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HOLONOMY DISPLACEMENT OF CURVES IN BUNDLE $\mathrm{SO}(N) \to \mathrm{SO}_0(1, N) \to \mathbb{H}^n$

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HOLONOMY DISPLACEMENT OF CURVES IN BUNDLE
$\mathrm{SO}(N) \to \mathrm{SO}_0(1, N) \to \mathbb{H}^n$

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

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Contents

Chapter 1

Introduction

Let $\mathrm{O}(1,n) = \{A \in \mathrm{GL}(n+1;\mathbb{R}) \mid A^t S A = S\}$, where $S = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{I}_n \end{pmatrix}$.

Let $\mathrm{SO}_0(1,n)$ be the identity component of $\mathrm{O}(1,n)$, which is also the identity component of $\mathrm{SO}(1,n)$, and consider a subgroup of $\mathrm{SO}_0(1,n)$ consisting of all matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, where $B \in \mathrm{SO}(n)$. Call the embedded subgroup $\mathrm{SO}(n)$ again.

Note the Lie algebra $\mathfrak{o}(1,n)$ is given by

$$\mathfrak{o}(1,n) = \{X \in \mathfrak{gl}(n+1;\mathbb{R}) | X^t S + SX = 0\}.$$

Now, think of a left-invariant metric on $\mathrm{SO}_0(1,n)$, induced from an inner product $\langle \cdot\, ,\ \cdot \rangle$ on the Lie algebra, $\mathfrak{so}(1,n)$, defined as follows :

$$\langle A, B \rangle = \frac{1}{2}\,\mathrm{trace}(A^t\, B) \qquad \text{for } A, B \in \mathfrak{so}(1,n).$$

If ϕ is a Killing-Cartan form, then

$$\phi(U,V) = -2(n-1)\langle U, V\rangle \qquad \text{for } U, V \in \mathfrak{o}(n) \subset \mathfrak{o}(1,n)\ ,$$

$$\phi(X,Y) = \quad 2(n-1)\langle X, Y\rangle \qquad \text{for } X, Y \in \mathfrak{o}(n)^{\perp} \subset \mathfrak{o}(1,n)\ ,$$

$$\phi(X,V) = 0 = \langle X, V\rangle \qquad \text{for } X \in \mathfrak{o}(n)^{\perp},\ V \in \mathfrak{o}(n)\ .$$

Note $\mathrm{Isom}_0\big(\mathrm{SO}_0(1,n)\big) = \mathrm{SO}_0(1,n) \times \mathrm{SO}(n)$, and with the Riemannian metric on $\mathrm{SO}_0(1,n) \times \mathrm{SO}(n)$ which makes the projection $\mathrm{SO}(n) \to \mathrm{SO}_0(1,n)/\mathrm{SO}(n)$ Riemannian, the quotient $\mathrm{SO}_0(1,n)/\mathrm{SO}(n)$ becomes isometric to the hyperbolic space $\mathbb{H}^n$.

If n=2, then it can be easily shown that for a given geodesic triangle in $\mathbb{H}^2$, the distance by the holonomy displacement of the boundary curve of the given geodesic triangle in the fiber is as same as the area of the triangle. Furthermore, the direction of the boundary curve of the given geodesic triangle in $\mathbb{H}^2$ will determine the direction of its holonomy displacement. All of these are dealt with in Chapter 4.

Can a similar result be obtained in a topological disk in $\mathbb{H}^n$?

If it is a geodesic triangle, something similar can be easily said from the result for the case n=2 and Fact 2, mentioned in Chapter 5. But, what can be done for a general disk in $\mathbb{H}^n$?

To answer this question, we intend to approximate the given disk with geodesic triangles, since there exists a unique totally geodesic triangle for any 3 different points in $\mathbb{H}^n$. And then we intend to construct a curve in the fiber by using the property for the case n=2. But how can we approximate it? Though each geodesic triangle and its boundary curve determine the direction of each holonomy displacement, some linear ordering of geodesic triangles and the induced ordering of their boundary curves may not represent the boundary curve of their union. If the given disk is contained in an isometrically embedded plane $\mathbb{H}^2$ in $\mathbb{H}^n$, something similar can be said from a curve in the fiber SO(n), made from the result for the case n=2 and Fact 2, mentioned in Chapter 5, since holonomy displacements are happening in the one-dimensional vertical subgroup. Though the different orderings of triangles give different curves in the vertical space, they will meet at the same ending point. So, with respect to any ordering, the holonomy displacement of the boundary curve of the given disk can be approximated. But in other cases, what can be obtained? Something similar could be done if the fiber SO(n) were abelian, which would make the ending points of any other different two curves in the fiber, induced from different linear orderings, be the same. But the fiber SO(n) is not abelian for $n \geq 3$. The difficult part is that not only the approximation of the area but also the linear ordering of the triangles on each step for the approximation of the boundary curve of the disk should be considered at the same time. This is one of the hardest parts in this paper, which is dealt with in Chapter 2 and Appendices A and B. Furthermore, can holonomy displacements by the lifts of piecewise geodesics approaching to the boundary of the given topological disk in the base space converge to the holonomy displacement by the lift of the boundary? It will be discussed in Section 5.3.

After the case n=2 is explained in Chapter 4, our following main result for the general case will be explained in Chapter 5.

Theorem 1.0.1 *Let $\pi : SO_0(1,n) \to \mathbb{H}^n$ be the Riemannian submersion given as before. Then, given a topological disk S, with smooth interior and with $\bar{e} = \pi(e)$ on its piecewise smooth boundary, in $\mathbb{H}^n$, there is a C^1- curve $f : [0,1] \to SO(n) \subset SO_0(1,n)$ with $f(0) = e$ such that*

- *$f(1) = f(0)^{-1}f(1) =$ the difference by the holonomy induced from the boundary of S in view of right multiplication*

- *the length of the curve f = the area of S.*

Corollary 1.0.2 *If S is a piecewise smooth disk in $\mathbb{H}^n$, then there is a piecewise C^1- curve $f : [0,1] \to SO(n) \subset SO_0(1,n)$ with the same properties of Theorem 1.0.1.*

Recall some definitions first. Let $\pi : M \to B$ denote a submersion, where M is a Riemannian manifold. The *horizontal distribution* of π is the orthogonal complement $\mathcal{H} = \mathcal{V}^{\perp}$ of the *vertical distribution* $\mathcal{V}$, defined to be the kernel of π_*, i.e., the collection of tangent spaces to fibers. If B is a Riemannian manifold, then π is called a *Riemannian submersion* if it is isometric when restricted to the horizontal distribution, i.e., $|\pi_* x| = |x|$ for all $x \in \mathcal{H}$. For a differentiable curve $c : [a,b] \to B$, a curve $\tilde{c} : [a,b] \to M$ is a *horizontal lift* of c if $\pi \circ \tilde{c} = c$ and $\dot{\tilde{c}}(t) \in \mathcal{H}_{\tilde{c}(t)}$ for each $t \in [a,b]$. Given $p \in \pi^{-1}\big(c(a)\big)$, the *holonomy displacement* of p associated to c is defined to be $\tilde{c}(b)$, where $\tilde{c}(a) = p$. In this paper, 'holonomy displacement' means 'holonomy displacement of e,' where e is the identity of $\mathrm{SO}_0(1,n)$. Note elements in $\pi^{-1}\big(c(a)\big)$ induce a map $h_c : \pi^{-1}\big(c(a)\big) \to \pi^{-1}\big(c(b)\big)$, which is called *holonomy diffeomorphism* assoicated to c. If c is a geodesic and $\gamma : (-\epsilon, \epsilon) \to \pi^{-1}\big(c(a)\big)$ is a differentiable curve, consider a *variation* $V : [a,b] \times (-\epsilon, \epsilon) \to M$ such that $V(a,s) = \gamma(s)$ for $s \in (-\epsilon, \epsilon)$ and that for each $s \in (-\epsilon, \epsilon)$, $t \mapsto V(t,s)$ is a horizontal lifting of c at $\gamma(s)$. Then, for each $s \in (-\epsilon, \epsilon)$, $t \mapsto V_* D_2(t,s)$ is a Jacobi field along a horizontal geodesic $t \mapsto V(t,s)$ called a *holonomy field.* A *polytope* is a piecewise totally geodesic surface, homeomorphic to a disk, whose boundary consists of piecewise geodesic curves.

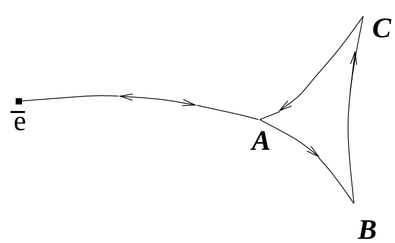
Chapter 2

Strategy for approximation

The approximation procedure in this paper is similar to that of 'Factorization Lemma', given by Lichnerowicz, *Theorie Globale des Connexions et des Groupes d'Holonomie* , [3, vol 1, p.284], so understanding the lemma will be helpful for this chapter. For the difference, focus on properties of triangles mentioned in number 6. The reason for introducing another approximation will be given in Subsection 5.4.7.

1. For any 3 points in $\mathbb{H}^n$, there exists a unique totally geodesic triangle with these vertices.

2. Let $\triangle ABC$ be a totally geodesic triangle in $\mathbb{H}^n$ and consider a piecewise geodesic from $\bar{e} = \pi(e)$ to A, where e is the identity of $\mathrm{SO}_0(1, n)$.

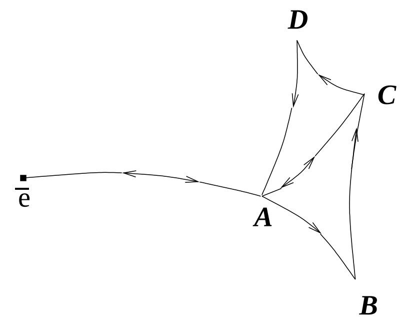


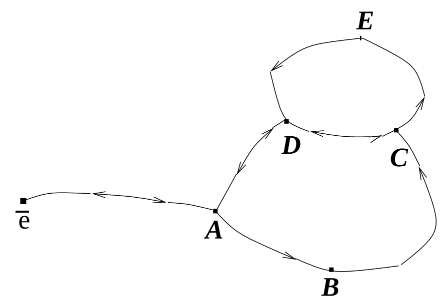
Then, it will be shown that the holonomy displacement of $\gamma = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CA} \cdot \overline{A\bar{e}}$ is $g \in$ SO(n), where

$$\text{the length of } \overline{eg} = \text{ the area of } \triangle ABC \, .$$

3. Let $\triangle ABC$ and $\triangle ACD$ be two given geodesic triangles in $\mathbb{H}^n$ and consider a piecewise geodesic curve from $\bar{e}$ to A.

Consider two curves $\gamma_1 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CA} \cdot \overline{A\bar{e}}$ and $\gamma_2 = \overline{\bar{e}A} \cdot \overline{AC} \cdot \overline{CD} \cdot \overline{DA} \cdot \overline{A\bar{e}}$. Then the holonomy displacement of $\gamma_1 * \gamma_2$ equals to that of $\gamma_3 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CD} \cdot \overline{DA} \cdot \overline{A\bar{e}}$.

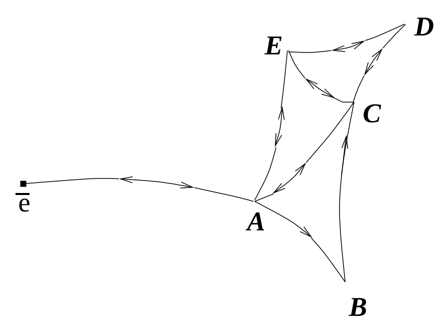




In general, if $\gamma_1 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CD} \cdot \overline{DA} \cdot \overline{A\bar{e}}$ and $\gamma_2 = \overline{\bar{e}A} \cdot \overline{AD} \cdot \overline{DC} \cdot \overline{CE} \cdot \overline{ED} \cdot \overline{DA} \cdot \overline{A\bar{e}}$ are two curves in $\mathbb{H}^n$, then the holonomy displacement of $\gamma_1 * \gamma_2$ equals to that of $\gamma_3 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CE} \cdot \overline{ED} \cdot \overline{DA} \cdot \overline{A\bar{e}}$.

4. What's the difficulty of the approximation?

Consider three given geodesic triangles $\triangle ABC$, $\triangle ACE$, $\triangle CDE$ in $\mathbb{H}^n$.



Then, for three curves $\gamma_1 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CA} \cdot \overline{A\bar{e}}$, $\gamma_2 = \overline{\bar{e}A} \cdot \overline{AC} \cdot \overline{CE} \cdot \overline{EA} \cdot \overline{A\bar{e}}$ and $\gamma_3 = \overline{\bar{e}A} \cdot \overline{AE} \cdot \overline{EC} \cdot \overline{CD} \cdot \overline{DE} \cdot \overline{EA} \cdot \overline{A\bar{e}}$, the horizontal lift of $\gamma_1 * \gamma_2 * \gamma_3$ equals to that of $\gamma_4 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CD} \cdot \overline{DE} \cdot \overline{EA} \cdot \overline{A\bar{e}}$, which relates to the boundary of the polygon $ABCDE$. But the horizontal lift of $\gamma_1 * \gamma_3 * \gamma_2$ equals to $\gamma_5 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CA} \cdot \overline{AE} \cdot \overline{EC} \cdot \overline{CD} \cdot \overline{DE} \cdot \overline{EA} \cdot \overline{AC} \cdot \overline{CE} \cdot \overline{EA} \cdot \overline{A\bar{e}}$, which does not relate to the boundary of the polygon $ABCDE$. Thus, for our object, the order of curves is important, which relates to the order of triangles.

5. Refer to the number 4.

Consider a curve $\tilde{\gamma}_3 = \overline{\bar{e}A} \cdot \overline{AC} \cdot \overline{CD} \cdot \overline{DE} \cdot \overline{EC} \cdot \overline{CA} \cdot \overline{A\bar{e}}$. Though the order $(\gamma_1,\ \gamma_2,\ \tilde{\gamma}_3)$ of curves relates to the order of the triangles, induced by the order $(\gamma_1,\ \gamma_2,\ \gamma_3)$ of curves, the horizontal lift of $\gamma_1 * \gamma_2 * \tilde{\gamma}_3$ equals to $\tilde{\gamma}_4 = \overline{\bar{e}A} \cdot \overline{AB} \cdot \overline{BC} \cdot \overline{CE} \cdot \overline{EA} \cdot \overline{AC} \cdot \overline{CD} \cdot \overline{DE} \cdot \overline{EC} \cdot \overline{CA} \cdot \overline{A\bar{e}}$, which does not relate to the boundary of the polygon $ABCDE$. Thus, it is also important how to make a curve that represents a given triangle. This problem in the construction of a curve for each triangle will be solved by introducing the starting point and the ending point of each triangle in Appendix A.

6. Instead of approximating a given topological disk in $\mathbb{H}^n$ directly, we will approximate D^2 by triangles, and approximate the given disk in $\mathbb{H}^n$ by the diffeomorphism from D^2 to it. In fact, in Appendix A, for each $n = 0, 1, 2, \cdots$, we will construct a subdivision D_n of the interval $[0, 1]$ and an ordered set A_n consisting of triangles having the following properties:

Property 1.) Given a non-first element L in A_n, the boundary of $\bigcup\{M \in A_n | M < L\}$ contains a side of L, which will be divided into two line segments in its barycentric subdivision, where one of two line segments will become a side of the first triangle and the other one will become a side of the second triangle in the barycentric subdivision of L.

Property 2.) Given $L \in A_n$, $\bigcup\{M \in A_n | M \leq L\}$ is diffeomorphic to the disk D^2.

Property 3.) Assume $L \in A_n$ and six triangles $M_1, M_2, \cdots, M_6 \in A_{n+1}$, obtained from the barycentric subdivision of L, follows the order of $i = 1, 2, \cdots 6$ in A_{n+1}. Then the starting points of M_1 and L are same. Also are the ending points of M_6 and L.

Property 4.) Assume $L, M \in A_n$ and that M is the next element of L in A_n for $n \geq 1$.

Then, The ending point of L and the starting point of M are same.

Furthermore, we can give one more property without loss of generality:

Property 5.) $\bigcup\{M \in A_n | M \leq L\}$ does not contain such a boundary point of D^2 that the the boundary of the given topological disk in $\mathbb{H}^n$ is not smooth at its image.

Chapter 3

Definitions, Triangles and Curves

All materials in this chapter will be dealt with in Appendix B concretely. And *'a constant speed curve'* in this paper means *'a piecewise constant speed curve.'*

3.1 Notations

$f * g : [0,1] \to \mathbb{H}^n$ is an ordinary juxtaposition of curves $f, g : [0,1] \to \mathbb{H}^n$. And, for a given curve $c : [0,1] \to \mathbb{H}^n$, $\bar{c}$ represents a curve whose direction is opposite to that of c, that is, $\bar{c} : [0,1] \to \mathbb{H}^n$ is given by $\bar{c}(t) = c(1-t)$.

3.2 Simplification γ of a curve $\mathbf{g} : [a,b] \to \mathbb{H}^n$

Given a curve $g : [a,b] \to S$, we can think of a curve $\gamma : [a,b] \to S$ whose direction is one-sided as follows :

If we can find $c, d, e \in (a,b)$ such that $a < c < d < e < b$ and $Im(g|_{[c,d]}) = Im(g|_{[d,e]})$ and that the directions of $g|_{[c,d]}$ and $g|_{[d,e]}$ are one-sided but opposite from each other, then we can think of the new curve $\tilde{g} : [a,b] \to D^2$ from the remaining part $g|_{[a,c]}$ and $g|_{[e,b]}$ by translating in the domain and rescaling as follows :

Note $g(c) = g(e)$.

Consider two curves $g_1 : [a,d] \to \mathbb{H}^n$ and $g_2 : [d,b] \to \mathbb{H}^n$ given by

$$g\left(\frac{c-a}{d-a}(t-a)+a\right) = g_1(t) \text{ for } t \in [a,d]$$

and

$$g\left(\frac{b-e}{b-d}(t-b)+b\right) = g_2(t) \text{ for } t \in [d,b],$$

and then let $\tilde{g} = g_1 * g_2$.

From a curve obtained by doing this work again and again and by reparametrizing it, we can think of a piecewise constant speed curve $\gamma : [a, b] \to S$ which we want.

3.3 The definition of $\mathbf{D_n, j_n, t_1^n, t_2^n}$

$$D_n = \left\{ \frac{1}{2} \cdot \frac{j}{6^n} \mid j = 0, 1, 2, \cdots, 6^n \right\} \bigcup$$

$$\left(\cup_{k=1}^{n} \left\{ \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \cdot \frac{j}{2^{k-1} \cdot 6^{n-k+1}} \mid j = 0, 1, 2, \cdots, 2^{k-1} \cdot 6^{n-k+1} \right\} \right)$$

Think of the usual order D_n and regard

$$0, \frac{1}{2} \cdot \frac{1}{6^n}, \frac{1}{2} \cdot \frac{2}{6^n}, \cdots, \frac{1}{2} = \frac{1}{2} \cdot \frac{6^n}{6^n}, \frac{1}{2} + \frac{1}{2^2} \cdot \frac{1}{2^0 \cdot 6^n}, \cdots \quad \in D_n$$

as 0th, 1st, 2nd, $\cdots$, 6^nth, 6^{n+1}th, $\cdots$ element, respectively.

Now, define functions

$$j_n : D_n \to \{0, 1, 2, 3, \cdots\}$$

$$t_1^n : \big(D_n - \{0\}\big) \cup \{1\} \to D_n$$

$$t_2^n : D_n - \{ \text{ the last element of } D_n \} \to D_n$$

as follows :

$j_n(s) = j$ for the j-th element $s \in D_n$.

$t_1^n(s)$ is the $(j-1)$-th element in D_n for a given j-th element $s \in D_n - \{0\}$ and $t_1^n(1)$ is the last element in D_n.

$t_2^n(s)$ is the $(j+1)$-th element in D_n for a given j-th element $s \in D_n - \{$ the last element of $D_n\}$.

3.4 Definition of $\gamma^{\mathbf{n}}_{\mathbf{t_0}}, \mathbf{c^n_{t_0}}, \mathbf{\bar{c}^n_{t_0}}, \mathbf{{}_1c^n_{t_0}}, \mathbf{{}_1\bar{c}^n_{t_0}}, \varphi^{\mathbf{n}}_{\mathbf{t_0}}$ and $\psi^{\mathbf{n}}_{\mathbf{t_0}}$ on the disk $\mathbf{D^2}$

Recall, from Properties mentioned later in Chapter 2, that the union U_i of triangles from 1st one to i-th one is diffeomorphic to a disk.

Let $n \in \{1, 2, 3, \cdots\}$ and $t_0 \in D_n$ be given.

With respect to the ordering of D_n, we will define $\gamma^n_{t_0}, c^n_{t_0}, \bar{c}^n_{t_0}$ and $\varphi^n_{t_0}$ inductively for each fixed n:

Case 1) t_0 is the first element in D_n , in fact, $t_0 = \frac{1}{2} \cdot \frac{1}{6^n}$

The orientation at the barycenter of $T_0 \in A_0$ will give the direction of the boundary curve of the first triangle in A_n.

Then

$$c^n_{t_0} : [0, 1] \to \{basepoint\} \subset D^2$$

$$\bar{c}^n_{t_0} : [0, 1] \to \{basepoint\} \subset D^2$$

$$\varphi^n_{t_0} : [0, 1] \to D^2$$

and

$$\gamma^n_{t_0} : [0, 1] \to D^2$$

can be thought, where $\varphi^n_{t_0}$ and $\gamma^n_{t_0}$ are the piecewise smooth boundary curve of the first triangle in A_n with piecewise constant speed and the direction of the boundary curve is induced from the given orientation.

Note $\gamma^n_{t_0}$ can be regarded as the simplification of $c^n_{t_0} * \varphi^n_{t_0} * \bar{c}^n_{t_0}$.

We will call $\gamma^n_{t_0}$ the *holonomy curve at time* $t = t_0$.

Now, consider the path from the basepoint to the ending point of the first triangle in n-step along the opposite direction of the holonomy curve $\gamma^n_{t_0}$ at $t = t_0$, which is a piecewise smooth curve with piecewise constant speed. Then from the path, we can define a piecewise smooth curve

$${}_1c^n_{t_0} : [0, 1] \to D^2$$

with piecewise constant speed. And its opposite direction can make us define

$${}_1\bar{c}^n_{t_0} : [0, 1] \to D^2.$$

Define a piecewise smooth curve

$$\psi^n_{t_0} : [0, 1] \to D^2$$

with piecewise constant speed as the boundary curve of the 1st triangle in the n-th step, where the curve is a loop at the ending point of the first triangle and the direction of the boundary curve is induced from the given orientation.

Case 2) t_0 is the j-th element in D_n, *i.e.*, $j_n(t_0) = j$, where $j \geq 2$

Let t_1 be the $(j-1)$-th element in D_n, *i.e.*, $t_1^n(t_0) = t_1$ and $j_n(t_1) = j-1$, where $j-1 \geq 1$.

Consider the path from the basepoint to the starting point of the j-th triangle in the n-th step along the opposite direction of the holonomy curve $\gamma_{t_1}^n$ at $t = t_1$, which is a piecewise smooth one with constant speed . Then from the path, we can define a piecewise smooth curve

$$c_{t_0}^n : [0,1] \to \partial U_{j-1} \subset D^2$$

with constant speed, where U_{j-1} is the union of triangle in A_n from the 1st one to the $(j-1)$-th one.

And its opposite direction can make us define

$$\bar{c}_{t_0}^n : [0,1] \to \partial U_{j-1} \subset D^2.$$

Define a piecewise smooth curve

$$\varphi_{t_0}^n : [0,1] \to D^2$$

with constant speed as the boundary curve of the j-th triangle in the n-th step, where the curve is a loop at the starting point of the triangle and the direction of the boundary curve is induced from the given orientation.

Now define a piecewise smooth curve

$$\gamma_{t_0}^n : [0,1] \to \partial U_j \subset D^2$$

with constant speed from the simplification of $\gamma_{t_1}^n * c_{t_0}^n * \varphi_{t_0}^n * \bar{c}_{t_0}^n$, where U_j is the union of triangle in A_n from the 1st one to the j-th one. The new curve will be also called the *holonomy curve at time* $t = t_0$.

Now, consider the path from the basepoint to the ending point of the j-th triangle in the n-th step along the opposite direction of the holonomy curve $\gamma_{t_0}^n$ at $t = t_0$, which is a piecewise smooth one with constant speed. Then from the path, we can define a piecewise smooth curve

$$_1c_{t_0}^n : [0,1] \to \partial U_j \subset D^2$$

with constant speed. And its opposite direction can make us define

$$_1\bar{c}_{t_0}^n : [0,1] \to \partial U_j \subset D^2.$$

Define a piecewise smooth curve

$$\psi_{t_0}^n : [0,1] \to D^2$$

with constant speed as the boundary curve of the j-th triangle in the n-th step, where the curve is a loop at the ending point of the j-th triangle and the direction of the boundary curve is induced from the given orientation.

3.5 The simplification of $\bar{\mathbf{c}}^{\mathbf{n}}_{\mathbf{t_0}} * {}_1\mathbf{c}^{\mathbf{n}}_{\mathbf{t_0}}$

For each $n \geq 1$ and $0 \neq t_0 \in D_n$, where t_0 is the $j_n(t_0)$-th element in D_n, the simplification of $\bar{c}^n_{t_0} * {}_1c^n_{t_0}$ is a curve along the boundary of $j_n(t_0)$-th triangle in A_n with opposite direction to the given orientation such that it starts from the starting point of the triangle and that its image consists of the following sets :

one point, one side, two sides or the boundary of the triangle.

3.6 The induced curves on the surface $S \subset \mathbb{H}^n$ and totally geodesic planes in $\mathbb{H}^n$

Let $\Phi : D^2 \to S$ be a given diffeomorphism. Then we can think of triangles in S induced from the barycentric subdivision on D^2 on each n-th step. We will use ' $\sim$ ' notation for the induced triangles and curves in S , that is ,

$$\tilde{T} = \Phi(T) \qquad \text{for} \quad T \in A_n$$

$$\tilde{A}_n = \{\Phi(T) \mid T \in A_n\}$$

and

$$\tilde{\gamma}^n_{t_0}, \tilde{c}^n_{t_0}, \tilde{\varphi}^n_{t_0}, \tilde{\bar{c}}^n_{t_0}, {}_1\tilde{c}^n_{t_0}, \tilde{\psi}^n_{t_0}, {}_1\tilde{\bar{c}}^n_{t_0},$$

which are piecewise smooth curves with constant speed such that

$$Im(\tilde{\gamma}^n_{t_0}) = Im(\Phi \circ \gamma^n_{t_0})$$

$$Im(\tilde{c}^n_{t_0}) = Im(\Phi \circ c^n_{t_0})$$

$$Im(\tilde{\varphi}^n_{t_0}) = Im(\Phi \circ \varphi^n_{t_0})$$

$$Im(\tilde{\bar{c}}^n_{t_0}) = Im(\Phi \circ \bar{c}^n_{t_0})$$

$$Im({}_1\tilde{c}^n_{t_0}) = Im(\Phi \circ {}_1c^n_{t_0})$$

$$Im(\tilde{\psi}^n_{t_0}) = Im(\Phi \circ \psi^n_{t_0})$$

$$Im({}_1\tilde{\bar{c}}^n_{t_0}) = Im(\Phi \circ {}_1\bar{c}^n_{t_0})$$

and whose direction relates to that of $\gamma^n_{t_0}, c^n_{t_0}, \varphi^n_{t_0}, \bar{c}^n_{t_0}, {}_1c^n_{t_0}, \psi^n_{t_0}, {}_1\bar{c}^n_{t_0}$, respectively.

Now with respect to each triangle in S, we can think of a totally geodesic triangle with same vertices in $\mathbb{H}^n$. So, each step will induce the similar concept , *i.e.* triangles and curves, on the induced pleated surface consisting of totally geodesic triangles and we'll use ' $\wedge$ ' notation for them. In other words, we can think of

$$\hat{T} \in \hat{A}_n,\ \hat{\gamma}^n_{t_0},\ \hat{c}^n_{t_0},\ \hat{\varphi}^n_{t_0},\ \hat{\bar{c}}^n_{t_0},\ {}_1\hat{c}^n_{t_0},\ \hat{\psi}^n_{t_0},\ {}_1\hat{\bar{c}}^n_{t_0}\ ,$$

where the curves $\hat{\gamma}^n_{t_0}, \hat{c}^n_{t_0}, \hat{\varphi}^n_{t_0}, \hat{\bar{c}}^n_{t_0}, {}_1\hat{c}^n_{t_0}, \hat{\psi}^n_{t_0}, {}_1\hat{\bar{c}}^n_{t_0}$ are piecewise geodesics in $\mathbb{H}^n$, induced from the boundaries of totally geodesic triangles $\hat{T}$, and are relating to the previous curves $\tilde{\gamma}^n_{t_0}, \tilde{c}^n_{t_0}, \tilde{\varphi}^n_{t_0}, \tilde{\bar{c}}^n_{t_0}, {}_1\tilde{c}^n_{t_0}, \tilde{\psi}^n_{t_0}, {}_1\tilde{\bar{c}}^n_{t_0}$ in S and $\gamma^n_{t_0}, c^n_{t_0}, \varphi^n_{t_0}, \bar{c}^n_{t_0}, {}_1c^n_{t_0}, \psi^n_{t_0}, {}_1\bar{c}^n_{t_0}$ in D^2.

Chapter 4

$\mathbf{SO}(2) \to \mathbf{SO}_0(1,2) \to \mathbb{H}^2$

For an ordered orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{o}(1,2)$, given by

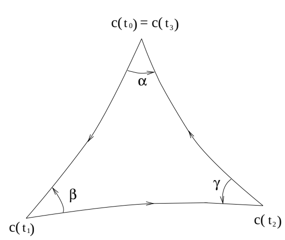
$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{and } E_3 = [E_1, E_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

let a triple (E_2, E_1, E_3) be the orientation of $\mathrm{SO}_0(1,2)$. Then, Fact(5), mentioned in Chapter 5, says that the induced orientation (E_2, E_1) on $\mathfrak{o}(2)^{\perp}$ is comparable to the counterclockwise orientation on $\mathbb{H}^2$.

For $t \in \mathbb{R}$, put

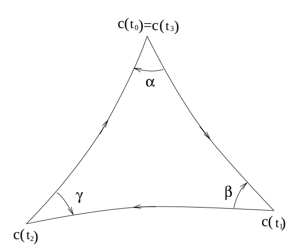
$$\Psi(t) \;=\; \exp(tE_3) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

Let $c : [t_0, t_3] \to \mathbb{H}^2$ be a simple-closed arc-length parameterized piecewise-smooth curve representing a geodesic triangle in $\mathbb{H}^2$ with the counterclockwise orientation:



$\alpha, \beta, \gamma > 0$

or



$\alpha, \beta, \gamma < 0$

More precisely, c is continuous on $[t_0, t_3]$ and smooth on $(t_0, t_1) \cup (t_1, t_2) \cup (t_2, t_3)$, where $c(t_0) = c(t_3)$, $c(t_1)$ and $c(t_2)$ are vertices of the given geodesic triangle.

Let

$$\begin{aligned}
&\alpha \text{ be the angle from } \dot{c}({t_0}^+) \text{ to } -\dot{c}({t_3}^-), \\
&\beta \text{ be the angle from } \dot{c}({t_1}^+) \text{ to } -\dot{c}({t_1}^-), \text{ and} \\
&\gamma \text{ be the angle from } \dot{c}({t_2}^+) \text{ to } -\dot{c}({t_2}^-)\,.
\end{aligned}$$

Then, either $\alpha, \beta, \gamma > 0$ or $\alpha, \beta, \gamma < 0$ holds.

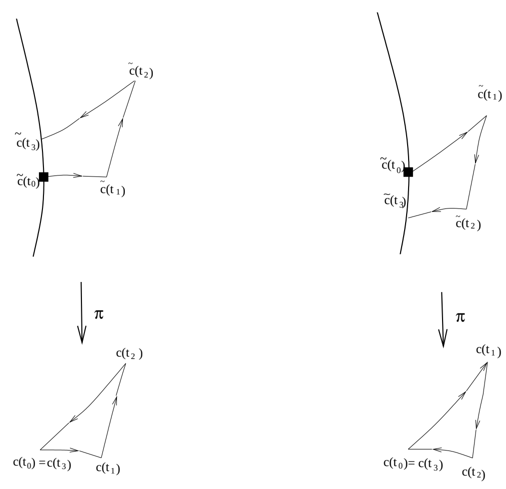
Lemma 4.0.1 *Under the above condition, let $\tilde{c} : [t_0, t_3] \to SO_0(1,2)$ be a horizontal lift of c. Then, the relation between the holonomy and the area of a geodesic triangle is given by*

$$\tilde{c}(t_0)^{-1} \cdot \tilde{c}(t_3) = \Big(\Psi\big(\pi - |\,\alpha + \beta + \gamma\,|\big)\Big)^{\delta} \quad ,$$

where

$$\delta = \begin{cases} \;\;1 & \text{if } \alpha, \beta, \gamma > 0 \\ -1 & \text{if } \alpha, \beta, \gamma < 0 \end{cases}$$

Furthermore, $\pi - |\,\alpha + \beta + \gamma\,|$ is the area of the geodesic triangle.



$\alpha, \beta, \gamma > 0$ or $\alpha, \beta, \gamma < 0$

Proof) Let $\pi :\ \mathrm{SO}_0(1,2) \to \mathbb{H}^2$ be the given Riemannian submersion. Recall Fact (5), mentioned in Chapter 5. For any $k \in \mathrm{SO}(2)$, the restriction $Ad_k|_{\mathfrak{o}(2)^\perp}$ of $Ad_k(\cdot) : \mathfrak{o}(1,2) \to \mathfrak{o}(1,2)$ to $\mathfrak{o}(2)^\perp$ is an automorphism of $\mathfrak{o}(2)^\perp$, which is projected to the action of K on $T_{\pi(e)}\mathbb{H}^2$ and the action is in fact a rotation if $n = 2$. For $-\dot{c}({t_3}^-)$ and its horizontal lift x at $\tilde{c}(t_0)$, find $A \in \mathfrak{o}(2)^\perp$ satisfying

$$L_{\tilde{c}(t_0)^{-1}\,*} x = A_e.$$

Then Fact (5) says that

$$L_{\tilde{c}(t_0)^{-1}}{}_* \dot{\tilde{c}}(t_0{}^+) = (Ad_{\Psi(\alpha)}A)_e.$$

And Fact (1), mentioned in Chapter 5, says that

$$\tilde{c}(t) = \tilde{c}(t_0) \cdot \exp\big((t - t_0)\, Ad_{\Psi(\alpha)}A\big) \quad \text{ for } t \in [t_0, t_1],$$

so

$$L_{\tilde{c}(t_1)^{-1}}{}_* \dot{\tilde{c}}(t_1{}^-) = (Ad_{\Psi(\alpha)}A)_e.$$

Now from Fact (5), we get

$$L_{\tilde{c}(t_1)^{-1}}{}_* \dot{\tilde{c}}(t_1{}^+) = \big(Ad_{\Psi(\beta)}(-Ad_{\Psi(\alpha)}A)\big)_e = (Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A)_e,$$

and from Fact (1)

$$\tilde{c}(t) = \tilde{c}(t_1) \cdot \exp\big((t - t_1)\, Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A\big) \quad \text{ for } t \in [t_1, t_2],$$

so

$$L_{\tilde{c}(t_2)^{-1}}{}_* \dot{\tilde{c}}(t_2{}^-) = (Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A)_e.$$

If we apply Fact (1) and (5) again, then we obtain

$$L_{c(t_2)^{-1}}{}_* \dot{\tilde{c}}(t_2{}^+) = \big(Ad_{\Psi(\gamma)}(-Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A)\big)_e = (Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)_e$$

and

$$\tilde{c}(t) = \tilde{c}(t_2) \cdot \exp\big((t - t_2)\, Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A\big) \quad \text{ for } t \in [t_2, t_3],$$

so

$$L_{\tilde{c}(t_3)^{-1}}{}_* \dot{\tilde{c}}(t_3{}^-) = (Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)_e,$$

that is,

$$-\dot{\tilde{c}}(t_3{}^-) = L_{\tilde{c}(t_3)}{}_{*e}(-Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)_e = L_{\tilde{c}(t_3)}{}_{*e}(Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)_e.$$

Therefore,

$$\begin{aligned}
\pi\big(\tilde{c}(t_0) \cdot \mathrm{e}^{tA}\big) &= \pi\big(\tilde{c}(t_3) \cdot \mathrm{e}^{t(Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)}\big) \\
&= \pi\big(\tilde{c}(t_0) \cdot \mathrm{e}^{tAd_{\tilde{c}(t_0)^{-1}\cdot\tilde{c}(t_3)\cdot\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A} \cdot \tilde{c}(t_0)^{-1} \cdot \tilde{c}(t_3)\big) \\
&= \pi\big(\tilde{c}(t_0) \cdot \mathrm{e}^{tAd_{\tilde{c}(t_0)^{-1}\cdot\tilde{c}(t_3)\cdot\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A}\big)
\end{aligned}$$

because

$$\tilde{c}(t_0)^{-1} \cdot \tilde{c}(t_3) \in \mathrm{SO}(2)$$

and

$$B \in \mathrm{so}(2)^{\perp} \text{ and } k \in \mathrm{SO}(2) \Rightarrow k \cdot e^{tB} \cdot k^{-1} = e^{tAd_k B}.$$

Thus, we get

$$A = Ad_{\tilde{c}(t_0)^{-1}\cdot\tilde{c}(t_3)\cdot\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A$$

and so

$$\tilde{c}(t_0)^{-1}\cdot\tilde{c}(t_3)\cdot\Psi\big(\pi+(\alpha+\beta+\gamma)\big) \;=\; \Psi(2n\pi) \qquad \text{for some } n\in\mathbb{Z}\,.$$

Therefore,

$$\begin{aligned}
\tilde{c}(t_0)^{-1}\cdot\tilde{c}(t_3) &= \Psi(2n\pi)\cdot\Big(\Psi\big(\pi+(\alpha+\beta+\gamma)\big)\Big)^{-1} \\
&= \Big(\Psi\big(\pi+(\alpha+\beta+\gamma)\big)\Big)^{-1} \\
&= \begin{cases} \Psi(-\pi-(\alpha+\beta+\gamma)) & \text{if } \alpha,\beta,\gamma>0 \\ \Big(\Psi\big(\pi+(\alpha+\beta+\gamma)\big)\Big)^{-1} & \text{if } \alpha,\beta,\gamma<0 \end{cases} \\
&= \begin{cases} \Psi(\pi-(\alpha+\beta+\gamma)) & \text{if } \alpha,\beta,\gamma>0 \\ \Big(\Psi\Big(\pi-\big((-\alpha)+(-\beta)+(-\gamma)\big)\Big)\Big)^{-1} & \text{if } \alpha,\beta,\gamma<0 \end{cases} \\
&= \Big(\Psi\big(\pi-|\alpha+\beta+\gamma|\big)\Big)^{\delta},\ \text{where } \delta = \begin{cases} 1 & \text{if } \alpha,\beta,\gamma>0 \\ -1 & \text{if } \alpha,\beta,\gamma<0. \end{cases}
\end{aligned}$$

Remark 4.0.2 *Recall that the induced orientation (E_2, E_1) on $\mathfrak{o}(2)^{\perp}$ is comparable to the counterclockwise one on $\mathbb{H}^2$. This lemma says that the orientation (E_2, E_1, E_3) of $SO_0(1,2)$ is comparable to the usual one (e_1, e_2, e_3) in the 3-dimensional Euclidean space $\mathbb{R}^3$.*

Chapter 5

Liftings in $\mathbf{SO}(n) \to \mathbf{SO}_0(1,n) \to \mathbb{H}^n$

This chapter is the proof of Theorem 1.0.1 and its Corollary 1.0.2.

5.1 Preliminaries on the Riemannian submersion $\pi : \mathbf{SO}_0(1,n) \to \mathbb{H}^n$

Let $\pi : \mathrm{SO}_0(1,n) \to \mathbb{H}^n$ be the given Riemannian submersion. In fact, this is the quotient of the isometric right translation by $\mathrm{SO}(n)$ and $\mathbb{H}^n$ is isometric to $\mathrm{SO}_0(1,n)/\mathrm{SO}(n)$.

Let $G = \mathrm{SO}_0(1,n)$, $K = \mathrm{SO}(n)$, and $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras, respectively. Then we have the following facts, see [KN; vol 2., Example 10.2] and [GW; Section 1.4, Section2.4].

Facts

1. For $X \in \mathfrak{k}^{\perp}$, $t \mapsto g\cdot \exp\,(tX) : (-\infty,\infty) \to G$ is a horizontal geodesic for any $g \in G$.

2. For $X, Y \in \mathfrak{k}^{\perp}$ with $0 \neq [X,Y] \in \mathfrak{k}$, let $\mathfrak{h}$ be a Lie subalgebra $\mathrm{Span}\{X, Y, [X,Y]\} \subset \mathfrak{g}$. Then its related subgroup H is isometric to $\mathrm{SO}_0(1,2)$. Furthermore, the riemannian submersion $\mathrm{SO}_0(1,n) \to \mathrm{SO}(n)$ can be restricted to $\big(H = \mathrm{SO}_0(1,2)\big) \to \mathrm{SO}(2)$.

3. Each fiber gK, $g \in G$, is totally geodesic. More precisely, for $U \in \mathfrak{k}$, $t \mapsto g\cdot \exp(tU) : (-\infty,\infty) \to gK \subset G$ is a vertical geodesic for any $g \in G$. Especially, if $g \in K$, then its image lies on $K = eK$. Furthermore,

for any piecewise smooth curve $c : [a,b] \to \mathbb{H}^n$, its induced holonomy $h_c : \pi^{-1}(c(a)) \to \pi^{-1}(c(b))$ is an isometry.

4. For any $k \in K$, the right translation $R_k : G \to G$ by $k, R_k(g) = gk$, is an isometry. Or, equivalently, $Ad_k : \mathfrak{g} \to \mathfrak{g}$ is a linear isometry for any $k \in K$.

5. The action of Ad_K on $\mathfrak{k}^\perp$ is projected to the action of K on $T_{\pi(e)}\mathbb{H}^n$, where e is the identity of G. More precisely, for $B \in \mathrm{SO}(n)$, for $k = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{k}$ and for a column vector $\xi \in \mathbb{R}^n$,
$$m = \begin{pmatrix} 0 & -\xi^t \\ \xi & 0 \end{pmatrix} \in \mathfrak{k}^\perp \quad \text{and} \quad Ad_k m = \begin{pmatrix} 0 & -(B\xi)^t \\ B\xi & 0 \end{pmatrix}.$$

Consider the following Lemma, which is the explanation of the holonomy isometry h_c in Fact 3 in terms of vector fields.

Lemma 5.1.1 *For any $U \in \mathfrak{k}$ and for any horizontal geodesic $\tilde{c} : [a,b] \to G$, $U \circ \tilde{c}$ is a holonomy field along $\tilde{c}$.*

Proof) Consider a vertical geodesic $\gamma : (-\epsilon, \epsilon) \to \tilde{c}K$ given by
$$\gamma(s) = \tilde{c}(a) \cdot \exp(sU),$$
and a variation $V(t,s) : [a,b] \times (-\epsilon, \epsilon) \to G$ defined by
$$V(t,s) = \tilde{c}(t) \cdot \exp(sU).$$
Then, for inclusions maps $i_s : [a,b] \to [a,b] \times (-\epsilon,\epsilon)$ and $j_t : (-\epsilon,\epsilon) \to [a,b] \times (-\epsilon,\epsilon)$ with $i_s(t) = (t,s) = j_t(s)$,
$$V \circ j_0 = \gamma, \quad V \circ i_0 = \tilde{c}$$
and
$$V \circ i_s \text{ is a horizontal geodesic with } \pi \circ V \circ i_s = \tilde{c}$$
since
$$\exp(sU) \in K, \qquad V \circ i_s(t) = V(t,s) = R_{\exp(sU)}\big(\tilde{c}(t)\big)$$
and
$$\text{the right multiplication } R_{\exp(sU)} \text{ is an isometry for each } s \in (-\epsilon,\epsilon).$$

So from

$$V(t,s) = \tilde{c}(t)\cdot \exp(s_0 U)\cdot \exp\big((s-s_0)U\big) = V(t,s_0)\cdot \exp\big((s-s_0)U\big),$$

we get

$$V_* D_2 \circ i_{s_0}(t) = L_{V(t,s_0)_*} U_e = U_{V(t,s_0)} = U \circ (V \circ i_{s_0})(t)$$

and that U is a holonomy field along $V \circ i_{s_0}$. Especially, $U \circ \tilde{c}$ is a holonomy field along a horizontal geodesic $\tilde{c} = V \circ i_0$.

5.2 Definition of $\bar{\mathbf{f}} : \bigcup_{\mathbf{m}=1}^{\infty} \mathbf{D_m} \to \mathbf{K} = \mathbf{SO(n)}, \bar{\mathbf{f}}_\mathbf{m} : \mathbf{D_m} \to \mathbf{SO(n)}$ and $\hat{\mathbf{f}}_\mathbf{m} : [\mathbf{0},\mathbf{1}] \to \mathbf{SO(n)}$ and the property of $\hat{\mathbf{f}}_\mathbf{m}$

5.2.1 Definition of $\bar{\mathbf{f}}, \bar{\mathbf{f}}_\mathbf{m}$

Let $\bar{f}(0) = e$. Fix $t_0 \in \bigcup_{m=1}^{\infty} D_m - \{0\}$. Then we can find a positive integer $n_0 = \min\{m_1 \mid m+1 \geq m_1 \Rightarrow t_0 \in D_m\}$.

Note that on the given surface S,

$$\tilde{\gamma}^n_{t_0} = \tilde{\gamma}^{n_0}_{t_0} \text{ and } \tilde{c}^n_{t^n_2(t_0)} = {}_1\tilde{c}^n_{t_0} = {}_1\tilde{c}^{n_0}_{t_0} = \tilde{c}^{n_0}_{t^{n_0}_2(t_0)}$$

for all $n \geq n_0$. So let

$$\tilde{\gamma}_{t_0} := \tilde{\gamma}^{n_0}_{t_0} \text{ and } {}_1\tilde{c}_{t_0} := {}_1\tilde{c}^{n_0}_{t_0}$$

Define

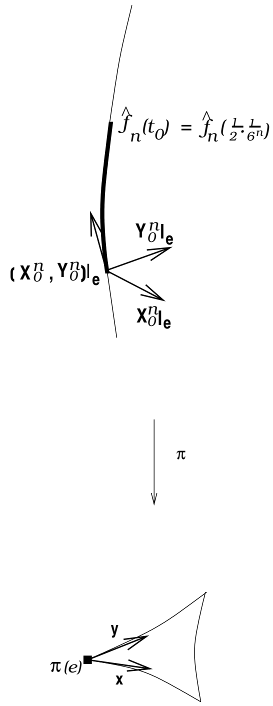
$$\bar{f}(t_0) := \text{ the value , at } t=1, \text{ of the horizontal lifting of } \tilde{\gamma}_{t_0} \text{ at } e\,.$$

Put $\bar{f}_n$ as the restriction $\hat{f}_n\,|_{\bigcup_{n=1}^{\infty} D_n}$ of $\hat{f}_n$, defined below, to D_n.

5.2.2 Definition of $\hat{\mathbf{f}}_\mathbf{n}$ and its property

Define a curve $\hat{f}_n : [0,1] \to K = \mathrm{SO}(n)$ with $\hat{f}_n(0) = e$ inductively as follows:

Step 1) Assume $t_0 \in D_n$ is the 1st element in D_n, in fact, $t_0 = \frac{1}{2}\cdot\frac{1}{6^n}$. Then, $t^n_1(t_0) = 0$.



Consider the first triangle in $\hat{A}_n$, its starting point and the horizontal lifting of

$$x := \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\varphi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\varphi}}^n_{t_0}(t)$$

and

$$y := -\lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\varphi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\varphi}}^n_{t_0}(t)$$

at e , respectively and find

$$X_0^n = X^n_{t_1^n(t_0)},\ Y_0^n = Y^n_{t_1^n(t_0)} \in \mathfrak{k}^\perp$$

with

$$\pi_* \ X_0^n \ |_e = \ \pi_* \ X^n_{t_1^n(t_0)} \ |_e = \ x$$

and

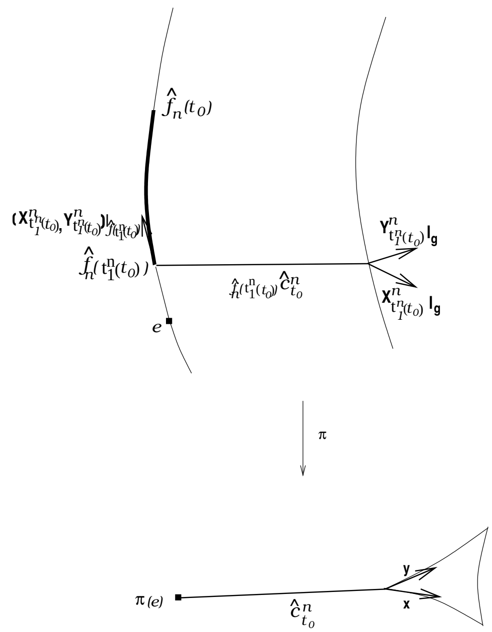
$$\pi_* \ Y_0^n \ |_e = \ \pi_* \ Y^n_{t_1^n(t_0)} \ |_e = \ y \ .$$

Then, define

$$\hat{f}_n(t) := \ \exp \Big(t \cdot \frac{(\text{Area of the 1st triangle in } \hat{A}_n)}{t_0 \cdot \mid [X_0^n, Y_0^n] \mid} \cdot [X_0^n, Y_0^n] \Big) \qquad \text{for } t \in [0, t_0] \ ,$$

which is a geodesic in $K = \mathrm{SO}(n)$ from Fact 3.

Step 2) Assume $t_0 \in D_n$ is the j-th element in D_n, where $j \geq 2$.



Note $t_1^n(t_0)$ is the $(j-1)$-th element in D_n, where $j-1 \geq 1$.

Let ${}_{\hat{f}_n(t_1^n(t_0))}\hat{c}_{t_0}^n : [0,1] \to \mathrm{SO}_0(1,n)$ be the horizontal lifting of $\hat{c}_{t_0}^n$ at $\hat{f}_n(t_1^n(t_0))$ and then consider the j-th triangle in $\hat{A}_n$, its starting point and the horizontal lifting of

$$x := \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\varphi}}_{t_0}^n(t) \mid} \cdot \dot{\hat{\varphi}}_{t_0}^n(t)$$

and

$$y := -\lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\varphi}}_{t_0}^n(t) \mid} \cdot \dot{\hat{\varphi}}_{t_0}^n(t)$$

at $g := {}_{\hat{f}_n(t_1^n(t_0))}\hat{c}_{t_0}^n(1)$, respectively and find

$$X_{t_1^n(t_0)}^n,\ Y_{t_1^n(t_0)}^n \in \mathfrak{k}^\perp$$

with

$$\pi_* \, X_{t_1^n(t_0)}^n \mid_g \ = \ x$$

and

$$\pi_* \, Y_{t_1^n(t_0)}^n \mid_g \ = \ y \, .$$

Then define

$$\hat{f}_n(t) \ :=$$
$$\hat{f}_n(t_1^n(t_0)) \cdot \ \exp\left(\left(t - t_1^n(t_0)\right) \cdot \frac{(\text{Area of } j\text{-th triangle in } \hat{A}_n)}{\left(t_0 - t_1^n(t_0)\right) \cdot \mid [X_{t_1^n(t_0)}^n, Y_{t_0^n(t_0)}^n] \mid} \cdot [X_{t_1^n(t_0)}^n, Y_{t_1^n(t_0)}^n]\right)$$

for $t \in [t_1^n(t_0), t_0]$.

Step3) $t_0 = 1$

Note $t_1^n(1)$ be the last element in D_n , in other words, $t_1^n(1) = \sum_{i=1}^{n+1} \frac{1}{2^i}$.

Then define

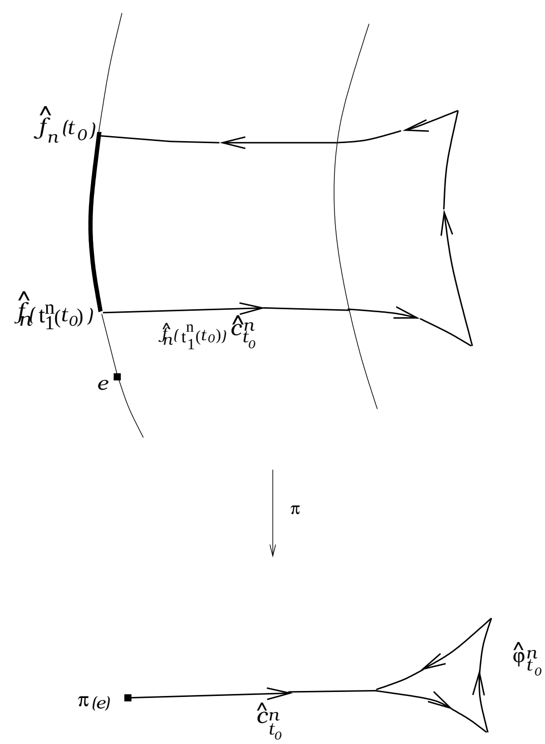
$$\hat{f}_n(t) := \hat{f}_n(t_1^n(1))$$

for $t \in [t_1^n(1), 1]$.

Now check the property of $\hat{f}_n$.

Assume $0 \neq t_0 \in D_n$ is a j-th element in D_n, where $j \geq 1$. Then $t_1^n(t_0)$ is the $(j-1)$-th elements in D_n , where $j - 1 \geq 0$, and from Facts, mentioned early in this chapter, and from the property in Chapter 4, we get

$$\begin{aligned} \hat{f}_n(t_0) = & \text{ the value , at } t = 1, \text{ of the horizontal lifting of } \hat{c}^n_{t_0} * \hat{\varphi}^n_{t_0} * \bar{\hat{c}}^n_{t_0} \text{ at } \hat{f}_n(t_1^n(t_0)) \\ = & \text{ the value , at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}^n_{t_0} \text{ at e .} \end{aligned}$$



Define, for any $g \in G$, $l_g : K \to G$ by $l_g(k) = gk$, which is an isometric imbedding of K onto the fiber gK.

And let ω and Ω be the connection form and the curvature form of the connection of the principal bundle $\pi : \mathrm{SO}_0(1, n) \to \mathbb{H}^n$, respectively.

Then, under the identification of T_eG and $\mathfrak{g}$, for $t \in (t_1^n(t_0), t_0)$ and $g = {}_{\hat{f}_n(t_1^n(t_0))}\hat{c}_{t_0}^n(1)$, which is the value, at $t = 1$, of the horizontal lifting of $\hat{c}_{t_0}^n$ at $\hat{f}_n(t_1^n(t_0))$,

$$\begin{aligned}
\omega\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t) \right) &= (l_{\hat{f}_n(t)*e})^{-1} \left(\tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot (\dot{\hat{f}}_n(t))^v \right) \\
&= (l_{\hat{f}_n(t)*e})^{-1} \left(\tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t) \right) \\
&= (l_{\hat{f}_n(t)*e})^{-1} \left(\tfrac{1}{|[X^n_{t_1^n(t_0)},\, Y^n_{t_1^n(t_0)}]|_{\hat{f}_n(t)}|} \cdot [X^n_{t_1^n(t_0)},\ Y^n_{t_1^n(t_0)}] \,|_{\hat{f}_n(t)} \right) \\
&= \tfrac{1}{|(l_{\hat{f}_n(t)*e})^{-1}([X^n_{t_1^n(t_0)},\, Y^n_{t_1^n(t_0)}]|_{\hat{f}_n(t)})|} \cdot (l_{\hat{f}_n(t)*e})^{-1} \left([X^n_{t_1^n(t_0)},\ Y^n_{t_1^n(t_0)}] \,|_{\hat{f}_n(t)} \right) \\
&= \tfrac{1}{|[X^n_{t_1^n(t_0)},\, Y^n_{t_1^n(t_0)}]|} \cdot\ [X^n_{t_1^n(t_0)},\ Y^n_{t_1^n(t_0)}] \\
&= \tfrac{-1}{|\Omega(X^n_{t_1^n(t_0)}|_g,\, Y^n_{t_1^n(t_0)}|_g)|} \cdot \Omega\left(X^n_{t_1^n(t_0)}\,|_g,\ Y^n_{t_1^n(t_0)}\,|_g\right)
\end{aligned}$$

and

$$\omega\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t) \right) |_e = \ L_{(\hat{f}_n(t))^{-1}*}\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t) \right)$$

Roughly speaking, the unit tangent vector $\frac{1}{|\dot{f}_n(t)|} \cdot \dot{f}_n(t)$, $t \in (t_1^n(t_0), t_0)$, is the negative of the unit curvature of the 2-dimensional horizontal plane

$$\hat{H}_g^n = \mathrm{Span}\{X^n_{t_1^n(t_0)}\,|_g,\ Y^n_{t_1^n(t_0)}\,|_g\}, \qquad \text{where } g = {}_{\hat{f}_n(t_1^n(t_0))}\ \hat{c}_{t_0}^n(1),$$

which projects to the tangent plane of the $j_n(t_0)$-th triangle in $\hat{A}_n$ at $\pi(g) = \hat{c}_{t_0}^n(1) =$ the starting point of the $j_n(t_0)$-th triangle in $\hat{A}_n$. And, the length of $\hat{f}_n\,|_{[t_1^n(t_0), t_0]}$ is the area of the $j_n(t_0)$-th triangle in $\hat{A}_n$.

5.3 The convergence of $\bar{\mathbf{f}}_{\mathbf{n}}(t_0)$ to $\bar{\mathbf{f}}(t_0)$

Recall

$$\bar{f}(t_0) = \text{ the value , at } t = 1, \text{ of the horizontal lifting of } \tilde{\gamma}_{t_0} \text{ at } e$$

and

$$\bar{f}_n(t_0) = \hat{f}_n(t_0) = \text{ the value , at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}_{t_0}^n \text{ at e .}$$

Consider our Riemannian submersion

$$\mathrm{SO}(n) \longrightarrow \mathrm{SO}_0(1, n) \longrightarrow \mathrm{SO}_0(1, n)/\mathrm{SO}(n).$$

This bundle has a global cross section $s : \mathbb{H} \to NA \subset G$, which comes from the Iwasawa decomposition NAK, where $K = \mathrm{SO}(n)$. That is, every element of G is uniquely written as nak, and the projection maps this to $naK \in \mathbb{H}$.

The cross section s provides us with a one-to-one correspondence between the space of all continuous piecewise C^k-curves in $\mathbb{H}^n$ and in $\mathrm{SO}_0(1,n)$, with initial points $\bar{e}$ and e, by

$$h \longleftrightarrow s \circ h.$$

By abusing notations, express $s \circ h$ by h. For a curve $h : [0,1] \to \mathbb{H}^n$, the unique horizontal lift $\tilde{h} : [0,1] \to \mathrm{SO}_0(1,n)$ is given by

$$h(t) \cdot a(t) = \tilde{h}(t)$$

for a unique curve $a(t)$ in $\mathrm{SO}(n)$. Such an $a(t)$ is obtained by solving the differential equation

$$\langle h^{-1}h' + a'a^{-1},\ V\rangle = 0 \tag{5.3–1}$$

for every $V \in \mathfrak{k}$, where $'$ means the derivative with respect to t. Note that the first entry $h^{-1}h' + a' \cdot a^{-1}$ is an element of the Lie algebra $\mathfrak{so}(1,n)$. The equation (5.3–1) comes about as follows. The curve $\tilde{h}(t)$ being horizontal implies the following equalities should hold.

$$\begin{aligned} 0 &= \langle (h(t)a(t))', (h(t)a(t))V\rangle \\ &= \langle h'(t)a(t) + h(t)a'(t), (h(t)a(t))V\rangle \\ &= \langle (h(t)a(t)) \left(a(t)^{-1}h(t)^{-1}h'(t)a(t) + a(t)^{-1}a'(t)\right), (h(t)a(t))V\rangle, \end{aligned}$$

for every $V \in \mathfrak{k}$, on the tangent space at $h(t)a(t)$. Since the metric on G is left-invariant, this implies

$$0 = \langle a(t)^{-1}h(t)^{-1}h'(t)a(t) + a(t)^{-1}a'(t), V\rangle,$$

for every $V \in \mathfrak{k}$, on the tangent space at e, $G_e = \mathfrak{g}$. Since this holds for all $V \in \mathfrak{k}$ and the multiplication by any element in K, especially $a(t)^{-1} \in K$, on the right-hand side is also an isometry, by taking conjugation by $a(t)$, the above is equivalent to the equality (5.3–1) above.

We examine the equalities (5.3–1) more closely. The equality holds for every $V \in \mathfrak{k}$ implies that $h(t)^{-1}h'(t) + a'(t)a^{-1}(t)$ does not have any vertical component. That is, $-a'(t)a^{-1}(t)$ is the vertical component of $h(t)^{-1}h'(t)$ so that

$$h(t)^{-1}h'(t) = -a'(t)a^{-1}(t) + X_1 \in \mathfrak{k} \oplus \mathfrak{k}^{\perp}.$$

is a vertical and horizontal splitting.

Let $g(t)$ be another path with a unique horizontal lift $\tilde{g}(t) = g(t)b(t)$, satisfying

$$0 = \langle g^{-1}g' + b'b^{-1}, V\rangle, \tag{5.3–2}$$

for every $V \in \mathfrak{k}$. Again, we have a splitting

$$g(t)^{-1}g'(t) = -b'(t)b^{-1}(t) + X_2 \in \mathfrak{k} \oplus \mathfrak{k}^{\perp}.$$

From

$$||h(t)^{-1}h'(t) - g(t)^{-1}g'(t)|| = ||a'(t)a^{-1}(t) - b'(t)b^{-1}(t)|| + ||X_1 - X_2||,$$

we get

$$||a'(t)a^{-1}(t) - b'(t)b^{-1}(t)|| \leq ||h(t)^{-1}h'(t) - g(t)^{-1}g'(t)||. \tag{5.3–3}$$

These are norms on the Lie algebra $\mathfrak{so}(1,n)$.

On the space of continuous piecewise C^k-curves ($k \geq 1$) in $\mathrm{SO}_0(1,n)$ with initial point e, we define a distance function by

$$\rho(h,g) = \int_0^1 ||h(t)^{-1} \cdot h'(t) - g(t)^{-1} \cdot g'(t)||\, dt.$$

Note that $h(t)^{-1} \cdot h'(t) \in \mathfrak{so}(1,n)$ and $||.||$ is the norm there. We argue that this is a metric. Suppose $\rho(h,g) = 0$. Then, by continuity (on each proper subinterval of $[0,1]$ if needed), $h(t)^{-1} \cdot h'(t) = g(t)^{-1} \cdot g'(t)$ for every t. Now we apply the similar statement of the following Lemma to the C^1-curves piece by piece to conclude $h(t) = g(t)$ for all $t \in [0,1]$ from the continuity of h and g and from translation by right multiplication if needed, see [KN], vol 1, p69. In fact, for $\tilde{h}(t) := h(t_0)^{-1}h(t_0 + t),\ t \in [0, t_1 - t_0]$, we get

$$\tilde{h}'(t) = h(t_0)^{-1}h'(t_0 + t),$$

$$\begin{aligned}
\tilde{h}(t)^{-1}\tilde{h}'(t) &= \tilde{h}(t)^{-1}h(t_0)^{-1}h'(t_0 + t) \\
&= h(t_0 + t)^{-1}h'(t_0 + t) \\
&= h(s)^{-1}h'(s) \quad \text{where } s = t_0 + t \in [t_0,\, t_1]
\end{aligned}$$

and $h(s) = h(t_0)\tilde{h}(s - t_0)$.

Lemma 5.3.1 *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra identified with $T_e(G)$. Let Y_t, $0 \leq t \leq 1$, be a continuous curve in $T_e(G)$. Then there exists in G a unique curve a_t of class C^1 such that $a_0 = e$ and $\dot{a}_t a_t^{-1} = Y_t$ for $0 \leq t \leq 1$.*

Let h be a curve in $\mathbb{H}^n$ (or in NA, by abuse of notation). The unique curve $a : [0,1] \to \mathrm{SO}(n)$ such that $h(t) \cdot a(t)$ is the horizontal lift of $h(t)$ will be called w_h.

For two curves h and g, the inequality (5.3–3) shows that $\rho(w_h, w_g) \leq \rho(h, g)$. Let $\mathfrak{P}$ be the space of all continuous piecewise C^k-curves on NA with the initial point e.

Proposition 5.3.2 *The map $\mathfrak{P} \longrightarrow G$ sending h to $w_h(1)$ is continuous. More precisely, let $h : [0,1] \to NA$ be a piecewise C^k-curve. For every $\epsilon > 0$, there exists $\delta > 0$ such that, if $g \in \mathfrak{P}$ and $\rho(h, g) < \delta$, then $d(e, w_h(1)^{-1} \cdot w_g(1)) = d(w_h(1), w_g(1)) < \epsilon$.*

Proof) For simplicity, we write $w_h(t)$, $w_g(t)$ by $a(t)$, $b(t)$, respectively. Note

$$0 = (bb^{-1})' = b'b^{-1} + b(b^{-1})' \tag{5.3–4}$$

Then, from $a(a^{-1}b)b^{-1} = e$,

$$\begin{aligned} a(a^{-1}b)'b^{-1} &= -a'(a^{-1}b)b^{-1} - a(a^{-1}b)(b^{-1})' \\ &= -a'a^{-1} - b(b^{-1})' \\ &= -a'a^{-1} + b'b^{-1} \quad \text{(from the equality (5.3–4))} \end{aligned}$$

Thus,

$$||a'a^{-1} - b'b^{-1}|| = ||a(a^{-1}b)'b^{-1}||.$$

Observe that $(a^{-1}b)' \in T_{a^{-1}b}(K)$. The left translation L_a and the right translation $R_{b^{-1}}$ maps this vector to a tangent vector at $T_e(K)$. However, both these translations are isometries so that they preserve the norms. We have,

$$||a'a^{-1} - b'b^{-1}|| = ||a(a^{-1}b)'b^{-1}|| = ||(a^{-1}b)'||.$$

Consequently, if

$$\int ||(a^{-1}b)'|| \; dt = \int ||a'a^{-1} - b'b^{-1}|| \; dt$$

is small, the arc-length of the path $a(t)^{-1}b(t)$ is small. Therefore, if $a(0)$ and $b(0)$ were close (or if $a(0) = b(0)$), then $a(1)$ and $b(1)$ are close. This finishes the proof from the inequality (5.3–3).

Remark 5.3.3 *The above proposition can be applied to a general Lie group with Iwasawa decomposition such that both left and right multiplications by K are isometries.*

Since

$$\bar{f}_n(t_0) = \text{ the value , at } t=1, \text{ of the horizontal lifting of } \hat{\gamma}^n_{t_0} \text{ at } e$$

and

$$\hat{\gamma}^n_{t_0} \text{ converges to } \tilde{\gamma}^{n_0}_{t_0} = \tilde{\gamma}_{t_0} \text{ as n goes to } \infty \text{ ,}$$

we get

$$\begin{aligned}\bar{f}(t_0) &= \text{ the value , at } t=1, \text{ of the horizontal lifting of } \tilde{\gamma}_{t_0} \text{ at } e \\ &= \lim_{n\to\infty} \bar{f}_n(t_0).\end{aligned}$$

5.4 Preliminaries for the main proof

Fix $t_0 \in \bigcup_{n=1}^{\infty} D_n - \{0\}$ and find a positive integer $n_0 = \min\,\{n_1 \mid n+1 \geq n_1 \Rightarrow t_0 \in D_n\}$.

Assume $n \geq n_0$.

Note t_0 is not the last element in D_n for $n \geq n_0$. Notice that with respect to totally geodesic triangles, $\hat{c}^n_{t_2^n(t_0)}(1) = {}_1\hat{c}^n_{t_0}(1) = {}_1\hat{c}^{n_0}_{t_0}(1) = \hat{c}^{n_0}_{t_2^{n_0}(t_0)}(1)$ for all $n \geq n_0$, which is the ending point of the $j_n(t_0)$-th triangle in $\hat{A}_n$ and also the starting point of the $(j_n(t_2^n(t_0)) = j_n(t_0)+1)$-th triangle in $\hat{A}_n$ for all $n \geq n_0$.

5.4.1 A new curve $\hat{\mathbf{c}}^{\mathbf{short}}_{\mathbf{t_0}}$ for the comparison of triangles

Define $\hat{c}^{short}_{t_0} : [0,1] \to \mathbb{H}^n$ as the shortest geodesic from $\pi(e) \in \mathbb{H}^n$ to $\hat{c}^n_{t_2^n(t_0)}(1) = \hat{c}^{n_0}_{t_2^{n_0}(t_0)}(1) = {}_1\hat{c}^{n_0}_{t_0}(1) = {}_1\hat{c}^n_{t_0}(1)$, in other words, to the starting point of $(j_n(t_2^n(t_0)) = j_n(t_0)+1)$ -th triangle in $\hat{A}_n$, which is also the ending point of $j_n(t_0)$-th triangle in $\hat{A}_n$. Consider its horizontal lift

$${}_e\hat{c}^{short}_{t_0} : [0,1] \to \mathrm{SO}_0(1,n)$$

at e.

5.4.2 The comparison of $(\mathbf{j_n}(\mathbf{t_2^n}(\mathbf{t_0})) = \mathbf{j_n}(\mathbf{t_0}) + \mathbf{1})$-th totally geodesic triangles

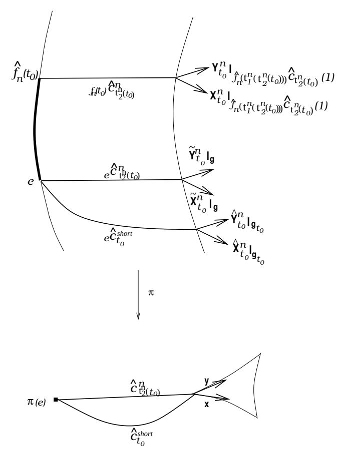
For each $n \geq n_0$, consider

$$ {}_e\hat{c}^n_{t_2^n(t_0)} := R_{(\hat{f}_n(t_0))^{-1}} \circ {}_{\hat{f}_n(t_0)}\hat{c}^n_{t_2^n(t_0)} : [0,1] \to \mathrm{SO}_0(1,n) \,, $$

which is the horizontal lifting of $\hat{c}^n_{t_2^n(t_0)}$ at e, that is ,

$$ \pi \circ {}_e\hat{c}^n_{t_2^n(t_0)} = \hat{c}^n_{t_2^n(t_0)} = \pi \circ {}_{\hat{f}_n(t_0)}\hat{c}^n_{t_2^n(t_0)} \,. $$

Note ${}_e\hat{c}^n_{t_2^n(t_0)}$ and ${}_{\hat{f}_n(t_0)}\hat{c}^n_{t_2^n(t_0)}$ are piecewise geodesics, since the right translation $R_k : G \to G$ by k is an isometry for any $k \in K = \mathrm{SO}(n)$ and $\hat{c}^n_{t_0}$ are piecewise geodesics.



Consider the $(j_n(t_2^n(t_0)) = j_n(t_0) + 1)$-th triangle in $\hat{A}_n$, its starting point and the horizontal lifting of

$$ x := \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) \mid} \cdot \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) $$

and

$$ y := -\lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) \mid} \cdot \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) $$

at ${}_e\hat{c}^n_{t_2^n(t_0)}(1) =: g$, respectively and find

$$ \tilde{X}^n_{t_0}, \tilde{Y}^n_{t_0} \in \mathfrak{k}^\perp $$

with

$$ \pi_* \tilde{X}^n_{t_0} \mid_g = x $$

and

$$\pi_* \tilde{Y}^n_{t_0} \mid_g = y \ .$$

Also consider the horizontal lifting of x and y at ${}_e\hat{c}^{short}_{t_0}(1) =: g_{t_0}$ and find

$$\hat{X}^n_{t_0}, \hat{Y}^n_{t_0} \in \mathfrak{k}^\perp$$

with

$$\pi_* \hat{X}^n_{t_0} \mid_{g_{t_0}} = x$$

and

$$\pi_* \hat{Y}^n_{t_0} \mid_{g_{t_0}} = y \ .$$

Note

$$_{\hat{f}_n(t_0)}\hat{c}^n_{t^n_2(t_0)} \ = \ _{\hat{f}_n(t^n_1(t^n_2(t_0)))}\hat{c}^n_{t^n_2(t_0)} \ ,$$

so

$$\begin{aligned}
\pi_* \ X^n_{t_0} \mid_{\hat{f}_n(t_0)\hat{c}^n_{t^n_2(t_0)}(1)} &= \pi_* \ X^n_{t^n_1(t^n_2(t_0))} \mid_{\hat{f}_n(t^n_1(t^n_2(t_0)))\hat{c}^n_{t^n_2(t_0)}(1)} \\
&= \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\varphi}}^n_{t^n_2(t_0)}(t) \mid} \cdot \dot{\hat{\varphi}}^n_{t^n_2(t_0)}(t) \\
&= x \\
&= \pi_* \tilde{X}^n_{t_0} \mid_g \\
&= \pi_* \tilde{X}^n_{t_0} \mid_{e\hat{c}^n_{t^n_2(t_0)}(1)} \quad ,
\end{aligned}$$

which implies

$$X^n_{t_0} = Ad_{(\hat{f}_n(t_0))^{-1}} \tilde{X}^n_{t_0}$$

from ${}_{\hat{f}_n(t_0)}\hat{c}^n_{t^n_2(t_0)} = R_{\hat{f}_n(t_0)} \circ {}_e\hat{c}^n_{t^n_2(t_0)}$.

Similarly,

$$Y^n_{t_0} = Ad_{(\hat{f}_n(t_0))^{-1}} \tilde{Y}^n_{t_0} \ .$$

And by considering a loop

$$\bar{\hat{c}}^{short}_{t_0} * \hat{c}^n_{t^n_2(t_0)} : [0,1] \to \mathbb{H}^n \ ,$$

where $\bar{\hat{c}}^{short}_{t_0} : [0,1] \to \mathbb{H}^3$ is given by

$$\bar{\hat{c}}^{short}_{t_0}(t) = \hat{c}^{short}_{t_0}(1-t) \ ,$$

and its horizontal lifting at ${}_e\hat{c}^{short}_{t_0}(1)$, we obtain

$$\tilde{X}^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1} \cdot {}_e\hat{c}^n_{t^n_2(t_0)}(1))^{-1}} \hat{X}^n_{t_0}$$

$$\tilde{Y}^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e\hat{c}^n_{t_2^n(t_0)}(1))^{-1}} \hat{Y}^n_{t_0} \,.$$

Then we get

$$X^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e\hat{c}^n_{t_2^n(t_0)}(1)\cdot \hat{f}_n(t_0))^{-1}} \hat{X}^n_{t_0}$$

$$Y^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e\hat{c}^n_{t_2^n(t_0)}(1)\cdot \hat{f}_n(t_0))^{-1}} \hat{Y}^n_{t_0} \,.$$

Since both $({}_e\hat{c}^{short}_{t_0}(1))^{-1} \cdot {}_e\hat{c}^n_{t_2^n(t_0)}(1)$ and $\hat{f}_n(t_0)$ are elements in $K = \mathrm{SO}(n)$, we get

$$[X^n_{t_0}, Y^n_{t_0}] = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e\hat{c}^n_{t_2^n(t_0)}(1)\cdot \hat{f}_n(t_0))^{-1}} [\hat{X}^n_{t_0}, \hat{Y}^n_{t_0}] \,.$$

Note $\hat{c}^n_{t_2^n(t_0)} = {}_1\hat{c}^n_{t_0}$. So we can rewrite $X^n_{t_0}$ and $Y^n_{t_0}$ as

$$X^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t_0))^{-1}} \hat{X}^n_{t_0}$$

$$Y^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t_0))^{-1}} \hat{Y}^n_{t_0} \,.$$

Since ${}_1\hat{c}^n_{t_0}$ converges to ${}_1\tilde{c}_{t_0}$ in $\mathbb{H}^n$ as n, the number of steps (not the dimension of $\mathbb{H}^n$) goes to ∞, ${}_e({}_1\hat{c})^n_{t_0}$ converges to ${}_e({}_1\tilde{c})_{t_0}$ in $\mathrm{SO}_0(1,n)$.

Since

$$\hat{f}_n(t_0) = \text{ the value , at } t=1, \text{ of the horizontal lifting of } \hat{\gamma}^n_{t_0} \text{ at } e$$

and

$$\hat{\gamma}^n_{t_0} \text{ converges to } \tilde{\gamma}^{n_0}_{t_0} = \tilde{\gamma}_{t_0} \text{ as n goes to } \infty \,,$$

we get, from Proposition 5.3.2,

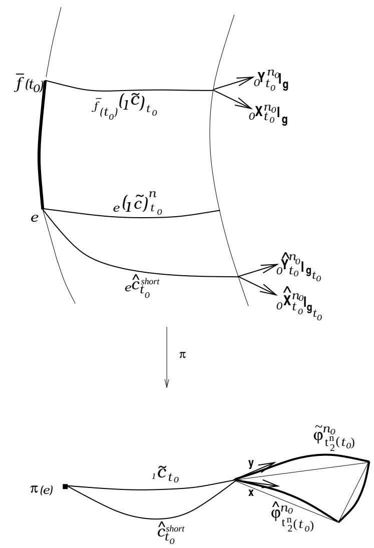
$$\begin{aligned} \bar{f}(t_0) &= \text{ the value , at } t=1, \text{ of the horizontal lifting of } \tilde{\gamma}_{t_0} \text{ at } e \\ &= \lim_{n\to\infty} \hat{f}_n(t_0). \end{aligned}$$

Then we also get

$${}_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0} = R_{\hat{f}_n(t_0)} \circ {}_e({}_1\hat{c})^n_{t_0} = {}_e({}_1\hat{c})^n_{t_0} \cdot \hat{f}_n(t_0),$$

which will converge to

$${}_e({}_1\tilde{c})_{t_0} \cdot \bar{f}(t_0) = R_{\bar{f}(t_0)} \circ {}_e({}_1\tilde{c})_t = {}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0} \,.$$



5.4.3 The comparison of $(\mathbf{j_n}(\mathbf{t_2^n}(\mathbf{t_0})) = \mathbf{j_n}(\mathbf{t_0}) + \mathbf{1})$-th triangles on the given surface S

Now, consider the $(j_{n_0}(t_2^{n_0}(t_0)) = j_{n_0}(t_0)+1)$-th triangle, lying on S, in $\tilde{A}_{n_0}$, its starting point and the horizontal lifting of

$$x := \lim_{t\to 0^+} \frac{1}{\mid \dot{\tilde{\varphi}}^{n_0}_{t_2^{n_0}(t_0)}(t) \mid} \cdot \dot{\tilde{\varphi}}^{n_0}_{t_2^{n_0}(t_0)}(t)$$

and

$$y := -\lim_{t\to 1^-} \frac{1}{\mid \dot{\tilde{\varphi}}^{n_0}_{t_2^{n_0}(t_0)}(t) \mid} \cdot \dot{\tilde{\varphi}}^{n_0}_{t_2^{n_0}(t_0)}(t)$$

at $g_{t_0} = {}_e\hat{c}^{short}_{t_0}(1)$ and at $g := {}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)$, respectively, and find

$${}_0\hat{X}^{n_0}_{t_0},\ {}_0\hat{Y}^{n_0}_{t_0},\ {}_0X^{n_0}_{t_0},\ {}_0Y^{n_0}_{t_0} \in \mathfrak{k}^{\perp}$$

with

$$\pi_*\, {}_0\hat{X}^{n_0}_{t_0} \mid_{g_{t_0}} = x = \pi_*\, {}_0X^{n_0}_{t_0} \mid_g$$

and

$$\pi_*\, {}_0\hat{Y}^{n_0}_{t_0} \mid_{g_{t_0}} = y = \pi_*\, {}_0Y^{n_0}_{t_0} \mid_g \ .$$

Then,

$$g = {}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1) = {}_e({}_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0) = g_{t_0}\cdot({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot{}_e({}_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0)$$

implies that

$${}_0X^{n_0}_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot{}_e({}_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0))^{-1}}\, {}_0\hat{X}^{n_0}_{t_0}$$

$${}_0Y^{n_0}_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot{}_e({}_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0))^{-1}}\, {}_0\hat{Y}^{n_0}_{t_0} \ .$$

5.4.4 The convergence of tangent planes induced by $(\mathbf{j_n}(\mathbf{t_2^n}(\mathbf{t_0})) = \mathbf{j_n}(\mathbf{t_0}) + \mathbf{1})$-th triangles at $\mathbf{t} = \mathbf{t_0}$ and the convergence of $\frac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t)$ under $\lim\limits_{\mathbf{n}\to\infty} \lim\limits_{\mathbf{t}\to\mathbf{t_0}^+}$

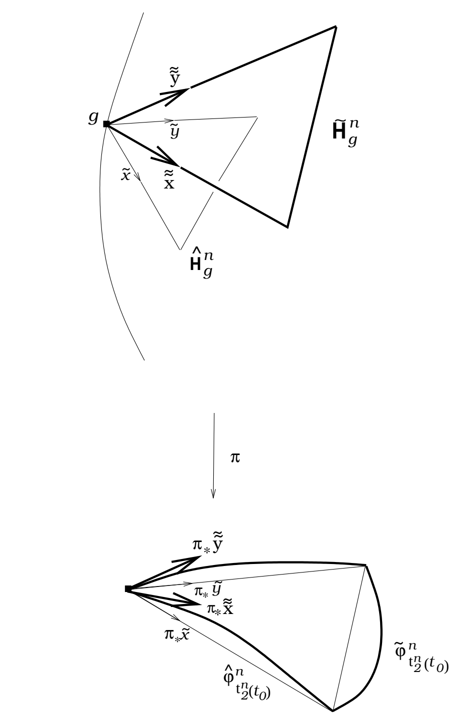
Now, for $g \in \pi^{-1}(\pi({}_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1)))$, let

$$\hat{H}^n_g := \mathrm{Span}\{\tilde{x}, \tilde{y}\} ,$$

where $\tilde{x}, \tilde{y}$ are horizontal vectors at g satisfying

$$\pi_* \, \tilde{x} = \lim_{t\to 0^+} \frac{1}{| \, \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) \, |} \cdot \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t)$$

$$\pi_* \, \tilde{y} = -\lim_{t\to 1^-} \frac{1}{| \, \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) \, |} \cdot \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) \, .$$



Also, for $g \in \pi^{-1}(\pi({}_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1)))$, let

$$\tilde{H}^n_g := \mathrm{Span}\{\tilde{\tilde{x}}, \tilde{\tilde{y}}\} ,$$

where $\tilde{\tilde{x}}, \tilde{\tilde{y}}$ are horizontal vectors at g satisfying

$$\pi_* \, \tilde{\tilde{x}} = \lim_{t\to 0^+} \frac{1}{| \, \dot{\tilde{\varphi}}^n_{t_2^n(t_0)}(t) \, |} \cdot \dot{\tilde{\varphi}}^n_{t_2^n(t_0)}(t)$$

$$\pi_* \, \tilde{\tilde{y}} = -\lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t) \mid} \cdot \dot{\hat{\varphi}}^n_{t_2^n(t_0)}(t).$$

Note, for $t \in (t_0,\ t_2^n(t_0)) = (t_1^n(t_2^n(t_0)),\ t_2^n(t_0))$, from Subsection 5.2.2,

$\omega\ \big(\ \frac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t)\ \big)$

$= \ (-1) \cdot \big(\$ the unit curvature of the 2-dimensional horizontal oriented tangent plane,

$\hat{H}^n_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1) \ = \ \hat{H}^n_{\hat{f}_n(t_1^n(t_2^n(t_0)))}\hat{c}^n_{t_2^n(t_0)}(1)$

$= \mathrm{Span}\{X^n_{t_1^n(t_2^n(t_0))} \mid_{\hat{f}_n(t_1^n(t_2^n(t_0)))}\hat{c}^n_{t_2^n(t_0)}(1),\ Y^n_{t_1^n(t_2^n(t_0))} \mid_{\hat{f}_n(t_1^n(t_2^n(t_0)))}\hat{c}^n_{t_2^n(t_0)}(1)\}$

$= \mathrm{Span}\{X^n_{t_0} \mid_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1),\ Y^n_{t_0} \mid_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1)\}$

at ${}_{\hat{f}_n(t_1^n(t_2^n(t_0)))}\hat{c}^n_{t_2^n(t_0)}(1) = {}_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1)$,

which projects to the tangent plane

of the $(j_n(t_2^n(t_0)) = j_n(t_0)+1)$-th triangle in $\hat{A}_n$

at $\pi({}_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1)) = {}_1\hat{c}^n_{t_0}(1) = \hat{c}^n_{t_2^n(t_0)}(1)$

$=$ the starting point of the $(j_n(t_2^n(t_0)) = j_n(t_0)+1)$-th triangle in $\hat{A}_n$

with respect to the connection of the principal bundle $\pi : \mathrm{SO}_0(1,n) \to \mathbb{H}^n\big)$

$= \frac{1}{|[X^n_{t_1^n(t_2^n(t_0))},\, Y^n_{t_1^n(t_2^n(t_0))}]|} \cdot \ [X^n_{t_1^n(t_2^n(t_0))},\ Y^n_{t_1^n(t_2^n(t_0))}]$

$= \frac{1}{|[X^n_{t_0},\, Y^n_{t_0}]|} \cdot \ [X^n_{t_0},\ Y^n_{t_0}]$

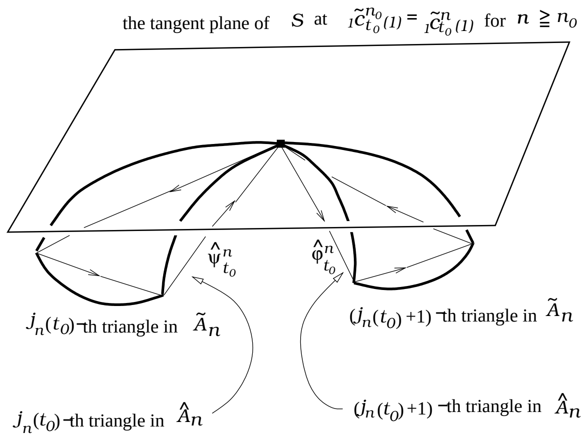
$= \frac{Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t_0))^{-1}}[\hat{X}^n_{t_0},\hat{Y}^n_{t_0}]}{|Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t_0))^{-1}}[\hat{X}^n_{t_0},\hat{Y}^n_{t_0}]|}$

$= Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t_0))^{-1}}\Big(\frac{1}{|[\hat{X}^n_{t_0},\hat{Y}^n_{t_0}]|} \cdot [\hat{X}^n_{t_0},\hat{Y}^n_{t_0}]\Big)$

$= (-1) \cdot Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t_0))^{-1}}\big($ the unit curvature of the 2-dimensional horizontal oriented tangent plane,

$\hat{H}^n_{{}_e\hat{c}^{short}_{t_0}(1)} = \mathrm{Span}\{\hat{X}^n_{t_0} \mid_{{}_e\hat{c}^{short}_{t_0}(1)}, \hat{Y}^n_{t_0} \mid_{{}_e\hat{c}^{short}_{t_0}(1)}\}\big)$

Note the tangent plane of the $(j_n(t_2^n(t_0)) = j_n(t_0)+1)$-th triangle in $\hat{A}_n$ at ${}_1\hat{c}^n_{t_0}(1) = \hat{c}^n_{t_2^n(t_0)}(1) = \hat{c}^{n_0}_{t_2^n(t_0)}(1) = {}_1\hat{c}^{n_0}_{t_0}(1) = \tilde{c}^{n_0}_{t_2^n(t_0)}(1) = {}_1\tilde{c}^{n_0}_{t_0}(1)$ for all $n \geq n_0$, the starting point of the $(j_n(t_2^n(t_0)) = j_n(t_0)+1)$-th triangle in $\hat{A}_n$ and the ending point of the $j_n(t_0)$-th triangle in $\hat{A}_n$ for all $n \geq n_0$ at the same time, which is also the starting point of the $(j_{n_0}(t_2^{n_0}(t_0)) = j_{n_0}(t_0)+1)$-th triangle, lying on S,



in $\tilde{A}_{n_0}$ and the ending point of the $j_{n_0}(t_0)$-th triangle in $\tilde{A}_{n_0}$ at the same time, will converge to the tangent plane of S at $\tilde{c}^{n_0}_{t_2^n(t_0)}(1) = {}_1\tilde{c}^{n_0}_{t_0}(1)$.

And, note, in general, if $\lim_{n\to\infty} g_n = g_{_0}$ in G and $\lim_{n\to\infty} X_n = X_0$ in $\mathfrak{g}$, then

$$\begin{aligned}
\lim_{n\to\infty} t \cdot Ad_{g_n} X_n &= \lim_{n\to\infty} \exp^{-1}(\exp(t \cdot Ad_{g_n} x_n)) \\
&= \lim_{n\to\infty} \exp^{-1}(g_n \cdot \exp(t \cdot X_n) \cdot {g_n}^{-1}) \\
&= \exp^{-1}(g_{_0} \cdot \exp(t \cdot X_0) \cdot {g_{_0}}^{-1}) \\
&= \exp^{-1}(\exp(t \cdot Ad_{g_{_0}} X_0)) \\
&= t \cdot Ad_{g_{_0}} X_0 \; .
\end{aligned}$$

Now, refer to previous three pictures. Then we get, for $t \in \big(t_0, t_2^n(t_0)\big)$,

$$\begin{aligned}
&\omega \left(\tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t) \right) \\
&= \tfrac{1}{|[X^n_{t_0},\, Y^n_{t_0}]|} \cdot \; [X^n_{t_0},\; Y^n_{t_0}] \\
&= Ad_{((_e\hat{c}^{short}_{t_0}(1))^{-1} \cdot {}_e({}_1\hat{c})^n_{t_0}(1) \cdot \hat{f}_n(t_0))^{-1}} \Big(\tfrac{1}{|[\hat{X}^n_{t_0}, \hat{Y}^n_{t_0}]|} \cdot [\hat{X}^n_{t_0}, \hat{Y}^n_{t_0}] \Big) \\
&= (-1) \cdot Ad_{((_e\hat{c}^{short}_{t_0}(1))^{-1} \cdot {}_e({}_1\hat{c})^n_{t_0}(1) \cdot \hat{f}_n(t_0))^{-1}} \big(\text{ the unit curvature of the 2-dimensional} \\
&\qquad\qquad \text{horizontal oriented tangent plane,} \\
&\qquad\qquad \hat{H}^n_{_e\hat{c}^{short}_{t_0}(1)} = \mathrm{Span}\{\hat{X}^n_{t_0} \mid_{_e\hat{c}^{short}_{t_0}(1)}, \hat{Y}^n_{t_0} \mid_{_e\hat{c}^{short}_{t_0}(1)}\} \big),
\end{aligned}$$

which will converge to

$$\begin{aligned}
&(-1) \cdot Ad_{((_e\hat{c}^{short}_{t_0}(1))^{-1} \cdot {}_e({}_1\tilde{c})_{t_0}(1) \cdot \bar{f}(t_0))^{-1}} \big(\text{ the unit curvature of the 2-dimensional} \\
&\qquad\qquad \text{horizontal oriented tangent plane,}
\end{aligned}$$

$$\tilde{H}^{n_0}_{{}_e\hat{c}^{short}_{t_0}(1)} = \mathrm{Span}\{{}_0\hat{X}^{n_0}_{t_0}\ |_{{}_e\hat{c}^{short}_{t_0}(1)},\ {}_0\hat{Y}^{n_0}_{t_0}\ |_{{}_e\hat{c}^{short}_{t_0}(1)}\}\Big)$$

$= Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot{}_e({}_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0))^{-1}}\left(\frac{1}{|[{}_0\hat{X}^{n_0}_{t_0},\ {}_0\hat{Y}^{n_0}_{t_0}]|}\cdot[{}_0\hat{X}^{n_0}_{t_0},\ {}_0\hat{Y}^{n_0}_{t_0}]\right)$
$= \frac{1}{|[{}_0X^{n_0}_{t_0},\ {}_0Y^{n_0}_{t_0}]|}\cdot\ [{}_0X^{n_0}_{t_0},\ {}_0Y^{n_0}_{t_0}]$

$=\ (-1)\cdot\Big($ the unit curvature of the 2-dimensional horizontal oriented tangent
plane, $\tilde{H}^{n_0}_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)}\ =\ \mathrm{Span}\{{}_0X^{n_0}_{t_0}\ |_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)},\ {}_0Y^{n_0}_{t_0}\ |_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)}\}$
at ${}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)$,
which projects to the tangent plane
of the $(j_{n_0}(t_2^{n_0}(t_0)) = j_{n_0}(t_0)+1)$-th triangle in $\tilde{A}_{n_0}$,
where $\quad n_0 =\ \min\ \{n_1\ |\ n+1\geq n_1 \Rightarrow t_0\in D_n\}$,
- so tangent to the given disk S -
at $\pi({}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)) = {}_1\tilde{c}_{t_0}(1) = \tilde{c}^{n_0}_{t_2^{n_0}(t_0)}(1)$
$=$ the starting point of the $(j_{n_0}(t_2^{n_0}(t_0)) = j_{n_0}(t_0)+1)$-th triangle in $\tilde{A}_{n_0}$
with respect to the connection of the principal bundle $\pi:\mathrm{SO}_0(1,n)\to\mathbb{H}^n\Big)$

under $\lim\limits_{n\to\infty}\lim\limits_{t\to t_0{}^+}$.

So, under the identification of T_eK with $\mathfrak{k}$,
$\lim\limits_{n\to\infty}\lim\limits_{t\to t_0{}^+} L_{(\hat{f}_n(t)^{-1})_*}\ \frac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)$
$=\ \lim\limits_{n\to\infty}\frac{1}{|\ [X^n_{t_0}, Y^n_{t_0}]\ |}\cdot[X^n_{t_0}, Y^n_{t_0}]$
$=\ \ \frac{1}{|\ [{}_0X^{n_0}_{t_0}, {}_0Y^{n_0}_{t_0}]\ |}\cdot[{}_0X^{n_0}_{t_0}, {}_0Y^{n_0}_{t_0}]$

$=\ (-1)\cdot\Big($ the unit curvature of the 2-dimensional horizontal oriented tangent
plane, $\tilde{H}^{n_0}_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)}$ at ${}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)$,
which projects to the tangent plane of S at $\pi({}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)) = {}_1\tilde{c}_{t_0}(1)$
with respect to the connection of the principal bundle $\pi:\mathrm{SO}_0(1,n)\to\mathbb{H}^n\Big)$

5.4.5 The convergence of $\frac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)$

To show

$$\lim_{n\to\infty}\lim_{t\to t_0{}^+} L_{(\hat{f}_n(t)^{-1})_*}\ \frac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\ =\ \lim_{n\to\infty}\lim_{t\to t_0{}^-} L_{(\hat{f}_n(t)^{-1})_*}\ \frac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\ ,$$

for $g \in \pi^{-1}(\pi({}_{\hat{f}_n(t_1^n(t_0))}({}_1\hat{c})^n_{t_0}(1))) = \pi^{-1}(\pi({}_{\hat{f}_n(t_0)}({}_1\hat{c})^n_{t_0}(1)))$, let

$$ {}_1\hat{H}^n_g := \text{Span}\{\tilde{x}, \tilde{y}\} , $$

where $\tilde{x}, \tilde{y}$ are horizontal vectors at g satisfying

$$ \pi_* \, \tilde{x} = \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) $$

$$ \pi_* \, \tilde{y} = - \lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) \, . $$

Also, for $g \in \pi^{-1}(\pi({}_{\hat{f}_n(t_1^n(t_0))}({}_1\hat{c})^n_{t_0}(1)))$, let

$$ {}_1\tilde{H}^n_g := \text{Span}\{\tilde{\tilde{x}}, \tilde{\tilde{y}}\} , $$

where $\tilde{\tilde{x}}, \tilde{\tilde{y}}$ are horizontal vectors at g satisfying

$$ \pi_* \, \tilde{\tilde{x}} = \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) $$

$$ \pi_* \, \tilde{\tilde{y}} = - \lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) \, . $$

Now, consider the horizontal lifting of

$$ z := \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) $$

and

$$ w := - \lim_{t \to 1^-} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) $$

at $g := {}_{\hat{f}_n(t_1^n(t_0))}({}_1\hat{c})^n_{t_0}(1) = {}_{\hat{f}_n(t_1^n(t_0))}\hat{c}^n_{t_2^n(t_0)}(1)$, respectively and find

$$ Z^n_{t_0}, \; W^n_{t_0} \in \mathfrak{k}^\perp $$

with

$$ \pi_* \, Z^n_{t_0} \mid_g = z $$

and

$$ \pi_* \, W^n_{t_0} \mid_g = w \, . $$

Also consider the horizontal lifting of z and w at ${}_e({}_1\hat{c})^n_{t_0}(1) = {}_e\hat{c}^n_{t_2^n(t_0)}(1) =: g$, respectively and find

$$ \tilde{Z}^n_{t_0}, \; \tilde{W}^n_{t_0} \in \mathfrak{k}^\perp $$

with

$$\pi_* \ \tilde{Z}^n_{t_0} \mid_g = z$$

and

$$\pi_* \ \tilde{W}^n_{t_0} \mid_g = w \ .$$

And consider the horizontal lifting of z and w at ${}_e\hat{c}^{short}_{t_0}(1) =: g_{t_0}$ and find

$$\hat{Z}^n_{t_0}, \ \hat{W}^n_{t_0} \in \mathfrak{k}^\perp$$

with

$$\pi_* \ \hat{Z}^n_{t_0} \mid_{g_{t_0}} = z$$

and

$$\pi_* \ \hat{W}^n_{t_0} \mid_{g_{t_0}} = w \ .$$

Note $\mathrm{Im}\varphi^n_{t_0} = \ \mathrm{Im}\psi^n_{t_0}$ is the boundary of a geodesic triangle in $\mathbb{H}^n$.

Then, from Facts, mentioned earlier in this chapter, and from the property in Chapter 4, we get

$$\begin{aligned}\hat{f}_n(t) &= \hat{f}_n(t^n_1(t_0)) \cdot \ \exp\left((t - t^n_1(t_0)) \cdot \frac{(\text{ Area of } j_n(t_0)\text{-th triangle in } \hat{A}_n)}{((t_0 - t^n_1(t_0)) \cdot \mid [X^n_{t^n_1(t_0)}, Y^n_{t^n_1(t_0)}] \mid)} \cdot [X^n_{t^n_1(t_0)}, Y^n_{t^n_1(t_0)}]\right) \\ &= \hat{f}_n(t^n_1(t_0)) \cdot \ \exp\left((t - t^n_1(t_0)) \cdot \frac{(\text{ Area of } j_n(t_0)\text{-th triangle in } \hat{A}_n)}{((t_0 - t^n_1(t_0)) \cdot \mid [Z^n_{t_0}, W^n_{t_0}] \mid)} \cdot [Z^n_{t_0}, W^n_{t_0}]\right)\end{aligned}$$

for $t \in [t^n_1(t_0), t_0]$.

Note

$$\begin{aligned}\pi_* \ Z^n_{t_0} \mid_{\hat{f}_n(t^n_1(t_0))}({}_1\hat{c})^n_{t_0}(1) &= \lim_{t \to 0^+} \frac{1}{\mid \dot{\hat{\psi}}^n_{t_0}(t) \mid} \cdot \dot{\hat{\psi}}^n_{t_0}(t) \\ &= z \\ &= \pi_* \ \tilde{Z}^n_{t_0} \mid_{e({}_1\hat{c})^n_{t_0}(1)} \ ,\end{aligned}$$

which implies

$$Z^n_{t_0} = Ad_{(\hat{f}_n(t^n_1(t_0)))^{-1}} \tilde{Z}^n_{t_0}$$

from ${}_{\hat{f}_n(t^n_1(t_0))}({}_1\hat{c})^n_{t_0} = R_{\hat{f}_n(t^n_1(t_0))} \circ {}_e({}_1\hat{c})^n_{t_0}$.

Similarly,

$$W^n_{t_0} = Ad_{(\hat{f}_n t^n_1(t_0))^{-1}} \tilde{W}^n_{t_0} \ .$$

And by considering a loop

$$\bar{\hat{c}}^{short}_{t_0} * {}_1\hat{c}^n_{t_0} : [0,1] \to \mathbb{H}^n \,,$$

where $\bar{\hat{c}}^{short}_{t_0} : [0,1] \to \mathbb{H}^n$ is given by

$$\bar{\hat{c}}^{short}_{t_0}(t) = \hat{c}^{short}_{t_0}(1-t) \,,$$

and its horizontal lifting at ${}_e\hat{c}^{short}_{t_0}(1)$, we obtain

$$\tilde{Z}^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1))^{-1}} \hat{Z}^n_{t_0}$$

$$\tilde{W}^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1))^{-1}} \hat{W}^n_{t_0} \,.$$

Then we get

$$Z^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t^n_1(t_0)))^{-1}} \hat{Z}^n_{t_0}$$

$$W^n_{t_0} = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t^n_1(t_0)))^{-1}} \hat{W}^n_{t_0} \,.$$

Since both $({}_e\hat{c}^{short}_{t_0}(1))^{-1} \cdot {}_e({}_1\hat{c})^n_{t_0}(1)$ and $\hat{f}_n(t^n_1(t_0))$ are elements in $K = \mathrm{SO}(n)$, we get

$$[Z^n_{t_0}, W^n_{t_0}] = Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t^n_1(t_0)))^{-1}} [\hat{Z}^n_{t_0}, \hat{W}^n_{t_0}] \,.$$

Note the tangent plane of the $j_n(t_0)$-th triangle in $\hat{A}_n$ at ${}_1\hat{c}^n_{t_0}(1) = \hat{c}^n_{t^n_2(t_0)}(1) = \hat{c}^{n_0}_{t^n_2(t_0)}(1) = {}_1\hat{c}^{n_0}_{t_0}(1) = \tilde{c}^{n_0}_{t^n_2(t_0)}(1) = {}_1\tilde{c}^{n_0}_{t_0}(1)$ for all $n \geq n_0$, the starting point of the $(j_n(t^n_2(t_0)) = j_n(t_0)+1)$-th triangle in $\hat{A}_n$ and also the ending point of the $j_n(t_0)$-th triangle in $\hat{A}_n$ for all $n \geq n_0$, which is also the starting point of the $(j_{n_0}(t^{n_0}_2(t_0)) = j_{n_0}(t_0)+1)$-th triangle, lying on S, in $\tilde{A}_{n_0}$ and also the ending point of the $j_{n_0}(t_0)$-th triangle in $\tilde{A}_{n_0}$, will converge to the tangent plane of S at $\tilde{c}^{n_0}_{t^n_2(t_0)}(1) = {}_1\tilde{c}^{n_0}_{t_0}(1)$, which implies that for $t \in \big(t^n_1(t_0), t_0\big)$

$$\begin{aligned}
&\omega\,\Big(\, \tfrac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t)\,\Big) \\
&= \tfrac{1}{|[Z^n_{t_0}, W^n_{t_0}]|} \cdot\ [Z^n_{t_0},\ W^n_{t_0}] \\
&= Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t^n_1(t_0)))^{-1}} \Big(\tfrac{1}{|[\hat{Z}^n_{t_0}, \hat{W}^n_{t_0}]|} \cdot [\hat{Z}^n_{t_0}, \hat{W}^n_{t_0}] \Big) \\
&= (-1)\cdot Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\hat{c})^n_{t_0}(1)\cdot \hat{f}_n(t^n_1(t_0)))^{-1}} \big(\text{ the unit curvature of the 2-dimensional} \\
&\qquad\qquad \text{horizontal oriented tangent plane,} \\
&\qquad\qquad {}_1\hat{H}^n_{{}_e\hat{c}^{short}_{t_0}(1)} = \mathrm{Span}\{\hat{Z}^n_{t_0}\,|_{{}_e\hat{c}^{short}_{t_0}(1)}, \hat{W}^n_{t_0}\,|_{{}_e\hat{c}^{short}_{t_0}(1)}\}\big)
\end{aligned}$$

will converge to

$$(-1)\cdot Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\tilde{c})_{t_0}(1)\cdot \bar{f}(t_0))^{-1}}\Big(\text{ the unit curvature of the 2-dimensional horizontal oriented tangent plane, } \tilde{H}^{n_0}_{{}_e\hat{c}^{short}_{t_0}(1)} = \mathrm{Span}\{{}_0\hat{X}^{n_0}_{t_0}\ |_{{}_e\hat{c}^{short}_{t_0}(1)},\ {}_0\hat{Y}^{n_0}_{t_0}\ |_{{}_e\hat{c}^{short}_{t_0}(1)}\}\Big)$$

$$= Ad_{(({}_e\hat{c}^{short}_{t_0}(1))^{-1}\cdot {}_e({}_1\tilde{c})_{t_0}(1)\cdot \bar{f}(t_0))^{-1}}\left(\tfrac{1}{|[{}_0\hat{X}^{n_0}_{t_0},\,{}_0\hat{Y}^{n_0}_{t_0}]|}\cdot[{}_0\hat{X}^{n_0}_{t_0},\ {}_0\hat{Y}^{n_0}_{t_0}]\right)$$
$$= \tfrac{1}{|[{}_0X^{n_0}_{t_0},\,{}_0Y^{n_0}_{t_0}]|}\cdot\ [{}_0X^{n_0}_{t_0},\ {}_0Y^{n_0}_{t_0}]$$

$= (-1)\cdot\big($ the unit curvature of the 2-dimensional horizontal oriented tangent plane, $\tilde{H}^{n_0}_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)} = \mathrm{Span}\{{}_0X^{n_0}_{t_0}\ |_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)},\ {}_0Y^{n_0}_{t_0}\ |_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)}\}$
at ${}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)$,
which projects to the tangent plane
of the $(j_{n_0}(t_2^{n_0}(t_0)) = j_{n_0}(t_0)+1)$-th triangle in $\tilde{A}_{n_0}$,
where $n_0 =$ min $\{n_1 \mid n+1 \geq n_1 \Rightarrow t_0 \in D_n\}$,
- so tangent to the given disk S -
at $\pi({}_{\bar{f}(t_0)}({}_1\tilde{c})_{t_0}(1)) = {}_1\tilde{c}_{t_0}(1) = \tilde{c}^{n_0}_{t_2^{n_0}(t_0)}(1)$
= the starting point of the $(j_{n_0}(t_2^{n_0}(t_0)) = j_{n_0}(t_0)+1)$-th triangle in $\tilde{A}_{n_0}$
with respect to the connection of the principal bundle $\pi : \mathrm{SO}_0(1,n) \to \mathbb{H}^n\big)$

Thus we get

$$\begin{aligned}
\lim_{n\to\infty}\lim_{t\to t_0^-} L_{(\hat{f}_n(t)^{-1})_*}\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\right) &= \lim_{n\to\infty}\frac{1}{|\ [Z^n_{t_0},W^n_{t_0}]\ |}\cdot[Z^n_{t_0},W^n_{t_0}] \\
&= \frac{1}{|\ [{}_0X^{n_0}_{t_0},{}_0Y^{n_0}_{t_0}]\ |}\cdot[{}_0X^{n_0}_{t_0},{}_0Y^{n_0}_{t_0}] \\
&= \lim_{n\to\infty}\lim_{t\to t_0^+} L_{(\hat{f}_n(t)^{-1})_*}\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\right) \quad .
\end{aligned}$$

5.4.6 Main Part

Define a function $s_n : D_n - \{0\} \to (0,\infty)$ as follows :
Given $t \in D_n - \{0\}$, assume t is the j-th element in D_n, *i.e.*, $j = j_n(t)$. Then,

$$
\begin{aligned}
s_n(t) :=& \sum_{i=1}^{j=j_n(t)} (\text{ the area of i-th triangle in } \hat{A}_n) \\
=& \text{ the area of the region surrounded by } \hat{\gamma}_t^n \text{ in the } n\text{-th step polytope.}
\end{aligned}
$$

Note in S, for $n \geq n_0$,

$$\text{the region surrounded by } \tilde{\gamma}_{t_0}^n \text{ in } S = \text{ the region surrounded by } \tilde{\gamma}_{t_0}^{n_0} \text{ in } S \ ,$$

so we get

$$\lim_{n\to\infty} s_n(t_0) = \text{ the area of the region surrounded by } \tilde{\gamma}_{t_0}^{n_0} \text{ in } S =: s(t_0) \ .$$

Thus, we obtain a function

$$s : \cup_{n=1}^{\infty} D_n - \{0\} \to (0, \infty) \ .$$

Now, induce a function

$$f_n \ : \ [0, \text{ the area of the } n\text{-step polytope}] \to K \ ,$$

which is the reparametrization of $\hat{f}_n$ with $|\, \dot{f}_n(t) \,| = 1$ on
$[0, \text{ the area of the } n\text{-step polytope}] \ -$
$\left\{ \sum_{i=1}^{j} (\text{ the area of the } i\text{-th triangle in } \hat{A}_n) \ | \ j = 1, 2, \cdots, |\hat{A}_n| \right\}$.
Then we get

$$f_n(s_n(t)) \ = \ \hat{f}_n(t) = \bar{f}_n(t) \qquad \text{for } t \in D_n - \{0\} \ .$$

Define a function

$$f : \{s(t) \mid t \in \cup_{n=1}^{\infty} D_n - \{0\}\} \to K$$

by

$$f(s(t)) = \bar{f}(t) \ .$$

Then we get

$$f(s(t_0)) = \bar{f}(t_0) = \lim_{n\to\infty} \bar{f}_n(t_0) = \lim_{n\to\infty} f_n(s_n(t_0)) \ .$$

Note, for $t_1 \in \cup_{n=1}^{\infty} D_n - \{0\}$,

$$
\begin{aligned}
\bar{f}(t_1) =& \lim_{n\to\infty} \bar{f}_n(t_1) \\
=& \lim_{n\to\infty} (\text{the value, at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}_{t_1}^n \text{ at } e) \\
=& \text{ the value, at } t = 1, \text{ of the horizontal lifting of } \tilde{\gamma}_{t_1} \text{ at } e \ .
\end{aligned}
$$

Since $\tilde{\gamma}_{t_1}$ converges to $\tilde{\gamma}_{t_0}$ as t_1 approaches t_0 in $\cup_{n=1}^{\infty} D_n$, Proposition 5.3.2 implies that $\bar{f}$ will be continuous on $\bigcup_{n=1}^{\infty} D_n - \{0\}$ and we can extend $\bar{f}$ on [0,1]. And from $f(s(t_0)) = \bar{f}(t_0)$, f will be continuous on $\left\{s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\}\right\}$.

Note s is continuous on $\bigcup_{n=1}^{\infty} D_n - \{0\}$ and so it can be extended on [0,1].

Since $\left\{s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\}\right\}$ is a dense subset of [0, the area of S], we can extend f on [0, the area of S] continuously. Call it f as well. Then we get

$$f \circ s = \bar{f} \text{ is continuous on } [0,1]$$

and

$$f(\text{ the area of } S) = \bar{f}(1) = \lim_{t \to 1, t \in \bigcup_{n=1}^{\infty} D_n} {}_e\tilde{\gamma}_t(1) = {}_e\tilde{\gamma}(1),$$

where $\tilde{\gamma} : [0,1] \to S$ is the boundary curve of S and ${}_e\tilde{\gamma}$ is its horizontal lifting at e.

Now, we show f is a C^1 curve.

Define a function F_n from
[0, the area of the n-step polytope] $-$
$$\left\{ \sum_{i=1}^{j} (\text{ the area of the } i\text{-th triangle in } \hat{A}_n) \mid j = 1, 2, \cdots, |\hat{A}_n| \right\}$$
to

$$\text{the unit sphere in } \mathfrak{k}$$

by

$$F_n(t) = L_{(f_n(t))^{-1}*} \dot{f}_n(t) .$$

And define a function

$$F : \left\{s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\}\right\} \to \text{ the unit sphere in } \mathfrak{k}$$

by

$$F(s(t_0)) = \lim_{n\to\infty} \lim_{t\to t_0^-} L_{(\hat{f}_n(t)^{-1})*} \left(\frac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t)\right) = \lim_{n\to\infty} \lim_{t\to t_0^+} L_{(\hat{f}_n(t)^{-1})*} \left(\frac{1}{|\dot{\hat{f}}_n(t)|} \cdot \dot{\hat{f}}_n(t)\right).$$

Then, F_n is constant on the interval

$$(0, \text{ the area of the first triangle in } \hat{A}_n)$$

and on the interval

$$\left(\sum_{i=1}^{j}(\text{the area of the } i\text{-th triangle in } \hat{A}_n)\ ,\ \sum_{i=1}^{j+1}(\text{the area of the } i\text{-th triangle in } \hat{A}_n)\right)$$

for each $j = 1, 2, \cdots, |\hat{A}_n|$, and

$$\begin{aligned}
F(s(t_0)) &= \lim_{n\to\infty}\lim_{t\to t_0{}^-} L_{(\hat{f}_n(t)^{-1})_*}\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\right) = \lim_{n\to\infty}\lim_{t\to t_0{}^+} L_{(\hat{f}_n(t)^{-1})_*}\left(\tfrac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\right) \\
&= \lim_{n\to\infty}\ \lim_{t\to s_n(t_0)^-} L_{(f_n(t))^{-1}{}_*}\ \dot{f}_n(t) \ = \lim_{n\to\infty}\ \lim_{t\to s_n(t_0)^+} L_{(f_n(t))^{-1}{}_*}\ \dot{f}_n(t) \\
&= \lim_{n\to\infty}\ \lim_{t\to s_n(t_0)}\ L_{(f_n(t))^{-1}{}_*}\ \dot{f}_n(t) \\
&= \lim_{n\to\infty}\ \lim_{t\to s_n(t_0)}\ F_n(t)\ .
\end{aligned}$$

Also

$F(s(t_0))$

$= \lim_{n\to\infty}\lim_{t\to t_0}\omega\left(\ \tfrac{1}{|\dot{\hat{f}}_n(t)|}\cdot\dot{\hat{f}}_n(t)\ \right)$

$= (-1)\cdot\Big($ the unit curvature of the 2-dimensional horizontal oriented tangent plane,

$$\tilde{H}^{n_0}_{e(_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0)} = \tilde{H}^{n_0}_{\bar{f}(t_0)}{}_{(_1\tilde{c})_{t_0}(1)}\ ,$$

which projects to the tangent plane of S at $(_1\tilde{c})_{t_0}(1)$. $\Big)$

Note paths $_1\tilde{c}_t$ on S gives us

$$\lim_{t\to t_0, t\in\cup_{n=1}^{\infty}D_n} {_1\tilde{c}_t(1)} \ = \ {_1\tilde{c}_{t_0}(1)}$$

and

$$\begin{aligned}
\lim_{t\to t_0, t\in\cup_{n=1}^{\infty}D_n} {}_{\bar{f}(t)}(_1\tilde{c})_t(1) &= \lim_{t\to t_0, t\in\cup_{n=1}^{\infty}D_n} {}_e(_1\tilde{c})_t(1)\cdot\bar{f}(t) \\
&= {}_e(_1\tilde{c})_{t_0}(1)\cdot\bar{f}(t_0) \\
&= {}_{\bar{f}(t_0)}(_1\tilde{c})_{t_0}(1)\ .
\end{aligned}$$

Then we get

$\lim_{t\to t_0,\ t\in\cup_{n=1}^{\infty}D_n} F(s(t))$

$= \lim_{t\to t_0,\ t\in\cup_{n=1}^{\infty}D_n} (-1)\cdot\Big($ the unit curvature of the 2-dimensional horizontal

oriented tangent plane, $\tilde{H}^{n_1(t)}_{\bar{f}(t)}{}_{(_1\tilde{c})_t(1)}$,

for some $n_1(t)\in\mathbb{N}$ depending on t,

whose projection is the tangent plane of the

surface S at $_1\tilde{c}_t(1)\,\Big)$

$= \ (-1) \cdot \big($ the unit curvature of the 2-dimensional horizontal oriented tangent plane, $\tilde{H}^{n_0}_{\bar{f}(t_0)({}_1\tilde{c})_{t_0}(1)}$,
whose projection is the tangent plane of the surface S at ${}_1\tilde{c}_{t_0}(1)\,\big)$

$= F(s(t_0))$.

So, we get

$$F : \{s(t) \mid t \in \cup_{n=1}^{\infty} D_n - \{0\}\} \to \text{ the unit sphere in } \mathfrak{k}$$

is a continuous function. Since $\{s(t) \mid t \in \cup_{n=1}^{\infty} D_n\}$ is a dense subset of $[0$, the area of $S]$, we can extend F on $[0$, the area of $S]$ continuously. Call it also F. Consider the C^1 curve

$$\alpha : [0,\ \text{the area of } S] \to K$$

satisfying

$$\begin{aligned} &\alpha(0) = e \qquad \text{and} \\ &L_{(\alpha(t)^{-1})_*}\dot{\alpha}(t) \ = \ F(t) \,. \end{aligned}$$

Note the function

$$f_n \ : \ [0,\ \text{the area of the } n\text{-step polytope}] \to K$$

can be regarded as the piecewise integral curve of

$$\dot{f}_n(t) = L_{(f_n(t))_*}(L_{(f_n(t))^{-1}{}_*}\dot{f}_n(t)) = L_{f_n(t)_*}F_n(t) \,,$$

or equivalently the piecewise solution of the ODE

$$L_{(\alpha_n(t))^{-1}{}_*}\dot{\alpha}_n(t) = F_n(t) \,.$$

Then

$$F(s(t_0)) = \lim_{n\to\infty} \lim_{t\to s_n(t_0)} L_{(f_n(t))^{-1}{}_*}\dot{f}_n(t) = \lim_{n\to\infty} \lim_{t\to s_n(t_0)} F_n(t)$$

implies that

$$\alpha(s(t_0)) = \lim_{n\to\infty} \lim_{t\to s_n(t_0)} \alpha_n(t) = \lim_{n\to\infty} \lim_{t\to s_n(t_0)} f_n(t) = \lim_{n\to\infty} f_n(s_n(t_0)) = f(s(t_0)) \,.$$

Since $\{s(t) \mid t \in \cup_{n=1}^{\infty} D_n\}$ is a dense subset of $[0$, the area of $S]$ and F is continuous on $[0$, the area of $S]$, we get

$$f = \alpha \text{ is a } C^1 \text{ curve on } [0,\ \text{the area of } S] \,.$$

Also, we obtain

$$\begin{aligned}\text{the length of the curve } f &= \text{the length of the curve } \alpha \\ &= \int_0^{\text{the area of } S} \mid \dot{\alpha}(t) \mid dt \\ &= \int_0^{\text{the area of } S} \mid F(t) \mid dt \\ &= \text{the area of } S \ ,\end{aligned}$$

which proves Theorem 1.0.1.

5.4.7 Remarks on Factorization Lemma

'Factorization Lemma', introduced by Lichnerowicz, *Theorie Globale des Connexions et des Groupes d'Holonomie* , [3, vol 1, p.284], can give us another sequence of piecewise smooth loops $\mu_m : [0,1] \to \mathbb{H}^n,\ m = 1, 2, \cdots$, with $\mu_m(0) = \pi(e)$ such that it converges to ∂S. And a similar way to make the sequence of curves $f_n : [0,1] \to K,\ n = 1, 2, \cdots$, can give us a sequence of curves $g_n : [0,1] \to K,\ n = 1, 2, \cdots$, with $g_n(0) = e$ such that $g_n(1)$ is the ending point of the horizontal lifting of μ_n at e and that the length of g_n is the area of the *polytope*, the union of totally geodesic triangles obtained in the construction of g_n. Since the sequence of the areas converges to the area of S, Prop 5.3.2 will say that $g_n(1)$ will converge to ${}_e\tilde{\gamma}(1)$ and that the distance from e to ${}_e\tilde{\gamma}(1)$ is less that equal to the area of S. But the sequence $\{g_n\}$ may not converge to some curve from e to ${}_e\tilde{\gamma}(1)$.

5.5 The proof of Corollary 1.0.2

Given a piecewise smooth disk S, consisting of m sub-disks with smooth interiors and piecewise smooth boundaries, pick the first smooth sub-disk S_1 with one vertex lying on the boundary ∂S. Then by induction, it can be shown that the sub-disks $S_1, \cdots, S_m$ can be ordered in such a way that

$$\bigcup_{i=1}^{j} S_i \text{ is homeomorphic to a disk for each } j = 1, \cdots, m.$$

By regarding the point of S_1 on the boundary as the basepoint, Theorem 1.0.1 and the similar arguments to those used in the construction of '$\hat{f}_n$' in Subsection 5.2.2 will give a piecewise smooth curve f, which we want.

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Appendices

Appendix A

About Triangles

For each $n = 0, 1, 2, \cdots$, all triangles inside $2^n \cdot 3$-gon will consist of two kinds of triangles, *interior* ones and *exterior* ones.

A.1 The definition of interior triangles and the definition of their starting points and ending points

Consider a regular triangle whose vertices lie on the boundary of the given disk D^2 and one of whose vertices is the base point of the disk. Call the triangle T_0. And the base point will be called its starting and ending point.

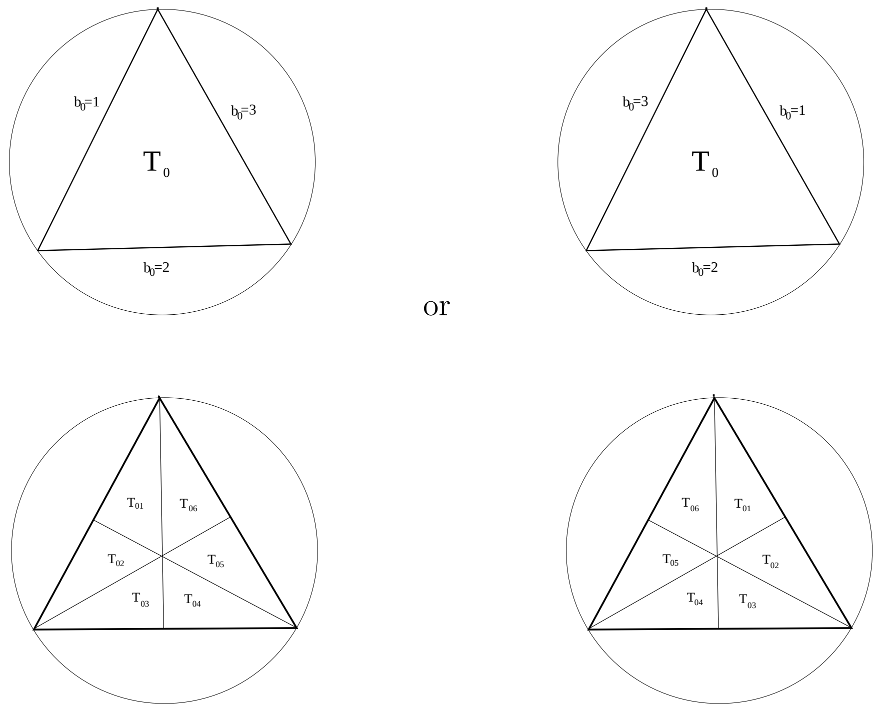
Now define triangles $T_{a_0 a_1 \cdots a_n}$ inductively as follows :

Case 1) $n = 1$:

The given orientation at the center of D^2 and the base point, or equivalently the starting and ending point of T_0, will give the order b_0 of sides of T_0, where $b_0 = 1.2.3$, in the counter-clockwise or clockwise order. For the barycentric subdivision of T_0, thinking of the triangle with the base point as its vertex and with one side lying on the first side of T_0 as the first triangle will give the order of triangles in the counter-clockwise or clockwise order. The i-th triangle will be called $T_{a_0 a_1}$, where $a_0 = 0$ and $a_1 = i$, for $i = 1, 2, \cdots, 6$.

For T_{01}, the base point, or equivalently the starting point of T_0, will be called the starting point of T_{01} and the barycenter of T_0 will be called the ending point of T_{01}.

For T_{0i}, where $i = 2, 3, 4, 5$, the barycenter of T_0 will be called the starting and ending point of T_{0i} for $i = 2, 3, 4, 5$.



For T_{06} , the barycenter of T_0 will be called the starting point of T_{06} and the base point , or equivalently the ending point of T_0 , will be called the ending point of T_{06}

Case 2) $n \geq 2$:

Let $L_{n-1} := T_{a_0 a_1 \cdots a_{n-2} a_{n-1}}$ be given, where $a_0 = 0$ and $a_1, \cdots, a_{n-1} \in \{1,2,3,4,5,6\}$. Let $L_{n-2} := T_{a_0 \cdots a_{n-2}}$ and assume the following properties:

- $M_j := T_{a_0 a_1 \cdots a_{n-2}\, j}$, $j \in \{1,2,3,4,5,6\}$, consists one of six triangles obtained by the barycentric subdivision of L_{n-2},

- L_{n-1} is also one of those, in other words,

$$L_{n-1} = T_{a_0 a_1 \cdots a_{n-2} a_{n-1}} = T_{a_0 a_1 \cdots a_{n-2}\, j_0} = M_{j_0}$$

for some $j_0 \in \{1,2,3,4,5,6\}$.

- common vertex of L_{n-2} and M_1 is the starting point of each of them,

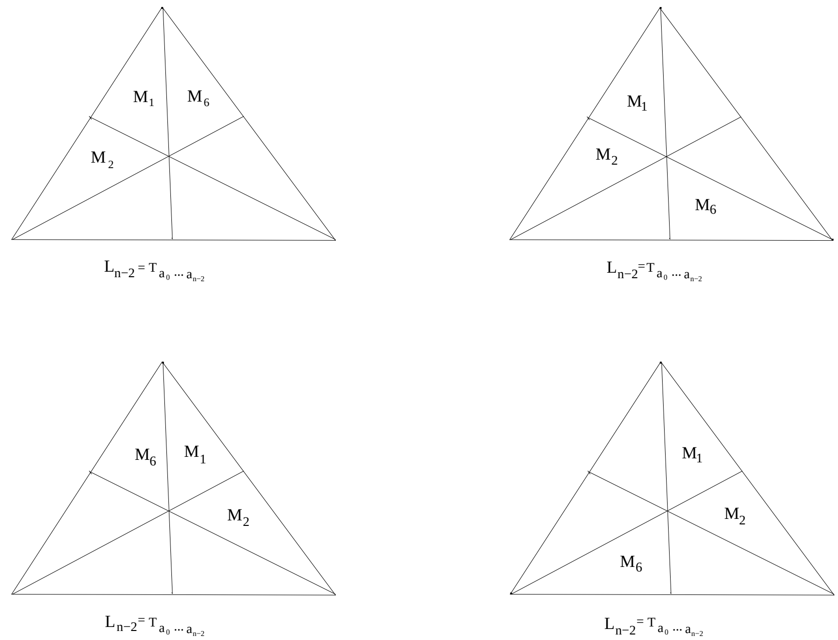
- the barycenter of L_{n-2} is the starting point of M_i for $i = 2,3,4,5,6$, and the ending point of M_i for $i = 1,2,3,4,5$,

- the common vertex of L_{n-2} and M_6 is the ending point of each of them,

- if the starting and the ending point of L_{n-2} are same, then they are the common vertex of L_{n-2} and M_6,

- if the starting and the ending point of L_{n-2} are different, then M_1 and M_6 are mutually opposite ones inside L_{n-2},

- one side of L_{n-2}, which contains a side of M_1, is divided into two line segments, each of which is one side of M_i for $i = 1,2$, respectively.

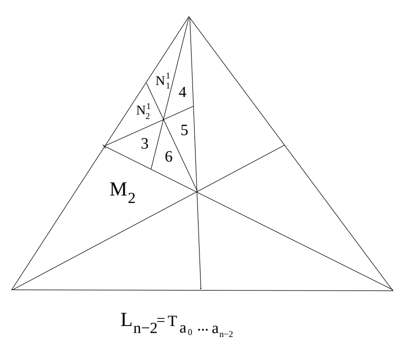


Notice all the above assumptions hold for n=2.

Note that the line segment connecting the barycenter and the starting point of L_{n-2} is one side of M_1 from the assumption that the common vertex of L_{n-2} and M_1 is the starting point of each of them.

Under the above assumptions, we have two choices such that the order of M_1 and M_2 is either the counter-clockwise order or the clockwise order with respect to the barycenter of L_{n-2} and the line segment connecting the barycenter and the starting point of L_{n-2}.

Subcase 2-1) $a_{n-1} = 1$, that is, $L_{n-1} = M_1 = T_{a_0 \cdots a_{n-2} 1}$:

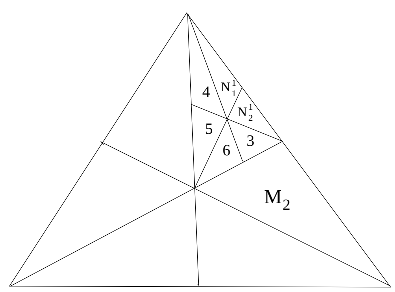


Assume the order of $L_{n-1} = M_1$ and $M_2 = T_{a_0 \cdots a_{n-2} 2}$ is the counter-clockwise order with respect to the barycenter of $L_{n-2} = T_{a_0 \cdots a_{n-2}}$ and the line segment connecting the barycenter and the starting point of L_{n-2}. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_1$, choose the triangle with a part of one side of L_{n-2} as its side and with the starting point and the barycenter of $L_{n-1} = M_1$ as its vertices, and call it $T_{a_0 \cdots a_{n-2}11}$ and let $N_1^1 := T_{a_0 \cdots a_{n-2}11}$. At the barycenter of $L_{n-1} = M_1$, consider the counter-clockwise order of the 6 triangles from the N_1^1. The 5 triangles from the next one of N_1^1 will be called

$$T_{a_0 \cdots a_{n-2}12},\ T_{a_0 \cdots a_{n-2}13},\ T_{a_0 \cdots a_{n-2}16},\ T_{a_0 \cdots a_{n-2}15},\ T_{a_0 \cdots a_{n-2}14}$$

in order. Let $N_i^1 := T_{a_0 \cdots a_{n-2}1i}$ for $i = 2, 3, 4, 5, 6$.

If the order of $L_{n-1} = M_1$ and M_2 is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_1$:



$L_{n-2} = T_{a_0 \cdots a_{n-2}}$

Recall the assumptions for $L_{n-1} = M_1$, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' and let $L_n := N_j^1,\ j = 1, \cdots, 6$.

Note the common vertex of $L_{n-1} = M_1$ and N_1^1 is the starting point of $L_{n-1} = M_1$ from the definition of N_1^1 . Now, call the vertex the starting point of N_1^1 . And call the barycenter of $L_{n-1} = M_1$ the starting point of N_i^1 for $i = 2, 3, 4, 5, 6$. Also, call the barycenter the ending point of N_i^1 for $i = 1, 2, 3, 4, 5$.

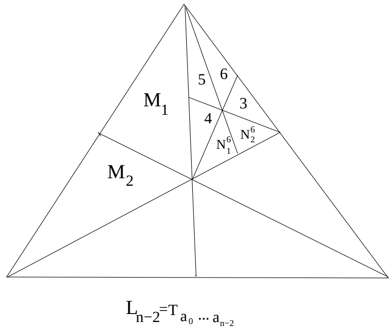
Note the common vertex of $L_{n-1} = M_1$ and N_6^1 is the barycenter of L_{n-2}, so the ending point of $L_{n-1} = M_1$ from the assumption for M_1. Call the vertex the ending point of N_6^1.

Note that the starting and the ending point of $L_{n-1} = M_1$ are different and the positions of N_1^1 and N_6^1 are mutually opposite inside $L_{n-1} = M_1$.

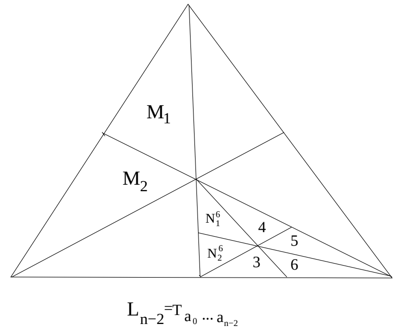
And the side of $L_{n-1} = M_1$, which contains a side of N_1^1, is divided into two line segments, each of which is one side of N_i^1 for $i = 1, 2$, respectively.

Subcase 2-2) $a_{n-1} = 6$, that is, $L_{n-1} = M_6 = T_{a_0 \cdots a_{n-2}6}$:

Assume the order of M_1 and M_2 is the counter-clockwise order with respect to the barycenter of L_{n-2} and the line segment connecting the barycenter and the starting point of L_{n-2}. From the assumptions, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' the vertex of $L_{n-1} = M_6$, which is also the barycenter of L_{n-2}, is the starting point of $L_{n-1} = M_6$. The



or

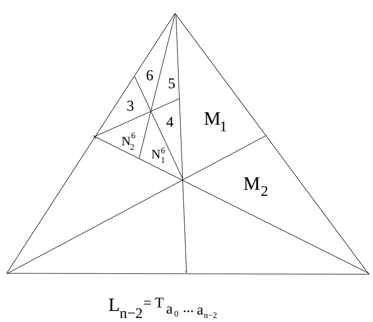


counter-clockwise angle of $L_{n-1} = M_6$ at its starting point determines its initial side and the terminal side. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_6$, choose the triangle with a part of the initial side of $L_{n-1} = M_6$ as its side and with the starting point and the barycenter of $L_{n-1} = M_6$ as its vertices, and call it $T_{a_0 \cdots a_{n-2} 61}$ and let $N_1^6 := T_{a_0 \cdots a_{n-2} 61}$. At the barycenter of $L_{n-1} = M_6$, consider the counter-clockwise order of the 6 triangles from the N_1^6. The 5 triangles from the next one of $N_1^6 = T_{a_0 \cdots a_{n-2} 61}$ will be called

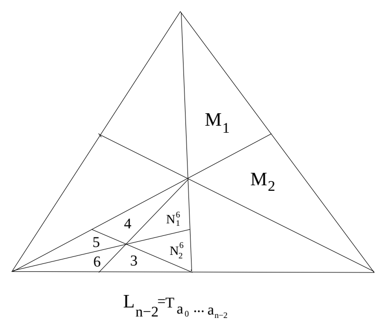
$$T_{a_0 \cdots a_{n-2} 62}, T_{a_0 \cdots a_{n-2} 63}, T_{a_0 \cdots a_{n-2} 66}, T_{a_0 \cdots a_{n-2} 65}, T_{a_0 \cdots a_{n-2} 64}$$

in order. Let $N_i^6 := T_{a_0 \cdots a_{n-2} 6i}$ for $i = 2, 3, 4, 5, 6$.

If the order of M_1 and M_2 is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_6$:



or



Recall the assumptions for $L_{n-1} = M_6$, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' and let $L_n := N_j^6,\ j = 1, \cdots, 6$.

Note the common vertex of $L_{n-1} = M_6$ and N_1^6 is the starting point of $L_{n-1} = M_6$ from the definition of N_1^6. Now, call the vertex the starting point of N_1^6. And call the barycenter of $L_{n-1} = M_6$ the starting point of N_i^6 for $i = 2, 3, 4, 5, 6$. Also, call the barycenter the ending point of N_i^6 for $i = 1, 2, 3, 4, 5$.

To consider the common vertex of $L_{n-1} = M_6$ and N_6^6, we have the following two possibilities :

The starting point and the ending point of L_{n-2} are same or different.

But in any possibilities, the common vertex of $L_{n-1} = M_6$ and N_6^6 is also the common vertex of L_{n-2} and $L_{n-1} = M_6$, so the ending point of $L_{n-1} = M_6$ from the assumption for M_6. Call the vertex the ending point of N_6^6.

Note that the starting and the ending point of $L_{n-1} = M_6$ are different and the positions of N_1^6 and N_6^6 are mutually opposite inside $L_{n-1} = M_6$.

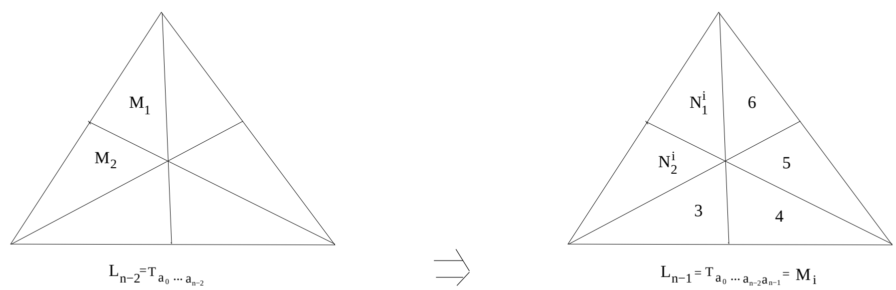
Notice the side of $L_{n-1} = M_6$, which contains a side of N_1^6, is divided into two line segments, each of which is one side of N_i^6 for $i = 1, 2$, respectively.

Subcase 2-3) $a_{n-1} \in \{2, 3\}$ or ($a_{n-1} \in \{4, 5\}$ and $a_{n-2} \in \{0, 2, 3, 4, 5\}$), that is,

$$L_{n-1} = M_i = T_{a_0 \cdots a_{n-2}\, i} \text{ for } \ i = 2, 3$$

or

$$L_{n-1} = M_i = T_{a_0 \cdots a_{n-2} i} \text{ for } \ i = 4, 5 \text{ and } a_{n-2} \in \{0, 2, 3, 4, 5\} :$$

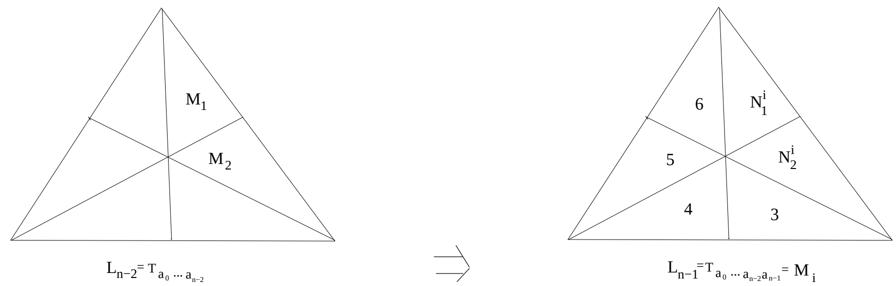


Let $i = a_{n-1}$.

Assume the order of M_1 and M_2 is the counter-clockwise order with respect to the barycenter of L_{n-2} and the line segment connecting the barycenter and the starting point of L_{n-2}. From the assumptions, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' the vertex of $L_{n-1} = M_i$, which is also the barycenter of L_{n-2}, is the starting point of $L_{n-1} = M_i$. The counter-clockwise angle of $L_{n-1} = M_i$ at its starting point determines the initial side and the terminal side. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_i$, choose the triangle with a part of the initial side of $L_{n-1} = M_i$ as its side and with the starting point and the barycenter of $L_{n-1} = M_i$ as its vertices, and call it $T_{a_0 \cdots a_{n-2} a_{n-1} 1}$, in other words, $T_{a_0 \cdots a_{n-2} i 1}$, and let $N_1^i := T_{a_0 \cdots a_{n-2} i 1}$. At the barycenter of $L_{n-1} = M_i$, consider the counter-clockwise order of the 6 triangles from the N_1^i. The 5 triangles from the next one of $N_1^i = T_{a_0 \cdots a_{n-2} a_{n-1} 1}$ will be called

$$T_{a_0 \cdots a_{n-2} a_{n-1} 2}, T_{a_0 \cdots a_{n-2} a_{n-1} 3}, T_{a_0 \cdots a_{n-2} a_{n-1} 4}, T_{a_0 \cdots a_{n-2} a_{n-1} 5}, T_{a_0 \cdots a_{n-2} a_{n-1} 6}$$

in order. Let $N_j^i := T_{a_0 \cdots a_{n-2}\, ij} = T_{a_0 \cdots a_{n-2} a_{n-1}\, j}$ for $j = 2, 3, 4, 5, 6$.



If the order of M_1 and M_2 is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_i = T_{a_0 \cdots a_{n-2} a_{n-1}}$:

Recall the assumptions for $L_{n-1} = M_i$, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' and let $L_n := N^i_j,\ j = 1, \cdots, 6$.

Note the common vertex of $L_{n-1} = M_i$ and N^i_1 is the starting point of $L_{n-1} = M_i$ from the definition of N^i_1. Now, call the vertex the starting point of N^i_1 . And call the barycenter of $L_{n-1} = M_i$ the starting point of N^i_j for $j = 2, 3, 4, 5, 6$. Also, call the barycenter the ending point of N^i_j for $j = 1, 2, 3, 4, 5$.

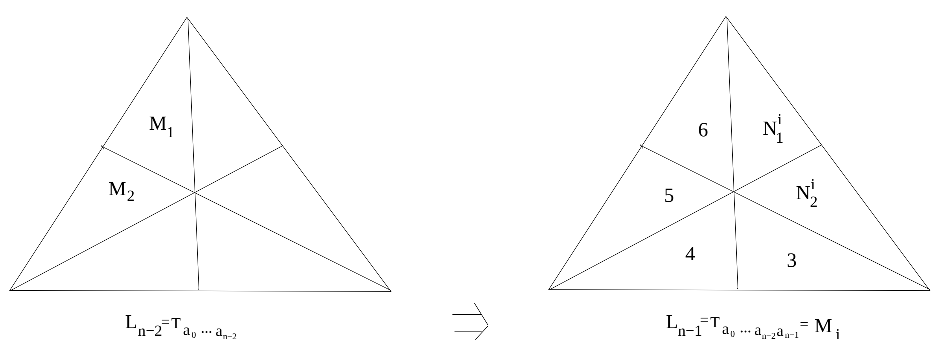
Note the common vertex of $L_{n-1} = M_i$ and N^i_6 is the starting point of $L_{n-1} = M_i$, so the barycenter of L_{n-2} and the ending point of $L_{n-1} = M_i$ from the assumption for M_i. Call the vertex the ending point of N^i_6.

Note that the starting and the ending point of $L_{n-1} = M_i$ are same and they are the common vertex of $L_{n-1} = M_i$ and N^i_6.

And the side of $L_{n-1} = M_i$, which contains a side of N^i_1, is divided into two line segments, each of which is one side of N^i_j for $j = 1, 2$, respectively.

Subcase 2-4) $a_{n-1} \in \{4, 5\}$ and $a_{n-2} \in \{1, 6\}$,
that is,

$$L_{n-1} = M_i = T_{a_0 \cdots a_{n-2} i} \text{ for } \ i = 4, 5 \text{ and } a_{n-2} \in \{1, 6\} :$$



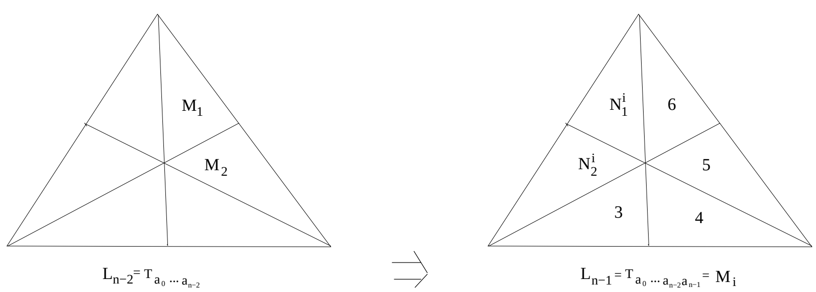
Let $i = a_{n-1}$.

Assume the order of M_1 and M_2 is the counter-clockwise order with respect to the barycenter of L_{n-2} and the line segment connecting the barycenter and the starting point of L_{n-2}. From the assumptions,lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' the vertex of $L_{n-1} = M_i$, which is also the barycenter of L_{n-2}, is the starting point of $L_{n-1} = M_i$. The clockwise angle of $L_{n-1} = M_i$ at its starting point determines the initial side and the terminal side. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_i$, choose the triangle with a part of the initial side of $L_{n-1} = M_i$ as its side and with the starting point and the barycenter of $L_{n-1} = M_i$ as its vertices, and call it $T_{a_0 \cdots a_{n-2} a_{n-1} 1}$, in other words, $T_{a_0 \cdots a_{n-2} i 1}$, and let $N_1^i := T_{a_0 \cdots a_{n-2} i 1}$. At the barycenter of $L_{n-1} = M_i$, consider the clockwise order of the 6 triangles from the N_1^i. The 5 triangles from the next one of $N_1^i = T_{a_0 \cdots a_{n-2} i 1} = T_{a_0 \cdots a_{n-2} a_{n-1} 1}$ will be called

$$T_{a_0 \cdots a_{n-2} a_{n-1} 2}, T_{a_0 \cdots a_{n-2} a_{n-1} 3}, T_{a_0 \cdots a_{n-2} a_{n-1} 4}, T_{a_0 \cdots a_{n-2} a_{n-1} 5}, T_{a_0 \cdots a_{n-2} a_{n-1} 6}$$

in order. Let $N_j^i := T_{a_0 \cdots a_{n-2}\, i j} = T_{a_0 \cdots a_{n-2} a_{n-1}\, j}$ for $j = 2, 3, 4, 5, 6$.

If the order of M_1 and M_2 is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_i$:



Recall the assumptions for $L_{n-1} = M_i$, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots,$' and let $L_n := N_j^i,\ j = 1, \cdots, 6$.

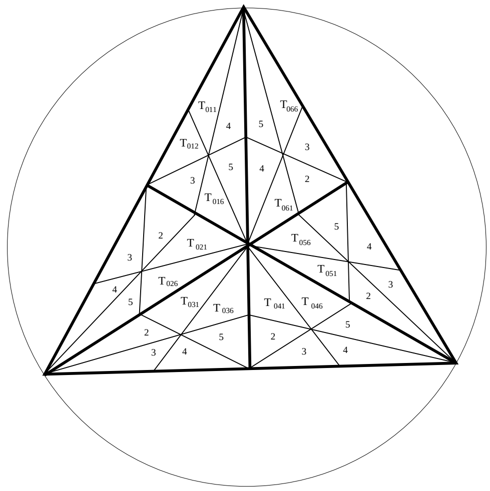
Note the common vertex of $L_{n-1} = M_i$ and N_1^i is the starting point of $L_{n-1} = M_i$ from the definition of N_1^i. Now, call the vertex the starting point of N_1^i. And call the barycenter of $L_{n-1} = M_i$ the starting point of N_j^i for $j = 2, 3, 4, 5, 6$. Also, call the barycenter the ending point of N_j^i for $j = 1, 2, 3, 4, 5$.

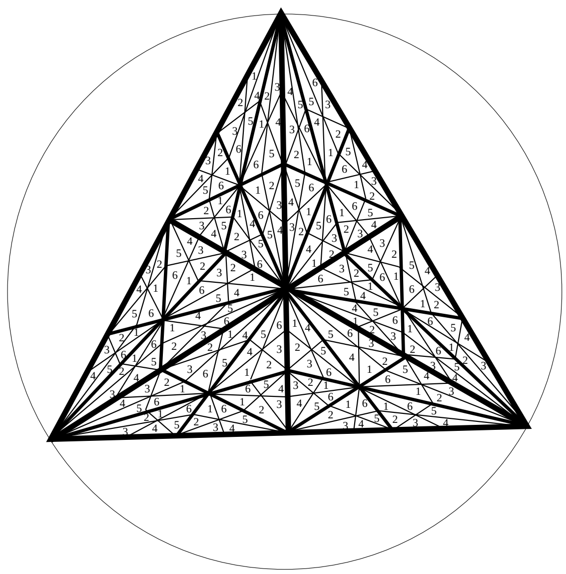
Note the common vertex of $L_{n-1} = M_i$ and N_6^i is the starting point of $L_{n-1} = M_i$, so the barycenter of L_{n-2} and the ending point of $L_{n-1} = M_i$ from the assumption for M_i. Call the vertex the ending point of N_6^i.

Note that the starting and the ending point of $L_{n-1} = M_i$ are same and they are the common vertex of $L_{n-1} = M_i$ and N_6^i.

And the side of $L_{n-1} = M_i$, which contains a side of N_1^i, is divided into two line segments, each of which is one side of N_j^i for $j = 1, 2$, respectively.

Under the counterclockwise orientation, interior triangles for $n = 2, 3$ will be given as follows :





Now let's review the definition of the starting points and ending points of triangles made right before as follows :

To begin with, note that the common vertex of $T_{a_0a_1\cdots a_{n-1}}$ and $T_{a_0a_1\cdots a_{n-1}1}$ is the starting point of $T_{a_0a_1\cdots a_{n-1}}$ and that the common vertex of $T_{a_0a_1\cdots a_{n-1}}$ and $T_{a_0a_1\cdots a_{n-1}6}$ is the ending point of $T_{a_0a_1\cdots a_{n-1}}$.

For $T_{a_0a_1\cdots a_{n-1}1}$, the starting point of $T_{a_0a_1\cdots a_{n-1}}$ is the common vertex with $T_{a_0a_1\cdots a_{n-1}1}$ and will be called the starting point of $T_{a_0a_1\cdots a_{n-1}1}$. And the barycenter of $T_{a_0a_1\cdots a_{n-1}}$ will be called the ending point of $T_{a_0a_1\cdots a_{n-1}1}$.

For $T_{a_0a_1\cdots a_{n-1}i}$, where $i = 2, 3, 4, 5$, the barycenter of $T_{a_0a_1\cdots a_{n-1}}$ will be called the starting and ending point of $T_{a_0a_1\cdots a_{n-1}i}$.

For $T_{a_0a_1\cdots a_{n-1}6}$, the barycenter of $T_{a_0a_1\cdots a_{n-1}}$ will be called the starting point of $T_{a_0a_1\cdots a_{n-1}6}$. And the ending point of $T_{a_0a_1\cdots a_{n-1}}$ is the common vertex with

$T_{a_0a_1\cdots a_{n-1}6}$ and will be called the ending point of $T_{a_0a_1\cdots a_{n-1}6}$.

To check whether we can define triangles inductively:

Recall the assumptions for $T_{a_0a_1\cdots a_{n-1}}$, lying between the phrase 'Case 2) $n \geq 2$' and the one 'Subcase 2-1) $a_{n-1} = 1, \cdots$.'

Note that the common vertex of $T_{a_0a_1\cdots a_{n-1}}$ and $T_{a_0a_1\cdots a_{n-1}1}$ is the starting point of each of them. The barycenter of $T_{a_0a_1\cdots a_{n-1}}$ is the starting point of $T_{a_0a_1\cdots a_{n-1}i}$ for $i = 2, 3, 4, 5, 6$, and the ending point of $T_{a_0a_1\cdots a_{n-1}i}$ for $i = 1, 2, 3, 4, 5$. The common vertex of $T_{a_0a_1\cdots a_{n-1}}$ and $T_{a_0a_1\cdots a_{n-1}6}$ is the ending point of each of them.

Notice that if the starting and the ending point of $T_{a_0\cdots a_{n-2}a_{n-1}}$ are same then $a_{n-1} \neq 1, 6$ and they are the common vertex of $T_{a_0\cdots a_{n-2}a_{n-1}}$ and $T_{a_0\cdots a_{n-2}a_{n-1}6}$. Also note that if the starting and the ending point of $T_{a_0\cdots a_{n-2}a_{n-1}}$ are different then $a_{n-1} \in \{1, 6\}$ and $T_{a_0\cdots a_{n-2}a_{n-1}1}$ and $T_{a_0\cdots a_{n-2}a_{n-1}6}$ are mutually opposite ones inside $T_{a_0\cdots a_{n-2}a_{n-1}}$.

And one side of $T_{a_0a_1\cdots a_{n-1}}$, which contains a side of $T_{a_0a_1a_2\cdots a_{n-1}1}$ is divided into two line segments, each of which is one side of $T_{a_0a_1\cdots a_{n-1}i}$, for $i = 1, 2$, respectively. Thus we can define triangles inductively.

A.2 The definition of exterior triangles and the definition of their starting points and ending points

A.2.1 The definition of $\mathbf{S_0^{b_0b_1\cdots b_n}}$

The given orientation at the center of D^2 and the base point, or equivalently the starting and ending point of T_0, will give the order b_0 of sides of T_0, where $b_0 = 1, 2, 3$, as explained early in 'Section A.1.'

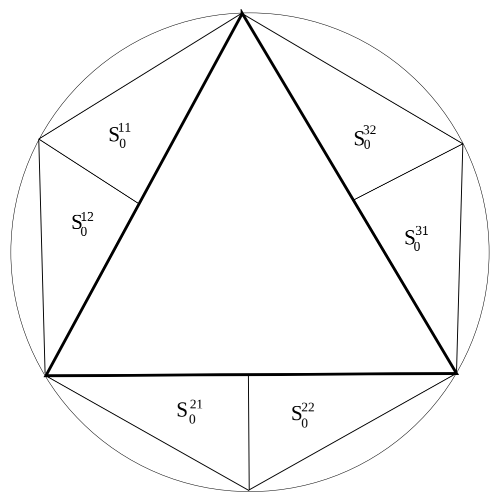
Case 1) $n = 1$:

From the given orientation at the center of D^2, consider the direction of each side of T_0, which will give the starting point and the ending point of each side.

For the side b_0 of T_0 and the (line) segment on the boundary of D^2 , which faces the side b_0 and has common terminal points with the side b_0 , consider the midpoint of the side b_0 and of the boundary segment, respectively. Then a given half of the side b_0, the straight line segment between the midpoint of the boundary segment and the common terminal point of the side b_0 and of the given half of the side b_0, and the straight line segment between the midpoint of the side b_0 and that of the boundary segment will determine a triangle, so we can obtain two triangles from each half of the side b_0. Let's call them $S_0^{b_01}$

and $S_0^{b_0 2}$, where for $S_0^{b_0 i}$, i is determined by the order with respect to the orientation at the center of D^2 and the line segment connecting the center of D^2 and the starting point of T_0.

Under the counterclockwise orientation, exterior triangles for $n = 1$ will be given as follows :



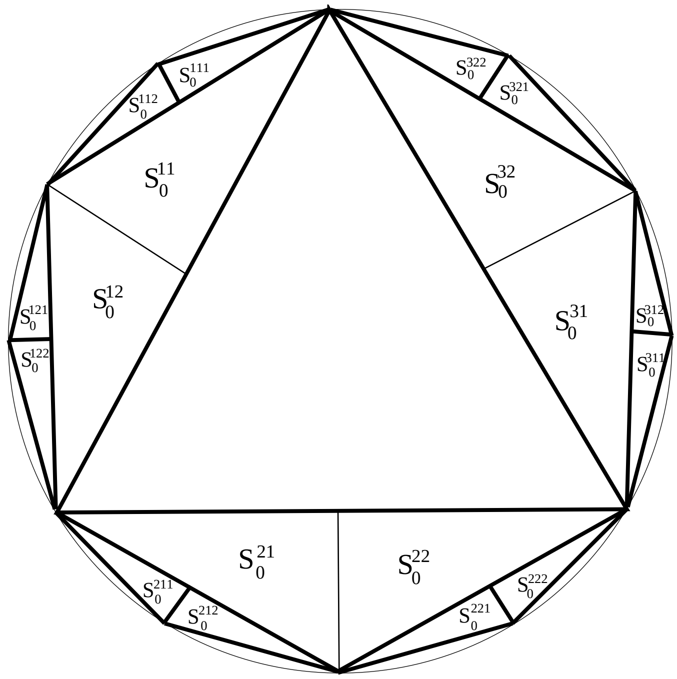
Case 2) $n \geq 2$:

Let $S_0^{b_0 b_1 \cdots b_{n-1}}$ be given.

The side of $S_0^{b_0 b_1 \cdots b_{n-1}}$, which faces the boundary of D^2, will give two triangles as follows :

Consider the direction of the side of $S_0^{b_0 b_1 \cdots b_{n-1}}$, which faces the boundary of D^2, and that of the line segment on the boundary of D^2, which is being faced by the side, respectively, from the orientation at the center of D^2 and the line segment connecting the center of D^2 and the starting point of T_0. Then we can think of the starting point, midpoint and ending point of the side of $S_0^{b_0 b_1 \cdots b_{n-1}}$, which faces the boundary of D^2, and those of the boundary segment, respectively. Now, refer to the construction of two triangles in 'case 1.' Then the triangle with the midpoints and common starting point of the side and the boundary segment as vertices will be called $S_0^{b_0 b_1 \cdots b_{n-1} 1}$ and the triangle with the midpoints and common ending point of the side and the boundary segment as vertices will be called $S_0^{b_0 b_1 \cdots b_{n-1} 2}$.

Under the counterclockwise orientation, exterior triangles for $n = 2, 3$ will be given as follows :



Now, define the starting point and the ending point of the triangles made right before as follows:

Let $n \geq 1$.

For $S_0^{b_0 b_1 \cdots b_{n-1} i}, i = 1, 2$, consider the direction of its side facing the boundary with respect to the orientation at the center of D^2.

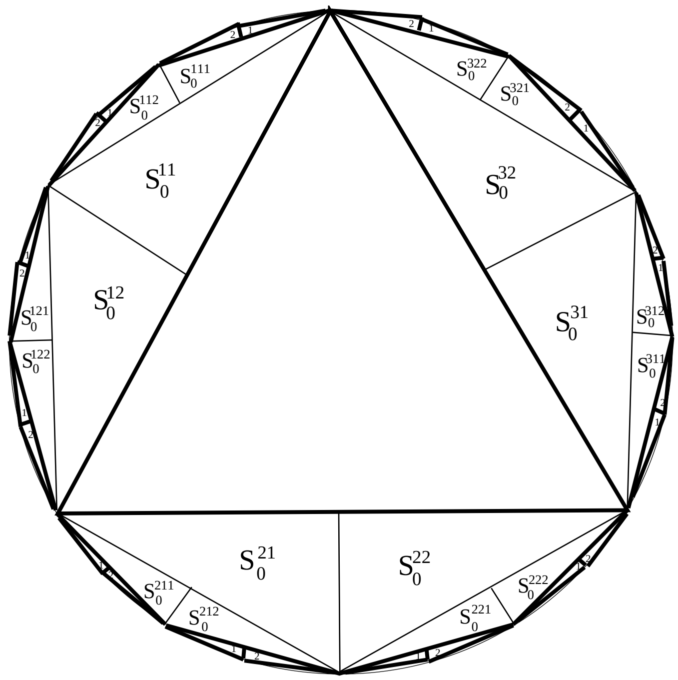
If n is odd, the ending point of the side, facing the boundary of D^2, will be called the starting point of $S_0^{b_0 b_1 \cdots b_{n-1} i}$ and the starting point of the side, facing the boundary of D^2, will be called the ending point of $S_0^{b_0 b_1 \cdots b_{n-1} i}$.

If n is even, the starting point of the side, facing the boundary of D^2, will be called the starting point of $S_0^{b_0 b_1 \cdots b_{n-1} i}$ and the ending point of the side, facing the boundary of D^2, will be called the ending point of $S_0^{b_0 b_1 \cdots b_{n-1} i}$.

A.2.2 The definition of $\mathbf{S_{a_0 a_1 \cdots a_m}^{b_0 b_1 \cdots b_k}}$

Let $1 \leq k < n$ be given. Let $m = n - k$. To define $S_{a_0 a_1 \cdots a_m}^{b_0 b_1 \cdots b_k}$, consider a triangle $\tilde{T}_0$ whose orientation is the opposite one of T_0 (, considering T_{014} might be helpful). Then the m-step barycentric subdivision makes us think of $\tilde{T}_{0 a_1 \cdots a_m}$, which is the mirror-symmetry of $T_{0 a_1 \cdots a_m}$ (, for example $T_{0141 a_1 \cdots a_m}$). Note the orientation of the triangle $\tilde{T}_{01}$ is the opposite one of T_{01} (,considering T_{0141} might be helpful), and its m-step barycentric subdivision $\tilde{T}_{01 a_2 \cdots a_m}$ is also the mirror-symmetry of $T_{01 a_2 \cdots a_m}$ (, for example $T_{0141 a_2 \cdots a_m}$).

We want to define $S_{a_0 a_1 \cdots a_m}^{b_0 b_1 \cdots b_k}$ as follows :



Case 1-1) k is odd and $b_k = 1$:

Consider the barycentric subdivision of $S_0^{b_0 b_1 \cdots b_k}$. By comparing it with that of T_{01}, define

- $S_{0j}^{b_0 b_1 \cdots b_k}$, which matches T_{01j} for $j \in \{1, 2, 3, 4\}$,
- $S_{05}^{b_0 b_1 \cdots b_k}$, which matches T_{016},
- $S_{06}^{b_0 b_1 \cdots b_k}$, which matches T_{015},

and their starting and ending points.

For $m \geq 2$, the respective identification of

$$S_{01}^{b_0 b_1 \cdots b_k}, S_{0j}^{b_0 b_1 \cdots b_k}, S_{04}^{b_0 b_1 \cdots b_k}, S_{06}^{b_0 b_1 \cdots b_k} \text{ with } T_{01}, T_0, \tilde{T}_0, \tilde{T}_{01},$$

where $j \in \{2, 3, 5\}$, and their m-step barycentric subdivision can make us define $S_{a_0 a_1 a_2 \cdots a_m}^{b_0 b_1 \cdots b_k}$, where $a_0 = 0$.

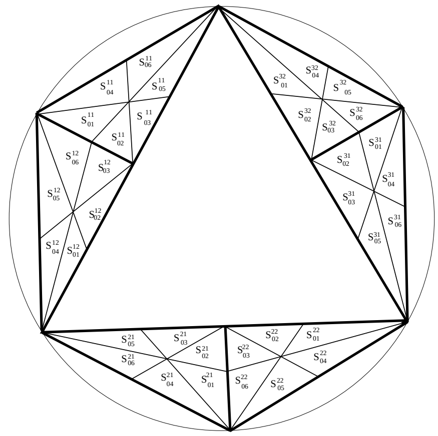
Case 1-2) k is odd and $b_k = 2$:

Identify $S_0^{b_0 b_1 \cdots b_k}$ with T_{01}, where the starting point and ending point of $S_0^{b_0 b_1 \cdots b_k}$ is also identified to those of T_{01}.

Consider the m-step barycentric subdivision of $S_0^{b_0 b_1 \cdots b_k}$ and T_{01} respectively. The identification, then, can make us define $S_{a_0 a_1 \cdots a_m}^{b_0 b_1 \cdots b_k}$ from $T_{01 a_1 \cdots a_m}$, where $a_0 = 0$.

Under the counterclockwise orientation, the triangles for $k = 1$ and $m = 1$ will be given as follows :

Case 2-1) k is even and $b_k = 1$:



By identifying $S_0^{b_0b_1\cdots b_k}$ with $\tilde{T}_{01}$, we can define $S_{a_0a_1\cdots a_m}^{b_0b_1\cdots b_k}$ from $\tilde{T}_{01a_1\cdots a_m}$, where $a_0 = 0$ (, for example $T_{0141a_1\cdots a_m}$) .

Case 2-2) k is even and $b_k = 2$:

Consider the barycentric subdivision of $S_0^{b_0b_1\cdots b_k}$. By comparing it with that of $\tilde{T}_{01}$, define

- $S_{0j}^{b_0b_1\cdots b_k}$, which matches $\tilde{T}_{01j}$ for $j \in \{1, 2, 3, 4\}$,
- $S_{05}^{b_0b_1\cdots b_k}$, which matches $\tilde{T}_{016}$,
- $S_{06}^{b_0b_1\cdots b_k}$, which matches $\tilde{T}_{015}$,

and their starting and ending points.

For $m \geq 2$, the respective identification of

$$S_{01}^{b_0b_1\cdots b_k}, S_{0j}^{b_0b_1\cdots b_k}, S_{04}^{b_0b_1\cdots b_k}, S_{06}^{b_0b_1\cdots b_k} \text{ with } \tilde{T}_{01}, \tilde{T}_0, T_0, T_{01},$$

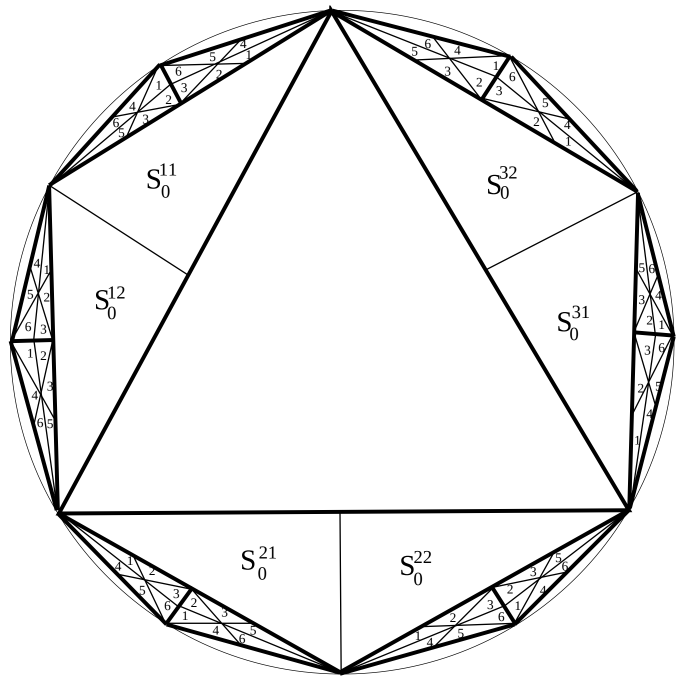
where $j \in \{2, 3, 5\}$, and their m-step barycentric subdivision can make us define $S_{a_0a_1a_2\cdots a_m}^{b_0b_1\cdots b_k}$, where $a_0 = 0$.

Under the counterclockwise orientation, the triangles for $k = 2$ and $m = 1$ will be given as follows :

A.3 The ordering of triangles in the $n-th$ step

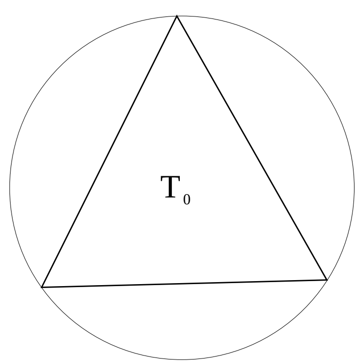
For $n = 1, 2, \cdots$, let

$$A_n = \{T_{a_0a_1\cdots a_n} \mid a_0 = 0, a_i \in \{1, 2, 3, 4, 5, 6\} \text{ for } i = 1, \cdots, n\} \bigcup$$
$$(\cup_{k+m=n, 1\leq k\leq n, 0\leq m\leq n-1}\{S_{c_0\cdots c_m}^{b_0b_1\cdots b_k} \mid b_0 \in \{1, 2, 3\}, b_i \in \{1, 2\} \text{ for } i = 1, \cdots k,$$
$$c_0 = 0, c_j \in \{1, \cdots, 6\} \text{ for } 1 \leq j \leq m \text{ if } m \geq 1\})$$



, which is regarded as the set of all triangles in the $n-th$ step.

Now refer to the following pictures for 0th, 1st, 2nd and 3rd step under the counterclockwise orientation :



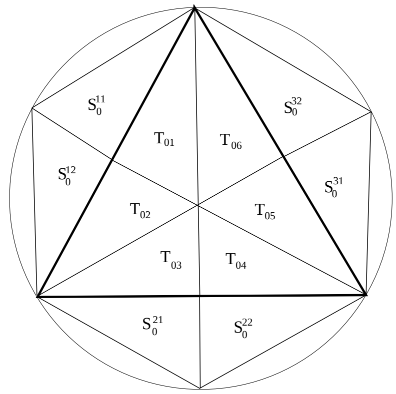
Case 1) $T_{a_0 \cdots a_n} < S^{b_0 b_1 \cdots b_k}_{c_0 \cdots c_m}$, where $k + m = n$

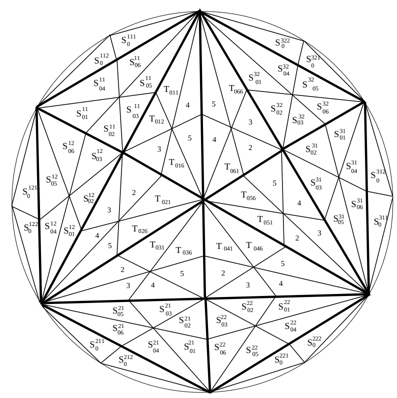
Case 2) $T_{a_0 \cdots a_n} < T_{b_0 \cdots b_n}$ if $(a_0, \cdots, a_n) < (b_0, \cdots, b_0)$ with respect to the dictionary order

Case 3) The order of $S^{b_0 b_1 \cdots b_k}_{a_0 \cdots a_m}$ and $S^{c_0 c_1 \cdots c_s}_{d_0 \cdots c_t}$, where $k + m = n = s + t$

Case 3-1) $k < s$:

$$S^{b_0 b_1 \cdots b_k}_{a_0 \cdots a_m} < S^{c_0 c_1 \cdots c_s}_{d_0 \cdots d_t}$$





Case 3-2) $k = s$ (so, $m = t$) and $(b_0, b_1, \cdots, b_k) < (c_0, c_1, \cdots, c_k)$ with respect to the dictionary order :

If k is odd, then $S^{b_0 b_1 \cdots b_k}_{a_0 \cdots a_m} > S^{c_0 c_1 \cdots c_k}_{d_0 \cdots d_m}$

If k is even, then $S^{b_0 b_1 \cdots b_k}_{a_0 \cdots a_m} < S^{c_0 c_1 \cdots c_k}_{d_0 \cdots d_m}$

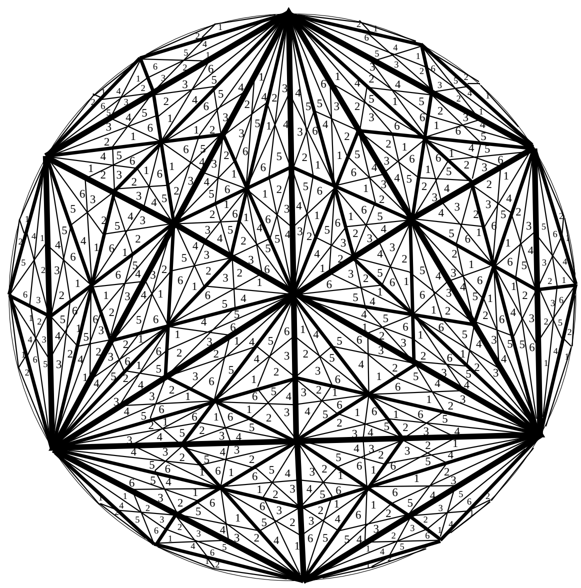
Case 3-3) $k = s$, $(b_0, b_1, \cdots, b_k) = (c_0, c_1, \cdots, c_k)$ and $(a_0, \cdots, a_m) < (d_0, \cdots, d_m)$ with respect to the dictionary order :

$$S^{b_0 b_1 \cdots b_k}_{a_0 \cdots a_m} < S^{c_0 c_1 \cdots c_k}_{d_0 \cdots d_m}$$

A.4 The properties of triangles in A_n

We can easily check the following three properties from the definition of triangles.

Property 1.) Given a non-first element L in A_n, the boundary of $\bigcup\{M \in A_n | M < L\}$ contains a side of L, which will be divided into two line segments in



its barycentric subdivision, where one of two line segments will become a side of the first triangle and the other one will become a side of the second triangle in the barycentric subdivision of L.

Property 2.) Given $L \in A_n$, $\bigcup\{M \in A_n | M \leq L\}$ is diffeomorphic to the disk D^2.

Property 3.) Assume $L \in A_n$ and six triangles $M_1, M_2, \cdots, M_6 \in A_{n+1}$, obtained from the barycentric subdivision of L, follows the order of $i = 1, 2, \cdots 6$ in A_{n+1}. Then the starting points of M_1 and L are same. Also are the ending points of M_6 and L.

And we also have the next property :

Property 4.) Assume $L, M \in A_n$ and that M is the next element of L in A_n for $n \geq 1$.

Then, The ending point of L and the starting point of M are same.

Proof)

Case 1) $L = T_{a_0 \cdots a_{n-1} a_n}$ for some $(a_0, \cdots, a_{n-1}, a_n)$

Subcase 1-1) $a_n \neq 6$

Note $M = T_{b_0 \cdots b_{n-1} b_n}$, where

$$b_i = a_i \text{ for } 0 \leq i < n \text{ and } b_n = a_n + 1.$$

Then inside the triangle $Ta_0 \cdots a_{n-1}$, the barycenter of $T_{a_0 \cdots a_{n-1}}$ is the ending point of $L = T_{a_0 \cdots a_{n-1} a_n}$ and the starting point of $M = T_{b_0 \cdots b_{n-1} b_n}$ at the same time.

Subcase 1-2) $a_n = 6$

If $a_0 = 0$ and $a_1 = \cdots = a_n = 6$, then the ending point of $L = T_{a_0 \cdots a_{n-1} a_n}$ is the ending point of T_{06} by induction and also the ending point of T_0, that is, the basepoint, which is the starting point of S_0^{32} and so the starting point of $M = S_0^{32}$ if $n = 1$ and the starting point of $M = S_{b_0 \cdots b_{n-1}}^{32}$ with $b_0 = 0$ and $b_1 = \cdots = b_{n-1} = 1$ if $n \geq 2$.

Now assume $n \geq 2$ and $a_i \neq 6$ for some i with $1 \leq i < n$.

We can find i_0 satisfying $1 \leq i_0 < n$, $a_{i_0} \neq 6$ and $a_i = 6$ for all $i_0 < i \leq n$. Then $M = T_{b_0 \cdots b_{n-1} b_n}$ satisfies

$$
\begin{aligned}
b_i &= a_i \text{ for all } 0 \leq i < i_0 \\
b_{i_0} &= a_{i_0} + 1 \\
b_i &= 1 \text{ for all } i_0 < i \leq n
\end{aligned}
$$

Note the ending point of $L = T_{a_0 \cdots a_{n-1} a_n}$ is the ending point of $T_{a_0 \cdots a_{i_0}}$ by induction.

Notice the starting point of $M = T_{b_0 \cdots b_{n-1} b_n}$ is the starting point of $T_{b_0 \cdots b_{i_0}}$ by induction.

Since $a_i = b_i$ for $0 \leq i < i_0$ and $b_{i_0} = a_{i_0} + 1$, the ending point of $L = T_{a_0 \cdots a_{i_0 - 1} a_{i_0}}$ is the barycenter of $T_{a_0 \cdots a_{i_0 - 1}}$, which is the starting point of $T_{a_0 \cdots a_{i_0 - 1} b_{i_0}} = T_{b_0 \cdots b_{i_0 - 1} b_{i_0}} = M$. Thus, we get

$$\text{the ending point of } L \text{ is the starting point of } M.$$

Case 2) $L = S_{c_0 \cdots c_m}^{b_0 b_1 \cdots b_k}$ where $k + m = n$.

Subcase 2-1) $m = 0$ and $k = n$ is odd.

Note $(b_0, b_1, \cdots, b_n) \neq (1, 1, \cdots, 1)$, because if $(b_0, b_1, \cdots, b_n) = (1, 1, \cdots, 1)$ then $L = S_0^{b_0 b_1 \cdots b_n} = S_0^{11 \cdots 1}$ is the last element in A_n.

If $n = 1$, then we can trivially obtain that the ending point of L is the starting point of M from the definition of triangles.

Assume $n \geq 2$.

If $M = S_0^{d_0 d_1 \cdots d_n}$, then we get $(b_0, b_1, \cdots, b_n) > (d_0, d_1, \cdots, d_n)$ and so

$$\text{either } (d_0 = b_0 - 1, d_1 = \cdots = d_n = 2 \text{ and } b_1 = \cdots = b_n = 1)$$

or

$$\exists i_0 \text{ with } 1 \leq i_0 \leq n \text{ such that}$$

$$\begin{aligned} d_i &= b_i \text{ for all } 0 \le i < i_0 \\ d_{i_0} &= 1, b_{i_0} = 2 \\ d_i &= 2, b_i = 1 \text{ for all } i_0 < i \le n \text{ if } 1 \le i_0 < n. \end{aligned}$$

In the first possibility, the side of L and the side of M, both of which faces the boundary, meet at one point where $S_0^{b_0 b_1} = S_0^{b_0 1}$ and $S_0^{d_0 d_1} = S_0^{d_0 2}$ meet.

In the second possibility, the side of L and the side of M, both of which faces the boundary, meet at one point which is contained in such line segment as the intersection of $S_0^{b_0 b_1 \cdots b_{i_0-1} b_{i_0}} = S_0^{b_0 b_1 \cdots b_{i_0-1} 2}$ and $S_0^{b_0 b_1 \cdots b_{i_0-1} 1} = S_0^{d_0 d_1 \cdots d_{i_0-1} d_{i_0}}$.

In any possibilities, the side of L and the side of M, both of which faces the boundary, meet at one point. Since n is odd, the point is the starting point of the side of L, facing the boundary, and the ending point of the side of M, facing the boundary, so

the ending point of L is the starting point of M.

Subcase 2-2) $m = 0$ and $k = n$ is even

Note $(b_0, b_1, \cdots, b_n) \neq (3, 2, \cdots, 2)$, because if $(b_0, b_1, \cdots, b_n) = (3, 2, \cdots, 2)$ then $L = S_0^{b_0 b_1 \cdots b_n} = S_0^{32\cdots 2}$ is the last element in A_n.

If $M = S_0^{d_0 d_1 \cdots d_n}$, then we get $(b_0, b_1, \cdots, b_n) < (d_0, d_1, \cdots, d_n)$ and so

$$\text{either } (d_0 = b_0 + 1, b_1 = \cdots = b_n = 2 \text{ and } d_1 = \cdots = d_n = 1)$$

or

$$\exists i_0 \text{ with } 1 \le i_0 \le n \text{ such that}$$

$$\begin{aligned} d_i &= b_i \text{ for all } 0 \le i < i_0 \\ d_{i_0} &= 2, b_{i_0} = 1 \\ d_i &= 1, b_i = 2 \text{ for all } i_0 < i \le n \text{ if } 1 \le i_0 < n \end{aligned} .$$

In the first possibility, the side of L and the side of M, both of which faces the boundary, meet at one point where $S_0^{b_0 b_1} = S_0^{b_0 2}$ and $S_0^{d_0 d_1} = S_0^{d_0 1}$ meet.

In the second possibility, the side of L and the side of M, both of which faces the boundary, meet at one point which is contained in such line segment as the intersection of $S_0^{b_0 b_1 \cdots b_{i_0-1} b_{i_0}} = S_0^{b_0 b_1 \cdots b_{i_0-1} 1}$ and $S_0^{b_0 b_1 \cdots b_{i_0-1} 2} = S_0^{d_0 d_1 \cdots d_{i_0-1} d_{i_0}}$.

In any possibilities, the side of L and the side of M, both of which faces the boundary, meet at one point. Since n is even, the above condition implies that the point is the ending point of the side of L, facing the boundary and the starting point of the side of M, facing the boundary, so

the ending point of L is the starting point of M.

Subcase 2-3) $m \geq 1$ and $c_m \neq 6$

If $m = 1$, then the barycenter is both the ending point of L and the starting point of M from the definition.

Assume $m \geq 2$. Note $L = S^{b_0 b_1 \cdots b_k}_{c_0 c_1 c_2 \cdots c_{m-1} c_m}$ and its next element M are inside the triangle $S^{b_0 b_1 \cdots b_k}_{c_0 c_1}$, which is one of the triangles obtained by the barycentric subdivision $S^{b_0 b_1 \cdots b_k}_{c_0} = S^{b_0 b_1 \cdots b_k}_{0}$.

Compare it with the proper one of $T_0, T_{01}, \tilde{T}_0$ and $\tilde{T}_{01}$. By referring to subcase 1-1, - by restricting it to the first triangle if needed-, we get

the ending point of L is the starting point of M.

Subcase 2-4) $m \geq 1$ and $c_m = 6$

Note $L = S^{b_0 b_1 \cdots b_k}_{c_0 c_1 \cdots c_m}$ is inside the triangle $S^{b_0 b_1 \cdots b_k}_{0}$, where $c_0 = 0$.

Assume $c_1 = c_2 = \cdots = c_{m-1} = c_m = 6$.

If $m = 1$, then the ending point of $L = S^{b_0 b_1 \cdots b_k}_{c_0 c_m} = S^{b_0 b_1 \cdots b_k}_{06}$ will be the ending point of $S^{b_0 \cdots b_k}_{06}$ tautologically. If $m \geq 2$, then compare $S^{b_0 b_1 \cdots b_k}_{c_0 c_1}$ with the proper one of $T_0, T_{01}, \tilde{T}_0$ and $\tilde{T}_{01}$. Then from the comparison, the ending point of L will be the ending point of $S^{b_0 \cdots b_k}_{066}$, which is also the ending point of $S^{b_0 \cdots b_k}_{06}$.

If $M = S^{d_0 d_1 \cdots d_s}_{a_0 \cdots a_t}$ with $a_0 = 0$, then we get
either

$$(k = s, a_1 = \cdots = a_t = 1 \text{ and } S^{d_0 d_1 \cdots d_s}_{01} \text{ is the next element of } S^{b_0 \cdots b_k}_{06} \text{ in } A_{k+1})$$

or

$$(s = k+1,\ t = m-1,\ a_i = 1 \text{ for } 1 \leq i \leq t \text{ in case of } m \geq 2 \text{ and}$$
$$S^{d_0 d_1 \cdots d_s}_{0} \text{ is the next element of } S^{b_0 \cdots b_k}_{06} \text{ in } A_{k+1}).$$

In the first possibility, $S^{d_0 d_1 \cdots d_s}_{0}$ will be also the next element of $S^{b_0 b_1 \cdots b_k}_{0}$ in A_k and so the ending point of $S^{b_0 b_1 \cdots b_k}_{0}$ will be the starting point of $S^{d_0 d_1 \cdots d_s}_{0}$ from subcase 2-1 and 2-2 , which implies

the ending point of $S^{b_0 b_1 \cdots b_k}_{06}$ will be the starting point of $S^{d_0 d_1 \cdots d_s}_{01}$.

In the second possibility, note one of k and s is odd and the other one is even, which implies that $S_0^{b_0b_1\cdots b_k}$ is the last element in A_k and that $S_0^{d_0d_1\cdots d_s}$ is the first element in the subset

$$\{S_0^{x_0x_1\cdots x_s} \mid x_0 \in \{1,2,3\}\ ,\ x_i \in \{1,2\} \text{ for } i = 1,\cdots s\}$$

of $A_s = A_{k+1}$. Also, notice that the ending point of $S_{06}^{b_0b_1\cdots b_k}$ is also the ending point of $S_0^{b_0b_1\cdots b_k}$ from the definition of triangles. By thinking of the side of $S_0^{b_0b_1\cdots b_k}$, which faces the boundary, and the side of $S_0^{d_0d_1\cdots d_s}$, which faces the boundary, we get

the ending point of $S_0^{b_0b_1\cdots b_k}$ will be the starting point of $S_0^{d_0d_1\cdots d_s}$,

so

the ending point of $S_{06}^{b_0b_1\cdots b_k}$ will be the starting point of $S_0^{d_0d_1\cdots d_s}$.

In any possibilities, the ending point of $S_{06}^{b_0\cdots b_k}$, which is also the ending point of L, is the starting point of its next element in A_{k+1}, which will be the starting point of M from $a_1 = \cdots = a_t = 1$ if $t \geq 1$. Thus, we get

the ending point of L is the starting point of M.

Now , assume $m \geq 2$ and $c_i \neq 6$ for some $1 \leq i < m$. From the comparison of $S_{c_0c_1}^{b_0b_1\cdots b_k}$ with the proper one of $T_0, T_{01}, \tilde{T}_0$ and $\tilde{T}_{01}$, we get L and M are inside the triangle $S_{c_0c_1}^{b_0\cdots b_k}$ and from subcase 1-2, we get

the ending point of L is the starting point of M.

Appendix B

About Curves

B.1 Notations

$f * g : [0,1] \to \mathbb{H}^n$ is an ordinary juxtaposition of curves $f, g : [0,1] \to \mathbb{H}^n$. And, for a given curve $c : [0,1] \to \mathbb{H}^n$, $\bar{c}$ represents a curve whose direction is opposite to that of c, that is, $\bar{c} : [0,1] \to \mathbb{H}^n$ is given by $\bar{c}(t) = c(1-t)$.

B.2 Simplification γ of a curve $\mathbf{g} : [\mathbf{a}, \mathbf{b}] \to \mathbb{H}^{\mathbf{n}}$

Given a curve $g : [a,b] \to S$, we can think of a curve $\gamma : [a,b] \to S$ whose direction is one-sided as follows :

If we can find $c, d, e \in (a,b)$ such that $a < c < d < e < b$ and $Im(g|_{[c,d]}) = Im(g|_{[d,e]})$ and that the directions of $g|_{[c,d]}$ and $g|_{[d,e]}$ are one-sided but opposite from each other, then we can think of the new curve $\tilde{g} : [a,b] \to D^2$ from the remaining part $g|_{[a,c]}$ and $g|_{[e,b]}$ by translating in the domain and rescaling as follows :

Note $g(c) = g(e)$.

Consider two curves $g_1 : [a,d] \to \mathbb{H}^n$ and $g_2 : [d,b] \to \mathbb{H}^n$ given by

$$g\left(\frac{c-a}{d-a}(t-a)+a\right) = g_1(t) \text{ for } t \in [a,d]$$

and

$$g\left(\frac{b-e}{b-d}(t-b)+b\right) = g_2(t) \text{ for } t \in [d,b],$$

and then let $\tilde{g} = g_1 * g_2$.

From a curve obtained by doing this work again and again, we can think of a constant speed curve $\gamma : [a,b] \to S$ which we want.

B.3 The definition of D_n, j_n, t_1^n, t_2^n

$$D_n = \left\{\frac{1}{2}\cdot\frac{j}{6^n} \mid j = 0,1,2,\cdots,6^n\right\} \bigcup$$
$$\left(\cup_{k=1}^{n}\left\{\sum_{i=1}^{k}\frac{1}{2^i}+\frac{1}{2^{k+1}}\cdot\frac{j}{2^{k-1}\cdot 6^{n-k+1}} \mid j = 0,1,2,\cdots,2^{k-1}\cdot 6^{n-k+1}\right\}\right)$$

Think of the usual order D_n and regard

$$0,\ \frac{1}{2}\cdot\frac{1}{6^n},\ \frac{1}{2}\cdot\frac{2}{6^n},\ \cdots,\ \frac{1}{2}=\frac{1}{2}\cdot\frac{6^n}{6^n},\ \frac{1}{2}+\frac{1}{2^2}\cdot\frac{1}{2^0\cdot 6^n},\ \cdots \quad \in D_n$$

as 0th, 1st, 2nd, $\cdots$, 6^nth, 6^{n+1}th, $\cdots$ element, respectively.
Now, define functions

$$j_n : D_n \to \{0,1,2,3,\cdots\}$$

$$t_1^n : D_n - \{0\} \to D_n$$

$$t_2^n : D_n - \{\text{ the last element of } D_n\} \to D_n$$

as follows :

$j_n(s) = j$ for the j-th element $s \in D_n$.

$t_1^n(s)$ is the $(j-1)$-th element in D_n for a given j-th element $s \in D_n - \{0\}$.

$t_2^n(s)$ is the $(j+1)$-th element in D_n for a given j-th element $s \in D_n - \{$ the last element of $D_n\}$.

B.4 Definition of $\gamma_{\mathbf{t_0}}^{\mathbf{n}}, \mathbf{c}_{\mathbf{t_0}}^{\mathbf{n}}, \bar{\mathbf{c}}_{\mathbf{t_0}}^{\mathbf{n}}, {}_{\mathbf{1}}\mathbf{c}_{\mathbf{t_0}}^{\mathbf{n}}, {}_{\mathbf{1}}\bar{\mathbf{c}}_{\mathbf{t_0}}^{\mathbf{n}}, \varphi_{\mathbf{t_0}}^{\mathbf{n}}$ and $\psi_{\mathbf{t_0}}^{\mathbf{n}}$ on the disk $\mathbf{D^2}$

Let $n \in \{1,2,3,\cdots\}$ and $t_0 \in D_n$ be given. With respect to the ordering of D_n, we'll define $\gamma_{t_0}^n, c_{t_0}^n, \bar{c}_{t_0}^n$ and $\varphi_{t_0}^n$ inductively for each fixed n:

Case 1) t_0 is the first element in D_n , in fact, $t_0 = \frac{1}{2}\cdot\frac{1}{6^n}$

The orientation at the barycenter of $T_0 \in A_0$ will give the direction of the boundary curve of the first triangle in A_n.

Then

$$c^n_{t_0} : [0,1] \to \{basepoint\} \subset D^2$$

$$\bar{c}^n_{t_0} : [0,1] \to \{basepoint\} \subset D^2$$

$$\varphi^n_{t_0} : [0,1] \to D^2$$

and

$$\gamma^n_{t_0} : [0,1] \to D^2$$

can be thought, where $\varphi^n_{t_0}$ and $\gamma^n_{t_0}$ are the piecewise smooth boundary curve of the first triangle in A_n with constant speed and the direction of the boundary curve is induced from the given orientation.

Note $\gamma^n_{t_0}$ can be regarded as the simplification of $c^n_{t_0} * \varphi^n_{t_0} * \bar{c}^n_{t_0}$.

We will call $\gamma^n_{t_0}$ the *holonomy curve at time* $t = t_0$.

Now, consider the path from the basepoint to the ending point of the first triangle in n-step along the opposite direction of the holonomy curve $\gamma^n_{t_0}$ at $t = t_0$, which is a piecewise smooth curve with constant speed. Then from the path, we can define a piecewise smooth curve

$${}_1c^n_{t_0} : [0,1] \to D^2$$

with constant speed. And its opposite direction can make us define

$${}_1\bar{c}^n_{t_0} : [0,1] \to D^2.$$

Define a piecewise smooth curve

$$\psi^n_{t_0} : [0,1] \to D^2$$

with constant speed as the boundary curve of the 1st triangle in the n-th step, where the curve is a loop at the ending point of the first triangle and the direction of the boundary curve is induced from the given orientation.

Case 2) t_0 is the j-th element in D_n , where $j \geq 2$

Let t_1 be the $(j-1)$-th element in D_n , where $j - 1 \geq 1$.

Consider the path from the basepoint to the starting point of the j-th triangle in the n-th step along the opposite direction of the holonomy curve $\gamma^n_{t_1}$ at $t = t_1$, which is a piecewise smooth one with constant speed . Then from the path, we can define a piecewise smooth curve

$$c^n_{t_0} : [0,1] \to \partial U_{j-1} \subset D^2$$

with constant speed, where U_{j-1} is the union of triangle in A_n from 1st one to $(j-1)$-th one.

And its opposite direction can make us define

$$\bar{c}^n_{t_0} : [0,1] \to \partial U_{j-1} \subset D^2.$$

Define a piecewise smooth curve

$$\varphi^n_{t_0} : [0,1] \to D^2$$

with constant speed as the boundary curve of the j-th triangle in the n-th step, where the curve is a loop at the starting point of the triangle and the direction of the boundary curve is induced from the given orientation.

Now define a piecewise smooth curve

$$\gamma^n_{t_0} : [0,1] \to \partial U_j \subset D^2$$

with constant speed from the simplification of $\gamma^n_{t_1} * c^n_{t_0} * \varphi^n_{t_0} * \bar{c}^n_{t_0}$, where U_j is the union of triangle in A_n from 1st one to j-th one. The new curve will be also called the *holonomy curve at time* $t = t_0$.

Now, consider the path from the basepoint to the ending point of the j-th triangle in the n-th step along the opposite direction of the holonomy curve $\gamma^n_{t_0}$ at $t = t_0$, which is a piecewise smooth one with constant speed. Then from the path, we can define a piecewise smooth curve

$${}_1c^n_{t_0} : [0,1] \to \partial U_j \subset D^2$$

with constant speed. And its opposite direction can make us define

$${}_1\bar{c}^n_{t_0} : [0,1] \to \partial U_j \subset D^2.$$

Define a piecewise smooth curve

$$\psi^n_{t_0} : [0,1] \to D^2$$

with constant speed as the boundary curve of the j-th triangle in the n-th step, where the curve is a loop at the ending point of the j-th triangle and the direction of the boundary curve is induced from the given orientation.

B.5 the simplification of $\bar{\mathbf{c}}^{\mathbf{n}}_{\mathbf{t_0}} * {}_{\mathbf{1}}\mathbf{c}^{\mathbf{n}}_{\mathbf{t_0}}$

For each $n \geq 1$ and $t_0 \neq 0$, where t_0 is the $j_n(t_0)$-th element in D_n, the simplification of $\bar{c}^n_{t_0} * {}_1c^n_{t_0}$ is a curve along the boundary curve of $j_n(t_0)$-th triangle in

A_n with opposite direction to the given orientation such that it starts from the starting point of the triangle and that its image consists of the following sets :

one point, one side, two sides or the boundary of the triangle.

Proof)

If n=1, then it can be easily checked.

Assume $n \geq 2$. If t_0 is greater than the maximum of $D_{n-1} - \{1\}$, then the above property can be easily checked.

Now assume $n \geq 2$ and t_0 is less than or equal to the maximum of $D_{n-1} - \{1\}$. Now find $\delta_n(t_0) \in D_{n-1}$ such that $t_1^{n-1}(\delta_n(t_0)) < t_0 \leq \delta_n(t_0)$, where $t_1^{n-1}(\delta_n(t_0))$ is the previous element of $\delta_n(t_0)$ in D_{n-1}. Then, the $j_n(t_0)$-th triangle in A_n is one of the barycentric subdivision of the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1}.

And find a value $\epsilon(j_n(t_0))$ such that , for the given $L = j_n(t_0)$-th triangle in A_n,

$$\text{if } L = T_{a_0 a_1 \cdots a_n}, \text{ then } \epsilon(j_n(t_0)) = a_n$$

and

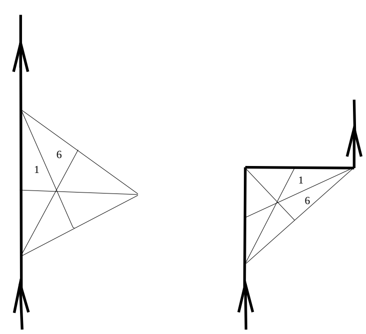
$$\text{if } L = S^{b_0 b_1 \cdots b_k}_{a_0 a_1 \cdots a_s}, \text{ where } n = k + s, \text{ then } \epsilon(j_n(t_0)) = a_s.$$

Assume $n \geq 2$ and that the property, mentioned early in this section, holds for $n - 1$.

Then we obtained the following result.

Case 1) Assume the image of the simplification of $\bar{c}^{n-1}_{\delta_n(t_0)} *_1 c^{n-1}_{\delta_n(t_0)}$ consists of one point.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$ and the outer triangle is the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1}.



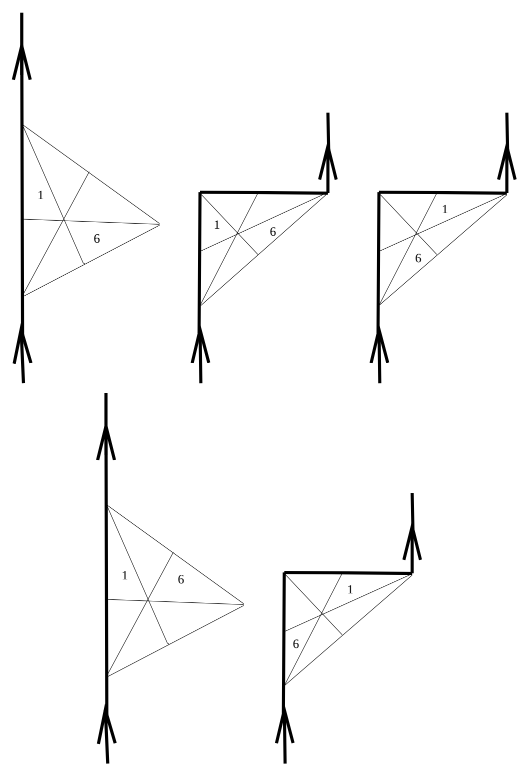
Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, mentioned in the Property 1 in the Section A.4 of the Chapter A, lying on the boundary curve $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, is from the common vertex of the

$j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1} with the second triangle of its barycentric subdivision to its common vertex with the first triangle of its barycentric subdivision , and

$$\begin{aligned}
\epsilon(j_n(t_0)) = 1, 6 &\Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of one side} \\
\epsilon(j_n(t_0)) = 2, 3 &\Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of one point} \\
\epsilon(j_n(t_0)) = 4, 5 &\Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of one point}
\end{aligned}$$

Case 2) Assume the image of the simplification of $\bar{c}^{n-1}_{\delta_n(t_0)} *_1 c^{n-1}_{\delta_n(t_0)}$ consists of one side.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$ and the outer triangle is the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1}. Don't forget that the ending point of $j_{n-1}(\delta_n(t_0))$ in A_{n-1} will lie on the image of $\gamma^{n-1}_{\delta_n(t_0)}$, even though it might not lie on the image of $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$.



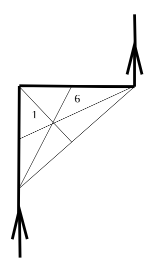
Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, mentioned in the Property 1 in the Section A.4 of the Chapter A, lying on the boundary curve $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, is from the common vertex of the

$j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1} with the second triangle of its barycentric subdivision to its common vertex with the first triangle of its barycentric subdivision , and

$$\begin{aligned}
&\epsilon(j_n(t_0)) = 1 \Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of one side}\\
&\epsilon(j_n(t_0)) = 2,3 \Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of one point}\\
&\epsilon(j_n(t_0)) = 4 \Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of the boundary}\\
&\epsilon(j_n(t_0)) = 5 \Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of either the boundary or one side}\\
&\epsilon(j_n(t_0)) = 6 \Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of either one side or two sides}
\end{aligned}$$

Remark B.5.1 *The last 2 pictures in the bottom seem to be possible under the induction hypothesis. But it might not happen in fact.*

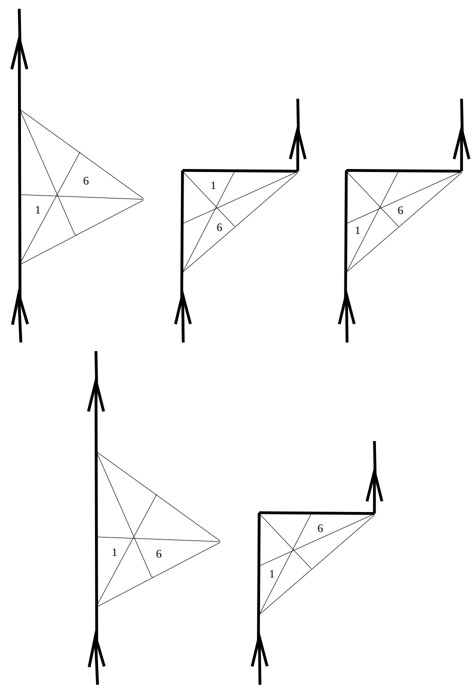
Remark B.5.2 *The following picture in the bottom can't happen from Property 2 in the Section A.4 of the Chapter A.*



Case 3) Assume the image of the simplification of $\bar{c}^{n-1}_{\delta_n(t_0)} *_1 c^{n-1}_{\delta_n(t_0)}$ consists of two sides.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of $\gamma^{n-1}_{\delta_n(t_0)}$ and the outer triangle is the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1}. Don't forget that the ending point of $j_{n-1}(\delta_n(t_0))$ in A_{n-1} will lie on the image of $\gamma^{n-1}_{\delta_n(t_0)}$, even though it might not lie on the image of $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$.

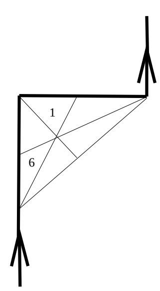
Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, mentioned in the Property 1 in the Section A.4 of the Chapter A, lying on the boundary curve $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, is from the common vertex of the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1} with the first triangle of its barycentric subdivision to its common vertex with the second triangle of its barycentric subdivision , and



$\epsilon(j_n(t_0)) = 1 \Rightarrow \vec{c}^{\,n}_{t_0} *_1 c^n_{t_0}$ consists of two sides
$\epsilon(j_n(t_0)) = 2, 3 \Rightarrow \vec{c}^{\,n}_{t_0} *_1 c^n_{t_0}$ consists of the boundary
$\epsilon(j_n(t_0)) = 4 \Rightarrow \vec{c}^{\,n}_{t_0} *_1 c^n_{t_0}$ consists of one point
$\epsilon(j_n(t_0)) = 5 \Rightarrow \vec{c}^{\,n}_{t_0} *_1 c^n_{t_0}$ consists of either one point or the boundary
$\epsilon(j_n(t_0)) = 6 \Rightarrow \vec{c}^{\,n}_{t_0} *_1 c^n_{t_0}$ consists of either two sides or one side

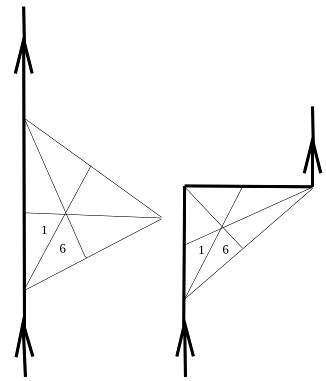
Remark B.5.3 *The last 3 pictures in the bottom seem to be possible under the induction hypothesis. But it might not happen in fact.*

Remark B.5.4 *The following picture in the bottom can't happen from Property 2 in the Section A.4 of the Chapter A.*



Case 4) Assume the image of the simplification of $\bar{c}^{n-1}_{\delta_n(t_0)} *_1 c^{n-1}_{\delta_n(t_0)}$ consists of the boudary.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$ and the outer triangle is the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1}.



Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, mentioned in the Property 1 in the Section A.4 of the Chapter A, lying on the boundary curve $\gamma^{n-1}_{t_1^{n-1}(\delta_n(t_0))}$, is from the common vertex of the $j_{n-1}(\delta_n(t_0))$-th triangle in A_{n-1} with the first triangle of its barycentric subdivision to its common vertex with the second triangle of its barycentric subdivision , and

$$\begin{aligned}
\epsilon(j_n(t_0)) = 1, 6 &\Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of two sides} \\
\epsilon(j_n(t_0)) = 2, 3 &\Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of the boundary} \\
\epsilon(j_n(t_0)) = 4, 5 &\Rightarrow \bar{c}^n_{t_0} *_1 c^n_{t_0} \text{ consists of the boundary}
\end{aligned}$$