \tag{36}$$

Let's take a closer look at the first term when calculating the zeta integral. Let

$$g \mapsto g\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}.$$

Note that

$$\det(g\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}) = \det(g),$$

and because $x \in \mathfrak{o}^\times$, we also have

$$W'(g\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}) = W'(g).$$

Thus, in the zeta integral, the first term becomes

$$\sum_{x\in\mathfrak{o}/\mathfrak{p}} \int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix}1 & x & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\varpi & & \\ & 1 & \\ & & \varpi\end{pmatrix})W'(g)|\det(g)|^s\,dg$$

$$=\sum_{x\in\mathfrak{o}/\mathfrak{p}} \int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix}\varpi & & \\ & 1 & \\ & & \varpi\end{pmatrix})W'(g)|\det(g)|^s\,dg$$

letting $g \to g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}$

$$=q\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix})W'(g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})|\det(g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})|^s\,dg$$

$$= q^{-s}\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix})qW'(g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})|\det(g)|^s\,dg. \tag{37}$$

Now we use equations (37) and (36) to determine that

$$\begin{aligned}Z(s,\beta_2 W, W') =& q^{-s}\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix})qW'(g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})|\det(g)|^s\,dg\\ &+q^{-s}\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix})W'(g\begin{pmatrix}\varpi & \\ & 1\end{pmatrix})|\det(g)|^s\,dg\end{aligned}$$

$$= q^{-s}\int_{N_2\backslash G_2} W(\begin{pmatrix} g & \\ & 1\end{pmatrix})\,[q(W'(g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})) + W'(g\begin{pmatrix}\varpi & \\ & 1\end{pmatrix})]\,|\det(g)|^s\,dg. \tag{38}$$

If we let

$$g \mapsto g\begin{pmatrix}1 & \\ x & 1\end{pmatrix},$$

then

$$W'(g\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}) \mapsto W'(g\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})$$

is the only change that occurs in the integral (38). This is because there exists some non-negative integer m for which W is right $\mathrm{K}(m)$-invariant and

$$\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{K}(m),$$

for any $m \in \mathbb{Z}_{\geq 0}$ and for any $x \in \mathfrak{o}$. Also,

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} = \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ x\varpi & 1 \end{pmatrix},$$

with

$$\begin{pmatrix} 1 & \\ x\varpi & 1 \end{pmatrix} \in \mathrm{K}_2,$$

and W' is right K_2-invariant. Lastly, because we have

$$\det(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}) = 1.$$

So far we have that

$$\mathrm{Z}(s, \beta_2\mathrm{W}, \mathrm{W}') = q^{-s} \int\limits_{\mathrm{N}_2\backslash\mathrm{G}_2} \mathrm{W}(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) \, [q\mathrm{W}'(g \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}) + \mathrm{W}'(g \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})] \, |\det(g)|^s \, dg.$$

Take $\sum\limits_{x \in \mathfrak{o}/\mathfrak{p}}$ of both sides of this equation. This results in multiplying the left hand side of the equation, and also

$$\mathrm{W}'(g \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix})$$

by q, since these terms have no x, and changing

$$\mathrm{W}'(g\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}) \mapsto \sum_{x\in\mathfrak{o}/\mathfrak{p}} \mathrm{W}'(g\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}).$$

By setting

$$(\pi(\theta)\mathrm{W}')(g) := \sum_{x\in\mathfrak{o}/\mathfrak{p}} \mathrm{W}'(g\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}) + \mathrm{W}'(g\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}),$$

we get $\mathrm{Z}(s, \beta_2\mathrm{W}, \mathrm{W}') = q^{-s}\mathrm{Z}(s, \mathrm{W}, \pi(\theta)\mathrm{W}')$.

Looking back at equation (15) for a Hecke operator of $\mathrm{GL}(2, \mathrm{F})$, we see that $\pi(\theta)$ is such a Hecke operator. Thus, $\pi(\theta)\mathrm{W}' = \lambda\mathrm{W}'$, for $\lambda \in \mathbb{C}$ the Hecke eigenvalue of τ related to α, the Satake parameter of W', via $\lambda = q^{\frac{1}{2}}(\alpha + \alpha^{-1})$. Therefore, we conclude that

$$\mathrm{Z}(s, \beta_2\mathrm{W}, \mathrm{W}') = \lambda q^{-s}\mathrm{Z}(s, \mathrm{W}, \mathrm{W}'). \tag{39}$$

The results of these relationships between the zeta integral and the level raising operators are summarized in the following lemma.

Lemma 4.5. *For* $\mathrm{W} \in \mathcal{W}(\pi, \psi)$ *and spherical vector* $\mathrm{W}' \in \mathcal{W}(\tau, \psi^{-1})$*, where both* π *and* τ *have trivial central character, the zeta integral has the following transformation properties:*

i.) $\mathrm{Z}(s, \beta_0 W, W') = \mathrm{Z}(s, W, W')$

ii.) $\mathrm{Z}(s, \beta_1 W, W') = q^{-2s}\mathrm{Z}(s, W, W')$

iii.) $\mathrm{Z}(s, \beta_2 W, W') = \lambda q^{-s}\mathrm{Z}(s, W, W')$*, where* $\lambda = q^{\frac{1}{2}}(\alpha + \alpha^{-1}) \in \mathbb{C}^\times$ *is the Hecke eigenvalue of* τ *and* α *is the Satake parameter of* W'*.*

Level raising in generic representations

In this section, we focus once again on irreducible, admissible, generic representations. For the zeta integrals that will come into play, we choose s such that $\mathrm{Re}(s)$ is large enough to ensure that the integral converges, and W' is chosen so that the zeta integral is non-zero.

Theorem 4.6. *Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GL}(3, \mathrm{F})$ with trivial central character. If $v \in \mathrm{V}^{\mathrm{K}(c(\pi))}$ is non-zero, then the space $\mathrm{V}^{\mathrm{K}(m)}$ for any integer $m \geq c(\pi)$ is spanned by the linearly independent vectors*

$$\beta_0^h \beta_1^i \beta_2^j v,$$

for $h, i, j \in \mathbb{Z}_{\geq 0}$ and where $h + i + j = m - c(\pi)$. In particular, the oldforms can be obtained by applying level raising operators to one newform and taking linear combinations.

Proof. As a result of Remark (4.4), we need only show that the elements in $\mathrm{V}^{\mathrm{K}(m)}$ that arise as the result of raising the level of the newform through repeated use of the level raising operators are linearly independent. To do so, we use the interplay found above between the zeta integral and the level raising operators, the isomorphism between V and its Whittaker model $\mathcal{W}$, as well as the Satake parameters described in section 3.6.

Let W_o be a new vector in $\mathcal{W}$, and assume

$$\sum_{\substack{h,i,j \in \mathbb{Z}_{\geq 0} \\ h+i+j = m - c(\pi)}} a_{hij} (\beta_0^h \beta_1^i \beta_2^j)(\mathrm{W}_o) = 0. \tag{40}$$

Applying the zeta integral to equation (40) gives us

$$\mathrm{Z}(s, \sum_{\substack{h,i,j\in\mathbb{Z}_{\geq 0}\\ h+i+j=m-c(\pi)}} a_{hij}(\beta_0^h\beta_1^i\beta_2^j)(\mathrm{W}_o), \mathrm{W}') = 0.$$

Therefore, we end up with

$$\sum_{\substack{h,i,j\in\mathbb{Z}_{\geq 0}\\ h+i+j=m-c(\pi)}} a_{hij}q^{-2si-sj}\lambda^j \mathrm{Z}(s, \mathrm{W}_o, \mathrm{W}') = 0. \tag{41}$$

By our choice of s and W', and because $\mathrm{W}_o \neq 0$ there is an open neighborhood, U_1, around s where the integral converges and is not zero. Now we use the fact that the zeta integral depends holomorphically on the Satake parameter α, i.e., there exists an open neighborhood around α in which the zeta integral converges and is not zero. Therefore, since the Hecke eigenvalue λ is equal to $q^{\frac{1}{2}}(\alpha + \alpha^{-1})$, there is an open neighborhood, U_2, around λ in which the integral converges and is not zero. Thus, equation (41) implies that

$$\sum_{\substack{h,i,j\in\mathbb{Z}_{\geq 0}\\ h+i+j=m-c(\pi)}} a_{hij}q^{-2si-sj}\lambda^j = \sum_{\substack{h,i,j\in\mathbb{Z}_{\geq 0}\\ h+i+j=m-c(\pi)}} a_{hij}(q^{-2s})^i(q^{-s}\lambda)^j = 0, \tag{42}$$

for every $s \in \mathrm{U}_1$ and every $\lambda \in \mathrm{U}_2$.

Now consider equation (42) to be a polynomial in the two variables $X = q^{-2s}$ and $Y = q^{-s}\lambda$, i.e., $Y = X^{\frac{1}{2}}\lambda$. Hence, we have

$$f(X, Y) = \sum_{\substack{h,i,j\in\mathbb{Z}_{\geq 0}\\ h=m-c(\pi)-i-j}} a_{hij}X^iY^j = 0,$$

for every $X \in \mathrm{U}_1'$, where U_1' depends on U_1, and for every $Y \in \mathrm{U}_2'$, where U_2' depends on both U_1 and U_2. Basically, we have a map $\mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C} : (X, \lambda) \mapsto (X, Y)$, and we choose an open neighborhood $\mathrm{U}_1' \times \mathrm{U}_2$ in the domain such that its image is in the open neighborhood $\mathrm{U}_1' \times \mathrm{U}_2'$. Consider the zero set of this polynomial, $\{(X, Y) \mid f(X, Y) = 0\}$. If $f(X, Y)$ is not the zero polynomial, then its zero set is known to be one-dimensional. This contradicts the previous statement that the sum equals zero on some open set $\mathrm{U}_1' \times \mathrm{U}_2'$ in $\mathbb{C}$. Hence, it must be that $a_{hij} = 0$ for every h, i, j. This proves the theorem. □

4.6 Non-generic representations

In the previous section, we considered generic representations arising from parabolic induction on the Borel subgroup. In this section, we make use of parabolic induction on P defined in section 4.1, along with the double coset decomposition put forward in lemma 4.1.

Let χ_1 and χ_2 be characters of $F^\times$, and let $\chi := \chi_1 \mathbf{1}_{\mathrm{GL(2,F)}} \times \chi_2$ have trivial central character. Set $\pi(\chi_1 \mathbf{1}_{\mathrm{GL(2,F)}} \times \chi_2) = \pi(\chi)$, the associated induced representation on $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\chi) =: V(\chi_1 \mathbf{1}_{\mathrm{GL(2)}} \times \chi_2) = \mathrm{V}(\chi)$. This is the set of all smooth functions $\phi : \mathrm{G} \to \mathbb{C}$ that have the transformation property

$$\phi(\begin{pmatrix} A & * \\ & u \end{pmatrix} g) = \chi_1(\det(A))\chi_2(u) \frac{|\det(A)|^{\frac{1}{2}}}{|u|} \phi(g),$$

for all $g \in \mathrm{G}$, all $A \in \mathrm{GL}(2, \mathrm{F})$, and all $u \in \mathrm{F}^\times$, where $*$ is a 2×1 column matrix with entries in F. Define γ_r as in equation (18).

Lemma 4.7. *With the above definitions, the following are equivalent.*

i.) χ *is unramified, i.e.,* $c(\chi_1) = 0 = c(\chi_2)$.

ii.) $\mathrm{V}^{\mathrm{K}(m)} \neq \{0\}$ *for every* $m \in \mathbb{Z}_{\geq 0}$.

iii.) $\mathrm{V}^{\mathrm{K}(m)} \neq \{0\}$ *for some* $m \in \mathbb{Z}_{\geq 0}$.

Proof. Clearly, if $c(\chi) = 0$, i.e, χ is unramified, then $\dim(\mathrm{V}^{\mathrm{K}(0)}) = 1$ which implies that $\mathrm{V}^{\mathrm{K}(m)} \neq 0$ for any m. Hence, $i)$ implies $ii)$. The implication from $ii)$ to $iii)$ is trivial. To prove that $iii)$ implies $i)$, take m such that $\mathrm{V}^{\mathrm{K}(m)} \neq 0$. Let $\phi \in \mathrm{V}^{\mathrm{K}(m)}$ be non-zero and r be such that $\phi(\gamma_r) \neq 0$. For $a \in \mathfrak{o}^\times$, we have

$$\begin{pmatrix} a & & \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{K}(m).$$

Thus,

$$\begin{aligned} \phi(\gamma_r) &= \phi(\gamma_r \begin{pmatrix} a & & \\ & 1 & \\ & & 1 \end{pmatrix}) \\ &= \phi(\begin{pmatrix} a & & \\ & 1 & \\ & & 1 \end{pmatrix} \gamma_r) \\ &= \chi_1(a)\phi(\gamma_r). \end{aligned}$$

Since $\phi(\gamma_r) \neq 0$, it must be that $\chi_1(a) = 1$ for every $a \in \mathfrak{o}^\times$, i.e. $c(\chi_1) = 0$. By virtue of the trivial central character, we know that $\chi_1^2\chi_2 = 1$. In particular, $\chi_1^2(a)\chi_2(a) = 1$ for every $a \in \mathfrak{o}^\times$. Hence, $\chi_2(a) = 1$ for every $a \in \mathfrak{o}^\times$, implying that χ_2 is unramified. □

As a result of the lemma, under the given conditions, we need only concern ourselves with the case where χ_1 and χ_2 are both unramified.

Dimension of $V^{K(m)}$ for group IIb

In order to determine the dimensions of $V^{K(m)}$ associated with the non-generic representations of G from group IIb table (2), we will make use of Reeder's formula for irreducible generic representations [Ree91], information that is known about the conductor of $\chi\mathrm{St}_{\mathrm{GL}(2)}$, and a deformation argument. By Reeder's paper, we have formula (21) for $\dim(V^{K(m)})$ if $\chi_1 \times \chi_2 \times \chi_3$ is irreducible and thus generic. We use the following deformation argument to show that formula (21) still holds if $\chi_1 \times \chi_2 \times \chi_3$ is not irreducible, i.e., in the full induced representations from group II and III.

Let $\chi = \chi_1 \times \chi_2 \times \chi_3$ be an irreducible, generic representation of G with trivial central character. So in particular, we have $\chi_1\chi_2\chi_3 \equiv 1$. Because the action of G on $V(\chi)$ is by right translation, and by utilizing the Iwasawa decomposition, another model for $\pi(\chi_1, \chi_2, \chi_3)$ is obtained by restricting functions in $V(\chi)$ to K. We denote this space of functions on K by $V(\chi)_K$. Hence, there is an injective map from $V(\chi)$ to $V(\chi)_K$ that commutes with the K-action. Therefore, $V(\chi) \simeq V(\chi)_K$ as K-modules. Now suppose we deform χ by modifying each χ_i, i=1,2,3, by multiplication with a power of an absolute value. Say $\chi' = |\cdot|^{s_1}\chi_1 \times |\cdot|^{s_2}\chi_2 \times |\cdot|^{s_3}\chi_3$. To preserve the trivial central character, it must be that $s_1 + s_2 + s_3 = 0$. Arguing as above we see that $V(\chi') \simeq V(\chi')_K$. Furthermore, we know $|k|^{s_i} = 1$ for all $k \in K$ and each $i = 1, 2, 3$. Hence, $V(\chi)_K = V(\chi')_K$. This implies that $V(\chi) \simeq V(\chi')$ as K-modules. Therefore, the dimension of the $V^{K(m)}$ associated to either of these representation spaces will be the same. Thus, for the full induced representations in group II and III from table (2) we can use formula (21) to compute $\dim(V^{K(m)})$.

In group IIa, $\pi = \chi_1\mathrm{St}_{\mathrm{GL}(2)} \times \chi_2$, and we know $c(\pi) = c(\chi_1\mathrm{St}_{\mathrm{GL}(2)}) + c(\chi_2)$. By lemma (4.7), we are only concerned with unramified characters in this case and by

[Sch02] we have $c(\chi \mathrm{St}_{\mathrm{GL}(2)}) = 1$ when χ is unramified. Therefore, we have

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = \binom{m+1}{2}. \tag{43}$$

Using the above argument and equations (21) and (43), we conclude that in group IIb

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = \binom{m+2}{2} - \binom{m+1}{2} = m+1. \tag{44}$$

Level raising operators

Notice that, in this case, a unit increase in level results in a unit increase in dimension. Thus, only two level raising operators are needed.

Claim 4.8. *Under the conditions of this section and if $\chi_2 \neq \chi_1|\cdot|^{\frac{3}{2}}$, all the oldforms can be obtained by applying the two level raising operators β_0 and β_1 to the newform and taking linear combinations. Equivalently, β_0 and β_1 combine to provide a surjective map from $\mathrm{V}^{\mathrm{K}(m)}$ to $\mathrm{V}^{\mathrm{K}(m+1)}$.*

Proof. If $\phi \in \mathrm{V}^{\mathrm{K}(m)}$, then ϕ is determined on the set of coset representatives, the γ_r's, from lemma 4.1. For $g \in \mathrm{K}$, and fixed $r \in \{0, 1, 2, \ldots, m\}$, try to define a non-zero function ϕ_r by

$$\phi_r(g) := \begin{cases} \chi_1(\det(A))\chi_2(x)\frac{|\det(A)|^{\frac{1}{2}}}{|x|} & \text{if } g \in \begin{pmatrix} A & * \\ & x \end{pmatrix}\gamma_r \mathrm{K}(m); \\ 0 & \text{if } g \notin \mathrm{P(F)}\gamma_r\mathrm{K}(m). \end{cases}$$

Clearly such ϕ_r are in $\mathrm{V}^{\mathrm{K}(m)}$, they are linearly independent, and there are exactly $m+1$ of them. Thus, they form a basis for $\mathrm{V}^{\mathrm{K}(m)}$.

By definition of β_0 and ϕ_r, we note that $(\beta_0\phi_r)(\gamma_s) = \delta_{rs}$ for every integer s such that $0 \leq s \leq m+1$, and every r, except when $r = m$ and $s = m+1$. In that case, we have $(\beta_0\phi_m)(\gamma_{m+1}) = 1$ because $\gamma_{m+1} \in \mathrm{K}(m)$ and ϕ_m is invariant under right multiplication by $\mathrm{K}(m)$.

Therefore, fixing $\{\phi_r\}_{r=0}^{m}$ as a basis for $\mathrm{V}^{\mathrm{K}(m)}$ gives the following $(m+1) \times m$ matrix for β_0 when $m > 0$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Now consider the second level raising operator. If $s \neq 0$, then, in general,

$$\begin{aligned} (\beta_1\phi_r)(\gamma_s) =& \phi_r(\gamma_s \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix}) \\ =& \phi_r(\begin{pmatrix} 1 & & \\ & 1 & \\ & \varpi^s & \varpi \end{pmatrix}) \\ =& \phi_r(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & \varpi^{s-1} & 1 \end{pmatrix} \\ =& \chi_1(1)\chi_2(\varpi) \left| \frac{1}{\varpi} \right| \phi_r(\gamma_{s-1}). \end{aligned}$$

Therefore,

$$(\beta_1\phi_r)(\gamma_s) = \begin{cases} q\chi_2(\varpi) & \text{if } s = r+1; \\ 0 & \text{if } s \neq r+1. \end{cases}$$

Consider the case where $s = 0$. We have

$$\begin{aligned}
(\beta_1\phi_r)(\gamma_0) =& \phi_r(\gamma_0 \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{pmatrix}) \\
=& \phi_r(\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \varpi \end{pmatrix}) \\
=& \phi_r(\begin{pmatrix} 1 & & \\ & \varpi & 1-\varpi \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \varpi - 1 \\ & & 1 \end{pmatrix}) \\
=& \chi_1(\varpi)\chi_2(1)|\varpi|^{\frac{1}{2}}\phi_r(\gamma_0).
\end{aligned}$$

Therefore,

$$(\beta_1\phi_r)(\gamma_0) = \begin{cases} q^{-\frac{1}{2}}\chi_1(\varpi) & \text{if } r = 0; \\ 0 & \text{if } r \neq 0. \end{cases}$$

Thus, for the same fixed basis, we have the following $(m+1) \times m$ matrix for β_1 when $m > 0$

$$\begin{pmatrix}
\chi_1(\varpi)q^{-\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\
\chi_2(\varpi)q & 0 & 0 & \cdots & 0 & 0 \\
0 & \chi_2(\varpi)q & 0 & \cdots & 0 & 0 \\
0 & 0 & \chi(\varpi)q & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \chi_2(\varpi)q & 0 \\
0 & 0 & 0 & \cdots & 0 & \chi_2(\varpi)q
\end{pmatrix}.$$

Combining these two matrices gives us the following $(m+1) \times (m+1)$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \chi_2(\varpi)q \end{pmatrix}.$$

This matrix has non-zero determinant for any $m \neq 0$. Thus, under these conditions, for a newform v, the vectors $\beta_0^i \beta_1^j v$ are linearly independent, for $i, j \in \mathbb{Z}_{\geq 0}$ with $i + j = m - c(\pi)$.

In the case where $m = 0$, we have $(\beta_0 \phi_0)(\gamma_i) = 1$, for $i = 0, 1$, since $\gamma_i \in \mathrm{K}$. Hence, our matrix with respect to β_0 looks like

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By our previous work, we know the matrix with respect to β_1 is:

$$\begin{pmatrix} \chi_1(\varpi)q^{-\frac{1}{2}} \\ \chi_2(\varpi)q \end{pmatrix}.$$

Thus, combining the matrices gives:

$$\begin{pmatrix} 1 & \chi_1(\varpi)q^{-\frac{1}{2}} \\ 1 & \chi_2(\varpi)q \end{pmatrix}.$$

Note that this has non-zero determinant if and only if $\chi_2 \neq \chi_1 |\cdot|^{\frac{3}{2}}$. Thus explaining why this extra condition needs to be in the hypothesis of this claim. □

Note that the condition $\chi_2 \neq \chi_1 |\cdot|^{\frac{3}{2}}$ agrees with the condition placed on group

II of table (2).

Suppose we would like to understand group IIIb from our table in section 2. Table (3) which states the relationship between the various components of $\nu\chi \times \chi \times \nu^{-1}\chi$ leads to the following short exact sequence

$$0 \to \mathrm{L}(\nu\chi, \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)}) \to \nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu^{-1}\chi \to \chi\mathbf{1}_{\mathrm{GL}(3)} \to 0. \tag{45}$$

By taking contragredients and replacing χ^{-1} with χ, we also get the short exact sequence

$$0 \to \chi\mathbf{1}_{\mathrm{GL}(3)} \to \nu^{-\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu\chi \to \mathrm{L}(\nu\chi, \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)}) \to 0. \tag{46}$$

Now check the condition $\chi_2 \neq \chi_1|\cdot|^{\frac{3}{2}}$. In the first sequence (45), we relate $\nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu^{-1}\chi$ to our representation $\chi_1\mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$, by setting $\chi_1 = \nu^{\frac{1}{2}}\chi$ and setting $\chi_2 = \nu^{-1}\chi$. Thus, we get $\chi_2 = \chi_1|\cdot|^{-\frac{3}{2}}$. Hence, our condition is satisfied which implies that we can use the two level raising operators to obtain all the old forms from the newform in the subrepresentation $\mathrm{L}(\nu\chi, \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)})$. But, in sequence (46), we would have $\chi_1 = \nu^{-\frac{1}{2}}\chi$ and $\chi_2 = \nu\chi$. This results in $\chi_2 = \chi_1|\cdot|^{\frac{3}{2}}$. This implies that in the subrepresentation $\chi\mathbf{1}_{\mathrm{GL}(3)}$, applying the level raising operators only results in scalar multiples of the newform. This is to be expected since if we let V be the representation space of $\chi\mathbf{1}_{\mathrm{GL}(3)}$, then $\dim \mathrm{V}^{\mathrm{K}(m)}$=1 for all $m \geq c(\pi)$.

Now we have only group IIIc to consider. Recall definition (24) of the parabolic subgroup $\mathrm{P}_{(1,2)}$, and the corresponding induction. Use table (3) to set up the short exact sequence

$$0 \to \mathrm{L}(\nu^{\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1}\chi) \to \nu\chi \times \nu^{-\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \to \chi\mathbf{1}_{\mathrm{GL}(3)} \to 0.$$

The representation we obtain from induction on $\mathrm{P}_{(1,2)}$ is of the form $\chi_1 \times \chi_2 \mathbf{1}_{\mathrm{GL}(2)}$. Relate it to the representation $\nu\chi \times \nu^{-\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)}$ by setting $\chi_1 = \nu\chi$ and $\chi_2 = \nu^{-\frac{1}{2}}\chi$. Thus, we have $\chi_2 = \nu^{-\frac{3}{2}}\chi_1$, which satisfies our condition and hence implies that all oldforms can be obtained by applying the two level raising operators to the newform and taking linear combinations.

Combining theorem (4.6) with claim (4.8) and the above information on groups IIIb, c, and d, leads to and proves the following theorem.

Theorem 4.9. *Let* (π, V) *be an irreducible, admissible representation of* $\mathrm{GL}(3, \mathrm{F})$ *with trivial central character. All the oldforms can be obtained by applying the level raising operators* β_0, β_1 *and* β_2 *to one newform and taking linear combinations.*

4.7 Dimensions of $\mathrm{V}^{\mathrm{K}(m)}$ for the representations of $\mathrm{GL}(3, \mathrm{F})$

Using what is already known about dimensions of some $\mathrm{V}^{\mathrm{K}(m)}$, the deformation argument, Reeder's dimension formula and a certain fact about the conductor of $\chi\mathrm{St}_{\mathrm{GL}(3)}$, we can now compute the dimensions of the remaining cases. The groups refer back to table (2). Note in all cases we are assuming that $m \geq c(\pi)$. In the case that $m < c(\pi)$, $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 0$.

Group I: In group I it is known [Ree91] that

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = \binom{m - c(\pi) + 2}{2}.$$

Group II: By lemma (4.7), we assume χ_1 and χ_2 are unramified. From the computations done in this paper, we know for group IIa

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = \binom{m + 1}{2},$$

while for group IIb

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = m + 1.$$

Group III: By setting $\chi' = \nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu^{-1}\chi$, we can use the deformation argument on the representation from group IIb to show that $\dim(\mathrm{V}^{\mathrm{K}(m)}) = m + 1$ for the representation $\nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)} \times \nu^{-1}\chi$. Similarly, setting $\chi' = \nu\chi \times \nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)}$ shows that $\dim(\mathrm{V}^{\mathrm{K}(m)}) = m + 1$ for the representation $\nu\chi \times \nu^{\frac{1}{2}}\chi\mathbf{1}_{\mathrm{GL}(2)}$. We know if (π_d, V_d) is a representation in group IIId, then

$$\dim(\mathrm{V}_d^{\mathrm{K}(m)}) = 1.$$

Thus, we have $\dim(\mathrm{V}_d^{\mathrm{K}(m)}) + \dim(\mathrm{V}_b^{\mathrm{K}(m)}) = m + 1$, for a representation (π_b, V_b) in group IIIb. Therefore,

$$\dim(\mathrm{V}_b^{\mathrm{K}(m)}) = m$$

for representations of the form $\mathrm{L}(\nu\chi, \nu^{-\frac{1}{2}}\chi\mathrm{St}_{\mathrm{GL}(2)})$.

Similarly we can compute

$$\dim(\mathrm{V}_c^{\mathrm{K}(m)}) = m,$$

for (π_c, V_c) a representation in group IIIc.

Finally, if (π_a, V_a) is a representation in group IIIa, we have

$$m + \dim(\mathrm{V}_a^{\mathrm{K}(m)}) = \binom{m+1}{2}.$$

Our final computation gives us

$$\dim(\mathrm{V}_a^{\mathrm{K}(m)}) = \binom{m}{2}.$$

Table 4: Iwahori-spherical representations of G with $\dim(\mathrm{V}^{\mathrm{K}(m)})$

	constituent of	representation	conductor	$\dim(\mathrm{V}^{\mathrm{K}(m)})$
I	$\chi_1 \times \chi_2 \times \chi_3$ (irreducible)		0	$\binom{m+2}{2}$
IIa	$\nu^{\frac{1}{2}}\chi_1 \times \nu^{-\frac{1}{2}}\chi_1 \times \chi_2$	$\chi_1 \mathrm{St}_{\mathrm{GL}(2)} \times \chi_2$	1	$\binom{m+1}{2}$
IIb	$(\chi_2 \neq \nu^{\pm\frac{3}{2}}\chi_1)$	$\chi_1 \mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$	0	$m+1$
IIIa		$\chi \mathrm{St}_{\mathrm{GL}(3)}$	2	$\binom{m}{2}$
IIIb	$\nu\chi \times \chi \times \nu^{-1}\chi$	$L(\nu\chi, \nu^{-\frac{1}{2}}\chi \mathrm{St}_{\mathrm{GL}(2)})$	1	m
IIIc		$L(\nu^{\frac{1}{2}}\chi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1}\chi)$	1	m
IIId		$\chi \mathbf{1}_{\mathrm{GL}(3)}$	0	1

In all cases we assume $m \geq c$, else $\dim(\mathrm{V}^{\mathrm{K}(m)}) = 0$.

We can also use Reeder's formula since IIIa is an irreducible, generic representation. To do so, we use the proposition on page 18 of [Roh94] and the Local Langlands' Conjecture to determine that $c(\chi \mathrm{St}_{\mathrm{GL}(3)}) = 2$ when χ is unramified. Therefore, Reeder's formula gives us

$$\dim(\mathrm{V}^{\mathrm{K}(m)}) = \binom{m+2-2}{2} = \binom{m}{2}.$$

The results presented in table (4) are for *Iwahori-spherical* representations which simply means that we are assuming that all the χ_i are unramified. If a representation is not Iwahori-spherical, Reeder's formula can be used to determine dimensions of the $\mathrm{V}^{\mathrm{K}(m)}$ for all the generic representations, and in the non-generic case we know by lemma (4.7) there are no fixed vectors.

5 Conclusion

Some of the fundamentals of representation theory of GL(2) transfer quite nicely to GL(3). One has to make adjustments to allow for the differences in the two groups, but the basic concepts seem to hold up. The concept of a conductor, and the existence of a relationship between the conductor and the dimension of our space $V^{K(m)}$ extends from GL(2) to GL(3). In the case of generic representations, there are similar zeta integrals and Whittaker models. Our results give level raising operators that act on the fixed vector spaces $V^{K(m)}$ of representation spaces V of $GL(3, F)$ just as there are those for such spaces in $GL(2, F)$. We also proved an oldforms theorem for $GL(3, F)$ showing that all oldforms can be obtained from a newform by applying certain level raising operators and taking linear combinations that relates to the oldforms theorem of $GL(2, F)$.

By using representation theory in general, some $GL(2, F)$ representation theory itself, and by extending some $GL(2, F)$ representation theory to $GL(3, F)$, we were able to compute the dimension of the $V^{K(m)}$ for all the non-supercuspidal representations of $GL(3, F)$. In the end, we were able to present our information in a complete table with all the pertinent information about the representations of GL(3).

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