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QUAN THUA TRAN
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SNOWFLAKE GROUPS WITH SUPER-EXPONENTIAL 2-DIMENSIONAL DEHN FUNCTIONS

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BY

Dr. Max Forester, Chair

Dr. Noel Brady

Dr. Darryl McCullough

Dr. Gerard Walschap

Dr. Deborah Trytten

For my dad, who could now challenge any man for the title of "World's Proudest Dad."

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Abstract

The objective of this paper is to combine the results of two papers to get a new result. The first paper [2] is called *Super-Exponential 2-Dimensional Dehn Functions* by Josh Barnard, Noel Brady and Pallavi Dani. In this paper, the authors construct groups whose 2-dimensional Dehn functions $\delta^{(2)}(x) \simeq \exp^n(x)$, where n is a natural number and $\exp^n(x)$ is a tower of exponentials of height n (i.e. $\exp^n(x) = e^{e^{\cdot^{\cdot^{\cdot^{e^x}}}}}$). The second paper [6] is called *Snowflake Groups, Perron-Frobenius Eigenvalues and Isoperimetric Spectra* by Noel Brady, Martin Bridson, Max Forester and Krishnan Shankar. In this paper, the authors construct groups whose k-dimensional Dehn function $\delta^k(x) \simeq x^{2\alpha}$ where $\alpha = log_\lambda(r)$ and λ is the Perron Frobenius eigenvalue of an irreducible non-negative integer matrix P and r is a natural number greater than every row sum of P. Notice that α can range over all rational numbers greater or equal to 1. By using the case when $k = 2$, we are able to recognize a common thread between the two constructions so that we can combine them to produce a new group whose 2-dimensional Dehn function $\delta^{(2)}(x) \simeq \exp^n(x^\alpha)$.

Chapter 1

Introduction

Historically, Dehn functions, developed by Max Dehn [8] in 1910, were originally used to answer the Word Problem: Is an arbitrary word equivalent to the identity? Recall that for any generating set $\mathcal{A}$ of a group G there exists a surjective homomorphism σ from $F(\mathcal{A})$, the free group with basis $\mathcal{A}$, onto G such that $G \cong F(\mathcal{A})/ker(\sigma)$. If we denote a normal generating set for the $ker(\sigma)$ by $\mathcal{R}$, then G can be simply represented by $\langle \mathcal{A} \mid \mathcal{R} \rangle$. This is called a *presentation of* G (it is called a *finite presentation* if $\mathcal{A}$ and $\mathcal{R}$ are both finite). In this context, the Word Problem can be restated as follows: For a group $G = \langle \mathcal{A} \mid \mathcal{R} \rangle$, is an element in $F(\mathcal{A})$ also an element in $ker(\sigma)$?

Now, given a finitely presented group $G = \langle \mathcal{A} \mid \mathcal{R} \rangle$ and $\sigma : F(A) \twoheadrightarrow G$ such that $G \cong F(\mathcal{A})/ker(\sigma)$, a *Dehn function* $\delta : \mathbb{N} \to \mathbb{N}$ is defined by

$$\delta(n) = \max\{\text{Area}(w) \mid w \in ker(\sigma), |w| \leq n\}$$

where

$$\text{Area}(w) = \min\{N \in \mathbb{N} \mid w \overset{free}{=} \prod_{i=1}^{N} x_i^{-1} r_i x_i,\ x_i \in F(\mathcal{A}),\ r_i \in \mathcal{R}^{\pm 1}\}$$

Intuitively, we can view any word $w \in ker(\sigma)$ as a path bounding a disc in the 2-complex determined by $\langle \mathcal{A} \mid \mathcal{R} \rangle$; Area($w$) as the area of the least-area disc with boundary label w (where unit areas are discs with boundary label $r \in \mathcal{R}^{\pm 1}$); and $\delta(n)$ as the maximum area over all least-area discs whose perimeter is less than or equal to n.

Now, given the Dehn function of $G = \langle \mathcal{A} \mid \mathcal{R} \rangle$, we can solve the Word Problem by an exhaustive approach. That is, given a word $w \in F(\mathcal{A})$, to determine if $w =_G 1$, compare w freely with all words of the form

$$\prod_{i=1}^{N} x_i^{-1} r_i x_i$$

where $x_i \in F(\mathcal{A})$, $r_i \in (R)^{\pm 1}$ and $N \leq \delta(|w|)$. It is always possible to choose each x_i so that $|x_i| \leq KN$ (K is the length of the longest relation in $\mathcal{R}$). This together with $N \leq \delta(|w|)$ allows us to solve the Word Problem in a finite amount of time. Thus Dehn functions give a very natural way of measuring the complexity of the Word Problem of a group

Dehn functions however have evolved from a tool to solve the Word Problem into an area of interest in and of itself. A key reason for this is that the Dehn function of a group is a quasi-isometry invariant, well defined up to coarse Lipschitz equivalence, $\simeq$.

In the 1990s, Mathematicians began to ask: Which (coarse Lipschitz equivalence) classes of functions arise as Dehn functions? Since there are only countably many finite presentations of groups, we know there can only be countably many coarse Lipschitz equivalence classes of Dehn functions. Thus the *isoperimetric spectrum*,

$$\text{IP} = \{\rho \in \mathbb{R}^+ \mid f(n) = n^\rho \textit{ is } \simeq \textit{ to a Dehn function}\}$$

is a countable set. Gromov [10], in 1987, proved there is a gap between 1 and 2 in the isoperimetric spectrum, i.e. IP$\cap(1, 2)$ is empty. This is due to the fact that every finitely presented group with sub-quadratic Dehn function is hyperbolic, and hence has linear Dehn function [10, 4, 11, 12]. Brady and Bridson [5] later showed that IP$\cap[2, \infty)$ is dense. More recently, Brady, Bridson, Forester and Shankar [6] gave examples of groups, called snowflake groups, that this dense set IP$\cap[2, \infty)$ contained all rational numbers greater than or equal to 2. But there are other types of functions that can arise as Dehn functions also. Baumslag Solitar groups have Dehn functions Lipschitz equivalence to exponential functions. As it stands today, Dehn functions $\succeq n^4$, have essentially all been classified. This is known as the *Sapir-Birget-Rips Theorem* [14].

Recent studies of Dehn functions have since moved toward higher-dimensional Dehn functions. If Dehn functions relate areas of discs (in a 2-complex of a group) in terms of their perimeter, 2-dimensional Dehn functions $\delta^{(2)}(x)$ relate volumes of balls (in a 3-complex of a group) in terms of their surface area x. An interesting fact about 2-dimensional Dehn functions, proved by Papasoglu [13], is that they are all bounded above by recursive functions (this is not true for regular Dehn functions).

A natural question to ask is: What Lipschitz equivalence classes of functions arise as 2-dimensional Dehn functions? Alonso, Bogley, Pride and Wang [1, 16, 17] partially answer this question by proving that the Lipschitz equivalence classes of x^ρ arise as 2-dimensional Dehn functions for infinitely many ρ in the interval $[1.5, 2)$. Brady and Bridson [5] and Bridson [9] further adds to this answer by proving IP$^{(2)}$, the set of exponents ρ, is dense in $[1.5, 2)$ and includes 2 and 3.

The more recent paper by Brady, Bridson, Forester and Shankar [6] adds another layer to this answer by asserting that $\mathrm{IP}^{(2)}$ is dense in $[1.5, \infty)$ and contains all rational numbers in this range. Beyond polynomial functions Barnard, Brady and Dani [2] give examples of groups that have super-exponential 2-dimensional Dehn functions. This paper serves to add yet another layer in answering this question by showing that for any natural number n and $\alpha = \log_\lambda(r)$ the Lipschitz equivalence classes of $\exp^n(x^\alpha)$ also arise as a 2-dimensional Dehn functions.

MAIN THEOREM 1.0.1. *Let $n \geq 1$ be an integer, P an irreducible non-negative integer square matrix with Perron-Frobenius eigenvalue $\lambda > 1$, r an integer greater than every row sum of P. There exists a group $SES_{n,P,r}$ of type $\mathcal{F}_3$ whose 2-dimensional Dehn function is given by*

$$\delta^{(2)}_{SES_{n,P,r}}(x) = \exp^n(x^\alpha)$$

where $\alpha = \log_\lambda(r)$.

1.1 Preliminaries

In this section, we give standard definitions and notations used throughout the paper as well as a proposition that is crucial in a later section.

DEFINITION 1.1.1. Given two functions $f, g : [0, \infty) \to [0, \infty)$ we define $f \preceq g$ to mean that there exists a positive constant C such that

$$f(x) \leq Cg(Cx) + Cx$$

for all $x \geq 0$. If $f \preceq g$ and $g \preceq f$ then f and g are said to be *coarse Lipschitz* *equivalent* (or simply *Lipschitz equivalent*), denoted $f \simeq g$.

DEFINITION 1.1.2. A group G is said to be of *type* $\mathcal{F}_k$ if it has a $K(G,1)$ with finite k-skeleton.

DEFINITION 1.1.3. If W is a compact k-dimensional manifold and X a CW complex, an *admissible map* is a continuous map $f : W \to X^{(k)} \subset X$ such that $f^{-1}(X^{(k)} - X^{(k-1)})$ is a disjoint union of open k-dimensional balls, each mapped by f homeomorphically onto a k-cell of X.

FACT 1.1.4. Let W be a compact manifold (smooth or piecewise-linear) of dimension k and let X be a CW complex. Then every continuous map $f : W \to X$ is homotopic to an admissible map. The proof of this fact is given [6] as Lemma 2.3.

DEFINITION 1.1.5. If $f : W \to X$ is admissible, then the *volume* of f, denoted $\mathrm{Vol}^k(f)$, to be the number of open k-balls in W mapping to k-cells of X.

DEFINITION 1.1.6. Given a group G of type $\mathcal{F}_{k+1}$, fix an aspherical CW complex X with fundamental group G and finite $(k+1)$-skeleton. Let $\widetilde{X}$ be the universal cover of X. If $f : S^k \to \widetilde{X}$ be an admissible map, then the *filling volume* of a f, denoted $\mathrm{FVol}(f)$ is given by:

$$\mathrm{FVol}(f) = \min\{\mathrm{Vol}^{k+1}(g) \mid g : B^{k+1} \to \widetilde{X} \text{ is an admissible map, } g|_{\partial B^{k+1}} = f\}$$

DEFINITION 1.1.7. Let G be a group of type $\mathcal{F}_3$ and let $\widetilde{X}$ be the universal cover of an aspherical CW complex with finite 3-skeleton and fundamental group G. Then

$$\delta_G^{(2)}(x) = \sup\{\mathrm{FVol}(f) \mid f : S^2 \to \widetilde{X} \text{ is an admissible map, } \mathrm{Vol}^2(f) \leq x\}$$

DEFINITION 1.1.8. A *graph* Γ is a set of vertices $V(\Gamma)$, a set of edges $E(\Gamma)$, maps $\partial_0, \partial_1 : E(\Gamma) \to V(\Gamma)$ which maps edges to their endpoints, and a fixed point free involution $e \mapsto \bar{e}$ of $E(\Gamma)$ such that $\partial_i \bar{e} = \partial_{1-i} e$. A *directed graph* is a choice of e for each each $\{e, \bar{e}\}$ in $E(\Gamma)$ where the direction of the edge is from $\partial_0(e)$ to $\partial_1(e)$. The set of choices of these edges e will be denoted by $E^*(\Gamma)$.

For the remainder of this paper, all diagrams of graphs will have directed edges, where each directed edge denotes the pair $\{e, \bar{e}\}$ in $E(\Gamma)$, but the direction of the edge denotes only the direction of e.

DEFINITION 1.1.9. The *star* of v in Γ, denoted $\mathrm{St}_\Gamma(v)$, is given by

$$\mathrm{St}_\Gamma(v) = \partial_0^{-1}(v)$$

.

DEFINITION 1.1.10. A *morphism* between graphs Γ and Γ' is a map $\phi : \Gamma \to \Gamma'$ taking vertices to vertices, edges to edges, and for $e \in E(\Gamma)$, $\phi(\partial_i e) = \partial_i \phi(e)$ $(i = 0, 1)$ and $\phi(\bar{e}) = \overline{\phi(e)}$.

Given a graph morphism ϕ, we have a *local map* $\phi_{(v)} : \mathrm{St}_\Gamma(v) \to \mathrm{St}_{\Gamma'}(\phi(v))$ for each $v \in V(\Gamma)$

DEFINITION 1.1.11. A *graph of groups* $\mathcal{G}$ over Γ, denoted $(\Gamma, \mathcal{G})$, is an assignment to each $v \in V(\Gamma)$, a (vertex) group G_v; and to each edge pair $\{e, \bar{e} \in E(\Gamma)$, an (edge) group $H_e = H_{\bar{e}}$; as well as for each $e \in E(\Gamma)$, monomorphisms $\mu_e : H_e \to G_{\partial_0(e)}$.

We will write $G_{v/e} = G_v / \mu_e(H_e)$ where $\partial_0(e) = v$ to denote the set of cosets of H_e in G_v.

DEFINITION 1.1.12. Let T be a maximal tree of Γ with edges only from $E^*(\Gamma)$. We define the *fundamental group* of $(\Gamma, \mathcal{G})$ to be obtained from the following presentation

$$\langle G_v, e\ (v \in V(\Gamma),\ e \in E^*(\Gamma)) \mid e\mu_{\bar{e}}(x)e^{-1} = \mu_e(x) \text{ for all } e \in E^*(\Gamma),\ x \in H_e \rangle \tag{1.1.13}$$

by adding relations $e = 1$ for each edge e in T. The generator e is called a *stable letter* of the graph of groups $(\Gamma, \mathcal{G})$.

A natural question to ask is, because T is chosen so arbitrarily, is this group is well defined? Proposition 20 of chapter I in [15] proves the affirmative. Because it does not matter which maximal tree we choose, we can leave out the maximal tree as a parameter in our denotation for the fundamental group, $\pi_1(\Gamma, \mathcal{G})$.

DEFINITION 1.1.14. An *HNN extension of (or over) a group* G is the fundamental group of a graph of groups whose graph is a single vertex v and an edge pair $\{e, \bar{e}\}$ with endpoints v and where G is the group assigned to v.

DEFINITION 1.1.15. Let $(\Gamma, \mathcal{G})$ and $(\Gamma', \mathcal{G}')$ be graphs of groups. By an *untwisted morphism between graphs of groups* $\Phi = (\phi, (\phi_v, \phi_e)) : (\Gamma, \mathcal{G}) \to (\Gamma', \mathcal{G}')$ we mean the following:

(a) a graph morphism $\phi : \Gamma \to \Gamma'$

(b) group homomorphisms

$$\phi_v : G_v \to G'_{\phi(v)} \quad \text{for } v \in V(\Gamma)$$
$$\phi_e : H_e \to H'_{\phi(e)} \quad \text{for } e \in E(\Gamma)$$

(c) and for each $e \in E(\Gamma)$ and $v = \partial_0 e$, the following diagram commutes:

$$\begin{array}{ccc} G_v & \xrightarrow{\phi_v} & G'_{\phi(v)} \\ {\scriptstyle \mu_e}\uparrow & & \uparrow{\scriptstyle \mu'_{\phi(e)}} \\ H_e & \xrightarrow[\phi_e]{} & H'_{\phi(e)} \end{array} \tag{1.1.16}$$

It is worth remarking that an untwisted morphism of graphs of groups is a special case of a morphism between graphs of groups in the sense of Bass [3].

A graph of groups morphism Φ induces the following map

$$\Phi_{v/e'} : \coprod_{e \in \phi_{(v)}^{-1}(e')} G_{v/e} \to G'_{v'/e'}$$

defined by $[g]_e \mapsto [\phi_v(g)]_{e'}$ for each $e \in \phi_{(v)}^{-1}(e')$ and $g \in G_v$.

Moreover, by Proposition 2.4 in [3], Φ induces a homomorphism $\Phi_* : \pi_1(\Gamma, \mathcal{G}) \to \pi_1(\Gamma', \mathcal{G}')$.

DEFINITION 1.1.17. We call Φ an *immersion* if each vertex group homomorphism ϕ_v is injective and each induced map $\Phi_{v/e'} : (\coprod_{e \in \phi_{(v)}^{-1}(e')} G_{v/e}) \to G'_{\phi(v)/e'}$ is injective (i.e. for each $v \in V(\Gamma)$ and $e' \in St_{\Gamma'}(\phi(v))$).

The following proposition is proved as Proposition 2.7 in [3] and will be used in Section 4.

PROPOSITION 1.1.18. *Let $(\Gamma, \mathcal{G})$ and $(\Gamma', \mathcal{G}')$ be graphs of groups and $\Phi : (\Gamma, \mathcal{G}) \to (\Gamma', \mathcal{G}')$ a graph of groups morphism. If Φ is an immersion then $\Phi_* : \pi_1(\Gamma, \mathcal{G}) \to \pi_1(\Gamma', \mathcal{G}')$ is injective.*

Chapter 2

Groups with Super-Exponential 2-Dimensional Dehn Functions

Barnard, Brady and Dani gives a way to construct groups with super-exponential 2-dimensional Dehn functions of height n [2].

$$\delta^{(2)}(x) \simeq e^{e^{e^{\cdot^{\cdot^{\cdot^{e^x}}}}}}$$

For simplicity, we denote this n-fold composition of exponential functions by $\exp^n(x)$. To achieve groups whose $\delta^{(2)}$ is super-exponential, the authors utilize a palindromic automorphism φ on $F_2 = \langle a, b \rangle$ defined by

$$a \mapsto aba \quad b \mapsto a$$

They begin with a group denoted $H_0 = F_2 \times F_2 \times F_2$. This begins a sequence of graphs of groups:

$$H_0 < G_0 < H_1 < G_1 < \cdots < H_{n-1} < G_{n-1} < H_n < G_n$$

in which each group in the sequence is defined inductively by its predecessor as Figure 2.0.1 indicates:

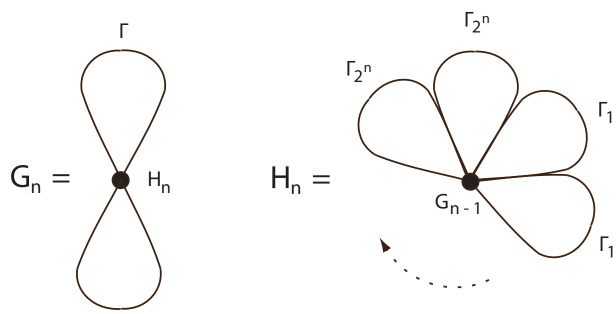

Figure 2.0.1:

where edge groups (Γ and Γ_k) and stable letters for H_n and G_n are defined as in Table 2.1:

Group	Edge groups	Stable letters
H_0		$\mathbf{a}_{01}$, $\mathbf{a}_{02}$, $\mathbf{y}$
G_0	$\langle \mathbf{a}_{01} \rangle \times \langle \mathbf{a}_{02} \rangle$	$\mathbf{u}_0$
H_1	$\langle \mathbf{a}_{01} \rangle \rtimes_\theta \langle \mathbf{u}_0, \mathbf{y} \rangle$ $\langle \mathbf{a}_{02} \rangle \rtimes_\theta \langle \mathbf{u}_0, \mathbf{y} \rangle$	$\mathbf{a}_{11}$ $\mathbf{a}_{12}$
G_1	$\langle \mathbf{a}_{11}, \mathbf{a}_{12} \rangle \times \langle \mathbf{u}_0 \rangle$	$\mathbf{u}_0$
H_2	$\langle \mathbf{a}_{11} \rangle \rtimes_\theta \langle \mathbf{u}_1, \mathbf{y} \rangle$ $\langle \mathbf{a}_{12} \rangle \rtimes_\theta \langle \mathbf{u}_1, \mathbf{y} \rangle$ $\langle \mathbf{u}_0^{-1}\mathbf{a}_{11} \rangle \rtimes_\theta \langle \mathbf{u}_1, \mathbf{a}_{01} \rangle$ $\langle \mathbf{u}_0^{-1}\mathbf{a}_{12} \rangle \rtimes_\theta \langle \mathbf{u}_1, \mathbf{a}_{02} \rangle$	$\mathbf{a}_{21}$ $\mathbf{a}_{22}$ $\mathbf{a}_{23}$ $\mathbf{a}_{24}$
G_2	$\langle \mathbf{a}_{21}, \mathbf{a}_{22}, \mathbf{a}_{23}, \mathbf{a}_{24} \rangle \times \langle \mathbf{u}_0 \rangle$	$\mathbf{u}_1$
H_2	$\langle \mathbf{a}_{21} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{y} \rangle$ $\langle \mathbf{a}_{22} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{y} \rangle$ $\langle \mathbf{a}_{23} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{a}_{01} \rangle$ $\langle \mathbf{a}_{24} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{a}_{02} \rangle$ $\langle \mathbf{u}_1^{-1}\mathbf{a}_{21} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{a}_{11} \rangle$ $\langle \mathbf{u}_1^{-1}\mathbf{a}_{22} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{a}_{12} \rangle$ $\langle \mathbf{u}_1^{-1}\mathbf{a}_{23} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{a}_{11} \rangle$ $\langle \mathbf{u}_1^{-1}\mathbf{a}_{24} \rangle \rtimes_\theta \langle \mathbf{u}_2, \mathbf{a}_{12} \rangle$	$\mathbf{a}_{31}$ $\mathbf{a}_{32}$ $\mathbf{a}_{33}$ $\mathbf{a}_{34}$ $\mathbf{a}_{35}$ $\mathbf{a}_{36}$ $\mathbf{a}_{37}$ $\mathbf{a}_{38}$
$\vdots$	$\vdots$	$\vdots$
H_n $n \geq 1$	$\langle \mathbf{a}_{(n-1)i} \rangle \rtimes_\theta \langle \mathbf{u}_{n-1}, \mathcal{L}_n(i) \rangle$ $1 \leq i \leq 2^{n-1}$ $\langle \mathbf{u}_{n-2}^{-1}\mathbf{a}_{(n-1)j} \rangle \rtimes_\theta \langle \mathbf{u}_{n-1}, \mathcal{L}_n(i) \rangle$ $2^{n-1} < i \leq 2^n, \ j = i - 2^{n-1}$	$\mathbf{a}_{ni}$ $1 \leq i \leq 2^{n-1}$ $\mathbf{a}_{ni}$ $2^{n-1} < i \leq 2^n$
G_n $n \geq 1$	$\langle \mathbf{a}_{nj} \rangle_{j=1}^{2^n} \times \langle \mathbf{u}_{n-1} \rangle$	$\mathbf{u}_n$

Table 2.1: Table of (super exponential) groups

Vector notation is used for the stable letters. That is, each bold faced vector denotes a free group of rank 2. For example $\mathbf{y}$ denotes the basis $\{y_1, y_2\}$, $\mathbf{u}_1$ denotes the basis $\{u_{11}, u_{12}\}$ and $\mathbf{a}_{23}$ denotes the basis $\{a_{231}, a_{232}\}$. Furthermore, an ordered list of k vectors describes an ordered basis of F_{2k}. For example, $\langle \mathbf{u}_0, \mathbf{y} \rangle$ denotes F_4 with ordered basis $\{u_{01}, u_{02}, y_1, y_2\}$. Also, product notation of vectors is used to denote coordinatewise multiplication of basis elements. For example, $\mathbf{u}_1^{-1}\mathbf{a}_{21}$ denotes the basis $\{u_{11}^{-1}a_{211}, u_{12}^{-1}a_{212}\}$. Lastly, $\mathcal{L}_n(i)$ denotes the i^{th} element of the ordered list $\mathcal{L}_n$ defined recursively as follows:

$$\mathcal{L}_1 = \{\mathbf{y}, \mathbf{y}\}$$
$$\mathcal{L}_2 = \{\mathcal{L}_1, \mathbf{a}_{01}, \mathbf{a}_{02}\}$$
$$\mathcal{L}_n = \{\mathcal{L}_{n-1}, \mathbf{a}_{(n-2)1}, \ldots, \mathbf{a}_{(n-2)2^{n-2}}, \mathbf{u}_{n-3}^{-1}\mathbf{a}_{(n-2)1}, \ldots, \mathbf{u}_{n-3}^{-1}\mathbf{a}_{(n-2)2^{n-2}}\}, \, n \geq 3$$

The edge maps for G_n are inclusion (as a subgroup) in one direction and $(\varphi * \varphi * \cdots * \varphi) \times \varphi$ in the other, where φ is the palindromic automorphism given above. And the edge maps for H_n are inclusion (as a subgroup) in one direction and $\varphi \times \mathrm{id}$, which acts by φ on the F_2 factor and by the identity on the F_4 factor; The map $\theta : F_4 \to Aut(F_2)$ is defined on the ordered basis $\{x_1, x_2, x_3, x_4\}$ as follows:

$$x_1 \mapsto \varphi, x_2 \mapsto \varphi, x_3 \mapsto \mathrm{id}, x_4 \mapsto \mathrm{id}$$

Next, we state the lemma that gives the structure of the edge groups in Table 2.1.

LEMMA 2.0.1. *For each $n > 0$, the following statements hold:*

1. *For $1 \leq i \leq 2^{n-1}$ the subgroup $\langle \boldsymbol{a}_{(n-1)i}, \boldsymbol{u}_{n-1}, \mathcal{L}_n(i)\rangle$ of G_{n-1} is isomorphic to $F_2 \rtimes_\theta F_4$, where F_2 has basis $\boldsymbol{a}_{(n-1)i}$ and F_4 has ordered basis $\{\boldsymbol{u}_{n-1}, \mathcal{L}_n(i)\}$ (θ defined as above).*

2. *For $1 \leq j \leq 2^{n-1}$ the subgroup $\langle \boldsymbol{u}_{n-2}^{-1}\boldsymbol{a}_{(n-1)i}, \boldsymbol{u}_{n-1}, \mathcal{L}_n(j + 2^{n-1})\rangle$ of G_{n-1} is isomorphic to $F_2 \rtimes_\theta F_4$, where F_2 has basis $\boldsymbol{u}_{n-2}^{-1}\boldsymbol{a}_{(n-1)i}$ and F_4 has ordered basis $\{\boldsymbol{u}_{n-1}, \mathcal{L}_n(j + 2^{n-1})\}$ (θ defined as above).*

3. *The subgroup $\langle \boldsymbol{a}_{n1}, \ldots, \boldsymbol{a}_{n2^n}, \boldsymbol{u}_{n-1}\rangle$ of H_n is isomorphic to $F_{2^{n+1}} \times F_2$, where the F_2 factor has basis $\boldsymbol{u}_{n-1}$.*

After defining our groups, we are ready to state the main theorem in [2].

THEOREM 2.0.2. *For every $n > 0$, there exist groups H_n and G_n of type $\mathcal{F}_3$, with $\delta^{(2)}_{H_n}(x) \simeq \exp^n(\sqrt{x})$ and $\delta^{(2)}_{G_n}(x) \simeq \exp^n(x)$.*

Chapter 3

Snowflake Groups

Brady, Bridson, Forester and Shankar show us in [6] how to construct snowflake groups, denoted by $G_{r,P}$, that have k-dimensional Dehn functions, $\delta^{(k)}_{G_{r,P}}(x) \simeq x^{2\alpha}$ where α is an element of a dense subset of $(1, \infty)$ containing the rational numbers greater than 1. We will be interested in the case for $k = 1$ and $k = 2$. A snowflake group, as defined in [6], is the fundamental group of a graph of groups determined by two parameters r and P. Let P be an irreducible non-negative square integer matrix with Perron-Frobenius eigenvalue $\lambda > 1$, and r an integer greater than every row sum of P. Then $\alpha = \log_\lambda r$. By ranging r and P, we see that indeed that α can be any rational number greater than 1.

Before we define $G_{r,P}$, we first define the group V_m that will end up being our vertex groups. The group V_m with $m \geq 2$ is defined as $m-1$ copies of $\mathbb{Z} \times \mathbb{Z}$ (the i^{th} copy having generators$\{a_i, b_i\}$) amalgamated so that we have the following relations:

$$b_1 = a_2 b_2, \quad b_2 = a_3 b_3, \quad \ldots, \quad b_{m-2} = a_{m-1} b_{m-1}.$$

We also define two new elements: $c = a_1 b_1$ and $a_m = b_{m-1}$. Then $a_1, \ldots, a_m$ generate V_m and the relation $a_1 \cdots a_m = c$ holds; thus c will be called the *diagonal element* of V_m. See Figure 3.0.1

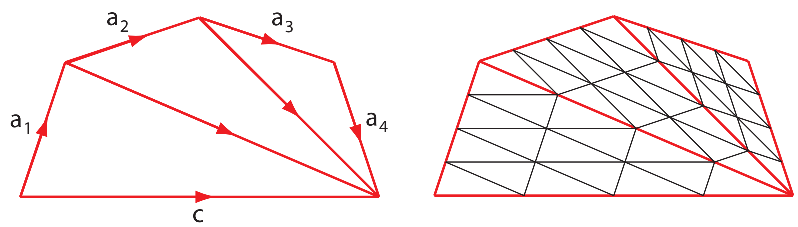

Figure 3.0.1: Some relations in V_4: $c = a_1 a_2 a_3 a_4$ and $c^3 = (a_1)^3 (a_2)^3 (a_3)^3 (a_4)^3$

If $m = 1$ then V_m is the infinite cyclic group $\langle a_1 \rangle$ and $c = a_1$. The following Lemma proved in [6] gives an area bound for V_m.

LEMMA 3.0.1. *(Area bound for V_m) Let $w(a_1, \ldots, a_{m-1}, b_1, \ldots, b_{m-1}, c)$ be a word representing the element x^N for some N, where x is a generator a_i, b_i or c. Let w be expressed as $w_1 \cdots w_k$ where each w_i is a power of a generator. Then $N \leq |w|$ and* $\text{Area}(wx^{-N}) \leq 3 \sum_{i<j} |w_i||w_j|$.

Now, to construct $G_{r,P}$, let $P = [p_{ij}]$ be a $R \times R$ matrix and r an integer greater than every row sum of P. Let $\mathcal{G}_{r,P}$ be the graph of groups whose underlying graph Γ has vertices $v_1, \ldots, v_R$ and p_{ij} directed edges from v_i to v_j. The vertex group G_{v_i} at v_i is defined to be V_{m_i} where $m_i = \sum_{j=1}^{R} p_{ij}$ (i.e. the sum of the entries of row i in P). Hence, there are $M = \Sigma_i^k m_i$ directed edges in Γ so that we can label the edges $\{e_1, \ldots, e_M\}$ and define two functions $\rho, \sigma : \{1, \ldots, M\} \to \{1, \ldots, R\}$ indicating the initial and terminal edges, respectively (i.e. e_i is a directed edge from $v_{\rho(i)}$ to $v_{\sigma(i)}$). Observe that these functions give the row and column of the matrix entry for e_i. Next, partition the set $\{1, \ldots, M\}$ as $\bigcup_i^R I_i$ where $I_i = \rho^{-1}(i)$ so that, by definition, I_i records the indices of the edges that emanate from v_i.

Since the number of standard generators of all the vertex groups add up to M, we will relabel these generators $\{a_1, \ldots, a_M\}$ in such a way that the standard generating set for G_{v_i} is $\{a_j \mid j \in I_i\}$.

Each edge group G_{e_i} of directed edge e_i is infinite cyclic. Let c_i be the diagonal element of G_{v_i}. Then the edge maps of G_{e_i} maps the generator of G_{e_i} to the elements $a_i^r \in G_{v_{\rho(i)}}$ and $c_{\sigma(i)} \in G_{v_{\sigma(i)}}$. The following Proposition 3.0.2 proven in [6] as Corollary 5.5 shows how geodesics in the edge group compares with geodesics in $G_{r,P}$.

PROPOSITION 3.0.2. *(Edge group distortion on $G_{r,P}$) Given r and P there is a positive constant D with the following property. If c is a diagonal element and w is a word in $G_{r,P}$ representing c^N then $|N| \leq D|w|^\alpha$.*

Let s_i be the stable letter associated to edge e_i. The fundamental group $G_{r,P}$ of $\mathcal{G}_{r,P}$ is obtained from the presentation

$$\langle G_{v_1}, \ldots, G_{v_R}, s_1, \ldots, s_M \mid s_i^{-1} a_i^r s_i = c_{\sigma(i)} \text{ for all } i \rangle$$

by adding relations $s_i = 1$ for each edge e_i in a maximal tree in Γ. We will continue to use the generating set $\{a_1, \ldots, a_M, s_1, \ldots, s_M\}$ for $G_{r,P}$ even though some of these generators are trivial.

Now, we define snowflake words from [6]. It is defined recursively on $|N| \in \mathbb{N}$ as follows. Let

$$N_0 = \frac{M(3r + r^2)}{r - M} + r$$

Let c be the diagonal element of a vertex group with the standard generating set $\{a_{i_1}, \ldots, a_{i_m}\}$. A word w^+ representing c^N is a *positive snowflake word* if either

1. $|N| \leq N_0$ and $w = a_{i_1}^N \cdots a_{i_m}^N$, or

2. $|N| > N_0$ and $w = (s_{i_1} u_1 s_{i_1}^{-1}) \cdots (s_{i_m} u_1 s_{i_m}^{-1})$ where each u_j is a positive snowflake word representing a power of the diagonal element c_j of a vertex group V_{m_j}.

A *negative snowflake word* w^- is defined similarly with the ordering of the terms representing powers of a_{i_j} reversed. A *snowflake disk* with diameter c^N is a disk whose boundary is $(w^+)(w^-)^{-1}$, see Figure 3.0.2.

REMARK 3.0.3. Brady et al. explain in [6] that for large enough $|N|$, the definition of snowflake words describes an iterated process of finding a path that represents c^N whose length is less than $|N|$.

The following proposition gives upper and lower bounds for the length of snowflake words representing c^N in terms of $|N|$ that will serve useful later in our paper. Its proof is given as Proposition 4.5 in [6].

PROPOSITION 3.0.4. *Given r and P there are positive constants C_0, C_1 with the following property. If c is the diagonal element of one of the vertex groups V_m and w is a snowflake word representing c^N then $C_0|w|^\alpha \leq |N| \leq C_1|w|^\alpha$, where $\alpha = log_\lambda(r)$ and λ is the Perron-Frobenius eigenvalue of P.*

We are now ready to state the theorem for the 1-dimensional Dehn function of a snowflake group:

THEOREM 3.0.5. *Let P be an irreducible non-negative integer matrix with Perron-Frobenius eigenvalue $\lambda > 1$, and r an integer greater than every row sum of P. Then there is a group $G_{r,P}$ that is finitely presented (i.e. of type $\mathcal{F}_2$) with Dehn function $\delta(x) \simeq x^{2\log_\lambda(r)}$.*

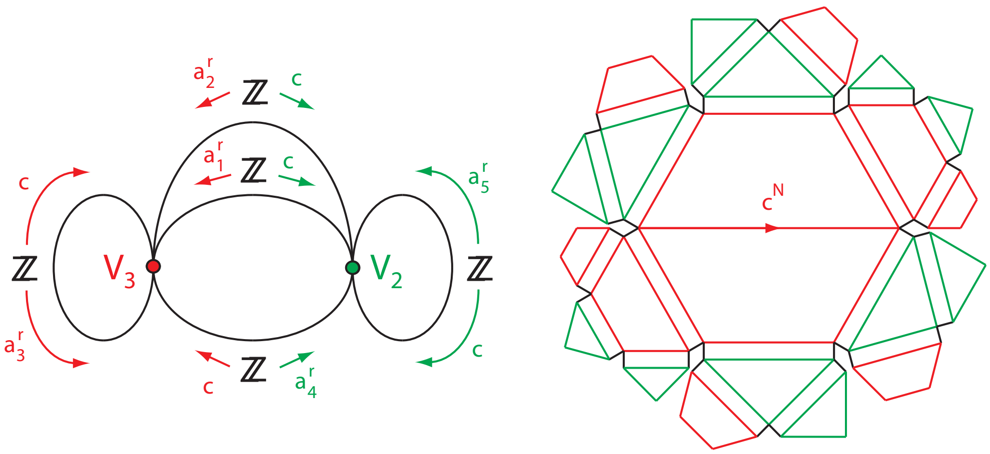

Figure 3.0.2: Example of a snowflake disk in $\widetilde{X}_{r,P}$ with $P = \left(\begin{smallmatrix}1 & 1\\ 2 & 1\end{smallmatrix}\right)$ and some integer $r > 3$.

Figure 3.0.2 shows an example of a snowflake disk imbedded in $\widetilde{X}_{r,P}$, the universal cover of the complex associated with $G_{r,P}$. To get snowflake balls, we define a monomorphism $\phi : G_{r,P} \to G_{r,P}$ that takes each a_i to a_i^r and each s_i to itself. Then the group $\Sigma G_{r,P}$ is defined to be the associated double HNN extension with stable letters u and v.

$$\Sigma G_{r,P} = \langle G_{r,P}, u, v \mid ugu^{-1} = \phi(g), vgv^{-1} = \phi(g),\ g \in G_{r,P}\rangle$$

We are now ready to state the theorem for the 2-dimensional Dehn function of a 3-dimensional snowflake group. Note, this is a special case of the actual theorem in [6] which states the theorem for general k-dimensional Dehn function of a $k+1$-dimensional snowflake group:

THEOREM 3.0.6. *Let P be an irreducible non-negative integer matrix with Perron-Frobenius eigenvalue $\lambda > 1$, and r an integer greater than every row sum of P. Then there is a group $\Sigma G_{r,P}$ of type $\mathcal{F}_3$ with 2-dimensional Dehn function $\delta^{(2)}(x) \simeq x^{2\log_\lambda(r)}$.*

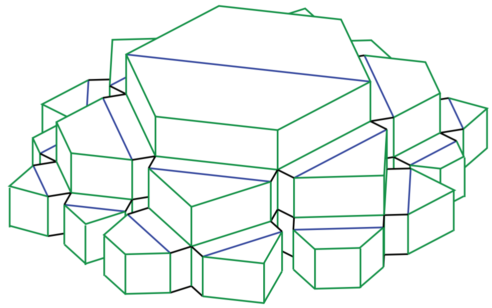

Figure 3.0.3: Example of a few layers of the upper hemisphere of a snowflake Ball B_j^3 imbedded in the universal cover of the complex associated with $\Sigma G_{r,P}$ with $P = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right)$ and some integer $r > 3$.

Now, we want to state a lemma from [6] that we will use in proving the lower bound of our balls. But before we state the lemma we want give some notation used in [6]. Figure 3.0.3 shows a few layers of the upper hemisphere of a B_j^3 ball where j denotes the height of a hemisphere of B_j^3. And for $i = 1, \ldots, j$, the disk B_i^2 denotes the 2-dimensional snowflake disk with diameter c^{r^i}.

LEMMA 3.0.7. *Given r and P there is a positive constant F_0 such that*

$$|\partial B_j^2| \leq \text{Area}(\partial B_j^3) \leq F_0 |\partial B_j^2|$$

for every j.

Chapter 4

Snowflake Groups with Super-Exponential 2-Dimensional Dehn Functions

We begin by constructing the sequence of groups up to H_n as in [2].

$$H_0 < G_0 < H_1 < \cdots < H_{n-1} < G_{n-1} < H_n$$

Next, we construct a snowflake group of type $\mathcal{F}_2$ as in [6]. Let P be an irreducible non-negative integer matrix with Perron-Frobenius eigenvalue $\lambda > 1$, r an integer number greater than every row sum of P, $\alpha = \log_\lambda r$, and let $G_{r,P}$ be the fundamental group of the graph of groups associated with r and P given in [6].

4.1 The groups $C_{n,P,r}$

Denote a vertex group of $G_{r,P}$ by V_m^* and define the group $C_{n,P,r}$ to be the fundamental group of the graph of groups represented by Figure 4.1.1 with vertex groups H_n and $G_{r,P}$:

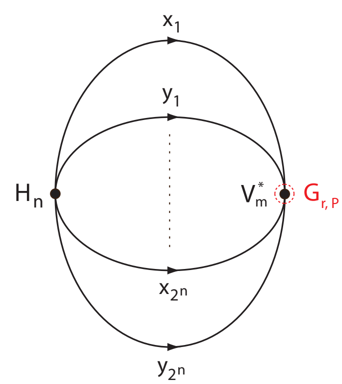

Figure 4.1.1: Graph of Groups with fundamental group $C_{n,P,r}$.

Where each edge group is infinite cyclic generated by $\langle a \rangle$. For each edge labeled x_i, y_i respectively, we have the following edge group monomorphisms:

$$\Phi_{x_i0} : \langle a \rangle \to H_n \qquad a \mapsto a_{ni1}$$

$$\Phi_{y_i0} : \langle a \rangle \to H_n \qquad a \mapsto u_{(n-1)1}^{-1} a_{ni1}$$

$$\Phi_{x_i1} : \langle a \rangle \to G_{r,P} \qquad a \mapsto c$$

$$\Phi_{y_i1} : \langle a \rangle \to G_{r,P} \qquad a \mapsto c$$

where c is the diagonal element in V_m^*. We denote the new stable letters of $C_{n,P,r}$ by x_i and y_i, respectively. That is, we add the following new relations $\{x_i^{-1} a_{ni1} x_i = c,\ y_i^{-1} u_{(n-1)1}^{-1} a_{ni1} y_i = c,\ x_1 \mid i = 1 \dots 2^n\}$.

4.2 The spaces $X_{C_{n,P,r}}$

A 3-complex $X_{C_{n,P,r}}$ whose fundamental group is $C_{n,P,r}$ can be constructed by forming a graph of spaces as follows. Start with an aspherical 3-complex K_{H_n} whose fundamental group is H_n as constructed in [2]; and an aspherical 2-complex $X_{r,P}$ whose fundamental group is $G_{r,P}$. Then attach annuli X_k and Y_k, one for each edge x_k, y_k, respectively in $\Gamma_{C_{n,P,r}}$. The two boundary curves of each X_k are attached to the edge labeled a_{nk1} in K_{H_n} and the diagonal edge labeled c in X_v^*, which is the subspace of $X_{r,P}$ associated with the subgroup V_m^* of $G_{r,P}$. The two boundary curves of each Y_k are attached to the edge labeled $u_{(n-1)1}^{-1}a_{nk1}$ in K_{H_n} and the edge labeled c in X_v^* of $X_{r,P}$. The resulting 3-complex $X_{C_{n,P,r}}$ has fundamental group $C_{n,P,r}$ and is aspherical because it is the total space of a graph of aspherical spaces. Therefore, $X_{C_{n,P,r}}$ is a 3-dimensional $K(C_{n,P,r}, 1)$ space.

The universal cover $\widetilde{X}_{C_{n,P,r}}$ is the union of copies of the universal covers $\widetilde{K}_{H_n}$ and $\widetilde{X}_{r,P}$ and infinite strips $\mathbb{R}^1 \times [-1, 1]$ covering each annuli X_k and Y_k. Each strip covering X_k is tiled by 2-cells whose boundary labels read $x_k^{-1}a_{nk1}x_kc^{-1}$; the two sides $\mathbb{R} \times \{\pm 1\}$ consist of edges labeled a_{nk1} and c respectively. And each strip covering Y_k is tiled by 2-cells whose boundary labels read $y_k^{-1}u_{(n-1)1}^{-1}a_{nk1}y_kc^{-1}$; the two sides $\mathbb{R} \times \{\pm 1\}$ consist of edges labeled $u_{(n-1)1}^{-1}a_{nk1}$ and c respectively. See Figure 4.2.1.

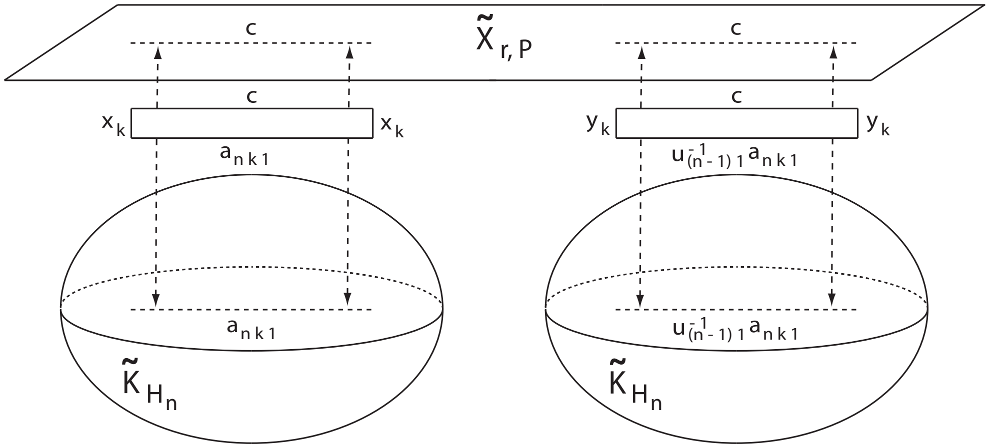

Figure 4.2.1: A local picture of $\widetilde{X}_{C_{n,P,r}}$

4.3 The subgroups $\bar{G}_{r,P}$

Next, we want to show there are subgroups $\bar{G}_{r,P}$, $W_{k,r,P}$ and $W'_{k,r,P}$ of $C_{n,P,r}$. In this section, we we will define $\bar{G}_{r,P}$ and prove that it is a subgroup of $C_{n,P,r}$. Consider $G_{r,P}$ and V_m^* from $C_{n,P,r}$ and for $k = 1, \ldots, 2^n$, define

$$S \quad = \quad \langle u_{(n-1)1} \rangle \times \langle a_{n11}, a_{n21}, a_{n31}, \ldots, a_{n2^n1} \rangle$$

Then $\bar{G}_{r,P}$ is the fundamental group of graph of groups represented in Figure 4.3.1.

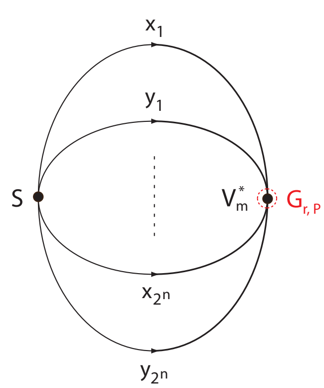

Figure 4.3.1: The graph of groups of $\bar{G}_{r,P}$

Where the edge groups are all infinite cyclic and the stable letters are x_i and y_i, respectively as shown. The edge maps are inclusion and are defined similar to those of the graph of groups of $C_{n,P,r}$ so that the following relations $\{x_i^{-1}a_{ni1}x_i = c,\ y_i^{-1}u_{(n-1)1}^{-1}a_{ni1}y_i = c,\ x_1 = 1 \mid i = 1, \ldots, 2^n\}$ exist inside $\bar{G}_{r,P}$.

To show $\bar{G}_{r,P}$ is a subgroup of $C_{n,P,r}$, we first prove the following lemma.

LEMMA 4.3.1. *There exists a retraction $r : H_n \to S$.*

Proof. First, note there is a retraction $f : G_{n-1} \to \langle u_{(n-1)1} \rangle$ defined by the trivial map on H_{n-1} and $u_{(n-1)2}$ and the identity map on $u_{(n-1)1}$. This is clear from the graph of groups description of G_{n-1} given by Figure 4.3.2.

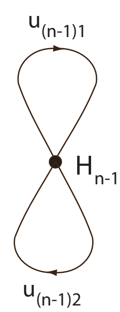

Figure 4.3.2: Graph of groups for G_{n-1}. The vertex group is H_{n-1}. The stable letters are $u_{(n-1)1}$ and $u_{(n-1)2}$.

Now, define $r : H_n \to S$ by f on G_{n-1}, the identity map on the stable letters a_{ni1} and the trivial map on the stable letters a_{ni2}, $i = 1, \ldots, 2^n$.

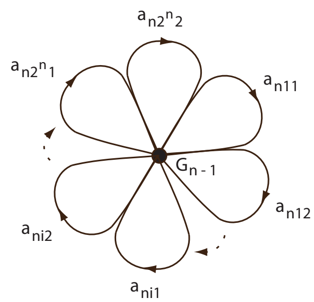

Figure 4.3.3: Graph of groups for H_n. The vertex group is G_{n-1}. The stable letters are a_{ni1} and a_{ni2} for $i = 1, \ldots, 2^n$.

To show that r is a retraction, we need to show that r preserves the relations of H_n (Figure 4.3.3 gives the graph of groups description of H_n). Since $r|_{G_{n-1}} = f$ is a retraction of G_{n-1} onto $\langle u_{(n-1)1} \rangle < S$ thus preserves all relations in $G_{n-1} < H_n$, we only need to check the relations of H_n involving the new stable letters a_{ni1} and a_{ni2}. They are listed as follows:

$a_{ni1}(a_{(n-1)i1})a_{ni1}^{-1}(\varphi(a_{(n-1)i1}))^{-1}$	$1 \leq i \leq 2^{n-1}$
$a_{ni1}(a_{(n-1)i2})a_{ni1}^{-1}(\varphi(a_{(n-1)i2}))^{-1}$	$1 \leq i \leq 2^{n-1}$
$a_{ni1}(u_{(n-2)1}^{-1}a_{(n-1)j1})a_{ni1}^{-1}(\varphi(u_{(n-2)1}^{-1}a_{(n-1)j1}))^{-1}$	$2^{n-1} < i \leq 2^n,\ j = i - 2^{n-1}$
$a_{ni1}(u_{(n-2)2}^{-1}a_{(n-1)j2})a_{ni1}^{-1}(\varphi(u_{(n-2)2}^{-1}a_{(n-1)j2}))^{-1}$	$2^{n-1} < i \leq 2^n,\ j = i - 2^{n-1}$
$a_{ni1}(u_{(n-1)1})a_{ni1}^{-1}(u_{(n-1)1})^{-1}$	$1 \leq i \leq 2^n$
$a_{ni1}(u_{(n-1)2})a_{ni1}^{-1}(u_{(n-1)2})^{-1}$	$1 \leq i \leq 2^n$
$a_{ni1}(\mathcal{L}_n(i)_1)a_{ni1}^{-1}(\mathcal{L}_n(i)_1)^{-1}$	$1 \leq i \leq 2^n$
$a_{ni1}(\mathcal{L}_n(i)_2)a_{ni1}^{-1}(\mathcal{L}_n(i)_2)^{-1}$	$1 \leq i \leq 2^n$

$a_{ni2}(a_{(n-1)i1})a_{ni2}^{-1}(\varphi(a_{(n-1)i1}))$	$1 \leq i \leq 2^{n-1}$
$a_{ni2}(a_{(n-1)i2})a_{ni2}^{-1}(\varphi(a_{(n-1)i2}))$	$1 \leq i \leq 2^{n-1}$
$a_{ni2}(u_{(n-2)1}^{-1}a_{(n-1)j1})a_{ni2}^{-1}(\varphi(u_{(n-2)1}^{-1}a_{(n-1)j1}))^{-1}$	$2^{n-1} < i \leq 2^n,\ j = i - 2^{n-1}$
$a_{ni2}(u_{(n-2)2}^{-1}a_{(n-1)j2})a_{ni2}^{-1}(\varphi(u_{(n-2)2}^{-1}a_{(n-1)j2}))^{-1}$	$2^{n-1} < i \leq 2^n,\ j = i - 2^{n-1}$
$a_{ni2}(u_{(n-1)1})a_{ni2}^{-1}(u_{(n-1)1})$	$1 \leq i \leq 2^n$
$a_{ni2}(u_{(n-1)2})a_{ni2}^{-1}(u_{(n-1)2})$	$1 \leq i \leq 2^n$
$a_{ni2}(\mathcal{L}_n(i)_1)a_{ni2}^{-1}(\mathcal{L}_n(i)_1)^{-1}$	$1 \leq i \leq 2^n$
$a_{ni2}(\mathcal{L}_n(i)_2)a_{ni2}^{-1}(\mathcal{L}_n(i)_2)^{-1}$	$1 \leq i \leq 2^n$

Where φ is the palindromic automorphism of a free group of rank 2 discussed in Section 2. Notice that these relations are all of the form: $a_{ni1}Xa_{ni1}^{-1}Y$ and $a_{ni2}Xa_{ni2}^{-1}Y$.

If $X \in \{a_{(n-1)i1}, a_{(n-1)i2}, u_{(n-2)1}^{-1}a_{(n-1)j1}, u_{(n-2)2}^{-1}a_{(n-1)j2}\} \subset H_{n-1}$, then $Y = \varphi(X)$ is an element of H_{n-1} and

$$r(a_{ni1})r(X)r(a_{ni1}^{-1})r(Y) = a_{ni1}1a_{ni1}^{-1}1 = 1$$

and

$$r(a_{ni2})r(X)r(a_{ni2}^{-1})r(Y) = (1)(1)(1)(1) = 1$$

as desired.

If $X = u_{(n-1)1}$, then $Y = u_{(n-1)1}^{-1}$ and

$$r(a_{ni1})r(X)r(a_{ni1}^{-1})r(Y) = a_{ni1}u_{(n-1)1}a_{ni1}^{-1}u_{(n-1)1}^{-1} = u_{(n-1)1}a_{ni1}a_{ni1}^{-1}u_{(n-1)1}^{-1} = 1$$

since $u_{(n-1)1}$ commutes with all a_{ni1} in S, and

$$r(a_{ni2})r(X)r(a_{ni2}^{-1})r(Y) = (1)u_{(n-1)1}(1)u_{(n-1)1}^{-1} = 1$$

as desired.

If $X = u_{(n-1)2}$, then $Y = u_{(n-1)2}^{-1}$ and $r(a_{ni1})r(X)r(a_{ni1}^{-1})r(Y) = a_{ni1}(1)a_{ni1}^{-1}(1) = 1$ and $r(a_{ni2})r(X)r(a_{ni2}^{-1})r(Y) = (1)(1)(1)(1) = 1$ as desired.

If $X \in \{\mathcal{L}_n(i)_1, \mathcal{L}_n(i)_2\} \subset H_{n-1}$, then $Y = X$ and $r(a_{ni1})r(X)r(a_{ni1}^{-1})r(Y) = a_{ni1}(1)a_{ni1}^{-1}(1) = 1$ and $r(a_{ni2})r(X)r(a_{ni2}^{-1})r(Y) = (1)(1)(1)(1) = 1$ as desired. □

Note that since S clearly includes into H_n and that r composed with this inclusion is the identity on S, S is a subgroup of H_n.

CLAIM 4.3.2. *$\bar{G}_{r,P}$ is a subgroup of $C_{n,P,r}$.*

Proof. We will prove this by showing there exists a retraction from $C_{n,P,r}$ to $\bar{G}_{r,P}$. To that end, let $R : C_{n,P,r} \to \bar{G}_{r,P}$ be a map defined by the identity on $G_{r,P}$, x_i and y_i ($i = 1, \ldots, 2^n$), and r from Lemma 4.3.1 on H_n. See Figure 4.3.4.

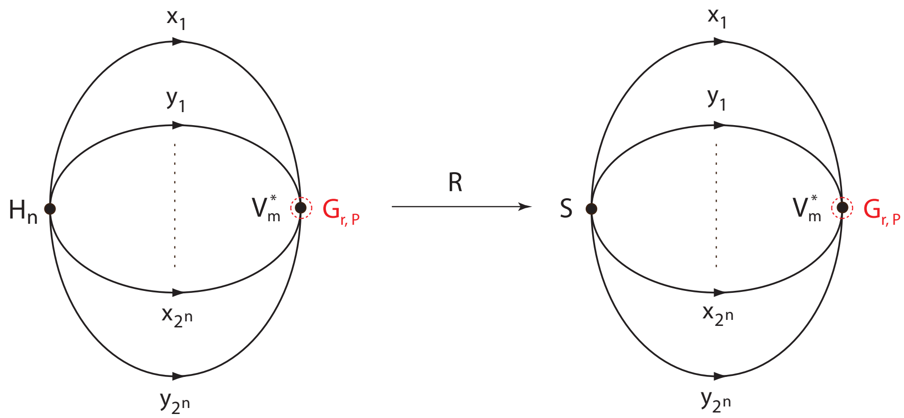

Figure 4.3.4:

To show that R is a retraction, we need to show that R preserves the relations of $C_{n,P,r}$. Since $R|_{G_{r,P}}$ is the identity on $G_{r,P} < \bar{G}_{r,P}$; and $R|_{H_n}$ is the retraction r from H_n onto $S < \bar{G}_{r,P}$, the only relations we need to check are the ones involving the new stable letters x_i and y_i. Those relations are precisely $\{x_i^{-1}a_{ni1}x_ic^{-1} = 1$, $y_i^{-1}u_{(n-1)1}^{-1}a_{ni1}y_ic^{-1} = 1$, $x_1 = 1 \mid i = 1, \ldots, 2^n\}$. A quick check shows that these relations hold:

$$
\begin{aligned}
R(x_i^{-1})R(a_{ni1})R(x_i)R(c^{-1}) &= x_i^{-1}a_{ni1}x_ic^{-1} &&= 1 \\
R(y_i^{-1})R(u_{(n-1)1}^{-1}a_{ni1})R(y_i)R(c^{-1}) &= y_i^{-1}u_{(n-1)1}^{-1}a_{ni1}y_ic^{-1} &&= 1 \\
R(x_1) &= x_1 &&= 1
\end{aligned}
$$

Now, let $i : \bar{G}_{r,P} \to C_{n,P,r}$ be the inclusion map (it is clearly a homomorphism). Then $R \circ i$ is the identity map on $\bar{G}_{r,P}$. This shows that i is injective and thus $\bar{G}_{r,P}$ is a subgroup of $C_{n,P,r}$. □

4.4 The subspaces $X_{\bar{G}_{r,P}}$

In this section we show that the subcomplex $X_{\bar{G}_{r,P}}$ of $X_{C_{n,P,r}}$ whose fundamental group is $\bar{G}_{r,P}$ is an aspherical 2-complex. It is constructed from an aspherical 2-complex $X_{r,P}$ whose fundamental group is $G_{r,P}$; and a 2-dimensional aspherical subcomplex K_S of K_{H_n} whose fundamental group is $S < H_n$. Note that K_S is 2^n copies of a 2-dimensional torus where the k^{th} copy is generated by the circles labeled by $u_{(n-1)1}$ and a_{nk1}. These tori are all attached along the generating circle labeled $u_{(n-1)1}$. Now attach annuli X_k and Y_k, one for each edge x_k, y_k, respectively in $\Gamma_{\bar{G}_{r,P}}$. The two boundary curves of each X_k are attached to the edge labeled a_{nk1} in K_S and the diagonal edge labeled c in X_v^*, which is the subspace of $X_{r,P}$ associated with the subgroup V_m^* of $G_{r,P}$. The two boundary curves of each Y_k are attached to the diagonal edge labeled $u_{(n-1)1}^{-1}a_{nk1}$ in K_S and the edge labeled c in X_v^* of $X_{r,P}$. The result is a 2-complex $X_{\bar{G}_{r,P}}$ whose fundamental group is $\bar{G}_{r,P}$ and is aspherical because it is the total space of a graph of aspherical spaces.

The universal cover $\widetilde{X}_{\bar{G}_{r,P}}$ is the union of copies of the universal covers $\widetilde{X}_{r,P}$ and $\widetilde{K}_S$ (which is the product space of a tree and $\mathbb{R}$) and infinite strips $\mathbb{R}^1 \times [-1,1]$ covering each annulus X_k and Y_k. Each strip covering X_k is tiled by 2-cells whose boundary labels read $x_k^{-1}a_{nk1}x_kc^{-1}$; the two sides $\mathbb{R} \times \{\pm 1\}$ consist of edges labeled a_{nk1} and c respectively. And each strip covering Y_k is tiled by 2-cells whose boundary labels read $y_k^{-1}u_{(n-1)1}^{-1}a_{nk1}y_kc^{-1}$; the two sides $\mathbb{R} \times \{\pm 1\}$ consist of edges labeled $u_{(n-1)1}^{-1}a_{nk1}$ and c respectively.

4.5 The subgroups $W_{k,r,P}$ and $W'_{k,r,P}$

In this section, we we will define $W_{k,r,P}$ and $W'_{k,r,P}$ and prove that they are a subgroups of $C_{n,P,r}$. Consider $G_{r,P}$ and V^*_m from $C_{n,P,r}$ and for $k = 1, \ldots, 2^n$ define

$$\begin{aligned} S_k &= \langle \mathcal{L}_{n+1}(k)_1 \rangle \times \langle a_{nk1} \rangle \\ T_k &= \langle \mathcal{L}_{n+1}(k+2^n)_1 \rangle \times \langle u^{-1}_{(n-1)1} a_{nk1} \rangle \end{aligned}$$

where $\mathcal{L}_{n+1}(k)$ is the k^{th} element of the ordered list $\mathcal{L}_{n+1}$ given in [2] and restated again in Section 2 of this paper. Since each element of this list is a free groups of rank 2, the subscript 1 in $\mathcal{L}_{n+1}(i)_1$ denotes the first generator of this free group.

Then $W_{k,r,P}$ and $W'_{k,r,P}$ are the fundamental group of the graph of groups given in Figure 4.5.1, respectively.

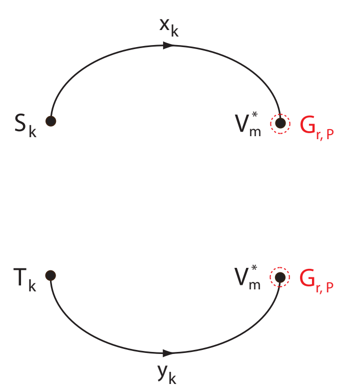

Figure 4.5.1: The graph of groups of $W_{k,r,P}$ (top right) and $W'_{k,r,P}$ (bottom right)

Where the edge groups are infinite cyclic and the stable letters are x_k and y_k, respectively as shown. The edge maps are inclusion and are defined similar to those of the graph of groups of $C_{n,P,r}$ so that the following relations $\{x_k^{-1} a_{nk1} x_i = c,\ x_k = 1\}$ are inside $W_{k,r,P}$; and the relations $\{y_k^{-1} u^{-1}_{(n-1)1} a_{nk1} y_k = c,\ y_k = 1\}$ exists inside $W'_{k,r,P}$.

Now, we want show the groups $W_{k,r,P}$ and $W'_{k,r,P}$ are subgroups of $C_{n,P,r}$. We begin by defining a graph Γ consisting of two vertices $\{v_0, v_1\}$ and a pair of edges $\{e, \bar{e}\}$ between them where e is the directed edge from v_0 to v_1. Next, we define a graph of groups $\mathcal{G}$ over Γ by assigning $\mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$ to v_0, $G_{r,P}$ to v_1 and $\mathbb{Z} = \langle e \rangle$ to the edge pair $\{e, \bar{e}\}$. The edge map $\mu_e : \langle e \rangle \to \langle a \rangle \times \langle b \rangle$ is defined by $e \mapsto a_{nk1}$; and the edge map $\mu_{\bar{e}} : \langle e \rangle \to G_{r,P}$ is defined by $e \mapsto c$ where c is the diagonal element of V_m^* in $G_{r,P}$. Now, for $k = 1, \ldots, 2^n$, define two sets of untwisted morphisms between graphs of groups $\{\Phi_k : (\Gamma, \mathcal{G}) \to (\Gamma_{C_{n,P,r}}, \mathcal{C}_{n,P,r})\}$ and $\{\Phi'_k : (\Gamma, \mathcal{G}) \to (\Gamma_{C_{n,P,r}}, \mathcal{C}_{n,P,r})\}$. See Figure 4.5.2:

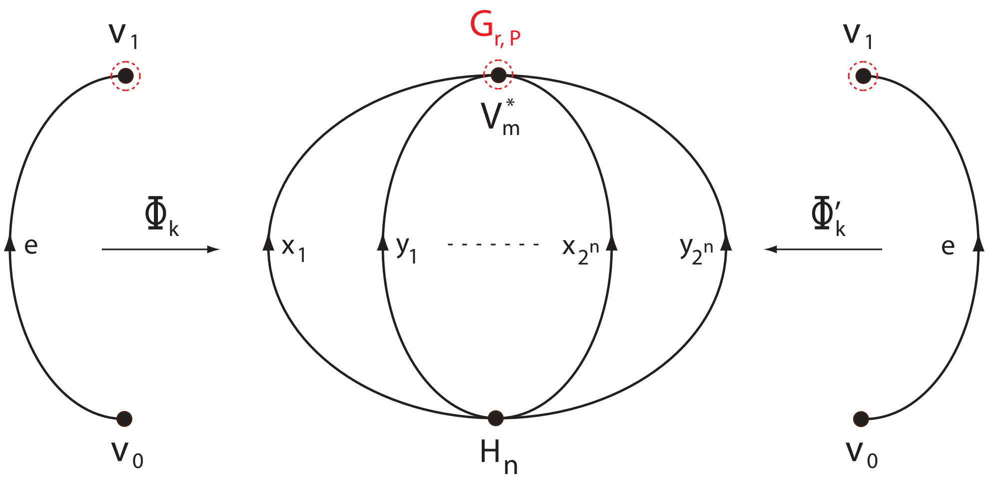

Figure 4.5.2:

with $\phi_k, \phi'_k : \Gamma \to \Gamma_{C_{n,P,r}}$ taking $v_0,\ v_1$ to the vertices in $\Gamma_{C_{n,P,r}}$ that are assigned to H_n and $G_{r,P}$ in $\mathcal{C}_{n,P,r}$, respectively; and the edge e is taken to the edge in $\Gamma_{C_{n,P,r}}$ associated with stable letter x_k by ϕ_k, and to the edge associated with stable letter y_k by ϕ'_k. The vertex group maps $(\phi_k)_{v_0}, (\phi'_k)_{v_0} : \langle a \rangle \times \langle b \rangle \to H_n$ are defined by $(\phi_k)_{v_0}(a) = a_{nk1},\ (\phi_k)_{v_0}(b) = \mathcal{L}_{n+1}(k)_1,\ (\phi'_k)_{v_0}(a) = u^{-1}_{(n-1)1} a_{nk1},\ (\phi_k)_{v_0}(b) = \mathcal{L}_{n+1}(k + 2^n)_1$. And let the vertex group maps $(\phi_k)_{v_1}, (\phi'_k)_{v_1} : G_{r,P} \to G_{r,P}$ be the identity on $G_{r,P}$. The edge group maps $(\phi_k)_e : \langle e \rangle \to \langle x_k \rangle$ maps e to x_k;

whereas the edge group maps $(\phi'_k)_e : \langle e \rangle \to \langle y_k \rangle$ maps e to y_k. We will finish showing Φ_k and Φ'_k are untwisted morphisms between graphs of groups in the claim below. But first, notice the images of Φ_k and Φ'_k are $W_{k,r,P}$ and $W'_{k,r,P}$, respectively. To show $W_{k,r,P}$ and $W'_{k,r,P}$ are subgroups of $C_{n,P,r}$, we want to show Φ_k and Φ'_k induce injective homomorphisms on fundamental groups. If we can show additionally that Φ_k and Φ'_k are immersions between graphs of groups, we can use Proposition 1.1.18 to show this injection.

CLAIM 4.5.1. *For $k = 1, \ldots, 2^n$, Φ_k and Φ'_k defined above are untwisted morphisms between graphs of groups. Moreover, they are immersions.*

Proof. First, we will show that Φ_k and Φ'_k are untwisted morphisms between graphs of groups. It is clear from the definition that ϕ_k and ϕ'_k are graph morphisms; and that $(\phi_k)_{v_1}$, $(\phi_k)_e$, $(\phi'_k)_{v_1}$ and $(\phi'_k)_e$ are all (injective) group homomorphisms.

To see that $(\phi_k)_{v_0}$ and $(\phi'_k)_{v_0}$ are injective group homomorphisms, we just need to show that the subgroups $\langle a_{nk1}, \mathcal{L}_{n+1}(k)_1 \rangle$ and $\langle u^{-1}_{(n-1)1} a_{nk1}, \mathcal{L}_{n+1}(k + 2^n)_1 \rangle$ in H_n is precisely $\langle a_{nk1} \rangle \times \langle \mathcal{L}_{n+1}(k)_1 \rangle$ and $\langle u^{-1}_{(n-1)1} a_{nk1} \rangle \times \langle \mathcal{L}_{n+1}(k + 2^n)_1 \rangle$ (i.e. they generate a $\mathbb{Z} \times \mathbb{Z}$ subgroup). By Lemma 2.0.1 these generators do generate a $\mathbb{Z} \times \mathbb{Z}$ subgroup in G_n. But since these generators are also elements of H_n, they generate a $\mathbb{Z} \times \mathbb{Z}$ subgroup in $H_n < G_n$. Therefore $(\phi_k)_{v_0}$ and $(\phi'_k)_{v_0}$ are injective group homomorphisms.

To finish showing that Φ_k is an untwisted morphisms of graphs of groups, we just need to prove that each square in Diagram 4.5.2 commutes:

$$
\begin{array}{ccc}
G_{v_1} = G_{r,P} & \xrightarrow{(\phi_k)_{v_1}} & G_{r,P} \\
\mu_{\bar{e}} \big\uparrow & & \big\uparrow \mu_{\bar{x}_k} \\
\langle e \rangle & \xrightarrow{(\phi_k)_e} & \langle x_k \rangle \\
\mu_e \big\downarrow & & \big\downarrow \mu_{x_k} \\
\langle a \rangle \times \langle b \rangle & \xrightarrow[(\phi_k)_{v_0}]{} & H_n
\end{array}
\tag{4.5.2}
$$

To see $(\phi_k)_{v_1} \circ \mu_{\bar{e}}(e) = \mu_{\bar{x}_k} \circ \phi_e(e)$:

$$
\begin{aligned}
(\phi_k)_{v_1} \circ \mu_{\bar{e}}(e) &= (\phi_k)_{v_1}(\mu_{\bar{e}}(e)) \\
&= (\phi_k)_{v_1}(c) \\
&= c \\
\mu_{\bar{x}_k} \circ (\phi_k)_e &= \mu_{\bar{x}_k}((\phi_k)_e(e)) \\
&= \mu_{\bar{x}}(x_k) \\
&= c
\end{aligned}
$$

where c is the diagonal element in $V_m^* < G_{r,P}$. To see $(\phi_k)_{v_0} \circ \mu_e(e) = \mu_{x_k} \circ \phi_e(e)$:

$$
\begin{aligned}
(\phi_k)_{v_0} \circ \mu_e(e) &= (\phi_k)_{v_0}(\mu_e(e)) \\
&= \phi_{v_{0,k}}(a) \\
&= a_{nk1} \\
\mu_{x_k} \circ (\phi_k)_e(e) &= \mu_{x_k}((\phi_k)_e(e)) \\
&= \mu_{x_k}(x_k) \\
&= a_{nk1}
\end{aligned}
$$

A similar proof shows that each square in Diagram 4.5.3 induced by Φ_k' commutes:

$$\begin{array}{ccc} G_{r,P} & \xleftarrow{(\phi_k')_{v_1}} & G_{r,P} = G_{v_1} \\ {\scriptstyle \mu_{\bar{y}_k}} \uparrow & & \uparrow {\scriptstyle \mu_{\bar{e}}} \\ \langle y_k \rangle & \xleftarrow{(\phi_k)_e} & \langle e \rangle \\ {\scriptstyle \mu_{y_k}} \downarrow & & \downarrow {\scriptstyle \mu_e} \\ H_n & \xleftarrow[(\phi_k')_{v_0}]{} & \langle a \rangle \times \langle b \rangle \end{array} \tag{4.5.3}$$

Therefore Φ_k' is also an untwisted morphism of graphs of groups.

Now, we show that Φ_k is an immersion. We need to show that each of following is injective:

$$\Phi_{v_1/x_i} : (\Pi_{e \in (\phi_k)_{v_1}^{-1}(x_i)} G_{v_1/e}) \to G_{r,P}/\mu_{\bar{x}_i}(\langle x_i \rangle)$$
$$\Phi_{v_0/x_i} : (\Pi_{e \in (\phi_k)_{v_0}^{-1}(x_i)} G_{v_0/e}) \to G_{r,P}/\mu_{x_k}(\langle x_i \rangle)$$

where for $i = 1, \ldots, 2^n$, x_i is the generator of the edge group assigned to edge $x_i \in \Gamma_{C_{n,P,r}}$. For $j = 0, 1$, because Γ has only one edge, $(\phi_k)_{(v_j)}^{-1} : St_{\Gamma_{C_{n,P,r}}}(\phi_k(v_j)) \to St_{\Gamma}(v_j)$ is only defined for edge $x_k \in E(\Gamma_{C_{n,P,r}})$ when $j = 0$ and $\bar{x}_k$ when $j = 1$. Therefore, the above maps respectively simplify to:

$$\Phi_{v_1/x_k} : G_{v_1/e} \to G_{r,P}/\mu_{\bar{x}_k}(\langle x_k \rangle)$$
$$\Phi_{v_0/x_k} : G_{v_0/e} \to H_n/\mu_{x_k}(\langle x_k \rangle)$$

Evaluating $G_{v_j/e}$ and edge maps further simplifies the maps respectively to:

$$\Phi_{v_1/x_k} : G_{r,P}/\langle c\rangle \to G_{r,P}/\langle c\rangle$$

$$\Phi_{v_0/x_k} : \langle a\rangle \times \langle b\rangle/\langle a\rangle \to H_n/\langle a_{nk1}\rangle$$

The first map is the identity map and therefore injective. The second map, because $\langle a\rangle \times \langle b\rangle \to H_n$ is injective (via $(\phi_k)_{v_0}$) and $a \mapsto a_{nk1}$, $\langle a\rangle \times \langle b\rangle/\langle a\rangle \to H_n/\langle a_{nk1}\rangle$ is injective as required. This shows Φ_k is an immersion.

A similar proof shows that Φ'_k is an immersion. □

Now Proposition 1.1.18 states that $(\Phi_k)_*$ and $(\Phi'_k)_*$ from $W_{k,r,p}$ and $W'_{k,r,p}$, respectively, into $C_{n,P,r}$ are injective homomorphisms. Hence $W_{k,r,p}$ and $W'_{k,r,p}$ are subgroups of $C_{n,P,r}$.

4.6 The subspaces $X_{W_{k,r,p}}$ and $X_{W'_{k,r,p}}$

In this sections we show that the subcomplex $X_{W_{k,r,p}}$ and $X_{W'_{k,r,p}}$ of $X_{C_{n,P,r}}$ whose fundamental group are $W_{k,r,p}$ and $W'_{k,r,p}$, respectively, are aspherical 2-complexes. Each is constructed from an aspherical 2-complex $X_{r,P}$ whose fundamental group is $G_{r,P}$; and a 2-dimensional asperical subcomplex K_{S_k} or K_{T_k} of K_{H_n} whose fundamental group is S_k or T_k, respectively. Note that K_{S_k} and K_{T_k} are each a 2-dimensional torus where K_{S_k} is generated by the circles labeled by $\mathcal{L}_{n+1}(k)_1$ and a_{nk1}; whereas K_{T_k} is generated by the circles labeled by $\mathcal{L}_{n+1}(k+2^n)_1$ and $u_{(n-1)1^{-1}a_{nk1}}$. Now attach annuli X_k and Y_k for the edges labeled x_k and y_k in Γ_{S_k} and Γ_{T_k}, respectively. The two boundary curves of X_k are attached to the edge labeled a_{nk1} in K_{S_k} and the diagonal edge labeled c in X^*_v, which is the subspace of $X_{r,P}$ associated with the subgroup V_m^* of $G_{r,P}$. The two boundary curves of each Y_k are attached to the diagonal edge labeled $u_{(n-1)1}^{-1}a_{nk1}$ in K_{T_k} and the edge labeled c in X_v^* of $X_{r,P}$. The result are 2-complexes $X_{W_{k,r,p}}$ and $X_{W'_{k,r,p}}$ whose fundamental groups are $W_{k,r,p}$ and $W'_{k,r,p}$, respectively. They are aspherical because it is the total space of a graph of aspherical spaces.

The universal cover of $X_{W_{k,r,p}}$ is the union of copies of the universal covers $\widetilde{X}_{r,P}$ and $\widetilde{K}_{S_k}$ (which is simply $\mathbb{R}^2$) and infinite strips $\mathbb{R}^1 \times [-1,1]$ covering each annuli X_k. Each strip covering X_k is tiled by 2-cells whose boundary labels read $x_k^{-1}a_{nk1}x_kc^{-1}$; the two sides $\mathbb{R} \times \{\pm 1\}$ consist of edges labeled a_{nk1} and c respectively.

The universal cover of $X_{W_{k,r,P}}$ is the union of copies of the universal covers $\widetilde{X}_{r,P}$ and $\widetilde{K}_{T_k}$ (which is $\mathbb{R}^2$) and infinite strips $\mathbb{R}^1 \times [-1,1]$ covering each annuli Y_k. Each strip covering Y_k is tiled by 2-cells whose boundary labels read $y_k^{-1}u_{(n-1)1}^{-1}a_{nk1}y_kc^{-1}$; the two sides $\mathbb{R} \times \{\pm 1\}$ consist of edges labeled $u_{(n-1)1}^{-1}a_{nk1}$ and c respectively.

4.7 The groups $SES_{n,P,r}$

Finally, we define a snowflake group whose 2-dimensional Dehn function is super-exponential of height n as follows:

DEFINITION 4.7.1. Let $n \geq 1$ be an integer, P an irreducible non-negative integer square matrix with Perron-Frobenius eigenvalue $\lambda > 1$, r an integer greater than every row sum of P, and let M be the sum of the integer entries of P. We define a group $SES_{n,P,r}$ to be a $(2^{n+1}+1)$-multiple HNN-extension of $C_{n,P,r}$ with edge groups $\bar{G}_{r,P}$, $W_{k,r,P}$ and $W'_{k,r,P}$, $k = 1, \ldots, 2^n$. See Figure 4.7.1:

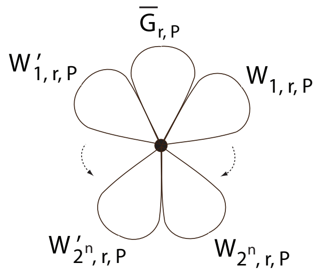

Figure 4.7.1: Graph of groups for $SES_{n,P,r}$. The vertex group is $C_{n,P,r}$ and the edge groups are labeled.

The edge maps for the edge associated with $\bar{G}_{r,P}$ are inclusion as a subgroup in one direction and $\phi : \bar{G}_{r,P} \to \bar{G}_{r,P}$ in the other, where for $i = 1, \ldots, 2^n$ and $j = 1, \ldots, M$, ϕ takes each $u_{(n-1)1}, a_{ni1}$ and a_j to $u^r_{(n-1)1}, a^r_{ni1}$ and a^r_j, respectively, and each x_i, y_i and s_j to themselves (Note, here we use $\{a_1, \ldots, a_M, s_1, \ldots, s_M\}$ as the generating set for $G_{r,P}$ as discussed in Section 3). The stable letter for this edge is z.

The edge maps for the edge associated with $W_{k,r,P}$ are inclusion as a subgroup in one direction and $\phi_k : W_{k,r,P} \to W_{k,r,P}$ in the other, where for $j = 1, \ldots, M$, ϕ_k takes a_{nk1} to a^r_{nk1}, each a_j to a^r_j, and each $\mathcal{L}_{n+1}(k)_1, x_i, y_i$ and s_j to themselves. The stable letters for these edges are b_k, respectively.

The edge maps for the edge associated with $W'_{k,r,P}$ are inclusion as a subgroup in one direction and $\phi'_k : W'_{k,r,P} \to W'_{k,r,P}$ in the other, where for $j = 1, \ldots, M$, ϕ_k takes $u^{-1}_{(n-1)1}a_{nk1}$ to $(u^{-1}_{(n-1)1}a_{nk1})^r$, each a_j to a^r_j, and each $\mathcal{L}_{n+1}(k+2^n)_1, x_i, y_i$ and s_j to themselves. The stable letters for these edges are d_k, respectively.

4.8 The spaces $X_{SES_{n,P,r}}$

In this section we will construct an aspherical 3-complex space $X_{SES_{n,P,r}}$ whose fundamental group is $SES_{n,P,r}$. Recall that $X_{\bar{G}_{r,P}}$, $X_{W_{k,r,P}}$ and $X_{W'_{k,r,P}}$ are all 2-dimensional aspherical subcomplexes of $X_{C_{n,P,r}}$. Also in this section, for M equal to the sum of all the entries of matrix P, we will use

$$\{a_1, \ldots, a_M, b_0, \ldots, b_{M-1}, s_1, \ldots, s_M\}$$

as the generating set for $G_{r,P}$ even though some of these generators are trivial.

First, we define cellular maps $\Phi_z : X_{\bar{G}_{r,P,}} \to X_{\bar{G}_{r,P,}}$ which induces $\phi : \bar{G}_{r,P,} \to \bar{G}_{r,P,}$. For $i = 1, \ldots, M$ and $k = 1, \ldots, 2^n$, it maps the one cells labeled s_i, x_k and y_k all homeomorphically to themselves; and a_i, b_{i-1}, a_{nk1} and $u_{(n-1)1}$ to themselves by degree r maps. Each 2-cell in $X_{\bar{G}_{r,P,}}$ is mapped in accordance with its boundary labels. This implies that the image of each triangular 2-cell has combinatorial area r^2; and the image of the remaining 2-cells (which have an s_i, x_k or y_k edge in their boundaries) have area r. Thus Φ_z adds a dimension on top of the 2-subcomplex $X_{\bar{G}_{r,P,}}$ by attaching a copy of $X_{\bar{G}_{r,P,}} \times [0,1]$ (with the product cell structure) to $X_{C_{n,P,r}}$ along $X_{\bar{G}_{r,P,}}$ as follows. The "bottom" side $X_{\bar{G}_{r,P,}} \times \{0\}$ is attached to $X_{\bar{G}_{r,P,}}$ by the identity map while the "top" side $X_{\bar{G}_{r,P,}} \times \{1\}$ is attached to $X_{\bar{G}_{r,P,}}$ via Φ_z. The vertical 1-cells of $X_{\bar{G}_{r,P,}} \times [0,1]$ are labeled z oriented from $X_{\bar{G}_{r,P,}} \times \{1\}$ to $X_{\bar{G}_{r,P,}} \times \{0\}$. From this construction, it is easy to see that each 1-cell and each 2-cell in $X_{\bar{G}_{r,P,}}$ gives rise to a 2-cell and 3-cell, respectively. Figure 4.8.1 shows all the possible types of 3-cells that Φ_z gives rise to. The top two 3-cells in the figure is given by $\Phi_z|_{X_{r,P}}$; the bottom left is given by $\Phi_z|_{X_S}$; and the two bottom right is given by Φ_z restricted to the edge spaces

X_k and Y_k, respectively.

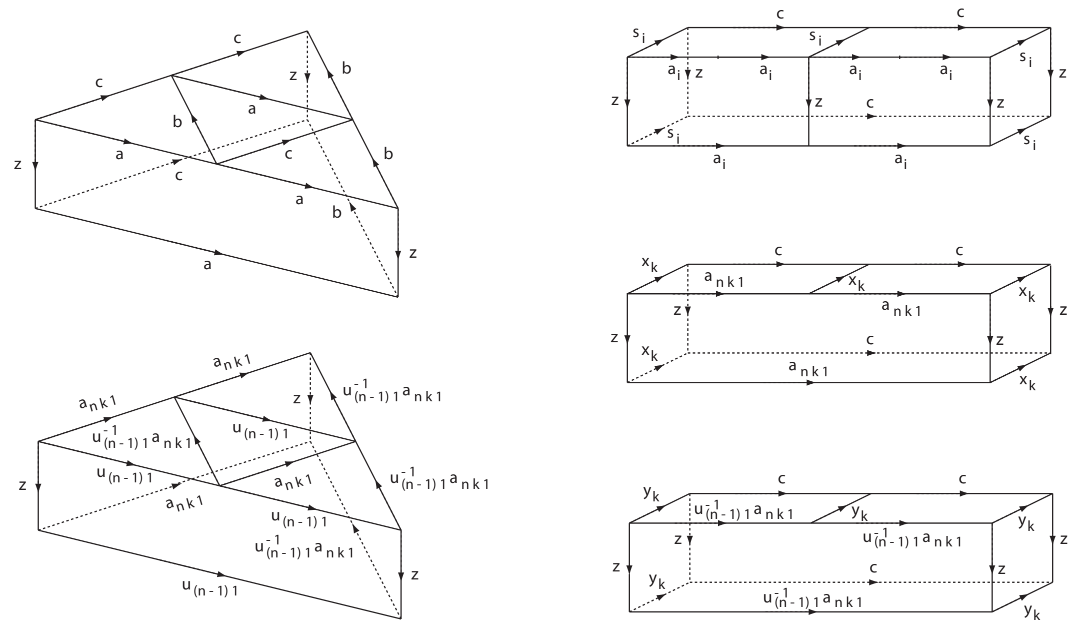

Figure 4.8.1: The above picture shows 3-cells in $X_{SES_{n,P,r}}$ given by Φ_z with $r = 2$. Here $a = a_i$, $b = b_i$ and $c = b_{i-1}$.

Next, we define cellular maps $\Phi_{b_k} : X_{W_{k,r,P,}} \to X_{W_{k,r,P,}}$ which induce homomorphisms $\phi_k : W_{k,r,P,} \to W_{k,r,P,}$. For $i = 1, \ldots, M$ it maps the one cells labeled s_i, x_k and $\mathcal{L}_{n+1}(k)_1$ all homeomorphically to themselves; and a_i and a_{nk1} to themselves by degree r maps. Each 2-cell in $X_{W_{k,r,P,}}$ is mapped in accordance with its boundary labels. This implies that the image of each triangular 2-cell has combinatorial area r^2; and the image of the remaining 2-cells (which have an s_i, x_k or $\mathcal{L}_{n+1}(k)_1$ edge in their boundaries) have area r. Thus Φ_{b_k} adds a dimension on top of the 2-subcomplex $X_{W_{k,r,P,}}$ by attaching a copy of $X_{W_{k,r,P,}} \times [0, 1]$ (with the product cell structure) to $X_{C_{n,P,r}}$ along $X_{W_{k,r,P,}}$ as follows. The"bottom" side $X_{W_{k,r,P,}} \times \{0\}$ is attached to $X_{W_{k,r,P,}}$ by the identity map while the "top" side $X_{W_{k,r,P,}} \times \{1\}$ is attached to $X_{W_{k,r,P,}}$ via Φ_{b_k}. The vertical 1-cells of $X_{W_{k,r,P,}} \times [0, 1]$ are labeled b_k oriented from $X_{W_{k,r,P,}} \times \{1\}$ to $X_{W_{k,r,P,}} \times \{0\}$. From this construction, it is easy to see that each 1-cell and each 2-cell in $X_{W_{k,r,P,}}$ gives rise to a 2-cell and 3-cell, respectively. Figure 4.8.2 shows all the possible types of 3-cells that Φ_{b_k} gives rise to. The two left 3-cells in the figure is given by $\Phi_{b_k}|_{X_{r,P}}$; the top right is given by $\Phi_{b_k}|_{X_{S_k}}$; and the bottom right is given by Φ_{b_k} restricted to the edge space X_k.

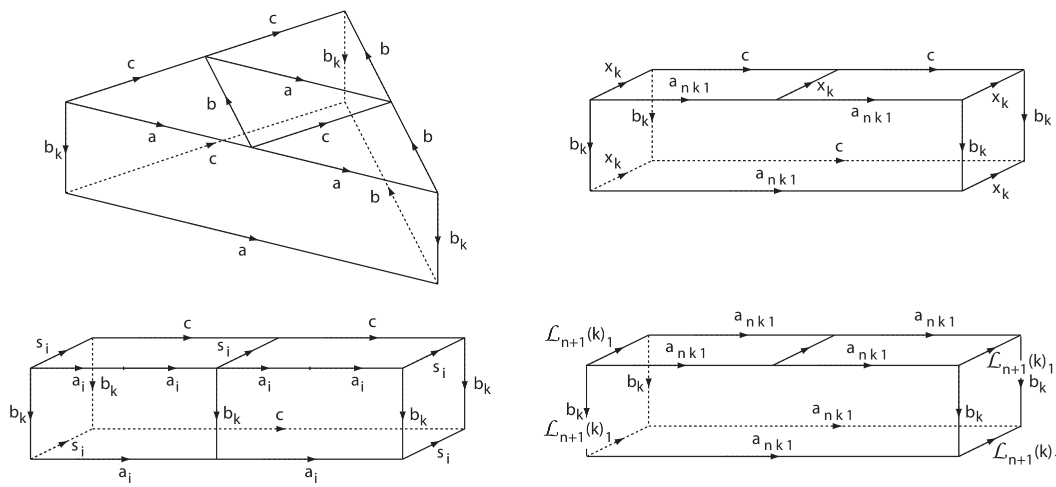

Figure 4.8.2: The above picture shows 3-cells in $X_{SES_{n,P,r}}$ given by Φ_{b_k} with $r = 2$. Here $a = a_i,\ b = b_i$ and $c = b_{i-1}$.

Now, we define cellular maps $\Phi_{d_k} : X_{W'_{k,r,P,}} \to X_{W'_{k,r,P,}}$ which induce homomorphims $\phi'_k : W'_{k,r,P,} \to W'_{k,r,P,}$. For $i = 1, \ldots, M$ it maps the one cells labeled s_i, y_k and $\mathcal{L}_{n+1}(k+2^n)_1$ all homeomorphically to themselves; and a_i and $u^{-1}_{(n-1)1}a_{nk1}$ to themselves by degree r maps. Each 2-cell in $X_{W'_{k,r,P,}}$ is mapped in accordance with its boundary labels. This implies that the image of each triangular 2-cell has combinatorial area r^2; and the image of the remaining 2-cells (which have an s_i, y_k or $\mathcal{L}_{n+1}(k+2^n)_1$ edge in their boundaries) have area r. Thus Φ_{d_k} adds a dimension on top of the 2-subcomplex $X_{W'_{k,r,P,}}$ by attaching a copy of $X_{W'_{k,r,P,}} \times [0,1]$ (with the product cell structure) to $X_{C_{n,P,r}}$ along $X_{W'_{k,r,P,}}$ as follows. The "bottom" side $X_{W'_{k,r,P,}} \times \{0\}$ is attached to $X_{W'_{k,r,P,}}$ by the identity map while the “top” side $X_{W'_{k,r,P,}} \times \{1\}$ is attached to $X_{W'_{k,r,P,}}$ via Φ_{d_k}. The vertical 1-cells of $X_{W'_{k,r,P,}} \times [0,1]$ are labeled d_k oriented from $X_{W'_{k,r,P,}} \times \{1\}$ to $X_{W'_{k,r,P,}} \times \{0\}$. From this construction, it is easy to see that each 1-cell and each 2-cell in $X_{W'_{k,r,P,}}$ gives rise to a 2-cell and 3-cell, respectively. Figure 4.8.3 shows all the possible types of 3-cells that Φ_{d_k} gives rise to. The two left 3-cells in the figure is given by $\Phi_{d_k}|_{X_{r,P}}$; the top right is given by $\Phi_{d_k}|_{X_{T_k}}$; and the bottom right is given by Φ_{d_k} restricted to the edge space Y_k.

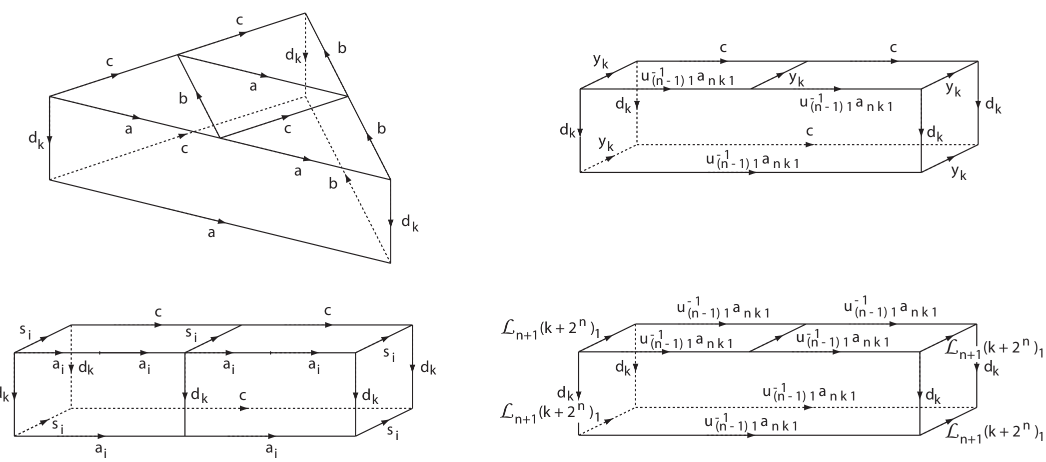

Figure 4.8.3: The above picture shows 3-cells in $X_{SES_{n,P,r}}$ given by Φ_{d_k} with $r = 2$. Here $a = a_i,\ b = b_i$ and $c = b_{i-1}$.

The 3-complex $X_{SES_{n,P,r}}$ has fundamental group $SES_{n,P,r}$. It is obtained by taking each mapping torus of the maps $\Phi_z,\ \Phi_{b_k}$ and Φ_{d_k} $(k = 1, \ldots, 2^n)$ and identifying them along the subcomplex $X_{\bar{G}_{r,P}}$, $X_{W_{k,r,P}}$ and $X_{W'_{k,r,P}}$, respectively of $X_{C_{n,P,r}}$. From this perspective, it is easy to see that $X_{SES_{n,P,r}}$ is aspherical since each $X_{\bar{G}_{r,P}}$, $X_{W_{k,r,P}}$ and $X_{W'_{k,r,P}}$ is aspherical.

Chapter 5

Upper Bounds

To show $SES_{n,P,r}$ is bounded above by $\exp^n(x^\alpha)$ we will repeatedly use the following proposition that is proved in [2].

PROPOSITION 5.0.1. *Let G be the fundamental group of a graph of groups with the following properties:*

1. *All the vertex groups are of type $\mathcal{F}_3$ and their 2-dimensional Dehn functions are bounded above by a superadditive increasing function f.*
2. *All the edge groups are of type $\mathcal{F}_2$ and their 1-dimensional Dehn functions are bounded above by a superadditive increasing function g.*

Then $\delta_G^{(2)}(x) \preceq (f \circ g)(x)$.

We will start with finding an upper bound for the 2-dimensional Dehn function of $C_{n,P,r}$ which is the fundamental group of the graph of groups with two vertex groups H_n and $G_{r,P}$ and $2^{(n+1)}$ edge groups each isomorphic to $\mathbb{Z}$. See Figure 5.0.1:

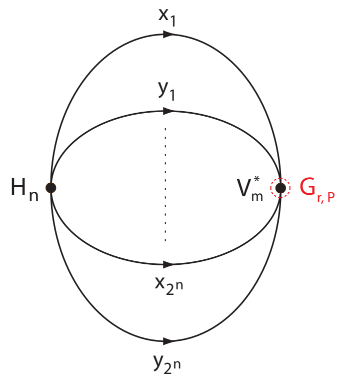

Figure 5.0.1:

The 2-dimensional Dehn function of the type $\mathcal{F}_3$ group H_n is proven in [2] to be bounded above by $\exp^n(\sqrt{x})$. Since the vertex snowflake group $\delta_{G_{r,P}}(x)$ is of type $\mathcal{F}_2$ and has no 3-cell in the complex induced by it, its 2-dimensional Dehn function is zero and thus bounded above by $\exp^n(\sqrt{x})$. Also each edge group is infinite cyclic and has no 2-cell in the complex associated with it so each edge group's 1-dimensional Dehn function is zero and thus bounded above by the superadditive increasing linear function $g(x) = x$. Therefore, by Proposition 5.0.1, $\delta^{(2)}_{C_{n,P,r}}(x) \preceq \exp^n(\sqrt{x})$.

Now, we can move on to find the upper bound for $SES_{n,P,r}$ which is the graph of groups of a single type $\mathcal{F}_3$ vertex group $C_{n,P,r}$ and edge groups $\bar{G}_{r,P}$, $W_{k,r,P}$ and $W'_{k,r,P}$, $k = 1, \ldots, 2^n$ as shown in Figure 5.0.2.

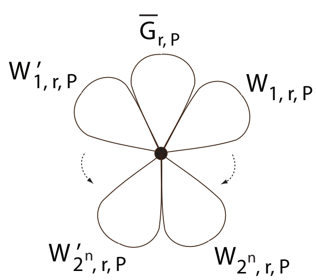

Figure 5.0.2:

We showed above that an upper bound for $\delta^{(2)}_{C_{n,P,r}}(x)$ is $\exp^n(\sqrt{x})$. Proposition 5.0.8 and its corollaries show that the 1-dimensional Dehn function of the edge groups $\bar{G}_{r,P}$, $W_{k,r,P}$ and $W'_{k,r,P}$ are bounded above by $x^{2\alpha}$. For simplicity of notation, we will relabel the generators of $S = \langle u_{(n-1)1} \rangle \times \langle a_{n11}, \ldots, a_{n2^n1} \rangle$ to be $\langle s \rangle \times \langle t_1, \ldots, t_{2^n} \rangle$, respectively.

LEMMA 5.0.2. *(Area bound for S) Let $w(s, t_1, \ldots, t_{2^n})$ be a word representing the element x^N for some N, where x is either s, t_i or $s^{-1}t_i$. Let w be expressed as $w_1 \cdots w_k$ w_i is a power of one of the generators $\{s, t_1, \ldots, t_{2^n}\}$. Then* $\text{Area}(wx^{-N}) \leq 3 \sum_{i<j} |w_i||w_j|$.

Proof. We may assume without loss of generality that either $x = s$, $x = t_1$ or $x = s^{-1}t_1$. Since s commutes with t_i in this group, we can successively transpose adjacent subwords w_i to obtain $v = s^m t$ for some m and where t is a word of the generators t_i. Each transposition of letters contributes 2 to the $\text{Area}(wv^{-1})$, so we have $\text{Area}(wv^{-1}) \leq 2 \sum_{i<j} |w_i||w_j|$.

In the case where $x = s$, $s^m t$ trivially reduces to s^N, and $\text{Area}(wx^{-N}) = \text{Area}(wv^{-1}) \leq 2 \sum_{i<j} |w_i||w_j| \leq 3 \sum_{i<j} |w_i||w_j|$ as desired.

Similarly, if $x = t_1$, $s^m t$ trivially reduces to t_1^N, and $\text{Area}(wx^{-N}) = \text{Area}(wv^{-1}) \leq 2 \sum_{i<j} |w_i||w_j| \leq 3 \sum_{i<j} |w_i||w_j|$ as desired.

Now, if $x = s^{-1}t_1$, $s^m t$ trivially reduces to $s^{-N}t_1^N$. Let I_s and I_{t_1} be the sets of indices for which w_i is a power of s and t_1, respectively. Then $\sum_{i \in I_s} |w_i| \geq N$, $\sum_{i \in I_{t_1}} |w_i| \geq N$, and therefore $\sum_{i<j} |w_i||w_j| \geq N^2 \geq N^2 - N = \text{Area}(vx^{-N})$. Then we have $\text{Area}(wx^{-N}) \leq \text{Area}(wv^{-1}) + \text{Area}(vx^{-N}) \leq 3 \sum_{i<j} |w_i||w_j|$ as desired. □

COROLLARY 5.0.3. *(Area bound for S_k) Let $w(\mathcal{L}_{n+1}(k)_1, a_{nk1})$ be a word* *representing the element* x^N *for some* N*, where* x *is either* $w(\mathcal{L}_{n+1}(k)_1$ *or* a_{nk1}*. Let* w *be expressed as* $w_1 \cdots w_k$ w_i *is a power of one of these generators. Then* $\mathrm{Area}(wx^{-N}) \leq 3\sum_{i<j}|w_i||w_j|$.

Proof. Relabel the generators s and t_1 with $\mathcal{L}_{n+1}(k)_1$ and a_{nk1}, respectively, and let $n = 0$ in the proof of the lemma above. Omit the case when $x = s^{-1}t_1$. □

COROLLARY 5.0.4. *(Area bound for* T_k*) Let* $w(\langle\mathcal{L}_{n+1}(k+2^n)_1, u_{(n-1)1}^{-1}a_{nk1})$ *be a word representing the element* x^N *for some* N*, where* x *is either* $\langle\mathcal{L}_{n+1}(k+2^n)_1$ *or* $u_{(n-1)1}^{-1}a_{nk1}$*. Let* w *be expressed as* $w_1 \cdots w_k$ w_i *is a power of one of these generators. Then* $\mathrm{Area}(wx^{-N}) \leq 3\sum_{i<j}|w_i||w_j|$.

Proof. Relabel the generators s and t_1 with $\mathcal{L}_{n+1}(k+2^n)_1$ and $u_{(n-1)1}^{-1}a_{nk1}$, respectively, and let $n = 0$ in the proof of the lemma above. Omit the case when $x = s^{-1}t_1$. □

CLAIM 5.0.5. *There exists a Lipschitz retraction from* $\bar{G}_{r,P}$ *onto* $G_{r,P}$*,* $k = 1, \ldots, 2^n$.

Proof. Let $r_G : \bar{G}_{r,P} \to G_{r,P}$ be a map defined by $x_i, y_i, s \mapsto 1$, the identity on $G_{r,P}$ and $t_i \mapsto c$ where c is the diagonal element in the vertex group V_m that is linked to S in $\bar{G}_{r,P}$. We need to check that r_G is a homomorphism, i.e. r_G preserves relations. The relations of $\bar{G}_{r,P}$ are the relations of $G_{r,P}$ together with $st_is^{-1}t_i^{-1} = 1$, $x_it_ix_i^{-1}c^{-1} = 1$, $y_is^{-1}t_iy_ic^{-1} = 1$ and $x_1 = 1$. Relations on $G_{r,P}$ are preserved since r_G is the identity on $G_{r,P}$. A quick check shows the other four types of relations are preserved.

$$\begin{aligned}
r_G(s)r_G(t_i)r_G(s^{-1})r_G(t_i^{-1}) &= (1)c(1)c^{-1} &= 1\\
r_G(x_i)r_G(t_i)r_G(x_i^{-1})r_G(c^{-1}) &= (1)c(1)c^{-1} &= 1\\
r_G(y_i)r_G(s^{-1})r_G(t_i)r_G(y_i)r_G(c^{-1}) &= (1)(1)c(1)c^{-1} &= 1\\
r_G(x_1) &= 1
\end{aligned}$$

The retraction r_G is Lipschitz since r_G either collapses unit edges to vertices or takes unit edges to unit edges. □

CLAIM 5.0.6. *There exists a Lipschitz retractions from $W_{k,r,P}$ and $W'_{k,r,P}$ onto $G_{r,P}$, $k = 1, \ldots, 2^n$.*

Proof. This proof is similar to the above proof of Claim 5.0.5. We will first prove the existence of a retraction r_W from $W_{k,r,P}$ onto $G_{r,P}$. The proof of the existence of a retraction r'_W from $W'_{k,r,P}$ onto $G_{r,P}$ will be exactly the same with a relabeling discussed at the end.

Now, let $r_W : W_{k,r,P} \to G_{r,P}$ be a map defined by $x_k, \mathcal{L}_{n+1}(k)_1 \mapsto 1$, the identity on $G_{r,P}$ and $a_{nk1} \mapsto c$ where c is the diagonal element in the vertex group V_m that is linked to S in $\bar{G}_{r,P}$. We need to check that r_W is a homomorphism, i.e. r_W preserves relations. The relations of $W_{k,r,P}$ are the relations of $G_{r,P}$ together with $\mathcal{L}_{n+1}(k)_1 a_{nk1} \mathcal{L}_{n+1}(k)_1^{-1} a_{nk1}^{-1} = 1$ and $x_k a_{nk1} x_k^{-1} c^{-1} = 1$. Relations on $G_{r,P}$ are preserved since r is the identity on $G_{r,P}$. A quick check shows the other two relations are preserved:

$$\begin{aligned}
r(\mathcal{L}_{n+1}(k)_1)r(a_{nk1})r(\mathcal{L}_{n+1}(k)_1^{-1})r(a_{nk1}^{-1}) &= (1)c(1)c^{-1} &= 1\\
r(x_k)r(a_{nk1})r(x_k^{-1})r(c^{-1}) &= (1)c(1)c^{-1} &= 1
\end{aligned}$$

The proof for the existence of a retraction from $W'_{k,r,P}$ is the same as the proof above with $W_{k,r,P}, r_W, x_k, \mathcal{L}_{n+1}(k)_1, a_{nk1}$ relabeled to $W'_{k,r,P}, r'_W, y_k, \mathcal{L}_{n+1}(k+2^n)_1, u_{(n-1)1}^{-1} a_{nk1}$, respectively.

The retractions r_W and r'_W are Lipschitz since they either collapse unit edges to vertices or take unit edges to unit edges. □

Proposition 3.0.2 together with Claim 5.0.5 gives:

PROPOSITION 5.0.7. *(Edge group distortion in $\bar{G}_{r,P}$, $W_{k,r,P}$ and $W'_{k,r,P}$) Given r and P there is a positive constant D with the following property. If c is a diagonal element and w is a word exclusively in $\bar{G}_{r,P}$, $W_{k,r,P}$ or $W'_{k,r,P}$ representing c^N then $|N| \leq D|w|^\alpha$.*

Proof. Let R be the Lipschitz retraction r_G, r_W or r'_W given in Claim 5.0.5 or Claim 5.0.6. Then $R(w)$ is a word in $G_{r,P}$ also representing c^N. By Proposition 3.0.2 and the fact that R is Lipschitz, we have

$$|N| \leq D|R(w)|^\alpha \leq D|w|^\alpha$$

as required. □

PROPOSITION 5.0.8. *(Area bound for $\bar{G}_{r,P}$) Given r and P there is a positive constant E with the following property. Let w be a word in $\bar{G}_{r,P}$ representing x^N for some N, where x is either a generator of the vertex groups $V_m = \langle a_1, \ldots, a_m \rangle$, a diagonal element $c = a_1 \cdots a_m \in V_m$, $s \in S$, $t_i \in S$ or $s^{-1}t_i \in S$. Then* $\text{Area}(wx^{-N}) \leq E|w|^{2\alpha}$.

Proof. We argue by induction on $|w|$. Let $E = (3/2)r^2D^2$ (D given in Proposition 5.0.7). Now, x is either in some vertex group V_m or in S.

First, suppose if $x \in V_m$. Then $x = a_j$ or $x = c$. Write w as $w_1 \cdots w_k$ where each w_i has the form $a_j^{N_i}$ or is a word beginning with $s_j^{\pm 1}$, x_j^{-1} or y_j^{-1} and ending with

$s_j^{\mp 1}$, x_j or y_j. In the latter case, w_i represents an element of the form $a_j^{N_i}$ or c^{N_i}, accordingly. Let I_a and I_c be the sets of indices for which these two cases occur, and let w' be the word obtained from w by replacing each subword w_i of this type with the appropriate word c^{N_i} or $a_j^{N_i}$. Then w' is a word in the standard generators of V_m (and the diagonal element c) representing x^N of length $\sum_i N_i$.

By Lemma 3.0.1, we have $\text{Area}(w'x^{-N}) \leq 3\sum_{i<j} N_i N_j$. To estimate each N_i, we use Proposition 5.0.7 as follows. If $i \in I_c$, then w_i represents c^{N_i} and Proposition 5.0.7 gives $N_i \leq D|w_i|^\alpha$. If $i \in I_a$ then $w_i = s_j u_i s_j^{-1}$ for some u_i representing $(c')^{N_i/r}$, where c' is the diagonal element of some vertex group $V_{m'}$ (this depends on the graph of groups construction). Then by the Proposition 5.0.7 we have $N_i/r \leq D(|w_i| - 2)^\alpha \leq D|w_i|^\alpha$, so $N_i \leq rD|w_i|^\alpha$. Finally if $i \notin (I_c \cup I_a)$ then $N_i = |w_i| \leq |w_i|^\alpha$. Putting these observations together we have

$$\text{Area}(w'x^{-N}) \leq 3r^2D^2 \sum_{i<j} |w_i|^\alpha |w_j|^\alpha \tag{5.0.9}$$

If $x \in S$. Then $x = s,\ x = t_j$ or $x = s^{-1}t_j$. Let c denote the diagonal element of the V_m vertex group that is linked to S. Write w as $w_1 \cdots w_k$ where each w_i has the form s^{N_i}, $t_j^{N_i}$ or $(s^{-1}t_j)^{N_i}$; or is a word of the form $x_{k_i} u_i x_{k_i}^{-1}$ or $y_{k_i} u_i y_{k_i}^{-1}$ representing elements of the form $t_{k_i}^{N_i}$ or $(s^{-1}t_{k_i})^{N_i}$, respectively. Here, u_i is a word representing c^{N_i}. Let I be the set of indices for which this latter case occurs, and let w' be the word obtained from w by replacing each subword w_i of this type with the appropriate word $t_j^{N_i}$ or $(s^{-1}t_j)^{N_i}$. Then w' is a word in the standard generators of S representing x^N of length $\sum_i N_i$.

By Lemma 5.0.2, we have $\text{Area}(w'x^{-N}) \leq 3\sum_{i<j} N_i N_j$. To estimate each N_i, we can use Proposition 5.0.7. If $i \in I$ then $w_i = x_j u_i x_j^{-1}$ or $w_i = y_j u_i y_j^{-1}$ where u_i is

a word representing c^{N_i}. Then by Proposition 5.0.7 we have $N_i \leq D(|w_i|-2)^\alpha \leq D|w_i|^\alpha$. Finally if $i \notin (I)$ then $N_i = |w_i| \leq |w_i|^\alpha$. Putting these observations together and the fact that $r \geq 1$, we have

$$\text{Area}(w'x^{-N}) \leq 3r^2D^2 \sum_{i<j} |w_i|^\alpha |w_j|^\alpha \tag{5.0.10}$$

Next we use the induction hypothesis and Proposition 5.0.7 to bound $\text{Area}(ww'^{-1})$.

First, if $x \in V_m$, $\text{Area}(ww'^{-1}) \leq \sum_{i\in I_c} \text{Area}(w_ic^{-N_i}) + \sum_{i\in I_a} \text{Area}(w_ia_j^{-N_i})$; if $x \in S$, $\text{Area}(ww'^{-1}) \leq \sum_{i\in I} \text{Area}(w_iz^{-N_i})$, where $z = t_j$ or $z = s^{-1}t_j$. We can simply combine these two inequalities together to get $\text{Area}(ww'^{-1}) \leq \sum_{i\in I_c} \text{Area}(w_ic^{-N_i}) + \sum_{i\in I_a} \text{Area}(w_ia_j^{-N_i}) + \sum_{i\in I} \text{Area}(w_iz^{-N_i})$.

If $i \in I_c$, then $w_i = s_j^{-1}u_is_j$, $w_i = x_j^{-1}u_ix_j$ or $w_i = y_j^{-1}u_iy_j$ where u_i represents $a_j^{rN_i}$, $t_j^{N_i}$ or $(s^{-1}t_j)^{N_i}$, respectively. Applying the induction hypothesis to u_i we have $\text{Area}(u_ia_j^{-rN_i})$, $\text{Area}(u_it_j^{-N_i})$ and $\text{Area}(u_i(s^{-1}t_j)^{-N_i})$ all less than or equal to $(3/2)r^2D^2(|w_i|-2)^{2\alpha}$. Each of the strips $s_j^{-1}a_j^{rN_i}s_jc^{-N_i}$, $x_j^{-1}t_j^{N_i}x_jc^{-N_i}$ and $y_j^{-1}(s^{-1}t_j)^{N_i}y_jc^{-N_i}$ has area $N_i \leq D|w_i|^\alpha$ by Proposition 5.0.7. Thus

$$\begin{aligned}\text{Area}(w_ic^{-N_i}) &\leq (3/2)r^2D^2(|w_i|-2)^{2\alpha} + D|w_i|^\alpha \\ &\leq (3/2)r^2D^2((|w_i|-2)^{2\alpha} + |w_i|^\alpha) \\ &\leq (3/2)r^2D^2|w_i|^{2\alpha}\end{aligned} \tag{5.0.11}$$

The last inequality above uses the fact that for numbers $x \geq 0$, one has $(x+2)^{2\alpha} \geq x^\alpha(x+2)^\alpha + 2^\alpha(x+2)^\alpha \geq x^{2\alpha} + (x+2)^\alpha$

If $i \in I_a$, then $w_i = s_ju_is_j^{-1}$ where u_i represents $(c')^{N_i/r}$, where c' is the diagonal element of some vertex group $V_{m'}$. Applying the induction hypothesis to u_i we have $\text{Area}(u_i(c')^{-N_i/r}) \leq (3/2)r^2D^2(|w_i|-2)^{2\alpha}$. The strip $s_j(c')^{N_i/r}s_j^{-1}a_j^{-N_i}$ has area $(N_i/r) \leq D(|w_i|-2)^{\alpha} \leq D(|w_i|-2)^{\alpha}$ by Proposition 5.0.7. Therefore

$$\begin{aligned} \text{Area}(w_i a_j^{-N_i}) &\leq (3/2)r^2D^2(|w_i|-2)^{2\alpha} + D(|w_i|-2)^{\alpha} \\ &\leq (3/2)r^2D^2((|w_i|-2)^{2\alpha} + (|w_i|-2)^{\alpha}) \\ &\leq (3/2)r^2D^2|w_i|^{2\alpha} \end{aligned} \tag{5.0.12}$$

If $i \in I$, then $w_i = x_j u_i x_j^{-1}$ or $w_i = y_j u_i y_j^{-1}$ where u_i represents c^{N_i}. Applying the induction hypothesis to u_i we have $\text{Area}(u_i c^{-N_i}) \leq (3/2)r^2D^2(|w_i|-2)^{2\alpha}$. The strip, $x_j c^{N_i} x_j^{-1} t_j^{-N_i}$ or $y_j c^{N_i} y_j^{-1}(s^{-1}t_j)^{-N_i}$, has area $N_i \leq D(|w_i|-2)^{\alpha}$ by Proposition 5.0.7. Thus, for $z = t_j$ or $z = s^{-1}t_j$,

$$\begin{aligned} \text{Area}(w_i z^{-N_i}) &\leq (3/2)r^2D^2(|w_i|-2)^{2\alpha} + D(|w_i|-2)^{\alpha} \\ &\leq (3/2)r^2D^2((|w_i|-2)^{2\alpha} + (|w_i|-2)^{\alpha}) \\ &\leq (3/2)r^2D^2|w_i|^{2\alpha} \end{aligned} \tag{5.0.13}$$

Combining (5.0.18) and (5.0.19) and (5.0.20) we then have

$$\text{Area}(ww'^{-1}) \leq \sum_{i \in I_c \cup I_a \cup I} (3/2)r^2D^2|w_i|^{2\alpha} \leq \sum_i (3/2)r^2D^2|w_i|^{2\alpha} \tag{5.0.14}$$

Finally, adding (5.0.16) or (5.0.17) with (5.0.21) we get the desired result:

$$\text{Area}(wx^{-N}) \leq (3/2)r^2D^2(\sum_i |w_i|^{\alpha})^2 \leq (3/2)r^2D^2(\sum_i |w_i|)^{2\alpha} = (3/2)r^2D^2|w|^{2\alpha}$$

□

COROLLARY 5.0.15. *(Area bound for $W_{k,r,P}$) Given r and P let E be the* *constant given in Proposition 5.0.8. Let w be a word in $W_{k,r,P}$ representing x^N for some N, where x is either a generator of the vertex groups $V_m = \langle a_1, \ldots, a_m \rangle$, a diagonal element $c = a_1 \cdots a_m \in V_m$, $a_{nk1} \in S_k$ or $\mathcal{L}_{n+1}(k)_1 \in S_k$. Then* $\mathrm{Area}(wx^{-N}) \leq E|w|^{2\alpha}$.

Proof. The proof of Proposition 5.0.8 above will hold here if we make the following modifications. Replace all instances of:

S	with	S_k
s	with	$\mathcal{L}_{n+1}(k)_1$
t_j, t_{k_i}	with	a_{nk1}
arbitrary x_j, x_{k_i}	with	fixed x_k
Lemma 5.0.2	with	Corollary 5.0.3

Also we need to delete all expressions that include y_j, y_{k_i} or $s^{-1}t_j$, and set $z = a_{nk1}$. We give the modified proof here in its entirety for ease of reading:

We argue by induction on $|w|$. Let $E = (3/2)r^2D^2$ (D given in Proposition 5.0.7). Now, x is either in some vertex group V_m or in S_k.

First, suppose if $x \in V_m$. Then $x = a_j$ or $x = c$. Write w as $w_1, \cdots, w_k$ where each w_i has the form $a_j^{N_i}$ or is a word beginning with $s_j^{\pm 1}$ or x_k^{-1} and ending with $s_j^{\mp 1}$ or x_k. In the latter case, w_i represents an element of the form $a_j^{N_i}$ or c^{N_i}, accordingly. Let I_a and I_c be the sets of indices for which these two cases occur, and let w' be the word obtained from w by replacing each subword w_i of this type with the appropriate word c^{N_i} or $a_j^{N_i}$. Then w' is a word in the standard generators of V_m (and the diagonal element c) representing x^N of length $\sum_i N_i$.

By Lemma 3.0.1, we have $\mathrm{Area}(w'x^{-N}) \leq 3\sum_{i<j} N_iN_j$. To estimate each N_i, we

use Proposition 5.0.7 as follows. If $i \in I_c$, then w_i represents c^{N_i} and Proposition 5.0.7 gives $N_i \leq D|w_i|^\alpha$. If $i \in I_a$ then $w_i = s_j u_i s_j^{-1}$ for some u_i representing $(c')^{N_i/r}$, where c' is the diagonal element of some vertex group $V_{m'}$ (this depends on the graph of groups construction). Then by the Proposition 5.0.7 we have $N_i/r \leq D(|w_i| - 2)^\alpha \leq D|w_i|^\alpha$, so $N_i \leq rD|w_i|^\alpha$. Finally if $i \notin (I_c \cup I_a)$ then $N_i = |w_i| \leq |w_i|^\alpha$. Putting these observations together we have

$$\text{Area}(w'x^{-N}) \leq 3r^2D^2 \sum_{i<j} |w_i|^\alpha |w_j|^\alpha \tag{5.0.16}$$

If $x \in S_k$. Then $x = \mathcal{L}_{n+1}(k)_1$ or $x = a_{nk1}$. Let c denote the diagonal element of the V_m vertex group that is linked to S_k. Write w as $w_1 \cdots w_k$ where each w_i has the form $\mathcal{L}_{n+1}(k)_1^{N_i}$ or $a_{nk1}^{N_i}$; or is a word of the form $x_k u_i x_k^{-1}$ representing elements of the form $a_{nk1}^{N_i}$. Here, u_i is a word representing c^{N_i}. Let I be the set of indices for which this latter case occurs, and let w' be the word obtained from w by replacing each subword w_i of this type with the appropriate word $a_{nk1}^{N_i}$. Then w' is a word in the standard generators of S_k representing x^N of length $\sum_i N_i$.

By Lemma 5.0.3, we have $\text{Area}(w'x^{-N}) \leq 3\sum_{i<j} N_i N_j$. To estimate each N_i, we can use Proposition 5.0.7. If $i \in I$ then $w_i = x_k u_i x_k^{-1}$ where u_i is a word representing c^{N_i}. Then by Proposition 5.0.7 we have $N_i \leq D(|w_i| - 2)^\alpha \leq D|w_i|^\alpha$. Finally if $i \notin (I)$ then $N_i = |w_i| \leq |w_i|^\alpha$. Putting these observations together and the fact that $r \geq 1$, we have

$$\text{Area}(w'x^{-N}) \leq 3r^2D^2 \sum_{i<j} |w_i|^\alpha |w_j|^\alpha \tag{5.0.17}$$

Next we use the induction hypothesis and Proposition 5.0.7 to bound $\text{Area}(ww'^{-1})$.

First, if $x \in V_m$, $\text{Area}(ww'^{-1}) \leq \sum_{i \in I_c} \text{Area}(w_i c^{-N_i}) + \sum_{i \in I_a} \text{Area}(w_i a_j^{-N_i})$; if $x \in S_k$, $\text{Area}(ww'^{-1}) \leq \sum_{i \in I} \text{Area}(w_i a_{nk1}^{-N_i})$. We can simply combine these two inequalities together to get

$$\text{Area}(ww'^{-1}) \leq \sum_{i \in I_c} \text{Area}(w_i c^{-N_i}) + \sum_{i \in I_a} \text{Area}(w_i a_j^{-N_i}) + \sum_{i \in I} \text{Area}(w_i a_{nk1}^{-N_i}).$$

If $i \in I_c$, then $w_i = s_j^{-1} u_i s_j$ or $w_i = x_k^{-1} u_i x_k$ where u_i represents $a_j^{rN_i}$ or $a_{nk1}^{N_i}$, respectively. Applying the induction hypothesis to u_i we have $\text{Area}(u_i a_j^{-rN_i})$ and $\text{Area}(u_i a_{nk1}^{-N_i})$ both less than or equal to $(3/2)r^2 D^2(|w_i| - 2)^{2\alpha}$. Each of the strips $s_j^{-1} a_j^{rN_i} s_j c^{-N_i}$ and $x_k^{-1} a_{nk1}^{N_i} x_k c^{-N_i}$ has area $N_i \leq D|w_i|^\alpha$ by Proposition 5.0.7. Thus

$$\begin{aligned} \text{Area}(w_i c^{-N_i}) &\leq (3/2)r^2 D^2(|w_i| - 2)^{2\alpha} + D|w_i|^\alpha \\ &\leq (3/2)r^2 D^2((|w_i| - 2)^{2\alpha} + |w_i|^\alpha) \\ &\leq (3/2)r^2 D^2 |w_i|^{2\alpha} \end{aligned} \tag{5.0.18}$$

The last inequality above uses the fact that for numbers $x \geq 0$, one has $(x+2)^{2\alpha} \geq x^\alpha(x+2)^\alpha + 2^\alpha(x+2)^\alpha \geq x^{2\alpha} + (x+2)^\alpha$

If $i \in I_a$, then $w_i = s_j u_i s_j^{-1}$ where u_i represents $(c')^{N_i/r}$, where c' is the diagonal element of some vertex group $V_{m'}$. Applying the induction hypothesis to u_i we have $\text{Area}(u_i(c')^{-N_i/r}) \leq (3/2)r^2 D^2(|w_i| - 2)^{2\alpha}$. The strip $s_j(c')^{N_i/r} s_j^{-1} a_j^{-N_i}$ has area $(N_i/r) \leq D(|w_i| - 2)^\alpha \leq D(|w_i| - 2)^\alpha$ by Proposition 5.0.7. Therefore

$$\begin{aligned} \text{Area}(w_i a_j^{-N_i}) &\leq (3/2)r^2 D^2(|w_i| - 2)^{2\alpha} + D(|w_i| - 2)^\alpha \\ &\leq (3/2)r^2 D^2((|w_i| - 2)^{2\alpha} + (|w_i| - 2)^\alpha) \\ &\leq (3/2)r^2 D^2 |w_i|^{2\alpha} \end{aligned} \tag{5.0.19}$$

If $i \in I$, then $w_i = x_k u_i x_k^{-1}$ where u_i represents c^{N_i}. Applying the induction hypothesis to u_i we have $\text{Area}(u_i c^{-N_i}) \leq (3/2)r^2 D^2(|w_i| - 2)^{2\alpha}$. The strip, $x_k c^{N_i} x_k^{-1} a_{nk1}^{-N_i}$ has area $N_i \leq D(|w_i| - 2)^\alpha$ by Proposition 5.0.7. Thus,

$$\begin{aligned} \text{Area}(w_i a_{nk1}^{-N_i}) &\leq (3/2)r^2 D^2(|w_i| - 2)^{2\alpha} + D(|w_i| - 2)^\alpha \\ &\leq (3/2)r^2 D^2((|w_i| - 2)^{2\alpha} + (|w_i| - 2)^\alpha) \\ &\leq (3/2)r^2 D^2 |w_i|^{2\alpha} \end{aligned} \tag{5.0.20}$$

Combining (5.0.18) and (5.0.19) and (5.0.20) we then have

$$\text{Area}(ww'^{-1}) \leq \sum_{i \in I_c \cup I_a \cup I} (3/2)r^2 D^2 |w_i|^{2\alpha} \leq \sum_i (3/2)r^2 D^2 |w_i|^{2\alpha} \tag{5.0.21}$$

Finally, adding (5.0.16) or (5.0.17) with (5.0.21) we get the desired result:

$$\text{Area}(wx^{-N}) \leq (3/2)r^2 D^2 (\sum_i |w_i|^\alpha)^2 \leq (3/2)r^2 D^2 (\sum_i |w_i|)^{2\alpha} = (3/2)r^2 D^2 |w|^{2\alpha}$$

□

COROLLARY 5.0.22. *(Area bound for $W'_{k,r,P}$) Given r and P let E be the constant given in Proposition 5.0.8. Let w be a word in $W'_{k,r,P}$ representing x^N for some N, where x is either a generator of the vertex groups $V_m = \langle a_1, \ldots, a_m \rangle$, a diagonal element $c = a_1 \cdots a_m \in V_m$, $u_{(n-1)1}^{-1} a_{nk1} \in T_k$ or $\mathcal{L}_{n+1}(k + 2^n)_1 \in T_k$. Then $\text{Area}(wx^{-N}) \leq E|w|^{2\alpha}$.*

Proof. Like Corollary 5.0.15, the proof of Proposition 5.0.8 above will hold here if we make the following modifications. Replace all instances of:

S	with	T_k
s	with	$\mathcal{L}_{n+1}(k+2^n)_1$
t_j	with	$u^{-1}_{(n-1)1}a_{nk1}$
arbitrary y_j, y_{k_i}	with	fixed y_k
Lemma 5.0.2	with	Corollary 5.0.4

Also we need to delete all expressions that include x_j, x_{k_i} or $s^{-1}t_j$, and set $z = u^{-1}_{(n-1)1}a_{nk1}$. $\square$

By setting $N = 0$ in the proposition and corollaries above:

$$\delta_{\bar{G}_{r,P}}(|w|),\ \delta_{W_{k,r,P}}(|w|),\ \delta_{W'_{k,r,P}}(|w|) \preceq |w|^{2\alpha}$$

for any word w representing 1 in $\bar{G}_{r,P}$, $W_{k,r,P}$, $W'_{k,r,P}$, respectively. Then because $\delta^{(2)}_{C_{n,P,r}}(x) \preceq \exp^n(\sqrt{x})$, by Proposition 5.0.1, we have proved the following result:

THEOREM 5.0.23. $\delta^{(2)}_{SES_{r,P}}(x) \preceq exp^n(x^\alpha)$ $\square$

Chapter 6

Lower Bounds

To find a lower bound for $\delta^{(2)}_{SES_{n,P,r}}$, we will use the following remarks given in [6].

REMARK 6.0.1. In order to establish the relation $f \preceq g$ between two non-decreasing functions, it suffices to consider relatively sparse sequences of integers. For if (n_i) is an unbounded sequence of integers for which there is a constant $C > 0$ such that $n_0 = 0$ and $n_{i+1} \leq Cn_i$ for all i, and if $f(n_i) \leq g(n_i)$ for all i, then $f \preceq g$. Indeed, given $x \in [0, \infty)$ there is an index i such that $n_i \leq x \leq n_{i+1}$, whence $f(x) \leq f(n_{i+1}) \leq g(n_{i+1}) \leq g(Cn_i) \leq g(Cx)$.

REMARK 6.0.2. Let X be a 3-dimensional aspherical space. Then a ball B in the universal cover $\widetilde{X}$ has the smallest volume among all balls with the same boundary label if it is an embedding in $\widetilde{X}$. For a detailed proof of this see Remark 2.9 in [7].

Thus, to show $\delta^{(2)}_{SES_{n,P,r}}(x) \preceq \exp^n(x^\alpha)$ where $\alpha = log_\lambda(r)$ and λ is the Perron-Frobenius eigenvalue of P, by Remark 6.0.1, we only need to construct a sequence of 3-balls $\{B_l\}$ whose surface area sequentially grows at most exponentially towards infinity. Also, to establish the inequality $\delta^{(2)}_{SES_{n,P,r}}(x) \leq A$, it is enough to give an embedded ball in $\widetilde{X}_{SES_{n,P,r}}$ with surface area x and volume greater or equal to A, by Remark 6.0.2. Here we use the fact that $X_{SES_{n,P,r}}$ is aspherical and 3-dimensional.

In the next section, we will construct a sequence of balls $\{B_l\}$ that embeds in in $X_{SES_{n,P,r}}$. In the following section, we will show that the surface area of each B_l grows at most exponentially towards infinity and that the volume of each B_l is bounded above by a function Lipschitz equivalent to $\exp^n(Area(B_l)^\alpha)$.

6.1 Construction

In this section, we will discuss how to construct a ball B_l in this sequence $\{B_l\}$. The ball B_l will be constructed in stages indexed in function notation by the subgroups $\{H_0, G_0, H_1, G_1, \ldots, H_{n-1}, G_{n-1}, H_n, C_{n,P,r}, SES_{n,P,r}\}$. For example $B_l(H_3)$ will denote the H_3-stage in the construction of B_l. The construction of B_l is finished at the last stage $SES_{n,P,r}$, thus $B_l = B_l(SES_{n,P,r})$. We begin with $B_l(H_0)$ defined to be a ball that sits inside the subspace $\widetilde{K}_{H_0}$ of the form $\varphi^N(a_{011}) \times \varphi^N(a_{021}) \times y_1$ (φ the palindromic automorphism). The ball $B_l(G_0)$ will be attained by adding 3-cells inside the subspace $\widetilde{K}_{G_0}$ to the surface of $B_l(H_0)$ as in [2]. The balls in the following stages up to H_{n-1} are defined similarly.

Now, construct a ball up to $B_l(H_{n-1})$ as in [2]. Figure 6.1.1 shows how this construction proceeds

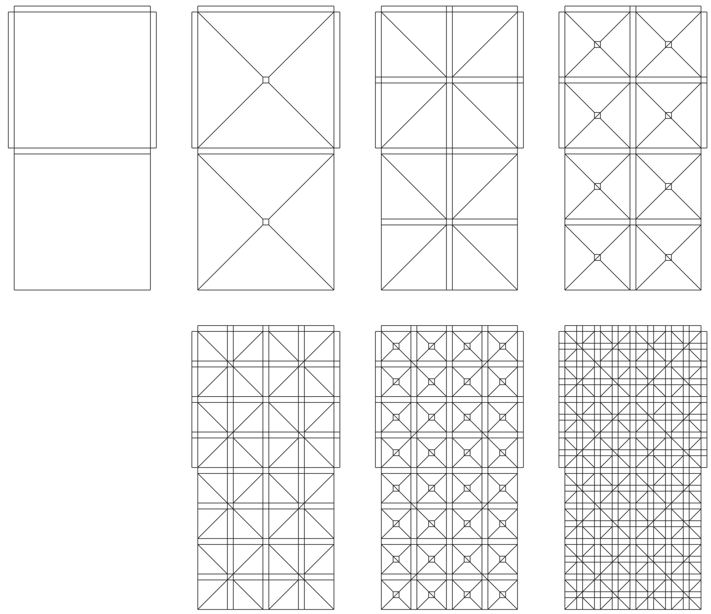

Figure 6.1.1: A schematic diagram of the surfaces of $B_l(H_0)$, $B_l(G_0)$, $B_l(H_1)$, $B_l(G_1)$, $B_l(H_2)$, $B_l(G_2)$ and $B_l(H_3)$, respectively

so that locally almost everywhere, the surface of $B_l(H_{n-1})$ looks like the picture in Figure 6.1.12:

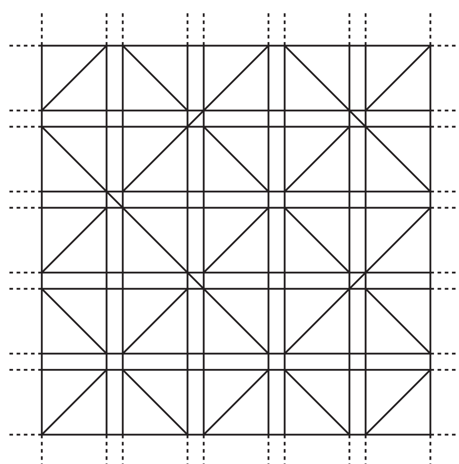

Figure 6.1.2:

The larger triangles and thin rectangles in the picture above are further subdivided as in Figure 6.1.3.

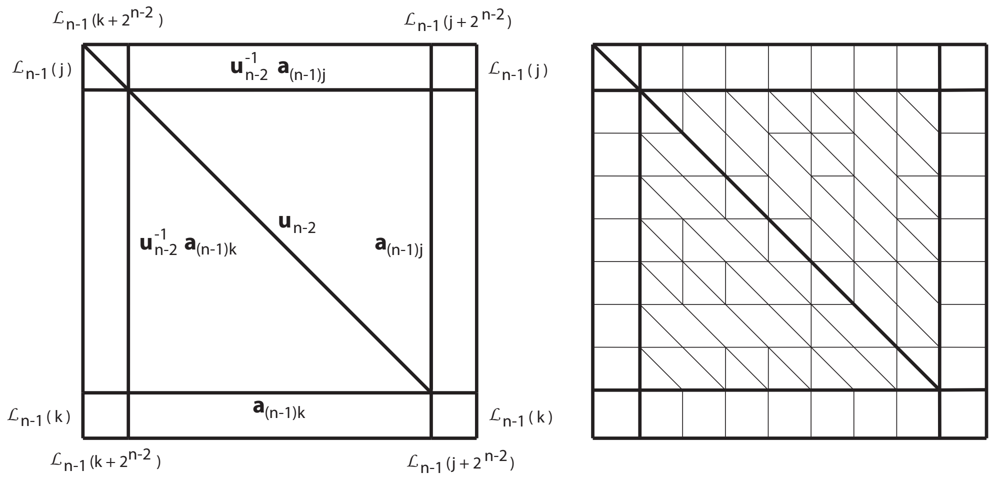

Figure 6.1.3:

Where each edge sits in the subspace associated with the F_2 subgroup denoted by its label. The strange subdivision of each large triangle is given in [2]. This is due to the fact that $\mathbf{u}_{n-2}$ commutes with $\mathbf{a}_{(n-1)i}$ and that the product $\mathbf{u}^{-1}_{n-2}\mathbf{a}_{(n-1)i}$ denotes only coordinate-wise multiplication of basis elements of $\mathbf{u}_{n-2} = \langle u_{(n-2)1}, u_{(n-2)2}\rangle$ and $\mathbf{a}_{(n-1)i} = \langle a_{(n-1)i1}, a_{(n-1)i2}\rangle$, respectively. That is $\mathbf{u}^{-1}_{n-2}\ \mathbf{a}_{(n-1)i}$ is the diagonal subgroup of $\mathbf{u}_{n-2}\times\mathbf{a}_{(n-1)i}$ with basis $\langle u^{-1}_{(n-2)1}a_{(n-1)i1}, u^{-1}_{(n-2)2}a_{(n-1)i2}\rangle$. Also note that the subdivision of the larger triangles and thin rectangles can be coarser or finer depending on the length of its sides. For example, the subdivision for triangles whose side lengths equal 1, 3, 7 and 17 are given in Figure 6.1.4:

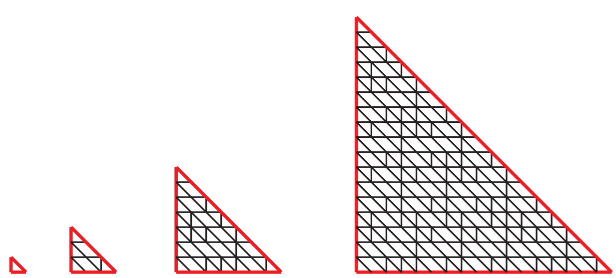

Figure 6.1.4:

In fact, the length of each side of these triangles on the surface of $B_l(H_{n-1})$ is $|\varphi^{r^l}(a_{(n-1)k1})|$.

REMARK 6.1.1. Up to now, we've constructed our balls in exactly the same way as in [2]. For the next two stages in the construction of B_l (the G_{n-1} and H_n stage), we will build our balls combinatorially the same way as in [2]. The only difference is in the choice of labels by group elements. In this way, our balls sit differently inside the subspaces $\widetilde{K}_{G_{n-1}}$ and $\widetilde{K}_{H_n}$, respectively, but will have the same surface area and volume as the balls constructed in [2].

Now, to construct $B_l(G_{n-1})$, we add onto the surface of $B_l(H_{n-1})$ blocks to cover the large triangles. But unlike the ones in [2], we will only use blocks whose vertical edges are labeled by only $u_{(n-1)1}$. See Figure 6.1.5. Note that in [2], at this stage, the authors used blocks whose vertical edges are labeled by both generators of $\mathbf{u}_{n-1} = \langle u_{(n-1)1}, u_{(n-1)2}\rangle$).

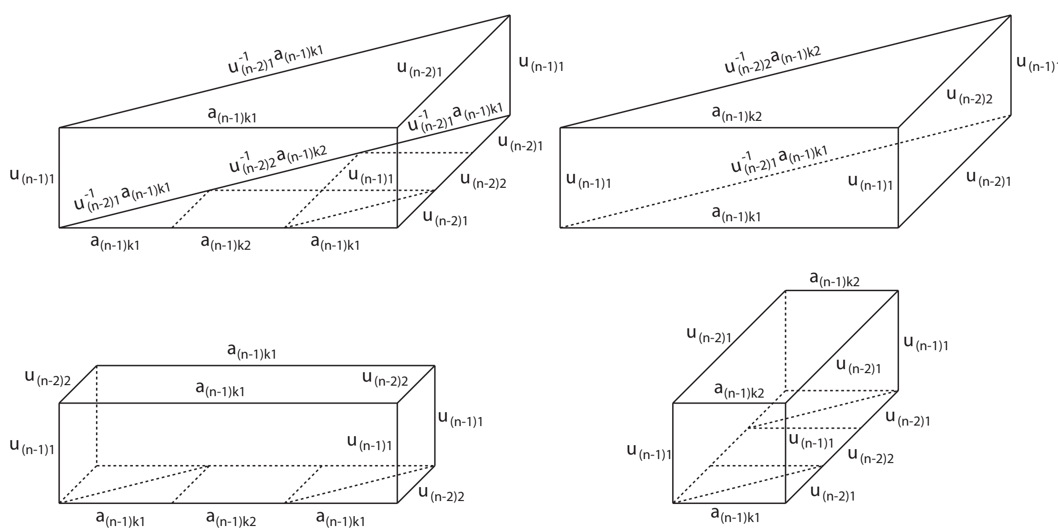

Figure 6.1.5:

As mentioned, these blocks are laid down to cover the large triangles in the same way they were in [2]. See Figure 6.1.6

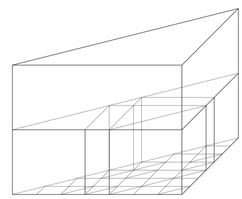

Figure 6.1.6:

Now, locally almost everywhere, the surface of $B_l(G_{n-1})$ looks like the diagram in Figure 6.1.7:

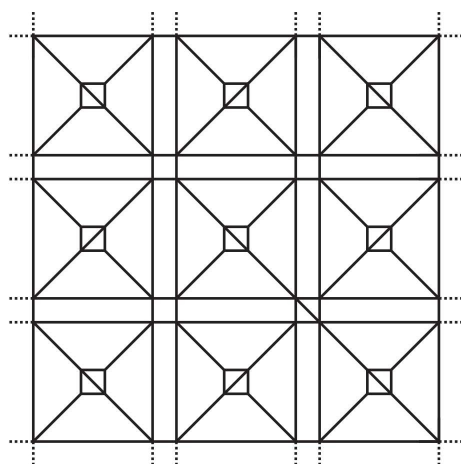

Figure 6.1.7:

The trapezoids and thin rectangular strips in Figure 6.1.7 are further subdivided as in the following diagram, see Figure 6.1.8.

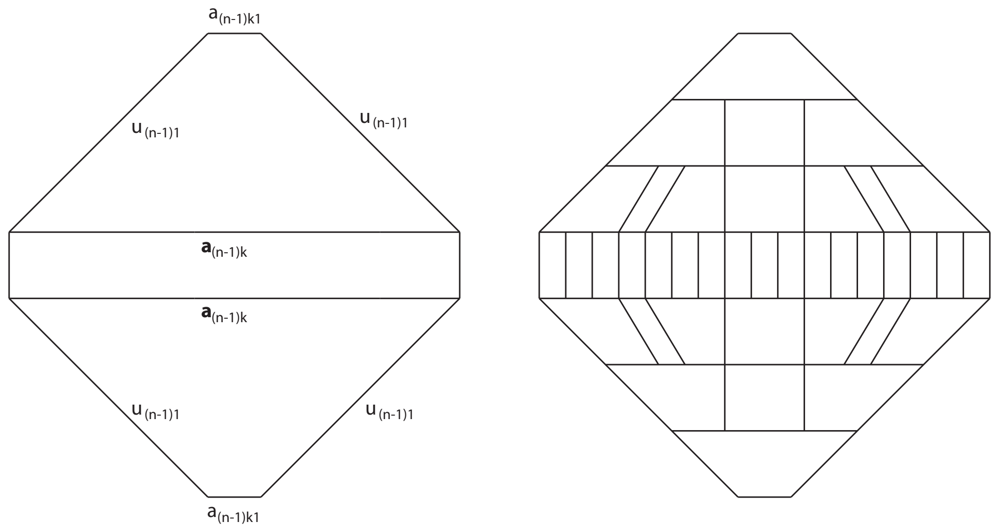

Figure 6.1.8:

The strange subdivision of each trapezoid is given in [2]. This is due to the palindromic automorphism $\varphi : \mathbf{a}_{(n-1)k} \rightarrow \mathbf{a}_{(n-1)k}$ in [2]. Also note that the subdivision of the trapezoids and thin rectangles can be coarser or finer depending on the length of its base. For example, the subdivision for trapezoids and strips whose base lengths equal 3, 7, 17 and 41 are given in Figure 6.1.9:

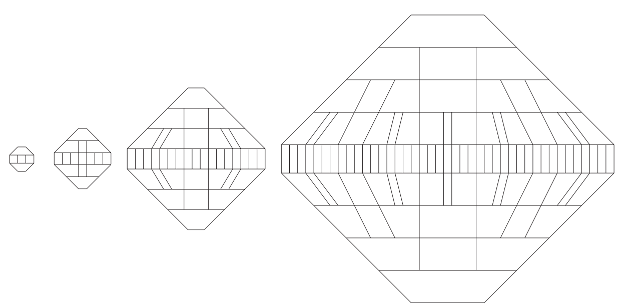

Figure 6.1.9:

In fact, because the triangular base of the prism in Figure 6.1.6 have side lengths $|\varphi^{r^l}(a_{(n-1)k1})|$, the height of the prism is r^l (i.e. the base of these trapezoids have length $|\varphi^{r^l}(a_{(n-1)k1})|$; the tops have unit length; and the sides, labeled by $u^{r^l}_{(n-1)1}$, have length is r^l).

Similar to the previous step, to construct $B_l(H_n)$, we continue to add to $B_l(G_{n-1})$ only the 3-cell blocks in K_{H_n} whose vertical edges are labeled only by a_{ni1}, $i = 1, \ldots, 2^n$ (i.e. we do not use blocks whose vertical edges are labeled by a_{ni2}). See Figure 6.1.10:

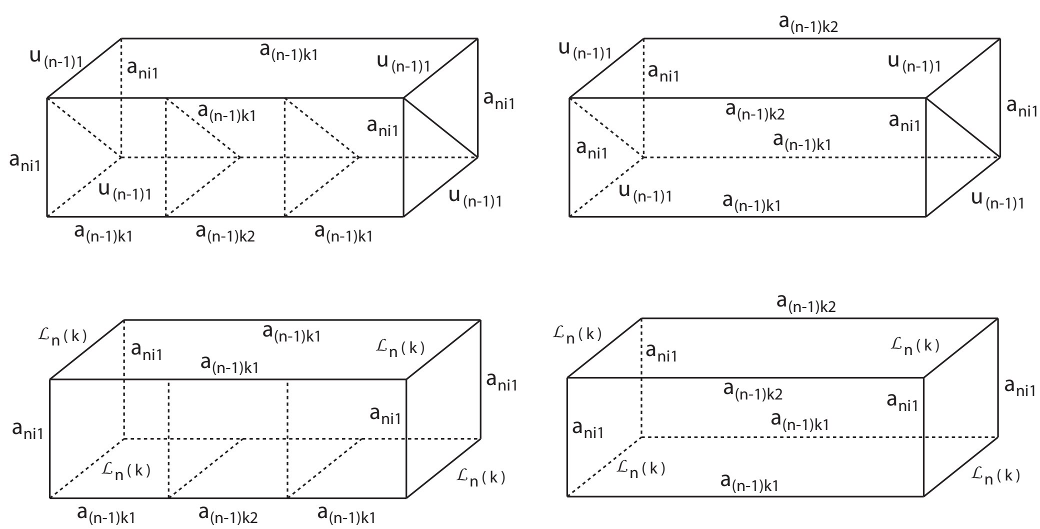

Figure 6.1.10:

These blocks are laid down to cover the trapezoids and rectangular strips in the same way they were in [2]. See Figure 6.1.11:

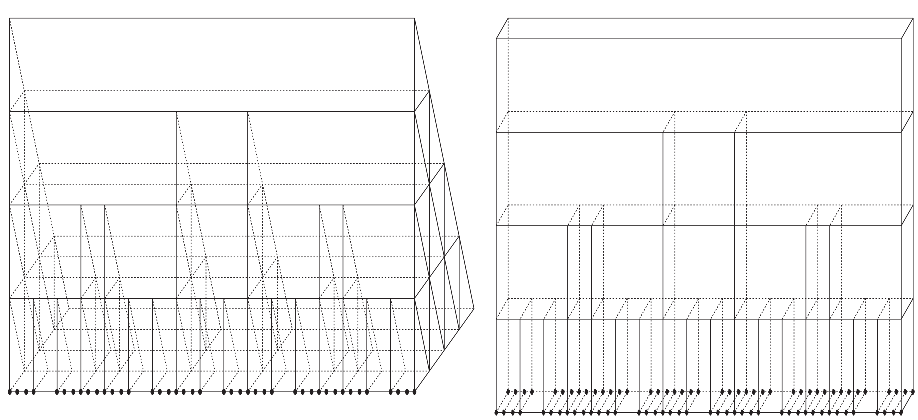

Figure 6.1.11:

Now, as before, locally, the surface of $B_l(H_n)$ look schematically like the diagram in Figure 6.1.12:

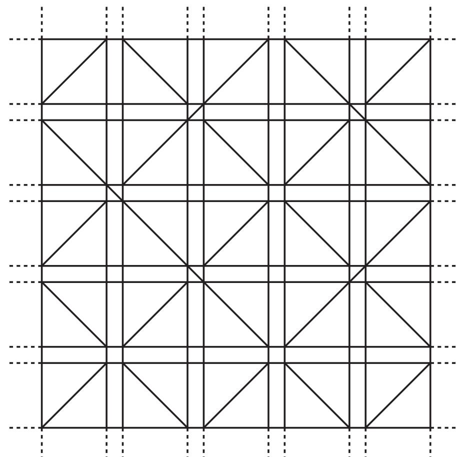

Figure 6.1.12:

The sides of the large triangles now all have length r^l. The difference now is that the large triangles are partitioned as in Figure 6.1.13:

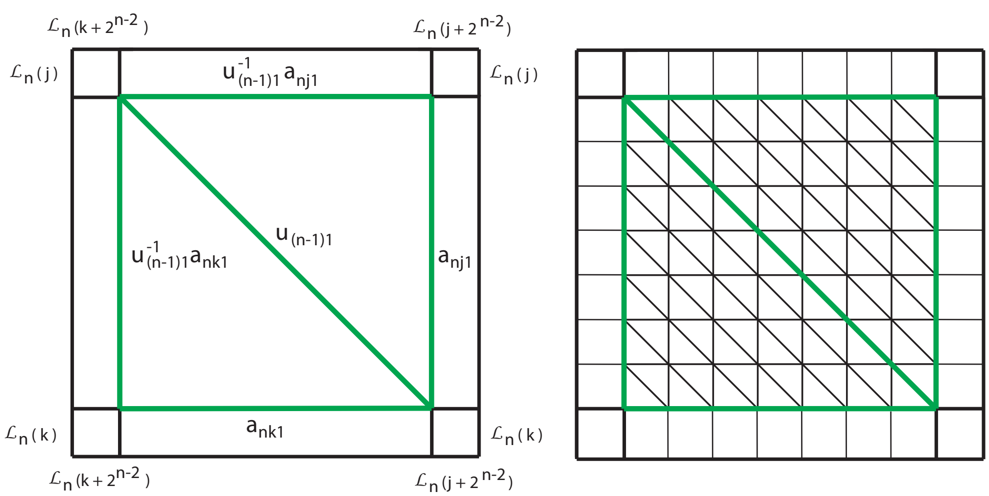

Figure 6.1.13:

Notice, the edges of these triangles span a $\mathbb{Z}^2$ subgroup. This subgroup will act like a V_2 vertex group, and is now compatible with a snowflake group. This is the reason we modify the original construction in [2].

Next, we will shift gears and consider snowflake groups $G_{r,P}$. We will use the generating set $\{a_1, \ldots, a_M, s_1, \ldots, s_M\}$ for $G_{r,P}$. We will call an edge labeled by a power of a_i on the boundary of a disk in $X_{r,P}$ a *fringe edge* and the length of a fringe edge to be the length of that a_i-syllable. Recall from Section 3 that if N is a power of r, then a positive snowflake word w^+ is a well defined string of s_i's and a_i's with all the a_i-syllables having length 1; and each a_i is part of a subword of the form $s_j a_{i_1} \cdots a_{i_m} s_j^{-1}$ where the letters $a_{i_1}, \ldots, a_{i_m}$ are all distinct. We define a *positive core snowflake word* to be the word attained from such a positive snowflake word w^+ by replacing each $s_j a_{i_1} \cdots a_{i_m} s_j^{-1}$ subword by a_j^r accordingly. This is essentially removing the outermost layer of a snowflake disk and pronouncing the boundary of this new disk (which we will call the *core* of a snowflake disk) to be its positive core snowflake word. See Figure 6.1.14. Note that the fringe edges of the core all have length r.

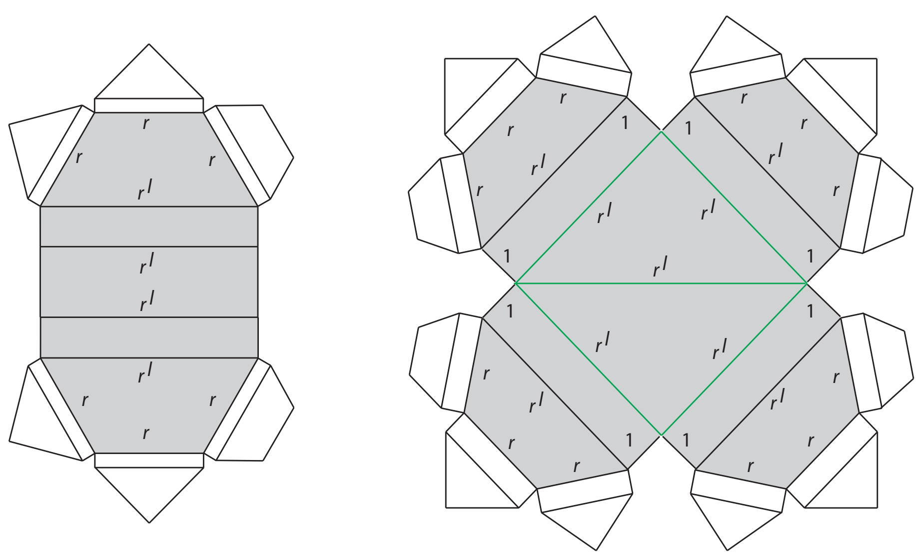

Figure 6.1.14: These snowflake disks have fringe edges all of length 1. The core of this disk of is given in gray and edge lengths in the core are labeled (here $l = 2$).

For the following examples, we will use

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Now, we will add 2-dimensional snowflake fins on top of the surface of our $B_l(H_n)$ along the a_{ni1} or $u^{-1}_{(n-1)1}a_{ni1}$ edge segments, each of length r^l. These fins consist of half of a core of a snowflake disk in $G_{r,P}$ with diameter labeled c^{r^l} and fringe edges of length r attached to the strips $x_i c^{r^l} x_i^{-1} a_{ni1}^{-r^l}$ or $y_i c^{r^l} y_i^{-1}(u^{-1}_{(n-1)1}a_{ni1})^{-r_l}$ along the path labeled c_{r^l}. See Figure 6.1.15

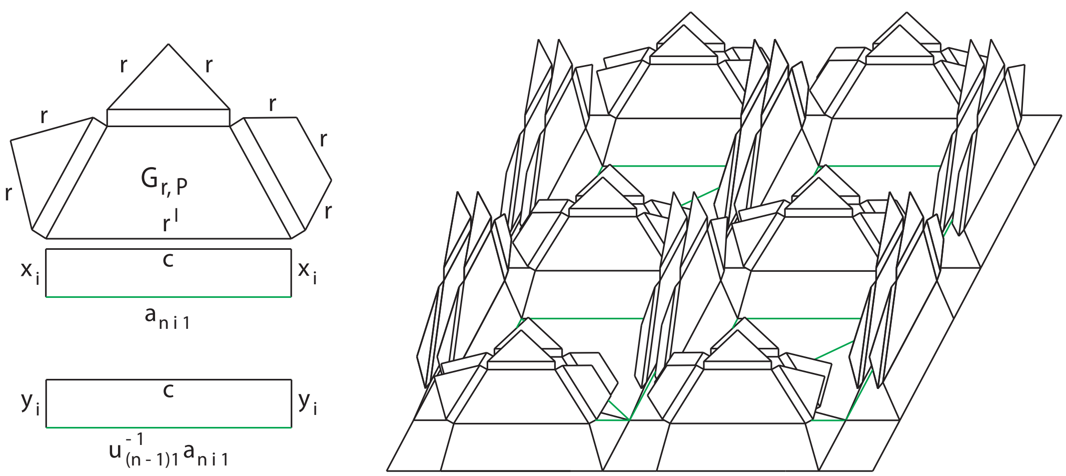

Figure 6.1.15: Snowflake fins (left) are added on top of the surface of an H_n ball along the a_{ni1} or $u^{-1}_{(n-1)1}a_{ni1}$ segments, each of length r^l. The labels around the half snowflake disk in $G_{r,P}$ indicates the length of those sides. It has diameter length r^l (here $l = 2$) and fringe edge length r.

We will denote this ball $B_l(C_{n,P,r})$ (even though it is not homeomorphic to a ball). On the surface of a $B_l(C_{n,P,r})$, we see copies of the three types of snowflake disks shown in Figure 6.1.16 which map to the subcomplexes associated with $W_{k,r,P}$, $W'_{k,r,P}$ and $\bar{G}_{r,P}$ respectively, and embed in their universal covers. We call these $W_{k,r,P}$-, $W'_{k,r,P}$- and $\bar{G}_{r,P}$-snowflake disks, respectively. Collectively, we will call them *modified snowflake disks* to differentiate them from the *standard snowflake disks* as constructed in [6]:

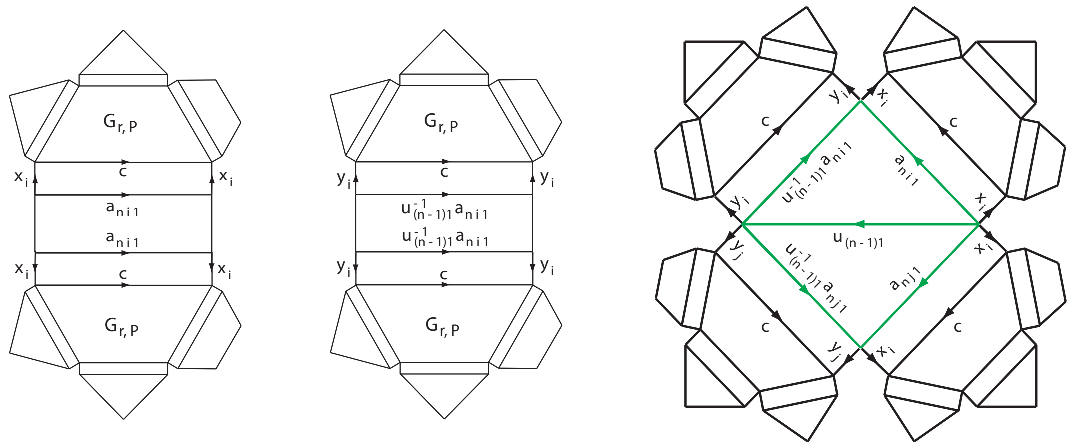

Figure 6.1.16: Snowflake disks in $\widetilde{X}_{W_{k,r,P}}$ (left), $\widetilde{X}_{W'_{k,r,P}}$ (center) and $\widetilde{X}_{\bar{G}_{r,P}}$ (right).

We will call the strip in the middle of each of the left and middle disks in Figure 6.1.16 the diameter strip (they are the strips labeled by a_{ni1} and $u^{-1}_{(n-1)1}a_{ni1}$, respectively on opposing sides); and the edge labeled by $u_{(n-1)1}$ in the middle of the right snowflake disk in Figure 6.1.16, we will call the diameter of the disk. Note that each of these snowflake disks has a diameter strip or diameter of length r^l and fringe edges of length r.

Though these disks are not quite $G_{r,P}$ snowflake disks, they are similar enough to them to build half of a snowflake ball on top of each of them as in [6]. That is, we can repeatedly use Φ_z, Φ_{b_k} and Φ_{d_k} on these snowflake disks that live in $\widetilde{X}_{\bar{G}_{r,P}}$, $\widetilde{X}_{W_{k,r,P}}$, and $\widetilde{X}_{W'_{k,r,P}}$, respectively ($k = 1, \ldots, 2^n$). Each use of these cellular maps on these snowflake disks creates a slab of blocks where the length of the diameter or diameter strip on the bottom side of the slab is r times the length of the diameter or diameter strip on the top side of the slab; and the fringe edges on the bottom side of each slab will have length r whereas the top side has fringe edge length 1. In this manner, the bottom side of the first slab will have a diameter or diameter strip length of r^l whereas the top side of the slab will have a diameter or diameter strip length of $r^{(l-1)}$. The second slab, laid on the top side of the first slab along its core will have a diameter or diameter strip length of r^{l-1} and r^{l-2} on its bottom and top side, respectively. We continue in this fashion until the diameter or diameter strip on the top side of the uppermost slab has length 1. We will call these half-snowflake balls $\bar{G}_{r,P}$-half snowflake balls, $W_{k,r,P}$-half snowflake balls and $W'_{k,r,P}$-half snowflake balls, respectively. Collectively, we will call them *modified half snowflake balls* to differentiate them from the *standard half snowflake balls* from [6].

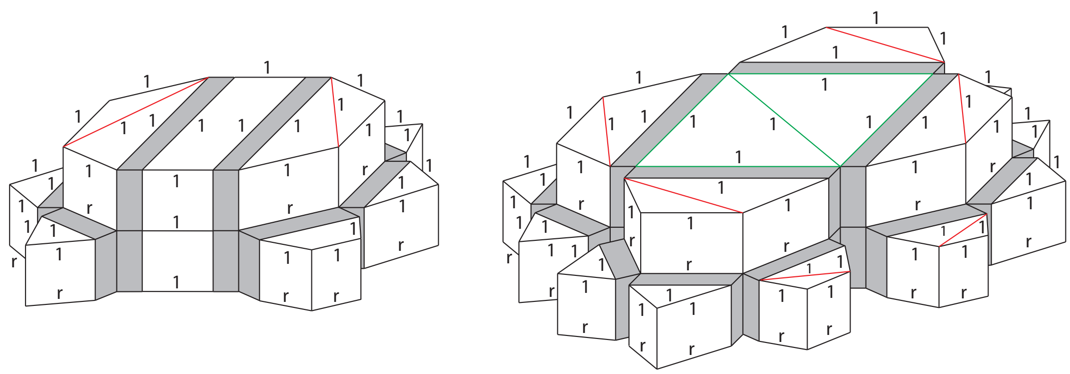

Figure 6.1.17: Diagram of snowflake balls stacked on atop snowflake disks in $W_{k,r,P}$ and $W'_{k,r,P}$ (left) and $\bar{G}_{r,P}$ (right). The labels indicate the length of each path. The height of each stack and the width of each grey 2-cell is 1.

We call our final ball (i.e. modified snowflake balls like the ones in Figure 6.1.17 stacked on top of $B_l(C_{n,P,r})$), $B_l(SES_{n,P,r}) = B_l$. Note that the snowflake fins in $B_l(C_{n,P,r})$ have been covered up on both sides and B_l is again homeomorphic to a ball. In the following section, we will show the volume of B_l is bounded below by $\exp^n(x^\alpha)$, where x is the surface area of the B_l.

6.2 Computations

To compute the volume of B_l in terms of its surface area Area(B_l), we note that it is at least the volume of $B_l(H_n)$. Recall that the surface of $B_l(H_n)$ is tesselated by triangles and rectangular strips (and tiny squares and triangles the area of which is a function of n, call it $h(n)$) as in Figure 6.1.12 where the sides of the triangles and the longer side of the strips all have length r^l. The numbers of these triangles and rectangular strips of length r^l in Figure 6.1.12 are functions of n, call them $T(n)$ and $R(n)$, respectively. Since schematically, for any l, there are $T(n) + R(n)$ triangles and strips, the surface area of $B_l(H_n)$ is Lipschitz equivalent to the area of one of these triangles, which is r^{2l} (more precisely, it is Lipschitz equivalent to the area of a triangle plus the area of a rectangular strip. But since the area of a rectangular strip is only r^l, this is absorbed to be Lipschitz equivalent to simply the area of a triangle). By definition of Lipschitz equivalence, this means there exists a constant C_0 such that $\text{Area}(B_l(H_n)) \geq C_0(C_0 r^l)^2 + C_0 r^l$. By setting a new constant $C_1 = C_0^{3/2}$, we can simplify this inequality to $\text{Area}(B_l(H_n)) \geq (C_1 r^l)^2$. By Theorem 2.0.2 and Remark 6.1.1, the volume of $B_l(H_n)$ and thus the volume of B_l is bounded below by a function Lipschitz equivalent to $\exp^n(\sqrt{\text{Area}(B_l(H_n))}) \geq \exp^n(C_1 r^l)$. That is, there exists a constant C_2 such that $\text{Vol}(B_l) \geq C_2 \exp^n(C_2 C_1 r^l) + C_2(C_1 r^l)^2$; and by setting $C = \min\{C_2, C_2 C_1\}$, we can simplify this inequality to:

$$\text{Vol}(B_l) \geq C \exp^n(C r^l) \tag{6.2.1}$$

In the next stage of the construction of B_l, the $C_{n,P,r}$ stage, we only added 2-cells, which we called snowflake fins, to the surface of $B_l(H_n)$. Thus the volume of

$B_l(C_{n,P,r})$ is unchanged from the volume of $B_l(H_n)$ and remains bounded below by $C\exp^n(Cr^l)$. The surface area of $B_l(C_{n,P,r})$ is increased from that of $B_l(H_n)$ due to the area added from the snowflake fins, but these 2-cells from the snowflake fins will be covered up in the next and final stage, stage-$SES_{n,P,r}$, so there is no need to compute its effect on our construction.

Since modified snowflake balls like the ones in Figure 6.1.17 use only a quarter of a standard snowflake ball (the kind of which that lives in $\widetilde{X}_{r,P}$), we give a corollary to Lemma 3.0.7 which gave bounds for the boundary of a standard snowflake ball in $\widetilde{X}_{r,P}$ with hemisphere of height j, denoted B^3_j. We define a quarter snowflake ball, $Q(B^3_j)$, to be the construction given in [6] of a standard half-snowflake ball, restricted to the core of half of a snowflake disk with diameter c^{r^j}. We define the *quarter-peel of a ball* B^3_j, denoted $\bar{\partial}Q(B^3_j)$, to be $Q(B^3_j) \cap \partial B^3_j$ (this the shaded part on the quarter snowflake ball of Figure 6.2.1 and Figure 6.2.2). The following corollary gives an upper bound for a quarter of the boundary of a standard snowflake ball B^3_j, that is $\bar{\partial}Q(B^3_j)$.

COROLLARY 6.2.2. *Given r and P, $\frac{1}{4}|\partial B^2_j| \leq \mathrm{Area}(\bar{\partial}Q(B^3_j)) \leq F_1|\partial B^2_j|$ for every j and $F_1 = \frac{1}{4}F_0$, (F_0 given in Lemma 3.0.7).*

Proof. By Lemma 3.0.7, there exists an F_0 such that $|\partial B^2_j| \leq \mathrm{Area}(\partial B^3_j) \leq F_0|\partial B^2_j|$ for every j. By the symmetry of B^3_j, $Area(\partial B^3_j) = 4Area(\bar{\partial}Q(B^3_j))$. Therefore

$$|\partial B^2_j| \leq 4\,\mathrm{Area}(\bar{\partial}Q(B^3_j)) = Area(\partial B^3_j) \leq F_0|\partial B^2_j|$$

for every j. Division by 4 gives the desired result. □

Now we give bounds for the surface area of the dome of a modified half snowflake ball. The *dome* of a modified half snowflake ball is the boundary of the half ball minus the modified snowflake disk base. Denote a modified half snowflake ball of diameter or diameter strip length N by $B_{\bar{G}_{r,P}}(N)$, $B_{W_{k,r,P}}(N)$ and $B_{W'_{k,r,P}}(N)$ depending on its type. And denote $\text{Dome}(B_{\bar{G}_{r,P}}(N))$, $\text{Dome}(B_{W_{k,r,P}}(N))$ and $\text{Dome}(B_{W'_{k,r,P}}(N))$ to be their domes.

CLAIM 6.2.3. *Given r, P and l, there exists constants E_0, E_1 such that*

$$E_0 r^{l/\alpha} \leq \text{Area}(\text{Dome}(B_{\bar{G}_{r,P}}(r^l))) \leq E_1 r^{l/\alpha}$$

where $\alpha = log_\lambda(r)$ and λ is the Perron-Frobenius eigenvalue of P..

Proof. From Figure 6.2.1, $B_{\bar{G}_{r,P}}(r^l)$ is constructed from 4 quarter snowflake balls of diameter length r^l around a central region. The dome $\text{Dome}(B_{\bar{G}_{r,P}}(r^l))$ as indicated in gray of Figure 6.2.1 is 4 quarter peels of B^3_l, denoted $\bar{\partial}Q(B^3_l)$, around the gray of the central region in Figure 6.2.1. Thus we want to compute 4 times the area of each quarter peel plus the area in gray of the central region of Figure 6.2.1.

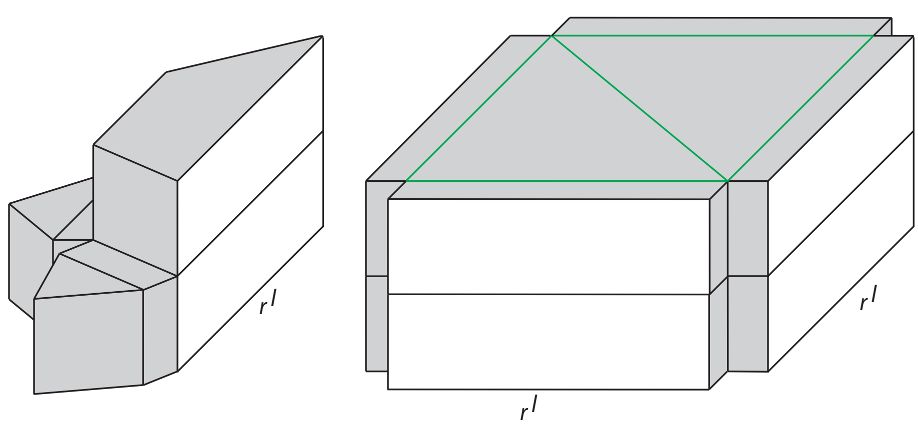

Figure 6.2.1: A $\bar{G}_{r,P}$-snowflake ball of diameter length r^l is constructed from 4 quarter snowflake balls (left) around the central region (right). The dome of the $\bar{G}_{r,P}$-snowflake ball is indicated in gray (here $l = 2$).

From the proof of Corollary 6.2.2, 4 times the area of each quarter peel, $\bar{\partial}Q(B_l^3)$, is $Area(\partial B_l^3)$ which is bounded below by $|\partial B_l^2|$ and above by $F_0|\partial B_l^2|$. Which, in turn is bounded below and above by $2(\frac{1}{C_1}r^l)^{1/\alpha}$ and $2(\frac{1}{C_0}r^l)^{1/\alpha}F_0$ by Proposition 3.0.4. The area in gray of the central region of Figure 6.2.1 is $8l$ (area of vertical strips) plus 4 (area of 4 rectangular strips on top) plus 2 (area of 2 triangles on top). Thus

$$E_0 r^{l/\alpha} \leq \text{Area}(\text{Dome}(B_{\bar{G}_{r,P}}(r^l))) \leq 2(\frac{1}{C_0}r^l)^{1/\alpha}F_0 + 8l + 6 \leq E_1 r^{l/\alpha}. \qquad (6.2.4)$$

Set $E_0 = 2(\frac{1}{C_1})^{1/\alpha}$ and $E_1 = 2F_0(\frac{1}{C_0})^{1/\alpha} + K_0$ (where $K_0 > 0$ is a constant that satisfies: $8l + 6 \leq K_0(r^{1/\alpha})^l$). □

CLAIM 6.2.5. *Given r, P, k and l, there exists constants E_0, E_2 such that*

$$E_0 r^{l/\alpha} \leq \text{Area}(\text{Dome}(B_{W_{k,r,P}}(r^l))) \leq E_2 r^l$$

and

$$E_0 r^{l/\alpha} \leq \text{Area}(\text{Dome}(B_{W'_{k,r,P}}(r^l))) \leq E_2 r^l$$

where $\alpha = log_\lambda(r)$ and λ is the Perron-Frobenius eigenvalue of P.

Proof. We prove the first set of inequalities first. From Figure 6.2.2, $B_{W_{k,r,P}}(r^l)$ is constructed from 2 quarter snowflake balls of with diameter length r^l on both ends of a central region. The dome $\text{Dome}(B_{W_{k,r,P}}(r^l))$ as indicated in gray of Figure 6.2.2 is 2 quarter peels of B_l^3, denoted $\bar{\partial}Q(B_l^3)$, on both ends of the gray of the central region in Figure 6.2.2. Thus we want to compute 2 times the area of each quarter peel plus the area in gray of the central region of Figure 6.2.2.

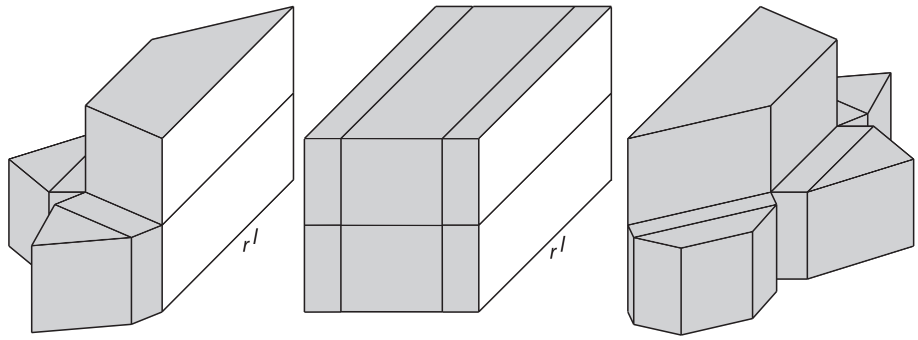

Figure 6.2.2: A $W_{k,r,P}$-snowflake ball with diameter strip length r^l is constructed from 2 quarter snowflake balls on both ends of the central region. The dome of the $W_{k,r,P}$-snowflake ball is indicated in gray (here $l = 2$).

From Corollary 6.2.2, each quarter peel, $\bar{\partial}Q(B_l^3)$ is bounded below by $\frac{1}{4}|\partial B_l^2|$ and above by $F_1|\partial B_l^2|$. Which, in turn is bounded below and above by $2(\frac{1}{C_1}r^l)^{1/\alpha}$ and $2(\frac{1}{C_0}r^l)^{1/\alpha}F_1$ by Proposition 3.0.4. The area in gray of the central region of Figure 6.2.1 is $6l$ (area of vertical rectangles) plus 3 (area of 3 rectangles on top). Thus,

$$E_0 r^{l/\alpha} \leq \text{Area}(\text{Dome}(B_{W_{k,r,P}}(r^l))) \leq 4F_1(\frac{1}{C_0}r^l)^{1/\alpha} + 6l + 3 \leq E_2 r^{l/\alpha}. \quad (6.2.6)$$

Set $E_0 = 2(\frac{1}{C_1})^{1/\alpha}$ and $E_2 = 4F_1(\frac{1}{C_0})^{1/\alpha} + K_1$ (where $K_1 > 0$ is a constant that satisfies: $6l + 3 \leq K_1(r^{1/\alpha})^l$).

The proof of the second set of inequalities is identical to the above since $W_{k,r,P}$-snowflake balls and $W'_{k,r,P}$-snowflake balls are combinatorially the same. $\square$

Putting these results together, we get lower and upper bounds for the surface area of $B_l(SES_{n,P,r}) = B_l$:

$$(T(n) + R(n))E_0 r^{l/\alpha} \leq \text{Area}(B_l) \leq T(n)E_1 r^{l/\alpha} + R(n)E_2 r^{l/\alpha} + h(n)$$

By setting constants $G_0 = T(n) + R(n)E_0$ and $G_1 = T(n)E_1 + R(n)E_2 + h(n)$ we can clean up this inequality:

$$G_0 r^{l/\alpha} \leq \text{Area}(B_l) \leq G_1 r^{l/\alpha} \tag{6.2.7}$$

It is also easy to see from here that our sequence of balls $\{B_l\}$ have surface areas that sequentially grows at most exponentially towards infinity since

$$\frac{\text{Area}(B_{l+1})}{\text{Area}(B_l)} \leq \frac{G_1 r^{(l+1)/\alpha}}{G_0 r^{l/\alpha}} = \frac{G_1}{G_0} r^{1/\alpha}$$

which is constant as required.

Now, from Equation 6.2.1, $\text{Vol}(B_l) \geq \text{Vol}(B_l(H_n)) \geq C \exp^n(Cr^l)$; and from Equation 6.2.7, we can deduce $r^l \geq (\frac{1}{G_1} \text{Area}(B_l))^\alpha$. Therefore:

$$\text{Vol}(B_l) \geq C \exp^n(C(\frac{1}{G_1} \text{Area}(B_l))^\alpha)$$

Which implies $\text{Vol}(B_l) \succeq \exp^n(\text{Area}(B_l)^\alpha)$. Thus by Remark 6.0.1, we have proved:

THEOREM 6.2.8. $\delta^{(2)}_{SES_{n,P,r}}(x) \succeq \exp^n(x^\alpha)$ □

The Main Theorem 1.0.1 now follows directly from Theorem 5.0.23 and Theorem 6.2.8.

Bibliography

[1] J. M. Alonso, W. A. Bogley, R. M. Burton, S. J. Pride, and X. Wang, *Second order Dehn functions of groups*, Quart. J. Math. Oxford Ser. (2), 49 (1998), pp. 1–30.

[2] J. Barnard, N. Brady, and P. Dani, *Super-exponential 2-dimensional Dehn functions*, To appear in Groups Geom. Dyn.

[3] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra, 89 (1993), pp. 3–47.

[4] B. H. Bowditch, *A short proof that a subquadratic isoperimetric inequality implies a linear one*, Michigan Math. J., 42 (1995), pp. 103–107.

[5] N. Brady and M. R. Bridson, *There is only one gap in the isoperimetric spectrum*, Geom. Funct. Anal., 10 (2000), pp. 1053–1070.

[6] N. Brady, M. R. Bridson, M. Forester, and K. Shankar, *Snowflake groups, Perron-Frobenius eigenvalues and isoperimetric spectra*, Geom. Topol., 13 (2009), pp. 141–187.

[7] N. Brady and M. Forester, *Density of isoperimetric spectra*, Geom. Topol., 14 (2010), pp. 435–472.

[8] M. R. Bridson, *The geometry of the word problem*, in Invitations to geometry and topology, vol. 7 of Oxf. Grad. Texts Math., Oxford Univ. Press, Oxford, 2002, pp. 29–91.

[9] ——, *Polynomial Dehn functions and the length of asynchronously automatic structures*, Proc. London Math. Soc. (3), 85 (2002), pp. 441–466.

[10] M. Gromov, *Hyperbolic groups*, in Essays in group theory, vol. 8 of Math. Sci. Res. Inst. Publ., Springer, New York, 1987, pp. 75–263.

[11] A. Y. Ol'shanskiĭ, *Hyperbolicity of groups with subquadratic isoperimetric inequality*, Internat. J. Algebra Comput., 1 (1991), pp. 281–289.

[12] P. Papasoglu, *On the sub-quadratic isoperimetric inequality*, in Geometric group theory (Columbus, OH, 1992), vol. 3 of Ohio State Univ. Math. Res. Inst. Publ., de Gruyter, Berlin, 1995, pp. 149–157.

[13] P. Papasoglu, *Isodiametric and isoperimetric inequalities for complexes and groups*, J. London Math. Soc. (2), 62 (2000), pp. 97–106.

[14] M. V. Sapir, J.-C. Birget, and E. Rips, *Isoperimetric and isodiametric functions of groups*, Ann. of Math. (2), 156 (2002), pp. 345–466.

[15] J.-P. Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[16] X. Wang, *Second order Dehn functions of finitely presented groups and monoids*, PhD Thesis, University of Glasgow, (1996).

[17] X. Wang and S. J. Pride, *Second order Dehn functions and HNN-extensions*, J. Austral. Math. Soc. Ser. A, 67 (1999), pp. 272–288. Group theory.