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DEDICATION

to

My parents

Rudra Nath Acharya and Kamala Acharya

For

Encouraging me to follow my dreams

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Index of Notations

$\mathbb{R}$ real numbers

$\mathbb{Z}$ integers

$\mathbb{N}$ natural numbers

$\mathbb{R}_+$ nonnegative real numbers

$\mathbb{R}_-$ negative real numbers

$\mathbb{Z}_+$ nonnegative integers

$\mathbb{Z}_-$ negative integers

$\mathbb{C}$ complex numbers

$\mathbb{C}^+$ upper half-plane

$\mathbb{C}^-$ lower half-plane

$L^2(\mathbb{R})$ square integrable functions on $\mathbb{R}$

$\ell^2(\mathbb{Z})$ square summable sequences on $\mathbb{Z}$

$B(\mathcal{H})$ space of all bounded linear operators on a Hilbert space $\mathcal{H}$

$\mathbb{H}$ space of Herglotz functions

$L^1_{loc}(\mathbb{R})$ space of locally integrable functions on $\mathbb{R}$

$\operatorname{tr} A$ sum of diagonal elements of a square matrix A

Abstract

The main purpose of this dissertation is to give an alternate proof of de Branges' theorem on canonical systems and to prove Remling's theorem on canonical systems.

In order to prove de Branges theorem, first we show that, in the limit-circle case, the defect index of a symmetric relation induced by a canonical system is constant on $\mathbb{C}$. Then this follows de Branges' theorem that a canonical system with $\operatorname{tr} H \equiv 1$ implies the limit-point case. As such, we develop spectral theory of a linear relation in a Hilbert space as a tool and use the theory to discuss spectral theory of a relation induced by a canonical system.

Next, we prove Remling's theorem on canonical systems. We follow the similar techniques of Remling from [14]. More precisely, we first prove Breimesser-Pearson theorem on canonical systems, following the similar techniques from [3]. Then, we present the proof of Remling's theorem on canonical systems. We also show the connection between Jacobi and Schrödinger equations and canonical systems.

Introduction

The Jacobi and Schrödinger equations are the fundamental equations in quantum mechanics which can be used to describe quantum dynamics of many-particle systems under the influence of a variety of forces. A Schrödinger equation in one dimensional space is given by

$$-y'' + V(x)y = zy, \ z \in \mathbb{C} \tag{0.0.1}$$

where a function $V : \mathbb{R} \to \mathbb{R}$ is called a potential. In the discrete case, a Jacobi equation is given by

$$a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) = zu(n), \ z \in \mathbb{C}. \tag{0.0.2}$$

Here $a(n)$ and $b(n)$ are bounded sequences of real numbers. The corresponding operators

$$T = -\frac{d^2}{dx^2} + V(x), \qquad J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-2} & b_{-1} & a_{-1} & & & \\ & & a_{-1} & b_0 & a_0 & & \\ & & & a_0 & b_1 & a_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

act on $L^2(\mathbb{R})$ and $\ell^2(\mathbb{Z})$ respectively. The equations (0.0.1) and (0.0.2) can be written as

$$Ty = zy \text{ and } Ju = zu$$

respectively and can be considered as eigenvalue equations. Here T and J act on infinite dimensional spaces as such the eigenvalue problem can become complicated. Indeed, the eigenvalue equations can have solutions which do not decay at infinity but instead are bounded or grow at infinity. The behavior at infinity crucially depends on the spectrum of the operators. This leads to the spectral theory of Schrödinger and Jacobi operators.

A canonical system is a family of differential equations of the form

$$Ju'(x) = zH(x)u(x), z \in \mathbb{C} \tag{0.0.3}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H(x)$ is a 2×2 positive semidefinite matrix whose entries are locally integrable. We also assume that there is no non-empty open interval I so that $H \equiv 0$ a.e. on I. The complex number $z \in \mathbb{C}$ involved in (0.0.3) is a spectral parameter. For fixed z, a function $u(., z) : [0, N] \to \mathbb{C}^2$ is called a solution if u is absolutely continuous and satisfies (0.0.3). Consider the Hilbert space

$$L^2(H, R_+) = \left\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} : \int_0^\infty f(x)^* H(x) f(x) dx < \infty \right\}$$

with an inner product $\langle f, g \rangle = \int_0^\infty f(x)^* H(x) g(x) dx$. Such canonical systems on the Hilbert space $L^2(H, R_+)$ have been studied by De snoo, Hassi, Remling and Winkler in [10, 11, 15, 18]. The Jacobi and Schrödinger equations can be written into canonical systems with appropriate choice of $H(x)$, see Section 3.5.

In addition, canonical systems are closely connected with the theory of de Branges spaces and the inverse spectral theory of one dimensional Schrödinger operators, see [15]. We always get positive Borel measures, as the spectral measures, from Schrödinger operators. However, it is not always possible to get a potential that defines a Schrödinger operator, from a given positive Borel measure. This situation has been dealt in the inverse spectral theory of Schrödinger operators.

There is a one to one correspondence between positive Borel measures and canonical systems with $\operatorname{tr} H(x) \equiv 1$, see [18]. As Schrödinger equations can be written into canonical systems, we believe that canonical systems with $\operatorname{tr} H(x) \equiv 1$ can be useful tools for inverse spectral theory of one dimensional Schrödinger operators. Thus, it is a natural context to consider the spectral theory of such systems and we believe that the extension of the theories form Jacobi and Schrödinger operators to canonical systems is to be of general interest.

For any $z \in \mathbb{C}$, the solution space of the canonical system (0.0.3) is a two dimensional vector space. For any $z \in \mathbb{C}^+$, we want to define a coefficient $m(z)$ such that $f(x,z) = u(x,z) + m(z)v(x,z) \in L^2(H, R_+)$ for any linearly independent solutions $u(x,z), v(x,z)$ of (0.0.3). This leads us defining Weyl m functions $m_N(z)$ on compact interval $[0, N]$. These are holomorphic functions which map upper half-plane to itself. Moreover, these are fractional linear transformations which describe Weyl circles say $C_N(z)$. The Weyl discs, consisting of $C_N(z)$ and the interior, have nested property. Therefore the sequence of Weyl circles $C_N(z)$ converges either to a point, called a limit-point case or to a circle, called a limit-circle case. For $z \in C^+$, it also follows from Weyl theory that in the limit-circle case all solutions of the system (0.0.3) are in $L^2(H, R_+)$ and in the limit-point case there is unique solution in $L^2(H, R_+)$. One of the main results in this text is that the canonical system (0.0.3) with $\operatorname{tr} H(x) \equiv 1$ prevails the limit-point case.

This results was first proved by L. de Branges in his paper [2] using function theoretic approach. We give an alternate proof of the result by using completely different approach.

Recently, in the spectral theory of Jacobi and Schrödinger operators, Remling's theorem, published in the *Annals of Math* in 2011 (see [14]), has been one of the most popular results. It has revealed some new fundamental properties of absolutely continuous spectrum of Jacobi and Schrödinger operators that changed the perspective of many mathematicians about the absolutely continuous spectrum. In this text, we will generalize Remling's theorem on canonical systems when $\operatorname{tr} H(x) \equiv 1$.

We may think of writing the system (0.0.3) in the form

$$H(x)^{-1}Ju' = zu$$

and consider as an operator on $L^2(H, R_+)$. But $H(x)$ is not invertible in general that prevents us to work as an eigen value problem of an operator. Instead, the system (0.0.3) induces a linear relation that may have a multi-valued part. Therefore, we will discuss spectral theory of such linear relation induced by (0.0.3) on $L^2(H, R_+)$. In order to do this, we need to develop a theory of a linear relation on any Hilbert space and then we use the theory to discuss spectral theory of a relation induced by the canonical system (0.0.3).

Organization of the text : In Chapter 1, we will discuss about some standard results from functional analysis, real analysis and complex analysis. More precisely, we will present basic definitions and state the theorems without giving proofs which can be found in any standard book on those areas. We will mainly focus on the spectral theory of a linear operator on a Hilbert space, basics of measure theory and some notion of Herglotz functions and related theorems.

Chapter 2 is devoted to prove de Branges theorem: a canonical system (0.0.3) with $\operatorname{tr} H(x) \equiv 1$ prevails the limit-point case. In order to prove this theorem, we develop a theory of linear relation in a Hilbert space as a tool following the analogous treatment of operator theory from [17]. In Section 2.1, we obtain the conditions for symmetric relations to have self-adjoint extensions. Then we discuss spectral theory of such self-adjoint relations in Section 2.2. We will use the theory to discuss spectral theory of a relation induced by the canonical system (0.0.3) which is used to prove de Branges theorem.

Our main goal in Chapter 3 is to prove Remling's theorem on canonical systems. We will use similar techniques from [14]. In Section 3.1, we discuss the Weyl theory of canonical systems following the analogous treatment of Weyl theory of Jacobi and Schrödinger operators. In Section 3.2, we consider the space of Hamiltonians $H(x)$ and a suitable topology on it so that the space is metrizable. Then we will define the ω limit set of a Hamiltonian which is an important object in the theorem. We will discuss the notion of reflectionless Hamiltonians and state the theorem in Section 3.3. We will also discuss basics of harmonic measures, and value distribution in Section 3.3 which are the basic tools there in. Then we prove Breimesser-Pearson theorem on canonical systems which is in fact the foundation for Remling's theorem. This, finally follows the proof of Remling's theorem on canonical systems. At the end, we will show the connection between Jacobi and Schrödinger operators and canonical systems in Section 3.5.

Chapter 1

Preliminaries

In this chapter, we present some basic definitions and some important theorems from functional analysis, real analysis and complex analysis. We assume that a reader has some basic knowledge on each subjects. We also believe that a reader can find the proof of the theorems that we mention here in any standard text book on above mentioned areas, for example [8, 9, 16, 17].

We will consider a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ in which the inner product $\langle f, g\rangle$ is linear in the second parameter and conjugate linear in the first parameter.

Let $T \in B(\mathcal{H})$, the adjoint of T is an operator $T^* : \mathcal{H} \to \mathcal{H}$ such that $\langle h, Tf\rangle = \langle T^*h, f\rangle$ for all $f, h \in \mathcal{H}$.

Definition 1.1. *Let $T \in B(\mathcal{H})$. We call T self-adjoint if $T = T^*$, unitary if $TT^* = T^*T = 1$ and normal if $TT^* = T^*T$.*

Notice that T is unitary precisely if T is invertible and $T^{-1} = T^*$ and the following statements are equivalent:

- T is unitary;
- T is bijective and $\langle Tf, Th\rangle = \langle f, h\rangle$ for all $f, h \in \mathcal{H}$;

- T is surjective and isometry (i.e. $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$).

Let $N(T) = \{f \in \mathcal{H} : Tf = 0\}$ and $R(T) = \{Tf : f \in \mathcal{H}\}$.

An operator $P \in B(\mathcal{H})$ is a projection on a closed subspace $M \subset \mathcal{H}$ precisely if $P^2 = P$ and $R(P) = M$. Observe that the following statements are equivalent:

- P is a projection;
- $I - P$ is a projection;
- $P^2 = P$ and $R(P) = N(P)^{\perp}$
- $P^2 = P$ and P is self-adjoint;
- $P^2 = P$ and P is normal.

Proposition 1.2. *Let P_1 and P_2 are the projections on a Hilbert space $\mathcal{H}$, then*

$$\|P_1 - P_2\| = \ max\ \{\rho_{12}, \rho_{21}\}, \quad where\ \rho_{ij} = \sup\{\|P_j h\| :\ h \in R(P_k)^{\perp}, \|h\| \leq 1\}.$$

Proposition 1.3. *If P and Q are orthogonal projection on a Hilbert space $\mathcal{H}$ such that $\|P - Q\| < 1$, then*

$$\dim R(P) = \dim R(Q), \ \big(\dim R(I - P) = \dim R(I - Q)\big).$$

Definition 1.4. *For $T \in B(\mathcal{H})$, define*

$$\rho(T) = \{z \in \mathbb{C} : T - z \textit{ is invertible in } B(\mathcal{H})\}$$

$$\sigma(T) = \mathbb{C} - \rho(T).$$

We call $\rho(T)$ a resolvent set of T and $\sigma(T)$ is the spectrum of T.

Call $z \in \mathbb{C}$ an eigenvalue of $T \in B(\mathcal{H})$ if there exists $f \in \mathcal{H}, f \neq 0$ such that $Tf = zf$. Denote the set of all eigenvalues of T by $\sigma_p(T)$; called the point spectrum of T. Notice that $\sigma_p(T) \subset \sigma(T)$. Now we would like to present spectral theorem of normal operators, one of the most fundamental theorems in the spectral theory.

Theorem 1.5 (Spectral representation of normal operators)**.** *Let $T \in B(\mathcal{H})$ be a normal operator. Then there exists a collection $\{\rho_\alpha : \alpha \in I\}$ of finite positive Borel measures on $\sigma(T)$ and a unitary map $U : \mathcal{H} \to \bigoplus_\alpha L^2(\sigma(T), d\rho_\alpha)$ so that*

$$UTU^{-1} = M_z, \quad (M_z f)_\alpha = z f_\alpha(z).$$

The minimal cardinality of the index set I is called the *spectral multiplicity* of T. If T is self-adjoint on a separable Hilbert space $\mathcal{H}$ then I is countable. The measures ρ_α are called the *spectral measures* and are not uniquely determined by T. The pair $(\{\rho_\alpha\}, U)$ is called the *spectral representation* of T. There are different versions of spectral theorem and the proofs of the theorems can be found in [8].

Definition 1.6. *For a self-adjoint operator T on a Hilbert space $\mathcal{H}$ and a vector $f \in \mathcal{H}$, the subspace*

$$C_T(f) = span\{(T-z)^{-1} f : z \in \mathbb{C}\}$$

is called the cyclic subspace of f. A vector f is cyclic if and only if $\overline{C_T(f)} = \mathcal{H}$.

If there is a cyclic vector f in a Hilbert space $\mathcal{H}$, then for any self-adjoint operator T on $\mathcal{H}$, there is a unique positive Borel measure ρ in Theorem 1.5; and also for all $z \in \mathbb{C}^+$,

$$\langle f, (T-z)^{-1} f \rangle = \int \frac{1}{x-z} d\rho(x).$$

This is of particular interest because Jacobi and Schrödinger operators have cyclic vectors therefore they have unique corresponding spectral measures. The following theorem provides a characterization of spectrum of a self-adjoint operator in terms of topological supports of corresponding spectral measures.

Theorem 1.7. *Let T be a self-adjoint operator. The spectrum of T is given by*

$$\sigma(T) = \overline{\cup_{n=1}^{N} \operatorname{supp} \rho_n}.$$

Moreover, T is bounded if and only if there is $r > 0, \sigma(T) \subset [-r, r]$ in which case,

$$\|T\| = \sup\{|x| : x \in \sigma(T)\}.$$

In particular, if T has a cyclic vector, then

$$\sigma(T) = \operatorname{supp} \rho.$$

Next, we want to mention the spectral theorem for compact operators.

Definition 1.8. *An operator T on $\mathcal{H}$ is called compact if $\overline{T(B)}$ is a compact set, where $B = \{f \in \mathcal{H} : \|f\| < 1\}$.*

Theorem 1.9. *Let $T \in B(\mathcal{H})$ be a compact, normal operator. Then $\sigma(T)$ is countable. Write $\sigma(T) - \{0\} = \{z_n\}$. Then each z_n is an eigenvalue of T of finite multiplicity: $1 \leq \dim N(T - z) < \infty$. Moreover, $z_n \to 0$ if z_n is infinite.*

We will use this theorem to show that the spectrum of self-adjoint relations induced by the canonical system (0.0.3) is a discrete set.

Recall that a Borel measure ρ on $\mathbb{R}$ is called *absolutely continuous* if $\rho(B) = 0$ for all Borel sets $B \subset \mathbb{R}$ of Lebesgue measure zero. By the Radon-Nikodym theorem, ρ is absolutely continuous if and only if $d\rho = f(t)dt$ for some density $f \in L^1_{loc}(\mathbb{R}),\ f \geq 0$. If ρ is supported by a Lebesgue null set, that is, there exists a Borel set $B \subset \mathbb{R}$ with $|B| = \rho(B^c) = 0$, then we say that ρ is *singular*. By

Lebesgue's decomposition theorem, any Borel measure ρ on $\mathbb{R}$ can uniquely be decomposed into absolutely continuous and singular parts:

$$\rho = \rho_{ac} + \rho_s.$$

The singular measure ρ_s can be further decomposed into singular continuous measure ρ_{sc} and pure point measure ρ_{pp}. Hence the decomposition becomes

$$d\rho = d\rho_{ac} + d\rho_{sc} + d\rho_{pp}.$$

This inspires the decomposition of spectrum into the absolutely continuous, singular continuous and pure point parts.

Definition 1.10. *If $(\{\rho_n\}_{n=1}^N, U)$ is a spectral representation of T the absolutely continuous, singular continuous and pure point spectra, denoted by $\sigma_{ac}(T), \sigma_{sc}(T)$ and $\sigma_{pp}(T)$ respectively and are defined by*

$$\sigma_*(T) = \overline{\cup_{n=1}^N \operatorname{supp} \rho_{n,*}}$$

where $*$ stands for ac, sc or pp .

Note that the three spectra need not be disjoint, but their union is all of $\sigma(T)$:

$$\sigma(T) = \sigma_{ac}(T) \cup \sigma_{sc}(T) \cup \sigma_{pp}(T).$$

Note that this decomposition is independent of choice of spectral representation.

Now, we want to mention one of the most important objects from complex analysis, called Herglotz function. Herglotz functions are holomorphic functions that map upper half-plane to itself. The theory of Herglotz functions play a key role in the spectral theory of Jacobi and Schrödinger operators. These functions have been widely use in the theory of value distribution as well, see [12].

Definition 1.11. *A holomorphic function $F : \mathbb{C}^+ \to \mathbb{C}^+$ is called a Herglotz function.*

These functions are also known as Nevanlinna or Pick functions. These functions are important as they have a connection with positive Borel measures on $\mathbb{R}$. This is due to Borel and Stieltjes.

By the Herglotz representation theorem, every Herglotz function $F \in \mathbb{H}$ has the integral representation of the form,

$$F(z) = a + \int_{\mathbb{R}_\infty} \Big(\frac{1+tz}{t-z}\Big) d\nu(t) \tag{1.0.1}$$

with $a \in \mathbb{R}$ and $\nu \neq 0$ is a finite, positive Borel measure on $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Here, we equip $\mathbb{R}_\infty$ with the topology of the 1-point compactification of $\mathbb{R}$. Both a and ν are uniquely determined by $F \in \mathbb{H}$.

If we let

$$b = \nu(\{\infty\}), \quad d\rho(t) = (1+t^2)\chi_{\mathbb{R}}(t) d\nu(t),$$

then (1.0.1) takes the form

$$F(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big) d\rho(t). \tag{1.0.2}$$

Theorem 1.12 (Stieltjes representation)**.** *Let F be a Herglotz function. Then there exists a unique positive measure ρ on $\mathbb{R}$ such that*

$$F(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1}\Big) d\rho(t), \quad z \in \mathbb{C}^+$$

with $\int \frac{1}{t^2+1} d\rho < \infty$ and numbers $a \in \mathbb{R}$, $b \geq 0$.

The proof of this theorem can be found in [1, 5].

If $F \in \mathbb{H}$, then $F(t) = \lim_{y \to 0+} F(t + iy)$ exists for almost every $t \in \mathbb{R}$.

We are particularly interested on this theorem because the Weyl m functions $m(z)$ corresponding to Jacobi and Schrödinger operators and canonical systems are Herglotz functions. More interestingly, the positive measure ρ in the integral representation of $m(z)$ from the above theorem play the same role as the spectral measures of corresponding operators.

Chapter 2

Theory of linear relations in Hilbert spaces.

In this chapter, we develop a theory of linear relations in Hilbert spaces as a tool. These linear relations were first studied by Coddington, Dijksma, and De Snoo, see [4, 6]. Our will concentrate on establishing the conditions for a symmetric relations to have self-adjoint extensions and the spectral theory of such self-adjoint relations. This in fact, is an analysis, analogous to the operator theory from [17].

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and denote by $\mathcal{H}^2$ the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. A linear relation $\mathcal{R} = \{(f,g) : f, g \in \mathcal{H}\}$ on $\mathcal{H}$ is a subspace of $\mathcal{H}^2$.
$D(\mathcal{R}) = \{f \in \mathcal{H} : \ (f,g) \in \mathcal{R}\}$ and $R(\mathcal{R}) = \{g \in \mathcal{H} : \ (f,g) \in \mathcal{R}\}$ are respectively defined as the domain and range of the relation $\mathcal{R}$.
$\mathcal{R}^{-1} = \{(g,f) : (f,g) \in \mathcal{R}\}$ denotes the inverse relation. The adjoint of $\mathcal{R}$ on $\mathcal{H}$ is a closed linear relation defined by

$$\mathcal{R}^* = \{(h,k) \in \mathcal{H}^2 : \ \langle g, h\rangle = \langle f, k\rangle \text{ for all } (f,g) \in \mathcal{R}\}.$$

A linear relation $\mathcal{S}$ is called symmetric if $\mathcal{S} \subset \mathcal{S}^*$ and self-adjoint if $\mathcal{S} = \mathcal{S}^*$. From now on we write relation to mean linear relation.

A relation $\mathcal{R}$ is called isometry if

$$\langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle \qquad \text{for all } (f_1, g_1), (f_2, g_2) \in \mathcal{R}$$

and $\mathcal{R}$ is unitary if it is isometry and $D(\mathcal{R}) = R(\mathcal{R}) = \mathcal{H}$.

Let $(z - \mathcal{R}) = \{(f, zf - g) : \ (f, g) \in \mathcal{R}\}, N(\mathcal{R}, z) = \{f : (f, zf) \in \mathcal{R}\}$ and $\mathcal{R}_z = R(z - \mathcal{R})$.

Lemma 2.1. *For any relation $\mathcal{R}$, on a Hilbert space $\mathcal{H}$, $N(\mathcal{R}^*, \bar{z}) = \mathcal{R}_z^{\perp}$.*

Proof.

$$\begin{aligned} u \in N(\mathcal{R}^*, \bar{z}) &\Leftrightarrow (u, \bar{z}u) \in \mathcal{R}^* \\ &\Leftrightarrow \text{ for any } (f, g) \in \mathcal{R}, \langle g, u \rangle = \langle f, \bar{z}u \rangle \\ &\Leftrightarrow \langle zf, u \rangle - \langle g, u \rangle = 0 \\ &\Leftrightarrow \langle zf - g, u \rangle = 0. \Leftrightarrow u \in R_z^{\perp}. \end{aligned}$$

□

2.1 Defect indices and self-adjoint extension

Let $\mathcal{R}$ be a relation on a Hilbert space $\mathcal{H}$. The set

$$\begin{aligned} \Gamma(\mathcal{R}) = \{z \in \mathbb{C} : \ &\text{there exists a } C(z) > 0 \text{ such that} \\ &\|(zf - g)\| \geq C(z)\|f\| \text{ for all } (f, g) \in \mathcal{R}\} \end{aligned}$$

is defined as the *regularity domain* of $\mathcal{R}$ and $S(\mathcal{R}) = \mathbb{C} - \Gamma(\mathcal{R})$ is defined as the *Spectral Kernal* of $\mathcal{R}$.

Theorem 2.2. *1. $z \in \Gamma(\mathcal{R})$ if and only if $(z-\mathcal{R})^{-1}$ is a bounded linear operator on $\mathcal{H}$.*

2. If $\mathcal{R}$ is symmetric, then $\mathbb{C} - \mathbb{R} \subset \Gamma(\mathcal{R})$.

3. $\Gamma(\mathcal{R})$ is open.

Proof. 1. If $z \in \Gamma(\mathcal{R})$ then for any $(f,g) \in \mathcal{R}$, there is a constat $C(z) > 0$ such that

$$\|(zf-g)\| \geq C(z)\|f\|.$$

This implies that $(z-\mathcal{R})^{-1}$ is a single valued relation. Clearly it is linear and bounded. Converse is obvious.

2. Suppose $\mathcal{R}$ is symmetric, then for any $z \in \mathbb{C} - \mathbb{R}, z = x + iy$ and $(f,g) \in \mathcal{R}$,

$$\|zf-g\|^2 = \|(xf-g)\|^2 + y^2\|f\|^2 \geq y^2\|f\|^2$$

which implies $z \in \Gamma(\mathcal{R})$.

3. Let $z_0 \in \Gamma(\mathcal{R})$. Then there is a constat $C(z_0) > 0$ such that $\|(z_0 f - g)\| \geq C(z_0)\|f\|$ for all $(f,g) \in \mathcal{R}$. If $z \in \mathbb{C}$ such that $|z - z_0| < C(z_0)$, then for all $(f,g) \in \mathcal{R}$,

$$\|(zf-g)\| \geq \|z_0 f - g\| - |z-z_0|\|f\| > \big(C(z_0) - |z-z_0|\big)\|f\|.$$

This implies that $z \in \Gamma(\mathcal{R})$. □

The subspace $\mathcal{R}_z^{\perp}$ is called the *defect space* of $\mathcal{R}$ and z. The cardinal number $\beta(\mathcal{R}, z) = \dim \mathcal{R}_z^{\perp}$ is called the *defect index* of $\mathcal{R}$ and z.

Theorem 2.3. *The defect index $\beta(\mathcal{R}, z)$ is constant on each connected subset of $\Gamma(\mathcal{R})$. If $\mathcal{R}$ is symmetric, then the defect index is constant on the upper and lower half-planes.*

Proof. Let Q_z denotes the orthogonal projection onto $\bar{\mathcal{R}}_z$. We first show that $\|Q_z - Q_{z_0}\| \to 0$ as $z \to z_0$, for any $z_0 \in \Gamma(\mathcal{R})$. Let $z_0 \in \Gamma(\mathcal{R})$, then there is a constant $C(z_0) > 0$ such that

$$\|f\| \leq C(z_0)\|z_0 f - g\|,$$

for all $(f, g) \in \mathcal{R}$. For $|z - z_0| < \frac{1}{2C(z_0)}$ and all $(f, g) \in \mathcal{R}$, we have,

$$\begin{aligned} \|f\| \leq C(z_0)\|z_0 f - g\| &\leq \Big(\|zf - g\| + |z - z_0|\|f\|\Big) \\ &\leq C(z_0)\|zf - g\| + \frac{1}{2}\|f\| \\ \therefore\ \ \|f\| &\leq C(z_0)\|zf - g\|. \end{aligned}$$

For $h \in R(Q_{z_0})^{\perp} = \mathcal{R}_{z_0}^{\perp}$,

$$\begin{aligned} \|Q_z h\| &= \sup\{|\langle h, zf - g\rangle| :\ zf - g \in \mathcal{R}_z,\ \|zf - g\| \leq 1\} \\ &= \sup\{|\langle h, (z - z_0)f\rangle| :\ zf - g \in \mathcal{R}_z,\ \|zf - g\| \leq 1\}. \\ \therefore\ \ \|Q_z h\| &\leq \|h\||z - z_0|C(z_0). \end{aligned}$$

Similarly for $h \in R(Q_z)^{\perp} = \mathcal{R}_z^{\perp}$

$$\|Q_{z_0} h\| \leq \|h\||z - z_0|C(z_0).$$

Now by Proposition 1.2,

$$\|Q_z - Q_{z_0}\| \leq |z - z_0|C(z_0).$$

Let P_z denotes an orthogonal projection onto $\bar{\mathcal{R}}_z^{\perp}$ then

$$\|p_z - p_{z_0}\| = \|Q_z - Q_{z_0}\| \leq |z - z_0|C(z_0) \to 0 \text{ as } z \to z_0.$$

Hence, if we choose $\epsilon > 0$ such that $\|p_z - p_{z_0}\| < 1$, for $|z - z_0| < \epsilon$ then by Proposition 1.3, $\dim \mathcal{R}_z^{\perp} = \dim \mathcal{R}_{z_0}^{\perp}$. It follows that

$$\beta(\mathcal{R}, z) = \beta(\mathcal{R}, z_0) \text{ for } |z - z_0| < \epsilon.$$

If $\mathcal{R}$ is symmetric, then the upper and lower half-planes are connected subsets of $\Gamma(\mathcal{R})$; therefore, the defect index is constant there. $\square$

Let $\mathcal{R}$ is a symmetric relation on a Hilbert space, for $z \in \mathbb{C}^+$, the defect index $m = \beta(\mathcal{R}, z)$ and for $w \in \mathbb{C}^-$, the defect index $n = \beta(\mathcal{R}, w)$ are written as a pair (m, n) and are called the *defect indices* of $\mathcal{R}$.

The Cayley trnsform of a symmetric relation $\mathcal{R}$ on $\mathcal{H}$ is defined by the relation

$$\mathcal{V} = \{(g + if, g - if) : \ (f, g) \in \mathcal{R}\}.$$

Then clearly $D(\mathcal{V}) = R(\mathcal{R} + i)$ and $R(\mathcal{V}) = R(\mathcal{R} - i)$.

Theorem 2.4. *If $\mathcal{R}$ be a Symmetric relation on $\mathcal{H}$ and $\mathcal{V}$ is the Cayley transform of $\mathcal{R}$ then,*

1. *$\mathcal{V}$ is isometry*
2. *$R(I - \mathcal{V}) = D(\mathcal{R})$ and $\mathcal{R} = \{(f - g, i(f + g)) : \ (f, g) \in \mathcal{V}\}$.*
3. *$\mathcal{R}$ is multi-valued if and only if $N(I - \mathcal{V}) \neq \{0\}$*

Proof. 1. Let $(u_1, v_1), (u_2, v_2) \in \mathcal{V}$ then $u_i = g_i + if_i$ and $v_i = g_i - if_i$, for $(f_i, g_i) \in \mathcal{R}, i = 1, 2$ then

$$\begin{aligned}
\langle u_1, u_2 \rangle &= \langle g_1 + if_1, g_2 + if_2 \rangle \\
&= \langle g_1, g_2 \rangle + \langle g_1, if_2 \rangle + \langle if_1, g_2 \rangle + \langle if_1, if_2 \rangle \\
&= \langle g_1, g_2 \rangle + i\langle g_1, f_2 \rangle - i\langle f_1, g_2 \rangle - i^2\langle f_1, f_2 \rangle \\
&= \langle g_1 - if_1, g_2 - if_2 \rangle \\
&= \langle v_1, v_2 \rangle.
\end{aligned}$$

2. This is clear.

3. Suppose $\mathcal{R}$ is multi-valued then there is $g \in \mathcal{H}, g \neq 0$ such that $(0, g) \in \mathcal{R}$.

It follows by definition of $\mathcal{V}$ that $(g, g) \in \mathcal{V}$. Hence, $g \in N(I - \mathcal{V})$. On the other hand , let $g \in N(I - \mathcal{V}), g \neq 0$ then $(g, g) \in \mathcal{V}$ then by 2, $(0, 2ig) \in \mathcal{R}$. Hence $\mathcal{R}$ is multi-valued . $\square$

Theorem 2.5. *A relation $\mathcal{V}$ on $\mathcal{H}$ is the Cayley transform of a symmetric relation $\mathcal{R}$ if and only if $\mathcal{V}$ has the following properties.*

1. *$\mathcal{V}$ is an isometric relation.*

2. *$R(I - \mathcal{V}) = D(\mathcal{R})$.*

The relation $\mathcal{R}$ is given by $\mathcal{R} = \{(f - g, i(f + g)) : \ (f, g) \in \mathcal{V}\}$.

Proof. If $\mathcal{V}$ is the Cayley transform of $\mathcal{R}$, then by Theorem 2.3.1, $\mathcal{V}$ satisfies the properties 1 and 2. Conversely suppose $\mathcal{V}$ has properties 1 and 2, we show that $\mathcal{R} = \{(f - g, i(f + g)) : \ (f, g) \in \mathcal{V}\}$ is a symmetric relation.
Suppose $(f_1 - g_1, i(f_1 + g_1)), \ (f_2 - g_2, i(f_2 + g_2)) \in \mathcal{R}$ then

$$\langle i(f_1 + g_1), (f_2 - g_2)\rangle = -i\Big(\langle f_1, f_2\rangle - \langle f_1, g_2\rangle + \langle g_1, f_2\rangle - \langle g_1, g_2\rangle\Big).$$

Since $\mathcal{V}$ is an isometry, for any $(f_1, g_1), (f_2, g_2) \in \mathcal{V}, \langle f_1, f_2\rangle = \langle g_1, g_2\rangle$. This implies that

$$\begin{aligned}\langle i(f_1 + g_1), (f_2 - g_2)\rangle &= -i\langle g_1, g_2\rangle + i\langle f_1, g_2\rangle - i\langle g_1, f_2\rangle + i\langle f_1, f_2\rangle \\ &= -i\Big(\langle g_1 - f_1, g_2\rangle + \langle g_1 - f_1, f_2\rangle\Big) \\ &= \langle f_1 - g_1, i(f_2 + g_2)\rangle.\end{aligned}$$

Hence $\mathcal{R}$ is symmetric. $\square$

Theorem 2.6. *A symmetric relation $\mathcal{R}$ is self-adjoint if and only if $\mathcal{V}$ is unitary.*

Proof. We show that the $\mathcal{R}$ is self-adjoint if and only if

$$R(\mathcal{R}+i) = R(\mathcal{R}-i) = \mathcal{H}.$$

Since $\mathcal{R}$ is symmetric we always have $\mathcal{R} \subset \mathcal{R}^*$. Let $(f,g) \in \mathcal{R}^*$ then $if - g \in \mathcal{H}$ and $R(\mathcal{R}-i) = \mathcal{H}$ implies that there is $(h,k) \in \mathcal{R}$ such that $k - ih = if - g$. So $i(f+h) = k+g$. So that $(f+h, i(f+h)) \in \mathcal{R}^*$. That is

$$(f+h) \in N(\mathcal{R}^*, i) = R((\mathcal{R}+i)^{\perp}) = \{0\}.$$

This implies $f = -h \in D(\mathcal{R})$. Hence $\mathcal{R}$ is self-adjoint.

Conversely suppose $\mathcal{R}$ is self-adjoint. Let

$$h \in R(\mathcal{R}-i)^{\perp} = N(\mathcal{R}^*, -i) = N(\mathcal{R}, -i).$$

So $(h, -ih) \in \mathcal{R}$. But

$$\langle -ih, h\rangle = \langle h, ih\rangle \Rightarrow i\langle h, h\rangle = -i\langle h, h\rangle.$$

Hence we must have $h = 0$. So $R(\mathcal{R}-i) = \mathcal{H}$. Similarly, $R(\mathcal{R}+i) = \mathcal{H}$. □

Theorem 2.7. *Let $\mathcal{R}$ be a closed symmetric relation on a Hilbert space $\mathcal{H}$ and $\mathcal{V}$ denotes its Cayley transform.*

1. *$\mathcal{V}'$ is the Cayley transform of a closed symmetric extension $\mathcal{R}'$ of $\mathcal{R}$ if and only if the following holds: There exists closed subspaces F_- of $R(\mathcal{R}-i)^{\perp}$ and F_+ of $R(\mathcal{R}+i)^{\perp}$ and an isometric relation $\tilde{\mathcal{V}}$ on $F_+ \oplus F_-$ for which*

$$\mathcal{V}' = \{(f+h, g+k) : (f,g) \in \mathcal{V}, (h,k) \in \tilde{\mathcal{V}}\} \text{ and}$$

$$D(\mathcal{V}') = R(\mathcal{R}'+i) = R(\mathcal{R}+i) \oplus F_+,$$

$$R(\mathcal{V}') = R(\mathcal{R}'-i) = R(\mathcal{R}-i) \oplus F_-.$$

The spaces F_+ and F_- have the same dimension.

2. *The relation $\mathcal{V}'$ in part 1 is unitary if and only if $F_- = R(\mathcal{R}-i)^\perp$ and $F_+ = R(\mathcal{R}+i)^\perp$.*

3. *$\mathcal{R}$ possess self-adjoint extension if and only if its defect indices are equal.*

Proof. 1. Suppose $\mathcal{V}'$ has the given form. Then $\mathcal{V}'$ is isometric relation, since for any $(f+h, g+k) \in \mathcal{V}'$ we have,

$$\|g+k\|^2 = \|g\|^2 + \|k\|^2 = \|f\|^2 + \|h\|^2 = \|f+h\|^2.$$

Hence we can define a symmetric extension $\mathcal{R}'$ such that $\mathcal{V}'$ is its Cayley transform. Conversely if $\mathcal{V}'$ is the Cayley transform of a symmetric extension $\mathcal{R}'$ of $\mathcal{R}$, then put $F_- = R(\mathcal{R}'-i) \ominus R(\mathcal{R}-i)$, $F_+ = R(\mathcal{R}'+i) \ominus R(\mathcal{R}+i)$ and $\tilde{\mathcal{V}} = \mathcal{V}'|_{F_+ \oplus F_-}$. Then we have the desired properties.

2. Here we have $\mathcal{V}'$ is unitary if and only if

$$D(\mathcal{V}') = R(\mathcal{V}') = \mathcal{H}.$$

That is, if and only if $F_- = R(\mathcal{R}-i)^\perp$ and $F_+ = R(\mathcal{R}+i)^\perp$.

3. By 1 and 2, $\mathcal{V}$ posses unitary extension if and only if there exists an isometry relation $\tilde{\mathcal{V}}$ onto $R(\mathcal{R}+i)^\perp \oplus R(\mathcal{R}-i)^\perp$. This happens if and only if

$$\dim(R(\mathcal{R}+i)^\perp) = \dim(R(\mathcal{R}-i)^\perp).$$

□

By definition of Cayley transform, it is clear that if $\mathcal{V}_1$ and $\mathcal{V}_2$ are the Cayley transforms of any two symmetric relations $\mathcal{R}_1$ and $\mathcal{R}_2$ then

$$\mathcal{R}_1 \subset \mathcal{R}_2 \text{ if and only if } \mathcal{V}_1 \subset \mathcal{V}_2.$$

Theorem 2.8. *Let $\mathcal{R}$ be a closed symmetric relation on a Hilbert space with defect indices (m, m).*

1. *$\mathcal{R}'$ be a symmetric extension of $\mathcal{R}$ if and only if the following holds: There are closed subspaces F_+ of $R(i+\mathcal{R})^\perp$ and F_- of $R(i-\mathcal{R})^\perp$ and an isometric mapping $\hat{\mathcal{V}}$ of F_+ onto F_- such that*

$$D(\mathcal{R}') = D(\mathcal{R}) + \{g + \hat{\mathcal{V}}g : g \in F_+\}.$$

2. *$\mathcal{R}'$ is self-adjoint if and only if $\mathcal{R}'$ is an $m-$dimensional extension of $\mathcal{R}$*

.

Proof. (1). Let $\mathcal{V}$ and $\mathcal{V}'$ be the Cayley transforms of the closed symmetric relation $\mathcal{R}$ and its symmetric extension $\mathcal{R}'$, respectively. By Theorem 2.7, there exist closed subspaces F_- of $R(\mathcal{R}-i)^\perp$ and F_+ of $R(\mathcal{R}+i)^\perp$ and an isometric relation $\tilde{\mathcal{V}}$ on $F_+ \oplus F_-$ for which

$$\mathcal{V}' = \{(f+h, g+k) : (f,g) \in \mathcal{V}, (h,k) \in \tilde{\mathcal{V}}\} \text{ and}$$

$$D(\mathcal{V}') = R(\mathcal{R}'+i) = R(\mathcal{R}+i) \oplus F_+,$$

$$R(\mathcal{V}') = R(\mathcal{R}'-i) = R(\mathcal{R}-i) \oplus F_-.$$

Then by definition of the Cayley transform we see that,

$$\begin{aligned} D(\mathcal{R}') &= R(I-\mathcal{V}') = (I-\mathcal{V}')D(\mathcal{V}') \\ &= (I-\mathcal{V}')R(i+\mathcal{R}') \\ &= (I-\mathcal{V}')(R(i+\mathcal{R}) \oplus F_+) \\ &= (I-\mathcal{V}')(D(\mathcal{V}) \oplus F_+) \\ &= (I-\mathcal{V})D(\mathcal{V}) + (I-\mathcal{V}')F_+ \\ &= D(\mathcal{R}) + \{g - \tilde{\mathcal{V}}g : g \in F_+\}. \end{aligned}$$

The converse is similar.

(2). By Theorem 2.7, $\mathcal{R}'$ is self-adjoint if and only if $\mathcal{V}'$ is unitary. This happens if and only if $F_+ = R(\mathcal{R}+i)^\perp$. So by 1, $\mathcal{R}'$ is self-adjoint if and only if it is a $m-$dimensional extension of $\mathcal{R}$. $\square$

Theorem 2.9. *Suppose $\mathcal{T}$ is a self-adjoint relation and suppose $z \in \Gamma(\mathcal{T})$ then*

$$\mathcal{H} = \{zf - g : \quad (f,g) \in \mathcal{T}\}.$$

Proof. We will show that $R(z-\mathcal{T}) = \{zf - g : \quad (f,g) \in \mathcal{T}\}$ is a closed subspace of $\mathcal{H}$. Since $z \in \Gamma(\mathcal{T})$, there is a constant $C(z)$ such that

$$\|zf - g\| \geq C(z)\|f\|.$$

Let $v_n \in R(z-\mathcal{T})$ and $v_n \to v$ in $\mathcal{H}$. Suppose $f_n \in D(\mathcal{T})$ such that $(f_n, g_n) \in \mathcal{T}$ and $v_n = zf_n - g_n$ so that $(f_n, v_n) \in z - \mathcal{T}$. But from above relation we have

$$\|v_n - v_m\| = \|z(f_n - f_m) - (g_n - g_m)\| \geq C\|f_n - f_m\|.$$

It follows that f_n is a Cauchy sequence in $\mathcal{H}$, and it converges to some f in $\mathcal{H}$. Hence $(f_n, v_n) \to (f, v)$. Since $\mathcal{T}$ is closed , $f \in D(\mathcal{T})$ and $(f,v) \in z - \mathcal{T}$ and $v \in R(z-\mathcal{T})$. Hence $R(z-\mathcal{T})$ is closed. So we have

$$\mathcal{H} = R(z-\mathcal{T}) \oplus R(z-\mathcal{T})^\perp.$$

We next show that $R(z-\mathcal{T})^\perp = \{0\}$. Let $h \in R(z-\mathcal{T})^\perp = N(\mathcal{T}, \bar{z})$ then $(h, \bar{z}h) \in \mathcal{T}$. But $0 = \|\bar{z}h - \bar{z}h\| \geq C(\bar{z})\|h\|$ implies $h = 0$ a. e. . $\square$

Let $\mathcal{T}$ be a self-adjoint relation on $\mathcal{H}$ and $z \in \Gamma(\mathcal{T})$. Define $T : \mathcal{H} \to \mathcal{H}$ by $T(zf - g) = f$. That is $T = (z-\mathcal{T})^{-1} = \{(zf - g, f) : \quad (f,g) \in \mathcal{T}\}$. Then T is a bounded linear operator since

$$\|T\| = \sup_{\|zf-g\|=1} \|T(zf-g)\| = \sup_{\|zf-g\|=1} \|f\| \leq \frac{1}{C(z)}$$

and $\mathcal{T}$ is given by

$$\mathcal{T} = \{(Tf, zTf - f) : \quad f \in \mathcal{H}\}.$$

2.2 Spectral theory of a linear relation

Definition 2.10. *Let $\mathcal{R}$ be a closed relation on a Hilbert space $\mathcal{H}$. We define*

$$\rho(\mathcal{R}) = \{z \in \mathbb{C} : \exists\, T \in B(\mathcal{H}), \mathcal{R} = \{(Tf, zTf - f) : \; f \in \mathcal{H}\}\}$$

to be the resolvent set and $\sigma(\mathcal{R}) = \mathbb{C} - \rho(\mathcal{R})$ to be the spectrum of $\mathcal{R}$.

Remark 2.11. *When a relation $\mathcal{R}$ is an operator on $\mathcal{H}$ then*

$$\rho(\mathcal{R}) = \{z \in \mathbb{C} : \; (z - \mathcal{R})^{-1} \in B(\mathcal{H})\}$$

Proof. Suppose $\mathcal{R}$ is an operator. If $z \in \mathbb{C}$ be such that $(z - \mathcal{R})^{-1} \in B(\mathcal{H})$ then take $T = (z - \mathcal{R})^{-1}$ so that for any $(f, g) \in \mathcal{R}, \; zf - g \in \mathcal{H}$ and

$$(f, g) = (T(zf - g), zT(zf - g) - (zf - g)).$$

Also for any $f \in \mathcal{H}$,

$$\begin{aligned} &(z - \mathcal{R})(z - \mathcal{R})^{-1} f = f \\ &\Rightarrow (z - \mathcal{R})Tf = f \\ &\Rightarrow \mathcal{R}Tf = zTf - f \\ &\Rightarrow (Tf, zTf - f) \in \mathcal{R}. \end{aligned}$$

Hence $\{z \in \mathbb{C} : \; (z - \mathcal{R})^{-1} \in B(\mathcal{H})\} \subset \rho(\mathcal{R})$. On the other hand , let $z \in \rho(\mathcal{R})$ want to show that $(z - \mathcal{R})$ is bijective. Let $f_1, f_2 \in \mathcal{H}$ such that $Tf_1 \neq Tf_2$. If $(z - \mathcal{R})Tf_1 = (z - \mathcal{R})Tf_2$ then this implies $f_1 = f_2$. This is not possible because T is an operator and $Tf_1 \neq Tf_2$. So $(z - \mathcal{R})$ is one to one. Also for any $f \in \mathcal{H}, \; Tf \in \mathcal{H}$ and $(z - R)Tf = zTf - zTf + f = f$. This implies that $(z - \mathcal{R})$ is onto. Now for any $f \in \mathcal{H}$,

$$\|(z - \mathcal{R})^{-1} f\| \leq \|Tf\| \leq C\|f\|.$$

This implies $(z - \mathcal{R})^{-1} \in B(\mathcal{H})$. $\square$

A complex number $z \in \mathbb{C}$ is called an *eigenvalue* of a relation $\mathcal{R}$ if there exists a $f \in \mathcal{H},\ f \neq 0$ such that $(f, zf) \in \mathcal{R}$. The set of all eigenvalues of $\mathcal{R}$ is called the *point spectrum* of $\mathcal{R}$ and is denoted by $\sigma_p(\mathcal{R})$.

Lemma 2.12. *For any closed relation $\mathcal{R}$ on a Hilbert space $\mathcal{H}$, $\sigma_p(\mathcal{R}) \subset \sigma(\mathcal{R})$.*

Proof. Let $z \in \sigma_p(\mathcal{R})$. Then there exists $f \in \mathcal{H},\ f \neq 0$ such that $(f, zf) \in \mathcal{R}$. Suppose $z \notin \sigma(\mathcal{R})$ then $z \in \rho(\mathcal{R})$ so there exists $T \in B(\mathcal{H})$ such that $\mathcal{R} = \{(Tf, zTf - f) : \ f \in \mathcal{H}\}$. Since $(f, zf) \in \mathcal{R}$ there is some $u \in \mathcal{H}$ such that $f = Tu$ and $zf = zTu - u$. This implies that $u = 0$ and hence $f = Tu = 0$. This contradicts the fact that $f \neq 0$. So $z \in \sigma(\mathcal{R})$. It follows that $\sigma_p(\mathcal{R}) \subset \sigma(\mathcal{R})$. □

Let $\mathcal{Z} = \{(0, g) \in \mathcal{R}\}$ and $Z = \{g : \ (0, g) \in \mathcal{R}\}$ be the multi-valued part of a relation $\mathcal{R}$. Clearly Z is a closed subspace of $\mathcal{H}$. Note that $D(\mathcal{R})$ is not dense if $\mathcal{R}$ is multi-valued. Now define the quotient space $\mathcal{H}_s = \mathcal{H}/Z$. We know that this quotient space is also a Hilbert space with the norm defined by

$$\|[f]\| = \inf_{g \in Z} \|f + g\|.$$

Define a relation $\mathcal{R}_s$ on $\mathcal{H}_s \oplus \mathcal{H}_s$ by $\mathcal{R}_s = \{([f], [g]) : (f, g) \in \mathcal{R}\}$. We consider the relation $\mathcal{R}_s$ as the restriction of $\mathcal{R}$ on $\mathcal{H}_s$. By natural isomorphism, the space $\mathcal{H}_s$ is identified as $\mathcal{H} \ominus Z$ and the relation $\mathcal{R}_s$ as $\mathcal{R} \ominus \mathcal{Z}$. Then clearly $\mathcal{R}_s$ is an operator on $\mathcal{H}_s$ with $D(\mathcal{R}_s) = D(\mathcal{R})$.

Theorem 2.13. *If $\mathcal{T}$ is a self-adjoint relation on $\mathcal{H}$ then*

$$S(\mathcal{T}) = \sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$$

.

Proof. Let $z \in \Gamma(\mathcal{T})$, then there exists a constant $C > 0$ such that

$$\|zf - g\| \geq C\|f\|, \quad \text{for all } (f, g) \in \mathcal{T}.$$

For such z, we can define $T = (z-\mathcal{T})^{-1}$ a bounded linear operator on $\mathcal{H}$ such that $\mathcal{T} = \{(Th, zTh - h) : \ h \in \mathcal{H}\}$. So $z \in \rho(\mathcal{T})$. On the other hand, let $z \in \rho(\mathcal{T})$ then there exists $T \in B(\mathcal{H})$ such that $\mathcal{T} = \{(Th, zTh - h) : \ h \in \mathcal{H}\}$. For any $(f, g) \in \mathcal{T}$, there is $h \in \mathcal{H}$ such that $Th = f$ and $zTh - h = g$. So

$$\|zf - g\| = \|zTh - zTh + h\| = \|h\| \geq C\|Th\| = C\|f\|$$

for some $C > 0$ and hence $z \in \Gamma(\mathcal{T})$. Hence, $S(\mathcal{T}) = \sigma(\mathcal{T})$.

Next assume that $z \in \Gamma(\mathcal{T}_s)$ then for any $([f], [g]) \in \mathcal{T}_s$, there exists a constant $C > 0$ such that

$$\|z[f] - [g]\| \geq C\|[f]\|.$$

For any $(f, g) \in \mathcal{T}$ we have

$$\|zf - g\| \geq \|z[f] - [g]\| \geq C\|[f]\| = C\|f\|.$$

Hence $z \in \Gamma(\mathcal{T})$. On the other hand suppose $z \in \Gamma(\mathcal{T})$ then there is a constant $C > 0$ such that

$$\|zf - g\| \geq C\|f\|.$$

For any $([f], [g]) \in \mathcal{T}_s$ we have

$$\|z[f]-[g]\| = \inf_{u\in Z} \|zf-g+u\| = \inf_{u\in Z} \Big(\|zf-g\|+\|u\|\Big) \geq \|zf-g\| \geq C\|f\| = C\|[f]\|.$$

This implies that $z \in \Gamma(\mathcal{T}_s)$. Thus $S(\mathcal{T}_s) = S(\mathcal{T})$. Hence $S(\mathcal{T}) = \sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$.

$\square$

Remark 2.14. *If $\mathcal{T}$ is a self-adjoint relation on $\mathcal{H}$ then $\sigma(\mathcal{T}) \subset \mathbb{R}$.*

Theorem 2.15. *Let $z \in \Gamma(\mathcal{T})$ and $T = (z - \mathcal{T})^{-1}$.*

1. *If $\lambda \in \Gamma(T)$ then $(z - \frac{1}{\lambda}) \in \Gamma(\mathcal{T})$.*

2. *If* $\lambda \in S(\mathcal{T})$ *then* $\frac{1}{z-\lambda} \in S(T)$.

3. $S(T) \subset \sigma(T)$.

Proof. (1). Let $\lambda \in \Gamma(T)$ then by definition there exists $C(\lambda) > 0$ such that

$$\|\lambda(zf - g) - f\| \geq C(\lambda)\|zf - g\| \text{ for all } (f, g) \in \mathcal{T}.$$

Note that $\lambda \neq 0$. For any $(f, g) \in \mathcal{T}$ we have,

$$\begin{aligned} \|(z - \frac{1}{\lambda})f - g\| &= \frac{1}{|\lambda|}\|z\lambda f - f - \lambda g\| \\ &= \frac{1}{|\lambda|}\|\lambda(zf - g) - f\| \\ &\geq \frac{C(\lambda)}{|\lambda|}\|zf - g\| \geq \frac{C(\lambda)C(z)}{|\lambda|}\|f\|. \end{aligned}$$

So $(z - \frac{1}{\lambda}) \in \Gamma(\mathcal{T})$.

(2). Let $\lambda \in S(\mathcal{T})$ and suppose $\frac{1}{z-\lambda} \notin S(T)$. Then $\frac{1}{z-\lambda} \in \Gamma(T)$. But by (1), $(z - \frac{1}{\frac{1}{z-\lambda}}) \in \Gamma(\mathcal{T})$. This implies that $\lambda \in \Gamma(\mathcal{T})$ which is a contradiction.

(3). Let $\lambda \in \rho(T)$ then $(\lambda - T)^{-1}$ is bounded and is defined on all of $\mathcal{H}$ then for any $(f, g) \in \mathcal{T}$

$$\begin{aligned} \|zf - g\| = \|(\lambda - T)^{-1}(\lambda - T)(zf - g)\| &\leq \|(\lambda - T)^{-1}\|\|\lambda(zf - g) - T(zf - g)\| \\ \Rightarrow \|\lambda(zf - g) - T(zf - g)\| &\geq \frac{1}{\|(\lambda - T)^{-1}\|}\|zf - g\|. \end{aligned}$$

$\Rightarrow \lambda \in \Gamma(T)$. This shows that $S(T) \subset \sigma(T)$. □

2.3 Spectral theory of a canonical system

Consider a relation $\mathcal{R}$ on $L^2(H, \mathbb{R}_+)$ given by

$$\mathcal{R} = \{(f, g) \in \left(L^2(H, \mathbb{R}_+)\right)^2 : \ f \in AC, Jf' = Hg\}.$$

Call this relation $\mathcal{R}$, a maximal relation. This relation is made up of pairs (f, g) of equivalence classes, such that there exists a locally absolutely continuous representative of f again denoted by f, and a representative of g, again denoted by g, such that $Jf' = Hg$ *a.e.* on $\mathbb{R}_+$. The adjoint relation $\mathcal{R}_0 = \mathcal{R}^*$ is called the minimal relation. These relations are discussed in details in [10, 11, 15, 18]. We will present some of the results about these relations from [11]. Let

$$\mathcal{R}_c = \{(f, g) \in \mathcal{R} : f \text{ has a compact support}\}.$$

Lemma 2.16 ([11])**.** *Let $[a, b] \subset \mathbb{R}_+$ be a compact interval. If $(\phi, \psi) \in \mathcal{R}_c$ and $\operatorname{supp} \phi \subset [a, b]$, then ψ satisfies:*

$$\operatorname{supp} H\psi \subset [a, b], \quad \int_a^b H(t)\psi(t)dt = 0. \tag{2.3.1}$$

Conversely, if the function $\psi \in L^2(H, \mathbb{R}_+)$ satisfies relation (2.3.1), then there exists ϕ, such that $(\phi, \psi) \in \mathcal{R}_c$ and $\operatorname{supp} \phi \subset [a, b]$.

Proof. If $(\phi, \psi) \in \mathcal{R}_c$, then $\phi, \psi \in L^2(H, \mathbb{R}_+)$ and $\phi' = -JH\psi$. Hence $\operatorname{supp} H\psi \subset [a, b]$. Since $\phi(a) = 0, \phi(x) = -\int_a^x JH(t)\psi(t)dt$. Moreover, $\phi(b) = 0$ implies $-\int_a^b JH(t)\psi(t)dt = 0 \Rightarrow \int_a^b H(t)\psi(t)dt = 0$. To see the converse, let $\psi \in L^2(H, \mathbb{R}_+)$ satisfying the (2.3.1) and define,

$$\phi(x) = -\int_a^x JH(t)\psi(t)dt.$$

Then $\operatorname{supp} \phi \subset [a, b]$ and $J\phi' = H\psi$. Hence $(\phi, \psi) \in \mathcal{R}_c$ and $\operatorname{supp} \phi \subset [a, b]$. □

Lemma 2.17 ([11])**.** *The relation $\mathcal{R}_c$ is symmetric and $\mathcal{R}_c^* = \mathcal{R}$.*

Proof. Let $(f, g) \in \mathcal{R}$ and $(\phi, \psi) \in \mathcal{R}_c$, then

$$\begin{aligned}
\langle g, \phi\rangle - \langle f, \psi\rangle &= \int_a^b g^* H\phi dt - \int_a^b f^* H\psi dt \\
&= \int_a^b (Jf')^* \phi dt - \int_a^b f^* J\phi' dt \\
&= - \int_a^b f'^* J\phi dt - \int_a^b f^* J\phi' dt \\
&= - f^* J\phi|_a^b + \int_a^b f^* J\phi' dt - \int_a^b f^* J\phi' dt \\
&= 0.
\end{aligned}$$

Hence $(f, g) \in \mathcal{R}_c^*$ and $\mathcal{R} \subset \mathcal{R}_c^*$. To show the reverse, let $(h, k) \in \mathcal{R}_c^*$ then $h, k \in L^2(H, \mathbb{R}_+)$. Let u be a solution of the equation $Ju' = Hk$. Suppose $[a, b]$ be a compact interval. For $\psi \in L^2(H, \mathbb{R}_+)$ satisfying (2.3.1), and ϕ as in Lemma 2.16 so that $(\phi, \psi) \in \mathcal{R}_c$, then $\langle h, \psi\rangle - \langle k, \phi\rangle = 0$. That is

$$\begin{aligned}
\int_a^b h(t)^* H(t)\psi(t) dt &= \int_a^b k(t)^* H(t)\phi(t) dt \\
&= - \int_a^b u'(t)^* J\phi(t) dt \\
&= \int_a^b u(t)^* J\phi' dt \\
&= \int_a^b u(t)^* H(t)\psi dt
\end{aligned}$$

$$\Rightarrow \int_a^b \big(h(t) - u(t)\big)^* H(t)\psi(t) dt = 0$$

By Lemma 2.16, the function $\psi(t)$ span the orthogonal complement of constants on $[a, b]$. Hence $h(t) - u(t)$ is equivalent to a constant on $[a, b]$. Therefore, h has a representative again denoted by h, which is absolutely continuous and satisfies $Jh' = Ju' = Hk$ a.e. on $[a, b]$. Suppose $I \subset [a, b]$ is a compact subinterval of positive type then the absolutely continuous representative does not depend on $I \subset [a, b]$. Since $[a, b]$ was arbitrary, it follows that $(h, k) \in \mathcal{R}$. Hence $\mathcal{R}_c^* = \mathcal{R}$, so that $\mathcal{R}_c$ is symmetric. □

Corollary 2.18 ([11]). *The relation $\mathcal{R}$ is closed and $\mathcal{R}_0$ is closed and symmetric.*

Proof. Since $\mathcal{R} = \mathcal{R}_c^*$ and $\mathcal{R}_0 = \mathcal{R}^*$, $\mathcal{R}$ and $\mathcal{R}_0$ are closed. To show $\mathcal{R}_0$ is symmetric, let $(h,k) \in \mathcal{R}_0$, then by definition, $\langle f,k\rangle - \langle g,h\rangle = 0$ for all $(f,g) \in \mathcal{R}$. So for $(\phi,\psi) \in \mathcal{R}_c \subset \mathcal{R}$ we see that $\langle \phi,k\rangle - \langle \psi,h\rangle = 0$ for all $(\phi,\psi) \in \mathcal{R}_c$. This implies that $(h,k) \in \mathcal{R}_c^* = \mathcal{R}$. So $\mathcal{R}_0 \subset \mathcal{R}_0^* = \mathcal{R}$. □

Lemma 2.19 ([11]). *For each $c \in \mathbb{C}^2$ there exists $(\phi,\psi) \in \mathcal{R}$ such that ϕ has compact support and $\phi(0+) = c$.*

Proof. Choose $[0,N]$ so that it contains an open subinterval of positive type. Then the matrix $\int_0^N H(t)dt$ is invertible. Hence there exists a vector $u \in \mathbb{C}^2$ such that

$$\Big(\int_0^N H(t)dt\Big)u = -Jc.$$

Define $\psi \in L^2(H,\mathbb{R}_+)$ by

$$\psi(t) = u,\ 0 \le t \le N,\ \psi(t) = 0,\ \ t > N,$$

and define ϕ by

$$\phi(x) = c - \int_0^x JH(t)\psi(x)dt.$$

Then ϕ is in $L^2(H,\mathbb{R}_+)$, is absolutely continuous and $\operatorname{supp}\phi \subset [0,N]$. Moreover, $\phi(0+) = c$ and $J\phi' = H\psi$ so that $(\phi,\psi) \in \mathcal{R}$. □

Lemma 2.20 ([11]). *Let $(f,g),(h,k) \in \mathcal{R}$. Then the following limit exists:*

$$\lim_{x\to\infty} h(x)Jf(x) = h(0+)Jf(0+) - [\langle f,k\rangle - \langle g,h\rangle]. \tag{2.3.2}$$

Lemma 2.21. *The minimal relation $\mathcal{R}_0$ is given by*

$$\mathcal{R}_0 = \{(f,g) \in \mathcal{R} : f(0+) = 0,\ \lim_{x\to\infty} f^*(x)Jh(x) = 0 \text{ for all } (h,k) \in \mathcal{R}\}.$$

Proof. By Lemma 2.20, we get

$$\{(f,g) \in \mathcal{R} : f(0+) = 0, \lim_{x\to\infty} f^*(x)Jh(x) = 0 \text{ for all } (h,k) \in \mathcal{R}\} \subset \mathcal{R}_0.$$

On the other hand let $(f,g) \in R_0$. By Lemma 2.19, for any $u \in \mathbb{C}^2$ there exists $(\phi,\psi) \in \mathcal{R}$ such that ϕ has compact support and $\phi(0+) = c$. So

$$\begin{aligned} 0 &= \langle f, \psi\rangle - \langle g, \phi\rangle \\ &= \lim_{x\to\infty} f^*(x)J\phi(x) - \phi(0+)Jf(0+) \\ &= uJf(0+). \end{aligned}$$

This implies that $f(0+) = 0$. This would also forces that

$$\lim_{x\to\infty} f^*(x)Jh(x) = 0 \text{ for all } (h,k) \in \mathcal{R}.$$

□

Remark 2.22. *For fixed $z \in \mathbb{C}$, the dimension of the solution space of system (0.0.3) is two.*

Lemma 2.23. *The defect index $\beta(\mathcal{R}_0)$ of the minimal relation $\mathcal{R}_0$ is equal to the number of linearly independent solutions of the system (0.0.3) of whose class lie in $L^2(H,\mathbb{R}_+)$.*

Proof. First we show that any two different solutions of the system (0.0.3) for some fixed $z \in \mathbb{C}$ belongs to the different class of functions. Let u and v be any two different solutions of the system (0.0.3), ie $Ju' = zHu$ and $Jv' = zHv$. Then $J(u'-v') = zH(u-v)$. Suppose u and v lie on the same class so that

$$H(u-v) = 0, \Rightarrow J(u'-v') = 0, \Rightarrow u - v = C,$$

some constat C. But $H(u-v) = HC = 0 \Rightarrow C = 0$. It follows that $u \equiv v$. Next we show that, any two solutions u, v are linearly independent if and only if their corresponding class $[u],[v]$ are linearly independent. Suppose $[u],[v]$ are linearly independent and let u,v be any representative of $[u],[v]$ respectively such that $au+bv=0$ for some $a,b\in\mathbb{R}$. Then $0=[au+bv]=a[u]+b[v]\Rightarrow a=0,b=0$. So u and v are linearly independent. Conversely suppose u and v are linearly independent. If $[u]=a[v]$ for some $a\in\mathbb{R}$ then $u=ah$ for some $h\in[v]$. This shows that h is also a solution of the system (0.0.3) that lie in $[v]$ but this implies that $h\equiv v$. Hence $u=av$ which is a contradiction. This completes the proof of the lemma. □

Remark 2.24. *In the limit-circle case, the defect indices of the minimal relation* $\mathcal{R}_0$ *are* $(2,2)$.

Since $\mathcal{R}_0$ has equal defect indices, by Theorem 2.7, it has self-adjoint extensions. In order to describe self-adjoint extensions first consider the system (0.0.3) on a large compact interval $[0,N]$ and define

$$\begin{aligned}\mathcal{T}^{\alpha,\beta}=\{(f,g)\in\mathcal{R}:\,& f_1(0)\sin\alpha+f_2(0)\cos\alpha=0,\\ & f_1(N)\sin\beta+f_2(N)\cos\beta=0,\quad \alpha,\beta\in(0,\pi]\}.\end{aligned}$$

Lemma 2.25. $\mathcal{T}^{\alpha,\beta}$ *is a self-adjoint relation.*

Proof. Clearly $\mathcal{T}^{\alpha,\beta}$ is a symmetric relation because of the boundary conditions at 0 and N. We will show that $\mathcal{T}^{\alpha,\beta}$ is a 2-dimensional extension of $\mathcal{R}_0$. Then by Theorem 2.8, $\mathcal{T}^{\alpha,\beta}$ is a self-adjoint relation. By Lemma 2.19, for $c=\begin{pmatrix}-\cos\alpha\\ \sin\alpha\end{pmatrix}$ and $w=\begin{pmatrix}-\cos\beta\\ \sin\beta\end{pmatrix}\in\mathbb{C}^2$ there exists ϕ_0 and ϕ_N in $D(\mathcal{R})$ such that $\phi_0(0+)=c$ and $\phi_N(N-)=w$ and the support of ϕ_0 and ϕ_N are contained in $[0,N]$. Clearly ϕ_0 , ϕ_N are linearly independent but ϕ_0 , ϕ_N are not in $D(\mathcal{R}_0)$. This shows that

$D(\mathcal{R}_0) \subset D(R_0) + L(\phi_0, \phi_N) \subset D(\mathcal{T}^{\alpha,\beta})$. Because of the boundary conditions at 0 and N, $D(\mathcal{T}^{\alpha,\beta})$ is a 2-dimensional restriction of $D(\mathcal{R})$. Hence $D(\mathcal{T}^{\alpha,\beta}) = D(\mathcal{R}_0) + L(\phi_0, \phi_N)$. Therefore, $\mathcal{T}^{\alpha,\beta}$ is a 2-dimensional extension of $\mathcal{R}_0$ so that $\mathcal{T}^{\alpha,\beta}$ is a self-adjoint relation. □

Let $u(x,z)$ and $v(x,z)$ be the solutions of the system (0.0.3) on $[0,N]$ with the initial values

$$u(0,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For $z \in C^+$ there is a unique $m(z)$ such that $f(x,z) = u(x,z) + m(z)v(x,z)$ satisfying

$$f_1(N,z)\sin\beta + f_2(N,z)\cos\beta = 0.$$

The coefficient $m(z)$ is called Weyl m function and is well explained in Chapter 3. Next, we describe the spectrum of $\mathcal{T}^{\alpha,\beta}$. Let

$$T(x,z) = \begin{pmatrix} u_1(x,z) & v_1(x,z) \\ u_2(x,z) & v_2(x,z) \end{pmatrix}, \quad T(0,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and define

$$w_\alpha(x,z) = \frac{1}{\sin\alpha + m(z)\cos\alpha} T(x,z) \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}$$

Lemma 2.26. *Using the notations above we have*

$$f(x,z)w_\alpha(x,\bar{z})^* - w_\alpha(x,z)f(x,\bar{z})^* = T(x,z)JT(x,\bar{z})^* = J.$$

Proof. Here,

$$
\begin{aligned}
&f(x,z)w_\alpha(x,\bar{z})^* \\
&= T(x,z)\begin{pmatrix} 1 \\ m(z) \end{pmatrix}(\cos\alpha, -\sin\alpha)\frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,\bar{z})^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,z)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ m(z)\cos\alpha & -m(z)\sin\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} JT(x,\bar{z})^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,z)\begin{pmatrix} \sin\alpha & \cos\alpha \\ m(z)\sin\alpha & m(z)\cos\alpha \end{pmatrix} JT(x,\bar{z})^*.
\end{aligned}
$$

$$
\begin{aligned}
&w_\alpha(x,z)f(x,\bar{z})^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,z)\begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}(1, m(z))T(x,\bar{z})^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,z)\begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}(1, m(z))\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} JT(x,\bar{z})^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,z)\begin{pmatrix} \cos\alpha & m(z)\cos\alpha \\ -\sin\alpha & -m(z)\cos\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} JT(x,\bar{z})^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha}T(x,z)\begin{pmatrix} -m(z)\cos\alpha & \cos\alpha \\ m(z)\sin\alpha & -\sin\alpha \end{pmatrix} JT(x,\bar{z})^*.
\end{aligned}
$$

Then

$$
\begin{aligned}
&f(x,z)w_\alpha(x,\bar z)^* - w_\alpha(x,z)f(x,\bar z)^* \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha} T(x,z) \begin{pmatrix} \sin\alpha + m(z)\cos\alpha & 0 \\ 0 & \sin\alpha + m(z)\cos\alpha \end{pmatrix} JT(x,\bar z)^*. \\
&= T(x,z)JT(x,\bar z)^* \\
&= J.
\end{aligned}
$$

□

Lemma 2.27. *Let* $z \in \Gamma(\mathcal{T}^{\alpha,\beta})$ *then* $(\mathcal{T}^{\alpha,\beta} - z)^{-1}$ *is a bounded linear operator and is defined by*

$$(\mathcal{T}^{\alpha,\beta} - z)^{-1}h(x) = \int_0^N G(x,t,z)H(t)h(t)dt,$$

where $G(x,t,z) = \begin{cases} f(x,z)w_\alpha(t,\bar z_0)^* & \text{if } 0 < t \le x \\ w_\alpha(t,\bar z)f(x,\bar z_0) & \text{if } x < t \le N. \end{cases}$

Proof. Let $y(x,z) = \int_0^N G(x,t,z)H(t)h(t)dt$. We show that $y(x,z)$ solves the inhomogeneous equation

$$Jy' = zHy - Hh$$

for a.e. $x > 0$. Here

$$y(x,z) = \int_0^x f(x,z)w_\alpha(t,\bar z)^*H(t)h(t)dt + \int_x^N w_\alpha(x,z)f(t,\bar z)^*H(t)h(t)dt$$

and $Jf' = zHf$, $Jw_\alpha' = zHw_\alpha$. Then on differentiation we get,

$$
\begin{aligned}
y'(x,z) = {}& f(x,z)w_\alpha(x,\bar z)^*H(x)h(x) + f'(x,z)\int_0^x w_\alpha(t,\bar z)^*H(t)h(t)dt \\
&- w_\alpha(x,z)f(x,\bar z)^*H(x)h(x) + w_\alpha'(x,z)\int_x^N f(t,\bar z)^*H(t)h(t)dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
Jy'(x,z) &= Jf(x,z)w_\alpha(x,\bar{z})^*H(x)h(x) + Jf'(x,z)\int_0^x w_\alpha(t,\bar{z})^*H(t)h(t)dt \\
&\quad -Jw_\alpha(x,z)f(x,\bar{z})^*H(x)h(x) + Jw_\alpha'(x,z)\int_x^N f(t,\bar{z})^*H(t)h(t)dt \\
&= Jf(x,z)w_\alpha(x,\bar{z})^*H(x)h(x) + zHf(x,z)\int_0^x w_\alpha(t,\bar{z})^*H(t)h(t)dt \\
&\quad -Jw_\alpha(x,z)f(x,\bar{z})^*H(x)h(x) + zHw_\alpha(x,z)\int_x^N f(t,\bar{z})^*H(t)h(t)dt. \\
&= J\Big(f(x,z)w_\alpha(x,\bar{z})^* - w_\alpha(x,z)f(x,\bar{z})^*\Big)Hh+ \\
&\quad zH\Big(\int_0^x f(x,z)w_\alpha(t,\bar{z})^*H(t)h(t)dt + \int_x^N w_\alpha(x,z)f(t,\bar{z})^*H(t)h(t)dt\Big) \\
&= JJHh + zHy \\
&= zHy - Hh.
\end{aligned}
$$

On the other hand, denote $g(x,z)$ as

$$g(x,z) = (\mathcal{T}^{\alpha,\beta} - z)^{-1}h(x)$$

then by Theorem 2.9, $h(x) = zu - v$ for some $(u,v) \in \mathcal{T}^{\alpha,\beta}$ so that $(g, zg-h) \in \mathcal{T}^{\alpha,\beta}$. So $g(x,z)$ also satisfies the inhomogeneous problem and $g(x,z) \in D(\mathcal{T}^{\alpha,\beta})$, it satisfies the boundary condition which implies that $g(0,z) = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} c(z)$ for some scalar $c(z)$. We have

$$y(0,z) = \frac{1}{\sin\alpha + m(z)\cos\alpha} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \langle f(x,\bar{z}), h\rangle$$

Now

$$\begin{aligned}
\langle f(.,\bar{z}),h\rangle &= \langle f(.,\bar{z}),h\rangle - \langle f(.,\bar{z}),zg\rangle + \langle f(.,\bar{z}),zg\rangle \\
&= \langle f(.,\bar{z}),h-zg\rangle + z\langle f(.,\bar{z}),g\rangle \\
&= -\int_0^N f(x,\bar{z})^* H(zg-h)dx + z\int_0^N f(x,\bar{z})^* Hgdx \\
&= -\int_0^N f(x,\bar{z})^* Jg'dx - \int_0^N f'(x,\bar{z})^* Jgdx \\
&= f(0,\bar{z})^* Jg(0,z) - f(N,\bar{z})^* Jg(N,z).
\end{aligned}$$

Since both $f(x,z)$ and $g(x,z)$ satisfies the same boundary condition at N, $f(N,\bar{z})^* Jg(N,z) = 0$. Now

$$f(0,\bar{z})^* Jg(0,z) = (1, m(z)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} c(z).$$

So

$$\begin{aligned}
y(0,z) &= \frac{1}{\sin\alpha + m(z)\cos\alpha} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} (1, m(z)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} c(z) \\
&= \frac{1}{\sin\alpha + m(z)\cos\alpha} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} (m(z), -1) \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} c(z) \\
&= \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} c(z) \\
&= g(0,z).
\end{aligned}$$

By uniqueness we must have, $g(x,z) = y(x,z)$. We have already shown in Theorem 2.2 that $(\mathcal{T}^{\alpha,\beta} - z)^{-1}$ is a bounded linear operator.

□

Now define a map $V : L^2(H, [0, N]) \to L^2(I, [0, N])$ by

$$Vy = H^{\frac{1}{2}}(x)y(x).$$

Here $H^{\frac{1}{2}}(x)$ is the unique positive semi-definite square root of $H(x)$. Then V is an isometry and hence maps $L^2(H, [0, N])$ unitarily onto the range $R(V) \subset L^2(I, [0, N])$. Define an integral operator $\mathcal{L}$ on $L^2(I, [0, N])$ as

$$(\mathcal{L}f)(x) = \int_0^N L(x,t)f(t)dt, \quad L(x,t) = H^{\frac{1}{2}}(x)G(x,t)H^{\frac{1}{2}}(t).$$

The kernel L is square integrable since

$$\begin{aligned}\int_0^N \int_0^N \|L^*L\|dxdt &\leq \int_0^N \int_0^N \|VG^*\|\|(VG)^*\|dxdt \\ &\leq \int_0^N \int_0^N \|G^*\|\|G\|dxdt < \infty.\end{aligned}$$

So $\mathcal{L}$ is a Hilbert-Schmidt operator and thus compact.
Since $L(x,t) = L^*(t,x), \mathcal{L}$ is also self-adjoint.

Lemma 2.28 ([15]). *Let $f \in L^2(I, [0, N]), \lambda \neq 0$, then the following statements are equivalent:*

1. $\mathcal{L}f = \lambda^{-1}f$.
2. *$f \in R(V)$, and the unique $y \in L^2(H, [0, N])$ with $Vy = f$ solves $(\mathcal{T}^{\alpha,\beta} - z)^{-1}y = \lambda y$.*

Proof. For all $g \in L^2(I, [0, N])$ we have,

$$(\mathcal{L}g)(x) = H^{\frac{1}{2}}(x)w(x) \text{ where } w(x) = \int_0^N G(x,t)H^{\frac{1}{2}}(t)g(t)dt,$$

lies in $L^2(H, [0, N])$. Then $R(\mathcal{L}) \subset R(V)$. Now if (1) holds then $f = \lambda\mathcal{L}f \in R(V)$. So $f = V(y)$ for unique $y \in L^2(H, [0, N])$ and

$$f(x) = H^{\frac{1}{2}}(x)y(x) = \lambda\mathcal{L}y(x) = \lambda H^{\frac{1}{2}}(x) \int_0^N G(x,t)H(t)y(t)dt$$

for $a.e. x \in [0, N]$. In other words,

$$H^{\frac{1}{2}}(x)\big(y(x) - \lambda \int_0^N G(x,t)H(t)y(t)dt\big) = 0.$$

Conversely if (2) holds,

$$\lambda y = \int_0^N G(x,t)H(t)y(t)dt$$

then $H^{\frac{1}{2}}(x)y = \frac{1}{\lambda}\int_0^N H^{\frac{1}{2}}(x)G(x,t)H(t)y(t)dt$.

□

Lemma 2.29. *Let $z \in \mathbb{C}$. For any $\lambda \neq z$, if $(f, \lambda f) \in \mathcal{T}^{\alpha,\beta}$ then f solves $(\mathcal{T}^{\alpha,\beta} - z)^{-1}y = \frac{1}{\lambda - z}y$. Conversely, if $y \in L^2(H, [0, N])$ and y solves $(T^{\alpha,\beta} - z)^{-1}y = \lambda y$ then $(y, (z + \frac{1}{\lambda})y) \in \mathcal{T}^{\alpha,\beta}$.*

Proof. Let $(f, \lambda f) \in \mathcal{T}^{\alpha,\beta}$ then $(f, \lambda f - zf) \in (\mathcal{T}^{\alpha,\beta} - z)$. It follows that

$$((\lambda - z)f, f) \in (\mathcal{T}^{\alpha,\beta} - z)^{-1} \Rightarrow \Big(f, \frac{1}{(\lambda - z)}\Big) \in (\mathcal{T}^{\alpha,\beta} - z)^{-1}.$$

This means that f solves

$$(\mathcal{T}^{\alpha,\beta} - z)^{-1}y = \frac{1}{\lambda - z}y.$$

Conversely suppose $y \in L^2(H, [0, N])$ and y solves

$$(\mathcal{T}^{\alpha,\beta} - z)^{-1}y = \lambda y.$$

That is $(y, \lambda y) \in (\mathcal{T}^{\alpha,\beta} - z)^{-1}$ so that $(\lambda y, y) \in (\mathcal{T}^{\alpha,\beta} - z)$. So there is $(f, g) \in \mathcal{T}^{\alpha,\beta}$ such that $\lambda y = f$ and

$$g - zf = y \Rightarrow g = y + z\lambda y.$$

Hence $(\lambda y, y + z\lambda y) \in \mathcal{T}^{\alpha,\beta}$. It follows that $\Big(y, (z + \frac{1}{\lambda}y)\Big) \in \mathcal{T}^{\alpha,\beta}$. □

By Lemma 2.28, we see that there is a one to one correspondence of eigenvalues (eigenfunctions) for the operator $\mathcal{L}$ and $(\mathcal{T}^{\alpha,\beta} - z)^{-1}$. As $\mathcal{L}$ is compact operator, it has only discrete spectrum consisting of only eigenvalues. Since $(\mathcal{T}^{\alpha,\beta} - z)^{-1}$ is unitarily equivalent with $\mathcal{L}\downharpoonright_{R(V)}$, that is

$$V^{-1}\mathcal{L}\downharpoonright_{R(V)} V = (\mathcal{T}^{\alpha,\beta} - z)^{-1},$$

$(\mathcal{T}^{\alpha,\beta} - z)^{-1}$ has only discrete spectrum consisting of only eigenvalues. Then by Theorem 2.15, $\mathcal{T}^{\alpha,\beta}$ has only discrete spectrum. By Lemma 2.29, the spectrum of $\mathcal{T}^{\alpha,\beta}$ consists only eigenvalues.

We would like to extend this idea over the half line $\mathbb{R}_+$. First note that we are considering the limit-circle case of the system (0.0.3). That implies that the deficect indices of $\mathcal{R}_0$ are $(2,2)$. Suppose $p \in D(\mathcal{R}) \setminus D(\mathcal{R}_0)$ such that $\lim_{x\to\infty} p(x)^* Jp(x) = 0$. Such function clearly exists.

Consider the relation

$$\begin{aligned}\mathcal{T}^{\alpha,p} = \{(f,g) \in \mathcal{R} : f_1(0)\sin\alpha + f_2(0,z)\cos\alpha = 0 \\ \text{and } \lim_{x\to\infty} f(x)^* Jp(x) = 0\}.\end{aligned}$$

Lemma 2.30. *$\mathcal{T}^{\alpha,p}$ defines a self-adjoint extension of $\mathcal{R}_0$.*

Proof. Clearly the relation $\mathcal{T}^{\alpha,p}$ is a symmetric relation. Also because of the boundary conditions, it is a 2-dimensional restriction of $\mathcal{R}$. On the other hand, let $u_0(x) = u(x)$ and $u_0(x) = 0$ near the neighborhood of ∞ ie $\lim_{x\to\infty} u_0(x) = 0$, and $p_0(x) = 0$ in the neighborhood of 0 and $p_0(x) = f(x)$ otherwise. Then $u_0, p_0 \notin D(\mathcal{R}_0)$ and are linearly independent. Moreover, $D(\mathcal{R}_0) + L(u_0, p_0) \subset D(\mathcal{T}^{\alpha,p})$. It follows that

$$D(\mathcal{T}^{\alpha,p}) = D(\mathcal{R}_0) + L(u_0, p_0).$$

So $D(\mathcal{T}^{\alpha,p})$ is a 2-dimensional extension of $D(\mathcal{R}_0)$. Hence $\mathcal{T}^{\alpha,p}$ is a self-adjoint relation. $\square$

Now we discuss the spectrum of $\mathcal{T}^{\alpha,p}$. Let $u(x,z)$ and $v(x,z)$ be two linearly independent solutions of the system (0.0.3) with

$$u(0,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $z \in \mathbb{C}^+$ and as above write $f(x,z) = u(x,z) + m(z)v(x,z) \in L^2(H,\mathbb{R}_+)$ satisfying $\lim_{x\to\infty} f(x,z)^* JP(x) = 0$. Let $T(x,z) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ and

$$w_\alpha(x,z) = \frac{1}{\sin\alpha + m(z)\cos\alpha} T(x,z) \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}.$$

Then as in Lemma 2.26 we have,

$$f(x,z)w_\alpha(x,\bar{z})^* - w_\alpha(x,z)f(x,\bar{z})^* = T(x,z)JT(x,\bar{z})^* = J.$$

Lemma 2.31. *Let $z \in \rho(\mathcal{T}^{\alpha,p})$ then the resolvent operator $(\mathcal{T}^{\alpha,p} - z)^{-1}$ is given by*

$$(\mathcal{T}^{\alpha,p} - z)^{-1}h(x) = \int_0^\infty G(x,t,z)H(t)h(t)dt$$

where $G(x,t,z) = \begin{cases} f(x,z)w_\alpha(t,\bar{z})^* & \text{if } 0 < t \le x \\ w_\alpha(t,\bar{z})f(x,\bar{z}) & \text{if } x < t \le \infty \end{cases}$

Proof. Let $y(x,z) = \int_0^\infty G(x,t,z)H(t)h(t)dt$ then y solves the inhomogeneous equation

$$Jy' = zHy - Hh.$$

This is clear by differentiating

$$y(x,z) = \int_0^x f(x,z)w_\alpha(t,\bar{z})^* H(t)h(t)dt + \int_x^\infty w_\alpha(x,z)f(t,\bar{z})^* H(t)h(t)dt.$$

On the other hand let $g(x,z) = (\mathcal{T}^{\alpha,p} - z)^{-1}h(x)$, then by Theorem 2.9, $h(x) = zu - v$ for some $(u,v) \in \mathcal{T}^{\alpha,p}$ so that $(g, zg - h) \in \mathcal{T}^{\alpha,p}$ and hence g satisfies the inhomogeneous equation. Since $g \in D(\mathcal{T}^{\alpha,p})$,

$$g_1(0,z)\sin\alpha + g_2(0,z)\cos\alpha = 0, \lim_{x\to\infty} g^*(x,z)Jp(x,z) = 0.$$

We also have $\lim_{x\to\infty} f^*(x,z)Jg(x,z) = 0$ and $g(0,z) = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} c(z)$ for some scalar $c(z)$. But also we have

$$y(0,z) = \frac{1}{(m(z)\cos\alpha + \sin\alpha)} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \langle f(\bar{z}), h\rangle.$$

Here

$$\begin{aligned} \langle f(\bar{z}), h\rangle &= \langle f(\bar{z}), h\rangle - \langle f(\bar{z}), zg\rangle - \langle \bar{z} f(\bar{z}), g\rangle \\ &= \langle f(\bar{z}), h + zg\rangle - \langle \bar{z} f(\bar{z}), g\rangle \\ &= f^*(0,\bar{z})Jg(0,z) - \lim_{x\to\infty} f^*(x,z)Jg(x,z) \\ &= f^*(0,\bar{z})Jg(0,z). \end{aligned}$$

Hence $y(0,z) = g(0,z)$. By uniqueness we have $y(x,z) = g(x,z)$. □

Lemma 2.32 ([15]). *Let $f \in L^2(I, R_+), \lambda \neq 0$, then the following statements are equivalent:*

1. *$\mathcal{L}f = \lambda^{-1} f$.*

2. *$f \in R(V)$, and the unique $y \in L^2(H, R_+)$ with $Vy = f$ solves $(\mathcal{T}^{\alpha,p} - z)^{-1} y = \lambda y$.*

Proof. Similar to the proof of Lemma 2.28. □

Lemma 2.33. *Let $z \in \mathbb{C}$. For any $\lambda \neq z$, if $(y, \lambda y) \in \mathcal{T}^{\alpha,p}$ then y solves $(\mathcal{T}^{\alpha,p} - z)^{-1} y = \frac{1}{\lambda - z} y$. Conversely, if $y \in L^2(H, R_+)$ and y solves $(T^{\alpha,p} - z)^{-1} y = \lambda y$ then $(y, (z + \frac{1}{\lambda})y) \in \mathcal{T}^{\alpha,p}$.*

Proof. Similar to the proof of Lemma 2.29. □

Now define a map $V : L^2(H, \mathbb{R}_+) \to L^2(I, \mathbb{R}_+)$ by $Vy = H^{\frac{1}{2}}(x) y(x)$. V is isometry and maps unitarily onto the range $R(V) \subset L^2(I, \mathbb{R}_+)$.

Define an integral operator $\mathcal{L}$ on $L^2(I, \mathbb{R}_+)$ by

$$(\mathcal{L}g)(x) = \int_0^\infty L(x,t) g(t) dt, \;\; L(x,t) = H^{\frac{1}{2}}(x) G(x,t,z) H^{\frac{1}{2}}(t).$$

Then as before the kernel $\mathcal{L}$ is square integrable. This means that

$$\int_0^\infty \int_0^\infty \parallel L^* L \parallel < \infty.$$

Hence $\mathcal{L}$ is a Hilbert Schmidt a operator and so is a compact operator.

Again by Lemma 2.32, we have a one to one correspondence of eigenvalues (eigenfunctions) for the operator $\mathcal{L}$ and $(\mathcal{T}^{\alpha,p} - z)^{-1}$. As $\mathcal{L}$ is compact operator, it has only discrete spectrum consisting of only eigenvalues converging to 0. Since $(\mathcal{T}^{\alpha,p} - z)^{-1}$ is unitarily equivalent with $\mathcal{L} \downharpoonright_{R(V)}$, that is $V^{-1} \mathcal{L} \downharpoonright_{R(V)} V = (\mathcal{T}^{\alpha,p} - z)^{-1}$, $(\mathcal{T}^{\alpha,p} - z)^{-1}$ has only discrete spectrum consisting of only eigenvalues. Then by Theorem 2.15, $\mathcal{T}^{\alpha,p}$ has only discrete spectrum. By Lemma 2.33, the spectrum of $\mathcal{T}^{\alpha,p}$ consists of only eigenvalues.

Theorem 2.34. *In the limit-circle case, the defect index $\beta(\mathcal{R}_0, z) = \dim N(\mathcal{R}, \bar{z})$ of $\mathcal{R}_0$ is constant on $\mathbb{C}$.*

Proof. Since $\mathcal{R}_0$ is a symmetric relation, by Theorem 2.3, the defect index $\beta(\mathcal{R}_0, z)$ is constant on upper and lower half-planes. In the limit-circle case, if z is in upper or lower half-planes, $\beta(\mathcal{R}_0, z) = 2$. Suppose $\beta(\mathcal{R}_0, \lambda) < 2$ for some $\lambda \in \mathbb{R}$.

Since $\Gamma(\mathcal{R}_0)$ is open, $\lambda \notin \Gamma(\mathcal{R}_0)$ and hence $\lambda \in S(\mathcal{R}_0)$. Since for each $\alpha \in (0, \pi]$, $\mathcal{T}^{\alpha,p}$ is self-adjoint extension of $\mathcal{R}_0$, $\lambda \in S(\mathcal{T}^{\alpha,p}) = \sigma(\mathcal{T}^{\alpha,p})$. In the limit-circle case, $\sigma(\mathcal{T}^{\alpha,p})$ consists of only eigenvalues. Therefore, λ is an eigenvalue for all boundary conditions α at 0. However, this is impossible unless $\beta(\mathcal{R}_0, \lambda) = 2$. □

Theorem 2.35. *A canonical system (0.0.3) with* $\operatorname{tr} H \equiv 1$ *prevails limit-point case.*

Proof. Suppose it prevails the limit-circle case. By Theorem 2.34, the defect index $\beta(\mathcal{R}_0, z) = \dim N(\mathcal{R}, \bar{z}) = 2$ for all $z \in \mathbb{C}$. In other words, for any $z \in \mathbb{C}$, all solutions of (0.0.3) are in $L^2(H, \mathbb{R}_+)$. In particular, the constant solutions $u(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of (0.0.3) when $z = 0$, are in $L^2(H, \mathbb{R}_+)$. But this is not possible because $\int_0^\infty u(x)^* H(x) u(x) dx + \int_0^\infty v(x)^* H(x) v(x) dx = \int_0^\infty \operatorname{tr} H(x) dx = \infty$. □

Chapter 3

Remling's theorem on canonical systems

In this chapter, we will prove Remling's theorem on canonical systems. We begin with the Weyl theory of a canonical system in the following section.

3.1 Weyl theory of a canonical system

Let u_α, v_α be the linearly independent solutions of (0.0.3) with the initial values

$$u_\alpha(0,z) = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \quad \text{and} \quad v_\alpha(0,z) = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}.$$

For $z \in \mathbb{C}^+$, want to define $m_\alpha(z)$ as the unique coefficient for which

$$f_\alpha = u_\alpha + m_\alpha(z) v_\alpha \in L^2(H, \mathbb{R}_+).$$

Consider a compact interval $[0, N]$ and let $z \in \mathbb{C}^+$, define the unique coefficient $m_N^\beta(z)$ such that $f(x,z) = u(x,z) + m_N^\beta(z) v(x,z)$ satisfying

$$f_1(N,z)\sin\beta + f_2(N,z)\cos\beta = 0.$$

It follows from boundary condition $f_1(N,z)\sin\beta + f_2(N,z)\cos\beta = 0$ at N that

$$m_N^\beta(z) = -\frac{u_1(N,z)\sin\beta + u_2(N,z)\cos\beta}{v_1(N,z)\sin\beta + v_2(N,z)\cos\beta}.$$

As z, N, β varies $m_N^\beta(z)$ becomes a funtion of these arguments, and since u_1, u_2, v_1, v_2 are entire functions of z, $m_N^\beta(z)$ is meromorphic function of z. Let

$$m_N^\beta(z) = -\frac{u_1 t + u_2}{v_1 t + v_2}, \quad t = \tan\beta, \quad t \in \mathbb{R} \cup \{\infty\}.$$

This is a fractional linear transformation. As a function of $t \in \mathbb{R}$, it maps real line to a circle. So for fixed $z \in C^+$, $C_N(z) = \{m_N^\beta(z) : 0 \le \beta < \pi\}$ is a circle. For any complex number $m \in \mathbb{C}$

$$m \in C_N(z) \Leftrightarrow \ \mathrm{Im}\ \frac{u_2 + m v_2}{u_1 + m v_1} = 0$$

From this identity, the equation of the circle $C_N(z)$ is given by

$$|m - c|^2 = r^2, \quad c = \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}, \quad \& \ r = \frac{1}{|W_N(\bar{v}, v)|}. \tag{3.1.1}$$

Suppose $f(x,z) = u(x,z) + m_N^\beta(z) v(x,z)$, then $m = m_N^\beta$ is an interior of C_N if and only if

$$|m - c|^2 < r^2 \Leftrightarrow \frac{W_N(\bar{f}, f)}{W_N(\bar{v}, v)} < 0 \tag{3.1.2}$$

Let us write $\tau y = zy$ if and only if $Jy' = zH(x)y$. Suppose f and g are the solutions of (0.0.3) then we have the following identity, called the *Green's Identity.*

$$\int_0^N (f^* H(x)\tau g - (\tau f)^* H(x) g(x))dx = W_0(\bar{f}, g) - W_N(\bar{f}, g) \tag{3.1.3}$$

Using the Green's identity

$$\begin{aligned}
W_N(\bar{f}, f) &= W_0(\bar{f}, f) - 2i \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx \\
&= W_0(\bar{u} + \bar{m}\bar{v}, u + mv) - 2i \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx \\
&= m - \bar{m} - 2i \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx \\
&= 2i \text{ Im } m(z) - 2i \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx.
\end{aligned}$$

$$W_N(\bar{f}, f) = 2i \text{ Im } m(z) - 2i \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx. \tag{3.1.4}$$

$$W_N(\bar{v}, v) = -2i \text{ Im}z \int_0^N v^*(x)H(x)v(x)dx.$$

$$\begin{aligned}
\frac{W_N(\bar{f}, f)}{W_N(\bar{v}, v)} &= \frac{2i \text{ Im } m(z) - 2i \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx}{-2i \text{ Im}z \int_0^N v^*(x)H(x)v(x)dx} \\
&= \frac{-\text{ Im } m(z) + \text{ Im}z \int_0^N f^*(x)H(x)f(x)dx}{\text{Im}z \int_0^N v^*(x)H(x)v(x)dx}.
\end{aligned}$$

Hence from (3.1.2) we see that $\frac{W_N(\bar{f},f)}{W_N(\bar{v},v)} < 0$ if and only if

$$\begin{aligned}
&\frac{-\text{ Im}m(z)}{\text{Im}z} + \int_0^N f^*(x)H(x)f(x)dx < 0. \\
&\Rightarrow \int_0^N f^*(x)H(x)f(x)dx < \frac{\text{Im}m(z)}{\text{Im}z}.
\end{aligned}$$

Thus it follows that m is an interior of C_N if and only if

$$\int_0^N f^*(x)H(x)f(x)dx < \frac{\text{Im}m(z)}{\text{Im}z} \tag{3.1.5}$$

and $m \in C_N(z)$ if and only if

$$\int_0^N f^*(x)H(x)f(x)dx = \frac{\text{Im } m(z)}{\text{Im } z}. \tag{3.1.6}$$

For $z \in \mathbb{C}^+$, the radius of the circle $C_N(z)$ is given by

$$r_N(z) = \frac{1}{|W_N(\bar{v}, v)|} = \frac{1}{2 \text{ Im } z \int_0^N v^*(x)H(x)v(x)dx}. \tag{3.1.7}$$

Now let $0 < N_1 < N_2 < \infty$. Then if m is inside or on C_{N_2}

$$\int_0^{N_1} f^*(x,z)H(x)f(x,z)dx < \int_0^{N_2} f(x,z)^*H(x)f(x,z)dx \leq \frac{\text{Im } m}{\text{Im } z}$$

and therefore m is inside C_{N_1}. Let us denote the interior of $C_N(z)$ by $\text{Int}C_N(z)$ and suppose $D_N(z) = C_N(z) \cup \text{Int}C_N(z)$. These are called the Wyle Disks. These Wyle Disks are nested, that is, $D_{N+\epsilon}(z) \subset D_N(z)$ for any $\epsilon > 0$, from the following identity

$$m \in D_N(z) \Leftrightarrow \int_0^N f^*(x)H(x)f(x)dx \leq \frac{\text{Im } m(z)}{\text{Im } z}.$$

From (3.1.7), we see that $r_N(z) > 0$, and $r_N(z)$ decreases as $N \to \infty$. So $\lim_{N\to\infty} r_N(z)$ exists and

$$\lim_{N\to\infty} r_N(z) = 0 \Leftrightarrow v \notin L^2(H, \mathbb{R}_+).$$

Thus for a given $z \in \mathbb{C}^+$ as $N \to \infty$ the circles $C_N(z)$ converges either to a circle $C_\infty(z)$ or to a point $m_\infty(z)$. If $C_N(z)$ converges to a circle, then its radius $r_\infty = \lim r_N$ is positive and (3.1.7) implies that $v \in L^2(H, \mathbb{R}_+)$. If $\tilde{m}_\infty$ is any point on $C_\infty(z)$ then $\tilde{m}_\infty$ is inside any $C_N(z)$ for $N > 0$. Hence

$$\int_0^N (u + \tilde{m}_\infty v)^* H(u + \tilde{m}_\infty v) < \frac{\text{Im } \tilde{m}_\infty}{\text{Im } z}$$

and letting $N \to \infty$ one sees that $f(x,z) = u + \tilde{m}_\infty v \in L^2(H, \mathbb{R}_+)$. The same argument holds if $\tilde{m}_\infty$ reduces to a point m_∞. Therefore, if $\text{Im } z \neq 0$, there always exists a solution of (0.0.3) of class $\in L^2(H, \mathbb{R}_+)$.

In the case $C_N(z) \to C_\infty(z)$ all solutions are in $L^2(H, \mathbb{R}_+)$ for Im $z \neq 0$ and this identifies the limit-circle case with the existence of the circle $C_\infty(z)$. Correspondingly, the limit-point case is identified with the existence of the point $m_\infty(z)$. In this case $C_N(z) \to m_\infty$ there results $\lim r_N = 0$ and (3.1.7) implies that v is not of class $L^2(H, \mathbb{R}_+)$. Therefore in this situation there is only one linearly independent solution of class $L^2(H, \mathbb{R}_+)$.

In the limit-circle case, $m \in C_N$ if and only (3.1.6) holds. Since $f(x,z) = u(x,z) + mv(x,z)$, it follows that m is on C_∞ if and only if

$$\int_0^\infty f(x,z)^* H f(x,z) dx = \frac{\mathrm{Im} m(z)}{\mathrm{Im} z}. \tag{3.1.8}$$

From (3.1.4), it follows that m is on the limit-circle if and only if $\lim\limits_{N\to\infty} W_N(\bar{f}, f) = 0$. For $z \in C^+$, the following theorem has been proved.

Theorem 3.1 ([4]). *1. The limit-point case* $(r_\infty = 0)$ *implies that (0.0.3) has precisely one* $L^2(H, R_+)$ *solution.*

2. The limit-circle case $(r_\infty > 0)$ *implies all solutions of (0.0.3) are in* $L^2(H, \mathbb{R}_+)$.

The identity (3.1.6) shows that $m_N^\beta(z)$ are holomorphic functions mapping upper half-plane to itself. The poles and zeros of these functions lie on the real line and are simple.

Theorem 3.2 ([4]). *In the limit-point case, the limit* $m_\infty(z)$ *is a holomorphic function mapping upper half-plane to itself.*

Proof. From (3.1.1) we see that the center and radius of the circle C_1 are continuous functions of z for Im $z > 0$. Thus , since C_N is interior to C_1 for $N > 1$, it follows that if z is restricted to compact subset $K \subset \mathbb{C}^+$ then the points $m_N^\beta(z)$ on C_N are uniformly bounded as $N \to \infty$. The functions $m_N^\beta(z)$ being meromorphic and bounded on K are analytic there. Hence by Cauchy's theorem, the functions m_N^β constitute an equicontinuous set on K, and m_N^β converges uniformly to m_∞. Being the uniform limit of analytic functions, m_∞ itself is analytic on K and hence on $\mathbb{C}^+$. Since m_∞ is inside C_N, it follows from (3.1.5) that $\text{Im } m_\infty > 0$ for $\text{Im } z > 0$. □

In limit-circle case, each circle $C_N(z)$ is traced by the points $m = m_N^\beta(z)$ as β ranges over $0 \leq \beta < \pi$ for fixed N and z. Let $z_0\mathbb{C}^+$ be fixed. A point $\tilde{m}_\infty(z_0)$ on the circle $C_\infty(z_0)$ is the limit-point of a sequence $m_{N_j}^{\beta_j}(z)$ with $N_j \to \infty$ as $j \to \infty$.

Theorem 3.3 ([4]). *Let $\tilde{m}_\infty(z_0)$ be a point on $C_\infty(z_0)$ and (N_j, β_j) a sequence such that $m_j(z_0) = m_{N_j}^{\beta_j}(z_0) \to \tilde{m}_\infty(z_0)$. Then for all $z \in \mathbb{C}^+$,*

$$\lim_{j \to \infty} m_j(z) = \tilde{m}_\infty(z)$$

and $\tilde{m}_\infty(z)$ is a holomorphic function mapping upper half-plane to itself.

Proof. Let

$$f_j(x, z) = u(x, z) + m_j(z)v(x, z). \tag{3.1.9}$$

Applying the Green's formula to $f_j(x, z)$ and $\bar{f}_j(x, z_0)$ and noting that both $f_j(x, z)$ and $\bar{f}_j(x, z_0)$ satisfies the boundary condition at N_j we get,

$$m_j(z) - m_j(z_0) = (z - z_0) \int_0^{N_j} \bar{f}_j(x, z_0)^* H f_j(x, z) dx. \tag{3.1.10}$$

Using (3.1.9) and (3.1.10),

$$m_j(z) = \frac{m_j(z_0) + (z - z_0) \int_0^{N_j} \bar{f}_j(x, z_0)^* H u(x, z) dx}{1 - (z - z_0) \int_0^{N_j} \bar{f}_j(x, z_0)^* H v(x, z) dx}. \tag{3.1.11}$$

In the limit-circle case all the solutions of (0.0.3) are in $L^2(H, \mathbb{R}_+)$. Therefore, as $j \to \infty$, the holomorphic function of z whose value at z is given by

$$\int_0^{N_j} \bar{f}_j(x, z_0)^* H u(x, z) dx = \int_0^{N_j} (\bar{u}(x, z_0) + \bar{m}_j(z_0)\bar{v}(x, z_0))^* H u(x, z) dx$$

which appears in the numerator of (3.1.11), tends to the limit

$$\int_0^{\infty} (\bar{u}(x, z_0) + \bar{\tilde{m}}_\infty(z_0)\bar{v}(x, z_0))^* H u(x, z) dx. \tag{3.1.12}$$

If z is restricted to some compact subset K of $\mathbb{C}^+$ the norm $\|u\|, \|v\|$ in $L^2(H, \mathbb{R}_+)$ are uniformly bounded in K, Thus by Schwarz inequality the integral in (3.1.12) are uniformly convergent in z over K. This implies that (3.1.12) defines an analytic function of z. The same is true for the integral in the denominator of 3.1.11. Thus, as $j \to \infty$, the holomorphic function m_j tends to a limit $\tilde{m}_\infty$ which is also a holomorphic function . The property Im $\tilde{m}_\infty(z) > 0,\ z \in \mathbb{C}^+$ follows from (3.1.8) that $\frac{\text{Im } m_\infty(z)}{\text{Im } z} > 0$. □

In Chapter 2, we showed that $\operatorname{tr} H \equiv 1$ implies the limit-point case. By a change of variable

$$t(x) = \int_0^x \operatorname{tr} H(s) ds. \tag{3.1.13}$$

a canonical system (0.0.3) can be reduced to a system with $\operatorname{tr} H \equiv 1$ which imply limit-point case. For if, $\widetilde{H}(t) = \frac{1}{\operatorname{tr} H(x)} H(x(t))$ so that $\operatorname{tr} \widetilde{H}(t) \equiv 1$. Further, let $u(x, z)$ be a solution of

$$Ju' = zHu$$

and put $\widetilde{u}(t, z) = u(x(t), z)$. Then $\widetilde{u}(t, z)$ solves

$$J\widetilde{u}' = z\widetilde{H}\widetilde{u}.$$

Their corresponding Weyl m functions on $[0, N]$ are related as follows,

$$\begin{aligned}\tilde{m}_N^{\beta}(z) &= -\frac{\tilde{u}_1(N,z)\sin\beta + \tilde{u}_2(N,z)\cos\beta}{\tilde{v}_1(N,z)\sin\beta + \tilde{v}_2(N,z)\cos\beta} \\ &= -\frac{u_1(x(N),z)\sin\beta + u_2(x(N),z)\cos\beta}{v_1(x(N),z)\sin\beta + v_2(x(N),z)\cos\beta} \\ &= m_{x(N)}^{\beta}(z)\end{aligned}$$

This shows that we get same Weyl circles even after changing the variable. The m function $\tilde{m}_N^{\beta}(z)$ of new system is obtained by changing the point of boundary condition from N to $x(N)$ of original system.

From now onward we will consider a canonical system with $\operatorname{tr} H \equiv 1$.

3.2 Space of Hamiltonians.

We need to consider the space of Hamiltonians and a suitable topology on it so that the space is compact. With the topology we have, we want to work with Weyl m functions of the canonical systems. We will again use the ideas from [14].

Let $M(\mathbb{R})$ denotes the set of Borel measures on $\mathbb{R}$. Consider the space

$$\mathcal{V}^C = \{\mu \in M(\mathbb{R}) : |\mu|(I) \leq C.\max\{|I|, 1\}\text{for all intervals } I \subset \mathbb{R}\}.$$

We would like to define a metric on $\mathcal{V}^C$. Let $C_c(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ with compact support, the continuous functions vanishing outside of a bounded interval. This space $C_c(\mathbb{R})$ is complete with respect to the $\|.\|_\infty$ norm. Pick a countable dense subset $\{f_n : \ n \in \mathbb{N}\} \subset C_c(\mathbb{R})$, the continuous functions of compact support, and let

$$\rho_n(\mu, \nu) = \Big| \int f_n d(\mu - \nu)(x) \Big|.$$

Define a metric d as

$$d(\mu,\nu)=\sum_{n=1}^{\infty}2^{-n}\frac{\rho_n(\mu,\nu)}{1+\rho_n(\mu,\nu)}.$$

Then $\left(\mathcal{V}^C,d\right)$ is a compact space. Let

$$\mathcal{V}^C_{2\times 2}=\Bigg\{\mu=\begin{pmatrix}\mu_{11} & \mu_{12}\\ \mu_{21} & \mu_{22}\end{pmatrix}\in M(\mathbb{R})^{2\times 2}:\mu_{ij}\geq 0 \text{ for } i=j,\ \mu_{ij}=\mu_{ji}\text{ for all } i,j\,,$$

$$\operatorname{tr}\mu(I)\leq C.\max\{|I|,1\},\quad \text{for all } I\subset\mathbb{R}\Bigg\}.$$

We now define a metric on $\mathcal{V}^C_{2\times 2}$. Let

$$\rho_n(\mu,\nu)=\sum_{1\leq i,j\leq 2}\Big|\int f_n d(\mu_{ij}-\nu_{ij})(x)\Big|.$$

Then define d as

$$d(\mu,\nu)=\sum_{n=1}^{\infty}2^{-n}\frac{\rho_n(\mu,\nu)}{1+\rho_n(\mu,\nu)}.$$

Clearly d is a metric on $\mathcal{V}^C_{2\times 2}$. Also we have $d(\mu_j,\mu)\to 0\Leftrightarrow \int f(x)d\mu_j\to\int f(x)d\mu$ for all $f\in C_c(\mathbb{R})$. We show that $\left(\mathcal{V}^C_{2\times 2},d\right)$ is compact.

Let $\mu_n=\begin{pmatrix}\mu^n_{11} & \mu^n_{12}\\ \mu^n_{21} & \mu^n_{22}\end{pmatrix}\in\mathcal{V}^C_{2\times 2}$ then, $|\mu^n_{ij}|(I)\leq C.\max\{|I|,1\}$ for all $I\subset\mathbb{R}$. So for each i,j, $\mu^n_{ij}\in\mathcal{V}^C$. Since $\left(\mathcal{V}^C,d\right)$ is compact, μ^n_{ij} has a convergent subsequence $\mu^{n_k}_{ij}$. Since the convergence is equivalent between the spaces $\left(\mathcal{V}^C_{2\times 2},d\right)$ and $\left(\mathcal{V}^C,d\right)$, we get a convergent subsequence μ_{n_j}.

We now consider a canonical system with measure as Hamiltonian,

$$Ju'=z\mu u,\qquad \mu\in\mathcal{V}^C_{2\times 2}. \tag{3.2.1}$$

If $I\subset\mathbb{R}$ is an compact interval and $B(I)$ denotes the space of all complex valued bounded Borel measurable functions on I.Then $B(I)$ is complete with respect to the metric given by $\rho(f,g) = ||f-g||_u$ where the norm on $B(I)$ is $||f||_u = \sup_{x\in I}|f(x)|$. Let $B(I)^2 = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} f_1, f_2 \in B(I) \right\}$. Clearly the space $B(I)^2$ is complete with respect to the metric given by $\rho(f,g) = ||f-g||_u$ where $||f||_u = \sup_{x\in I}|f_1(x)| + \sup_{x\in I}|f_2(x)|$. Let $f \in B(I)^2$, we call f a solution to the equation (3.2.1) if and only if

$$J(u(x) - u(a+)) = z\int_{(a,x)} d\mu(t)u(t) \quad \text{if } x \geq a \geq 0 \text{ and}$$

$$J(u(x) - u(a-)) = -z\int_{(x,a)} d\mu(t)u(t) \quad \text{if } x \leq a \leq 0.$$

In order to show the existence of a solution of the system (3.2.1), define a map on $B(I)^2$

$$Tu(x) = u(0) - zJ\int_0^x \mu(t)u(t) \leq 0.$$

and show that T is a contraction mapping.

$$\begin{aligned}
\|Tu - Tv\|_u &= \sup_{x\in I}\left| z\int_0^x (u_1 - v_1)d\mu_{11} + (u_2 - v_2)d\mu_{12}\right| \\
&\quad + \sup_{x\in I}\left| z\int_0^x (u_1 - v_1)d\mu_{21} + (u_2 - v_2)d\mu_{22}\right| \\
&\leq \sup_{x\in I}\Big[|z|\int_0^x |u_1 - v_1|d\mu_{11} + |z||u_2 - v_2|d\mu_{12} \\
&\quad + \sup_{x\in I}|z|\int_0^x |u_1 - v_1|d\mu_{21} + |z||u_2 - v_2|d\mu_{22}\Big] \\
&\leq \frac{c}{2}\sup_{x\in I}\Big[\sup_{t\in[0,x]}|u_1 - v_1| + \sup_{t\in[0,x]}|u_2 - v_2|\Big] \\
&\quad + \frac{c}{2}\sup_{t\in[0,x]}\Big[\sup_{t\in[0,x]}|u_1 - v_1| + \sup_{t\in[0,x]}|u_2 - v_2|\Big] \\
&\leq \frac{c}{2}\Big[2\sup_{x\in I}|u_1 - v_1| + 2\sup_{x\in I}|u_2 - v_2|\Big] \\
&= c\|u - v\|_u.
\end{aligned}$$

This shows that T is a contraction mapping, so it has a unique fixed point say $u(x)$ in $B(I)^2$ such that $Tu(x) = u(x)$. So there is a solution in $B(I)^2$ that satisfy $u(x) = u(0) - zJ\int_0^x \mu(t)u(t)$.

Now consider the space

$$\mathcal{V}_{2\times 2} = \left\{\mu \in M(\mathbb{R})^{2\times 2} : d\mu = H(x)dx, H(x) \geq 0, \operatorname{tr} H(x) \equiv 1, H(x) \in L^1_{\text{loc}}\right\}.$$

Then $(\mathcal{V}_{2\times 2}, d)$ is compact. For if $\mu_n \to \mu$ then $\mu \in \mathcal{V}_{2\times 2}$ since

$$\int f(x)\Big((0,1)d\mu_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (1,0)d\mu_n \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big) \to \int f(x)\Big((0,1)d\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (1,0)d\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big)$$

$$\Rightarrow \int f(x)d(\mu_{11} + \mu_{22}) = \int f(x)d(x) \text{ for all } f(x) \in C_c(\mathbb{R}).$$

So $(\mu_{11} + \mu_{22})(B) = |B|$ for any Borel set B which implies that$\mu_{11} + \mu_{22}, \mu_{22}$ are absolutely continuous measures and hence μ_{ij} are all absolutely continuous. Hence $\mu \in \mathcal{V}_{2\times 2}$. Moreover, $\operatorname{tr}\mu \equiv 1$.

As already seen that $\operatorname{tr} H(x) \equiv 1$ implies the limit-point case. This means that for $z \in \mathbb{C}^+$ there exist (unique up to a factor) solutions $f_\pm(x,z) = u(x,z) \pm m_\pm(z)v(x,z)$ of (3.2.1) such that $f_- \in L^2(H, \mathbb{R}_-), f_+ \in L^2(H, \mathbb{R}_+)$ where $u(x,z)$ and $v(x,z)$ are any two linearly independent solutions of (3.2.1). Let $x \in \mathbb{R}$, and consider boundary conditions at x, $u_1(x,z) = v_2(x,z) = 1, v_1(x,z) = u_2(x,z) = 0$, the Titchmarsh-Weyl m functions of the system (3.2.1) are alternately defined as $m_\pm(x,z) = \pm\frac{f_{\pm 2}(x,z)}{f_{\pm 1}(x,z)}$. Recall that $m_\pm(x,z)$ are Herglotz functions. So the boundary values of these m functions are defined by $m_\pm(x,t) \equiv \lim_{y\to 0} m_\pm(x, t + iy)$.

Definition 3.4. *Let $A \subset \mathbb{R}$ be a Borel set. We call a Hamiltonian $\mu \in \mathcal{V}_{2\times 2}$*

reflectionless on A *if*

$$m_+(x,t) = -\overline{m_-(x,t)} \tag{3.2.2}$$

for almost every $t \in A$ *and for some* $x \in \mathbb{R}$.

The set of reflectionless hamiltonian on A is denoted by $\mathcal{R}(A)$. Notice that the equation (3.2.2) is independent of the choice of boundary condition and the choice of a point. Suppose (3.2.2) is true for a boundary condition α at 0.

$$v_\alpha(0,z) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \text{ and } u_\alpha(0,z) = \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}.$$

Let $m_+^\alpha(z)$ be such that $f(x,z) = u_\alpha(x,z) + m_+^\alpha(z)v_\alpha(x,z) \in L^2(H,\mathbb{R}_+)$. Suppose $T_\alpha(x,z) = \begin{pmatrix} u_{\alpha_1}(x,z) & v_{\alpha_1}(x,z) \\ u_{\alpha_2}(x,z) & v_{\alpha_2}(x,z) \end{pmatrix}$ with $T_\alpha(0,z) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$ and $T_\beta(x,z) = \begin{pmatrix} u_{\beta_1}(x,z) & v_{\beta_1}(x,z) \\ u_{\beta_2}(x,z) & v_{\beta_2}(x,z) \end{pmatrix}$ with $T_\beta(0,z) = \begin{pmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{pmatrix}$.

Then $T_\alpha(x,z) = T_\beta(x,z) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$, where $\gamma = \beta - \alpha$.

Here $m_+^\alpha(z) \in \mathbb{C}$ is a unique number such that

$$T_\alpha(x,z) \begin{pmatrix} 1 \\ m_+^\alpha(z) \end{pmatrix} \in L^2(H,\mathbb{R}_+)$$

$$\Rightarrow \; T_\beta(x,z) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 \\ m_+^\alpha(z) \end{pmatrix} \in L^2(H,\mathbb{R}_+)$$

$$\Rightarrow \; T_\beta(x,z) \begin{pmatrix} \cos\gamma + m_+^\alpha(z)\sin\gamma \\ -\sin\gamma + m_+^\alpha(z)\cos\gamma \end{pmatrix} \in L^2(H,\mathbb{R}_+)$$

$$\Rightarrow \; (\cos\gamma + m_+^\alpha(z)\sin\gamma) T_\beta(x,z) \begin{pmatrix} 1 \\ \frac{-\sin\gamma + m_+^\alpha(z)\cos\gamma}{\cos\gamma + m_+^\alpha(z)\sin\gamma} \end{pmatrix} \in L^2(H,\mathbb{R}_+).$$

Since $m_+^\beta(z)$ be the unique coefficient such that

$T_\beta(x,z)\begin{pmatrix}1\\ m_+^\beta(z)\end{pmatrix}\in L^2(H,\mathbb{R}_+)$ we must have,

$m_+^\beta(z)=\frac{-\sin\gamma+m_+^\alpha(z)\cos\gamma}{\cos\gamma+m_+^\alpha(z)\sin\gamma}$

$\Rightarrow\ m_+^\beta(z)=\begin{pmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & \cos\gamma\end{pmatrix}m_+^\alpha(z)$. On the other hand,

$$T_\alpha(x,z)\begin{pmatrix}1\\ -m_-^\alpha(z)\end{pmatrix}\in L^2(H,\mathbb{R}_-)$$

$$\Rightarrow T_\beta(x,z)\begin{pmatrix}\cos\gamma & \sin\gamma\\ -\sin\gamma & \cos\gamma\end{pmatrix}\begin{pmatrix}1\\ -m_-^\alpha(z)\end{pmatrix}\in L^2(H,\mathbb{R}_-),\ \gamma=\beta-\alpha$$

$$\Rightarrow T_\beta(x,z)\begin{pmatrix}\cos\gamma-m_-^\alpha(z)\sin\gamma\\ -\sin\gamma-m_-^\alpha(z)\cos\gamma\end{pmatrix}\in L^2(H,(-\infty,0])$$

$$\Rightarrow (\cos\gamma-m_-^\alpha(z)\sin\gamma)T_\beta(x,z)\begin{pmatrix}1\\ \frac{-\sin\gamma-m_-^\alpha(z)\cos\gamma}{\cos\gamma-m_-^\alpha(z)\sin\gamma}\end{pmatrix}\in L^2(H,\mathbb{R}_-)$$

$$\Rightarrow\ -m_-^\beta(z)=\frac{-\sin\gamma-m_-^\alpha(z)\cos\gamma}{\cos\gamma-m_-^\alpha(z)\sin\gamma}=\begin{pmatrix}\cos\gamma & \sin\gamma\\ -\sin\gamma & \cos\gamma\end{pmatrix}m_-^\alpha(z).$$

Let

$$P_+(0,z)=\begin{pmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & \cos\gamma\end{pmatrix}\text{ and }P_-(0,z)=\begin{pmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & \cos\gamma\end{pmatrix},$$

so that

$$m_-^\beta(z)=P_-(0,z)m_-^\alpha(z),\ m_+^\beta(z)=P_+(0,z)m_+^\alpha(z)\text{ and}$$
$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}P_+(0,z)=P_-(0,z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Now

$$\begin{aligned}
-\overline{m_+^\beta(t)} =& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{m_+^\beta(t)} \\
=& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_+(0,z)\overline{m_+^\alpha(t)} \\
=& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_+(0,z)(-m_-^\alpha(t)) \\
=& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_+(0,z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m_-^\alpha(t) \\
=& P_-(0,z)m_-^\alpha(t) \\
=& m_+^\beta(t) \\
\Rightarrow m_+^\beta(t) =& -\overline{m_-^\beta(t)}.
\end{aligned}$$

Similarly, equation (3.2.2) is independent of the choice of the point. Suppose

$$T_0(x,z) = \begin{pmatrix} u_1(x,z) & v_1(x,z) \\ u_2(x,z) & v_2(x,z) \end{pmatrix}$$

be solutions with the boundary conditions at 0. Then

$$T_0(x,z) = T_a(x,z) \begin{pmatrix} u_1(a,z) & v_1(a,z) \\ u_2(a,z) & v_2(a,z) \end{pmatrix}.$$

Suppose $m_\pm(0,z) \in \mathbb{C}$ be the unique coefficients such that

$$f_\pm(x,z) = u(x,z) \pm m_\pm(0,z)v(x,z) \in L^2(H, \mathbb{R}_\pm).$$

In another way, $T_0(x,z)\begin{pmatrix}1\\ \pm m_\pm(0,z)\end{pmatrix}\in L^2(H,\mathbb{R}_\pm)$.

$$\Rightarrow\ T_a(x,z)\begin{pmatrix}u_1(a,z) & v_1(a,z)\\ u_2(a,z) & v_2(a,z)\end{pmatrix}\begin{pmatrix}1\\ \pm m_\pm(0,z)\end{pmatrix}\in L^2(H,\mathbb{R}_\pm).$$

$$\Rightarrow\ m_\pm(a,z)=\frac{u_2(a,z)\pm m_\pm(0,z)v_2(a,z)}{u_1(a,z)\pm m_\pm(0,z)v_1(a,z)}.$$

$$=\begin{pmatrix}v_2(a,z) & \pm u_2(a,z)\\ \pm v_1(a,z) & u_1(a,z)\end{pmatrix}m_\pm(0,z).$$

Let $T_\pm(z)=\begin{pmatrix}v_2(a,z) & \pm u_2(a,z)\\ \pm v_1(a,z) & u_1(a,z)\end{pmatrix}$, then $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}T_+(z)=T_-(z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

Now

$$\begin{aligned}-\overline{m_+(a,t)}=&\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\overline{m_+(a,t)}\\ =&\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}T_+(z)\overline{m_+(0,t)}\\ =&\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}T_+(z)(-m_-(0,t))\\ =&\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}T_+(z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}m_-(0,t)\\ =&T_-(z)m_-(0,t)\\ =&m_-(a,t)\end{aligned}$$

$$\Rightarrow\ m_+(a,t)=-\overline{m_-(a,t)}.$$

Now by the Herglotz representation theorem the Weyl m functions $m(x,z)$ have unique integral representation of the form,

$$m(x,z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1} \Big) d\nu(t), \quad z \in \mathbb{C}^+$$

for some positive Borel measure ν on $\mathbb{R}$ with $\int \frac{1}{t^2+1} d\nu < \infty$ and numbers $a \in \mathbb{R}$, $b \geq 0$. We call the measure ν in above integral representation of $m(z) = m(0,z)$ as *spectral measure* of the system (0.0.3).

The shift by x of a measure μ on $\mathbb{R}$, denoted by $S_x\mu$, is defined by

$$\int_{\mathbb{R}} f(t) d(S_x\mu) = \int_{\mathbb{R}} f(t-x) d\mu(t).$$

For $\mu \in \mathcal{V}_{2\times 2}$, $d\mu = H(t)dt$ then this reduces to the shift map $(S_xH)(t) = H(x+t)$.

Definition 3.5. *The ω limit set of the Hamiltonian $\mu \in \mathcal{V}_{2\times 2}$ under the shift map is defined as,*

$$\omega(\mu) = \{\nu \in \mathcal{V}_{2\times 2} : \text{ there exist } x_n \to \infty \text{ so that } d(S_{x_n}\mu, \nu) \to 0\}.$$

Note that $\omega(\mu) \subset \mathcal{V}_{2\times 2}$ is compact, non-empty and S is a homeomorphism on $\omega(\mu)$. Moreover, $\omega(\mu)$ is connected.

3.3 Remling's theorem on canonical systems

We are now ready to state the Remling's theorem on canonical systems on $\mathbb{R}_+$.

Theorem 3.6 (Remling's Theorem)**.** *Let $\mu \in \mathcal{V}_{2\times 2}$ be a (half line) Hamiltonian, and let Σ_{ac} be the essential support of the absolutely continuously part of the spectral measure. Then*

$$\omega(\mu) \subset \mathcal{R}(\Sigma_{ac}).$$

In order to prove this theorem we use the techniques from [13, 14].

Let $\mu \in \mathcal{V}_{2\times 2}$ is a whole line Hamiltonian. We write $\mu_\pm$ for the restrictions of μ to $\mathbb{R}_\pm$.

Denote the set of restrictions by $\mathcal{V}_\pm = \{\mu_\pm : \mu \in \mathcal{V}_{2\times 2}\}$ and $M_\pm = m_\pm^\mu(0,z)$ so $\{M_\pm = m_\pm^\mu(0,z)\} \subset \mathbb{H}$.

Lemma 3.7. *The maps $\mathcal{V}_\pm \longmapsto \mathbb{H}$, $\mu_\pm \longmapsto M_\pm = m_\pm^\mu(0,z)$ are homeomorphism onto their images.*

Proof. We have $\mu_+ = H_+(x)dx$. By Theorem 1 in [18], for every canonical system with Hamiltonian H_+ with $\operatorname{tr} H(x)_+ \equiv 1$ there is unique $m_+(0,z)$. Conversely for every $m_+ \in \mathbb{H}$ there exists a unique Hamiltonian H_+ on $\mathbb{R}_+$ such that m_+ is a Weyl coefficient of the canonical system corresponding to H_+. So $\mu_+ \mapsto M_+$ is one-to-one. Next we show that the map is homeomorphism. Suppose $\mu_n \to \mu$ in $\mathcal{V}_+$. That is $H_n(x)dx \to H(x)dx$ for some Hamiltonian $H_n(x), H(x)$. Let u_n be the solutions of canonical systems with Hamiltonian $H_n(x)$. Let K be a compact subset of $\mathbb{C}^+$ contained in a ball $B(0,R)$. We claim that u_n has convergent subsequence on $[0,N]$. Suppose a subinterval $[0,\eta]$ be such that $\eta = \frac{1}{8R}$. Define the operators $T_n : C[0,\eta] \longrightarrow C[0,\eta]$ by

$$T_n u(x) = -zJ\int_0^x H_n(t)u(t)dt.$$

Since

$$\begin{aligned}\|T_n\| = &\sup_{\|u\|_\infty = 1} \| - zJ\int_0^x H_n(t)u(t)dt\| \\ \le &|z|\|u\|_\infty \int_0^x |H_n(t)|dt \\ \le &\ R4\eta = R4\frac{1}{8R} = \frac{1}{2},\end{aligned}$$

$\|T_n\|$ are uniformly bounded. So the Neumann series $(1 - T_n)^{-1} = \sum_{k=0}^{\infty} T_n^k$ is convergent. Here $u_n(x) = (1 - T_n)^{-1}(u_0), \quad u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\|u_n\| \leq \|(1 - T_n)^{-1}\|\|u_0\| = \|(1 - T_n)^{-1}\| \leq \sum_{k=0}^{\infty} \|T_n\|^k = \sum_{k=0}^{\infty}(\frac{1}{2})^k = 2$. So $\{u_n(x) = n \in \mathbb{N}\}$ is uniformly bounded in n on $[0, \eta]$ and locally uniformly in z. Similar argument shows that u_n remains bounded on $[\eta, \eta + p]$ so that u_n are eventually bounded uniformly on $[0, N]$. Moreover, u_n are equicontinuous. Let $\epsilon > 0$ be given. Since u_n are solutions for the system (0.0.3) we have,

$$u_n(x) - u_n(x_0) = -zJ \int_{x_0}^{x} H_n(t)u_n(t)dt.$$

$$\begin{aligned} \|u_n(x) - u_n(x_0)\| &\leq |z|\|u_n\| \int_{x_0}^{x} |H_n(t)|dt \\ &= |z|\|u_n\|4\eta\|x - x_0\| \\ &\leq R2.4\eta\|x - x_0\|. \end{aligned}$$

Let $\delta = \frac{\epsilon}{8R\eta}$ then $\|u_n(x) - u_n(x_0)\| < \epsilon$, if $\|x - x_0\| < \delta$ for all n . By Arzella-Ascolli Theorem $\{u_n\}$ has convergent subsequence say $u_{n_j} \to u$. We show that u satisfies the canonical system corresponding to $H(x)$.

$$\begin{aligned} u_{n_j}(x) - u_{n_j}(0) &= -zJ \int_0^x H_{n_j}(t)u_{n_j}(t)dt \\ &= -zJ \int_0^x H_{n_j}\big((t)u_{n_j}(t) - u(t)\big)dt - zJ \int_0^x H_{n_j}(t)u(t)dt. \end{aligned}$$

Since $\| -zJ \int_0^x H_{n_j}\big((t)u_{n_j}(t) - u(t)\big)dt \| \leq |z|\|H_{n_j}\|_{L_1(0,x)}\|u_{n_j} - u\|$,
$\lim_{j\to\infty} -zJ \int_0^x H_{n_j}\big((t)u_{n_j}(t) - u(t)\big)dt = 0$. Hence, taking the limit as $j \to \infty$ we get, $u(x) - u(0) = \int_0^x H(t)u(t)dt$. So $\mu_n \to \mu$ and so $u_n \to u \Rightarrow m_+^{\mu_n}(0, z) \to m_+(0, z)$. This proves the continuity of the map on the interval $[0, N]$. Inverse of a continuous map on compact set is also continuous. Hence the map is homeomorphic. Exactly, the same way $\mu_- \longleftrightarrow M_-$ is also a homeomorphism. □

For $z = x + iy \in \mathbb{C}^+$, $\omega_z(S) = \frac{1}{\pi}\int_S \frac{y}{(t-x)^2+y^2}dt$, denotes the harmonic measure in the upper half-plane. For any $G \in \mathbb{H}$ and $t \in \mathbb{R}$ we define $\omega_{G(t)}(S)$ as the limit

$$\omega_{G(t)}(S) = \lim_{y \to 0+} \omega_{G(t+iy)}(S).$$

This limit exists almost everywhere on $\mathbb{R}$ because the map $z \mapsto \omega_z(S)$ is a non-negative harmonic function on $\mathbb{C}^+$. Note that if $\text{Im } G(t) > 0$, this limit coincides with the direct definition of $\omega_{G(t)}(S)$ where we just substitute $G(t)$:

$$\begin{aligned}\omega_{G(t)}(S) &= \lim_{y \to 0+} \omega_{G(t+iy)}(S) \\ &= \lim_{y \to 0+} \frac{1}{\pi}\int_S \frac{\text{Im } G(t+iy)}{(u - \text{ Re } G(t+iy))^2 + (\text{ Im } G(t+iy))^2}du.\end{aligned}$$

Then by dominated convergence theorem we have

$$\omega_{G(t)}(S) = \frac{1}{\pi}\int_S \frac{\text{Im } G(t)}{(u - \text{ Re } G(t))^2 + (\text{ Im } G(t))^2}du.$$

On the other hand, if $G(t)$ is real, then

$$\lim_{y \to 0+} \omega_{G(t+iy)}(S) = \begin{cases} 1 & : G(t) \in \bar{S} \\ 0 & : G(t) \notin \mathring{S} \end{cases}. \tag{3.3.1}$$

So for a nice set S, $\omega_{G(t+iy)}(S)$ is essentially $\chi_S(G(t))$ if $G(t) \in \mathbb{R}$.

Lemma 3.8 ([13])**.** *Let $A \subset \mathbb{R}$ be a Borel set with $|A| < \infty$. Then*

$$\lim_{y \to 0+} \sup_{F \in \mathcal{H}; S \subset \mathbb{R}} \left| \int_A \omega_{F(t+iy)}(S)dt - \int_A \omega_{F(t)}(S)dt \right| = 0$$

.

Proof. First observe that

$$\omega_{F(z)}(S) = \int_{-\infty}^{\infty} \omega_{F(u)}(S)d\omega_z(u) \tag{3.3.2}$$

for all Borel sets $S \subset \mathbb{R}$ and $z \in \mathbb{C}^+$. For this observation, it suffices to show that both sides of (3.3.2) are bounded, non-negative harmonic function of $z \in \mathbb{C}^+$ with the same boundary values $\omega_{F(t)}(S)$ for $a.e$ $t \in \mathbb{R}$. Clearly they are bounded and non-negative. For any $z = t + iy \in \mathbb{C}^+$, $\lim_{y \to 0+} \omega_{F(z)}(S) = \omega_{F(t)}(S)$ $a.e.$ $t \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \omega_{F(u)}(S) d\omega_z(u) = \int_{-\infty}^{\infty} \omega_{F(u)}(S) \frac{1}{\pi} \frac{y}{(u-t)^2 + y^2} du = \omega_{F(t)}(S).$$

Since $z \mapsto \omega_{F(z)}(S)$ is non-negative harmonic function, both sides of (3.3.2) are identical. For fixed F the statement follows from (3) of Theorem 3.10. By (3.3.2),

$$\begin{aligned}
\int_A \omega_{F(t)}(S) dt &= \int_A \Big[\int_{-\infty}^{\infty} \omega_{F(u)}(S) d\omega_{t+iy)}(u) \Big] dt \\
&= \int_A \int_{-\infty}^{\infty} \omega_{F(u)}(S) \frac{1}{\pi} \frac{y}{(u-t)^2 + y^2} du dt \\
&= \int_{-\infty}^{\infty} \omega_{F(u)}(S) \int_A \frac{1}{\pi} \frac{y}{(u-t)^2 + y^2} dt du \\
&= \int_{-\infty}^{\infty} \omega_{u+iy}(A) \omega_{F(u)}(S) du.
\end{aligned}$$

Therefore,

$$\Big| \int_A \omega_{F(t+iy)}(S) dt - \int_A \omega_{F(t)}(S) dt \Big| = \Big| \int_{-\infty}^{\infty} \omega_{F(t)}(S)(\omega_{t+iy}(A) - \chi_A(t)) dt \Big| ......(I)$$

. We know that, $0 \leq \omega_{F(t)}(S) \leq 1$. For $t \in A^c$, $\omega_{t+iy}(A) - \chi_A(t) \geq 0$ and for $t \in A$, $\omega_{t+iy}(A) - \chi_A(t) \leq 0$.

$$\begin{aligned}
I &\leq \int_A (\omega_{t+iy}(A) - \chi_A(t)) dt + \int_{A^c} (\omega_{t+iy}(A) - \chi_A(t)) dt \\
&= \Big| \int_A (\omega_{t+iy}(A) - 1) dt + \int_{A^c} (\omega_{t+iy}(A)) dt \Big| \\
&\leq \max \Big\{ \int_{A^c} (\omega_{t+iy}(A)) dt, \int_A (\omega_{t+iy}(A^c)) dt \Big\} \\
&= \max_{B = A, A^c} \int_B (\omega_{t+iy}(B^c)) dt.
\end{aligned}$$

But by Fubini

$$\int_A (\omega_{t+iy}(A^c))dt = \int_A \int_{A^c} \frac{1}{\pi} \frac{y^2}{(u-t)^2+y^2} du dt = \int_{A^c} (\omega_{t+iy}(A))dt.$$

Hence $I \leq \epsilon_A(y) = \int_{A^c} (\omega_{t+iy}(A))dt$, a quantity that is independent of both F and S. Next we show that $\epsilon_A(y) \to 0$ as $y \to 0^+$. By Lebesgue's differentiation theorem we have that $|A^c \cap (t-h, t+h)| = \circ(h)$ for *a.e.* $t \in A$. For such a t, we obtain that

$$\begin{aligned}\omega_{t+iy}(A^c) &\leq \frac{1}{\pi} \int_{A^c \cap (t-Ny, t+Ny)} \frac{y}{(s-t)^2+y^2} ds + \frac{1}{\pi} \int_{|s-t| \geq Ny} \frac{y}{(s-t)^2+y^2} ds \\ &= N \circ (1) + 1 - \frac{2}{\pi} \arctan N, \ \ y \to 0+ .\end{aligned}$$

By taking y small enough and noting that $N > 0$ is arbitrary, we see that $\omega_{t+iy}(A^c) \to 0$ for *a.e.* $t \in A$ and thus $\epsilon_A(y) \to 0$. This completes the proof. □

Definition 3.9. *If $F_n, F \in \mathbb{H}$, we say that $F_n \to F$ in value distribution if*

$$\lim_{n \to \infty} \int_A \omega_{F_n(t)}(S) dt = \int_A \omega_{F(t)}(S) dt \tag{3.3.3}$$

for all Borel set $A, S \subset \mathbb{R}, |A| < \infty$.

Notice that if the limit in the value distribution exists, it is unique : Suppose $F_n \to F$ and $F_n \to G$ in value distribution . That is
$\lim_{n \to \infty} \int_A \omega_{F_n(t)}(S) dt = \int_A \omega_{F(t)}(S) dt$ and $\lim_{n \to \infty} \int_A \omega_{F_n(t)}(S) dt = \int_A \omega_{G(t)}(S) dt$
for all Borel sets $A, S \subset \mathbb{R}, |A| < \infty$. Then for $A = B(r,t), \ r > 0$. Let $f(t) = \omega_{F(t)}(S) - \omega_{G(t)}(S)$ we have $\int_A f(t) dt = 0$. Since $f \in L^1_{\text{loc}}$, by Lebesgue differentiation theorem for almost every $t \in \mathbb{R}$, $\lim_{r \to 0} \frac{1}{|B(r,t)|} \int_{B(r,t)} f(s) ds = f(t)$. This implies that for fixed S, $\omega_{F(t)}(S) = \omega_{G(t)}(S)$ a. e. and hence $F(t) = G(t)$ for a. e. on $\mathbb{R}$. Since Herglotz functions are uniquely determined by their boundary values on a set of positive measure, it follows that $F = G$ on $\mathbb{C}^+$.

Theorem 3.10 ([13]). *Suppose $F_n, F \in \mathbb{H}$, and let a_n, a, and ν_n, ν be the associated numbers and measures, respectively, from the integral representation of Herglotz function. Then the following are equivalent:*

1. *$F_n(z) \to F(z)$ uniformly on compact subsets of $\mathbb{C}^+$;*

2. *$a_n \to a$ and $\nu_n \to \nu$ weak * on $\mathcal{M}(\mathbb{R}_\infty)$, that is,*

$$\lim_{n\to\infty} \int_{\mathbb{R}_\infty} f(t)d\nu_n(t) = \int_{\mathbb{R}_\infty} f(t)d\nu(t)$$

   for all $f \in C(\mathbb{R}_\infty)$;

3. *$F_n \to F$ in value distribution.*

4. *3.3.3 hold for all open, bounded intervals $A = (a, b), S = (c, d)$.*

Proof. $(1 \Leftrightarrow 2)$: Observe that $F(i) = a + i\nu(\mathbb{R}_\infty)$, so if (1) holds then $a_n \to a$ and the ν_n form a bounded sequence in $\mathcal{M}(\mathbb{R}_\infty)$. By the Banach-Alaoglu Theorem, we can extract a weak * convergence subsequence $\nu_{n_j} \to \mu$. We can then pass to the limit in the Herglotz representation 1.0.1 of the F_{n_j} and use the uniqueness of such representation that to conclude that $\mu = \nu$. In particular, this is the only limit point of ν_n and thus it was not necessary to pass a subsequence.
Conversely, suppose (2) holds. Since for fixed $z \in \mathbb{C}$, $f(t) = \frac{1+tz}{1-z}$ is continuous function on $\mathbb{R}_\infty$, in the Herglotz representation we can pass to the limit and get the pointwise convergence. In order to see the locally uniform convergence, we use the the normal family argument and we obtain (1).
$(4 \to 3)$: Suppose (4) holds. We show that if (3.3.3) holds for for all $A = (a, b)$ and fixed S then (3.3.3) holds for all Borel sets A of finite Lebesgue measure. Fix S, and to simplify the notation, abbreviate $\omega_{F_n(t)}(S) = \omega_n,\ \omega_{F(t)}(S) = \omega$.

Suppose that (3.3.3) holds for all $A = (a, b)$. Then if we are given disjoint intervals I_j with $|\cup I_j| < \infty$, then by dominated and monotone convergence,

$$\int_{\cup I_j} \omega_n dt = \sum_j \int_{I_j} \omega_n dt \to \sum_j \int_{I_j} \omega dt = \int_{\cup I_j} \omega_n dt$$

because $0 \leq \omega_n \leq 1$, thus $0 \leq \int_{I_j} \omega_n dt \leq |I_j|$ and $\sum |I_j| < \infty$. Let A be a Borel set of finite measure. Then by the regularity of Lebesgue measure, for given $\epsilon > 0$ we can find disjoint open intervals I_j so that

$$A \subset \cup I_j, \ \ |\cup I_j \setminus A| < \epsilon.$$

Then

$$\int_{\cup I_j} \omega_n dt - \epsilon < \int_A \omega_n dt \leq \int_{\cup I_j} \omega_n dt,$$
$$\int_A \omega dt \leq \int_{\cup I_j} \omega dt < \int_A \omega dt + \epsilon.$$

As above we have,

$$\int_{\cup I_j} \omega_n dt \to \int_{\cup I_j} \omega dt.$$

So

$$\limsup \int_A \omega_n dt \leq \limsup \int_{\cup I_j} \omega_n dt = \int_{\cup I_j} \omega dt < \int_A \omega dt + \epsilon,$$
$$\liminf \int_A \omega_n dt \geq \liminf \int_{\cup I_j} \omega_n dt - \epsilon = \int_{\cup I_j} \omega dt - \epsilon \geq \int_A \omega dt - \epsilon.$$

We see that $\liminf, \limsup \int_A \omega_n dt$ both differ from $\int_A \omega dt$ by at most ϵ, but $\epsilon > 0$ was arbitrary, so we obtain that

$$\int_A \omega_n dt \to \int_A \omega dt$$

as desired.

$(1 \to 3)$: Given $F \in \mathbb{H}$ let

$$F^{(y)}(z) = \frac{1 + yF(z)}{y - F(z)} \qquad (y \in \mathbb{R}_\infty).$$

$F^{(y)} \in \mathbb{H}$ since

$$\frac{(1+yF(z))(y-\overline{F(z)})}{|y-F(z)|^2} = \frac{y - y|F(z)|^2 + y^2 F(z) - \overline{F(z)}}{|y-F(z)|^2},$$

$$\text{Im } F^{(y)}(z) = \frac{(1+y^2)\text{ Im } F(z)}{|y-F(z)|^2} > 0.$$

Next we observe the *Spectral Averaging formula* from [3]. If $A, S \subset \mathbb{R}$ are Borel sets, $|A| < \infty$, then

$$\int_A \omega_{F(t)}(S)dt = \int_S \rho^{(y)}(A)\frac{dy}{1+y^2}$$

where $d\rho^{(y)}(t) = (1+t^2)\chi_{\mathbb{R}}(t)d\nu^{(y)}$, and $\nu^{(y)}$ is the measure from the Herglotz representation of $F^{(y)}$.

$$\begin{aligned}
\int_A \omega_{F(t)}(S)dt &= \int_A \lim_{u\to 0^+} \int_S \frac{1}{\pi}\frac{\text{Im } F(t+iu)}{(y - \text{ Re } F)^2 + (\text{ Im } F)^2}dydt \\
&= \int_A \lim_{u\to 0^+} \int_S \frac{1}{\pi} \text{ Im } F^{(y)}(t+iu)\frac{dy}{(1+y^2)}dt \\
&\left(\text{since Im } F^{(y)}(z) = \frac{(1+y^2)\text{ Im } F(z)}{|y-F(z)|^2}\right) \\
&= \int_S \lim_{u\to 0^+} \int_A \frac{1}{\pi} \text{ Im } F^{(y)}(t+iu)\frac{dy}{(1+y^2)}dt, \quad (\text{ by Fubini Theorem }) \\
&= \int_S \lim_{u\to 0^+} \int_{\mathbb{R}} \frac{1}{\pi} \text{ Im } F^{(y)}(t+iu)\chi_A\frac{dy}{(1+y^2)}dt \\
&= \int_S \rho^{(y)}(A)\frac{dy}{1+y^2}.
\end{aligned}$$

Since $(4 \to 3)$, we may assume that $A = (a, b)$. Let $R = \max(|a|, |b|)$. Then

$$\rho^{(y)}(A) \le (1+R^2)\nu^{(y)}(A) \le (1+R^2)\nu^{(y)}(\mathbb{R}_\infty) = (1+R^2)\text{ Im } F^{(y)}(i).$$

*Claim*1 :

$$\rho_n^{(y)}(A) \le C \quad (n \in \mathbb{N}, y \in \mathbb{R}_\infty)$$

Proof of the Claim 1 :

$$\begin{aligned}
\rho_n^{(y)}(A) &\leq (1+R^2)\ \text{Im}\ F_n^{(y)}(i) \\
&= (1+R^2)\frac{(1+y^2)\ \text{Im}\ F_n(i)}{|y-F_n(i)|^2} \\
&= (1+R^2)\frac{(1+y^2)}{(y-a_n)^2+(\ \text{Im}\ F_n(i))^2}\ \text{Im}\ F_n(i) \\
&= (1+R^2)\frac{(1+y^2)}{(y-a_n)^2+(\nu_n(\mathbb{R}_\infty))^2}\nu_n(\mathbb{R}_\infty).
\end{aligned}$$

Since $\nu_n(\mathbb{R}_\infty) \to \nu(\mathbb{R}_\infty)$, $\nu_n(\mathbb{R}_\infty)$ form a bounded sequence and $\frac{(1+y^2)}{(y-a_n)^2+(\nu_n(\mathbb{R}_\infty))^2}$ is a continuous function of y on $\mathbb{R}_\infty$, $\rho_n^{(y)}(A)$ is uniformly bounded.

Since $F_n^{(y)} \to F^{(y)}$ locally uniformly, we have the weak * convergence of the measures by $(1 \Leftrightarrow 2)$ and thus,

$$\rho_n^{(y)}(A) \to \rho^{(y)}(A) \tag{3.3.4}$$

except for those values of y for which $\rho^{(y)}(\{a,b\}) \neq 0$. By *Claim* 1, (3.3.4) and the *Spectral Averaging formula*; using the dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\int_A \omega_{F_n(t)}(S)dt &= \lim_{n\to\infty}\int_S \rho_n^{(y)}(A)\frac{dy}{1+y^2} \\
&= \int_S \rho_n^{(y)}(A)\frac{dy}{1+y^2} \\
&= \int_A \omega_{F(t)}(S)dt.
\end{aligned}$$

Finally, we show that $(4 \to 1)$. In order to show this we will use the compactness and uniqueness. For this, pick a subsequence, denoting again by F_n for convenience, that converges locally uniformly to G. Here, either $G \in \mathbb{H}$ or $G \equiv a \in \mathbb{R}_\infty$. The second case is not possible because: If, $F_n \to a \in \mathbb{R}$ then for every $R > 0$,

$$\rho_n^{(y)}([-R,R]) \to 0 \quad (n \to \infty),$$

uniformly in $|y-a| \geq \delta > 0$. Therefore by Spectral Averaging formula,

$$\int_{-R}^{R} \omega_{F_n(t)}((a-R,a-\delta)\cup(a+\delta,a+R)) = 0$$

for almost every $t \in (-R, R)$. This is not possible if $F(t) \equiv \lim F(t+iy) \in \mathbb{C}^+$ and if $F(t)$ exists and is real, then, since $R, \delta > 0$ are arbitrary, it follows that $F(t) = a$. In other words, $F(t) = a$ almost everywhere, but this is not a possible boundary value of an $F \in \mathbb{H}$. Similarly we can show that it is also not possible to have $|F_n| \to \infty$. In fact we can also work with $G_n = -\frac{1}{F_n}$ and run the exact argument again. Thus $F_n \to G \in \mathbb{H}$, uniformly on compact sets. But by $(1 \to 3)$, $F_n \to G$ in value distribution, and since such a limit is unique, $G = F$. Now every subsequence of $\{F_n\}$ has a locally uniformly convergent sub-subsequence, but the corresponding limit can only be F, so in fact $F_n \to F$ locally uniformly without the need of passing to a subsequence and hence we obtain (1). □

3.4 Breimesser-Pearson theorem on canonical systems

In this section we will prove Breimesser-Pearson theorem on canonical systems. We will follow the similar techniques from [3].

Theorem 3.11 (Breimesser-Pearson)**.** *Consider a half-line canonical system. Let Σ_{ac} denotes the essential support of absolutely continuous part of the spectral measure then for any $A \subset \Sigma_{ac}, |A| < \infty$ and $S \subset \mathbb{R}$, we have*

$$\lim_{N \to \infty} \Big(\int_A \omega_{m_-(N,t)}(-S)dt - \int_A \omega_{m_+(N,t)}(S)dt \Big) = 0.$$

Moreover, the convergence is uniform in S.

Here $m_-(N,t) = -\frac{v_2(N,z)}{v_1(N,z)}$ is a m function on $[0, N]$.

The hyperbolic distance of two points $w, z \in \mathbb{C}^+$ is defined as

$$\gamma(w, z) = \frac{|w - z|}{\sqrt{\operatorname{Im} w}\sqrt{\operatorname{Im} z}}.$$

For fixed, $S \subset \mathbb{R}$ the map $z \mapsto \omega_z(S)$ is a positive harmonic function on $\mathbb{C}^+$. This function has a harmonic conjugate $\alpha(z)$ so that $F(z) = \alpha(z) + i\omega_z(S)$ is a Herglotz function. The relation between the harmonic measure and the hyperbolic distance is given by

$$|\omega_w(S) - \omega_z(S)| \leq \frac{|\omega_w(S) - \omega_z(S)|}{\sqrt{\omega_w(S)}\sqrt{\omega_z(S)}} \leq \gamma(F(w), F(z)) \leq \gamma(w, z). \tag{3.4.1}$$

Lemma 3.12. *Let $u(., z), v(., z)$ be the solutions of the canonical system (0.0.3), subject to the condition $u(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let w be any constant such that $Im\ w \geq 0$, for any $N > 0$, and all $z \in \mathbb{C}^+$, we have the estimate,*

$$\gamma\Big(- \frac{v_2(N, z)}{v_1(N, z)}, - \frac{u_2(N, z) + \bar{w}v_2(N, z)}{u_1(N, z) + \bar{w}v_1(N, z)}\Big) \leq \frac{1}{\sqrt{I(I + 1)}},$$

where $I = I(N, z)$ is the integral defined by $I(N, z) = (\ Imz) \int_0^N Im(u^* Hv)dx$.

Proof. Denote the wronskian $W_N(f, g) = f_1(N)g_2(N) - f_2(N)g_1(N)$. Using the Greens's identity we have,

$$\int_0^N v^* Hv dx = \frac{1}{2i\ \text{Im} z} W_N(v, \bar{v}), \tag{3.4.2}$$

$$\int_0^N \text{Im}(u^* Hv)dx = -\frac{1}{2\ \text{Im} z}\Big(1 - \ \text{Re} W_N(\bar{u}, v)\Big) = \frac{1}{2\ \text{Im} z}\Big(1 - \ \text{Re} W_N(u, \bar{v})\Big), \tag{3.4.3}$$

$$|W(u, \bar{v})|^2 = 1 - W(u, \bar{u})W(v, \bar{v}). \tag{3.4.4}$$

Now at $x = N$ we have,

$$\begin{aligned}
&\gamma^2\Big(-\frac{v_2}{v_1}, -\frac{u_2+\bar{w}v_2}{u_1+\bar{w}v_1}\Big)\\
&= \frac{\big|-\frac{v_2}{v_1}+\frac{u_2+\bar{w}v_2}{u_1+\bar{w}v_1}\big|^2}{\text{Im}\,\big(-\frac{v_2}{v_1}\big)\ \text{Im}\big(-\frac{u_2+\bar{w}v_2}{u_1+\bar{w}v_1}\big)}\\
&= \frac{1}{\frac{\big|v_1(u_1+\bar{w}v_1)\big|^2}{\frac{1}{2i}(-\frac{v_2}{v_1}+\frac{\bar{v}_2}{\bar{v}_1})\frac{1}{2i}(.....)}}\\
&= \frac{1}{-\frac{1}{4}|v_1(u_1+\bar{w}v_1)|^2\frac{(-v_2\bar{v}_1+v_1\bar{v}_2)}{|v_1|^2}\frac{\big(-(u_2+\bar{w}v_2)(\bar{u}_1+w\bar{v}_1)+(u_1+\bar{w}v_1)(\bar{u}_2+w\bar{v}_2)\big)}{|u_1+\bar{w}v_1|^2}}\\
&= -\frac{4}{W(v,\bar{v})W(u+\bar{w}v,\bar{u}+w\bar{v})}.
\end{aligned}$$

Therefore,

$$\gamma^2\Big(-\frac{v_2}{v_1}, -\frac{u_2+\bar{w}v_2}{u_1+\bar{w}v_1}\Big) \leq -\frac{4}{W(v,\bar{v})W(u+\bar{w}v,\bar{u}+w\bar{v})}.$$

Let w be real. The denominator on the right side is of the form $A + Bw + Cw^2$, where $A \geq 0, C \geq 0$ and B is real. The denominator has minimum value $A - \frac{B^2}{4C}$. Hence,

$$\begin{aligned}
\gamma^2 &\leq \frac{4}{-W(v,\bar{v})W(u,\bar{u}) - \frac{\big(W(v,\bar{v})(W(u,\bar{v})-W(\bar{u},v))\big)^2}{4(-W(v,\bar{v})^2)}}\\
&= \frac{4}{-W(v,\bar{v})W(u,\bar{u}) - \frac{\big(2i\ \text{Im}(W(u,\bar{v})\big)^2}{-4}}\\
&\leq \frac{-4}{W(v,\bar{v})(W(u,\bar{v}) + \ \text{Im}(W(u,\bar{v})\big)^2}.
\end{aligned}$$

Using equation (3.4.4) we get,

$$\begin{aligned}
\gamma^2 &\leq -\frac{4}{1-|W(u,\bar{v})|^2 + (\ \text{Im}(W(u,\bar{v})\big)^2)}\\
&= \frac{-4}{1-(\text{Re}W(u,\bar{v}))^2}.
\end{aligned}$$

Here,

$$1 - (\mathrm{Re}W(u,\bar{v}))^2 = (1 - (\mathrm{Re}W(u,\bar{v})))(1 + (\mathrm{Re}W(u,\bar{v})))$$
$$= \Big(- 2\ \mathrm{Im}z \int_0^N \ \mathrm{Im}(u^* Hv)dx\Big)\Big(1 + 2\ \mathrm{Im}z \int_0^N \ \mathrm{Im}(u^* Hv)dx\Big).$$

Therefore,

$$\gamma^2 \leq \frac{1}{I(1+I)} \text{ where } I = \ \mathrm{Im}z \int_0^N \ \mathrm{Im}(u^* Hv)dx.$$

If w is not real, $w = \ \mathrm{Re}w + iY, Y > 0$ then $u - iYv$ is also a solution and we have,

$$|W(u - iYv, \bar{v})|^2 = 1 - W(u - iYv, \bar{u} + iY\bar{v})W(v,\bar{v}).$$

Also from above equation,

$$\gamma^2 \leq \frac{-4}{W(v,\bar{v})W(u,\bar{u}) + (\ \mathrm{Im}(W(u,\bar{v}))^2) + Y^2 W(v,\bar{v})^2 + 2iY\mathrm{Re}W(u,\bar{v})W(v,\bar{v})}$$
$$\leq \frac{-4}{W(u - iYv, \bar{u} + iY\bar{v})W(v,\bar{v}) + \Big(\ \mathrm{Im}W(u - iYv, \bar{v})\Big)^2}.$$

Since the equation (3.4.4) is valid for $u - iYv$ we get,

$$\gamma^2\Big(- \frac{v_2}{v_1}, -\frac{u_2 + \bar{w}v_2}{u_1 + \bar{w}v_1}\Big)$$
$$\leq \frac{-4}{1 - \big(\ \mathrm{Re}W(u - iYv, \bar{v})\big)^2}$$
$$= \frac{-4}{\big(1 + \ \mathrm{Re}W(u - iYv, \bar{v})\big)\big(1 - \ \mathrm{Re}W(u - iYv, \bar{v})\big)}$$
$$= \frac{-4}{\Big(1 + \ \mathrm{Re}\big(W(u,\bar{v}) - iYW(v,\bar{v})\big)\Big)\Big(1 - \ \mathrm{Re}\big(W(u,\bar{v}) - iYW(v,\bar{v})\big)\Big)}$$
$$= \frac{-4}{\big(1 - \ \mathrm{Re}W(u,\bar{v}) - Y\ \mathrm{Im}W(v,\bar{v})\big)\big(1 + \ \mathrm{Re}W(u,\bar{v}) + Y\ \mathrm{Im}W(v,\bar{v})\big)}$$
$$= \frac{-4}{\Big(- 2\ \mathrm{Im}z \int_0^N \ \mathrm{Im}(u^* Hv)dx - \frac{Y}{i}W(v,\bar{v})\Big)\Big(2\ \mathrm{Im}z \int_0^N \ \mathrm{Im}(u^* Hv)dx + 2 + \frac{Y}{i}W(v,\bar{v})\Big)}$$
$$= \frac{1}{I'(I'+1)},$$

where $I' = \text{Im}z \int_0^N \text{Im}(u^*Hv)dx + 2 + \frac{Y}{2i}W(v, \bar{v})$. Notice that $I' \geq I$ since $W(v, \bar{v}) = 2i \text{ Im}z \int_0^N v^*Hvdx \geq 0$. Hence the lemma is proved for general case.

□

Corollary 3.13. *With the notation above, we have*

$$\lim_{N\to\infty} \gamma\Big(-\frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z)+\bar{w}v_2(N,z)}{u_1(N,z)+\bar{w}v_1(N,z)}\Big) = 0$$

Proof. From above lemma we have

$$\gamma\Big(-\frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z)+\bar{w}v_2(N,z)}{u_1(N,z)+\bar{w}v_1(N,z)}\Big) \leq \frac{1}{\sqrt{I(I+1)}},$$

where $I = I(N, z)$ is the integral defined by $I(N, z) = (\text{ Im}z) \int_0^N Im(u^*Hv)dx$. Want to show that $I \to \infty$ as $N \to \infty$. We have,

$$\int_0^N v^*Hvdx = \frac{1}{2i \text{ Im}z} W_N(v, \bar{v})$$

$$\int_0^N \text{ Im}(u^*Hv)dx = -\frac{1}{2i \text{ Im } z}\big(1 - \text{ Re}W_N(u, \bar{v})\big).$$

Now lets look at the ratio

$$\begin{aligned}\frac{2 \text{ Im}z \int_0^N \text{ Im}(u^*Hv)dx + 1}{2i \text{ Im}z \int_0^N v^*Hvdx} &= \frac{W_N(u,\bar{v}) + W_N(\bar{u}, v)}{2iW_N(v,\bar{v})} \\ &= \frac{W_N(u,\bar{v})}{2iW_N(v,\bar{v})} - \frac{W_N(\bar{u}, v)}{2iW_N(\bar{v}, v)} \\ &= \text{ Im}C(z)\end{aligned}$$

where $C(z)$ is the center of a Weyl circle. Since $C(z)$ is a continuous function of z, it is uniformly bounded on a compact subset of $\mathbb{C}^+$. So,

$$\int_0^N \text{ Im } (u^*Hv)dx + 1 = \text{ Im}C \int_0^N v^*Hvdx \to \infty \text{ as } N \to \infty.$$

This implies that $I \to \infty$ as $n \to \infty$. □

We are now ready to prove Theorem 3.11. We follow the similar approach for the proof of Theorem 3.11 as in [3].

Proof of Theorem 3.11 : Let $A \subset \Sigma_{ac}$, $|A| < \infty$ and let $\epsilon > 0$ be given. We first define a partition $A = A_0 \cup A_1 \cup A_2, ... \cup A_N$ of disjoint subsets such that $|A_0| < \epsilon, A_j$ is bounded for $j \geq 1$. We also require that $m_+(t) \equiv \lim_{y \to 0+} m_+(t + iy)$ exists and $m_+(t) \in \mathbb{C}^+$ on $\bigcup_{j=1}^N A_j$. To find A_j's with these properties, first of all put all $t \in A$ for which $m_+(t)$ does not exist or does not lie in $\mathbb{C}^+$ into A_0. Then pick (sufficiently large) compact subset $K \subset \mathbb{C}^+, K' \subset \mathbb{R}$ so that $A_0 = \{t \in A : m_+(t) \notin K \text{ or } t \notin K'\}$ satisfies $|A_0| < \epsilon$. Subdivide K into finitely many subsets of hyperbolic diameter less than ϵ, then take the inverse images under m_+ of these subsets, and finally intersect with K' to obtain the A_j for $j \geq 1$. It is then true that $m_+(N, t)$ exists and lies in $\mathbb{C}^+$ for arbitrary $N \in \mathbb{R}$ if $t \in \bigcup_{j=1}^N A_j$. Moreover, we need $m_j \in \mathbb{C}^+$ such that

$$\gamma(m_+(t), m_j) < \epsilon, \tag{3.4.5}$$

such m_j can be defined as $m_j = m_+(t_j)$ for any fixed $t_j \in A_j$. By Lemma 3.8, there is a number $y > 0$ such that , for arbitrary Herglotz function F, for any Borel subset S of $\mathbb{R}$ and for all $j = 1, 2, ..., n$ we have the estimate

$$\Big| \int_{A_j} \omega_{F(t+iy)}(S)dt - \int_{A_j} \omega_{F(t)}(S)dt \Big| \leq \epsilon |A_j|. \tag{3.4.6}$$

We can define y for each value of j ; so y is a function of j. However, by taking the minimum value of $y(j)$ as j runs from 1 to n we my assume y is independent of j. Let $M_j(N, z) = \frac{u_2(N,z)+\bar{m}_j v_2(N,z)}{u_1(N,z)+\bar{m}_j v_1(N,z)}$ for any $z \in \mathbb{C}^+$. We shall complete the proof of the theorem by showing that, for $j \geq 1$,

(i): $\int_{A_j} w_{m_+(N,t)}(S)dt$ is close to the integral $\int_{A_j} \omega_{\overline{M_j}(N,t)}(S)dt$
where $M_j(N, t) = \frac{u_2(N,t)+\bar{m}_j v_2(N,t)}{u_1(N,t)+\bar{m}_j v_1(N,t)}$ and that

(ii): $\int_A \omega_{m_-(N,t)}(-S)dt$ is close to the same integral for all N sufficiently large.

Proof of (i): We have

$$m_+(N,t) = \frac{u_2(N,t) + m_+(t)v_2(N,t)}{u_1(N,t) + m_+(t)v_1(N,t).}$$

Hence, for fixed N and t, the mapping from $m_+(t)$ to $m_+(N,t)$ is a Mobius transformation with real coefficients and discriminant $u_1v_2 - v_1u_2 = 1$. and γ is invariant under Mobius transformations. Now from (3.4.5) we see that

$$\gamma\Big(m_+(N,t), \frac{u_2(N,t) + m_jv_2(N,t)}{u_1(N,t) + m_jv_1(N,t)}\Big) \leq \epsilon \text{ for } j \geq 1 \text{ and } t \in A_j.$$

By equation (3.4.1) we see that,

$$\Big|\omega_{m_+(N,t)}(S) - \omega_{M_j(N,t)}(S)\Big| \leq \epsilon,$$

and integration with respect to t over A_j gives the estimate

$$\Big|\int_{A_j} \omega_{m_+(N,t)}(S)dt - \int_{A_j} \omega_{\overline{M_j}(N,t)}(S)dt\Big| \leq \epsilon|A_j|. \tag{3.4.7}$$

This holds for all $j = 1, 2, ...n$.

Proof of (ii): For $j \geq 1$, define the subset A_j^y of $\mathbb{C}^+$, consisting of all $z \in \mathbb{C}^+$ of the form $z = t + iy$, for $t \in A_j$. Thus A_j^y is the translation of A_j by distance y above the real z-axis. Since A_j is bounded, A_j^y is contained in a compact subset of $\mathbb{C}^+$. Hence by Corollary 3.13, there is a positive number N_0 such that for $j \geq 1, N \geq N_0$ and $z \in A_j^y$ we have the estimate

$$\gamma\Big(- \frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z) + \bar{m}_jv_2(N,z)}{u_1(N,z) + \bar{m}_jv_1(N,z)}\Big) \leq \epsilon. \tag{3.4.8}$$

As in the case of y we may choose N_0 to be independent of j. Let $m_-(N,z) = -\frac{v_2(N,z)}{v_1(N,z)}$. Following the similar argument to that in the proof of (i), for any

$z = t + iy$ we have the estimate

$$\Big| \int_{A_j} \omega_{m_-(N,z)}(-S)dt - \int_{A_j} \omega_{-M_j(N,z)}(-S)dt \Big| \leq \epsilon |A_j|,$$

valid for $j \geq 1$ and $N \geq N_0$. Now by Lemma 3.8, equation (3.4.6) we have,

$$\Big| \int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{-M_j(N,t)}(-S)dt \Big| \leq 3\epsilon |A_j|.$$

Now using the identity $\omega_{-w}(S) = \omega_{\bar{w}}(S)$

$$\Big| \int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{\overline{M_j}(N,t)}(S)dt \Big| \leq 3\epsilon |A_j|, \tag{3.4.9}$$

which holds for all $j \geq 1$ and $N \geq N_0$ and completes the proof of (ii). Combining the inequalities (3.4.7) and (3.4.9) now yields, for $j \geq 1$ and $N \geq N_0$,

$$\Big| \int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{m_+(N,t)}(S)dt \Big| \leq 4\epsilon |A_j|. \tag{3.4.10}$$

Noting that A_0 was chosen such that $|A_0| \leq \epsilon |A|$ we now have for all $N \geq N_0$,

$$\begin{aligned} &\Big| \int_{A} \omega_{m_-(N,t)}(-S)dt - \int_{A} \omega_{m_+(N,t)}(S)dt \Big| \\ &\leq \sum_{j=0}^{n} \Big| \int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{m_+(N,t)}(S)dt \Big| \\ &\leq 2|A_0| + 4\epsilon \sum_{j=0}^{n} |A_j| \leq \epsilon |A_j| \leq 6\epsilon |A|. \end{aligned}$$

Since ϵ was arbitrary, the theorem follows.

Proof of Theorem 3.6:

Let $\nu \in \omega(\mu)$. Then there exists a sequence $x_n \to \infty$ such that $d(S_{x_n}\mu, \nu) \to 0$. Then by Lemma 3.7 we have that

$$m_\pm(x_n, z) \to M_\pm(z) \ \ (n \to \infty),$$

uniformly on compact subset of $\mathbb{C}^+$. Here $M_\pm(z) = m_\pm^\nu(0,z)$ are the m functions of the whole line Hamiltonian ν. By Theorem 3.10 we see that

$$m_\pm(x_n, z) \to M_\pm(z) \ \ (n \to \infty),$$

in value distribution. That is

$$\lim_{n\to\infty} \int_A \omega_{m_\pm(x_n,t)}(S)dt = \int_A \omega_{M_\pm(t)}(S)dt$$

for all Borel sets $A, S \subset \mathbb{R}, |A| < \infty$. Also by Theorem 3.11 we have

$$\int_A \omega_{M_-(t)}(-S)dt = \int_A \omega_{M_+(t)}(S)dt.$$

By Lebesgue differentiation theorem,

$$\omega_{M_-(t)}(-S) = \omega_{M_+(t)}(S) \tag{3.4.11}$$

for $t \in \Sigma_{ac}\mathbb{N}$, $|N| = 0$ and all intervals S with rational end points. We can also assume that $M_\pm(t) = \lim_{y\to 0+} M(t+iy)$ exists for these t. If $M_-(t) \in \mathbb{R}$, then, by choosing small intervals about this value for $-S$, we see that $M_+(t) = -M_-(t)$. If $M_-(t) \in \mathbb{C}$, $M_-(t) = u + iv$ then $-\overline{M_-}(t) = -u + iv$, we can define $\omega_{M_-(t)}$ directly as

$$\begin{aligned}\omega_{M_-(t)}(-S) &= \int_{(-S)} \frac{v}{(t-u)^2+v^2}dt \\ &= -\int_{(S)} \frac{v}{(t+u)^2+v^2}dt \\ &= \omega_{-\overline{M_-}(t)}(S).\end{aligned}$$

By (3.4.11) we get,

$$M_+(t) = -\overline{M_-(t)}. \tag{3.4.12}$$

In the case when $M_-(t) \in \mathbb{R}$ we already have $M_+(t) = -M_-(t)$. So (3.4.12) holds for almost every $t \in \Sigma_{ac}$, that is $\nu \in \mathcal{R}(\Sigma_{ac})$.

3.5 Relation between Schrödinger and Jacobi equations and canonical systems

In this section we will show the connection between Schrödinger and Jacobi equations and canonical systems.

3.5.1 Reduction of a Schrödinger equation into a canonical system

Let

$$-y'' + V(x)y = zy \tag{3.5.1}$$

be a Schrödinger equation. Suppose $u(z, z)$ and $v(x, z)$ are the linearly independent solutions of (3.5.1), satisfying some boundary condition α at 0. Then $u_0 = u(x, 0)$ and $v_0 = v_0(x, 0)$ are the solutions of $-y'' + V(x)y = 0$. Let

$$H(x) = \begin{pmatrix} u_0^2 & u_0 v_0 \\ u_0 v_0 & v_0^2 \end{pmatrix}$$

then the Schrödinger equation (3.5.1) is equivalent with the canonical system

$$Jy' = zHy \tag{3.5.2}$$

If y solves equation (3.5.1) then $U(x, z) = T^{-1}(x) \begin{pmatrix} y(x, z) \\ y'(x, z) \end{pmatrix}$ solves a canonical system (3.5.2). Here

$$\begin{aligned}
U(x,z) &= T^{-1}(x)\begin{pmatrix} y(x,z) \\ y'(x,z) \end{pmatrix} \\
&= \begin{pmatrix} v_0' & -v_0 \\ -u_0' & u_0 \end{pmatrix}\begin{pmatrix} y(x,z) \\ y'(x,z) \end{pmatrix} \\
&= \begin{pmatrix} v_0'y - v_0y' \\ -u_0'y + u_0y' \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
zH(x)U(x,z) &= z\begin{pmatrix} u_0^2 & u_0v_0 \\ u_0v_0 & v_0^2 \end{pmatrix}\begin{pmatrix} v_0'y - v_0y' \\ -u_0'y + u_0y' \end{pmatrix} \\
&= z\begin{pmatrix} u_0^2v_0'y - u_0^2v_0y' - u_0v_0u_0'y + u_0^2v_0y' \\ u_0v_0v_0'y - u_0v_0^2y' - v_0^2u_0'y + v_0^2u_0y' \end{pmatrix} \\
&= z\begin{pmatrix} u_0y(u_0v_0' - u_0'v_0') \\ v_0y(u_0v_0' - u_0'v_0') \end{pmatrix} \\
&= z\begin{pmatrix} u_0y \\ v_0y \end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
JU' &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} (v_0'y - v_0y')' \\ (-u_0'y + u_0y')' \end{pmatrix} \\
&= \begin{pmatrix} -u_0''y + u_0y'' \\ -v_0''y + v_0y'' \end{pmatrix} \\
&= z\begin{pmatrix} u_0y \\ v_0y \end{pmatrix}.
\end{aligned}$$

Alternative Approach : Let

$$-y'' + V(x)y = z^2 y \tag{3.5.3}$$

be a Schrödinger equation such that $-\frac{d^2}{dx^2} + V(x) \geq 0$ and $y(x,z)$ be its solution. Then $y_0 = y(x,0)$ be a solution of $-y'' + V(x)y = 0$. Let $W(x) = \frac{y_0'}{y_0}$ then $W^2(x) + W'(x) = V(x)$ so that equation (3.5.3) becomes

$$-y'' + (W^2 + W')y = z^2 y. \tag{3.5.4}$$

Claim that the equation (3.5.4) is equivalent with the Dirac system

$$Ju' = \begin{pmatrix} z & W \\ W & z \end{pmatrix} u. \tag{3.5.5}$$

If y is a solution of (3.5.4) then $u = \begin{pmatrix} y \\ -\frac{1}{z}(-y' + Wy) \end{pmatrix}$ is a solution of (3.5.5).

Also if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of (3.5.5) then u_1 is a solution of (3.5.4). Next we show that the Dirac system (3.5.5) is equivalent with the Canonical System

$$Ju'(x) = zH(x)u(x), \quad H(x) = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}. \tag{3.5.6}$$

For if u is a solution of (3.5.5) then $T_0 u$, where $T_0 = \begin{pmatrix} e^{-\int_0^x W(t)dt} & 0 \\ 0 & e^{\int_0^x W(t)dt} \end{pmatrix}$ is a solution of (3.5.6).

If we consider a Schrödinger equation of the form,

$$-y'' + (W^2 - W')y = z^2 y \tag{3.5.7}$$

then it is equivalent with the Dirac system

$$Ju' = \begin{pmatrix} z & -W \\ -W & z \end{pmatrix} u. \tag{3.5.8}$$

In other words, if y is a solution of Schrödinger equation (3.5.7) then $u = \begin{pmatrix} zy \\ y' + Wy \end{pmatrix}$ is a solution of the Dirac system (3.5.8). Conversely, if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of the Dirac system (3.5.8) then u_1 is a solution to the Schrödinger equation (3.5.7).

Proof. Let y is a solution of the Schrödinger equation (3.5.7) then for $u = \begin{pmatrix} zy \\ y' + Wy \end{pmatrix}$ we have

$$u' = \begin{pmatrix} zy' \\ y'' + Wy' + W'y \end{pmatrix} = \begin{pmatrix} zy' \\ W^2y - z^2y + Wy' \end{pmatrix}.$$

Then $Ju' = \begin{pmatrix} z^2y - W^2y - Wy' \\ zy' \end{pmatrix}$. On the other hand,

$$\begin{pmatrix} z & -W \\ -W & z \end{pmatrix} \begin{pmatrix} zy \\ y' + Wy \end{pmatrix} = \begin{pmatrix} z^2y - W^2y - Wy' \\ zy' \end{pmatrix}.$$

Conversely, let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of the Dirac system 3.5.8 then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} z & -W \\ -W & z \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

From this equation we get $u_1' = -Wu_1 + zu_2$ and $-u_2' = zu_1 - Wu_2$. Then

$$\begin{aligned} u_1'' &= -W'u_1 - Wu_1' + zu_2' \\ &= -W'u_1 - W(-Wu_1 + zu_2) + zu_2' \\ &= -W'u_1 + W^2u_1 - zWu_2 + z(-zu_1 + Wu_2) \\ &= (W^2 - W')u_1 - z^2u_1 \end{aligned}$$

$$-u_1'' + (W^2 - W')u_1 = z^2u_1.$$

□

The Dirac system (3.5.8) is equivalent with the canonical system,

$$Ju'(x) = zH(x)u(x) \tag{3.5.9}$$

where $H(x) = \begin{pmatrix} e^{-2\int_0^x W(t)dt} & 0 \\ 0 & e^{2\int_0^x W(t)dt} \end{pmatrix}$. If u is a solution of the Dirac system (3.5.8) then $y = T_0 u$, $\;T_0 = \begin{pmatrix} e^{\int_0^x W(t)dt} & 0 \\ 0 & e^{-\int_0^x W(t)dt} \end{pmatrix}$ is a solution of the canonical system (3.5.9). Conversely if u is a solution of the canonical system (3.5.9) then $T_0^{-1}u$ is a solution of the Dirac system (3.5.8).

3.5.2 Reduction of a Jacobi equation into a canonical system.

Let a Jacobi equation be

$$a(n)u(n+1) + a(n-1)u(n) + b(n)u(n) = zu(n). \tag{3.5.10}$$

This equation can be written as

$$\begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{z-b(n)}{a(n)} \end{pmatrix} \cdot \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}$$

$$= [B(n) + zA(n)] \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}$$

where $B(n) = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{-b(n)}{a(n)} \end{pmatrix}$ and $A(n) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a(n)} \end{pmatrix}$. Suppose $p(n,z)$ and $q(n,z)$ be solutions of (3.5.10) such that $p(0,z) = 1, p(1,z) = 1$ and $q(0,z) = 0, q(1,z) = 1$. So that $p_0(n) = p(n,0)$ and $q_0(n) = q(n,0)$ be solutions of equation (3.5.10) when $z = 0$. Then

$$\begin{pmatrix} p_0(n) \\ p_0(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{-b(n)}{a(n)} \end{pmatrix} \begin{pmatrix} p_0(n-1) \\ p_0(n) \end{pmatrix}.$$

(similar expression for $q_0(n)$.) Let $T(n) = \begin{pmatrix} p_0(n-1) & q_0(n-1) \\ p_0(n) & q_0(n) \end{pmatrix}, T(1) = 1$. Then $T(n+1) = B(n)T(n)$. Define $U(n,z) = T(n+1)^{-1}Y(n,z)$, $Y(n,z) = \begin{pmatrix} p_(n-1,z) & q_(n-1,z) \\ p_(n,z) & q_(n,z) \end{pmatrix}$. Then $U(n,z)$ solves an equation of the form

$$J\big(U(n+1,z) - U(n,z)\big) = zH(n)U(n,z) \tag{3.5.11}$$

where $H(n) = JT(n+1)^{-1}A(n)T(n)$.

$$\Big[\text{Proof: } zH(n)U(n,z) = zJT(n+1)^{-1}A(n)T(n)T(n)^{-1}Y(n,z)$$
$$= zJT(n+1)^{-1}A(n)Y(n,z)$$
$$= zJT(n+1)^{-1}\big(Y(n+1,z) - B(n)Y(n,z)\big)$$
$$= J\big(T(n+1)^{-1}Y(n+1,z) - T(n+1)^{-1}B(n)Y(n,z)\big)$$
$$= J\big(T(n+1)^{-1}Y(n+1,z) - T(n)^{-1}Y(n,z)\big)$$
$$= J\big(U(n+1,z) - U(n,z)\big)\Big].$$

Suppose for each $n \in \mathbb{Z}$, on $(n, n+1)$, H has the form

$$H(x) = h(x)P_\phi, \qquad P_\phi = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

for some $\phi \in [0, \pi)$ and some $h \in L_1(n, n+1), h \geq 0$ (we may choose $h(x) \equiv 1$ on $(n, n+1)$ for each $n \in \mathbb{Z}$). Then the canonical system (0.0.3) becomes

$$u'(x) = -zh(x)JP_\phi u(x).$$

Since the matrices on the right-hand side commute with one another for different values of x, the solution is given by

$$u(x) = \exp\Big(-z\int_a^x h(t)dtJP_\phi\Big)u(a).$$

However, $P_\phi JP_\phi = 0$, we see that the exponential terminates and we get

$$u(x) = \Big(1 - z\int_a^x h(t)dtJP_\phi\Big)u(a). \tag{3.5.12}$$

Clearly equation (3.5.12) is equivalent with the equation (3.5.11).

3.5.3 Relation between Weyl m functions

We next observe the relation between the Weyl m functions for Shrodinger equations and canonical systems.

Lemma 3.14. *For $z \in \mathbb{C}^+$, let $m_s(z), m_c(z)$ denote the Weyl m functions corresponding to the Schrödinger equation (3.5.1) and the canonical system (3.5.2) respectively. Then $m_s(z) = m_c(z)$.*

Proof. Let $T_s(x,z) = \begin{pmatrix} u(x,z) & v(x,z) \\ u'(x,z) & v'(x,z) \end{pmatrix}$ and $T_c(x,z) = \begin{pmatrix} u_1(x,z) & v_1(x,z) \\ u_2(x,z) & v_2(x,z) \end{pmatrix}$ are the transfer matrices corresponding to the Schrödinger equation (3.5.1) and the canonical system (3.5.2) respectively. Let $T_0(x) = T_s(x,0)$ then in (3.5.2), $H(x) = T_0^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0$. Here $m_s(z)$ is such that $(1,0)T_s(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} \in L^2(R_+)$ and $m_c(z)$ is such that $T_c(x,z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} \in L^2(H, R_+)$. Note that here, $T_s(x,z) = T_0(x)T_c(x,z)$

It follows that,

$$\int_0^\infty (1, \bar{m}_s) T_s^*(x,z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_s(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_s) T_c^*(x,z) T_0^*(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0(x) T_c(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_s) T_c^*(x,z) H T_c(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_s) T_c^*(x,z) H T_c(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty.$$

Since the Weyl m function $m_c(z)$ is uniquely defined we must have $m_s(z) = m_c(z)$. □

Theorem 3.15. *Let $\omega(V)$ and $\omega(H)$ be ω limit set corresponding to a Schrödinger equation (3.5.1) and its canonical system (3.5.2) respectively. Then if $W \in \omega(V)$ then $K \in \omega(H)$ where K is the Hamiltonian corresponding to a canonical system of the Schrödinger equation with W as potential. Conversely, if $K \in \omega(H)$ then K is a Hamiltonian for a canonical system of a Schrödinger equation for some potential $W \in \omega(V)$.*

Proof. Suppose $W \in \omega(V)$ then by definition of ω limit set there exists a sequence $x_n \to \infty$ such that $V(x + x_n) \to W$. Then the corresponding Weyl m functions also converge, ie $m_s^{V_n}(z) \to m_s^W(z)$. Let H_n be the Hamiltonian of the canonical system obtained from the Schrödinger equation with the potential $V(x+x_n)$ then $H_n = H(x + x_n)$ then by Lemma 3.14 $m_s^{V_n}(z) = m_c^{H_n}(z)$. and $m_s^W(z) = m_c^H(z)$. Now apply the change of variable by (3.1.13) and obtain $\widetilde{H}_n$ and the corresponding $m$function is $m_c^{\widetilde{H}_n}(z)$. After the change of variable the corresponding Weyl m functions are the same up to the change of the point of boundary condition. So the convergence of $m_s^{V_n}(z) = m_c^{H_n}(z)$ implies the convergence of $m_c^{\widetilde{H}_n}(z)$. It follows that $m_c^{\widetilde{H}_n}(z) \to m_s^W(z)$. But by Lemma 3.7 $m_s^W(z) = m_c^{\widetilde{H}}(z)$ where $m_c^{\widetilde{H}}(z)$ is the Weyl $m$function for some Hamiltonian $\widetilde{H}$. It follows that $m_c^{\widetilde{H}_n}(z) \to m_c^{\widetilde{H}}(z)$. Again by Lemma 3.7, we get $\widetilde{H}_n \to \widetilde{H}$ using the change of variable on the canonical system with Hamiltonian $\widetilde{H}$ we obtain a Hamiltonian K such that $m_c^{\widetilde{H}}(z) = m_c^K(z)$ up to the change of point of boundary condition. It follows that $H_n \to K$ and so $K \in \omega(H)$. Converse is similar. □

Lemma 3.16. *For $z \in \mathbb{C}^+$, let $m_s(z^2), m_c(z)$ denote the Weyl m functions corresponding to the Schrödinger equation (3.5.7) and the canonical system (3.5.9)* *respectively. Then* $m_s(z^2) = zm_c(z)$.

Proof. Note that, since $H(x) = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}$, $f \in L^2(H, \mathbb{R}_+)$ if and only if

$$\int_0^\infty |f_1|^2 e^{2\int_0^x W(t)dt} dx < \infty, \ \int_0^\infty |f_2|^2 e^{-2\int_0^x W(t)dt} dx < \infty.$$

Let $T_s(x, z^2), T_d(x, z)$ and $T_c(x, z)$ denote the transfer matrices of the Schrödinger equation (3.5.4), the Dirac system (3.5.5) and the canonical system (3.5.6) respectively. Then,

$$\begin{aligned} T_s(x, z^2) &= \begin{pmatrix} u(x, z^2) & v(x, z^2) \\ u'(x, z^2) & v'(x, z^2) \end{pmatrix}, \\ T_d(x, z) &= \begin{pmatrix} u(x, z^2) & zv(x, z^2) \\ \frac{u'(x,z^2) - W(x)u(x,z^2)}{z} & v'(x, z) - W(x)v(x, z), \end{pmatrix} \\ T_c(x, z) &= T_0 T_d(x, z). \end{aligned}$$

It follows that

$$T_d(x, z) = \begin{pmatrix} z & 0 \\ -W & 1 \end{pmatrix} T_s(x, z^2) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix}.$$

So $T_d(x, z) = T_0^{-1} T_c(x, z)$ and

$$T_s(x, z^2) = \frac{1}{z} \begin{pmatrix} 1 & 0 \\ W & z \end{pmatrix} T_d(x, z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we have,

$$\int_0^\infty (1, \bar{m}_c(z)) T_c^*(x,z) H(x) T_c(x,z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_c(z)) T_c^*(x,z) \Big[T_0^{-1}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0(x)^{-1} +$$

$$T_0(x)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_0(x)^{-1} \Big] T_c(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_c(z)) T_d^*(x,z) T_0(x) T_0(x)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

$$T_0(x)^{-1} T_0(x) T_d(x,z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_c) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} T_s^*(x, z^2) \begin{pmatrix} \bar{z} & W \\ 0 & 1 \end{pmatrix} .$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ -W & 1 \end{pmatrix} T_s(x, z^2) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty \begin{pmatrix} \frac{1}{z} & \bar{m}_c \end{pmatrix} T_s^*(x, z^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_s(x, z^2) \begin{pmatrix} \frac{1}{z} \\ m_c(z) \end{pmatrix} dx < \infty.$$

Since the Weyl m function $m_c(z)$ is uniquely defined we must have

$$m_s(z^2) = z m_c(z).$$

□

Suppose

$$H_+ = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}, \quad H_- = \begin{pmatrix} e^{-2\int_0^x W(t)dt} & 0 \\ 0 & e^{2\int_0^x W(t)dt} \end{pmatrix}$$

in the canonical system (3.5.6) and (3.5.9) respectively. The following lemma shows the relation between their Weyl m functions.

Lemma 3.17. *If m_{c_+} and m_{c_-} are the Weyl m functions corresponding to the canonical systems (3.5.6) and (3.5.9) respectively then $m_{c_+} = \frac{-1}{m_{c_-}}$.*

Proof. Notice that,

$$-JH_+J = H_-.$$

For if

$$\begin{aligned} -JH_+J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-2\int_0^x W(t)dt} & 0 \\ 0 & e^{2\int_0^x W(t)dt} \end{pmatrix} \\ &= H_-. \end{aligned}$$

u is a solution of

$$Ju' = zH_+u$$

if and only if Ju is a solution of

$$Ju' = zH_-u.$$

For if

$$\begin{aligned} J[Ju]' &= JJu' \\ &= zJH_+u \\ &= zJH_+J^{-1}Ju \\ &= -zJH_+JJu \\ &= zH_-[Ju]. \end{aligned}$$

Let $T_{c_+}(x)$ and $T_{c_-}(x)$ be the transfer matrices and m_{c_+} and m_{c_-} are the Weyl m functions of the canonical systems with the Hamiltonians H_+ and H_- respectively. Then $T_{c_-}(x) = -JT_{c_+}(x)J$ and

$$\int_0^\infty (1, \bar{m}_{c_-})T^*_{c_-}(x)H_-T_{c_-}(x) \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_{c_-})(-JT_{c_+}(x)J)^*H_-(-JT_{c_+}(x)J) \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (1, \bar{m}_{c_-})(-J)T^*_{c_+}(x)(-JH_-J)T_{c_+}(x)J \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty (\bar{m}_{c_-}, -1)T^*_{c_+}(x)(-JH_-J)T_{c_+}(x) \begin{pmatrix} m_{c_-} \\ -1 \end{pmatrix} dx < \infty.$$

$$\Rightarrow \int_0^\infty \begin{pmatrix} 1, & \frac{-1}{\bar{m}_{c_-}} \end{pmatrix} T^*_{c_+}(x)H_+T_{c_+}(x) \begin{pmatrix} 1 \\ \frac{-1}{\bar{m}_{c_-}} \end{pmatrix} dx < \infty.$$

Since m_{c_+} is the unique coefficient such that

$$\int_0^\infty (1, \bar{m}_{c_+})T^*_{c_+}(x)H_+T_{c_+}(x) \begin{pmatrix} 1 \\ m_{c_+} \end{pmatrix} dx < \infty$$

we have

$$m_{c_+} = \frac{-1}{m_{c_-}}.$$

□

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