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ROOTS OF
DEHN TWISTS

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DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My parents and family

For

Encouraging me to pursue my dreams

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Chapter 1

Introduction

1.1 Mapping class groups

The *mapping class group* Mod(F) of an orientable surface F is defined to be the group of isotopy classes of orientation-preserving self-diffeomorphisms on F. The mapping class group has been one of the central objects in the field of 2-dimensional geometric topology. It has been widely studied since it also plays an important role in several other fields including Teichmüller theory and algebraic geometry, where it is called the *modular group*.

The study of mapping class groups was started by Max Dehn [3, 4] and Jakob Nielsen [17, 18, 19] in the 1920s. Dehn tried to understand the mapping class groups by addressing such questions as the existence of a finite set of generators. For this purpose, he studied the action of Mod(F) on the isotopy classes of curves on the surface, which he called the *arithmetic field*. In the process, he introduced a basic element of Mod(F) called a *Dehn twist*.

Regarding S^1 as $\mathbb{R}/\mathbb{Z}$, a *Dehn twist* of an annulus $S^1 \times I$ is a homeomorphism $h : S^1 \times I \longrightarrow S^1 \times I$ defined by $h(x, s) = (x + s, s)$. A Dehn twist t_C about a simple closed curve C on a surface F is defined to be the map h on an annular neighborhood $S^1 \times I$ of C and the identity elsewhere. The effect of a Dehn twist on an arc transverse to the curve C is shown in the figure below.

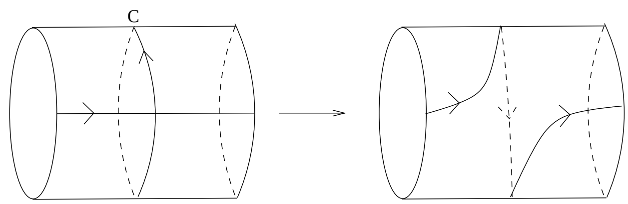

Figure 1.1: A Dehn twist

Dehn showed that Mod(F) is generated by finitely many Dehn twists.

Nielsen, on the other hand, tried to analyze the elements in Mod(S) by using techniques in hyperbolic geometry. The works of Dehn and Nielsen were later revisited by Harvey [8, 9], who put a natural simplicial structure on Dehn's arithmetic field and called it the *complex of curves*. William Thurston, in his theory of surface diffeomorphisms [22, 23], extended the work of Nielsen and brought it to a complete form (see section 1.6).

1.2 The Fuchsian group viewpoint

The classical approach to the theory of surface diffeomorphisms was by studying the action of discrete groups on Riemann surfaces. One of the fundamental results in this theory is the Uniformization Theorem.

Theorem 1.2.1 (The Uniformization Theorem). *If a Riemann surface is homeomorphic to a sphere then it is conformally equivalent to the Riemann sphere. Any Riemann surface F that is not homeomorphic to a sphere is conformally equivalent to a quotient of the form $\mathbb{C}/G$, or $\mathbb{H}^2/G$, where G is a discrete group* *of conformal isometries acting without fixed points on $\mathbb{C}$, or on $\mathbb{H}^2$. Further, G is holomorphic to the fundamental group of F.*

The only Riemann surfaces of the form $\mathbb{C}/G$ are the plane $\mathbb{C}$, the punctured plane $\mathbb{C} \setminus \{0\}$, and the tori. Consequently, every Riemann surface F of genus $g \geq 2$ is of the form $\mathbb{H}^2/G$ for some group G of isometries of the hyperbolic plane $\mathbb{H}^2$. Any group of isometries of $\mathbb{H}^2$ is a subgroup of $\mathrm{SL}(2, \mathbb{R})$. A *Fuchsian group* is a discrete group of isometries of $\mathbb{H}^2$, or a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$.

Consider a Riemann surface F of genus $g \geq 2$. Since $\mathbb{H}^2$ is the universal cover of F, we have an unramified holomorphic covering map $p : \mathbb{H}^2 \to F$. Let Γ be the group of deck transformations of this covering space. As an abstract group, Γ is isomorphic to $\pi_1(F)$ and p induces a homeomorphism $\widetilde{p} : \mathbb{H}^2/\Gamma \to F$. $\widetilde{p}$ induces a $\mathbb{H}^2/\Gamma$ structure on F and one usually says that the Fuchsian group Γ *uniformizes* F. Also, every group of automorphisms G of F lifts to a group of automorphisms $\widetilde{G}$ of $\mathbb{H}^2$ with $G = \widetilde{G}/\Gamma$.

To obtain a presentation for $\widetilde{G}$, we consider a fundamental region for the covering $\widetilde{p} : \mathbb{H}^2/\Gamma \to F$. The edges of a fundamental polygonal region generate $\widetilde{G}$. We modify this region using cutting and gluing techniques to obtain a new fundamental polygonal region R with edges a $\{x_1, \ldots, x_r, a_1, b_1, \ldots a_{\widetilde{g}}, b_{\widetilde{g}}\}$ for some $\widetilde{g}$, satisfying $x_i^{m_i} = 1$, for some m_i, and $x_1 x_2 \ldots x_r [a_1, b_1][a_2, b_2] \ldots [a_{\widetilde{g}}, b_{\widetilde{g}}] = 1$. Thus we obtain the following presentation for $\widetilde{G}$.

$$\widetilde{G} = \langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_{g'}, b_{g'} \mid x_i^{m_i} = 1,\ x_1 \ldots x_r [a_1, b_1] \ldots [a_{g'}, b_{g'}] = 1 \,\rangle.$$

The tuple $(\widetilde{g}; x_1, \ldots, x_r)$ is called the *signature* of $\widetilde{G}$.

If $G = \widetilde{G}/\Gamma$, then the signature $(\widetilde{g}; x_1, \ldots, x_r)$ provides the topological features of the action of G on F. F/G has genus g' and the $m_1, \ldots, m_r$ provide the ramification data for the projection $F \to F/G$. The hyperbolic area of the fundamental region is given by the Gauss-Bonnet formula

$$\mu(\widetilde{G}) = 2\pi \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right) .$$

If $\widetilde{G}'$ is a subgroup of $\widetilde{G}$ of finite index, then we have the Riemann-Hurwitz formula $[\widetilde{G} : \widetilde{G}'] = \mu(\widetilde{G}')/\mu(\widetilde{G})$.

1.3 Thurston's orbifold viewpoint

Thurston, in the 1970s, coined the term *n-dimensional orbifold or n-orbifold* for a Hausdoff, paracompact space that is locally homeomorphic to the quotient space of $\mathbb{R}^n$ by a finite G-action. Thurston's orbifold theory gave a topological perspective to the theory of groups actions on surfaces that mirrors the classical algebraic viewpoint of Fuchsian groups.

If $p : Y \to X$ is a continuous mapping between the topological spaces underlying two orbifolds, then p is said to be a *orbifold covering* if every point $x \in X$ has an orbifold chart $\phi : D^n/G \hookrightarrow X$ that is evenly covered by $p|_{p^{-1}(\phi(D^n/G))}$. The *cone points* are the points of the orbifold that in a local chart are fixed by some nontrivial element of G. In an orientable 2-orbifold, the cone points are isolated.

If $p : Y \longrightarrow X$ is an orbifold covering, then the group of covering transformations $G_X(Y)$ acts properly discontinuously on Y and $Y/G_X(Y)$ covers X via the map induced by p. Also, the quotient map $Y \longrightarrow Y/G_X(Y)$ is an orbifold covering. Conversely, if G acts properly discontinuously on the orbifold Y, then $Y \to Y/G$ is an orbifold covering whose group of covering translations is G. A covering of an orbifold is said to be *regular* if the induced covering $Y/G_X(Y) \to X$

is an isomorphism.

The usual proof of existence of universal covers can be adapted to show that any orbifold X has a universal covering $\widetilde{X} \to X$ that is regular. This covering has a group of covering translations called the *orbifold fundamental group* of X, $\pi_1^{orb}(X)$. All connected coverings of X come, up to isomorphism, by dividing $\widetilde{X}$ by a subgroup of $\pi_1^{orb}(X)$, and a connected cover is regular if and only if this subgroup is a normal subgroup.

Thurston showed that the quotient of an n-manifold Y by a properly discontinuous G-action is a n-orbifold $X = Y/G$, and the quotient map $p : Y \longrightarrow X$ is a regular orbifold covering. This action of G on Y gives the following exact sequence

$$1 \longrightarrow \pi_1(Y) \longrightarrow \pi_1^{orb}(X) \xrightarrow{\rho} G \longrightarrow 1 ,$$

where ρ has a torsion-free kernel, and is obtained by lifting path representatives of elements of $\pi_1^{orb}(X)$.

Suppose that we have a closed orientable surface F of genus g and G acts properly discontinuously on F preserving orientation. Then the projection $p : F \to \mathcal{O} = F/G$ is a regular orbifold covering. Denote by $\widetilde{g}$ the genus of $\mathcal{O}$. To obtain a presentation for the orbifold fundamental group $\pi_1^{orb}(\mathcal{O})$, let $\{x_1, x_2, \ldots, x_k\}$ denote the finite set of cone points of $\mathcal{O}$. Fixing a base point x for $\mathcal{O}$, let α_i be a loop based at x and going around x_i, as shown in Figure 1.2.

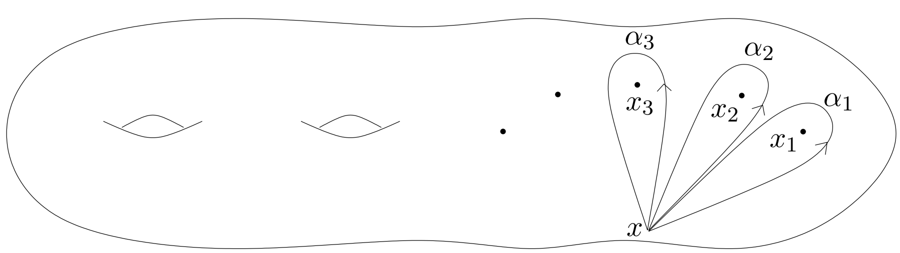

Figure 1.2: The orbifold $\mathcal{O}$

If $2\pi/n$ is the rotation angle of the G-action around each point of $p^{-1}(x_i)$, then $\alpha_i^{n_i} = 1$ in $\pi_1^{orb}(\mathcal{O})$. In Figure 1.2, the product $\alpha_1\alpha_2\cdots\alpha_k$ is homotopic to a loop that bounds a disk D containing the cone points, and for which the closure $S = \overline{F \setminus D}$ is a 2-manifold of some genus $\widetilde{g}$ and ∂S represents $\alpha_1\alpha_2\cdots\alpha_k$. Letting $a_1, b_1, \ldots, a_{\widetilde{g}}, b_{\widetilde{g}}$ be standard generators of $\pi_1 S$ chosen so that ∂S represents $\prod[a_i, b_i]$, we have the following presentation for $\pi_1^{orb}(\mathcal{O})$:

$$\pi_1^{orb}(\mathcal{O}) = \langle \alpha_1, \ldots, \alpha_k, a_1, b_1, \ldots, a_{\widetilde{g}}, b_{\widetilde{g}} \mid$$
$$\alpha_1^{n_1} = \cdots = \alpha_k^{n_k} = 1,\ \alpha_1 \cdots \alpha_k = \prod_{1=1}^{\widetilde{g}} [a_i, b_i]\ \rangle.$$

A 2-orbifold is *good* if its universal covering orbifold is a surface. We state the following theorem of Thurston on good 2-orbifolds.

Theorem 1.3.1. *Every good 2-orbifold without boundary is isomorphic to the quotient of S^2, E^2 or H^2 by some discrete group of isometries.*

In particular, if F is a surface and G is finite group acting on F, then F/G is a 2-orbifold. In fact, every $\mathcal{O} = F/G$ has a finite surface covering as stated in the following theorem.

Theorem 1.3.2. *Every good, compact 2-orbifold without boundary is finitely covered by a surface.*

1.3.1 The Riemann-Hurwitz equation

Suppose that we have a good compact 2-orbifold $\mathcal{O}$ with underlying surface S and k cone points of orders n_i, $1 \le i \le k$. Then $\mathcal{O}$ is finitely covered by a surface F. If d is the degree of the covering, then we can naturally define the Euler number of $\mathcal{O}$ by the equation $\chi(F) = d\chi(\mathcal{O})$. Let $D = D_1 \cup D_2 \cup \ldots \cdots \cup D_k$ be the disjoint union of open disks containing the cone points $\{x_1, x_2, \ldots, x_k\}$. If $W = S \setminus D$, then $\chi(S) = \chi(D) + \chi(W) = k + \chi(W)$. Since $p : F \to \mathcal{O}$ is a d-fold covering, the projection $p^{-1}(W) \to W$ is a d-fold covering space so that $\chi(p^{-1}(W)) = d\chi(W)$. Now the pre-image in F of D_i is d/n_i disks, so we obtain

$$\chi(F) = d\chi(W) + \sum_{i=1}^{k} d/n_i \ .$$

But $\chi(F) = d\chi(\mathcal{O})$, so by dividing the equation by d and substituting $\chi(W) = \chi(S) - k$, we obtain

$$\chi(\mathcal{O}) = \chi(S) - \sum_{i=1}^{k} (1 - 1/n_i) \ ,$$

which is known as the Riemann-Hurwitz equation. As in case of Fuchsian groups, here the tuple $(\widetilde{g}; n_1, \ldots, n_k)$ is called the *signature* of the orbifold.

1.4 The Wiman-Harvey upper bound

In the late nineteenth century, Hurwitz [10] showed that the group of automorphisms of a compact Riemann surface of genus g is finite if $g \geq 2$, and obtained the best possible bound $84(g-1)$ for the order of such a group. About the same time, Wiman [24] improved on this bound for a cyclic group, by showing that the maximum possible order for an automorphism is $2(2g + 1)$. Harvey [7], in 1964, used Fuchsian groups to find the the minimum genus g of a surface which admits an automorphism of order n and Wiman's result was a direct consequence of Harvey's theorem. In this section, we give a proof of this Wiman-Harvey result using Thurston's orbifold viewpoint. We will use one of the number-theoretic results of Harvey proved in [7], given as Lemma 1.4.5 below.

Lemma 1.4.1. *Let $p : F \to \mathcal{O}$ be an orbifold covering with cyclic covering group C_n, and suppose that $\mathcal{O}$ has signature $(0; n_1, \ldots, n_k)$. Then the least common multiple of the orders of any $k-1$ cone points is n.*

Proof. Let $m_i = \text{lcm}\{n_1, \ldots, \widehat{n_i}, \ldots n_k\}$, and let $m = \text{lcm}\{n_1, \ldots, n_k\}$. First we show that $m_i = m$ for all i. Since $m_i | m$ and $n_j \mid m_i$ for $j \neq i$, it suffices to show that $n_i | m_i$. Let α_j be the generator of $\pi_1^{orb}(\mathcal{O})$ going around the cone point of order n_j. From Thurston's orbifold theory (Section 1.3), since the surface part of $\mathcal{O}$ is a sphere, $\pi_1^{orb}(\mathcal{O})$ will have a presentation of the form

$$\pi_1^{orb}(\mathcal{O}) = \langle \alpha_1, \ldots, \alpha_k | \alpha_1^{n_1} = \cdots = \alpha_k^{n_k} = \alpha_1 \alpha_2 \cdots \alpha_k = 1 \rangle,$$

and we have an exact sequence

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1^{orb}(\mathcal{O}) \stackrel{\rho}{\longrightarrow} C_n \longrightarrow 1 \ .$$

Since $\alpha_1 \alpha_2 \cdots \alpha_k = 1$, we note that

$$\alpha_1 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_n = (\alpha_{i+1} \cdots \alpha_n)^{-1} \alpha_i^{-1} (\alpha_{i+1} \cdots \alpha_n). \tag{1.1}$$

Moreover, ρ is a homomorphism and the kernel of ρ is torsion-free, which implies that $\rho(\alpha_1)\rho(\alpha_2) \ldots \rho(\alpha_i) \ldots \rho(\alpha_k) = 1$, where $|\rho(\alpha_j)| = n_j$ in C_n. This shows that

$$|\rho(\alpha_1)\rho(\alpha_2) \ldots \widehat{\rho(\alpha_i)} \ldots \rho(\alpha_k)| = |\rho(\alpha_i)^{-1}| = n_i \ .$$

Consider the subgroup H of C_n generated by $\{\rho(\alpha_j) | j \neq i\}$. The fact that the order of cyclic group is the least common multiple of the orders of the elements in any generating set would imply that

$$|H| = \mathrm{lcm}\{\rho(\alpha_1), \ldots, \widehat{\rho(\alpha_i)}, \ldots, \rho(\alpha_k)\} = \mathrm{lcm}\{n_1, \ldots, \widehat{n_i}, \ldots, n_k\} = m_i \ .$$

Since $\rho(\alpha_1)\rho(\alpha_2)\ldots\widehat{\rho(\alpha_i)}\ldots\rho(\alpha_k) \in H$, we have that $n_i \mid m_i$. Finally, Equation 1.1 would imply that $\rho(\alpha_i) \in C_n$, and from the exact sequence, we have that ρ is surjective. Consequently, $\{\rho(\gamma_j)\}$ generates C_n, giving $m = n$. □

Lemma 1.4.2. *Suppose that F is a closed oriented surface of genus $g \geq 2$. Let $p : F \to \mathcal{O}$ be an orbifold covering with cyclic covering group C_n, and suppose that $\mathcal{O}$ has signature $(\widetilde{g}; n_1, \ldots, n_k)$. Then $\widetilde{g} = 0$ whenever $n > 2g - 2$.*

Proof. From the Riemann-Hurwicz equation, we have that

$$\frac{2g-2}{n} = 2\widetilde{g} - 2 + k - \sum_{i=1}^{k} \frac{1}{n_i} \ .$$

Since $n > 2g - 2$ and $n_i \geq 2$, we get that

$$1 > 2\widetilde{g} - 2 + k - \frac{k}{2} = 2\widetilde{g} - 2 + \frac{k}{2} \ .$$

Suppose we assume for the sake of contradiction that $\widetilde{g} \geq 1$. Then we would have $1 > \frac{k}{2}$, giving $k < 2$. If $k = 0$, then would have $g = \widetilde{g} = 1$, which contradicts our hypothesis. If $k = 1$, then by Lemma 1.4.1, we would have that $n_1 = n$, so the Riemann-Hurwitz equation takes the form

$$\frac{2g-2}{n} = 2\widetilde{g} - 1 - \frac{1}{n} \ ,$$

giving $n = \frac{2g-1}{2\widetilde{g}-1} \leq 2g - 1$, which is impossible. □

Lemma 1.4.3. *Suppose that* $\{n_1, \ldots, n_k\}$ *is a collection of integers such that* $n_i \geq 2$*, and let* $n = \operatorname{lcm}\{n_1, \ldots, n_k\}$*. If* n *is even, then* $\sum_{i=1}^{k} \frac{1}{n_i} \leq \frac{k-1}{2} + \frac{1}{r}$*, where*

(i) $r = n$ *if* $4 \mid n$*, and*

(ii) $r = \frac{n}{2}$ *otherwise.*

Proof. We may assume that $n_i \leq n_j$ for $i \leq j$. Let r be the smallest integer greater than or equal to 2 such that $\operatorname{lcm}(2, r) = n$. Then $r = n$ if $4 \mid n$, and $r = \frac{n}{2}$ otherwise. We start by showing that

$$\sum_{i=1}^{k} \frac{1}{n_i} \leq \sum_{i=1}^{k-1} \frac{1}{n_i} + \frac{1}{r} .$$

First, we establish this inequality for $k = 2$. Consider the sum $\frac{1}{n_1} + \frac{1}{n_2}$. Suppose that $n_1 = 2$. Since $\operatorname{lcm}\{n_1, n_2\} = n$, we have $n_2 \geq r$. Consequently,

$$\frac{1}{n_1} + \frac{1}{n_2} \leq \frac{1}{2} + \frac{1}{r} .$$

Suppose that $n_1 = 3$. Then we would have that

$$\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{3} + \frac{1}{n_2} ,$$

where n_2 is even and $n_2 \geq 4$. If $3 \mid n_2$, then $n_2 = n \geq r$, so we would have

$$\frac{1}{3} + \frac{1}{n_2} < \frac{1}{2} + \frac{1}{r} .$$

If $\gcd(3, n_2) = 1$, then $n_2 = \frac{n}{3}$. We consider the cases when $4 \mid n$ and $4 \nmid n$

separately. When $4 \mid n$, we would have that $n \geq 12$, so

$$\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{3} + \frac{3}{n} = \left(\frac{1}{3} + \frac{2}{n}\right) + \frac{1}{n} \leq \frac{1}{2} + \frac{1}{n} = \frac{1}{2} + \frac{1}{r} \ .$$

When $4 \nmid n$, we have that $n \geq 6$, and

$$\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{3} + \frac{3}{n} = \left(\frac{1}{3} + \frac{1}{n}\right) + \frac{2}{n} \leq \frac{1}{2} + \frac{2}{n} = \frac{1}{2} + \frac{1}{r} \ .$$

Suppose that $n_1 \geq 4$. Then $n_2 \geq 4$, which implies that

$$\frac{1}{n_1} + \frac{1}{n_2} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < \frac{1}{2} + \frac{1}{r} \ .$$

In general, for any k, we consider $\frac{1}{n_{k-1}} + \frac{1}{n_k}$. By a similar argument, we can show that

$$\frac{1}{n_{k-1}} + \frac{1}{n_k} \leq \frac{1}{2} + \frac{1}{r} \ .$$

Since $n_i \geq 2$, the lemma follows from the fact that

$$\sum_{i=1}^{k-1} \frac{1}{n_i} + \frac{1}{r} \leq \frac{k-1}{2} + \frac{1}{r} \ .$$

$\square$

Lemma 1.4.4. *Suppose that F is a closed oriented surface of genus $g \geq 2$. Let $p : F \to \mathcal{O}$ be an orbifold covering with cyclic covering group C_n, and suppose that $\mathcal{O}$ has signature $(0; n_1, \ldots, n_k)$. Then $k = 3$ whenever*

(i) n is even and $n > 4g$, or

(ii) n is odd and $n > 3g - 3$.

Proof. We know from Lemma 1.4.1 that $\text{lcm}\{n_1, \ldots, n_k\} = n$. Let $n = {p_1}^{a_1}{p_2}^{a_2} \ldots {p_r}^{a_r}$ be a prime factorization for n, and let $p = \min\{p_1, \ldots, p_k\}$. For an orbifold with signature $(0; n_1, \ldots, n_k)$, the Riemann-Hurwicz equation gives

$$\frac{2-2g}{n} = 2 - k + \sum_{i=1}^{k} \frac{1}{n_i} \ .$$

If $k \leq 2$, then we would have that $\frac{2-2g}{n} > 0$, which is impossible.

If n is even, then $p = 2$, and from Lemma 1.4.3

$$\frac{2-2g}{n} = 2 - k + \sum_{i=1}^{k} \frac{1}{n_i} \leq 2 - k + \frac{k-1}{2} + \frac{2}{n} \ ,$$

giving

$$\frac{2g}{n} \geq \frac{k-3}{2} \ .$$

If $k \neq 3$, then $k \geq 4$ giving $n \leq 4g$, which proves (i).

If n is odd, then $p \geq 3$. Since $n_i \geq p$, we have that

$$\frac{2-2g}{n} \leq 2 - k + \frac{k}{p} \ ,$$

giving

$$k \leq \left(\frac{2p}{p-1}\right)\left(\frac{g-1}{n} + 1\right) \ .$$

If $k \geq 4$, then we would have that

$$n \leq \frac{g-1}{1-\frac{2}{p}} \leq 3g - 3 \ ,$$

and (ii) holds. □

We will state the following result of Harvey [7] without proof.

Lemma 1.4.5. *Suppose that a positive integer N has a prime decomposition $N = {p_1}^{r_1}{p_2}^{r_2}\cdots{p_k}^{r_k}$, where $p_i < p_j$ for $i < j$ and each $r_i > 0$. Let E denote the set of all integer triples (n_1, n_2, n_3) satisfying* $\operatorname{lcm}(n_1, n_2) = \operatorname{lcm}(n_2, n_3) = \operatorname{lcm}(n_1, n_3) = N$ *and let* $\Delta(E) = \max_{(a,b,c)\in E}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$. *Then*

(i) $\Delta(E) = \frac{1}{N} + \frac{1}{p_1} + \frac{p_1}{N}$ *if* $r_1 = 1$, *and N not prime;*

(ii) $\Delta(E) = \frac{1}{N} + \frac{1}{p_1} + \frac{1}{N}$ *if* $r_1 > 1$.

Theorem 1.4.6. *The maximum order n for a cyclic automorphism of a surface of genus $g \geq 2$ is $4g + 2$.*

Proof. Let F be a hyperbolic surface of genus $g \geq 2$. The maximum order must be at least $4g + 2$, since there always exists a C_{4g+2}-action on F with orbifold signature $(0; 2, 2g + 1, 4g + 2)$.

Suppose for contradiction that there is a homeomorphism of F with finite order $n > 4g+2$. Regarding it as a C_n-action, let $F \to F/C_n = \mathcal{O}$ be the orbifold covering map, and $(\widetilde{g}; n_1, \ldots, n_k)$ be a signature of $\mathcal{O}$. Using Lemmas 1.4.2 and 1.4.4, we may assume that $k = 3$ and $\widetilde{g} = 0$. The Riemann-Hurwitz equation would then take the form

$$\frac{2-2g}{n} = -1 + \sum_{i=1}^{3} \frac{1}{n_i} \ .$$

Let $n = {p_1}^{r_1}{p_2}^{r_2}\cdots{p_k}^{r_k}$, where $p_i < p_j$ for $i < j$. If $r_1 > 1$ and n is not prime, then Lemmas 1.4.1 and 1.4.5 tell us that

$$\sum_{i=1}^{3} \frac{1}{n_i} \leq \frac{2}{n} + \frac{1}{p_1} \ ,$$

which upon simplification gives

$$n \leq \frac{2g}{1 - \frac{1}{p_1}} \leq 4g \ .$$

If $r_1 = 1$, then from Lemmas 1.4.1 and 1.4.5 we would have that

$$\sum_{i=1}^{3} \frac{1}{n_i} \leq \frac{1}{n} + \frac{1}{p_1} + \frac{p_1}{n} \ .$$

Since $n > 4g + 2$, we have that

$$4g + 2 < \frac{2g + p_1 - 1}{1 - \frac{1}{p_1}} = p_1 \left(\frac{2g}{p_1 - 1} + 1 \right) \ ,$$

that is,

$$2g \left(\frac{2 - p_1}{p_1 - 1} \right) > 2 - p_1.$$

which is clearly not true when $p_1 = 2$. If $p_1 > 2$, then we must have $g < \frac{p_1 - 1}{2}$. But

$$\sum_{i=1}^{3} \frac{1}{n_i} \leq \frac{1}{n} + \frac{1}{p_1} + \frac{p_1}{n} \ ,$$

giving

$$\frac{2 - 2g}{n} \leq -1 + \frac{1}{n} + \frac{1}{p_1} + \frac{p_1}{n} \ ,$$

that is,

$$g \geq \left(\frac{p_1 - 1}{2} \right) \left(\frac{n}{p_1} - 1 \right) \geq \frac{p_1 - 1}{2} \ ,$$

which is a contradiction when $n \neq p_1$. If $n = p_1$, then the fact that $n_i \mid n$ would imply that each $n_i = n$, so the Riemann-Hurwitz equation takes the form

$$\frac{2 - 2g}{n} = -1 + \frac{3}{n} \ ,$$

giving $n = 2g + 1$, which contradicts our hypothesis.

□

1.5 Teichmüller space

A *marking* on a surface F is a pair (S, ϕ) where S is a hyperbolic surface and $\phi : F \to S$ is a diffeomorphism. Two markings are (S_1, ϕ_1) and (S_2, ϕ_2) on a surface F of genus g are said to be *equivalent* if $\phi_2 \circ {\phi_1}^{-1}$ is homotopic to an isometry. The *Teichmüller space of* F, denoted by $\mathrm{Teich}(F)$, is the space of all equivalent markings on F. In other words, the Teichmüller space of F is the space of all marked hyperbolic structures on F.

We can obtain a natural topology on $\mathrm{Teich}(F)$ by putting Fenchel-Nielsen coordinates on it. To describe these coordinates, we start by gluing pairs of pants in the pants decomposition of F along their boundaries with certain twist parameters. Since each point in $\mathrm{Teich}(F)$ defines a marked hyperbolic structure on F, we can put coordinates by taking the length of the gluing curves under the hyperbolic metric together with their twisting parameters. A closed orientable surface F has $3g - 3$ disjoint curves in its pants decomposition. Therefore any point in $\mathrm{Teich}(F)$ can be described by $6g-6$ coordinates and it is a classical result of R. Fricke and F. Klein [6] that for a surface F of genus $g \geq 2$, $\mathrm{Teich}(F) \cong \mathbb{R}^{6g-6}$.

$\mathrm{Mod}(F)$ acts isometrically and properly discontinuously on $\mathrm{Teich}(F)$, and this action is defined in the following manner: for any $[h] \in \mathrm{Mod}(F)$ and $[(S, \phi)] \in \mathrm{Teich}(F)$, $[h] \cdot [(S, \phi)] = [(S, \phi \circ h^{-1})]$. The space thus obtained by taking the quotient $\mathrm{Teich}(F)/\mathrm{Mod}(F)$, is called the *moduli space of* F, denoted by $\mathcal{M}(F)$.

1.6 Nielsen-Thurston Classification

As mentioned before, Thurston completed the work initiated by Nielsen in his classification theorem for surface homeomorphisms.

Theorem 1.6.1 (Nielsen-Thurston Classification)**.** *Each $g \in \mathrm{Mod}(F)$ has a representative f satisfying exactly one of the following:*

1. *f has finite order.*

2. *f has infinite order and $f(\mathcal{C}) = \mathcal{C}$ for some collection $\mathcal{C}$ of disjoint nonisotopic curves in S.*

3. *f is pseudo-Anosov: There exists a transverse pair of measured singular foliations on F, F_s (stable) and F_u (unstable), and a real number $\lambda > 1$ such that the foliations are preserved by f and their transverse measures are multiplied by $1/\lambda$ and λ.*

Thurston's classification is also related to the action of $\mathrm{Mod}(F)$ on $\mathrm{Teich}(F)$. Thurston introduced measured laminations and showed that the space of projectivised measured laminations $\mathcal{PML}(F)$ can be regarded as a boundary of $\mathrm{Teich}(F)$, giving a compactification of $\mathrm{Teich}(F)$. A key feature of this boundary is that the action of $\mathrm{Mod}(F)$ on $\mathrm{Teich}(F) \cong \mathbb{R}^{6g-6}$ extends naturally to an action on $\mathcal{PML}(F) \cong S^{6g-5}$, giving an action on $\mathrm{Teich}(F) \cup \mathcal{PML}(F) \cong B^{6g-6}$. The type of an element $f \in \mathrm{Mod}(F)$ in the Thurston classification is related to its fixed points when acting on this compactification of $\mathrm{Teich}(F)$:

1. If f is periodic, then there is a non-empty fixed-point set within $\mathrm{Teich}(F)$; these points correspond to hyperbolic structures on F whose isometry group contains an element isotopic to f;

2. If f is pseudo-Anosov, then f has no fixed points in Teich(F) but has a pair of fixed points on the Thurston boundary $\mathcal{PML}(F)$; these fixed points correspond to the stable and unstable foliations of F preserved by f.

3. For some reducible mapping classes f, there is a single fixed point on the Thurston boundary $\mathcal{PML}(F)$. In general, unlike the isometries of hyperbolic space, reducible classes can even fix infinitely many points on $\mathcal{PML}(F)$, for example, a Dehn twist t_C about a curve C fixes every foliation in the complement of C. Keckhoff showed in [13] that for a handlebody M^3 of genus g with boundary surface F, the closure in $\mathcal{PML}(F)$ of simple closed curves in F which bound disks in M^3 is of measure 0.

1.7 Some earlier results on roots

In 2008, C. Bonatti and L. Paris [1] studied the roots in the mapping class groups. Let F be a compact oriented surface of genus g with boundary ∂F with a finite set of punctures P, and let Mod(F, P) denote the mapping class group relative to the boundary of (F, P). They proved that if $g = 1$ and $\partial F \neq \emptyset$, then each $f \in$ Mod(F) has at most one m-root up to conjugation for all $m \geq 1$. However, if $g \geq 2$, then there exist non-conjugate elements $f, g \in$ Mod(F, P) such that $f^2 = g^2$.

In the same paper, they also derived some results for pseudo-Anosov elements. They showed that if $\partial F \neq \emptyset$, then each pseudo-Anosov element $f \in$ Mod(F, P)has at most one m-root for all $m > 1$, but if $\partial F = \emptyset$, then there exist two non-conjugate pseudo-Anosov elements $f, g \in$ Mod(F) such that $f^m = g^m$ for some $m \geq 2$. Finally, they showed that an element of a pure subgroup G of Mod(F) can have at most one root of degree m in G.

In 2009, D. Margalit and and S. Schleimer [15] showed that for a surface f of genus $g \geq 2$, every Dehn twist in $\mathrm{Mod}(F)$ has a nontrivial root. For Dehn twists about separating curves, the fact is well-known: if C is a separating curve then a square root of the left Dehn twist t_C is obtained by twisting one side of C through an angle of π. For a nonseparating curve C, they gave a geometric and algebraic construction of a root of degree $2g - 1$. They also gave roots of several other analogues of Dehn twists like half-twists and Nielsen transformations. In particular, they noted that their roots of Dehn twists in $\mathrm{Mod}(F)$, adapted to once-punctured surfaces, provide examples of "geometric" roots of Nielsen transformations in $\mathrm{Out}(F_n)$.

1.8 Roots of Dehn twists

A natural question concerning mapping class groups is whether a Dehn twist t_C can have a root. We have seen that it is easy to find examples of roots for Dehn twists about separating curves. However, in the case of nonseparating curves, existence of a root is not obvious. As mentioned earlier, D. Margalit and S. Schleimer [15] showed the existence of such roots by constructing roots of degree $2g + 1$ in the surface of genus $g + 1 \geq 2$. The natural questions were whether there exist roots of other degrees, and whether we could classify them.

These questions motivated me to look deeper into this subject. My first research work (on nonseparating curves) [16], a collaborative effort with my thesis adviser Dr. Darryl McCullough, was a direct outcome of this pursuit. The main theorem said that given a genus g and a degree n, the Dehn twist has a root of degree n if and only if there exists a collection of integers satisfying some simple identities mod n. Its proof made extensive use of Thurston's theory of orbifolds (see W. Thurston [22, Chapter 13]).

A number of applications were obtained from the main theorem by elementary considerations. An immediate consequence was the following corollary.

Corollary 1.8.1. *Suppose that t_{g+1} has a root of degree n. Then*

(a) n *is odd.*

(b) $n \leq 2g + 1$.

This corollary shows that Margalit-Schleimer roots always have the maximum degree among the roots of t_{g+1} for a given genus.

I have continued this work by examining the case of a Dehn twist about a separating curve C. Although the existence of some roots is obvious in this case, there is much to be understood about their possible degrees and other behaviors. This work constitutes the remainder of this dissertation.

In ongoing work, I have been investigating a generalization of roots, the fractional powers of t_C. These are homeomorphisms h such that some h^n equals some t_C^ℓ, $\ell \neq 0$, in $\mathrm{Mod}(F)$. In contrast to the case of roots ($\ell = 1$), fractional powers may interchange the sides of C, so new phenomena occur. This work will be detailed in future publications.

Chapter 2

Roots of Dehn twists about separating curves

2.1 Introduction

Let F be a closed orientable surface of genus $g \geq 2$ and C be a simple closed curve in F. Let t_C denote a left handed Dehn twist about C.

When C is a nonseparating curve, the existence of roots of t_C is not so apparent. In their paper [15], D. Margalit and S. Schleimer showed the existence of such roots by finding elegant examples of roots of t_C whose degree is $2g+1$ on a surface of genus $g+1$. This motivated an earlier collaborative work with D. McCullough [16] in which we derived necessary and sufficient conditions for the existence of a root of degree n. As immediate applications of the main theorem in the paper, we showed that roots of even degree cannot exist and that $n \leq 2g+1$. The latter shows that the Margalit-Schleimer roots achieve the maximum value of n among all the roots for a given genus.

Suppose that C is a curve that separates F into subsurfaces $\widetilde{F}_i$ of genera g_i for

$i = 1, 2$. It is evident that roots of t_C exist. As a simple example, for the closed orientable surface of genus 2, we can obtain a square root of the Dehn twist t_C by rotating one of the subsurfaces on either side of C by an angle π, producing a half-twist near C. As in the case for nonseparating curves, a natural question is whether we can give necessary and sufficient conditions for the existence of a degree n root of t_C. In this chapter, we derive such conditions and apply them to obtain information about the possible degrees. We use Thurston's orbifold theory [22, Chapter 13] to prove the main result. A good reference for this theory is P. Scott [21].

We start by defining a special class of C_n-actions called *nestled* (n, ℓ)*-actions*. These C_n-actions have a distinguished fixed point and the points fixed by some nontrivial element of C_n form $\ell + 1$ orbits. The equivalency of two such actions will be given by the existence of a conjugating homeomorphism that also satisfies an additional condition on their distinguished fixed points. Two equivalence classes of actions will form a *compatible pair* if the turning angles of their representative actions around their distinguished fixed points add up to $2\pi/n$. The key topological idea in our theory is defining nestled (n_i, ℓ_i)-actions on the subsurfaces $\tilde{F}_i$ for $i = 1, 2$ so that they form a compatible pair, thus giving a root of degree $n = \mathrm{lcm}(n_1, n_2)$. Conversely, for each root of degree n, we reverse this argument to produce a corresponding compatible pair.

In Section 2.4, we introduce the abstract notion of a *data set* of degree n. As in the case of nonseparating curves, a *data set of degree* n is basically a tuple that encodes the essential algebraic information required to describe a nestled action. We show that equivalence classes of nestled (n, ℓ)-actions actually correspond to data sets, that is, each class has a corresponding data set representation. Data sets D_i of degree n_i, for $i = 1, 2$ form a *data set pair* (D_1, D_2) when they satisfy the formula $\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \bmod n$, where the turning angles at the centers of the disks are $\frac{2\pi k_i}{n_i} \bmod 2\pi$. In Theorem 2.5.2, we show that this number-theoretic condition is an algebraic equivalent of the compatibility condition for actions, thus proving that data set pairs correspond bijectively to conjugacy classes of roots. This theorem is essentially a translation of our topological theory of roots to the algebraic language of data sets.

As an immediate application of Theorem 2.5.2, we show the existence of a root of degree $\mathrm{lcm}(4g_1, 4g_2+2)$, and in Section 2.6, we give calculation of roots in low-genus cases. In Section 2.7, we obtain some bounds on the orders of spherical nestled actions, that is, nestled actions whose quotient orbifolds are topologically spheres. For example, we prove that all nestled (n, ℓ)-actions for $n \geq \frac{2}{3}(2g-1)$ have to be spherical. Finally, in Section 2.8, we use the main theorem and the results obtained in Section 2.7 to derive bounds on n. We show that in general, $n \leq 4g^2 + 2g$ and for any positive integer N, $n \leq 4g^2 + (4-2N)g + \frac{(N-2)^2}{4}$ whenever both $g_i > N + 3$.

2.2 Nestled (n, ℓ)-actions

An action of a group G on a topological space X is defined as a homomorphism $h : G \to \mathrm{Homeo}(X)$. Since we are interested only in C_n-actions, we will fix a generator t for C_n and identify the action with the isotopy class of the homeomorphism $h(t)$ in $\mathrm{Mod}(X)$. In this section, we introduce nestled (n, ℓ)-actions and give an example for such an action. These actions will play a crucial role in the theory we will develop for roots of Dehn twists.

Definition 2.2.1. An orientation-preserving C_n-action on a surface F of genus at least 1 is said to be a *nestled (n, ℓ)-action* if either $n = 1$, or $n > 1$ and:

(i) the action has at least one fixed point,

(ii) some fixed point has been selected as the distinguished fixed point, and

(iii) the points fixed by some nontrivial element of C_n form $\ell + 1$ orbits.

This is equivalent to the condition that the quotient orbifold has $\ell+1$ cone points, one of which is a distinguished cone point of order n.

A nestled (n, ℓ)-action is said to be *trivial* if $n = 1$, that is, if it is the action of the trivial group on F. In this case only, we allow a cone point of order 1 in the quotient orbifold. The distinguished cone point can then be any point in F, and we require $\ell = 0$.

Definition 2.2.2. Assume that F has a fixed orientation and fixed Riemannian metric. Let h be a nestled-(n, ℓ) action on F with a distinguished fixed point P. The *turning angle* $\theta(h)$ for h is the angle of rotation of the induced isomorphism h_* on the tangent space T_P, in the direction of the chosen orientation.

Example 2.2.3 (Margalit-Schleimer, [15])**.** Rotate a regular $(4g + 2)$-gon with opposite sides identified about its center P through an angle $\frac{2\pi(g+1)}{(2g+1)}$. Identifying the opposite sides of P, we get a C_{2g+1}-action h on S_g with three fixed points denoted by P, x and y. Since the quotient orbifold has three cone points of order $2g + 1$, this defines a nestled $(2g + 1, 2)$-action on S_g. If we choose P as the distinguished fixed point for the action h, then $\theta(h) = \frac{2\pi(g+1)}{(2g+1)}$.

Remark 2.2.4. Every nestled (n, ℓ)-action has an invariant disk around its distinguished fixed point. Let F be a closed oriented surface with a fixed Riemannian metric ρ, and let h be a nestled (n, ℓ)-action on F with a distinguished fixed point

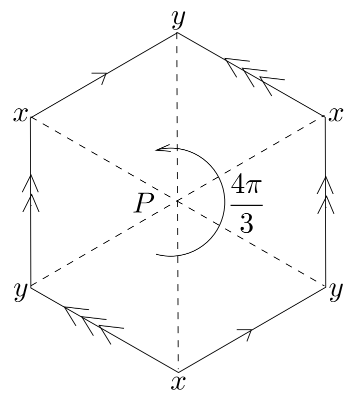

Figure 2.1: A nestled $(2g+1, 2)$-action for $g = 1$.

P. Consider the Riemannian metric $\bar{\rho}$ defined by

$$\langle v, w \rangle_{\bar{\rho}} = \frac{1}{n} \sum_{i=1}^{n} \langle {h^i}_*(v), {h^i}_*(w) \rangle_{\rho} ,$$

where $v, w \in T_P F$. Under this metric $\bar{\rho}$, h is an isometry. Since there exists $\epsilon > 0$ such that $exp_P : B_\epsilon(0) \subset T_P F \to B_\epsilon(P) \subset F$ is a diffeomorphism, h preserves the disk $B_\epsilon(P)$.

Definition 2.2.5. Two nestled (n, ℓ)-actions h and h' on F with distinguished fixed points P and P' are *equivalent* if there exists an orientation-preserving homeomorphism $t\colon F \to F$ such that

(i) $t(P) = P'$.

(ii) tht^{-1} is isotopic to h' relative to P'.

Remark 2.2.6. By definition, equivalent nestled (n, ℓ)-actions h and h' on F are conjugate in $\mathrm{Mod}(F)$. Since conjugate homeomorphisms have the same fixed point data, we have that $\theta(h) = \theta(h')$.

2.3 Compatible pairs and roots

Suppose that C is a curve that separates a surface F of genus g into two subsurfaces. As mentioned earlier, the central idea is defining compatible nestled actions on the subsurfaces that "fit together" to give a degree n root of the Dehn twist t_C. We will show in Theorem 2.3.4 that compatible pairs of equivalent actions correspond bijectively to conjugacy classes of roots of t_C.

Notation 2.3.1. Suppose that C separates a closed orientable surface F into subsurfaces of genera g_1 and g_2, where $g_1 \geq g_2$. Let F_i denote the closed surface obtained by coning the subsurface of genus g_i. We will think of F as $(F_1, C)\#(F_2, C)$, that is, the surface obtained by taking the connected sum of the F_i along C. For convenience, we will denote this by $F = F_1\#_C F_2$.

Definition 2.3.2. Equivalence classes $[h_i]$ of nestled (n_i, ℓ_i)-actions h_i on closed oriented surfaces F_i for $i = 1, 2$ are said to form a *compatible pair* $([h_1], [h_2])$ if $\theta(h_1) + \theta(h_2) = 2\pi/n \bmod 2\pi$.

The integer $n = lcm(n_1, n_2)$ is called the *degree* of the compatible pair. We may treat $([h_1], [h_2])$ as an unordered pair, since $([h_2], [h_1])$ is a compatible pair if and only if $([h_1], [h_2])$ is.

Lemma 2.3.3. *Let F be a compact orientable surface, possibly disconnected. If $h : F \to F$ is a homeomorphism such that h^n is isotopic to id_F, then h is isotopic to a homeomorphism j with $j^n = id_F$.*

Proof. When F is connected, this is the Nielsen-Kerchkoff theorem [11, 12, 19]. Suppose that F is not connected. We may asssume that h acts transitively on the set of components $F_1, F_2, ..., F_\ell$ of F. Choose notation so that $h\,|_{F_i}\colon F_i \to F_{i+1}$ and $h\,|_{F_{\ell-1}}\colon F_{\ell-1} \to F_1$. Since $h^n = (h^l)^{n/\ell} \simeq id_F$, the Nielsen-Kerchkoff theorem implies that $h^\ell |_{F_1} \simeq j_1$ where j_1 is a homeomorphism on F_1 with ${j_1}^{n/\ell} = id_{F_1}$. Therefore, $id_{F_1} \simeq j_1 \circ (h^\ell |_{F_1})^{-1}$ via an isotopy K_t. Define an isotopy H_t of h by $H_t |_{F_i} = h$ for $1 \leq i \leq \ell - 2$ and $H_t |_{F_{\ell-1}} = K_t \circ h |_{F_{\ell-1}}$. Then, $H_1 |_{F_{\ell-1}} = K_1 \circ h = j_1 \circ (h^\ell |_{F_1})^{-1} \circ h$. We see that $(H_1 |_{F_i})^\ell = h^i \circ (j_1 \circ h^{1-\ell}) \circ h^{\ell-1-i} = h^i \circ j_1 \circ h^{-i}$ and $(H_1 |_{F_i})^n = (H_1 |_{F_i}{}^\ell)^{n/\ell} = h^i \circ {j_1}^{n/\ell} \circ h^{-i} = h^i \circ h^{-i} = id_{F_i}$. The required homeomorphism is $j = H_1$. □

Theorem 2.3.4. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Then the conjugacy classes in* $\mathrm{Mod}(F)$ *of roots of t_C of degree n correspond to the compatible pairs* $([h_1], [h_2])$ *of equivalence classes of nestled (n_i, ℓ_i)-actions h_i on F_i of degree n.*

Proof. We will first prove that every root of degree n yields a compatible pair of $([h_1], [h_2])$ of degree n.

Fix a closed annulus neighborhood N of C. Let $\widetilde{F_i}$ for $i = 1, 2$ be the components of $\overline{G - N}$, and denote the genus of $\widetilde{F_i}$ by g_i . We fix coordinates on F so that the subsurface $\widetilde{F_1}$ is to the left of C as shown in Figure 2.2. By isotopy we may assume that $t_C(C) = C$, $t_C(N) = N$, and $t_C|_{\widetilde{F_i}} = id_{\widetilde{F_i}}$ for $i = 1, 2$.

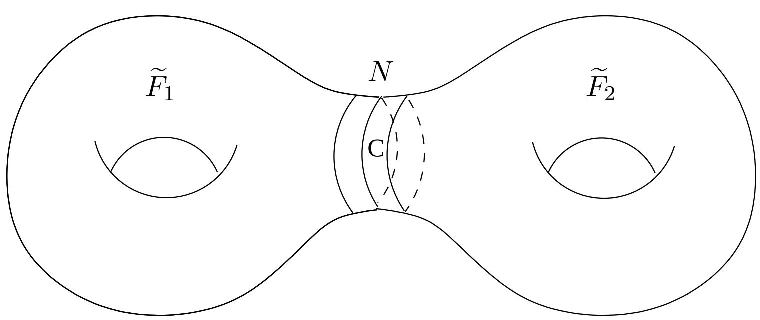

Figure 2.2: The surface F with the separating curve C and the tubular neighborhood N of C.

Suppose that h is an n^{th} root of t_C. We have $t_C \simeq h t_C h^{-1} \simeq t_{h(C)}$, which implies that $h(C)$ is isotopic to C. Changing h by isotopy, we may assume that

h preserves C and takes N to N. Put $\widetilde{h_i} = h|_{\widetilde{F_i}}$ for $i = 1, 2$. Since $h^n \simeq t_C$ and both preserve C, there is an isotopy from h^n to t_C preserving C and hence one taking N to N at each time. That is, $\widetilde{h_1}^n$ is isotopic to $id_{\widetilde{F_1}}$ and $\widetilde{h_2}^n$ is isotopic to $id_{\widetilde{F_2}}$. By Lemma 2.3.3, $\widetilde{h_i}$ is isotopic to a homeomorphism whose n^{th} power is $id_{\widetilde{F_i}}$ for $i = 1, 2$. So we may change $\widetilde{h_i}$ and hence h by isotopy to assume that $\widetilde{h_i}^n = id_{\widetilde{F_i}}$ for $i = 1, 2$.

Let n_i be the smallest positive integer such that $\widetilde{h_i}^{n_i} = id_{\widetilde{F_i}}$ for $i = 1, 2$. Let $s = lcm(n_1, n_2)$. Clearly, $s|n$ since $n_i|n$. Also, $h^s = id_{\widetilde{F_1} \cup \widetilde{F_2}}$ which implies that $h^s = {t_C}^d$ for some integer d. Hence, $(h^s)^{n/s} = ({t_C}^d)^{n/s}$ i.e. $h^n = {t_C}^{dn/s}$. We get, $t_C = {t_C}^{dn/s}$ which implies that $dn/s = 1$ since no higher power of t_C is isotopic to t_C. Hence, $d = 1$ and $n = s = lcm(n_1, n_2)$.

Assume for now that h does not interchange the sides of C. We fill in the boundary circles of $\widetilde{F_1}$ and $\widetilde{F_2}$ with disks to obtain the closed orientable surfaces F_1 and F_2 with genera g_1 and g_2 . We then extend $\widetilde{h_i}$ to a homeomorphism h_i on F_i by coning. Thus h_i defines a C_{n_i} action on F_i where $n_i|n$, $C_{n_i} = \langle h_i \mid h_i^{n_i} = 1\rangle$ for $i = 1, 2$ and $lcm(n_1, n_2) = n$. Since the homeomorphism h_i fixes the center point P_i of the disk $\overline{F_i - \widetilde{F_i}}$, we choose P_i as the distinguished fixed point for h_i. So h_i defines a nestled (n_i, ℓ_i)-action on F_i for some ℓ_i.

The orientation on F restricts to orientations on the F_i, so that we may speak of rotation angles $\theta(h_i)$ for h_i. Then the rotation angle $\theta(h_i) = 2\pi k_i/n_i$ for some k_i with $\gcd(k_i, n_i) = 1$. As seen in Figure 2.3, the difference in turning angles equals $2\pi k_2/n_2 - (-2\pi k_1/n_1) = 2\pi/n$, giving $\theta(h_1) + \theta(h_2) \equiv 2\pi/n \bmod 2\pi$. That is, (h_1, h_2) is a compatible pair.

Suppose now that h interchanges the sides of C. Then h must be of even order, say $2n$, and h^2 preserves the sides of C and is of order n. Since the actions of $h^2|_{\widetilde{F_i}}$ on the $\widetilde{F_i}$ are conjugate by $h|_{\widetilde{F_1} \cup \widetilde{F_2}}$, these actions will induce conjugate

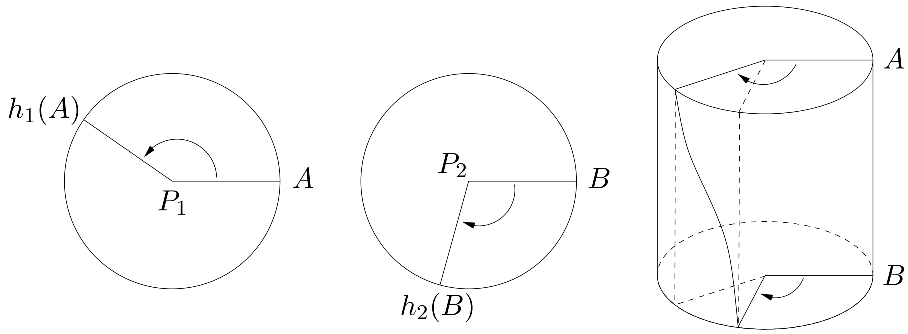

Figure 2.3: The local effect of h_1 and h_2 on disk neighborhoods of P_1 and P_2 in F_1 and F_2, and the effect of h on the neighborhood N of C in F. Only the boundaries of the disk neighborhoods are contained in $\widetilde{F}_i$, where they form the boundary of N. The rotation angle $\theta(h_1)$ is $2\pi k_1/n_1$ and the angle $\theta(h_2)$ is $2\pi k_2/n_2 = 2\pi(1/n - k_1/n_1)$.

C_n-actions on the coned surfaces F_i. Consequently, these induced actions will have the same turning angles at the centers P_i of the coned disks of F_i. For this compatible pair of nestled (n_i, ℓ_i)-actions, say (h_1, h_2), associated with h^2, we must have $\theta(h_1) = \theta(h_2) = \pi/n$ and $n_1 = n_2 = n$. If we extend to N using a simple left-handed twist, the twisting angle is $2\pi k/n$, and consequently $h^{2n} = t_C^{2k}$. Other extensions will differ from this by full twists, giving $h^{2n} = t_C^{2k+2jn}$ for some integer j. In any case, h^{2n} cannot equal t_C. This proves that h cannot reverse the sides of C.

Suppose that we have roots h and h' that are conjugate in $\mathrm{Mod}(F)$, that is, there exists $t \in \mathrm{Mod}(F)$ such that $h' = t \circ h \circ t^{-1}$. Then $(h')^n = t \circ h^n \circ t^{-1}$, that is, $t_C = t \circ t_C \circ t^{-1} = t_{t(C)}$. This shows that C and $t(C)$ are isotopic curves. Changing t by isotopy, we may assume that $t(C) = C$ and $t(N) = N$. Let t_i, h_i and h'_i respectively denote the extensions of $t|_{\widetilde{F}_i}$, $h|_{\widetilde{F}_i}$ and $h'|_{\widetilde{F}_i}$ to F_i by coning.

Assume for now that t does not exchange the sides of C. Since t, h and h' all preserve N, we may assume that the isotopy from $t \circ h \circ t^{-1}$ to h' preserves N, and consequently each $t_i \circ h_i \circ t_i^{-1}$ is isotopic to h'_i preserving P_i. Since t_i

takes P_i to P_i, h_i and h_i' are equivalent as nestled (n_i, ℓ_i)-actions on F_i, so h and h' produce the same compatible pair $([h_1], [h_2])$.

Suppose that t exchanges the sides of C. Then $g_1 = g_2$, $h_{3-i}' \simeq t_i \circ h_i \circ {t_i}^{-1}$ and $t_i(P_i) = P_{3-i}$. So the actions h_1 and h_2' are equivalent, as are the actions h_1' and h_2. Therefore, the (unordered) compatible pairs for the two roots are the same.

Conversely, given a compatible pair $([h_1], [h_2])$ of equivalence classes of nestled (n_i, ℓ_i)-actions, we can reverse the argument to produce a root h. For let P_i denote the distinguished fixed point of h_i and let p_i denote the corresponding cone point of order n_i in the quotient orbifold $\mathcal{O}_i$. By Remark 2.2.4, there exists an invariant disk D_i for h_i around p_i. Removing D_i produces the surfaces $\widetilde{F}_i$, and attaching an annulus N produces the surface F of genus g. Condition (ii) on compatible pairs ensures that the rotation angles work correctly to allow an extension of $h_1|_{\widetilde{F}_1} \cup h_2|_{\widetilde{F}_2}$ to an h with h^n being a single Dehn twist about C.

It remains to show that the resulting root h of t_C is determined up to conjugacy in the mapping class group of F. Suppose that $h_i' \in [h_i]$. Let P_i' denote the distinguished fixed point for h_i', and let D_i' be an invariant disk for h_i' around P_i'. Removing the $D_i's$ produces surfaces $\widetilde{F}_i' \cong F_i$, for $i = 1, 2$, and attaching an annulus N' with a $1/n^{th}$ twist, extends $h_1'|_{\widetilde{F}_1'} \cup h_2'|_{\widetilde{F}_2'}$ to a homeomorphism h' on a surface $F' \cong F$ of genus g. Since $h_i' \in [h_i]$, by definition, there exists t_i such that $t_i(P_i) = P_i'$ and $t_i \circ h_i \circ {t_i}^{-1} \simeq h_i'$ rel P_i' via an isotopy H_i in $\mathrm{Mod}(F_i')$. Since h_i and h_i' have finite order and are conjugate up to isotopy by t_i, we may assume that $t_i(D_i) = D_i'$ and, identifying F and F' using t, that the isotopy H_i from $t_i \circ h_i \circ {t_i}^{-1}$ to h_i' is relative to D_i. With respect to this identification, we choose a $k : N \to N$ such that $h'|_N = k \circ h|_N \circ k^{-1}$. Now define $t : F \to F$ by $t|_{\widetilde{F}_i} = h_i|_{\widetilde{F}_i}$, and $t|_N = k$. Then $h' \simeq t \circ h \circ t^{-1}$ via an isotopy H given by $H|_{\widetilde{F}_i} = H_i|_{\widetilde{F}_i}$, and

$H|_N = id_N$. □

2.4 Nested (n, ℓ)-actions and data sets

In this section, we introduce the language of data sets of degree n in order to algebraically encode equivalence classes of nestled (n, ℓ)-actions. We will prove that the equivalence classes of nestled (n, ℓ)-actions actually correspond to the possible data sets.

Definition 2.4.1. A *data set* for F is a tuple $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$ where n, $\widetilde{g}$ and the x_i are integers, a is a residue class modulo n, and each c_i is a residue class modulo x_i, such that

(i) $n \geq 1$, $\widetilde{g} \geq 0$, each $x_i > 1$, and each x_i divides n.

(ii) $\gcd(a, n) = \gcd(c_i, x_i) = 1$.

(iii) $a + \sum_{i=1}^{\ell} \frac{n}{x_i} c_i \equiv 0 \bmod n$.

The number n is called the *degree* of the data set. If $n = 1$, then we require that $a = 1$, and the data set is $D = (1, \widetilde{g}, 1;)$. The integer g defined by

$$g = \widetilde{g}n + \frac{1}{2}(1 - n) + \frac{1}{2} \sum_{i=1}^{\ell} \frac{n}{x_i}(x_i - 1)$$

is called the *genus* of the data set. We consider two data sets to be the same if they differ by reordering the pairs $(c_1, x_1), \ldots, (c_\ell, x_\ell)$.

Remark 2.4.2. For any data set $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_\ell, x_\ell))$, $\operatorname{lcm}\{x_1, x_2, \ldots, x_n\} = n$. To see this, put $k = \operatorname{lcm}\{x_1, x_2, \ldots, x_\ell\}$. Since each

$x_i \mid n$, $k \mid n$. So it remains to show that $n \mid k$. Condition (iii) implies that

$$\frac{ak}{k} + \sum_{i=1}^{\ell} \frac{n(k/x_i)}{k} c_i \equiv 0 \bmod n \ .$$

Multiplying by k we get

$$ak + n \sum_{i=1}^{\ell} (k/x_i) c_i \equiv 0 \bmod n \ .$$

Since $gcd(a, n) = 1$, we have $n \mid k$.

Notation 2.4.3. For a nestled nestled-(n, ℓ) action h on a closed orientable surface F of genus g, we will use the following notation throughout this section. Let $\mathcal{O}$ be the quotient orbifold for the action and let $\widetilde{g}$ be the genus of its underlying 2-manifold. Let P be the distinguished fixed point of h and let p be the cone point in $\mathcal{O}$ of order n that is its image in $\mathcal{O}$. Let $p_1, \ldots, p_\ell$ be the other possible cone points of $\mathcal{O}$, if any.

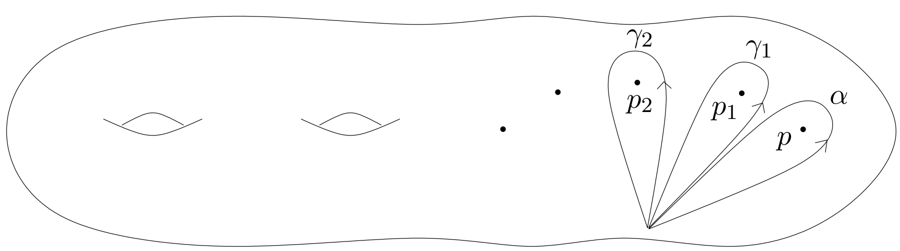

Figure 2.4: The orbifold $\mathcal{O}$

Figure 2.4 shows a generator α of the orbifold fundamental group $\pi_1^{orb}(\mathcal{O})$ that goes around the point p, and generators $\gamma_i, 1 \leq i \leq \ell$ going around p_i. Let a_j and b_j, $1 \leq j \leq \widetilde{g}$ be standard generators of the "surface part" of $\mathcal{O}$, chosen to give the following presentation of $\pi_1^{orb}(\mathcal{O})$:

$$\pi_1^{orb}(\mathcal{O}) = \langle \alpha, \gamma_1, \ldots, \gamma_\ell, a_1, b_1, \ldots, a_{\widetilde{g}}, b_{\widetilde{g}} \mid$$

$$\alpha^n = \gamma_1^{x_1} = \cdots = \gamma_\ell^{x_\ell} = 1,\ \alpha\gamma_1 \cdots \gamma_\ell = \prod_{1=1}^{\widetilde{g}} [a_i, b_i] \rangle.$$

With this notation, we are ready to establish the key property of data sets.

Proposition 2.4.4. *Data sets of degree* n *and genus* g *correspond to equivalence classes of nestled* (n, ℓ)*-actions on closed orientable surfaces of genus* g*.*

Proof. From orbifold covering space theory [22], we have the following exact sequence:

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1^{orb}(\mathcal{O}) \xrightarrow{\rho} C_n \longrightarrow 1\ .$$

The homomorphism ρ is obtained by lifting path representatives of elements of $\pi_1^{orb}(\mathcal{O})$— these do not pass through the cone points so the lifts are uniquely determined.

For $1 \leq i \leq l$, the preimage of p_i consists of n/x_i points cyclically permuted by h, where x_i is the order of the stabilizer of each point in the preimage of p_i. Each of the points has stabilizer generated by h^{n/x_i}. Its rotation angles must be the same at all points of the orbit, since its action at one point is conjugate by a power of h to its action at each other point. So the rotation angle at each point is of the form $2\pi c_i'/x_i$, where c_i' is a residue class modulo x_i and $\gcd(c_i', x_i) = 1$. Lifting the γ_i, we have that $\rho_1(\gamma_i) = h^{(n/x_i)c_i}$ where $c_i c_i' \equiv 1 \bmod x_i$.

Finally, we have $\rho(\prod_{i=1}^{\widetilde{g}} [a_i, b_i]) = 1$, since C_n is abelian, so

$$1 = \rho_i(\alpha\gamma_1 \cdots \gamma_\ell) = t^{a + (n/x_1)c_1 + \cdots + (n/x_i)c_i}$$

giving

$$a+\sum_{i=1}^{\ell}\frac{n}{x_i}c_i \equiv 0 \bmod n\ .$$

The fact that the data set D has genus equal to g follows easily from the multiplicativity of the orbifold Euler characteristic for the orbifold covering $F \to \mathcal{O}$:

$$\frac{2-2g}{n} = 2-2\widetilde{g}+\left(\frac{1}{n}-1\right)+\sum_{i=1}^{\ell}\left(\frac{1}{x_i}-1\right) \tag{2.1}$$

Thus, h gives a data set $D=(n,\widetilde{g},a;(c_1,x_1),\ldots,(c_\ell,x_\ell))$ of degree n and genus g.

Consider another nestled (n,ℓ)-action h' in the equivalence class of h with a distinguished fixed point P'. Then by definition there exists an orientation-preserving homeomorphism $t\in \mathrm{Mod}(F)$ such that $t(P)=P'$ and $th't^{-1}$ is isotopic to h relative to P. Therefore, the two actions will have the same fixed point data and hence produce the same data set D.

Conversely, given a data $D=(n,\widetilde{g},a;(c_1,x_1),\ldots,(c_\ell,x_\ell))$, we can reverse the argument to produce an equivalence class of a nestled (n,ℓ)-action h on a surface F of genus g. We construct the orbifold $\mathcal{O}$ and representation $\rho\colon \pi_1^{orb}(\mathcal{O})\to C_n$. Any finite subgroup of $\pi_1^{orb}(\mathcal{O})$ is conjugate to one of the cyclic subgroups generated by α or a γ_i, so condition (ii) in the definition of the data set ensures that the kernel of ρ is torsionfree. Therefore the orbifold covering $F\to\mathcal{O}$ corresponding to the kernel is a manifold, and calculation of the Euler characteristic shows that F has genus g.

It remains to show that the resulting action on F is determined up to our equivalence in $\mathrm{Mod}(F)$. Suppose that two actions h and h' on F with distinguished fixed points P and P' have the same data set D. D encodes the fixed-point data of the periodic transformations h. By a result of J. Nielsen [19] (see also A. Edmonds [5, Theorem 1.3]), h and h' have to be conjugate by an orientation-preserving homeomorphism t. As in the proof of Theorem 1.1 in [16], t may be chosen so that it preserves $t(P) = P'$. Thus D determines h up to equivalence. □

Proposition 2.4.4 enables us to view equivalence classes of nestled (n, ℓ)-actions simply as data sets.

Notation 2.4.5. We will denote a data set of degree n and genus g by $D_{n,g,i}$, where i is an index. The trivial data set $D = \{1, g, 1; \}$, for any g, will be denoted by $D_{1,g}$.

Example 2.4.6. The following are examples of data sets that represent nestled $(n, 2)$-actions, for every $g \geq 1$ and n equal to $2g + 1$, $4g$ and $4g + 2$:

(i) $D_{2g+1,g,1} = (2g + 1, 0, 1; (g, 2g + 1), (g, 2g + 1))$.

(ii) $D_{4g,g,1} = (4g, 0, 1; (1, 2), (2g - 1, 4g))$.

(iii) $D_{4g+2,g,1} = (4g + 2, 0, 1; (1, 2), (g, 2g + 1))$.

Remark 2.4.7. For the data set $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_n, x_\ell))$ associated with a nestled (n, ℓ)-action, Equation 2.1 in the proof of Proposition 2.4.4 gives the following inequality

$$\frac{1 - 2g}{n} = -(\ell - 1) - 2\widetilde{g} + \sum_{i=1}^{\ell} \frac{1}{x_i} \leq -(\ell - 1) + \sum_{i=1}^{\ell} \frac{1}{x_i}\ . \tag{2.2}$$

Remark 2.4.8. There exists no non-trivial action with $\ell = 0$. Suppose that we assume the contrary. Using Notation 2.4.3, we have

$$\pi_1^{orb}(\mathcal{O}) = \langle \alpha, a_1, b_1, \ldots, a_{\widetilde{g}}, b_{\widetilde{g}}\ |\alpha^n = 1, \alpha = \prod_{j=1}^{\widetilde{g}} [a_j, b_j]\ \rangle\ .$$

Since C_n is abelian, $\rho(\alpha) = \rho(\prod_{j=1}^{\widetilde{g}}[a_j, b_j]) = 1$, which is impossible since ρ has torsion free kernel.

2.5 Data set pairs and roots

By Theorem 2.3.4, each conjugacy class of a root of t_C in Mod(F) corresponds to a compatible pair $([h_1], [h_2])$ of (equivalence classes of) nestled actions, and by Proposition 2.4.4, such a pair determines a pair (D_1, D_2) of data sets. To determine which pairs arise, we must replace the geometric compatibility condition in Theorem 2.3.4 by an algebraic compatibility condition on the corresponding data sets.

Definition 2.5.1. Two data sets $D_1 = (n_1, \widetilde{g}_1, a_1; (c_{11}, x_{11}), \ldots, (c_{1\ell}, x_{1\ell}))$ and $D_2 = (n_2, \widetilde{g}_2, a_2; (c_{21}, x_{21}), \ldots, (c_{2m}, x_{2m}))$ are said to form a *data set pair* (D_1, D_2) if

$$\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \bmod n \tag{2.3}$$

where $n = \text{lcm}(n_1, n_2)$ and $a_i k_i \equiv 1 \bmod n_i$. Note that although the k_i are only defined modulo n_i, the expressions $\frac{n}{n_i}k_i$ are well-defined modulo n. The integer n is called the *degree* of the data set pair and $g = g_1 + g_2$ is called the *genus* of the data set pair. We consider (D_1, D_2) to be an unordered pair, that is, (D_1, D_2) and (D_2, D_1) are equivalent as compatible pairs.

We can now reformulate Theorem 2.3.4 in terms of data sets.

Theorem 2.5.2. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Then, data set pairs (D_1, D_2) of degree n and genus g, where D_1 is a data set of genus g_1 and D_2 is a data set of genus g_2, correspond to the conjugacy classes in $Mod(F)$ of roots of t_C of degree n.*

Proof. Let h denote the conjugacy class of a root of t_C of degree n with compatible pair representation $([h_1],[h_2])$. From Proposition 2.4.4, the h_i correspond to data sets $D_i = (n_i, \widetilde{g}_i, a_i; (c_{i1}, x_{i1}), \ldots, (c_{i\ell_i}, x_{1\ell_i}))$. So it suffices to show that the geometric condition $\theta(h_1) + \theta(h_2) = 2\pi/n$ in Definition 2.3.2 is equivalent to the condition $\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \bmod n$ in Definition 2.5.1.

As in the proof of Proposition 2.3.4, let P_i denote the center of the filling disk of the subsurface $\widetilde{F}_i$ of genus g_i. Choosing P_i as the distinguished fixed point of h_i, we get that $\theta(h_i) = 2\pi k_i/n_i$, where $\gcd(k_i, n_i) = 1$ and $a_i k_i \equiv 1 \bmod n_i$. Since $h^n = t_C$, the left-hand twisting angle along N is $2\pi/n$, which equals $2\pi k_2/n_2 - (-2\pi k_1/n_1) = 2\pi/n$, giving $\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \bmod n$. The converse is just a matter of reversing the argument. □

Corollary 2.5.3. *Suppose that $F = F_1 \#_C F_2$. Then there always exists a root of the Dehn twist t_C about C of degree lcm$(4g_1, 4g_2 + 2)$.*

Proof. As in Theorem 2.5.2, let $\widetilde{F}_i$ denote the subsurfaces obtained by cutting F along C, and let F_i denote the surfaces obtained by adding disks to the F_i. Let $n_1 = 4g_1$ and $n_2 = 4g_2 + 2$. From Example 2.4.6, for any residue class a_i modulo n_i with $\gcd(a_i, n_i) = 1$, the data set $D_1 = (n_1, 0, a_1; (-a_1, 2g_1), (a_1, 4g_1))$ defines a nestled $(n_1, 2)$-action on a surface F_1 of genus g_1, and the data set $D_2 = (n_2, 0, a_2; (a_2, 2), (a_2 g_2, 2g_2 + 1))$ defines a nestled $(n_2, 2)$-action on F_2 of genus g_2.

Let k_i denote the inverse of a_i modulo n_i and let $n = \mathrm{lcm}(n_1, n_2)$. We will now show that the a_i can be selected so that Equation 2.3 is satisfied. In other words, this will prove that D_1 and D_2 form a data set pair (D_1, D_2). Since $\frac{n}{n_1}$

and $\frac{n}{n_2}$ are relatively prime, there always exist integers p and q such that

$$\frac{n}{n_1}p + \frac{n}{n_2}q = 1 \ .$$

In particular, since $\frac{n}{n_1}$ and $\frac{n}{n_2}$ are not both odd, by [16, Lemma 7.1], p and q can be chosen so that $gcd(p, n_1) = gcd(q, n_2) = 1$. Let k_1 be the residue class of p modulo n_1 and let k_2 be the residue class of q modulo n_2. Taking modulo n, we get

$$\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \bmod n \ .$$

Therefore, by Theorem 2.5.2, there exists a root of t_C of order $lcm(4g_1, 4g_2 + 2)$. □

Corollary 2.5.4. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that M denotes the maximum degree of a root of the Dehn twist t_C about C. Then $2g^2 + 2g \leq M$.*

Proof. If g is even, then Corollary 2.5.3 with $g_1 = g_2 = \frac{g}{2}$ gives a root of degree $\text{lcm}(2g, 2g+1) = 2g(2g+1)$. If g is odd, then $g_1 = \frac{g+1}{2}$ and $g_2 = \frac{g-1}{2}$ gives a root of degree $\text{lcm}(2(g+1), 2g) \geq 2g(g+1)$. □

2.6 Classification of roots for the closed orientable surfaces of genus 2 and 3

2.6.1 Surface of genus 2

Let F denote the closed orientable surface of genus 2. Up to homeomorphism, there is a unique curve C that separates F into two subsurfaces of genus 1.

Given a root of t_C, the process described in the proof of Theorem 2.5.2 produces orientation-preserving C_{n_i} actions on the tori F_i for $i = 1, 2$ with $n = lcm(n_1, n_2)$.

If a cyclic group C_n acts faithfully on a surface F fixing a point x_0, then the map $C_n \longrightarrow Aut(\pi_1(F, x_0))$ is a monomorphism [2, Theorem 2, p.43]. We also know that the group of orientation-preserving automorphisms $Aut^+(\pi_1(F_i, x_0)) \cong Aut^+(\mathbb{Z} \times \mathbb{Z}) \cong SL(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Since any element of finite order of an amalgamated product $A *_C B$ is conjugate into one of the groups A or B [14], it can only be of order 2, 3, 4 or 6. Taking the least common multiple of any two of these orders gives 12 as the only other possibility for the order of a root of t_C. We summarize these inferences in the following corollary.

Corollary 2.6.1. *Let F be the closed orientable surface of genus 2 and C a separating curve in F. Then a root of a Dehn twist t_C about C can only be of degree 2, 3, 4, 6, or 12.*

Given below are the data set pairs that represent each conjugacy class of roots.

For $n = 2$:

(i) $(D_{2,1,1}, D_{1,1})$, where $D_{2,1,1} = (2, 0, 1; (1, 2), (1, 2), (1, 2))$.

For $n = 3$:

(i) $(D_{3,1,1}, D_{1,1})$, where $D_{3,1,1} = (3, 0, 1; (1, 3), (1, 3))$.

(ii) $(D_{3,1,2}, D_{3,1,2})$, where $D_{3,1,2} = (3, 0, 2; (2, 3), (2, 3))$.

For $n = 4$:

(i) $(D_{4,1,1}, D_{1,1})$, where $D_{4,1,1} = (4, 0, 1; (1, 2), (1, 4))$.

(ii) $(D_{4,1,2}, D_{2,1,1})$, where $D_{4,1,2} = (4, 0, 3; (1, 2), (3, 4))$.

For $n = 6$:

(i) $(D_{6,1,1}, D_{1,1})$, where $D_{6,1,1} = (6, 0, 1; (1, 2), (1, 3))$.

(ii) $(D_{6,1,2}, D_{3,1,1})$, where $D_{6,1,2} = (6, 0, 5; (1, 2), (2, 3))$.

(iii) $(D_{3,1,2}, D_{2,1,1})$.

For $n = 12$:

(i) $(D_{6,1,2}, D_{4,1,1})$.

(ii) $(D_{4,1,2}, D_{3,1,1})$.

It can be shown using elementary calculations that these are the only possible roots for the various orders. For example, when $n = 12$, the condition $\mathrm{lcm}(n_1, n_2) = 12$ would imply that the set $\{n_1, n_2\}$ can be either $\{6, 4\}$ or $\{4, 3\}$. When $n_1 = 6$ and $n_2 = 4$, the data set pair condition gives $2k_1 + 3k_2 \equiv 1 \bmod 12$. Since k_i is a residue modulo n_i, the only possible solution to this equation is $k_1 = 5$ and $k_2 = 1$. This would imply that $a_1 = 5$ and $a_2 = 1$ since a_i is the inverse of k_i modulo n_i. Geometrically, this represents the root h of t_C whose twisting angle on one side is $2\pi k_1/n_1 = 5\pi/3$ and on the other side of C is $2\pi k_2/n_2 = \pi/2$. Each data set D_i in the data set pair (D_1, D_2) is then uniquely determined by condition *(iii)* (for data sets) and the formula for calculating the genus g_i. Similar calculations can be used to determine all the data set pairs for the surface of genus 3.

2.6.2 Surface of genus 3

Up to homeomorphism, the surface of genus $g = 3$ has a unique curve that separates the surface into two subsurfaces of genera 2 and 1.

Given below are the data set pairs that represent roots of various degrees. For $n = 2$:

(i) $(D_{1,2}, D_{2,1,1})$.

(ii) $(D_{2,2,1}, D_{1,1})$, where $D_{2,2,1} = (2, 0, 1; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))$.

(iii) $(D_{2,2,2}, D_{1,1})$, where $D_{2,2,2} = (2, 1, 1; (1, 2))$.

For $n = 3$:

(i) $(D_{1,2}, D_{3,1,1})$.

(ii) $(D_{3,2,1}, D_{1,1})$, where $D_{3,2,1} = (3, 0, 1; (2, 3), (2, 3), (1, 3))$.

(iii) $(D_{3,2,2}, D_{1,1})$, where $D_{3,2,2} = (3, 0, 2; (1, 3), (1, 3), (2, 3))$.

For $n = 4$:

(i) $(D_{1,2}, D_{4,1,1})$.

(ii) $(D_{4,2,1}, D_{1,1})$, where $D_{4,2,1} = (4, 0, 1; (1, 2), (1, 2), (3, 4))$.

(iii) $(D_{4,2,2}, D_{4,1,1})$, where $D_{4,2,2} = (4, 0, 3; (1, 2), (1, 2), (2, 4))$.

For $n = 5$:

(i) $(D_{5,2,1}, D_{1,1})$, where $D_{5,2,1} = (5, 0, 1; (1, 5), (3, 5))$.

(ii) $(D_{5,2,2}, D_{1,1})$, where $D_{5,2,2} = (5, 0, 1; (2, 5), (2, 5))$.

For $n = 6$:

(i) $(D_{1,2}, D_{6,1,2})$.

(ii) $(D_{6,2,1}, D_{1,1})$, where $D_{6,2,1} = (6, 0, 1; (2, 3), (1, 6))$.

(iii) $(D_{2,2,1}, D_{3,1,2})$.

(iv) $(D_{2,2,2}, D_{3,1,2})$.

(v) $(D_{3,2,2}, D_{2,1,1})$.

(vi) $(D_{3,2,1}, D_{6,1,2})$.

(vii) $(D_{6,2,2}, D_{3,1,1})$, where $D_{6,2,2} = (6, 0, 5; (1, 3), (5, 6))$.

For $n = 8$:

(i) $(D_{8,2,1}, D_{1,1})$, where $D_{8,2,1} = (8, 0, 1; (1, 2), (3, 8))$.

(ii) $(D_{8,2,2}, D_{2,1,1})$, where $D_{8,2,2} = (8, 0, 5; (1, 2), (7, 8))$.

(iii) $(D_{8,2,3}, D_{4,1,1})$, where $D_{8,2,3} = (8, 0, 7; (1, 2), (5, 8))$.

(iv) $(D_{8,2,4}, D_{4,1,2})$, where $D_{8,2,4} = (8, 0, 3; (1, 2), (1, 8))$.

For $n = 10$:

(i) $(D_{10,2,1}, D_{1,1})$, where $D_{10,2,1} = (10, 0, 1; (1, 2), (2, 5))$.

(ii) $(D_{5,2,3}, D_{2,1,1})$, where $D_{5,2,3} = (5, 0, 3; (1, 5), (1, 5))$.

(iii) $(D_{5,2,4}, D_{2,1,1})$, where $D_{5,2,4} = (5, 0, 3; (3, 5), (4, 5))$.

For $n = 12$:

(i) $(D_{4,2,2}, D_{3,1,1})$.

(ii) $(D_{3,2,1}, D_{4,1,2})$.

(iii) $(D_{4,2,1}, D_{6,1,2})$.

(iv) $(D_{6,2,2}, D_{4,1,1})$.

For $n = 15$:

(i) $(D_{5,2,5}, D_{3,1,2})$, where $D_{5,2,5} = (5, 0, 3; (1, 5), (1, 5))$.

(ii) $(D_{5,2,6}, D_{3,1,2})$, where $D_{5,2,6} = (5, 0, 3; (3, 5), (4, 5))$.

For $n = 20$:

(i) $(D_{5,2,5}, D_{4,1,1})$, where $D_{5,2,5} = (5, 0, 4; (4, 5), (2, 5))$.

(ii) $(D_{5,2,6}, D_{4,1,1})$, where $D_{5,2,6} = (5, 0, 4; (3, 5), (3, 5))$.

(iii) $(D_{10,2,1}, D_{4,1,2})$, where $D_{10,2,1} = (10, 0, 7; (1, 2), (4, 5))$.

For $n = 24$:

(i) $(D_{8,2,4}, D_{3,1,2})$.

(ii) $(D_{8,2,3}, D_{6,1,1})$.

For $n = 30$:

(i) $(D_{10,2,2}, D_{3,1,1})$, where $D_{10,2,2} = (10, 0, 9; (1, 2), (3, 5))$.

(ii) $(D_{5,2,7}, D_{6,1,2})$, where $D_{5,2,7} = (5, 0, 1; (1, 5), (3, 5))$.

(iii) $(D_{5,2,8}, D_{6,1,2})$, where $D_{5,2,8} = (5, 0, 1; (2, 5), (2, 5))$.

2.7 Spherical nestled actions

A *spherical action* is simply a nestled (n, ℓ)-action whose quotient orbifold is topologically a sphere. We will show in Proposition 2.7.3 that nestled (n, ℓ)-actions must be spherical when n is sufficiently large relative to g. This means that in order to derive bounds on n, it suffices to restrict our attention to spherical actions. We will also derive several other results on spherical actions which we will use in later sections.

Definition 2.7.1. A non-trivial nestled (n, ℓ)-action is said to be *spherical* if the underlying manifold of its quotient orbifold is topologically a sphere.

Example 2.7.2. The actions in Examples 2.1 and 2.4.6 are spherical actions.

Proposition 2.7.3. *If $n > \frac{2}{3}(2g-1)$, then every nestled (n, ℓ)-action on F is spherical.*

Proof. Let $D = (n, \widetilde{g}, a; (c_1, x_1), \ldots, (c_n, x_\ell))$ be the data set associated with a nestled (n, ℓ)-action on F. Equation 2.2 gives

$$\widetilde{g} = \frac{1}{2} + \frac{2g-1}{2n} - \frac{\ell}{2} + \frac{1}{2}\sum_{i=1}^{\ell} \frac{1}{x_i} \ , \tag{2.4}$$

Each $x_i \geq 2$, and by Remark 2.4.8, we must have $\ell \geq 1$, so this becomes

$$\widetilde{g} \leq \frac{1}{2} + \frac{2g-1}{2n} - \frac{\ell}{4} \leq \frac{1}{4} + \frac{2g-1}{2n} \ .$$

That is, $\widetilde{g} \geq 1$ can hold only when $n \leq (4g-2)/3$. □

Remark 2.7.4. There exists no spherical nestled (n, ℓ)-action with $\ell = 1$. Suppose we assume on the contrary that $\ell = 1$. Then, Equation 2.1 would imply that

$$\frac{1-2g}{n} = \frac{1}{x_1} \ .$$

This is impossible since $x_1 > 0$ and $g \geq 1$.

Proposition 2.7.5. *Suppose that a surface F of genus g has a spherical nestled (n, ℓ)-action. Write the prime factorization of n as $n = p^a {q_1}^{a_1} \cdots {q_k}^{a_k}$ where*

$p^a > {q_i}^{a_i}$ for each $i \geq 1$, and write q for $\min\{p, q_1, \ldots, q_k\}$. If

$$n > \frac{2g-1}{2 - \frac{2}{q} - \frac{1}{p^a}} \ ,$$

then $\ell = 2$.

Proof. Each $x_i \geq q$, and by Proposition 2.4.2, at least one $x_i \geq p^a$. Using Equation 2.4 we have

$$0 = \frac{1}{2} + \frac{2g-1}{2n} - \frac{\ell}{2} + \frac{1}{2}\sum_{i=1}^{\ell} \frac{1}{x_i} \leq \frac{1}{2} + \frac{1}{2p^a} + \frac{2g-1}{2n} - \frac{\ell}{2} + \frac{\ell - 1}{2q}$$
$$\ell \leq 1 + \frac{q}{(q-1)p^a} + \frac{q}{q-1}\left(\frac{2g-1}{n}\right)$$

The right-hand side of the latter inequality is less than 3 when the inequality in the proposition holds. Therefore, by Remark 2.7.4, $\ell = 2$. □

Corollary 2.7.6. *Suppose that a surface F of genus g has a spherical nestled (n, ℓ)-action, $\ell \geq 2$.*

(i) If $n = 2$, then $\ell = 2g + 1$. In particular, there does not exist a spherical nestled $(2, 2)$-action.

(ii) If $n = 3$, then $\ell = g + 1$. There exists a spherical nestled $(3, 2)$-action if and only if $g = 1$.

(iii) If n is even, $n \geq 4$, and $n > \frac{4}{3}(2g - 1)$, then $\ell = 2$.

(iv) If n is odd, $n \geq 5$, and $n > \frac{15}{17}(2g - 1)$, then $\ell = 2$.

Proof. For (i), an Euler characteristic calculation shows that $\ell = 2g + 1$ when $n = 2$. These are exactly the hyperelliptic actions.

For (ii), when $n = 3$, an Euler characteristic calculation shows that $\ell = g + 1$, and as seen in Section 2.6, there is a nestled $(3, 2)$-action on the torus.

For (iii), suppose first that $n = 6$. In Proposition 2.7.5 we have $q = 2$ and $p^a = 3$, giving the conclusion that if $6 > \frac{3}{2}(2g - 1)$, then $\ell = 2$. The condition $6 > \frac{3}{2}(2g - 1)$ holds exactly when $g \leq 2$, so (iii) is true in this case. One can check that there exist nestled $(6, 2)$-actions exactly when $g \leq 2$. For the cases of (iii) other than $n = 6$, we have $q = 2$ and $p^a \geq 4$, and Proposition 2.7.5 gives the result.

For (iv), we have $q \geq 3$ and $p^a \geq 5$. Again Proposition 2.7.5 gives the result. □

2.8 Bounds on the degree of a root

In this section, we use Theorem 2.5.2 and the results derived in Section 2.7 to derive some results on the degree n of a root. Among the results derived is an upper bound and a stable upper bound for n.

Remark 2.8.1. It is a well known fact [7] that the maximum order for an automorphism of a surface of genus g is $4g + 2$. In Example 2.4.6, we showed that a nestled action of order $4g + 2$ always exists.

Proposition 2.8.2. *There exists no nestled* $(4g + 1, \ell)$*-action.*

Proof. By Proposition 2.7.3, a nestled $(4g+1, \ell)$-action must be spherical, and by Proposition 2.7.5, $\ell = 2$. Therefore, Equation 2.1 in the proof of Theorem 2.5.2 simplifies to give

$$\frac{2g + 2}{4g + 1} = \frac{1}{x_1} + \frac{1}{x_2} \ .$$

Without loss of generality, we may assume that $x_1 \leq x_2$. Since $x_i \mid 4g+1$, $x_i \geq 3$. If $x_1 = 3$, then

$$x_2 = \frac{3(4g+1)}{2g+5} = 3\left(2 - \frac{9}{2g+5}\right) .$$

Since $x_2 = 3$ is the only integer solution for x_2, Proposition 2.4.2 would imply that $n = 3$ which contradicts that fact that $n = 4g+1$. If $x_1 \geq 4$, then we would have that

$$\frac{1}{2} < \frac{2+2g}{4g+1} = \frac{1}{x_1} + \frac{1}{x_2} \leq \frac{1}{2} ,$$

which is not possible. □

Proposition 2.8.3. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Let (D_1, D_2) be a data set pair corresponding to a root of t_C of degree n, and let n_i be the degree of D_i for $i = 1, 2$. Then the n_i cannot both satisfy $n_i \equiv 2 \bmod 4$.*

Proof. Suppose for contradiction that both n_i satisfy $n_i \equiv 2 \bmod 4$. Let a_i denote the a-value of D_i, and let k_i denote the inverse of a_i modulo n_i. Since $\gcd(k_i, n_i) = 1$, the k_i must be odd. Also the fact that $\gcd(n_1, n_2) = 2k$ for some odd integer k implies that $\frac{n}{n_i}$ is odd. From Equation 2.3 for the data set pair (D_1, D_2), we must have that

$$\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2 \equiv 1 \bmod n ,$$

which is impossible since $\frac{n}{n_1}k_1 + \frac{n}{n_2}k_2$ and n are even. □

Proposition 2.8.4. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that $M(g_1, g_2)$ denotes the maximum degree of a root of the Dehn twist t_C about C. Then $M(g_1, g_2) \leq 16g_1g_2 + 4(2g_1 - g_2) - 2$.*

Proof. Let n be the order of a root of t_C, given by a data set pair (D_1, D_2). We have $n = \mathrm{lcm}(n_1, n_2)$, where n_i is the degree of D_i. By Remark 2.8.1, each $n_i \leq 4g_i + 2$. By Proposition 2.8.2, neither $n_i = 4g_i + 1$, and by Proposition 2.8.3, we cannot have both $n_1 = 4g_1 + 2$ and $n_2 = 4g_2 + 2$. If both $n_1 = 4g_1$ and $n_2 = 4g_2$, then $\mathrm{lcm}(n_1, n_2) = 4\,\mathrm{lcm}(g_1, g_2) \leq 4g_1g_2 \leq 16g_1g_2 + 4(2g_1 - g_2) - 2$. In general, since $g_1 \geq g_2$, we have that $M(g_1, g_2) \leq \max\{(4g_1 + 2)(4g_2 - 1), (4g_1 - 1)(4g_2 + 2)\} = 16g_1g_2 + 4(2g_1 - g_2) - 2$. □

Notation 2.8.5. We will denote the upper bound $16g_1g_2 + 4(2g_1 - g_2) - 2$ derived in Proposition 2.8.4 by $U(g_1, g_2)$.

Theorem 2.8.6. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that n denotes the degree of a root of the Dehn twist t_C about C. Then $n \leq 4g^2 + 2g$.*

Proof. Since $g_2 = g - g_1$, we have that $16g_1g_2 + 4(2g_1 - g_2) - 2 = -16g_1^2 + g_1(16g + 12) - (4g + 2)$, which has its maximum when $g_1 = \frac{1}{8}(4g + 3)$. The fact that g_1 is an integer implies that when g is even, $g_1 = g_2 = g/2$, and when g is odd, $g_1 = (g + 1)/2$ and $g_2 = (g - 1)/2$. So Proposition 2.8.4 tells us that when g is even, $n \leq M(g/2, g/2) \leq 4g^2 + 2g - 2$, and when g is even, $n \leq M((g + 1)/2, (g - 1)/2) \leq 4g^2 + 2g$. □

Notation 2.8.7. We will denote the upper bound $4g^2 + 2g$ derived in Theorem 2.8.6 by $U(g)$.

For $2 \leq g \leq 35$, Table 1 gives the realizable maximum degrees of root, $m(g)$ (coming from compatible pairs of spherical nestled $(n, 2)$-actions) and the upper bound $U(g)$. The last column gives the ratio $m(g)/U(g)$. These computations were made using software [20] written for the GAP programming language.

g	$m(g)$	$U(g)$	$m(g)/U(g)$
2	12	20	0.60
3	30	42	0.71
4	42	72	0.58
5	90	110	0.81
6	126	156	0.81
7	210	210	1.00
8	240	272	0.88
9	330	342	0.96
10	390	420	0.93
11	462	506	0.91
12	546	600	0.91
13	570	702	0.81
14	714	812	0.88
15	798	930	0.86
16	858	1056	0.81
17	966	1190	0.81
18	1122	1332	0.84
19	1254	1482	0.85
20	1326	1640	0.81
21	1518	1806	0.84
22	1650	1980	0.83
23	1794	2162	0.83
24	1950	2352	0.83
25	2046	2550	0.80
26	2262	2756	0.82
27	2418	2970	0.81
28	2550	3192	0.80
29	2730	3422	0.80
30	2958	3660	0.81
31	3162	3906	0.81
32	3306	4160	0.79
33	3570	4422	0.81
34	3774	4692	0.80
35	3990	4970	0.80

Table 2.1: The data seems to indicate that for large genera the ratio $m(g)/U(g)$ stabilizes to the 0.79-0.82 range.

Proposition 2.8.8. *Suppose that we have a spherical nestled (n, ℓ)-action on a surface F of genus g, where n is a positive odd integer. Then $n \leq 3g + 3$.*

Proof. From Remark 2.7.4, we have that $\ell \neq 1$. When $\ell \geq 2$, the proposition follows from Corollary 2.7.6. Let $D = (n, 0, a; (c_1, x_1), (c_2, x_2))$ be a data set for the nestled $(n, 2)$-action on F. Since n is odd and $x_i \mid n$, we have that $x_i \geq 3$. If $x_1 \geq 3$, then Remark 2.4.2 implies that $x_2 \geq \frac{n}{3}$. So Equation 2.2 gives the inequality

$$\frac{1 - 2g}{n} \leq -1 + \frac{1}{3} + \frac{3}{n} ,$$

which upon simplification gives $n \leq 3g + 3$. □

Corollary 2.8.9. *Suppose that we have a spherical nestled $(4g - N, 2)$-action on a F of genus g, where N is a positive odd integer. Then $g \leq N + 3$.*

Theorem 2.8.10. *Let $F = F_1 \#_C F_2$ be a closed oriented surface of genus $g \geq 2$. Suppose that $M(g_1, g_2)$ denotes the maximum order of a root of the Dehn twist t_C about C. Then given a positive odd integer N, we have that $M(g_1, g_2) \leq 16g_1g_2 + 4(2g_1 - Ng_2) - 2N$ whenever both $g_i > N + 3$.*

Proof. By Remark 2.8.1, each $n_i \leq 4g_i + 2$. From Propositions 2.8.2 and 2.8.3, we know that $n_i \neq 4g_i + 1$ and that n_i cannot both be $4g_i + 2$. Suppose that the n_i are not both even. If $\ell_i > 2$, then from Corollary 2.7.6 we have that $n_i \leq \frac{15}{17}(2g_i - 1)$. If $\ell_i = 2$, then Corollary 2.8.9 tells us that for all $g_i > N + 3$, there exists no spherical nestled $(4g_i - N, 2)$-action on F. In particular, if $g_i > N + 3$, then from Proposition 2.7.3, $n_i \leq \frac{2}{3}(2g_i - 1) \leq \frac{15}{17}(2g_i - 1)$. So for all ℓ, if $g_i > N + 3$, then $n_i \leq \frac{15}{17}(2g_i - 1)$. We can see that $\frac{15}{17}(2g_i - 1) \leq 4g_i - N$ whenever $g_i \geq \frac{1}{38}(17N - 15)$. Therefore, if $g_i > \max\{N+3, \frac{1}{38}(17N-15)\} = N+3$, then we have that $M(g_1, g_2) \leq \max\{(4g_1 - N)(4g_2 + 2), (4g_1 + 2)(4g_2 - N)\} =$

$16g_1g_2 + 4\max\{(2g_1 - Ng_2), (2g_2 - Ng_1)\} - 2N = 16g_1g_2 + 4(2g_1 - Ng_2) - 2N$.

Suppose that both the n_i are even. Then from Propositions 2.8.2 and 2.8.3, we have that $M(g_1, g_2) \leq \text{lcm}(4g_1 + 2, 4g_2) \leq 8g_1g_2 + 4g_2$. We need to show that $8g_1g_2 + 4g_2 \leq 16g_1g_2 + 4(2g_1 - Ng_2) - 2N$. Since $g_1 > N + 3$, $(16g_1g_2 + 4(2g_1 - Ng_2) - 2N) - (8g_1g_2 + 4g_2) = 8g_1g_2 + 8g_1 - 4(N+1)g_2 - 2N > 8g_1g_2 + 8g_1 + 4(g_1 - 2)g_2 + 2(g_1 - 3) = 12g_1g_2 + 10g_1 - 8g_2 - 6 > 0$. □

Notation 2.8.11. We will denote the upper bound $16g_1g_2 + 4(2g_1 - Ng_2) - 2N$ derived in Theorem 2.8.10 by $U(g_1, g_2, N)$.

Example 2.8.12. When $N = 11$, if both $g_i > 14$, then from Theorem 2.8.10, $M(g_1, g_2) \leq U(g_1, g_2, 11) = 16g_1g_2 + 4(2g_1 - 11g_2) - 22$. For genera pairs (g_1, g_2) with $30 \leq g_1 + g_2 \leq 35$, Table 2 gives the values of the realizable maximum degree $m(g_1, g_2)$ (coming from compatible spherical nestled $(n, 2)$-actions), the upper bound $U(g_1, g_2)$ (derived in Proposition 2.8.4), and the stable upper bound $U(g_1, g_2, N)$.

g	$(g1, g2)$	$m(g_1, g_2)$	$U(g_1, g_2, 11)$	$U(g_1, g_2)$
30	$(15, 15)$	2790	3038	3658
31	$(16, 15)$	3162	3286	3906
32	$(16, 16)$	3264	3498	4158
32	$(17, 15)$	3162	3534	4154
33	$(17, 16)$	3570	3762	4422
33	$(18, 15)$	3534	3782	4402
34	$(17, 17)$	3570	3990	4690
34	$(18, 16)$	3774	4026	4686
34	$(19, 15)$	3534	4030	4650
35	$(18, 17)$	3990	4270	4970
35	$(19, 16)$	3876	4290	4950
35	$(20, 15)$	3690	4278	4898

Table 2.2: For $N = 11$, this data illustrates the stable bound $U(g_1, g_2, 11)$ and the upper bound $U(g_1, g_2)$. When $g = 32$, we can see that both $U(16, 16, 11)$ and $U(17, 15, 11)$ are significantly closer to $m(32) = 3306$ when compared with $U(32)$.

Bibliography

[1] Christian Bonatti and Luis Paris. Roots in the mapping class groups. *Proc. Lond. Math. Soc. (3)*, 98(2):471–503, 2009.

[2] P. E. Conner and Frank Raymond. Deforming homotopy equivalences to homeomorphisms in aspherical manifolds. *Bull. Amer. Math. Soc.*, 83(1):36–85, 1977.

[3] M. Dehn. Die Gruppe der Abbildungsklassen. *Acta Math.*, 69(1):135–206, 1938. Das arithmetische Feld auf Flächen.

[4] Max Dehn. *Papers on group theory and topology.* Springer-Verlag, New York, 1987. Translated from the German and with introductions and an appendix by John Stillwell, With an appendix by Otto Schreier.

[5] Allan L. Edmonds. Surface symmetry. I. *Michigan Math. J.*, 29(2):171–183, 1982.

[6] Robert Fricke and Felix Klein. *Vorlesungen über die Theorie der automorphen Funktionen. Band 1: Die gruppentheoretischen Grundlagen. Band II: Die funktionentheoretischen Ausführungen und die Andwendungen*, volume 4 of *Bibliotheca Mathematica Teubneriana, Bände 3.* Johnson Reprint Corp., New York, 1965.

[7] W. J. Harvey. Cyclic groups of automorphisms of a compact Riemann surface. *Quart. J. Math. Oxford Ser. (2)*, 17:86–97, 1966.

[8] W. J. Harvey. Geometric structure of surface mapping class groups. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 255–269. Cambridge Univ. Press, Cambridge, 1979.

[9] W. J. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 245–251. Princeton Univ. Press, Princeton, N.J., 1981.

[10] A. Hurwitz. Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.*, 41(3):403–442, 1892.

[11] Steven P. Kerckhoff. The Nielsen realization problem. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):452–454, 1980.

[12] Steven P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[13] Steven P. Kerckhoff. The measure of the limit set of the handlebody group. *Topology*, 29(1):27–40, 1990.

[14] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.

[15] Dan Margalit and Saul Schleimer. Dehn twists have roots. *Geom. Topol.*, 13(3):1495–1497, 2009.

[16] Darryl McCullough and Kashyap Rajeevsarathy. Roots of dehn twists. *Geometriae Dedicata*, pages 1–13, 2010. 10.1007/s10711-010-9541-4.

[17] Jakob Nielsen. Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. II. *Acta Math.*, 53(1):1–76, 1929.

[18] Jakob Nielsen. Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. III. *Acta Math.*, 58(1):87–167, 1932.

[19] Jakob Nielsen. Abbildungsklassen endlicher Ordnung. *Acta Math.*, 75:23–115, 1943.

[20] Kashyap Rajeevsarathy. *GAP Software*. available at: http://www.math.ou.edu/~kashyap/n2.g.

[21] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[22] William P. Thurston. *The Geometry and Topology of Three-Manifolds*. notes available at: http://www.msri.org/communications/books/gt3m/PDF.

[23] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[24] A. Wiman. Ueber die hyperelliptischen curven and diejenigen vom geschlechte $p = 3$, welche eindeutigen transformationen in sich zulassen. *Bihang Kongl. Svenska Vetenskaps-Akademiens Handilgar (Stockholm 1895-6)*.