PIPELINES WITH IDEALIZED TERMINATIONS
SUBJECTED TO STATISTICAL INPUTS

By

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LIST OF SYMBOLS

The symbols listed below are briefly defined and any descriptive units refer to the gravitational (technical) system. F is a force (usually in pounds), L is a length (usually in feet), and T is time (usually in seconds).

a	The velocity of propagation of the pressure wave; LT^{-1}.
A	Area; L^2 .
c	The limit for the validity of the inverse Laplace transform; T^{-1} .
C	The coefficient of capacitance in a pipe; $L^4\, F^{-1}$.
D	The diameter of the pipe; L .
f	The Darcy-Weisbach resistance coefficient.
$h(t)$	Impulse response of linear system.
$H(x, j\omega)$	Linear system Fourier transfer function for system with variables of x and t .
j	$\sqrt{-1}$.
K'	The fluid system modulus of elasticity including wall effect; FL^{-2} .
ℓ	Length; L.
L	The coefficient of inertia of the fluid; $FL^{-6}T^2$.
n	The exponent of $\overline{q}$ for mean flow.
N_r	The Reynolds number.
ℓ	The point in a continuous pipe line where the flow source is considered to be located.
p	A dependent variable in a continuous system; the transient pressure; FL^{-2}.

$p(x,t)$ — The transient pressure at the point x and at time t ; FL^{-2}.

p_t — The instantaneous pressure at any point x and time t ; FL^{-2} .

$\bar{p}$ — The pressure which establishes the mean flow $\bar{q}$; FL^{-2}.

$\bar{p}_f$ — That part of $\bar{p}$ which is a loss due to the flow $\bar{q}$ in the line; FL^{-2}.

$P(x,s)$ — The transformed transient pressure at the point x ; $FL^{-2}T$.

$p_{rms}(x,\omega_c)$ — The rms value for p at the point x and for the discrete frequency ω_c ; FL^{-2}.

$p_{rms}(\ell,\omega_c)$ — The rms value for p at the point ℓ and for the discrete frequency ω_c ; FL^{-2} .

$\phi_{pp}(x,j\omega)$ — The spectral density function of the random pressure fluctuation at the point x .

$\phi_{pp}(\ell,j\omega)$ — The spectral density function of the random pressure fluctuation at the source of disturbance ℓ .

q — A dependent variable in a continuous system; the transient flow rate; L^3T^{-1} .

$q(x,t)$ — The transient flow rate at the point x and the time t ; L^3T^{-1} .

q_t — The instantaneous volume flow rate; L^3T^{-1} .

$\bar{q}$ — The mean flow rate; L^3T^{-1} .

$Q(x,s)$ — The transformed transient flow rate at the point x ; L^3 .

ℓ — The point in the system at which the disturbance flow occurs.

R — The coefficient of functional resistance; $FL^{-6}T$. Also the receiving end of the pipeline.

s — The Laplace transform operator; the complex transform plane; T^{-1}.

t — An independent variable; time; T .

x — An independent variable; distance from the receiving end; L .

y — An independent variable; distance from the flow source; L.

$Z(x,s)$ — The generalized impedence at any point x ; $FL^{-5}T$.

Z_c — The characteristic impedance; $FL^{-5}T$.

α — The spatial attenuation coefficient; L^{-1} .

$\alpha(\infty)$ — The limiting value of α for increasing values of ω ; L^{-1}.

β — The spatial phase coefficient in Equations (24), (25), and (26) only; L^{-1} .

$\beta(\infty)$ — The limiting function of β for increasing values of ω ; L^{-1} .

γ — The propagation coefficient; L^{-1} .

μ — The viscosity of the fluid; $FL^{-2}T$.

ρ — The density of the fluid; $FL^{-4}T$.

τ — A dummy variable of integration; T .

ω — The complex part of s ; T^{-1} .

ω_c — A specific value of ω ; T^{-1} .

ω_n — The values of ω for which the nth maximum occurs at a point in the pipeline.

INTRODUCTION

Traditionally systems have been analyzed with load inputs which are expressible as deterministic mathematical functions of time (e.g., step, ramp, sine, etc.). Such a function will be called a conventional time function. A function not expressible as a deterministic function of time is called a random time function. Random time functions are characterized by controlling physical mechanisms of such complexity as to make human prediction impossible (e.g., wind gusts, particle motion in turbulent flow, etc.).

A random physical phenomena can frequently be described statistically. Such a description, while not purporting to exactly express a random function, does state its most probable form. If the random input of a linear system has an adequate statistical description, it is possible to predict the probable output of the system. An analysis of this nature can be carried out either in the time or frequency domains.

The purpose of this report is an analytical investigation of a pipeline system when subjected to random pressure fluctuations. Three idealized terminations are considered when acted upon by two separate random inputs.

This report attempts to unite the work from two fields. It does not propose to extend either the fundamental theory of transient conditions in pipelines or the tools of analysis employed in random function theory.

CHAPTER I

HYDRAULIC TRANSIENTS IN A PIPELINE SYSTEM

1.0 General.

The discussion following comes in a large part from references [1]*, [2], and [3] by Waller. The terminology and symbols introduced in the discussion of pipelines also comes from these references. The author feels that the fundamental theory of transient flow in pipelines is well established [4, 5], and has been thoroughly confirmed by experimentation [6, 7]. The mode selected for describing the system was chosen primarily because of its simplicity and direct application to this report.

1.1 The General Pipeline System

The mathematical development for a pipeline from the continuity equation, bulk modulus considerations, and the Navier-Stokes equations leads to two simultaneous partial differential equations. Solution of these equations is complicated by the presence of a nonlinear frictional resistance term. This difficulty can be eliminated by analyzing the flow condition as a variable fluctuation about some mean flow state. To this end, define

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*Numbers in brackets refer to references in the bibliography.

$$q_t = \overline{q} + q$$
$$p_t = \overline{p} + p \;, \qquad (1)$$

where

q_t is the total instantaneous volume flow rate (ft^3/sec);

$\overline{q}$ is the steady state flow rate which would exist in the absence of disturbance;

q is the flow due to disturbance;

p_t is the total instantaneous pressure (lbs/ft^2);

$\overline{p}$ is the steady state pressure which would exist in the absence of disturbance;

p is the pressure due to disturbance.

The resulting equations then are [2]

$$L \frac{\partial q}{\partial t} + Rq + \frac{\partial p}{\partial x} = 0 \;,$$

and (2)

$$C \frac{\partial q}{\partial t} + 0 + \frac{\partial q}{\partial x} = 0$$

where L, R, and C are parameters of the pipeline, x and t are the independent variables of direction and time, and p and q, as defined in Equation (1), are the dependent variables.

The parameters are defined as

$$R = \frac{n\,\overline{p}_f}{\ell \overline{q}} \left[1 + \frac{(n-1)}{2!}\left(\frac{q}{\overline{q}}\right) + \frac{(n-1)(n-2)}{3!}\left(\frac{q}{\overline{q}}\right)^2 + \cdots \right], \qquad (3)$$

for turbulent flow;

$$R = \frac{32\mu}{AD^2}\;, \text{ for viscous flow;} \qquad (3.a)$$

$$L = \frac{\rho}{A} ; \tag{4}$$

$$C = \frac{A}{K'} \tag{5}$$

where

$\overline{p}_f$ is the portion of the mean pressure $\overline{p}$ required to overcome frictional resistance and maintain the steady state flow rate;

ℓ is the length of the pipeline;

ρ is the density of the liquid in the pipeline;

A is the cross-sectional area of the pipe; and

K' is the bulk modulus of the liquid corrected for yield of the pipe.

A relationship useful in finding C is the equation for the velocity of propagation of liquid-borne sound. This relationship is given by

$$a^2 = \frac{1}{LC} = \frac{K'}{\rho} \tag{6}$$

where a is the velocity.

Figure 1 below is definitive of the pipeline system considered in this report.

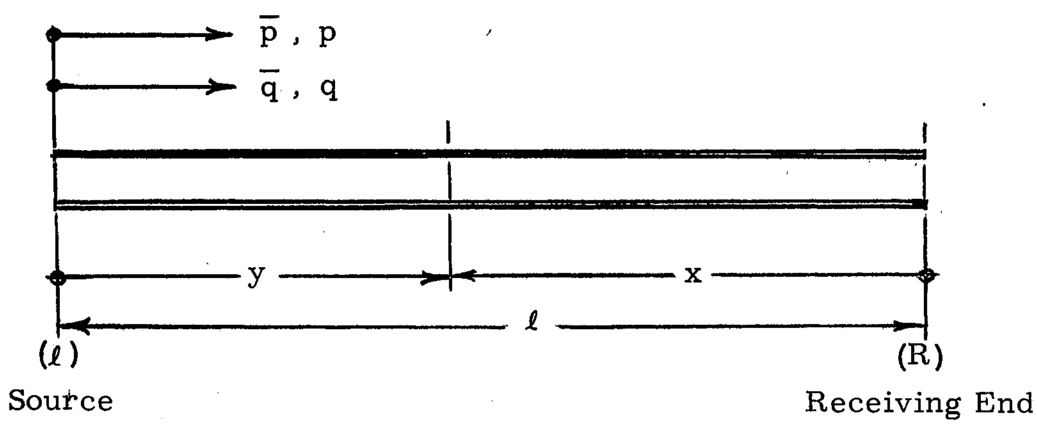


Figure 1

Schematic Diagram of the General Pipeline System

The source of mean and disturbing flow and pressure is at ℓ . The line is terminated at R , the receiving end.

It has been previously demonstrated [8] that the Laplace domain solution for Equations (2) with the medium initially relaxed (i.e., $p(x,0) = q(x,0) = 0$) is

$$P(x,s) = P(R,s)\cosh\gamma x + Z_c\,Q(R,s)\sinh\gamma x;$$
$$Q(x,s) = Q(R,s)\cosh\gamma x + \frac{P(R,s)}{Z_c}\sinh\gamma x \quad (7)$$

when referenced from the receiving end R , and

$$P(y,s) = P(\ell,s)\cosh\gamma y - Z_c\,Q(\ell,s)\sinh\gamma y;$$
$$Q(y,s) = Q(\ell,s)\cosh\gamma y - \frac{P(\ell,s)}{Z_c}\sinh\gamma y \quad (8)$$

when referenced from the source of flow ℓ . By definition,

$$\gamma^2 = (sC)(R + sL); \quad (9)$$

$$Z_c^2 = \frac{(R + sL)}{sC}. \quad (10)$$

γ will be termed the propagation coefficient, while Z_c is the characteristic impedance.

This report uses capitalization to denote a transformed relationship for the dependent variables. This may be shown by

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

for the one sided Laplace transform.

1.2 Idealized Terminations of the General Pipeline System.

Equations (7) and (8) governing the general system become much simpler for certain idealized terminations of the pipeline at R . This section will consider three such terminations.

If at $x = 0$ the line is terminated in an open end, it has been verified experimentally that $p(R, t) = 0$ [6] and, consequently, $p(R, s) = 0$. For this specialized termination Equation (7) reduces to

$$P(x, s) = Z_c Q(R, s) \sinh \gamma x ; \tag{11}$$

$$Q(x, s) = Q(R, s) \cosh \gamma x . \tag{12}$$

For $x = \ell$, in Equation (11),

$$P(\ell, s) = Z_c Q(R, s) \sinh \gamma \ell . \tag{13}$$

Substituting for $Q(R, s)$ from Equation (13) into Equation (11),

$$P(x, s) = P(\ell, s) \frac{\sinh \gamma x}{\sinh \gamma \ell} . \tag{14}$$

Equation (14) states the transformed relationship between the pressure a distance x from the receiving end R in terms of the system parameters and the distrubing pressure at the source ℓ.

If at $x = 0$ the line is terminated in a closed end, obviously $q(0, t) = 0$; consequently, $Q(0, s) = 0$. For this termination the general equations reduce to

$$P(x, s) = P(R, s) \cosh \gamma x ; \tag{15}$$

$$Q(x, s) = \frac{P(R, s)}{Z_c} \sinh \gamma x . \tag{16}$$

For $x = \ell$ in Equation (15),

$$P(\ell, s) = P(R, s) \cosh \gamma \ell . \tag{17}$$

Substituting from (17) into (15),

$$P(x, s) = P(\ell, s) \frac{\cosh \gamma x}{\cosh \gamma \ell} . \tag{18}$$

Equation (18) gives the transformed relationships between the pressure a distance x from R, in terms of the system parameters, and the disturbing pressure.

The third idealized termination is the infinitely terminated pipeline and occurs when the receiving end R is far removed from the source of disturbance. This case will be approached from the point impedence concept. Defining the point impedence of a pipeline as

$$Z(x, s) = \frac{P(x, s)}{Q(x, s)} ,$$

one obtains from Equations (7) and (8)

$$Z(x, s) = Z_c \frac{Z(R, s) + Z_c \tanh \gamma x}{Z_c + Z(R, s) \tanh \gamma x} , \tag{19}$$

and

$$Z(y, s) = Z_c \frac{Z(\ell, s) - Z_c \tanh \gamma y}{Z_c - Z(\ell, s) \tanh \gamma y} . \tag{20}$$

If x is set equal to ℓ in Equation (19),

$$Z(\ell, s) = Z_c \frac{Z(R, s) + Z_c \tanh \gamma \ell}{Z_c + Z(R, s) \tanh \gamma \ell} . \tag{21}$$

It has been shown [1] that γ, the propagation coefficient, is a complex term of the form

$$\gamma = \alpha + j\beta . \tag{22}$$

If ℓ is sufficiently large, $\sinh \alpha\ell = \cosh \alpha\ell$, and $\tanh \alpha\ell = 1$. For this condition, Equation (21) yields

$$Z(\ell, s) = Z_c .$$

Substituting into Equation (20), one obtains

$$Z(y, s) = Z_c . \tag{23}$$

This relationship is in terms of y and is not dependent on the relative distance between the disturbance and the point of consideration.

If the result of Equation (23) is substituted into Equation (8),

$$P(y, s) = P(\ell, s) \cosh \gamma y - P(\ell, s) \sinh \gamma y ,$$

or

$$P(y, s) = P(\ell, s)\, e^{-\gamma y} .$$

Finally, since $\mathbf{P}(x, s) = \mathbf{P}(y, s)$,

$$\mathbf{P}(x, s) = \mathbf{P}(\ell, s)\, e^{-\gamma y} . \tag{24}$$

The governing relationships for three physical idealizations have now been established.

Equations (14), (18), and (24) govern respectively the open, closed, and infinitely terminated pipelines. These equations state the transformed relationships between the disturbing pressure, the system parameters, and the pressure at some point in the line. Pressure fluctuation is of primary importance in pipeline systems; therefore, the equations developed are for pressure.

CHAPTER II

LINEAR SYSTEMS SUBJECTED TO RANDOM INPUTS

2.0 General.

The material following is available in several excellent sources. The author drew heavily from Newton, Gould, and Kaiser [9] in the physical interpretation of spectral density. Aseltine [10] and Truxal [11] present competent developments of the necessary derivations, as does Chang [12] .

The description of random inputs is frequently approached from the theory of probability. For the problems involved in this report, this approach is unjustified and unnecessary. For this reason, the developments following are not pursued from such a basis. Persons interested in this approach are referred to Laning and Battin [13].

This section is obviously not original with the author. It is felt, however, that the relative newness of random function theory as applied to systems justifies its inclusion within this report.

2.1 Description of Random Processes in the Time Domain.

The functional statement of a random process in time can proceed from various basic statistical concepts. Averaging processes comprise the necessary statistical theory for this report.

The time average is not unfamiliar and will be introduced first.

Consider the emission $v(t)$ from one of N identical function generators. Mathematically the time average for this function may be stated as

$$\overline{v(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(t)\, dt$$

where T is the period of time.

A somewhat more difficult concept is the ensemble average. Consider the emission $v_i(t_0)$ of all N function generators at a given time t_0. The ensemble average of the emission of the generators can be defined as

$$\widetilde{v(t)} = \frac{v_1(t_0) + v_2(t_0) + \cdots + v_i(t_0) + \cdots + v_N(t_0)}{N} .$$

The ensemble average is often called the statistical average.

Statistical functions may be roughly classified as stationary or nonstationary. A stationary random function may be defined as a random function whose statistical description does not vary with time. For a physical phenomena this would mean that the complex physical mechanism controlling the function generators was time invariant.

If the output of identical function generators is controlled by the same underlying mechanism, and the output is stationary, the ergodic hypothesis states that the time averaged emission for a single function generator, considered over all time, would be equal to the ensemble average of an infinite number of like function generators at a discrete time t_0. From intuition, this hypothesis would appear reasonable for most physical phenomena. An ergodic process may then be defined as a random stationary process whose time average equals its ensemble average. All ergodic processes are stationary, but all stationary processes are not necessarily ergodic.

Random processes in the time domain are described by the correlation function. The correlation function describing two random time functions $v(t)$ and $u(t)$ is defined by

$$\phi_{vu}(\tau) = \overline{v(t)\ u(t+\tau)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(t)\, u(t+\tau)\, dt \qquad (25)$$

and is called the cross-correlation function of $v(t)$ and $u(t)$. The correlation function for $v(t)$ with itself is defined similarly as

$$\phi_{vv}(\tau) = \overline{v(t)\, v(t+\tau)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(t)\, v(t+\tau)\, dt\ , \qquad (26)$$

and is called the autocorrelation function of $v(t)$. This report will concern itself only with the autocorrelation function. It should be noted that if $\tau = 0$,

$$\phi_{vv}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(t)^2\, dt \qquad (27)$$

which is the mean square value of $v(t)$. It can be shown that 10 the autocorrelation function is an even function of τ and is maximum for $\tau = 0$. For $v(t)$ an ergodic process,

$$\phi_{vv}(\tau) = \overline{v(t)\, v(t+\tau)} = \widetilde{v(t)\, v(t+\tau)}\ .$$

The significance of the autocorrelation function is that it serves as the mathematical statement of a random function in time. The correlation function of a random input can be used for analysis operations, by means of the convolution integral, in exactly the same fashion as conventional time functions.

2.2 Description of a Random Process in the Frequency Domain

The previous section developed the correlation function as the describing relationship for random functions in time. As would be expected, a corresponding relationship in the frequency domain is obtained from

$$\Phi_{VV}(s) = \int_{-\infty}^{\infty} \phi_{VV}(\tau)\, e^{-s\tau}\, d\tau . \tag{28}$$

This relationship defines the spectral density function $\Phi_{VV}(s)$ as the two sided Laplace transform of the autocorrelation function $\phi_{VV}(\tau)$. The spectral density function is significant in that it serves as the mathematical statement of a random function in the frequency domain.

The inverse relationship also holds. Thus,

$$\phi_{VV}(\tau) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \Phi_{VV}(s)\, e^{st}\, ds \quad . \tag{29}$$

If s is replaced by $j\omega$, the Fourier inversion integral results;

$$\phi_{VV}(\tau) = \frac{1}{2\pi} \int_{-j\infty}^{j\infty} \Phi_{VV}(j\omega)\, e^{j\omega\tau}\, d\omega \quad . \tag{30}$$

When $\tau = 0$,

$$\phi_{VV}(0) = \frac{1}{2\pi} \int_{-j\infty}^{j\infty} \Phi_{VV}(j\omega)\, d\omega \quad , \tag{31}$$

but

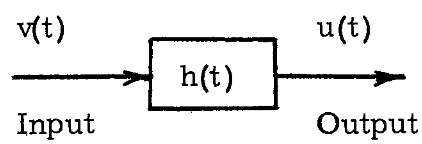
$$\phi_{VV}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} v(t)^2\, dt ,$$

the mean square value of the function $v(t)$. This indicates that the spectral density function can be considered a mean-square-value density spectrum. The mean square value per unit frequency for $v(t)$ is found [9] for a discrete frequency ω_c by dividing the spectral density ϕ_{vv} by 2π. The author recommends reference [14] by Cheng for readers desiring additional insight into frequency spectrum distributions.

In summary, the spectral density function is the mathematical statement of a random time function in the frequency domain. It can be employed in a manner very similar to the Laplace transform of a conventional time function.

2.3 Analysis of Linear Systems by Correlation Functions.

A linear system with a random input may be represented by



where $h(t)$, the weighting function, is the time response of the system due to a unit impulse input. $v(t)$ and $u(t)$ are random time functions. The block diagram above represents, from the convolution integral, the relationship

$$u(t) = \int_{-\infty}^{\infty} h(t_1)\, v(t - t_1)\, dt_1 \qquad (32)$$

or

$$u(t + \tau) = \int_{-\infty}^{\infty} h(t_2)\, v(t + \tau - t_2)\, dt_2 \quad . \qquad (33)$$

From the previous definition of the correlation function,

$$\phi_{uu}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, u(t + \tau)\, dt \quad . \qquad (34)$$

Substitution from Equation (32) and (33) for $u(t)$ and $u(t + \tau)$ yields

$$\phi_{uu}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\int_{-\infty}^{\infty} h(t_1)\, v(t - t_1)\, dt_1 \right]\left[\int_{-\infty}^{\infty} h(t_2) v(t + \tau - t_2) dt_2 \right] dt \,. \tag{35}$$

If the sequence of integration is altered and integration is performed first with respect to time,

$$\phi_{uu}(\tau) = \left[\int_{-\infty}^{\infty} h(t_1)\, dt_1 \right]\left[\int_{-\infty}^{\infty} h(t_2)\, dt_2 \right] \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t - t_1)\, v(t + \tau - t_2)\, dt \quad . \tag{36}$$

By definition, however,

$$\phi_{vv}(\tau + t_1 - t_2) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t - t_1)\, v\left[(t - t_1) + (\tau + t_1 - t_2) \right] dt \,. \tag{37}$$

Substituting from Equation (37) into (36),

$$\phi_{uu}(\tau) = \int_{-\infty}^{\infty} h(t_1)\, dt_1 \int_{-\infty}^{\infty} h(t_2)\, \phi_{vv}(\tau + t_1 - t_2)\, dt_2 \quad . \tag{38}$$

Equation (38) above is a mathematical statement predicting the output autocorrelation function of a linear system as a function of the unit impulse response of the system and the autocorrelation function of the input. This relationship holds for any linear system.

2.4 Analysis of Linear Systems by Spectral Density.

The mode of analysis for a random function has been developed for the time domain in Section 2.3. As with conventional functions, it is frequently more convenient to analyze random functions in the Laplace domain.

If both sides of Equation (38) are transformed by the two sided Laplace operation,

$$\int_{-\infty}^{+\infty}\phi_{uu}(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} h(t1)dt_1 \int_{-\infty}^{+\infty} h(t_2)\phi_{vv}(\tau + t_1 - t_2)dt_2\right]e^{-s\tau}d\tau \quad (39)$$

Equation (39) can be altered by changing the order of integration and manipulating the exponential term to the equivalent expression.

$$\int_{-\infty}^{+\infty}\phi_{uu}(\tau)e^{-s\tau} = \left[\int_{-\infty}^{+\infty} h(t_1)e^{st_1}\, dt_1\right]\left[\int_{-\infty}^{+\infty} h(t_2)e^{-st_2}\right] \left[\int_{-\infty}^{+\infty} \phi_{vv}(\tau + t_1 - t_2)\, e^{-s(\tau+t_1-t_2)}\, dt_2\right] . \quad (40)$$

From Equation (28) and the fundamental definition of the two sided Laplace transformation, Equation (40) is seen to be equivalent to the relationship

$$\phi_{uu}(s) = H(s)\, H(-s)\, \phi_{vv}(s) \quad (41)$$

If s is replaced by $j\omega$ in Equation (41), the Fourier solution is obtained;

$$\phi_{uu}(j\omega) = H(j\omega)\, H(-j\omega)\, \phi_{vv}(j\omega) = |H(j\omega)|^2\, \phi_{vv}(j\omega) . \quad (42)$$

Equation (41) is the fundamental relationship used to analyze random time functions. It states the frequency domain relationship between the spectral density $\phi_{vv}(s)$ of the random input function v, the system transfer function $H(s)$ which is the transform of the system weighting function $h(t)$, and the spectral density $\phi_{uu}(s)$ of the system output u. Equation (41) corresponds to the conventional time relationship; $U(s) = H(s)\, V(s)$. Equations (41) and (42) are basic and applicable to any linear system.

It was shown in Section 2.2 that spectral density is a mean-square-value frequency spectrum distribution. This is the only information available from frequency domain analysis. This analysis gives results for the steady state only, and reveals nothing of the transient conditions.

CHAPTER III

SPECTRAL DENSITY ANALYSIS OF PIPELINES WITH IDEALIZED TERMINATIONS

3.0 General.

The two preceding sections have reviewed pertinent topics in the fields of transient pipeline flow and basic random time function theory. The following sections will employ portions of both of these fields. Basic equations governing the specially terminated pipelines will be evolved which will govern these systems when subjected to random inputs. The analysis will proceed from a spectral density approach as opposed to the correlation function approach.

3.1 Open End Termination.

From Equation (11), the governing relationship for the open end line is

$$P(x, s) = P(\ell, s) \frac{\sinh \gamma(s)x}{\sinh \gamma(s)\ell}$$

where $[\sinh \gamma(s)x / \sinh \gamma(s)\ell]$ is the system transfer function $H(s)$. Considering the discussion of spectral density analysis and Equation (42),

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega) \frac{\sinh \gamma(j\omega)x \ \sinh \gamma(-j\omega)x}{\sinh \gamma(j\omega)\ell \ \sinh \gamma(-j\omega)\ell} . \qquad (43)$$

This relationship may be considerably simplified by operation on the system parameters $\gamma(j\omega)$ and $\gamma(-j\omega)$. From Equation (9),

$$\gamma(j\omega) = \left[j\omega C(j\omega L + R)\right]^{1/2} ;$$

or

$$\gamma(j\omega) = \left[-CL\omega^2 + jCR\omega\right]^{1/2} .$$

From fundamental complex variable theory,

$$- CL\omega^2 + jCR\omega = Me^{j\theta}$$

where

$$M = C\omega \left[L^2\omega^2 + R^2\right]^{1/2}$$

and

$$\theta = \tan^{-1}\left(\frac{R}{-L\omega}\right) . \qquad (44)$$

The parameter $\gamma(j\omega)$ is now defined by the relationship,

$$\gamma(j\omega) = M^{1/2}\left[\cos\left(\frac{\theta + 2K\pi}{2}\right) + j \sin\left(\frac{\theta + 2K\pi}{2}\right)\right]$$

where $K = 0, 1$. The resultant two values of $\gamma(j\omega)$ are

$$\gamma(j\omega)_0 = - \gamma(j\omega)_1 = M^{1/2}\left[\cos\left(\frac{\theta}{2}\right) + j \sin\left(\frac{\theta}{2}\right)\right]$$

where the subscripts refer to the respective values of K . A similar development on the function $\gamma(-j\omega)$ yields

$$\gamma(-j\omega)_0 = - \gamma(- j\omega)_1 = M^{1/2}\left[\cos\left(\frac{\theta}{2}\right) - j \sin\left(\frac{\theta}{2}\right)\right] .$$

This report will denote

$$\alpha = M^{1/2} \cos\left(\frac{\theta}{2}\right) , \qquad (45)$$

and

$$\beta = M^{1/2} \sin\left(\frac{\theta}{2}\right) . \qquad (46)$$

Operation on Equation (44) yields

$$\cos\theta = \frac{- L\omega}{\sqrt{R^2 + L^2\omega^2}}$$

which when introduced into the trigonometric relationships for $\cos(\theta/2)$ and $\sin(\theta/2)$ results in

$$\cos\frac{\theta}{2} = \left[\frac{\sqrt{L^2\omega^2 + R^2} - L\omega}{2\sqrt{L^2\omega^2 + R^2}}\right]^{1/2} ;$$

$$\sin\frac{\theta}{2} = \left[\frac{\sqrt{L^2\omega^2 + R^2} + L\omega}{2\sqrt{L^2\omega^2 + R^2}}\right]^{1/2} .$$

This yields from Equations (45) and (46)

$$\alpha = \sqrt{\frac{C\omega}{2}}\left[\sqrt{L^2\omega^2 + R^2} - L\omega\right]^{1/2} ; \tag{47}$$

$$\beta = \sqrt{\frac{C\omega}{2}}\left[\sqrt{L^2\omega^2 + R^2} + L\omega\right]^{1/2} . \tag{48}$$

The functions $\gamma(j\omega)$ and $\gamma(-j\omega)$ can then be stated as

$$\gamma(j\omega) = \pm(\alpha + j\beta) ; \tag{49}$$

$$\gamma(-j\omega) = \pm(\alpha - j\beta) \tag{50}$$

with α and β defined by Equations (47) and (48). The negative sign can be dropped* as it would never occur in a passive physical system. Equation (45) may now be stated as

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega)\frac{\sinh(\alpha + j\beta)x \sinh(\alpha - j\beta)x}{\sinh(\alpha + j\beta)\ell \sinh(\alpha - j\beta)\ell} . \tag{51}$$

This equation can be further simplified by now considering the term

$$\sinh(\alpha + j\beta)\sinh(\alpha - j\beta) .$$

From the hyperbolic identities,

$$\sinh(\alpha + j\beta) = \sinh\alpha\cos\beta + j\cosh\alpha\sin\beta$$

and

$$\sinh(\alpha - j\beta) = \sinh\alpha\cos\beta - j\cosh\alpha\sin\beta ,$$

---

*See Section (3.3) for confirmation of this statement.

$$\sinh(\alpha + j\beta)\sinh(\alpha - j\beta) = \left[\frac{\cosh 2\alpha - \cos 2\beta}{2}\right]. \tag{52}$$

The substitution of Equation (52) into Equation (51) results in

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega)\left[\frac{\cosh 2\alpha x - \cos 2\beta x}{\cosh 2\alpha\ell - \cos 2\beta\ell}\right]. \tag{53}$$

This equation will be employed in the solution of the example problems.

3.2 Closed End Termination.

The relationship governing the closed end pipeline termination, from Equation (15), is

$$P(x, s) = P(\ell, s)\frac{\cosh \gamma x}{\cosh \gamma \ell} .$$

This system when subjected to random inputs is described by

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega)\left[\frac{\cosh \gamma(j\omega)x \cosh \gamma(-j\omega)x}{\cosh \gamma(j\omega)\ell \cosh \gamma(-j\omega)\ell}\right]. \tag{54}$$

Substitutions from Equation (49) and (50) into Equation (54) yield

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega)\left[\frac{\cosh(\alpha + j\beta)x \cosh(\alpha - j\beta)x}{\cosh(\alpha + j\beta)\ell \cosh(\alpha - j\beta)\ell}\right] . \tag{55}$$

By a development very much similar to the steps previously employed in simplifying the term $\sinh(\alpha + j\beta)\sinh(\alpha - j\beta)$ it can be shown that

$$\cosh(\alpha + j\beta)\cosh(\alpha - j\beta) = \left[\frac{\cosh 2\alpha + \cos 2\beta}{2}\right]. \tag{56}$$

Equation (56) when substituted into Equation (55) results in

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega)\left[\frac{\cosh 2\alpha x + \cos 2\beta x}{\cosh 2\alpha\ell + \cos 2\beta\ell}\right] . \tag{57}$$

Equation (57) will be used in solving example problems to follow.

3.3 Infinitely Terminated Pipeline.

The describing mathematical relationship governing the infinitely terminated pipeline is, from Equation (24),

$$P(x, s) = P(\ell, s)\, e^{-\gamma y} . \tag{58}$$

For $P(\ell, s)$ a random function

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega)\, e^{-y[\gamma(j\omega) + \gamma(-j\omega)]} .$$

Substituting from Equation (49) and (50),

$$\phi_{pp}(x, j\omega) = \phi_{pp}(\ell, j\omega) e^{-2\alpha y} . \tag{59}$$

This simplified relationship will be employed in the solution of example problems.

The sign of the exponent in the above equation illustrates why the negative sign was dropped in Equations (49) and (50). A negative α would result in a positive exponent, indicating that pressure is an increasing exponential function of y . This would obviously be incorrect since the pipe is a passive system and tends to damp the input pressure rather than amplifying it.

CHAPTER IV

PROBLEM PRESENTATIONS

4.0 General.

This chapter contains selected examples of specially terminated pipelines designed to illustrate the equations developed in Chapter III. Idealized inputs to be used in solution of the examples are also presented.

Unfortunately, an adequate solution of a single pipeline entails a large number of laborious numerical operations. In addition, due to the small numbers and diverse operations involved, it becomes difficult to obtain a measure of accuracy that is both reasonable and convenient. These factors combine to make a computer solution desirable.

R , the coefficient of resistance, L , the coefficient of inertia, and C , the coefficient of capacitance, are required for a computer solution as is ℓ , the length of the pipeline. The parameter R is subject to large fluctuation and is discussed extensively by Waller [6] . Recent discussion of L and C may be found in [15] by Parmakian.

4.1 Spectral Distributions for $\phi_{pp}(\ell, j\omega)$.

Two distributions will be considered in this section and will serve as inputs to the specially terminated pipelines. As has been previously stated, the inputs are purposely idealized in an attempt to illustrate the nature of the system response.

The limited nature of this report precludes any discussion of means for actually computing spectral densities from a trace or recording of a random function. A discussion of this nature can be found in Chang [12]. The following discussion then assumes that by some means or another the spectral density for the input pressure has been obtained.

The first input considered is a clipped white, or completely random, distribution. This input, while physically improbable, is frequently used for illustrative purposes. It should clearly show the response of the system to the frequencies considered. Figure 2 below defines this input.

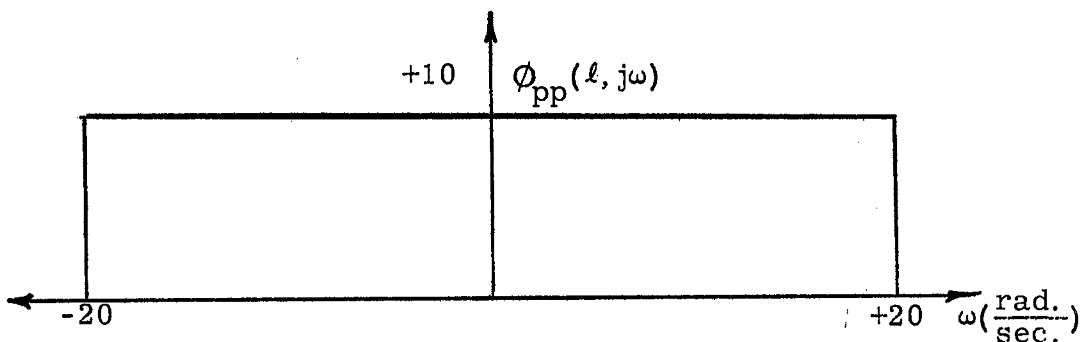


Figure 2

Input 1

The second input is Gaussian, a frequently encountered physical distribution. The governing relationship for this input is

$$\phi_{pp}(\ell, j\omega) = 10\, e^{-\frac{\omega^2}{100}}$$

and is shown in Figure 3.

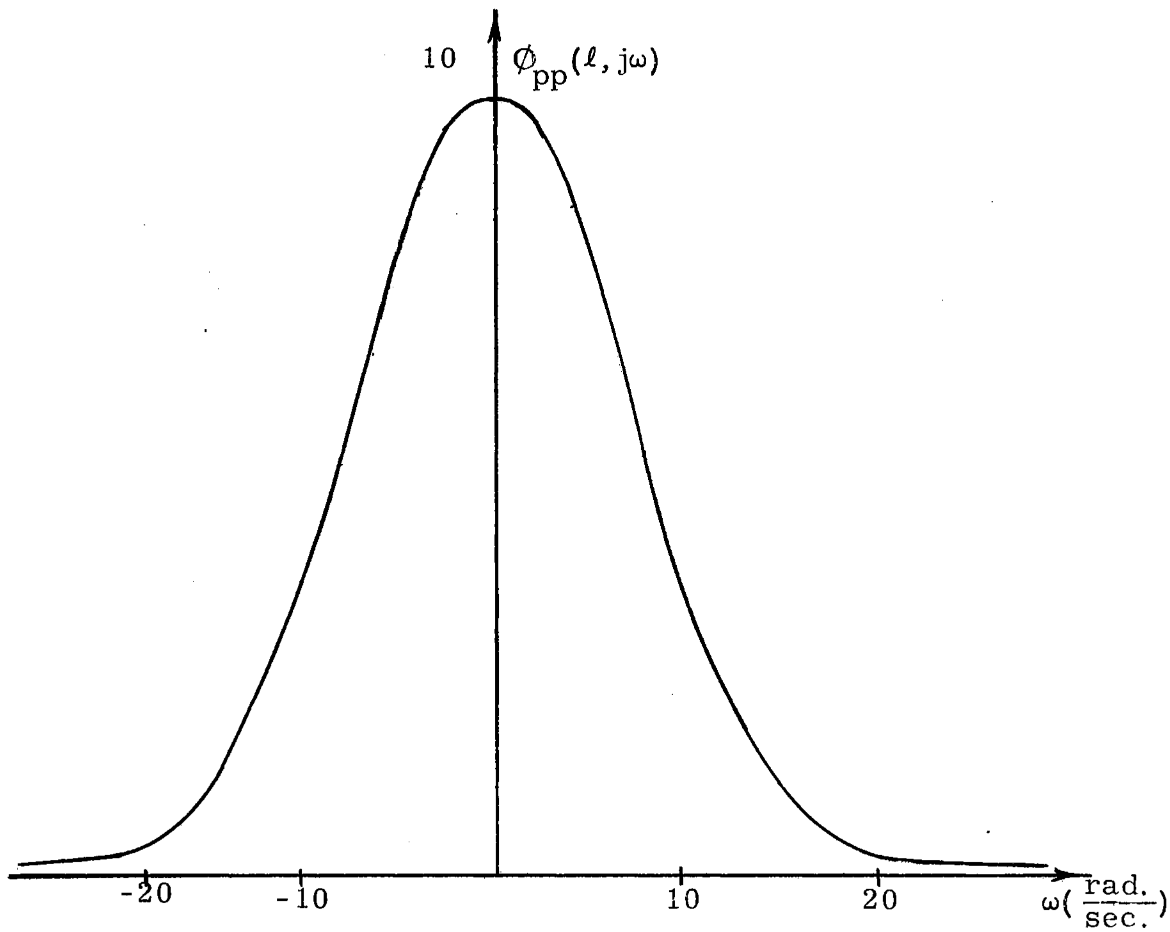


Figure 3

Input 2

4.2 An Open End Pipeline.

Consider an open end commercial steel pipeline carrying water at 70° F. The following information is known:

$\bar{q}$ = 0.5 cfs.;

D = 3 in.;

μ = 2.05×10^{-5} lb. sec. ft.$^{-2}$;

ρ = 1.936 slug ft.$^{-3}$;

ℓ = 2000 ft.

Rather than become involved in additional parameters, the acoustical velocity, a , will be set equal to 4000 feet per second for all three pipelines, although this would not be so. The preceeding information is sufficient to calculate the parameters L and C . The additional information following is required to compute R :

1) the Reynolds number, $N_r = 241,000$;

2) the Darcy resistance coefficient , $f = 0.018$;

3) $n = 1.80$

Equation (3), (4), and (6) plus the known value of ℓ give, respectively,

$$R = 26.7 \text{ lb. sec. ft.}^{-6};$$

$$L = 39.4 \text{ lb. sec.}^{2} \text{ ft.}^{-4};$$

$$C = 15.85 \times 10^{-10} \text{ ft.}^{4} \text{ lb.}^{-1};$$

$$\ell = 2000 \text{ ft.}$$

R is obtained as a first term approximation of the series in Equation (3). Since $\bar{q} \gg q$, this would appear to be a good approximation and has been experimentally verified [6] to be so.

4.3 A Closed-End Pipeline.

Consider a closed end commercial steel pipeline containing water at 70° F. The following information is given:

$$\bar{q} = 0.0 \text{ ft.}^{3} \text{ sec.}^{-1};$$

$$D = 3 \text{ in.};$$

$$\mu = 2.05 \times 10^{-5} \text{ lb. sec. ft.}^{-2};$$

$$\rho = 1.936 \text{ slugs ft.}^{-3};$$

$$\ell = 2000 \text{ ft.};$$

$$a = 4000 \text{ ft. sec.}^{-1}.$$

This comprises sufficient information to calculate the parameters R , L , and C . Thus; from Equations (3.a), (4), and (6) and the known value of ℓ , one obtains, respectively,

$$R = 0.215 \text{ lb. sec. ft.}^{-6} ;$$

$$L = 39.4 \text{ lb. sec.}^{2} \text{ ft.}^{-4} ;$$

$$C = 15.85 \times 10^{-10} \text{ ft.}^{4} \text{ lb.}^{-1} ;$$

$$\ell = 2000 \text{ ft.}$$

Equation (3.a) was used in computing R since for the closed end pipeline laminar flow is assumed to exist.

4.4 An Infinitely Terminated Pipeline.

Consider a pipeline carrying water at 70° F. The following information is known:

$$\bar{q} = 2 \text{ cfs.} ;$$

$$D = 10 \text{ in.} ;$$

$$\mu = 2.05 \times 10^{-5} \text{ lb. sec. ft.}^{-2} ;$$

$$\rho = 1.936 \text{ slugs ft.}^{-3} ;$$

$$\ell = 50 \text{ mi.} = 264,000 \text{ ft.} ;$$

$$a = 4000 \text{ ft. sec.}^{-1} .$$

Additional information required to calculate the parameter R is:

1) the Reynolds number, $N_r = 289,000$;

2) the Darcy resistance coefficient, $f = 0.018$;

3) $n = 1.80$.

Equations (3), (4), and (6) plus the known value of ℓ give, respectively,

$$R = 0.228 \text{ lb. sec. ft.}^{-6} ;$$

$$L = 3.55 \text{ lb. sec.}^{2} \text{ ft.}^{-4} ;$$

$$C = 1.76 \times 10^{-8} \text{ ft.}^{4} \text{ lb.}^{-1} ;$$

$$\ell = 264,000 \text{ ft.}$$

4.5 Comparative Example Closed End Pipeline.

One of the primary aims of this report is a comparison of the solutions of two pipelines which are identical except for termination differences. A comparison of this nature for the open and closed terminations is impossible since the closed end pipeline of Section 4.3 has a different R from the open end pipeline of Section 4.2. To remedy this, a closed end pipeline is proposed which has identical parameters R , L , C and ℓ as does the open end pipeline of Section 4.2. Such a pipeline would be very difficult to realize physically.

CHAPTER V

PROBLEM SOLUTIONS

5.0 General.

There are two general types of problem solutions in this report. The first type consists of obtaining the spectral density at a given point in one of the specially terminated pipelines when subjected to one of the inputs. Consider such a solution, namely, the open end pipeline with Input 1 at $x = 0.5\ell$. Section 4.2 contains the required pipeline data. Equations (47) and (48) give, respectively, α and β, while Equation (53) provides the defining relationship for the open end pipeline. If the known data is substituted into Equation (53), the relationship reduces to

$$\phi_{pp}(0.5\ell, j\omega) = \phi_{pp}(\ell, j\omega)\, F(\omega) .$$

The solution proceeds by substituting a sequence of values for ω into the right side of this equation and obtaining their corresponding values of $\phi_{pp}(0.5\ell, j\omega)$. For this particular case these values constitute a solution for the first type.

A second type of solution requires the calculation of values for $\phi_{pp}(x, \omega_c)$ at characteristic points along the pipeline for a single frequency ω_c. The results of this calculation may be considered a mean squared density distribution in x for a constant value of ω. This is analogous to the preceding type of solution where a mean square distribution in ω was calculated with x held constant. For the preceding pipeline a solution of this nature might be the calculation of $\phi_{pp}(x, \omega_c)$ at x equal to $.1\ell$, $.2\ell$, etc. for $\omega_c = 6$ rad./sec. Numerical computation is the same for both types of solutions.

Selected solutions of both types are in the figures of the following section. It was stated in Section 2.2 that the mean square frequency distribution (per unit frequency)* is the square root of this quotient. Ordinates of the figures in this section are normalized rms values.

5.1 Computer Solution.

The computer program on the following page illustrates the general nature of an I.B.M. 650 Fortran program for a partial pipeline solution. The program is presented for illustrative purposes only and does not necessarily represent the optimum program. The nature of the program is such that on the indices J, M, and N the computer selects, respectively, the pipeline, the input and the point of consideration (value of x) in the pipeline. The index I causes a sequential increase in ω.

The order of operation is such that the computer selects the pipeline by reading the parameters R, L, C, and ℓ. It then selects the input and value for x and proceeds with a cyclic problem solution by computing frequency spectrums for each point considered.

5.2 Solution Presentation.

Ordinates in the following figures are normalized rms values of p for a given frequency. Normalization was carried out by dividing

*This is to be understood throughout the remainder of this report.

```
C 0000 0 COMPUTER SOLUTION
C 0000 0 IN THIS PROGRAM LENGTH =EL
C 0000 0 L=D,ALPHA=A,BETA=B,
C 0000 0 INPUTS=PHIN,OUTPUTS=PHIO
     1 0 DIMENSION A(3,20),B(3,20),
     1 1 PHIN(3,20),PHIO(10,20),S1(10,
     1 2 20),S2(10,20)
   101 0 ZERO=0.0
     2 0 DO 5 I=1,20
     3 0 W=I
     4 0 PHIN(1,I)=10.
     5 0 PHIN(2,I)=10.*EXPEF(-(W**2.)/
     5 1 100.)
     7 0 DO 36 J=1,3
     8 0 READ,R,EL,C,D
     9 0 DO 34 M=1,3
    10 0 DO 30 N=1,10
    11 0 DO 30 I=1,20
   111 0 W=I
    12 0 IF (M-1) 13,13,24
    13 0 P=N-1
    14 0 IF (N-1) 15,15,17
    15 0 A(J,I)=W*((C*D/2.)*(((1.+(R/(
    15 1 D*W))**2.)**.5)-1.))**.5
    16 0 B(J,I)=W*((C*D/2.)*(((1.+(R/(
    16 1 D*W))**2.)**.5)+1.))**.5
    17 0 Q=COSHF(A(J,I)*P*EL/5.)
    18 0 X=COSF(B(J,I)*P*EL/5.)
    19 0 Y=COSHF(2.*A(J,I)*EL)
    20 0 Z=COSF(2.*B(J,I)*EL)
    21 0 S1(N,I)=(Q-X)/(Y-Z)
    22 0 S2(N,I)=(Q+X)/(Y+Z)
    24 0 GO TO (25,27,29),J
    25 0 PHIO(N,I)=PHIN(M,I)*S1(N,I)
    26 0 GO TO 30
    27 0 PHIO(N,I)=PHIN(M,I)*S2(N,I)
    28 0 GO TO 30
    29 0 PHIO(N,I)=PHIN(M,I)*EXPEF(-(A(
    30 0 CONTINUE
    31 0 DO 32 I=1,20
    32 0 DUMMY=PRNTF(PHIO(1,I),PHIO(2,
    32 1 I),PHIO(3,I),PHIO(4,I),PHIO(5
    32 2 ,I),PHIO(6,I),PHIO(7,I),PHIO(
    32 3 8,I),PHIO(9,I),ZERO)
    33 0 DO 34 I=1,20
    34 0 DUMMY=PRNTF(PHIO(10,I),ZERO,
    34 1 PHIN(M,I),ZERO,ZERO,ZERO,ZERO,
    34 2 ZERO,ZERO,ZERO)
    35 0 DO 36 I=1,20
    36 0 DUMMY=PRNTF(A(J,I),B(J,I),
    36 1 ZERO,ZERO,ZERO,ZERO,ZERO,
    36 2 ZERO,ZERO,ZERO)
    37 0 END
```

Figure 4

Fortran Program for Pipeline Solution

all rms values by the input rms value, i.e., by $p_{rms}(\ell)$, the rms value for p at the source (ℓ).

Figures 5, 6, 7 and 8 are plots of computed values obtained in partial solution of the pipelines presented in Sections 4.1 and 4.5. These pipelines differ only in termination, being respectively open and closed.

Figure 9 illustrates the marked effect that the parameter R has on the magnitude of the solution. It is a normalized p_{rms} frequency distribution at $x = 0$ for the pipeline presented in Section 4.2. Except for different values of R, the closed end pipelines introduced in Sections (4.2) and (4.5) are identical.

Figure 10 illustrates the effect of a variation in system inputs and is for the pipeline presented in Section 4.1 when subjected to Input 2. Figure 6 shows the same situation for Input 1. The figure shows the normalized p_{rms} frequency distribution at $x = 0$.

Figure 11 illustrates the form of the solution for the infinitely terminated pipeline presented in Section 4.3. The open and closed end pipelines of Sections (4.2) and (4.5) are also shown for comparison of the three terminations at the same input frequency of 8 rad./sec. The infinitely terminated pipeline shown in Figure 11, although idealized, would be physically realizable. This would be true also for the pipelines in Figures (9) and (5).

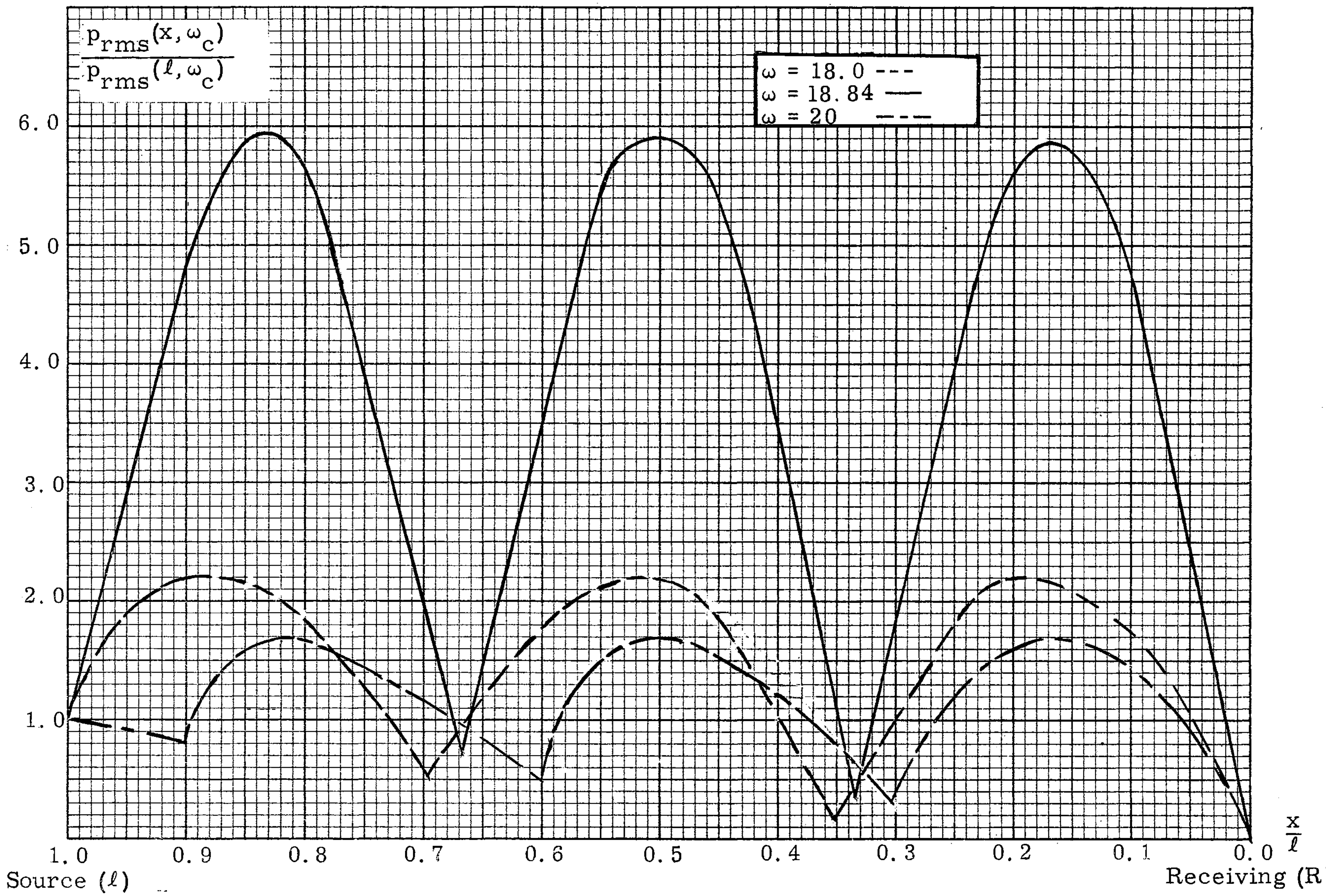


Figure 5

Normalized Distance Distribution for Open End Pipeline

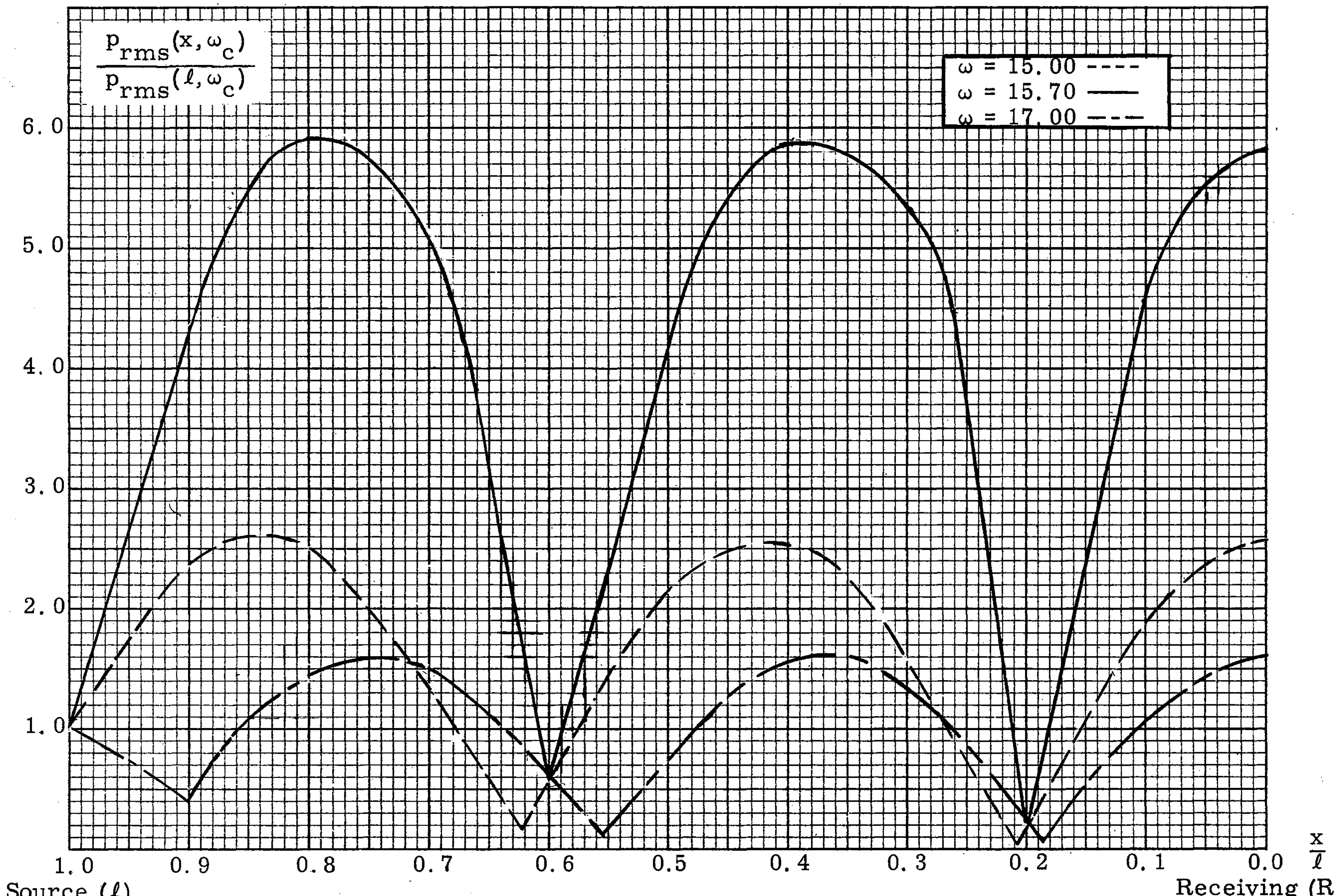


Figure 6

Normalized Distance Distribution for Closed End Pipeline

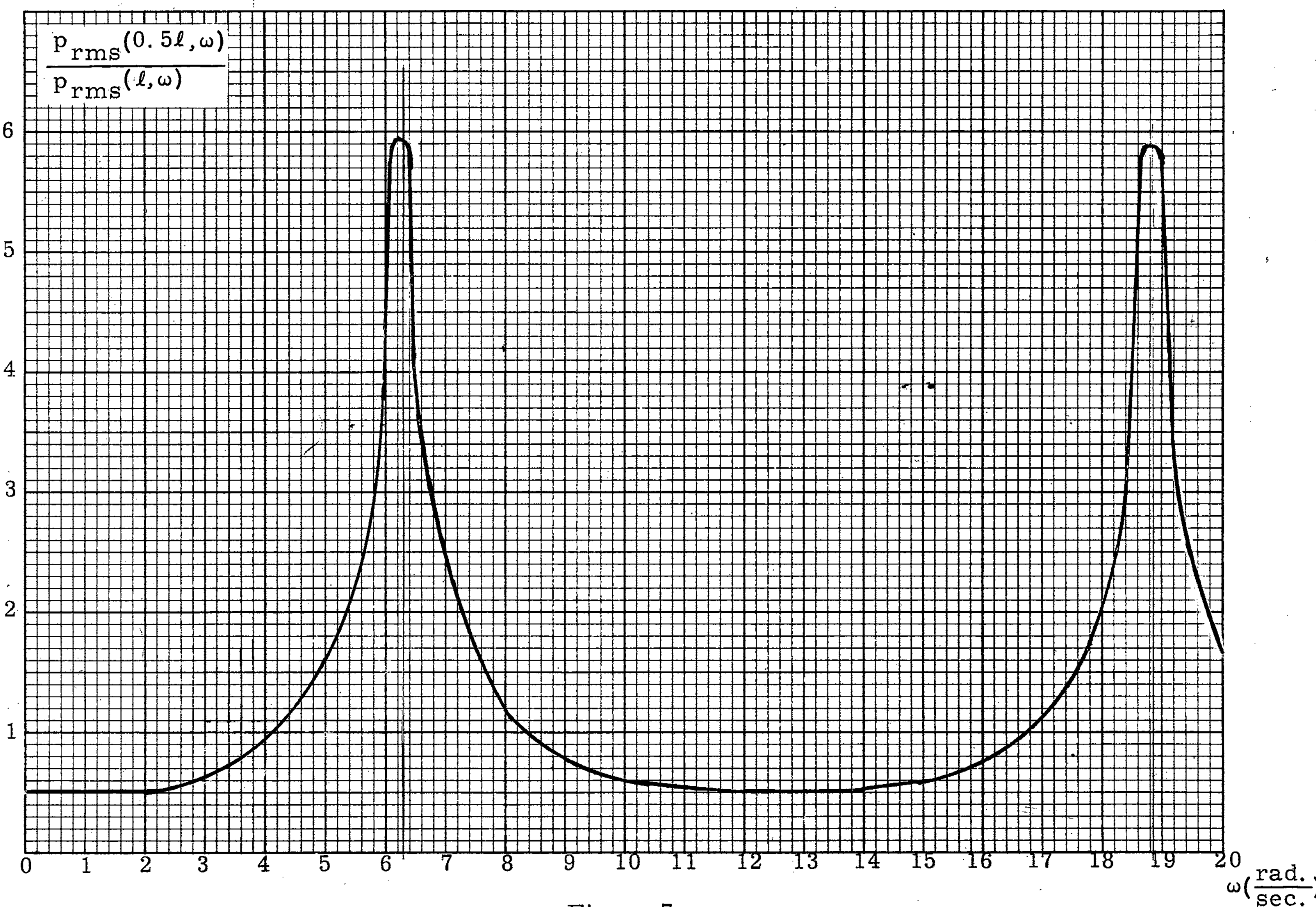


Figure 7

Normalized Frequency Distribution for Open End Pipeline at $x = 0.5\ell$

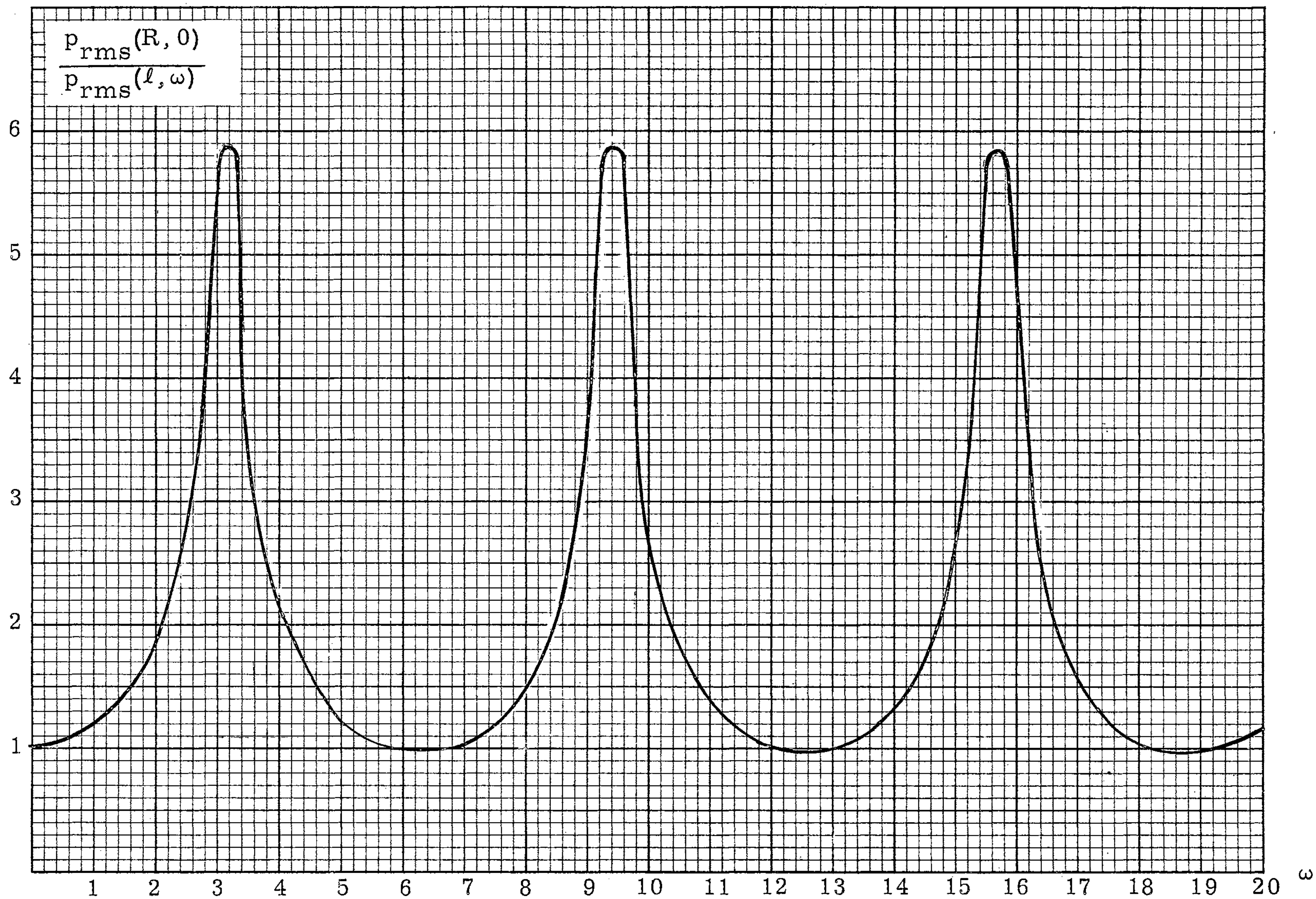


Figure 8

Normalized Frequency Distribution for Closed End Pipeline at x = 0

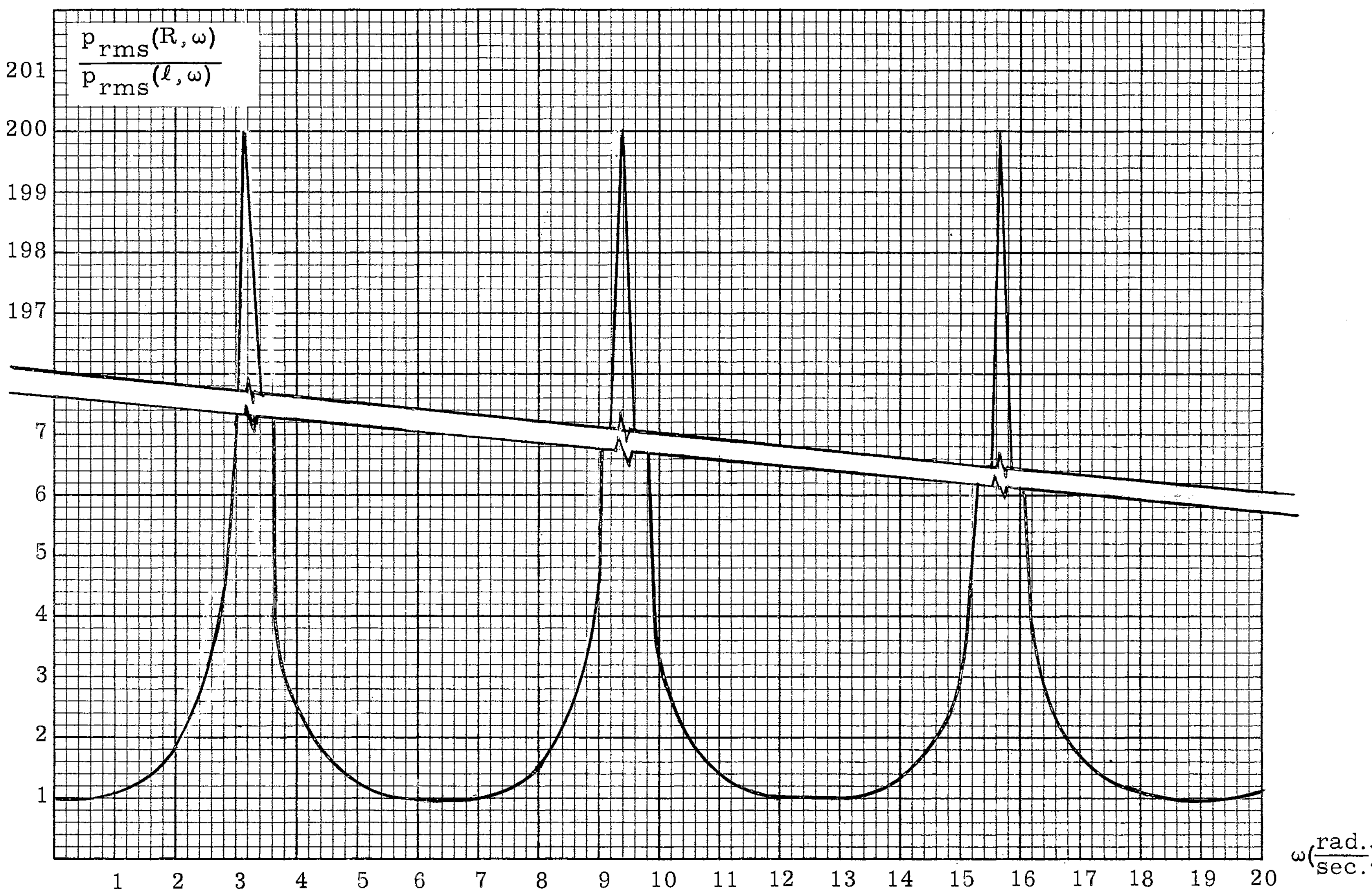


Figure 9

Normalized Frequency Distribution for Closed End Pipeline at $x = 0$, $R = 0.215$ lb. sec. ft.$^{-6}$

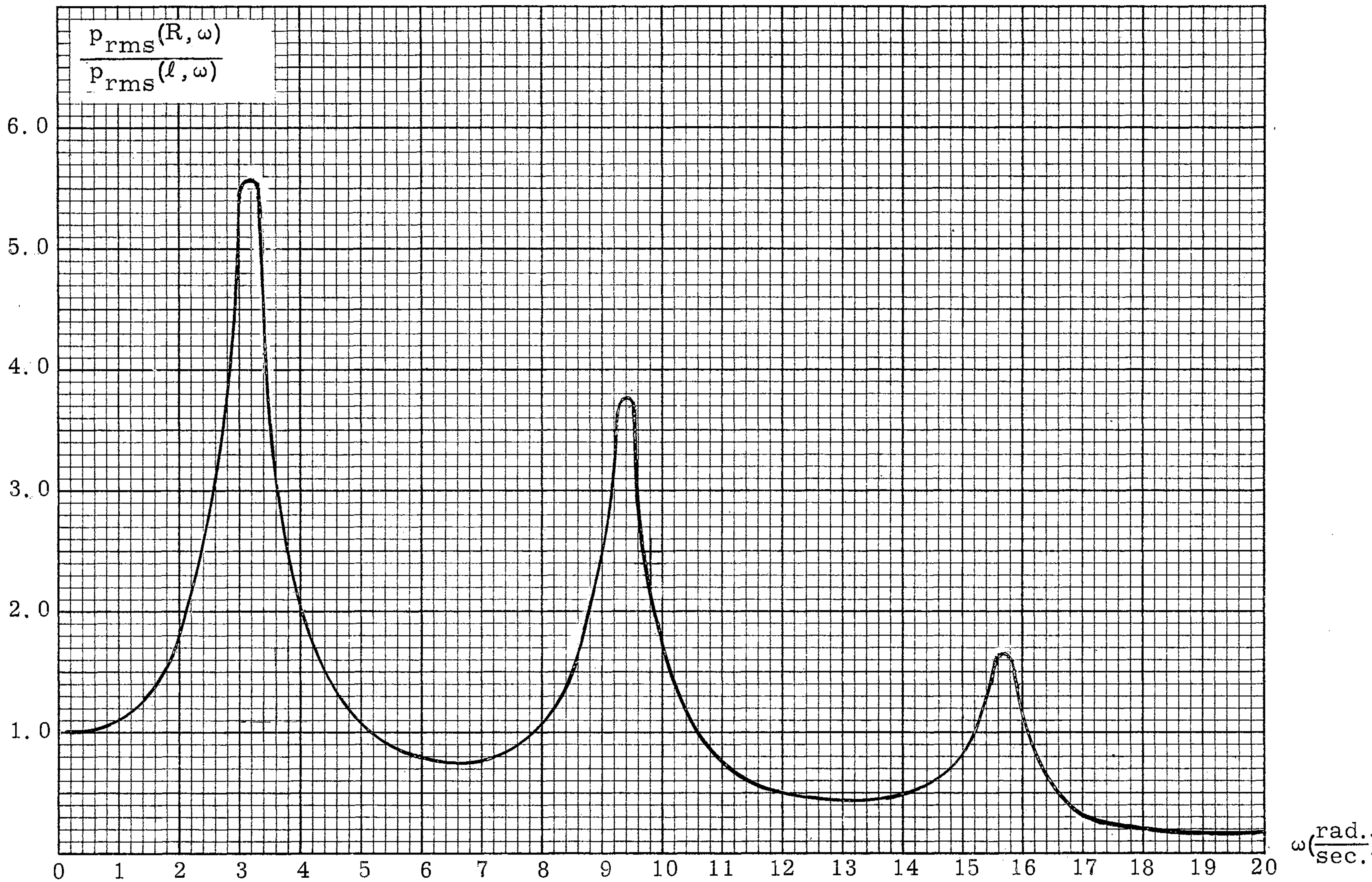


Figure 10

Normalized Frequency Spectrum for Closed End Pipeline at x = 0 for Input 2

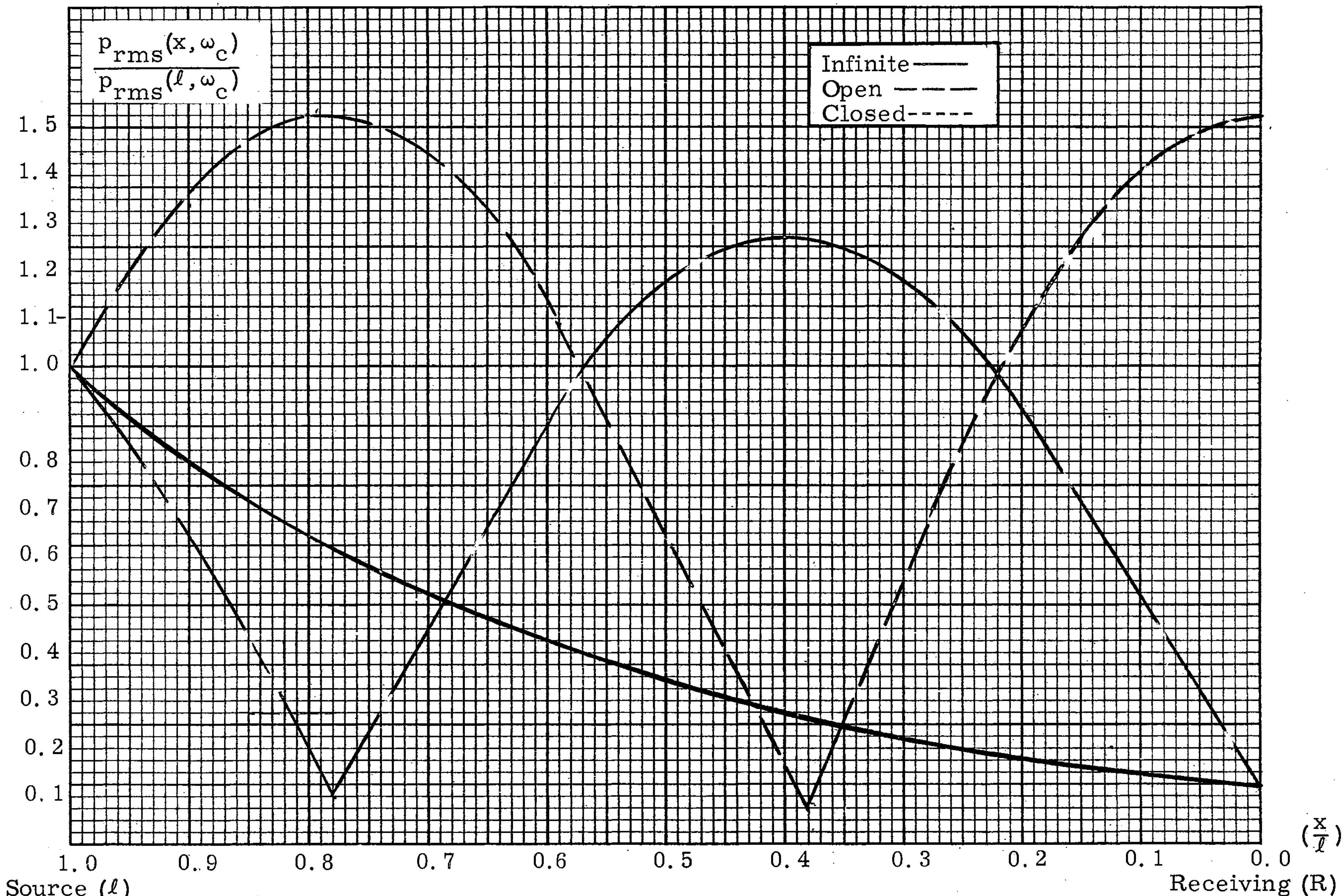


Figure 11

Normalized Distance Distribution for Three Idealized Terminations, $\omega = 8(\frac{\text{rad.}}{\text{sec.}})$

CHAPTER VI

DISCUSSION OF RESULTS

6.0 General.

The discussion in the following sections analyzes and interprets the solutions of Chapter V. It begins with a study of the behavior of the coefficients α and β as functions of the angular frequency ω and the system parameters R , L , C , ℓ , and a .

The factor $\|H(x, j\omega)\|^2$ [see Equation (42)] is analyzed, rather than the defining equations, for the various pipelines. This approach is employed since a consideration of the ordinates of the figures in Chapter V shows that

$$\frac{p_{rms}(x, j\omega)}{p_{rms}(\ell, j\omega)} = \frac{\frac{1}{2\pi} \quad \phi(x, j\omega)}{\frac{1}{2\pi} \quad \phi(0, j\omega)} = |H(x, j\omega)| \quad .$$

For simplicity, $|H(x_c, j\omega)|^2$ rather than $|H(x_c, j\omega)|$ is discussed, and in this report is called the spectral transfer function.

The primary information desired in a study of spectral transfer functions is a prediction of the nature and occurrences of maximums, since maximum pressure fluctuations are critical factors in pipeline systems. This is accomplished in the following sections by first analyzing $\|H(x, j\omega)\|^2$ as a function of $\alpha(\omega)$ and $\beta(\omega)$, and as a result, as a function of ω . $\|H(x, j\omega)\|^2$ is also analyzed as a function of x .

Characteristics of the various pipeline solutions are interpreted on the basis of this analysis. Variations in the form of the solutions as a result of different terminations, inputs, and values of R are discussed.

6.1 Discussion of α and β.

In the following discussion, the attenuation coefficient α is considered a function of ω alone. In the limit as ω approaches infinity, Equation (41), the defining relationship for $\alpha(\omega)$, assumes the indeterminate form $(\frac{0}{0})$, and by L'Hospitals rule can be reduced to

$$\lim_{\omega \to \infty} \alpha(\omega) = \alpha(\infty) = \left[\frac{CR^2}{4L}\right]^{\frac{1}{2}} = \frac{R}{2aL} . \qquad (60)$$

Experience has shown that $\alpha(\omega)$ very rapidly approaches $\alpha(\infty)$ and that the rate at which it converges to its final value is largely determined by R. This is illustrated by the pipelines in Chapter IV. The pipelines in Section 4.4 and 4.5 had, respectively, R equal to 26.7 and .215 with $\alpha(\infty)$ approximately equal to 8.46×10^{-5} and 6.821×10^{-7}. The pipelines were identical in other respects. $\alpha(\omega)$ for the pipeline with R = 26.7 had $\alpha(8) = 8.4595 \times 10^{-5}$, while for R = 0.215, $\alpha(1) = 6.8209 \times 10^{-7}$ (for R in lbs. sec. ft.$^{-6}$, α in ft.$^{-1}$).

β, the phase coefficient, is also considered a function of ω alone, and for large values of ω is seen from Equation (48) to approach the linear function

$$\beta(\infty) = \omega\,[CL]^{\frac{1}{2}} = \frac{\omega}{a} . \qquad (61)$$

The rate at which $\beta(\omega)$ approaches the linear expression in Equation (61) is dependent on R, although $\beta(\infty)$ itself is not a function of R.

6.2 Open End Pipeline.

From Equation (53), the spectral transfer function for the open end pipeline is given by

$$\left| H(x, j\omega) \right|^2 = \frac{\cosh 2\alpha x - \cos 2\beta x}{\cosh 2\alpha \ell - \cos 2\beta \ell} \quad . \tag{62}$$

Equation (62) plots as a three dimensional surface with independent variables x and ω. Formal differentiation to obtain the occurrence of maximums becomes extremely involved. Furthermore, the surface has a large number of local maximums and would require additional involved differentiation to obtain the "maximum" maximums or true maximums. Obviously these true maximums for pressure fluctuation are of fundamental importance in a pipeline system. The location of these true maximums involves a consideration of the interrelated variables x and ω and is approached in the following paragraphs by first identifying the true maximum and then demonstrating the required values of x and ω for which it would occur. The terms "maximum" maximum, true maximum, and, simply, maximum are used interchangeably in the following material.

For $\alpha = 0$, the numerator in Equation (62) has a maximum and minimum, respectively, of two and zero, as does the denominator. It is seen that the magnitude of the function is heavily dependent on the denominator and approaches infinity for $2\beta\ell = n\pi$ where $n = 0, 2, 4,$ etc. As compared to the denominator, the numerator has a minor role in determining the occurrences of maximums. In a true physical system, α is never equal to zero. It was pointed out in Section 6.1, however, that α is generally a very small number, and it would be possible for $2\alpha\ell$ to also be a very small number. If this is the case,

$|H(x, j\omega)|^2$, while not infinite, could have very large maximum values.

Consider $|H(x_c, j\omega)|^2$, which is a function of ω alone at x_c, some constant value of x. It was demonstrated in Section 4.1 that for increasing values of ω, $\alpha(\omega)$ rapidly converges to $\frac{R}{2aL}$, while $\beta(\omega)$ approaches $(\frac{\omega}{a})$. Assuming that this state exists, the denominator in Equation (62) is plotted in Figure 12. This plot shows a negative displaced cosine curve at $\cosh 2\alpha\ell$ with minimums at $(\cosh 2\alpha\ell - 1)$ and a period of $(\frac{ax}{\ell})$, the minimums occurring for $\omega = \frac{an\pi}{2\ell}$ where $n = 0, 2, 4,$ etc. The plot of the numerator shows maximums of $(\cosh 2\alpha x_c + 1)$ and a period of $(\frac{a\pi}{x_c})$, the maximums occurring for $\omega = (n + 1)\frac{a\pi}{2x_c}$ with n as before.

The true maximum value for $|H(x_c, j\omega)|^2$ at a given point x_c occurs for a value of ω which causes simultaneously, a maximum in the numerator and minimum in the denominator and is clearly

$$|H(x_c, j\omega)|^2 = \frac{\cosh 2\alpha x_c + 1}{\cosh 2\alpha\ell - 1} \quad . \tag{63}$$

For a given value of x_c the true maximum may or may not occur. Consider Figure 12 and assume that this maximum does occur for some value of $\omega = \omega_1$. For the numerator then, $\omega_1 = \frac{a\pi}{2x_c} + N\frac{a\pi}{x_c}$ where N is the number of full periods $(\frac{a\pi}{x_c})$ between $\omega = 0$ and $\omega = \omega_1$. Similarly for the denominator, $\omega_1 = M\frac{a\pi}{\ell}$ where M is the number of full periods $(\frac{a\pi}{\ell})$ between $\omega = 0$ and $\omega = \omega_1$. Equating the two,

$$\frac{a\pi}{2x_c} + N\frac{a\pi}{x_c} = \frac{Ma\pi}{\ell} ,$$

and

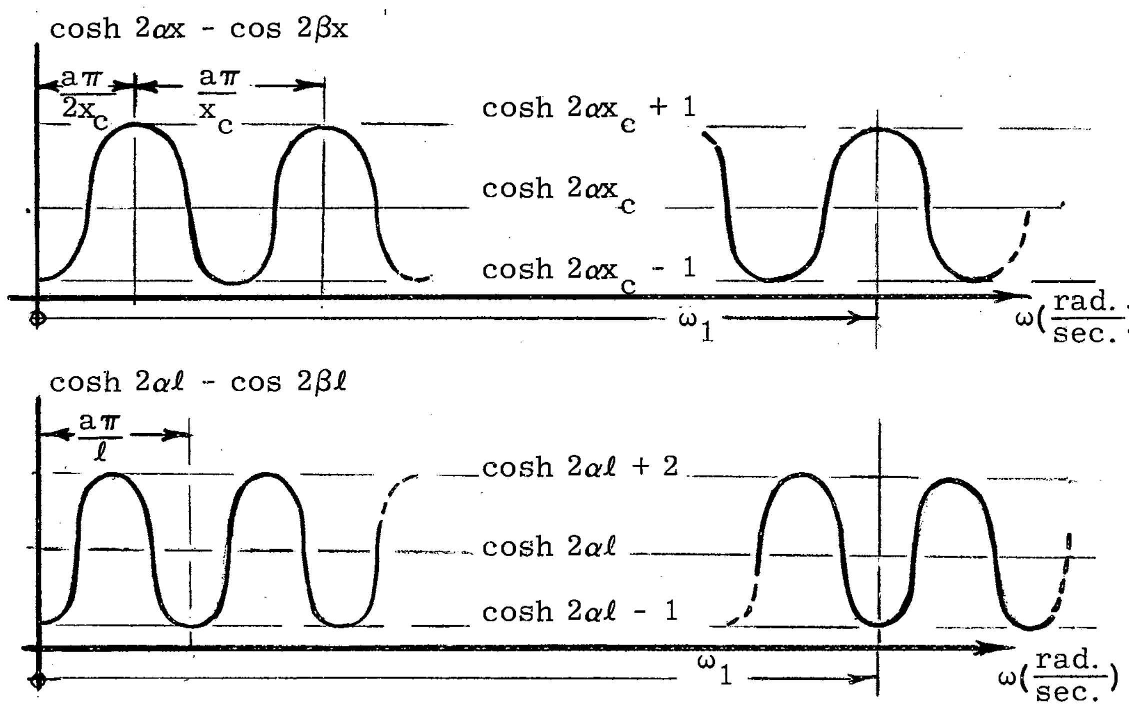


Figure 12

Numerator and Denominator of $|H(x_c, j\omega)|^2$

$$N = \left[\frac{2M(\frac{x_c}{\ell}) - 1}{2}\right]$$

where M and N are by definition integers. If the ratio $(\frac{x_c}{\ell})$ is such that some integer value of M causes an integer value for N , a true maximum is possible at $x = x_c$. If the ratio of $(\frac{x_c}{\ell})$ is such that no integer value of M exists which yields an integer value for N , a true maximum can not occur. Maximums occur for $(\frac{x_c}{\ell} = \frac{B}{2M})$ where M = 1, 2, 3, etc. and B = 1, 3, · · · , 2M - 1 . The values of M correspond to the modes of vibration, while the values of B refer to the peaks of the modes. The first true maximum occurs for $\omega_1 = M(\frac{a\pi}{\ell})$ and is repeated after a period of $2\omega_1 = 2M(\frac{a\pi}{\ell})$.

This analysis is verified in Figure 7. For this case $(\frac{x_c}{\ell}) = \frac{1}{2}$ and for M = 1, N = 0. The first maximum occurs at $\frac{a\pi}{\ell} = 6.28$ rad./sec. The period is seen to be $\frac{2a\pi}{\ell} = 12.56$ rad./sec. The second maximum occurs for $M = 3$, $N = 1$ at $\omega_2 = 18.84$ rad./sec. It should be noted that the "maximum" maximum value does not change at a given point x_c with increasing frequency. The maximum at $\omega_2 = 18.84$ rad./sec. for $\frac{x_c}{\ell} = 0.5$ is also shown in Figure 5.

The occurrence of integer multiples of π as characteristic frequencies is an individual characteristic of the pipeline considered and is not the general case, because for this pipeline $2\ell = a$.

The preceding discussion of the spectral transfer function is intended to demonstrate primarily the effects of frequency variations; however, they reveal much of the functional relationship of x . Equation (63), which defines the maximum for $|H(x_c, j\omega)|^2$, shows that the maximum values are increasing functions of x_c . Figure 5, which shows $|H(x, j\omega_c)|$ corresponding to the third mode of vibration at $\omega_c = 18.84$ rad./sec., demonstrates this. The locus of both maximum and minimum values in Figure 5 increases from right to left. Extending the discussion that developed from Figure 12, it is seen that for a given value of M , the largest maximum occurs when $B = 2M - 1$ and $\frac{x_c}{\ell} = \frac{2M - 1}{M}$.

Frequencies lying to both sides of $\omega = 18.84$ rad./sec. in Figure 5 demonstrate the effect, along the pipeline, of increasing frequencies. It is seen that higher harmonics result for increasing frequencies. Figure 5 shows the beginning of transition from third harmonic to fourth.

6.3 Closed End Pipeline.

Discussion of the closed end pipeline is much the same as for the open end and is consequently condensed.

Equation (57) gives for the closed end pipeline,

$$|H(x, j\omega)|^2 = \frac{\cosh 2\alpha x + \cos 2\beta x}{\cosh 2\alpha \ell + \cos 2\beta \ell} \tag{65}$$

The spectral transfer function is seen to have the same possible maximum for the closed end pipeline as for the open end pipeline.

If the spectral transfer function is considered as a function of frequency alone at x_c, some constant value of x, an analysis of the nature pursued in the preceding section can be followed in determining ratios of $(\frac{x_c}{\ell})$ for which true maximums can occur and the frequencies at which they do occur.

The following defining relationship for ω_1, the frequency at which a true maximum occurs in the closed end pipeline is evaluated:

$$N = \left[\frac{x_c}{\ell}(M + 1)\right]. \tag{66}$$

A true maximum may exist for $(\frac{x_c}{\ell}) = \frac{B}{2M+1}$ where $M = 0, 1, 2,$ etc. and $B = 0, 2, \cdots, M-1$ with M and N as previously defined. The first maximum occurs in the frequency spectrum at $\omega_1 = \frac{a\pi}{2\ell}(2M + 1)$ and is repeated after a period of $2\omega_1 = \frac{a\pi}{\ell}(2M + 1)$. For the closed end pipeline $M + 1$ corresponds to the harmonic modes of vibration, while B corresponds to the peaks in the modes when moving from right to left down the pipeline.

Figure 6 and 8 illustrate the correctness of this analysis. In Figure 8, $\frac{x_c}{\ell} = 0.0$. The first maximum would be expected for the initial value of $M = 0$. The initial maximum occurs for $\frac{a\pi}{2\ell}(2M + 1) =$ 3.14 rad./sec. and is repeated after a period of $\frac{a\pi}{\ell}(2M + 1) = 6.28$ rad./sec. The third maximum occurs for $M = 2$ at $\omega_3 = \frac{a\pi}{2\ell}(2M + 1)$ = 15.70 rad./sec. This maximum occurs for the third harmonic mode $(M + 1 = 3)$ and is seen in Figure 6 at $(\frac{x_c}{\ell}) = 0.0$ $(B = 0)$. The value of B = 2 would correspond to the next peak to the left at $\frac{x_c}{\ell} = 0.4$.

Study of Figures 6 and 8 reveals that the open and closed end pipelines behave similarly in several respects. First, Figure 8 demonstrates that at a given point x_c, maximums do not change for increasing frequencies. Figure 6 shows that the closed end pipeline tends to damp maximum pressure surges as they proceed away from the source of disturbance, and that increasing frequencies result in higher harmonics.

The primary difference in the open and closed terminations is illustrated by Figures 5 and 6. The receiving end R is always a mode in the open end pipeline. In the closed end pipeline for harmonic modes, R is always a point of maximum fluctuation. Maximums for the two pipelines never coincide, making it difficult to compare frequency differences.

6.4 The Infinitely Terminated Pipeline.

Equation (59) yields for the infinitely terminated pipeline

$$|H(x, j\omega)|^2 = e^{-2\alpha y} .$$

This relatively simple relationship is illustrated by Figure 11. Typically, in an infinite pipeline R is small, and $\alpha(\omega)$ converges to $\alpha(\infty)$ very rapidly. Consequently, there is practically no variation in the form of this solution for different frequencies.

Figure 11 shows an interesting contrast in the form of the solutions for the three general types of terminations at $\omega = 8$ rad./sec. The maximums and minimums for the open and closed pipelines nearly coincide for this frequency.

6.5 Input and Values of R.

Figures 10 and 8 show the solution differences which occurred, respectively, when Input 2 and Input 1 were inputs to the pipeline of Section 4.5. The solutions differ only in magnitude, reflecting the linearity of the system. Figure 10 is of interest in that Input 2 is a more physically realistic input than Input 1. It is known that for very high frequencies, the pressure fluctuations at $x = 0$ decrease. On the basis of the results in Section 6.3, this can only be explained by decreasing pressure fluctuations at the source for increasing frequencies as illustrated by the Gaussian Input.

Figure 9 is for the pipeline of Section 4.3 where $R = 0.215$ lb. sec. ft.$^{-6}$ and demonstrates the marked influence that R has on the magnitude of the solutions. It may be compared with Figure 6 which shows the same pipeline with $R = 26.7$ lb. sec. ft.$^{-6}$. Discussion in the first parts of Sections 6.1 and 6.2 should clarify this phenomena. It should be noted that the solutions differ in magnitude but not in phase.

CHAPTER VII

SUMMARY AND CONCLUSIONS

The purpose of this report is an analytical investigation of specialized pipelines systems when subjected to random pressure fluctuations. The investigation proceeds as follows. The governing partial differential equations for transient conditions in pipelines are introduced and by Laplace transform analysis are reduced, using boundary conditions, to a pair of transformed equations in x and s . These equations are simplified for idealized pipelines with open, closed, and infinite terminations, and are altered in form to relate the transformed relationships between pressure at the source of disturbance and at some other point in the pipeline.

The fundamental relationships governing linear systems when subjected to random inputs are introduced in Chapter II. On the bases of statistical definitions introduced in that chapter, the correlation and spectral density functions are shown to serve, respectively, as the time and frequency domain description of random functions. Equations are developed to show analytical techniques using both of these descriptions.

Chapter III demonstrates that the transformed relationships of Chapter I can be fitted to the analytical techniques evolved in Chapter II. Equations relating the spectral density of the input pressure to the spectral density of pressure at a point x are developed for the three ideally terminated pipelines.

Idealized pipelines and inputs presented in Chapter IV are designed to illustrate the nature of the solutions for the various terminations in terms of the variables, frequency ω and distance x . Solutions to particular cases are presented in Chapter V in the form of normalized rms distance and frequency distributions.

The results presented in Chapter V are analyzed and discussed in Chapter VI by correlating predicted results from analysis of the spectral transfer function to particular figures in Chapter V. Comparison is made on the basis of variations in termination, input, and value of the parameter R .

Analysis is presented in Chapter VI which makes possible prediction of the location (value of x) of true maximum pressure fluctuations and the frequencies at which these true maximums would occur. The correctness of this analysis is verified by comparison with example problems. Conclusions are drawn on the basis of this analysis which relate the magnitudes of maximum pressure fluctuations to changes in x and ω .

This report clearly illustrates the applicability of spectral density analysis to specially terminated pipelines. It is conceivable that for some investigations the location of maximum pressure fluctuations and the frequency at which they would be expected to occur could be of more significance than the time response of the system. The form of analysis presented in this report could then be superior to conventional analysis.

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