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COMPARISON THEOREMS, GEOMETRIC INEQUALITIES, AND APPLICATIONS TO p-HARMONIC GEOMETRY

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY

Dr. Shihshu Walter Wei, Chair

Dr. John Albert

Dr. Alan Roche

Dr. May Yuan

Dr. Meijun Zhu

DEDICATION

to

My parents

Luo, Li and Li, Jianwei

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Abstract

In this dissertation, we consider three aspects: comparison theorems on complete manifold which posses a pole, geometric inequalities on complete manifolds, and the applications of inequalities to p-harmonic geometry. More precisely, we first derive a comparison theorem of the matrix-valued Riccati equation with certain initial conditions, and then use this as a tool to obtain Hessian comparison theorem on manifolds with nonnegative curvatures. We study Hardy type inequality, weighted Hardy inequality and weighted Sobolev inequality via Hessian comparison theorems. One of the main results in this dissertation is the Caffarelli-Kohn-Nirenberg type inequality on Cartan-Hadamard manifolds, which is an extension of the the result in Caffarelli-Kohn-Nirenberg's paper [6]. Furthermore, we also discuss some L^p version of Caffarelli-Kohn-Nirenberg type inequalities on punched manifolds and point out a possible value of the constant. Finally, we study Liouville theorems of p-harmonic functions, p-harmonic morphisms, and weakly conformal maps, with assumption only on curvature and q-energy growth. As further applications we obtain Picard type theorems in p-harmonic geometry.

Chapter 1

Introduction and Statements of Main Results

In this chapter, we introduce the history, motivation, background and main results of this thesis.

1.1 History, Motivation and Background

On the 10th of June 1854 Georg Friedrich Bernhard Riemann (1826-1866) gave his famous “Habilitationsvortrag”(probationary lecture) in the Colloquium of the Philosophical Faculty at Göttingen. In his important talk “Über die Hypothesen, welche der Geometrie zu Grunde liegen” (“On the hypotheses that lie at the foundation of geometry”), he introduced (what is now called) an n-dimensional Riemannian manifold and its curvature tensor.

In Riemannian geometry, sectional curvatures of a Riemannian manifold M have strong influences on other geometric features of M. As Riemannian manifolds with constant sectional curvature are the simplest, it is natural to discuss general manifolds via the study of manifolds with constant sectional curvature (the model). One of the important parts is the comparison theorems on manifolds. From comparison theorems, various quantities such as volume, diameter, and the first eigenvalue are bounded by the corresponding quantities of the model (cf. [42]). For example, Toponogov’s theorem affords a characterization of sectional curvature in terms of how “fat” geodesic triangles appear when compared to their Euclidean counterparts; Rauch comparison theorem roughly states that for large curvature, geodesics tend to converge, while for small (or negative) curvature, geodesics tend to spread; Hessian comparison theorem roughly says that the larger the curvature, the smaller the Hessian of the distance function.

Inequalities play an important role in almost all branches of mathematics as well as in other areas of science and engineering. We derived geometric inequalities on manifolds (e.g. Proposition 3.1, Theorem 3.11), and we also proved weighted Hardy and weighted Sobolev inequalities (Theorem 3.7,Theorem 3.8) on Cartan-Hadamard manifolds. We extend important inequalities, such as Hardy's inequality (first published in 1920 [23]) and Caffarelli-Kohn-Nirenberg inequality (published in 1984 [6]) from Euclidean spaces to general Riemannian manifolds In fact, we pioneered the use of Hessian comparison theorem to prove generalized Caffarelli-Kohn-Nirenberg type inequalities and its L^2 and L^p versions on various complete manifolds under curvature assumptions. The technique of Caffarelli-Kohn-Nirenberg is to use the rotational symmetry of the Euclidean spaces to reduce an inequality in high dimension to that in one dimension. This does not seem to carry over to general manifolds. To overcome this difficulty, we employ the weighted Hardy inequality and weighted Sobolev inequality to prove generalized Caffarelli-Kohn-Nirenberg type inequalities on Cartan-Hadamard manifolds(cf. [29]).

In recent years, p-harmonic geometry has become an active research field, since p-harmonic maps are natural generalizations of geodesics, minimal submanifolds, conformal maps, analytic functions on the complex plane $\mathbb{C}$, harmonic map, etc. A great deal of work has been done by B. White [52], R. Hardt and F.-H. Lin [22], S. Luckhaus [**?**] from the view point of geometric measure theory, and by S.W. Wei and others from the view point of differential geometry [43], [44], [51], [46]. In particular, S.-C. Chang, J.-T. Chen and S.W. Wei showed in [10] a Liouville type theorem for p-harmonic function via inequalities and energy functional. This motivates us to study the application of inequalities to p-harmonic geometry (like Liouville type theorems, Picard type theorems, and etc.).

1.2 Main results

In this section, we describe the main results presented in this thesis into the following categories:

A. Comparison Theorems.

Let E be a vector space with an inner product $\langle\, ,\, \rangle$, $S(E)$ be the space of self-adjoint linear endomorphisms of E , and $R_i:\ (0,\, t_i) \to S(E)$ be continuous functions with maximal $t_i \in (0,\, \infty]$ $(i = 1,\, 2)$. We say

$$R_1 \le R_2$$

if

$$\langle R_1(t)\,(x), x\rangle \le \langle R_2(t)\,(x), x\rangle$$

for every $t \in (0,\, t_0)$ and every $x \in E$, where $t_0 = \min\{t_1,\, t_2\}$.

Theorem 2.1 . *Let $R_i:\ (0,\, t_i) \to S(E)$ be smooth with $0 \le R_1 \le R_2$. Let $S_1:\ (0,\, t_1) \to S(E)$ be a solution of the Riccati equation*

$$S_1' + S_1^2 + R_1 = 0$$

with maximal $t_1 \in (0,\, \infty]$. Let $S_2:\ (0,\, t_2) \to S(E)$ satisfy the following *inequality*

$$S_2' + S_2^2 + R_2 \leq 0$$

with maximal $t_2 \in (0, \infty]$. *Define* $U := S_2 - S_1$ *and assume that* $\limsup_{t \to 0^+} U(t) \leq 0$. *Then* $t_2 \leq t_1$ *and* $S_2 \leq S_1$ *on* $(0, t_2)$.

Theorem 2.3 . *If the radial curvatures* K *of* M *satisfy for some* $c \in [0,1]$ *and all* $r > 0$

$$0 \leq K \leq \frac{c(1-c)}{r^2}$$

then we have

$$\frac{c}{r}|X|^2 \leq Hess_r(X,\, X) \leq \frac{1}{r}|X|^2, \qquad X \in T_xM \backslash \mathbb{R}\nabla r(x)$$

$$Hess_r(X,\, X) = 0, \qquad X \in \mathbb{R}\nabla r(x).$$

Application: See Theorem 3.13.

B. Geometric Inequalities.

Theorem 3.3 . *Let* M *be a complete* n*-manifold with sectional curvature* $Sec^M \leq 0$ *and with* $n > p > 1$. *Given a fixed point* $x_0 \in M$, *and let* r *be the distance form* x_0. *Then for every* $u \in C_0^\infty(M)$ *the following inequality holds:*

$$\left(\frac{n-p}{p}\right)^p \int_M \frac{|u|^p}{r^p} dv \leq \int_M |\nabla u|^p\, dv.$$

Theorem 3.4 . *Let* M *be a complete Riemannian* n*-manifold with a pole* x_0. *If* $Ric^M \geq 0$ *and* $2 \leq n < p$, *then for every* $u \in C_0^\infty(M)$ *and* $\frac{u}{r} \in L^p(M)$, *one* *has*

$$(\frac{p-n}{p})^p \int_M \frac{|u|^p}{r^p} dv \leq \int_M |\nabla u|^p \, dv,$$

where r is the distance function from the pole of M.

Theorem 3.7(Weighted Hardy Inequality) **.** *Let M be an n-dimensional Cartan-Hadamard manifold. Let x_0 be a fixed point and r be the distance from x_0. Then for every $u \in C_0^\infty(M)$, the following inequality holds:*

$$\left(\frac{n+\alpha-p}{p}\right)^p \int_M r^\alpha \frac{|u|^p}{r^p} dv \leq \int_M r^\alpha |\nabla u|^p dv,$$

where dv is the volume element on M, $1 \leq p < \infty$ and $n+\alpha-p > 0$.

Theorem 3.8 (Weighted Sobolev Inequality) **.** *Let M be an n-dimensional Cartan-Hadamard manifold. Let x_0 be a fixed point and r be the distance from x_0. Then for every $u \in C_0^\infty(M)$, the following inequality holds:*

$$\left(\int_M r^{\alpha p^*} |u|^{p^*} dv\right)^{\frac{1}{p^*}} \leq C \left(\int_M r^{\alpha p} |\nabla u|^p dv\right)^{\frac{1}{p}},$$

where dv is the volume element on M, $1 \leq p < n$, $\frac{\alpha-1}{n} + \frac{1}{p} > 0$, $p^* = \frac{np}{n-p}$ and C is a positive constant independent of u.

Theorem 3.9 (Generalized Caffarelli-Kohn-Nirenberg type Inequality) **.** *Let M be an n-dimensional Cartan-Hadamard manifold. Let x_0 be a fixed point and r be the distance from x_0. Suppose there exists a constant $\tilde{C}$ such that*

$$Area(\partial B_r(x_0)) \leq \tilde{C} r^{n-1}.$$

Let p, q, s, α, β, γ, σ, a be fixed real numbers satisfying

$$q,\ s \geq 1, \quad 1 \leq p < n, \quad 0 \leq a \leq 1,$$

$$\frac{1}{s}+\frac{\gamma}{n}>0, \quad \frac{1}{p}+\frac{\alpha}{n}>0, \quad \frac{1}{q}+\frac{\beta}{n}>0,$$

where

$$\gamma = a\sigma + (1-a)\beta.$$

There exists a positive constant C *such that the following inequality holds for all* $u \in C_0^\infty(M)$

$$\|r^\gamma u\|_{L^s} \leq C\|r^\alpha |\nabla u|\|_{L^p}^a \, \|r^\beta u\|_{L^q}^{1-a}$$

if the following relations hold:

$$\frac{1}{s}+\frac{\gamma}{n} = a(\frac{1}{p}+\frac{\alpha-1}{n}) + (1-a)(\frac{1}{q}+\frac{\beta}{n}).$$

$$\alpha - \sigma \geq 0, \qquad \textit{if } a>0,$$

$$\alpha - \sigma \leq 1, \qquad \textit{if } a>0 \textit{ and } \frac{1}{s}+\frac{\gamma}{n} = \frac{1}{p}+\frac{\alpha-1}{n}.$$

Theorem 3.11 . *Let* M *be a complete noncompact Riemannian* n*-manifold. Then for every* $x_0 \in M$*, every* $u \in C_0^\infty(M\setminus\{x_0\})$*, and every* $a, b \in \mathbb{R}$*, with* $a+b \neq 1$*, the following inequalities hold:*

(i) *For* $p \geq 2$*,*

$$\frac{1}{p}\int_M \frac{a+b-r\Delta r}{r^{a+b+1}}\,|u|^p\,dv \leq \left(\int_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}} \left(\int_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}}.$$

(ii) *For* $1<p<2$*,*

$$\frac{1}{p}\int_M \frac{a+b-r\Delta r}{r^{a+b+1}}(|u|^2+\delta)^{\frac{p}{2}}dv \leq \left(\int_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}} \left(\int_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}}$$

where $\delta > 0$, dv is the volume element of M, r is the distance to x_0, and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. In particular, if $Ric^M \geq 0$ and $a + b + 1 \geq n$, then

$$\frac{(a+b+1)-n}{p}\int_M \frac{|u|^p}{r^{a+b+1}}\,|u|^p\,dv \leq \left(\int_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}}\left(\int_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}}.$$

Theorem 3.12 . *Let M be an n-dimensional Cartan-Hadamard manifold. Then for every $x_0 \in M$, every $u \in C_0^\infty(M\setminus\{x_0\})$, and every $a, b \in \mathbb{R}$, with $a + b + 1 \leq n$, the following inequality holds:*

$$\frac{n-(a+b+1)}{p}\int_M \frac{|u|^p}{r^{a+b+1}}dv \leq \left(\int_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}}\left(\int_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}}$$

where dv is the volume element of M, r is the distance to x_0, and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.13 . *Let M be an n-dimensional manifold with a pole of radial curvature $0 \leq K \leq \frac{c(1-c)}{r^2}$, where $c \in [0, 1]$. Then for every $u \in C_0^\infty(M)$ and every $a, b \in \mathbb{R}$ with $c(n-1) - (a+b) \geq 0$, the following inequality holds:*

$$\frac{cn-(a+b+c)}{p}\int_M \frac{|u|^p}{r^{a+b+1}}dv \leq \left(\int_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}}\left(\int_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}}$$

where dv is the volume element of M, r is the distance to x_0, and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

C. Applications to p-harmonic Geometry

A C^2 function $u : M \to \mathbb{R}$ is said to be p-harmonic (resp. p-superharmonic, and p-subharmonic) in a storng sense if its p-Laplacian $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (resp. ≤ 0 , and ≥ 0). A function $u : M \to \mathbb{R}$ is said to be p-harmonic (

resp. p-superharmonic, and p-subharmonic) in a weak sense if its p-Laplacian $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (resp. ≤ 0 , and ≥ 0) in the sense of distributions.

Theorem 4.11 (Liouville Theorem for p-harmonic functions)**.** *Let M be a complete noncompact Riemannian n-manifold with a pole, and non-positive radial curvature. Suppose that* $Ric^M \geq -\tau\frac{(n-2)^2}{4r^2}$ *a.e. , where τ is a constant satisfying*

$$\begin{aligned} \tau &< \frac{4\left(q-1+\kappa+b\right)}{q^2}, \\ \textit{in which}\, \kappa &= \min\{\frac{(p-1)^2}{n-1}, 1\} \textit{ and } b = \min\{0, (p-2)(q-p)\}. \end{aligned}$$

Let $u \in C^3(M)$ be a p-harmonic function in a weak sense for $p \in \{2\} \cup [4, \infty)$, and in a strong sense for $p \in (1,2) \cup (2,4)$, with finite q-energy $E_q(u) = \int_M |du|^q dv$, for p and q satisfying one of the following:

(1) $p = 2$ *and* $q > \frac{n-2}{n-1}$,

(2) $p = 4$, $q > 1$ *and* $q - 1 + \kappa + b > 0$,

(3) $p > 2$, $p \neq 4$, *and either* $\max\left\{1, p-1-\frac{\kappa}{p-1}\right\} < q \leq p - \frac{(p-4)^2 n}{4(p-2)}$ *, or both* $q > 2$ *and* $q - 1 + \kappa + b > 0$.

Then u is constant. If p and q satisfy

(4) $1 < p < 2$ *and* $q > 2$,

then u does not exist.

Theorem 4.12. *Let N be a Riemannian $(n+1)$-manifold, M be a stable minimal hypersurface in N, and ν be a unit normal vector to M, such that the length $|A|$ of the second fundamental form of M in N satisfying $|A|^2 + Ric^N(\nu) > 0$ a.e.. Suppose* $Ric^M \geq -\tau(|A|^2 + Ric^N(\nu))$ *, where τ is as in Theorem 4.11. Let $u \in C^3(M)$ be a p-harmonic function with finite q-energy,* *for p and q as in Theorem 4.11. Then the same conclusion as in Theorem 4.11 holds.*

Theorem 4.14 (Liouville Theorem for p-harmonic morphisms)**.** *Let M be as in Theorem 4.11 or in Theorem 4.12. Suppose $Ric^M \geq -\tau\frac{(n-2)^2}{4r^2}$, where τ is as in Theorem 4.11. If $u \in C^3(M)$ is a p-harmonic morphism $u : M \to \mathbb{R}^k$, with finite q-energy, for p and q as in Theorem 4.11. Then the same conclusion as in Theorem 4.11 holds.*

Theorem 4.15 (Liouville Theorem for weakly conformal maps)**.** *Let M be as in Theorem 4.11 or in Theorem 4.12, in which $p = n$ in Theorem 4.11. If $u : M \to \mathbb{R}^n$ is a weakly conformal map with finite q-energy, for n and q satisfying one of the following:*

(1) $n = 2$ *and* $q > 0$,

(2) $n = 4$, $q > 1$ *and* $q + b > 0$,

(3) $n > 2$, $n \neq 4$, *and either* $\frac{n(n-2)}{n-1} < q \leq n - \frac{(n-4)^2 n}{4(n-2)}$, *or both* $q > 2$ *and* $q + b > 0$,

then u is a constant.

Theorem 4.16(Picard Theorem for p-harmonic morphisms)**.** *Let M be as in Theorem 4.11 or Theorem 4.12. Suppose that $u \in C^3(M)$ is a p-harmonic morphism $u : M \to \mathbb{R}^k \backslash \{y_0\}$, and the function $x \mapsto |u(x) - y_0|^{\frac{p-n}{p-1}}$ has finite q-energy where $p \neq n$, for p and q satisfying one of the following:* (1), (2), *and* (3) *as in Theorem 4.11. Then u is constant. For p and q satisfying* (4) *as in Theorem 4.11, then u does not exist.*

Theorem 4.17(Picard Theorem for weakly conformal maps)**.** *Let M be as in Theorem 4.11 or in Theorem 4.12, in which $p = n$ in* (4.5)*. Suppose that $u : M \to \mathbb{R}^n \backslash \{y_0\}$ is a weakly conformal map and the function $x \mapsto \log |u(x) - y_0|$ has finite q-energy, for n and q satisfying one of the following:* (1), (2), *and* (3) *as in Theorem 4.15. Then u is constant.*

Chapter 2

Comparison Theorems

We denote $T_{x_0}M$ the tangent space to M at $x_0 \in M$. A pole is a point $x_0 \in M$ such that the exponential map $exp_{x_0} : T_{x_0}M \to M$ is a diffeomorphism. Furthermore, if M possess a pole, M is complete. Given such a manifold M with a pole x_0, for any point $x \in M$, there is a unique geodesic γ emanating from the pole x_0 such that $\gamma(t) = x$. Let $r(x)$ be the distance from x_0 to x, then ∇r is a vector field defined on $M \backslash \{x_0\}$ such that for any $x \in M \backslash \{x_0\}$, $\nabla r\,(x)$ is the unit vector tangent to the unique geodesic joining x_0 to x and pointing away from x_0. A radial plane is a plane π which contains $\nabla r(x)$ in the tangent space T_xM. By the radial curvature K of a manifold with a pole, we mean the restriction of the sectional curvature function to all the radial planes. We define $K(t)$ to be the radial curvature of M at x for any x such that $r(x) = t$. Let (M, g) be a manifold with a pole x_0. Then r is a smooth function on $M \backslash \{x_0\}$. The Hessian of r by definition the second covariant differential $Hess_r$ of r, i.e.,

$$Hess_r(X,\, Y) = X(Yr) - (\nabla_X Y)r,$$

for all vector X, Y on M. It is a symmetric tensor. Let a tensor $g - dr \otimes dr = 0$ on the radial direction, and is just the metric tensor g on the orthogonal complement of ∇r. The Hessian comparison theorem roughly says that the larger the curvature, the smaller the Hessian of the distance function. We recall the following Hessian comparison theorem on manifolds with nonpositive radial curvature:

Theorem A. *(cf. [19]) (i) If* $-\alpha^2 \le K(r) \le -\beta^2$ *with* $\alpha > 0,\ \beta > 0,$ *then*

$$\beta \coth(\beta r)[g - dr \otimes dr] \le Hess_r \le \alpha \coth(\alpha r)[g - dr \otimes dr]$$

(ii) If $-\frac{a}{1+r^2} \le K(r) \le 0$ *with* $a \ge 0,$ *then*

$$\frac{1}{r}[g - dr \otimes dr] \le Hess_r \le \frac{1 + \sqrt{1+4a}}{2r}[g - dr \otimes dr]$$

(iii) If $-Ar^{2q} \le K(r) \le -Br^{2q}$ *with* $A \ge B > 0$ *and* $q > 0,$ *then*

$$B_0 r^q [g - dr \otimes dr] \le Hess_r \le (\sqrt{A} \coth \sqrt{A}) r^q [g - dr \otimes dr]$$

for $r \ge 1,$ *where* $B_0 = \min\{1, -\frac{q+1}{2} + [B + (\frac{q+1}{2})^2]^{1/2}\}.$

Greene and Wu obtain the above comparison theorem via Jacobi equations. As Jacobi equations are related to Riccati equations, We are interested in obtaining Hessian comparison theorems for manifolds with nonnegative radial curvatures via Riccati equations.

Let E be a vector space with an inner product $\langle\, ,\, \rangle$, $S(E)$ be the space of self-adjoint linear endomorphisms of E, and $R_i : (0,\, t_i) \to S(E)$ be continuous functions with maximal $t_i \in (0, \infty]$ $(i = 1,\, 2)$. We say

$$R_1 \le R_2$$

if

$$\langle R_1(t)\,(x), x\rangle \le \langle R_2(t)\,(x), x\rangle$$

for every $t \in (0,\, t_0)$ and every $x \in E$, where $t_0 = \min\{t_1,\, t_2\}$.

In [17], Eschenburg and Heintze gave a short prove for the comparison theory of the matrix valued Riccati equation with singular initial value. We weaken their initial condition, extend their comparison class of Riccati equations and obtain the following theorem:

Theorem 2.1. *Let $R_i : (0,\, t_i) \to S(E)$ be smooth with $0 \le R_1 \le R_2$. Let $S_1 : (0,\, t_1) \to S(E)$ be a solution of the Riccati equation*

$$S_1' + S_1^2 + R_1 = 0$$

with maximal $t_1 \in (0,\, \infty]$. Let $S_2 : (0,\, t_2) \to S(E)$ satisfy the following inequality

$$S_2' + S_2^2 + R_2 \le 0$$

with maximal $t_2 \in (0,\, \infty]$. Define $U := S_2 - S_1$ and assume that $\limsup_{t \to 0^+} U(t) \le 0$. Then $t_2 \le t_1$ and $S_2 \le S_1$ on $(0,\, t_2)$.

We fix a basis of a vector space, then any linear endomorphism of the vector space can be represented by a matrix. For the simplicity, we now consider the operators as matrices.

Proof: Let $t_0 = \min\{t_1,\, t_2\}$. Denote $X = -\frac{1}{2}(S_1 + S_2)$ and $Y = R_1 - R_2$. By the ricatti equation $S_1' + S_1^2 + R_1 = 0$ and the inequality $S_2' + S_2^2 + R_2 \le 0$, U satisfies

$$U' \le X \cdot U + U \cdot X + Y. \tag{2.1}$$

Since $S_j' \le -R_j \le 0$ $(j = 1, 2)$, then for any fixed $t^* \in (0,\, t_0)$ we have

$$\int_t^{t^*} S_j' \le \int_t^{t^*} -R_j \le 0,$$

which imply

$$S_j(t) \geq S_j(t^*) \qquad\qquad \text{for any } t \in (0,\, t^*).$$

That is, S_j is bounded from below near 0. Hence X is bounded from above near 0, i.e. there exists $c \in \mathbb{R}$ such that $X \leq c \cdot I$.

Let $g : (0,\, t_0) \to End(E)$ be a nonsingular solution of the homogeneous equation

$$g' = X \cdot g. \tag{2.2}$$

In fact, we could use all the elements g_{ij} of g to form a new vector v and all the elements X_{ij} of X to form a new matrix A such that the following homogeneous equation holds

$$v' = A \cdot v. \tag{2.3}$$

Then by the existence and uniqueness of homogeneous equation, once the initial condition is given, there exists a unique solution of (2.3). In other words, there is a unique solution of (2.2).

Once the initial value $g(s_0)$ where $s_0 \in (0,\, t_0)$ with $g(s_0)$ nonsingular is given, it is easy to show the solution of (2.2) is nonsingular. To show this claim, we consider the following initial value problem:

$$\bar{g} = -\bar{g} \cdot X, \quad \bar{g}(s_0) = g(s_0)^{-1}, \tag{2.4}$$

where $\bar{g} : (0,\, t_0) \to End(E)$. It has a unique solution and also satisfies $(\bar{g}g)' = 0$. Therefore, we get $\bar{g}(t)g(t) = \bar{g}(s_0)g(s_0) = I$, for any $t \in (0,\, t_0)$, i.e. $\bar{g}$ is the inverse of g.

Now let $U = g \cdot V \cdot g^T$, where $V : (0,\, t_0) \to S(E)$ satisfies

$$V' \leq g^{-1} \cdot Y \cdot (g^{-1})^T. \tag{2.5}$$

Then

$$\begin{aligned} U' &= g' \cdot V \cdot g^T + g \cdot V' \cdot g^T + g \cdot V \cdot (g^T)' \\ &\leq X \cdot g \cdot V \cdot g^T + g \cdot (g^{-1} \cdot Y \cdot (g^{-1})^T) \cdot g^T + g \cdot V \cdot (X \cdot g)^T \\ &= X \cdot g \cdot V \cdot g^T + Y + g \cdot V \cdot (X \cdot g)^T \\ &= X \cdot U + Y + U \cdot X \end{aligned}$$

That is, U is a solution of (2.1).

Since $Y \leq 0$, then $V' \leq 0$ on $(0,\, t_0)$. Next we have to show that $\limsup_{t\to 0^+} V(t) \geq 0$. Since $V' \leq 0$ on $(0,\, t_0)$, then either $\lim_{t\to 0^+} V(t)$ exists, or $\lim_{t\to 0^+} V(t) = \infty$ which means

$$\limsup_{t\to 0^+} V(t) \;=\; \lim_{t\to 0^+} V(t).$$

Note that

$$\langle Vx,\, x\rangle \;=\; \langle g^{-1} \cdot U \cdot (g^{-1})^T x,\, x\rangle \;=\; \langle U \cdot (hx),\, hx\rangle$$

for any $x \in E$, where $h = (g^{-1})^T$. Consider the function $f = \|hx\|^2$,

$$f' = 2\langle h'x,\, hx\rangle = -2\langle X \cdot (hx),\, hx\rangle \geq \langle c \cdot I \cdot (hx),\, hx\rangle = -2cf.$$

Then

$$
\begin{aligned}
& \int_t^{t^*} \frac{f'}{f} \geq \int_t^{t^*} -2c \\
\Rightarrow\quad & \ln f(t^*) - \ln f(t) \geq -2c(t^* - t) \\
\Rightarrow\quad & \ln f(t) \leq \ln f(t^*) + 2c(t^* - t) \\
\Rightarrow\quad & \ln f(t) \leq \ln f(t^*) + 2ct^*
\end{aligned}
$$

for any $t \in (0,\, t^*)$. That is f is bounded near 0.

Therefore, there exists a sequence $s_k \to 0^+$ such that $h(s_k)x$ converges to some $y \in E$ as $k \to \infty$. Then we have

$$
\begin{aligned}
\lim_{t\to 0^+} \langle Vx,\, x\rangle &= \lim_{k\to\infty} \langle U \cdot (h(s_k)x),\, h(s_k)x\rangle \\
&\leq \langle \limsup_{t\to 0^+} u(t)y,\, y\rangle \\
&\leq 0
\end{aligned}
$$

Now from $\lim_{t\to 0^+} V(t) \leq 0$ and $V' \leq 0$, we get $V \leq 0$ and hence $U \leq 0$. Thus $S_1 \geq S_2$ on $(0,\, t_0)$.

If $t_1 < t_2$, we have $S_1(t) \sim \frac{1}{t-t_1} I + O(t)$ near $t = t_1$. As $t \to t_1^-$, $S_1 \to -\infty$. However, S_2 is finite on $(0,\, t_1)$. We get a contradiction. Hence $t_0 = t_2 \leq t_1$.

□

Let $S_i \,:\, (0,\, t_i) \to S(E)$ $(i = 1,\, 2)$. If $R_1(t) = 0$, $S_1(t) = \frac{1}{t} I + O(t)$ as $t \to 0^+$, then

$$S_1(t) = \frac{1}{t} I,$$

where I is the identity linear transformation, is the solution of

$$S_1' + S_1^2 + R_1 = 0 \quad \text{with} \quad t_1 = \infty\,.$$

Similarly, if $R_2(t) = \frac{c(1-c)}{t^2} I$, where $0 < c < 1$, $S_2(t) = \frac{c}{t} I + O(t)$ as $t \to 0^+$, then

$$S_2(t) = \frac{c}{t} I,$$

is the solution of

$$S_2' + S_2^2 + R_2 = 0 \quad \text{with} \quad t_2 = \infty.$$

Then the following corollary holds immediately:

Corollary 2.2. *If* $0 \le R \le \frac{c(1-c)}{t^2} I$ *and* $S : (0, \infty) \to S(E)$ *is a solution of*

$$S' + S^2 + R = 0$$

satisfying $S(t) = \frac{1}{t} I + O(t)$ *as* $t \to 0^+$, *then*

$$\frac{c}{t} I \le S(t) \le \frac{1}{t} I.$$

Let M be a manifold which posses a pole x_0. Let S be the shape operator of geodesic balls in M (cf. [35]), i.e $S : T_x M \backslash \mathbb{R} \nabla r(x) \to T_x M \backslash \mathbb{R} \nabla r(x)$ with $S(v) = \nabla_v \nabla r$. Then we have

$$\nabla_{\nabla r} S + S^2 + R = 0,$$

where $R : T_x M \backslash \mathbb{R} \nabla r(x) \to T_x M \backslash \mathbb{R} \nabla r(x)$ is the radial curvature given by $R(v) = R(v, \nabla r) \nabla r$.

Since $Hess_r(X, X) = \langle S(X), X \rangle$ and the radial curvature K of M is given by $K(v) := \langle R(v), v \rangle$, we have the following theorem as an application of the above comparison theorem in differential equation:

Theorem 2.3. *If the radial curvatures K of M satisfy for some $c \in [0,1]$ and all $r > 0$*

$$0 \le K \le \frac{c(1-c)}{r^2}$$

then we have

$$\frac{c}{r}|X|^2 \le Hess_r(X,\, X) \le \frac{1}{r}|X|^2, \qquad X \in T_xM \backslash \mathbb{R}\nabla r(x)$$

$$Hess_r(X,\, X) = 0, \qquad X \in \mathbb{R}\nabla r(x)$$

There are some applications of comparison theorems: one is geometric inequalities, which will be shown in Chapter 3, and the other is the monotonicity results studied in [15].

Chapter 3

Geometric inequalities

3.1 Preliminaries

A Cartan-Hadamard manifold is a complete simply-connected Riemannian manifold of nonpositive sectional curvature. The theorem of Cartan-Hadamard states that if M is a Cartan-Hadamard manifold, and $x \in M$, then the exponential map $exp_x : T_xM \to M$ is a diffeomorphism. Thus every point of a Cartan-Hadamard manifold is a pole.

Without curvature assumption, we derive geometric inequalities on manifolds with a pole for functions $u \in C_0^\infty(M)$.

Proposition 3.1. *[49] Let M be a complete Riemannian n-manifold with a pole x_0. For every $u \in C_0^\infty(M)$, every $\epsilon > 0$, and every $\delta > 0$, with $\delta < d_0$, one has the following:*

$$\begin{aligned} &\left| - \int_{\partial B_\delta(x_0)} \frac{r}{r^p+\epsilon} |u|^p dS + \int_{M \backslash B_\delta(x_0)} \frac{(r^p+\epsilon)(r\Delta r+1) - p\, r^p}{(r^p+\epsilon)^2} \, |u|^p \, dv \right| \\ \le\ & p \Bigg(\int_{M \backslash B_\delta(x_0)} \Big(\frac{|u|^{p-1} r}{r^p+\epsilon} \Big)^{\frac{p}{p-1}} dv \Bigg)^{\frac{p-1}{p}} \Bigg(\int_{M \backslash B_\delta(x_0)} |\nabla u|^p \, dv \Bigg)^{\frac{1}{p}} \end{aligned} \tag{3.1}$$

where $d_0 = \max_{x \in \mathrm{Spt}\, u} \mathrm{dist(x_0,\, x)}$, $\mathrm{Spt\, u}$ is the support of u, $\mathrm{dist(x_0,\, x)}$ is the distance from x_0 to x, $\partial B_\delta(x_0)$ denotes the C^1 boundary of the geodesic ball $B_\delta(x_0)$ centered at x_0 with radius $\delta > 0$, r is the distance from x_0, Δr is the Laplacian of r, dS and dv are the volume element of $\partial B_\delta(x_0)$ and M respectively.

Proof: We first fix $\delta > 0$ and consider $I := p \int\limits_{M\setminus B_\delta(x_0)} \left\langle |u|^{p-2}u\frac{r\nabla r}{r^p+\epsilon}, \nabla u\right\rangle dv\,,$ for any given $\epsilon > 0$. Then it follows that

$$\begin{aligned} I &= \int\limits_{M\setminus B_\delta(x_0)} \operatorname{div}\left(\frac{r\nabla r}{r^p+\epsilon}|u|^p\right) dv - \int\limits_{M\setminus B_\delta(x_0)} \frac{\operatorname{div}(r\nabla r)}{r^p+\epsilon}\,|u|^p\,dv \\ &\quad + \int\limits_{M\setminus B_\delta(x_0)} \frac{pr^p}{(r^p+\epsilon)^2}\,|u|^p\,dv, \end{aligned} \tag{3.2}$$

for every $u \in C_0^\infty(M)$. By the divergence theorem, and the fact that the unit outward normal vector ν on $\partial B_\delta(x_0)$ is $-\nabla r$, the first term on the right hand side of (3.2) satisfies

$$\begin{aligned} \int\limits_{M\setminus B_\delta(x_0)} \operatorname{div}\left(\frac{r\nabla r}{r^p+\epsilon}|u|^p\right) dv &= \int\limits_{B_R(x_0)\setminus B_\delta(x_0)} \operatorname{div}\left(\frac{r\nabla r}{r^p+\epsilon}|u|^p\right) dv \\ &= -\int\limits_{\partial B_\delta(x_0)} \left\langle \frac{r\nabla r}{r^p+\epsilon}|u|^p,\, \nu\right\rangle dS \\ &= \int\limits_{\partial B_\delta(x_0)} \frac{r}{r^p+\epsilon}|u|^p dS \end{aligned} \tag{3.3}$$

where $B_R(x_0)$ is a geodesic ball centered at x_0 with radius $R > d_0$ and Spt u $\subset B_R(x_0) \subset M$.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on M such that $e_1 = \nabla r$. Denote ∇ the Riemannian connection on M. Then $\nabla_{\nabla r}\nabla r = 0$ in M and the Hessian of r is given by $(\nabla_{e_i}dr)(e_i) = \nabla_{e_i}(dr(e_i)) - dr(\nabla_{e_i}e_i)$. Furthermore, off $B_\delta(x_0)$

$$
\begin{aligned}
\operatorname{div}(\nabla r) &= \langle \nabla_{\nabla r}\nabla r, \nabla r\rangle + \sum_{i=2}^{n} \langle \nabla_{e_i}(\nabla r), e_i\rangle \qquad (3.4)\\
&= \sum_{i=2}^{n} (\nabla_{e_i} dr)(e_i)\\
&= \sum_{i=2}^{n} Hess_r(e_i, e_i)
\end{aligned}
$$

where $Hess_r$ is the Hessian of r.

Note that

$$
\nabla(r^p+\epsilon)^{-1} = -(r^p+\epsilon)^{-2} p r^{p-1} \nabla r \qquad (3.5)
$$

Substituting (3.3)-(3.5) into (3.2), one has

$$
-\int_{\partial B_\delta(x_0)} \frac{r}{r^p+\epsilon}|u|^p dS + \int_{M\setminus B_\delta(x_0)} \frac{(r^p+\epsilon)\left[\sum_{i=2}^{n} r Hess_r(e_i,e_i)+1\right] - p\, r^p}{(r^p+\epsilon)^2} |u|^p\, dv = -I
$$

In view of Hölder inequality and the fact $\sum_{i=2}^{n} r Hess_r(e_i,e_i)+1 = r\Delta r + 1$, one obtains the desired (3.1). □

Based on this proposition, we obtain the following geometric inequalities, which have simpler forms on M. Here we allow the values of the integrals on the right hand sides to be $+\infty$.

Proposition 3.2. *[12] Let M be a complete Riemannian n-manifold with a pole x_0.*

(i) For every $u \in C_0^\infty(M)$, and every $\epsilon > 0$, the following inequality holds:

$$
\left| \int_M \frac{(r^p+\epsilon)(r\Delta r+1) - p\, r^p}{(r^p+\epsilon)^2} |u|^p\, dv \right| \le p\left(\int_M \frac{|u|^p}{r^p} dv\right)^{\frac{p-1}{p}} \left(\int_M |\nabla u|^p\, dv\right)^{\frac{1}{p}}. \qquad (3.6)
$$

(ii) For every $u \in C_0^\infty(M)$, every $\epsilon > 0$, and for every $\delta > 0$, with $\delta < d_0$,

one has the following:

$$\int\limits_{M\setminus B_\delta(x_0)} \frac{p\,r^p-(r^p+\epsilon)(r\Delta r+1)}{(r^p+\epsilon)^2}\,|u|^p\,dv \le p\Big(\int\limits_M \frac{|u|^p}{r^p}dv\Big)^{\frac{p-1}{p}}\Big(\int\limits_M |\nabla u|^p\,dv\Big)^{\frac{1}{p}}. \tag{3.7}$$

Proof: Note that the right hand side of (3.1) is less than or equal to $p\Big(\int_M \frac{|u|^p}{r^p}dv\Big)^{\frac{p-1}{p}}\Big(\int_M |\nabla u|^p\,dv\Big)^{\frac{1}{p}}$. As δ tends to zero, $\int\limits_{\partial B_\delta(x_0)} \frac{r}{r^p+\epsilon}|u|^p dS$ tends to zero, and hence the left hand side of (3.1) tends to $\Big|\int_M \frac{(r^p+\epsilon)(r\Delta r+1)-p\,r^p}{(r^p+\epsilon)^2}\,|u|^p\,dv\Big|$ as $\delta\to 0$. This proves (i).

On the other hand, the left hand side of (3.1) is greater than or equal to

$$\begin{aligned}
&\int\limits_{\partial B_\delta(x_0)} \frac{r}{r^p+\epsilon}|u|^p dS + \int\limits_{M\setminus B_\delta(x_0)} \frac{-(r^p+\epsilon)(r\Delta r+1)+p\,r^p}{(r^p+\epsilon)^2}\,|u|^p\,dv\\
\ge\ &\int\limits_{M\setminus B_\delta(x_0)} \frac{-(r^p+\epsilon)(r\Delta r+1)+p\,r^p}{(r^p+\epsilon)^2}\,|u|^p\,dv
\end{aligned}$$

for every $u\in C_0^\infty$, for every $\epsilon>0$, and for every $0<\delta<d_0$. This proves (ii).

$\square$

3.2 Hardy Type Inequalities on Complete Manifolds

Hardy's inequality is an important inequality in mathematics, which was first published in 1920 (cf. [23]) in the one dimensional case:

$$\int_0^\infty \Big(\frac{|F|}{x}\Big)^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty |F'|^p dx.$$

Later on, it has been extended to higher dimensions and there have been lots of research concerning the higher dimensional extension on the Euclidean space (e.g. [4], [18], [41]), in particular, sharp inequalities (cf. [5]) as well as improved versions. In recent years, some attention has been paid to Hardy's inequality in sub-Riemannian spaces (e.g. [20]). However, there is less literature for a general Riemannian manifold.

We first discuss whether there are Hardy type inequalities on manifolds with a pole. That is, whether the following inequality

$$\left|\frac{n-p}{p}\right|^p \int_M \frac{|u|^p}{r^p} dv \leq \int_M |\nabla u|^p \, dv, \tag{3.8}$$

holds for every $u \in C_0^\infty(M)$.

Theorem 3.3. *[49] Let M be a complete n-manifold with sectional curvature $Sec^M \leq 0$ and with $n > p > 1$. Given a fixed point $x_0 \in M$, and let r be the distance from x_0. Then for every $u \in C_0^\infty(M)$ the following inequality holds:*

$$\left(\frac{n-p}{p}\right)^p \int_M \frac{|u|^p}{r^p} dv \leq \int_M |\nabla u|^p \, dv. \tag{3.9}$$

Proof: M is a complete n-manifold with sectional curvature $\mathrm{Sec}^M \leq 0$, then by Cartan-Hadamard Theorem, any point in M is a pole. For any fixed point $x_0 \in M$, in view of Theorem 3.2(i) and Hessian comparison theorem, one obtains

$$\begin{aligned} & \int_M \frac{(n-p)r^p + n\epsilon}{(r^p+\epsilon)^2} \, |u|^p \, dv \qquad (3.10) \\ \leq \;\; & p\left(\int_M \frac{(r^p)^{\frac{1}{p-1}}}{(r^p+\epsilon)^{\frac{p}{p-1}}} \, |u|^p \, dv\right)^{\frac{p-1}{p}} \left(\int_M |\nabla u|^p \, dv\right)^{\frac{1}{p}} \end{aligned}$$

For sufficiently small $\epsilon > 0$, one has

$$\begin{aligned}
\int_M \frac{(n-p)r^p + n\epsilon}{(r^p+\epsilon)^2}\,|u|^p\,dv &\geq \int_M \frac{(n-p)r^p + n\epsilon - p\epsilon}{(r^p+\epsilon)^2}\,|u|^p\,dv \\
&\geq (n-p)\int_M \frac{(r^p+\epsilon)^{\frac{1}{p-1}}}{(r^p+\epsilon)^{\frac{p}{p-1}}}\,|u|^p\,dv \\
&\geq (n-p)\int_M \frac{(r^p)^{\frac{1}{p-1}}}{(r^p+\epsilon)^{\frac{p}{p-1}}}\,|u|^p\,dv.
\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), one has

$$\frac{n-p}{p}\Big(\int_M \frac{(r^p)^{\frac{1}{p-1}}}{(r^p+\epsilon)^{\frac{p}{p-1}}}\,|u|^p\,dv\Big)^{\frac{1}{p}} \leq \left(\int_M |\nabla u|^p\,dv\right)^{\frac{1}{p}}. \tag{3.12}$$

Letting $\epsilon \to 0$, one obtains the desired (3.9). □

Surprisingly, (3.8) does not hold in general for smooth function u with compact support in a complete Riemannian n-manifold with a pole x_0 and with nonnegative Ricci curvature. The following is a counter example (cf. [12]).

We choose $M = \mathbb{R}^n$ and $u \in C_0^\infty(\mathbb{R}^n)$ to be a standard smooth cutoff function in $\mathbb{R}^n$ with $0 \leq u \leq 1, u \equiv 1$ on $B_a(0)$, $u \equiv 0$ off $B_{2a}(0)$, and $|\nabla u| \leq C$ in $B_{2a}(0)\backslash\overline{B_a(0)}$, for some constants a and C. If $p > n$, then via coarea formula, the left hand of (3.8)

$$\begin{aligned}
\left(\tfrac{p-n}{p}\right)^p \textstyle\int_M \frac{|u|^p}{r^p}dv &\geq \left(\tfrac{p-n}{p}\right)^p \textstyle\int_{B_a(0)} \frac{1}{r^p}dv \\
&= \left(\tfrac{p-n}{p}\right)^p \lim_{\epsilon\to 0} \textstyle\int_\epsilon^a \int_{\partial B_r(0)} \frac{1}{r^p}dSdr \\
&= \left(\tfrac{p-n}{p}\right)^p n\omega_n \lim_{\epsilon\to 0} \textstyle\int_\epsilon^a r^{n-p-1}dr \\
&= \infty
\end{aligned}$$

where dS is the volume element of $\partial B_r(0)$, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. On the other hand the right hand of (3.8)

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dv \le C^p \omega_n (2^n - 1) a^n < \infty$$

Consequently, (3.8) does not hold for $u \in C_0^\infty(M)$ in general.

To obtain the Hardy type inequality on complete Riemannian manifolds with a pole and with nonnegative Ricci curvature, we need the Laplacian comparison theorem (cf. [19], [37]) and the essential condition that $\frac{u}{r} \in L^p(M)$.

Theorem 3.4. *[12] Let M be a complete Riemannian n-manifold with a pole x_0 . If $Ric^M \ge 0$ and $2 \le n < p$, then for every $u \in C_0^\infty(M)$ and $\frac{u}{r} \in L^p(M)$, one has*

$$\Big(\frac{p-n}{p}\Big)^p \int_M \frac{|u|^p}{r^p} dv \le \int_M |\nabla u|^p \, dv, \tag{3.13}$$

where r is the distance function from the pole of M.

Proof: In view of Theorem 3.2(ii) , and the Laplacian comparison theorem, for every $u \in C_0^\infty(M)$, for every $\epsilon > 0$, and for every $\delta > 0$, with $\delta < d_0$,

$$\int\limits_{M \setminus B_\delta(x_0)} \frac{p\, r^p - (r^p + \epsilon) n}{(r^p + \epsilon)^2} |u|^p \, dv \le p \Big(\int\limits_M \frac{|u|^p}{r^p} dv \Big)^{\frac{p-1}{p}} \Big(\int\limits_M |\nabla u|^p \, dv \Big)^{\frac{1}{p}} \tag{3.14}$$

In particular, for every $\epsilon < \frac{d_0^p (p-n)}{n}$, we choose $\delta = \delta_0(\epsilon)$ defined to be $\left(\frac{\epsilon n}{p-n}\right)^{\frac{1}{p}}$, then $0 < \delta_0(\epsilon) < d_0$, and (3.14) takes the form of

$$\begin{aligned} & \int_M \frac{pr^p - n(r^p + \epsilon)}{(r^p + \epsilon)^2} \left| u \, \chi_{M \setminus B_{\delta_0(\epsilon)}(x_0)} \right|^p dv \\ \le \;\; & p \Big(\int\limits_M \frac{|u|^p}{r^p} dv \Big)^{\frac{p-1}{p}} \Big(\int\limits_M |\nabla u|^p \, dv \Big)^{\frac{1}{p}} \end{aligned} \tag{3.15}$$

where $\chi_{M\setminus B_{\delta_0(\epsilon)}(x_0)}$ is the characteristic function on $B_{\delta_0(\epsilon)}(x_0)$.

Since $\frac{pr^p-n(r^p+\epsilon)}{(r^p+\epsilon)^2}\left|u\chi_{M\setminus B_{\delta_0(\epsilon)}(x_0)}\right|^p \geq 0$, we apply monotone convergence theorem to the left hand side of (3.15) by letting $\epsilon \to 0$, we get the desired inequality for $u \in C_0^\infty(M)$ with $\frac{u}{r} \in L^p(M)$.

$\square$

As immediate application of the Hardy type inequalities, we obtain the following topological application via the same idea as in Proposition 5.1 in [45].

Theorem 3.5. *[49] Let M be a complete Riemmanian n-manifold. If M supports inequality* (3.9) *with $n > p$ for every $u \in C_0^\infty(M)$, then M is not compact.*

Proof: If M *were* compact, then substituting $u \equiv 1$ into (3.9) we would have $\int\limits_M \frac{|u|^p}{r^p}dv = 0$, or $u = 0$ *a.e.* This is a contradiction.

$\square$

Since geometric inequalities are linked to topology, and since curvature is related to topology, we have the following geometric application:

Theorem 3.6. *[49] Let M be a complete Riemannian n-manifold with $n > p$, and $x_0 \in M$. If M supports inequality* (3.9) *for every $u \in C_0^\infty(M)$, then there does not exists a constant $\tau > 0$ such that the Ricci curvature $Ric^M \geq \tau$.*

Proof: Suppose on the contrary, then by Bonnet-Myers' Theorem (cf. [3, 33]), M would be compact. This contradicts Theorem 3.5.

$\square$

3.3 Geometric inequalities on Cartan-Hadamard manifolds

3.3.1 Weighted Hardy inequality and Weighted Sobolev inequality

In [24], Hardy and Littlewood first gave a one-dimensional weighted Hardy inequality and proved the constant is sharp via Bliss lemma (cf. [1]). After that, plenty of work has been done on weighted Hardy inequalities in Euclidean spaces (e.g. [38], [39]). Employing the Divergence theorem and the Hessian comparison theorem, we obtained the following weighted Hardy inequality on Cartan-Hadamard manifolds.

Theorem 3.7 (Weighted Hardy Inequality)**.** *[29] Let M be an n-dimensional Cartan-Hadamard manifold. Let x_0 be a fixed point and r be the distance from x_0. Then for every $u \in C_0^\infty(M)$, the following inequality holds:*

$$\left(\frac{n+\alpha-p}{p}\right)^p \int_M r^\alpha \frac{|u|^p}{r^p} dv \le \int_M r^\alpha |\nabla u|^p dv, \tag{3.16}$$

where dv is the volume element on M, $1 \le p < \infty$ and $n+\alpha-p > 0$.

Proof: Let $u = (r^2+\epsilon)^{\frac{\beta}{2}}\psi$, where $\psi \in C_0^\infty(M)$ and $\beta < 0$. We have

$$|\nabla u| = |\frac{\beta}{2}(r^2+\epsilon)^{\frac{\beta}{2}-1}\psi\nabla r^2 + (r^2+\epsilon)^{\frac{\beta}{2}}\nabla\psi|.$$

Since for $1 \le p < \infty$, the following inequality is valid:

$$|v+w|^p - |v|^p \ge p|v|^{p-2}\langle v,\, w\rangle\,,$$

for any $v,\, w \in V$, where V is a vector space with the inner product $\langle\,,\,\rangle$. This yields

$$
\begin{aligned}
r^{\alpha}|\nabla u|^{p} &\geq pr^{\alpha}|\frac{\beta}{2}(r^2+\epsilon)^{\frac{\beta}{2}-1}\psi\nabla r^2|^{p-2}\left\langle\frac{\beta}{2}(r^2+\epsilon)^{\frac{\beta}{2}-1}\psi\nabla r^2,\, (r^2+\epsilon)^{\frac{\beta}{2}}\nabla\psi\right\rangle \\
&\quad +r^{\alpha}|\frac{\beta}{2}(r^2+\epsilon)^{\frac{\beta}{2}-1}\psi\nabla r^2|^{p} \\
&= p\beta|\beta|^{p-2}r^{\alpha+p-1}(r^2+\epsilon)^{(\frac{\beta}{2}-1)(p-2)+\beta-1}\psi|\psi|^{p-2}\left\langle\nabla\psi,\, \nabla r\right\rangle \\
&\quad +|\beta|^{p}r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}.
\end{aligned}
$$

Integrating the above inequality over M and applying the divergence theorem, we obtain

$$
\begin{aligned}
\int_M r^{\alpha}|\nabla u|^{p}dv &\geq |\beta|^{p}\int_M r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}dv \\
&\quad +|\beta|^{p-1}\int_M r^{\alpha+p-1}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p+1}\Delta r|\psi|^{p}dv \\
&\quad +(\alpha+p-1)|\beta|^{p-1}\int_M r^{\alpha+p-2}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p+1}|\psi|^{p}dv \\
&\quad +(\beta p-2p+2)|\beta|^{p-1}\int_M r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}dv.
\end{aligned}
$$

By the Hessian comparison theorem, $r\Delta r \geq n-1$, then

$$
\begin{aligned}
\int_M r^{\alpha}|\nabla u|^{p}dv &\geq |\beta|^{p}\int_M r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}dv \\
&\quad +(n-1)|\beta|^{p-1}\int_M r^{\alpha+p-2}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}r^2|\psi|^{p}dv \\
&\quad +(\alpha+p-1)|\beta|^{p-1}\int_M r^{\alpha+p-2}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}r^2|\psi|^{p}dv \\
&\quad +(\beta p-2p+2)|\beta|^{p-1}\int_M r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}dv \\
&= |\beta|^{p}\int_M r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}dv \\
&\quad +(n+\alpha-p+\beta p)|\beta|^{p-1}\int_M r^{\alpha+p}(r^2+\epsilon)^{\left(\frac{\beta}{2}-1\right)p}|\psi|^{p}dv.
\end{aligned}
$$

Let $\beta = \frac{p-\alpha-n}{p} < 0$, then $n+\alpha-p+\beta p = 0$, and we obtain

$$\int_M r^{\alpha}|\nabla u|^p dv \geq \left(\frac{n+\alpha-p}{p}\right)^p \int_M r^{\alpha+p}(r^2+\epsilon)^{-p}|u|^p dv.$$

Since $r^{\alpha+p}(r^2+\epsilon)^{-p}|u|^p \geq 0$ and $\int_M r^{\alpha}\frac{|u|^p}{r^p}dv < \infty$ if $n+\alpha-p > 0$, we apply monotone convergence theorem to the right hand side of the above inequality by letting $\epsilon \to 0$ and we get the desired (3.16) for $u \in C_0^{\infty}(M)$.

□

Sobolev inequalities, also called Sobolev imbedding theorems, are very popular in partial differential equations or in the calculus of variations, and have been investigated by a great number of authors (cf. [40],[31]). In geometric analysis, the Sobolev inequality plays an important role as well. For instance, it is well known that the isoperimetric inequality is equivalent to the Sobolev inequality on manifold M. It is also shown that if M is a complete n-dimensional Riemannian manifold and the Sobolev inequalities holds on M, then the geodesic ball has maximal volume growth (cf. [36]). On Cartan-Hadamard manifolds, the following Sobolev inequality holds (cf. [27], [13], [26]):

Theorem B. *Let M be an n-dimensional Cartan-Hadamard manifold. For any $u \in C_0^{\infty}(M)$, the following inequality holds:*

$$\left(\int_M |u|^{p^*} dv\right)^{\frac{1}{p^*}} \leq C\left(\int_M |\nabla u|^p dv\right)^{\frac{1}{p}}, \tag{3.17}$$

where $1 \leq p < n$, $p^* = \frac{np}{n-p}$ and C is a positive constant independent of u.

Similar to the weighted Hardy inequality, there is a weighted Sobolev inequality on Cartan-Hadamard manifolds.

Theorem 3.8 (Weighted Sobolev Inequality)**.** *[29] Let M be an n-dimensional Cartan-Hadamard manifold. Let x_0 be a fixed point and r be the distance from x_0. Then for every $u \in C_0^\infty(M)$, the following inequality holds:*

$$\left(\int_M r^{\alpha p^*}|u|^{p^*}dv\right)^{\frac{1}{p^*}} \leq C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}, \tag{3.18}$$

where dv is the volume element on M, $1 \leq p < n$, $\frac{\alpha-1}{n}+\frac{1}{p} > 0$, $p^* = \frac{np}{n-p}$ and C is a positive constant independent of u.

Throughout the proof, C denotes a constant, depending on the parameters n, α, p, whose value may change from line to line.

Proof: It is clear that if $\alpha = 0$, (3.18) is just the Sobolev inequality. If $\alpha \neq 0$, since $\frac{\alpha-1}{n}+\frac{1}{p} > 0$, then $\int_M r^{\alpha p^*}|u|^{p^*}dv < \infty$ and $\int_M r^{\alpha p}|\nabla u|^p dv < \infty$ for any $u \in C_0^\infty(M)$. Note that for any $\epsilon > 0$ and $u \in C_0^\infty(M)$, $(r^2+\epsilon)^{\frac{\alpha}{2}}u \in C_0^\infty(M)$. Apply (3.17), we have

$$\left(\int_M |(r^2+\epsilon)^{\frac{\alpha}{2}}u|^{p^*}dv\right)^{\frac{1}{p^*}} \leq C\left(\int_M |\nabla\left((r^2+\epsilon)^{\frac{\alpha}{2}}u\right)|^p dv\right)^{\frac{1}{p}}.$$

Since

$$\nabla\left((r^2+\epsilon)^{\frac{\alpha}{2}}u\right) = (r^2+\epsilon)^{\frac{\alpha}{2}}\nabla u + \frac{\alpha}{2}(r^2+\epsilon)^{\frac{\alpha}{2}-1}u\nabla r^2,$$

Then by Minkowski inequality

$$\begin{aligned}
&\left(\int_M |\nabla\left((r^2+\epsilon)^{\frac{\alpha}{2}}u\right)|^p dv\right)^{\frac{1}{p}} \\
\leq\ &\left(\int_M |(r^2+\epsilon)^{\frac{\alpha}{2}}\nabla u|^p dv\right)^{\frac{1}{p}} + \left(\int_M |\frac{\alpha}{2}(r^2+\epsilon)^{\frac{\alpha}{2}-1}u\nabla r^2|^p dv\right)^{\frac{1}{p}}.
\end{aligned}$$

If $\alpha < 0$,

$$\left(\int_M |(r^2+\epsilon)^{\frac{\alpha}{2}}\nabla u|^p dv\right)^{\frac{1}{p}} \leq \left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}.$$

And by the weighted Hardy inequality

$$\begin{aligned}\left(\int_M |\frac{\alpha}{2}(r^2+\epsilon)^{\frac{\alpha}{2}-1}u\nabla r^2|^p dv\right)^{\frac{1}{p}} &\leq |\alpha|\left(\int_M r^{\alpha p}\frac{|u|^p}{r^p}dv\right)^{\frac{1}{p}}\\ &\leq C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}.\end{aligned}$$

Combine the above two inequalities, we obtain

$$\left(\int_M |(r^2+\epsilon)^{\frac{\alpha}{2}}u|^{p^*}dv\right)^{\frac{1}{p^*}} \leq C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}.$$

Letting $\epsilon \to 0$, we obtain the desired (3.18) by the monotone convergence theorem.

If $\alpha > 0$, then $\left(\int_M |\nabla\left((r^2+\epsilon)^{\frac{\alpha}{2}}u\right)|^p dv\right)^{\frac{1}{p}} \geq \left(\int_M r^{\alpha p^*}|u|^{p^*}dv\right)^{\frac{1}{p^*}}$ obviously. On the other hand,

$$\int_M |(r^2+\epsilon)^{\frac{\alpha}{2}}\nabla u|^p dv \leq 2^{\frac{\alpha p}{2}}\int_M \left(r^{\alpha p}+\epsilon^{\frac{\alpha p}{2}}\right)|\nabla u|^p dv,$$

And by the weighted Hardy inequality

$$\begin{aligned}\int_M |\frac{\alpha}{2}(r^2+\epsilon)^{\frac{\alpha}{2}-1}u\nabla r^2|^p dv &\leq |\alpha|^p\int_M (r^2+\epsilon)^{\frac{\alpha p}{2}}\frac{|u|^p}{r^p}dv\\ &\leq 2^{\frac{\alpha p}{2}}|\alpha|^p\int_M \left(r^{\alpha p}+\epsilon^{\frac{\alpha p}{2}}\right)\frac{|u|^p}{r^p}dv\\ &\leq C\int_M r^{\alpha p}|\nabla u|^p dv + C\epsilon^{\frac{\alpha p}{2}}\int_M |\nabla u|^p dv.\end{aligned}$$

Combine the above three inequalities we have

$$\left(\int_M r^{\alpha p^*}|u|^{p^*}dv\right)^{\frac{1}{p^*}} \leq C\left(\int_M \left(r^{\alpha p}+\epsilon^{\frac{\alpha p}{2}}\right)|\nabla u|^p dv\right)^{\frac{1}{p}}.$$

Let $\epsilon \to 0$, we obtain the desired (3.18).

$\square$

3.3.2 Generalized Caffarelli-Kohn-Nirenberg Type Inequalities

In 1984, Caffarelli-Kohn-Nirenberg obtained a class of first order interpolation inequalities with weights on Euclidean spaces (cf. [6]).

Theorem. *Let p, q, r, α, β, γ, σ, a be fixed real numbers satisfying*

$$p,\, q \geq 1, \quad r > 0, \quad 0 \leq a \leq 1, \tag{3.19}$$

$$\frac{1}{r} + \frac{\gamma}{n} > 0, \quad \frac{1}{p} + \frac{\alpha}{n} > 0, \quad \frac{1}{q} + \frac{\beta}{n} > 0, \tag{3.20}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{3.21}$$

Then there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(M)$

$$\|r^\gamma u\|_{L^r} \leq C\|r^\alpha |\nabla u|\|_{L^p}^a \, \|r^\beta u\|_{L^q}^{1-a} \tag{3.22}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = a(\frac{1}{p} + \frac{\alpha - 1}{n}) + (1-a)(\frac{1}{q} + \frac{\beta}{n}). \tag{3.23}$$

(this is dimensional balance),

$$\alpha - \sigma \geq 0, \qquad \textit{if } a > 0, \tag{3.24}$$

and

$$\alpha - \sigma \le 1, \qquad \textit{if } a > 0 \textit{ and } \frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha - 1}{n}. \tag{3.25}$$

These inequalities include many results such as Hardy inequality and Sobolev inequality. In 1986, C. S. Lin extended their result to higher order derivatives (cf. [30]). Recently, a special case of Caffarelli-Kohn-Nirenberg type inequality on sub-Riemannian manifold was proved in [21] via Hardy inequality and Sobolev inequality. Unlike Caffarelli-Kohn-Nirenberg's procedure, we obtain Caffarelli-Kohn-Nirenberg type inequalities on Cartan-Hadamard manifolds by employing the weighted Sobolev inequality and weighted Hardy inequality.

Theorem 3.9. *[29] Let M be an n-dimensional Cartan-Hadamard manifold. Let x_0 be a fixed point and r be the distance from x_0. Suppose there exists a constant $\tilde{C}$ such that*

$$Area(\partial B_r(x_0)) \le \tilde{C} r^{n-1}. \tag{3.26}$$

Let p, q, s, α, β, γ, σ, a be fixed real numbers satisfying

$$q,\ s \ge 1, \quad 1 \le p < n, \quad 0 \le a \le 1, \tag{3.27}$$

$$\frac{1}{s} + \frac{\gamma}{n} > 0, \quad \frac{1}{p} + \frac{\alpha}{n} > 0, \quad \frac{1}{q} + \frac{\beta}{n} > 0, \tag{3.28}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{3.29}$$

Then there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(M)$

$$\|r^\gamma u\|_{L^s} \le C \|r^\alpha |\nabla u|\|_{L^p}^a \, \|r^\beta u\|_{L^q}^{1-a} \tag{3.30}$$

if the following relations hold:

$$\frac{1}{s}+\frac{\gamma}{n}=a(\frac{1}{p}+\frac{\alpha-1}{n})+(1-a)(\frac{1}{q}+\frac{\beta}{n}). \tag{3.31}$$

$$\alpha-\sigma\geq 0, \qquad \textit{if } a>0, \tag{3.32}$$

$$\alpha-\sigma\leq 1, \qquad \textit{if } a>0 \textit{ and } \frac{1}{s}+\frac{\gamma}{n}=\frac{1}{p}+\frac{\alpha-1}{n}. \tag{3.33}$$

Throughout the proof, C denotes a constant, depending on the parameters, whose value may change from line to line.

Proof: M is a Cartan-Hadamard manifold. Then (3.27)-(3.28) tell us that $\|r^{\gamma}u\|_{L^s}$, $\|r^{\alpha}|\nabla u|\|_{L^p}$, $\|r^{\beta}u\|_{L^q}<\infty$.

If $a=0$, then (3.30) holds obviously. So we only need to treat the case $0<a\leq 1$.

Case I: $a=1$**.**

When $a=1$, (3.29) and (3.32)-(3.33) imply

$$\alpha-1\leq\gamma=\sigma\leq\alpha, \quad \frac{1}{s}+\frac{\gamma}{n}=\frac{1}{p}+\frac{\alpha-1}{n}.$$

Let $p^*=\frac{np}{n-p}$. Then $p\leq s\leq p^*$ and there exists $t\in[0,1]$ such that

$$s=tp+(1-t)p^*=\frac{p(n-tp)}{n-p},$$

and

$$\sigma s=ns(\frac{1}{p}+\frac{\alpha-1}{n})-n=\alpha s-tp=\alpha\left(tp+(1-t)p^*\right)-tp.$$

Apply Hölder inequality, weighted Hardy's inequality (3.16) and weighted

Sobolev inequality (3.18), we obtain

$$
\begin{aligned}
& \left(\int_M r^{\gamma s}|u|^s dv\right)^{\frac{1}{s}} \\
= & \left(\int_M r^{\alpha(tp+(1-t)p^*)-tp}|u|^{tp+(1-t)p^*} dv\right)^{\frac{1}{s}} \\
\leq & \left(\int_M (r^{\alpha tp-tp}|u|^{tp})^{\frac{1}{t}} dv\right)^{\frac{t}{s}} \left(\int_M (r^{\alpha(1-t)p^*}|u|^{(1-t)p^*})^{\frac{1}{1-t}} dv\right)^{\frac{1-t}{s}} \\
\leq & \; C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{t}{s}} \left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{p^*}{ps}(1-t)} \\
= & \; C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{ps}(tp+(1-t)p^*)} \\
= & \; C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}
\end{aligned}
$$

This is the desired (3.30) for $a = 1$.

Case II: $0 < a < 1$ **and** $0 \leq \alpha - \sigma \leq 1$**.**

Since $0 \leq \alpha - \sigma \leq 1$, then it is easy to check $p \leq \left(\frac{1}{p} + \frac{\alpha-\sigma-1}{n}\right)^{-1} \leq p^*$. An argument similar to the Case I shows that there exists $t \in [0, 1]$ such that

$$\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{n}\right)^{-1} = \frac{p(n - tp)}{n - p},$$

and

$$\sigma\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{n}\right)^{-1} = \alpha\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{n}\right)^{-1} - tp.$$

Hence,

$$\int_M r^{\sigma\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}}|u|^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}} dv \leq C\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}}. \tag{3.34}$$

By (3.29) and (3.31), $\frac{1}{s} = a\left(\frac{1}{p} + \frac{\alpha-\sigma-1}{n}\right) + \frac{1-a}{q}$. For $s = 1$, apply Hölder inequality

$$
\begin{aligned}
& \left(\int_M r^{\gamma s}|u|^s dv\right)^{\frac{1}{s}} \\
= & \int_M r^{a\sigma+(1-a)\beta}|u|^{a+(1-a)} dv \\
\leq & \left(\int_M \left(r^{a\sigma}|u|^a\right)^{\frac{1}{a}\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}} dv\right)^{a\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)} \left(\int_M \left(r^{(1-a)\beta}|u|^{1-a}\right)^{\frac{q}{1-a}} dv\right)^{\frac{1-a}{q}} \\
= & \left(\int_M r^{\sigma\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}}|u|^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}} dv\right)^{a\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)} \left(\int_M r^{\beta q}|u|^q dv\right)^{\frac{1-a}{q}}
\end{aligned}
$$

Combine (3.34) and the above inequality, we obtain the desired (3.30). For $s > 1$, $1 = a\left(\frac{1}{p} + \frac{\alpha-\sigma-1}{n}\right) + \frac{1-a}{q} + \frac{s-1}{s}$. Then apply Hölder inequality

$$
\begin{aligned}
& \int_M r^{\gamma s}|u|^s dv \\
= & \int_M r^{a\sigma+(1-a)\beta+\gamma(s-1)}|u|^{a+(1-a)+(s-1)} dv \\
\leq & \left(\int_M \left(r^{a\sigma}|u|^a\right)^{\frac{1}{a}\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}} dv\right)^{a\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)} \left(\int_M \left(r^{(1-a)\beta}|u|^{1-a}\right)^{\frac{q}{1-a}} dv\right)^{\frac{1-a}{q}} \\
& \left(\int_M \left(r^{\gamma(s-1)}|u|^{s-1}\right)^{\frac{s}{s-1}} dv\right)^{\frac{s-1}{s}} \\
= & \left(\int_M r^{\sigma\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)^{-1}}|u|^{\left(\frac{1}{p}-\frac{1+\sigma}{n}\right)^{-1}} dv\right)^{a\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{n}\right)} \left(\int_M r^{\beta q}|u|^q dv\right)^{\frac{1-a}{q}} \\
& \left(\int_M r^{\gamma s}|u|^s dv\right)^{\frac{s-1}{s}}
\end{aligned}
$$

Combine (3.34) and the above inequality, we obtain the desired (3.30).

Case III: $0 < a < 1$ **and** $\alpha - \sigma > 1$**.**

The idea of proving Case III follows [6]. (3.33) tells us that $\frac{1}{s} + \frac{\gamma}{n} \neq \frac{1}{p} + \frac{\alpha-1}{n}$. Setting $A = \|r^\alpha|\nabla u|\|_{L^p}$ and $B = \|r^\beta u\|_{L^q}$, then (3.30) can be written as

$$\|r^\gamma u\|_{L^s} \leq CA^a B^{1-a}.$$

Rescaling u such that $A^a B^{1-a} = 1$, our goal becomes to show $\|r^\gamma u\|_{L^s}$ is bounded by a constant. From now on, we assume $A^a B^{1-a} = 1$, since this normalization may be achieved by scaling.

To investigate our goal, we introduce a smooth compactly-supported function $\xi(x)\,(0 \leq \xi(x) \leq 1)$ on M with the properties

$$\xi(x) = \begin{cases} 1 & \text{if } r(x) < \frac{1}{2}, \\ 0 & \text{if } r(x) > 1. \end{cases}$$

We have already checked that for $\sigma = \alpha$ and $\sigma = \alpha - 1$, (3.30) holds. Hence, we conclude that

$$\int_M r^{\delta m} |u|^m dv \leq C \quad \text{and} \quad \int_M r^{\epsilon k} |u|^k dv \leq C \tag{3.35}$$

where δ, ϵ, m, k satisfy

$$\begin{aligned} \delta &= b\alpha + (1-b)\beta \\ \frac{1}{m} &= \frac{b}{p} + \frac{1-b}{q} - \frac{b}{n} \\ \epsilon &= d(\alpha - 1) + (1-d)\beta \\ \frac{1}{k} &= \frac{d}{p} + \frac{1-d}{q} \end{aligned} \tag{3.36}$$

for some choice of b and d, $0 \leq b,\ d \leq 1$, and provided that

$$\frac{1}{m} + \frac{\delta}{n} > 0, \qquad \frac{1}{k} + \frac{\epsilon}{n} > 0. \tag{3.37}$$

Obviously,

$$
\begin{aligned}
\frac{1}{s}+\frac{\gamma}{n} &= a(\frac{1}{p}+\frac{\alpha-1}{n})+(1-a)(\frac{1}{q}+\frac{\beta}{n}) \\
\frac{1}{m}+\frac{\delta}{n} &= b(\frac{1}{p}+\frac{\alpha-1}{n})+(1-b)(\frac{1}{q}+\frac{\beta}{n}) \\
\frac{1}{k}+\frac{\epsilon}{n} &= d(\frac{1}{p}+\frac{\alpha-1}{n})+(1-d)(\frac{1}{q}+\frac{\beta}{n})
\end{aligned}
$$

If $\frac{1}{p}+\frac{\alpha-1}{n}<\frac{1}{q}+\frac{\beta}{n}$, then take $b<a<d$, otherwise take $d<a<b$ such that

$$\frac{1}{k}+\frac{\epsilon}{n}<\frac{1}{s}+\frac{\gamma}{n}<\frac{1}{m}+\frac{\delta}{n} \tag{3.38}$$

A direct computation shows that

$$
\begin{aligned}
\frac{1}{s}-\frac{1}{m} &= (a-b)(\frac{1}{p}-\frac{1}{q}-\frac{1}{n})+\frac{a}{n}(\alpha-\sigma) \\
\frac{1}{s}-\frac{1}{k} &= (a-d)(\frac{1}{p}-\frac{1}{q})+\frac{a}{n}(\alpha-\sigma-1)
\end{aligned}
$$

Since $a>0$ and $\alpha-\sigma>1$, then $0<\frac{a}{n}(\alpha-\sigma-1)<\frac{a}{n}(\alpha-\sigma)$. Therefore if $|b-a|$ and $|a-d|$ are sufficiently small, then (3.37) holds and $\frac{1}{m}<\frac{1}{s}$, $\frac{1}{k}<\frac{1}{s}$.

Meanwhile, Fubini theorem and (3.26) show that

$$
\begin{aligned}
\int_{B_1(x_0)} r^{\frac{(\gamma-\epsilon)ks}{k-s}}dv &\leq \int_0^1 r^{\frac{(\gamma-\epsilon)ks}{k-s}}\text{Area}(\partial B_r(x_0))dr \\
&\leq C\int_0^1 r^{\frac{(\gamma-\epsilon)ks}{k-s}}r^{n-1}dr \\
&\leq C
\end{aligned} \tag{3.39}
$$

and

$$\begin{aligned}
\int_{M\setminus B_{\frac{1}{2}}(x_0)} r^{\frac{(\gamma-\delta)ms}{m-s}} dv &\le \int_{\frac{1}{2}}^{\infty} r^{\frac{(\gamma-\delta)ms}{m-s}} \mathrm{Area}(\partial B_r(x_0))dr \qquad (3.40)\\
&\le C\int_{\frac{1}{2}}^{\infty} r^{\frac{(\gamma-\epsilon)ks}{k-s}} r^{n-1}dr\\
&\le C
\end{aligned}$$

Hence, we obtain the following inequalities by applying Hölder inequality

$$\begin{aligned}
\left(\int_M r^{\gamma s}\xi|u|^s dv\right)^{\frac{1}{s}} &\le \left(\int_M r^{\epsilon k}|u|^k dv\right)^{\frac{1}{k}} \left(\int_{B_1(x_0)} r^{\frac{(\gamma-\epsilon)ks}{k-s}} dv\right)^{\frac{1}{s}-\frac{1}{k}} \qquad (3.41)\\
&\le C\left(\int_M r^{\epsilon k}|u|^k dv\right)^{\frac{1}{k}},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_M r^{\gamma s}(1-\xi)|u|^s dv\right)^{\frac{1}{s}} \qquad (3.42)\\
\le\ &\left(\int_M r^{\delta m}|u|^m dv\right)^{\frac{1}{m}} \left(\int_{M\setminus B_{\frac{1}{2}}(x_0)} r^{\frac{(\gamma-\delta)ms}{m-s}} dv\right)^{\frac{1}{s}-\frac{1}{m}}\\
\le\ &C\left(\int_M r^{\delta m}|u|^m dv\right)^{\frac{1}{m}}.
\end{aligned}$$

Combining (3.35), (3.41) and (3.42), we deduce that

$$\|r^\gamma u\|_{L^s} \le C$$

□

The following theorem gives us a sharp constant for (3.30).

Theorem 3.10. *[50] Let M be an n-dimensional Cartan-Hadamard manifold. Let $s > p$, $1 < p < n$ and α, β be fixed real numbers satisfying*

$$\frac{1}{p}+\frac{\alpha}{n}, \quad \frac{p-1}{p(s-1)}\left(1+\frac{\beta}{n}\right), \quad \frac{1}{s}+\frac{\gamma}{n}>0 \tag{3.43}$$

where

$$\gamma=\frac{1}{s}(\alpha-1)+\frac{p-1}{ps}\beta \tag{3.44}$$

Then for any point x_0, any $u \in C_0^\infty(M)$, the following inequality holds:

$$\int_M r^{\gamma s}|u|^s dv \leq \frac{s}{n+\gamma s}\left(\int_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}\left(\int_M r^{\beta}|u|^{\frac{p(s-1)}{p-1}} dv\right)^{\frac{p}{p-1}}. \tag{3.45}$$

where dv is the volume element of M , r is the distance to x_0.

The inequality is sharp when $M=\mathbb{R}^n$ with the assumption that $n+\beta<\left(1-\alpha+\frac{\beta}{p}\right)\frac{(s-1)p}{s-p}$. This has been discussed in [53].

Proof: As the sectional curvature of M is non-positive, we know that $\Delta r^2 \geq 2n$ by the hessian comparison theorem. Start with $\int_M r^{\gamma s}|u|^s dv$, and apply the divergence theorem, we have

$$\begin{aligned}
\int_M r^{\gamma s}|u|^s dv &\leq \frac{1}{2n}\int_M r^{\gamma s}|u|^s \Delta r^2 dv \\
&= \frac{1}{2n}\int_M \left(\operatorname{div}(r^{\gamma s}|u|^s\nabla r^2)-\langle\nabla(r^{\gamma s}|u|^s),\, \nabla r^2\rangle\right) dv \\
&= -\frac{1}{2n}\int_M \langle \gamma s r^{\gamma s-1}\nabla r|u|^s + r^{\gamma s}s|u|^{s-2}u\nabla u,\, 2r\nabla r\rangle dv \\
&= -\frac{1}{n}\int_M \left(\gamma s r^{\gamma s}|u|^s + \langle r^{\gamma s+1}s|u|^{s-2}u\nabla u,\, \nabla r\rangle\right) dv
\end{aligned} \tag{3.46}$$

Combine the like terms, one obtains

$$\left(1+\frac{\gamma s}{n}\right)\int_M r^{\gamma s}|u|^s dv \leq -\frac{1}{n}\int_M \langle r^{\gamma s+1}s|u|^{s-2}u\nabla u,\, \nabla r\rangle dv \tag{3.47}$$

Since $\gamma = \frac{1}{s}(\alpha - 1) + \frac{p-1}{ps}$, then $\alpha + \frac{p-1}{p}\beta = \gamma s + 1$. Apply the Höler's inequality,

$$\begin{aligned}
& \left(1+\tfrac{\gamma s}{n}\right) \int\limits_M r^{\gamma s}|u|^s dv \\
\leq\ & -\tfrac{1}{n}\int\limits_M \langle r^{\gamma s+1} s|u|^{s-2}u\nabla u,\ \nabla r\rangle dv \\
\leq\ & \tfrac{s}{n}\left(\int\limits_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}\left(\int\limits_M r^{\frac{p-1}{p}\beta\frac{p}{p-1}}|u|^{(s-1)\frac{p}{p-1}}|\nabla r|^{\frac{p}{p-1}} dv\right)^{\frac{p-1}{p}} \\
=\ & \tfrac{s}{n}\left(\int\limits_M r^{\alpha p}|\nabla u|^p dv\right)^{\frac{1}{p}}\left(\int\limits_M r^{\beta}|u|^{\frac{p(s-1)}{p-1}} dv\right)^{\frac{p-1}{p}}
\end{aligned} \tag{3.48}$$

Then (3.45) follows immediately.

□

3.4 Weighted-norm Inequalities for Functions with Compact Support in $M\backslash\{x_0\}$

Let M be a complete Riemannian n-manifold. For any $p \in M$, giving a vector $X \in T_pM$, let $\gamma(t)$ be the unique geodesic starting from p along the direction X. When t is small, we have $\exp_p(tX) = \gamma(t)$ for $t > 0$, and γ is the unique minimal geodesic joining p and $\exp_p(tX)$.

Let

$$t_0 = \sup\{t > 0 : \gamma \text{ is the unique minimal geodesic joining } p \text{ and } \gamma(t)\}.$$

If $t_0 < \infty$, then $\gamma(t_0)$ is called a cut point of p. The set of all cut points of p is called the cut locus of p (denoted by $\mathrm{Cut}(p)$).

If we denote $S_p = \{X \in T_pM : \|X\| = 1\}$, it is clear that for any $X \in S_p$ there can be at most one cut point on the geodesic $\exp_p(tX)$, $t > 0$. If $\exp_p(t_0X) = q$ is a cut point of p then we set $\mu(X) = d(p, q)$, the geodesic distance between p and q. If there is no cut point we set $\mu(X) = \infty$.

Define

$$E_p = \{tX : 0 \leq t < \mu(X),\ X \in S_p\}$$

Then it can be shown that $\exp_p : E_p \to \exp_p(E_p)$ is a diffeomophism. Also

$$M = \exp_p(E_p) \cup \mathrm{Cut}(p).$$

$\mathrm{Cut}(p)$ has n-dimensional measure zero.

If $\mathrm{Cut}(p) = \emptyset$, it is clear that p is a pole in M. If $\mathrm{Cut}(p) \neq \emptyset$, notice that E_p is a star-shaped domain of T_pM. Hence one can construct a family of smooth star-shaped domains $E_p^\epsilon \subset E_p$ such that $\lim\limits_{\epsilon\to 0} E_p^\epsilon = E_p$ in the sense that $\bigcup\limits_{\epsilon>0} E_p^\epsilon = E_p$. Let $\Omega_\epsilon = \exp_p(E_p^\epsilon)$.

It is important to note that the function $r(x) = d(x,p)$ is smooth on $M \backslash (\mathrm{Cut}(p) \cup \{p\})$ and the function satisfies

$$|\nabla r| = 1 \ \text{ on } \ M\backslash(\mathrm{Cut}(p) \cup \{p\}).$$

Theorem 3.11. *[50] Let M be a complete noncompact Riemannian n-manifold. Then for every $x_0 \in M$, every $u \in C_0^\infty(M\backslash\{x_0\})$, and every $a, b \in \mathbb{R}$, with $a+b \neq 1$, the following inequalities hold:*

(i) For $p \geq 2$,

$$\frac{1}{p}\int_M \frac{a+b-r\Delta r}{r^{a+b+1}}\,|u|^p\,dv \leq \left(\int_M \frac{|u|^p}{r^{aq}} dv\right)^{\frac{1}{q}} \left(\int_M \frac{|\nabla u|^p}{r^{bp}} dv\right)^{\frac{1}{p}}. \tag{3.49}$$

(ii) For $1 < p < 2$,

$$\frac{1}{p}\int_M \frac{a+b-r\Delta r}{r^{a+b+1}}(|u|^2+\delta)^{\frac{p}{2}} dv \leq \left(\int_M \frac{|u|^p}{r^{aq}} dv\right)^{\frac{1}{q}} \left(\int_M \frac{|\nabla u|^p}{r^{bp}} dv\right)^{\frac{1}{p}} \tag{3.50}$$

where $\delta > 0$, dv is the volume element of M, r is the distance to x_0, and p, $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. In particular, if $Ric^M \geq 0$ and $a + b + 1 \geq n$,

$$\frac{(a+b+1)-n}{p}\int\limits_M \frac{|u|^p}{r^{a+b+1}}\,|u|^p\,dv \leq \left(\int\limits_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}} \left(\int\limits_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}}. \qquad (3.51)$$

Proof: Given any fixed point x_0 in M, let $\mathrm{cut}(x_0)$ be the cut locus of x_0. If $\mathrm{cut}(x_0) \neq \emptyset$, let $\Omega_\epsilon = \exp_{x_0}(E^\epsilon_{x_0})$ and $\Omega = \exp_{x_0}(E_{x_0})$. Then $\lim\limits_{\epsilon \to 0}\Omega_\epsilon = \Omega$, and for $\forall x \in \Omega_\epsilon \backslash \{x_0\}$, there exists a unique normal geodesic linking x to x_0. Thus, ∇r is well defined in $\Omega_\epsilon \backslash \{x_0\}$, and $|\nabla r| = 1$ a.e. in Ω_ϵ.

For $p \geq 2$, for every $u \in C_0^\infty(M \backslash \{x_0\})$, consider $II := p\int\limits_{\Omega_\epsilon} \left\langle |u|^{p-2}u\frac{\nabla r}{r^{a+b}}, \nabla u\right\rangle dv$. Then it follows from the Green's formula that

$$\begin{aligned} II &= \frac{1}{1-(a+b)}\int\limits_{\Omega_\epsilon} \left\langle \nabla |u|^p,\, \nabla r^{1-(a+b)}\right\rangle dv \\ &= -\frac{1}{1-(a+b)}\Big(\int\limits_{\Omega_\epsilon} |u|^p \Delta r^{1-(a+b)} dv - \int\limits_{\partial\Omega_\epsilon} |u|^p \frac{\partial r^{1-(a+b)}}{\partial \nu} dS\Big), \end{aligned}$$

where ν is the outward unit normal vector of $\partial\Omega_\epsilon$.

Since $a + b \neq 1$ which implies $1 - (a + b) \neq 0$, then $\frac{1}{1-(a+b)} \frac{\partial r^{1-(a+b)}}{\partial \nu} > 0$ on $\partial\Omega_\epsilon$. One obtains

$$\begin{aligned} & \tfrac{1}{1-(a+b)}\int\limits_{\Omega_\epsilon} \left\langle \nabla |u|^p,\, \nabla r^{1-(a+b)}\right\rangle dv \\ \geq\ & -\tfrac{1}{1-(a+b)}\int\limits_{\Omega_\epsilon} |u|^p \Delta r^{1-(a+b)} dv \\ =\ & -\tfrac{1}{1-(a+b)}\int\limits_{\Omega_\epsilon} |u|^p \left(\tfrac{1-(a+b)}{r^{a+b}}\Delta r + (1-(a+b))(-(a+b))\tfrac{|\nabla r|^2}{r^{a+b+1}}\right) dv \\ =\ & \int\limits_{\Omega_\epsilon} |u|^p \tfrac{a+b-r\Delta r}{r^{a+b+1}}\, dv. \end{aligned} \qquad (3.52)$$

On the other hand, Hölder inequality shows

$$\begin{aligned} |II| &\leq p\left(\int\limits_{\Omega_\epsilon}\left|\frac{|u|^{p-2}u\nabla r}{r^a}\right|^q dv\right)^{\frac{1}{q}}\left(\int\limits_{\Omega_\epsilon}\left|\frac{\nabla u}{r^b}\right|^p dv\right)^{\frac{1}{p}} \\ &= p\Bigg(\int\limits_{\Omega_\epsilon}\frac{|u|^p}{r^{aq}}dv\Bigg)^{\frac{1}{q}}\Bigg(\int\limits_{\Omega_\epsilon}\frac{|\nabla u|^p}{r^{bp}}dv\Bigg)^{\frac{1}{p}}. \end{aligned} \tag{3.53}$$

Combine (3.52) and (3.53), one obtains

$$\int\limits_{\Omega_\epsilon}|u|^p\frac{a+b-r\Delta r}{r^{a+b+1}}\,dv \leq \Bigg(\int\limits_{\Omega_\epsilon}\frac{|u|^p}{r^{aq}}dv\Bigg)^{\frac{1}{q}}\Bigg(\int\limits_{\Omega_\epsilon}\frac{|\nabla u|^p}{r^{bp}}dv\Bigg)^{\frac{1}{p}}. \tag{3.54}$$

Since $\mathrm{cut}(x_0)$ is a measure zero set and $u \in C_0^\infty(M\setminus\{x_0\})$, then let $\epsilon \to 0$, the desired (3.49) follows.

If $\mathrm{cut}(x_0) = \emptyset$, then consider $\bar{II} := p\int\limits_M \left\langle |u|^{p-2}u\frac{\nabla r}{r^{a+b}}, \nabla u\right\rangle dv\,.$

By the divergence theorem, and $|\nabla r| = 1$ a.e., one has

$$\begin{aligned} \bar{II} &= \tfrac{1}{1-(a+b)}\int\limits_M \left\langle \nabla|u|^p,\, \nabla r^{1-(a+b)}\right\rangle dv \\ &= \tfrac{1}{1-(a+b)}\int\limits_M \left(\mathrm{div}(|u|^p\nabla r^{1-(a+b)}) - |u|^p\mathrm{div}(\nabla r^{1-(a+b)})\right)\,dv \\ &= -\tfrac{1}{1-(a+b)}\int\limits_M |u|^p\mathrm{div}(\nabla r^{1-(a+b)})\,dv \\ &= -\tfrac{1}{1-(a+b)}\int\limits_M |u|^p\left(\tfrac{1-(a+b)}{r^{a+b}}\Delta r + (1-(a+b))(-(a+b))\tfrac{|\nabla r|^2}{r^{a+b+1}}\right)dv \\ &= \int\limits_M |u|^p\tfrac{a+b-r\Delta r}{r^{a+b+1}}dv \end{aligned} \tag{3.55}$$

Similarly, Hölder's inequality shows

$$\begin{aligned} |\bar{II}| &\leq p\left(\int\limits_M\left|\frac{|u|^{p-2}u\nabla r}{r^a}\right|^q dv\right)^{\frac{1}{q}}\left(\int\limits_M\left|\frac{\nabla u}{r^b}\right|^p dv\right)^{\frac{1}{p}} \\ &= p\Bigg(\int\limits_M\frac{|u|^p}{r^{aq}}dv\Bigg)^{\frac{1}{q}}\Bigg(\int\limits_M\frac{|\nabla u|^p}{r^{bp}}dv\Bigg)^{\frac{1}{p}} \end{aligned} \tag{3.56}$$

Combine (3.55) and (3.56), one obtains the desired (3.49).

For the case $1 < p < 2$, if $\mathrm{cut}(x_0) \neq \emptyset$, in case that $u \equiv 0$ on a subset of Ω_ϵ, we consider $II_1 := p\int\limits_{\Omega_\epsilon} \left\langle (|u|^2+\delta)^{\frac{p-2}{2}} u \frac{\nabla r}{r^{a+b}}, \nabla u \right\rangle dv$, where $\delta > 0$.

Then it follows from the Green's formula that

$$
\begin{aligned}
II_1 &= \frac{1}{1-(a+b)} \int\limits_{\Omega_\epsilon} \left\langle \nabla(|u|^2+\delta)^{\frac{p}{2}},\, \nabla r^{1-(a+b)} \right\rangle dv \\
&= -\frac{1}{1-(a+b)} \Big(\int\limits_{\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \Delta r^{1-(a+b)} dv - \int\limits_{\partial\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \frac{\partial r^{1-(a+b)}}{\partial \nu} dS \Big),
\end{aligned}
$$

Similarly, one obtains

$$
\begin{aligned}
&\frac{1}{1-(a+b)} \int\limits_{\Omega_\epsilon} \left\langle \nabla(|u|^2+\delta)^{\frac{p}{2}},\, \nabla r^{1-(a+b)} \right\rangle dv \\
\geq\ & -\frac{1}{1-(a+b)} \int\limits_{\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \Delta r^{1-(a+b)} dv \\
=\ & -\frac{1}{1-(a+b)} \int\limits_{\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \left(\frac{1-(a+b)}{r^{a+b}} \Delta r + (1-(a+b))(-(a+b)) \frac{|\nabla r|^2}{r^{a+b+1}} \right) dv \\
=\ & \int\limits_{\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \frac{a+b-r\Delta r}{r^{a+b+1}} dv
\end{aligned} \tag{3.57}
$$

And since $1 < p < 2$,

$$
\begin{aligned}
|II| &\leq p \left(\int\limits_{\Omega_\epsilon} \left| \frac{(|u|^2+\delta)^{\frac{p-2}{2}} u \nabla r}{r^a} \right|^q dv \right)^{\frac{1}{q}} \left(\int\limits_{\Omega_\epsilon} \left| \frac{\nabla u}{r^b} \right|^p dv \right)^{\frac{1}{p}} \\
&= p \Big(\int\limits_{\Omega_\epsilon} \frac{(|u|^2+\delta)^{\frac{(p-2)q}{2}} |u|^q}{r^{aq}} dv \Big)^{\frac{1}{q}} \Big(\int\limits_{\Omega_\epsilon} \frac{|\nabla u|^p}{r^{bp}} dv \Big)^{\frac{1}{p}} \\
&\leq p \Big(\int\limits_{\Omega_\epsilon} \frac{|u|^p}{r^{aq}} dv \Big)^{\frac{1}{q}} \Big(\int\limits_{\Omega_\epsilon} \frac{|\nabla u|^p}{r^{bp}} dv \Big)^{\frac{1}{p}}
\end{aligned} \tag{3.58}
$$

Combine (3.57) and (3.58), one obtains

$$
\int\limits_{\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \frac{a+b-r\Delta r}{r^{a+b+1}} dv \leq \Big(\int\limits_{\Omega_\epsilon} \frac{|u|^p}{r^{aq}} dv \Big)^{\frac{1}{q}} \Big(\int\limits_{\Omega_\epsilon} \frac{|\nabla u|^p}{r^{bp}} dv \Big)^{\frac{1}{p}} \tag{3.59}
$$

Let $\epsilon \to 0$, the desired (3.50) follows.

If $\mathrm{cut}(x_0)=\emptyset$, consider $I\bar{I}_1 := p\int\limits_M \left\langle (|u|^2+\delta)^{\frac{p-2}{2}} u\frac{\partial}{r^{a+b}}, \nabla u\right\rangle dv$.

By the divergence theorem, and $|\nabla r|=1$ a.e., one has

$$\begin{aligned}
I\bar{I}_1 &= \tfrac{1}{1-(a+b)}\int\limits_M \left\langle \nabla(|u|^2+\delta)^{\frac{p}{2}}, \nabla r^{1-(a+b)}\right\rangle dv \\
&= \tfrac{1}{1-(a+b)}\int\limits_M \left(\mathrm{div}((|u|^2+\delta)^{\frac{p}{2}}\nabla r^{1-(a+b)}) - (|u|^2+\delta)^{\frac{p}{2}}\mathrm{div}(\nabla r^{1-(a+b)})\right) dv \\
&= -\tfrac{1}{1-(a+b)}\int\limits_M (|u|^2+\delta)^{\frac{p}{2}}\mathrm{div}(\nabla r^{1-(a+b)})\, dv \\
&= \int\limits_{\Omega_\epsilon} (|u|^2+\delta)^{\frac{p}{2}} \tfrac{a+b-r\Delta r}{r^{a+b+1}} dv
\end{aligned} \tag{3.60}$$

Hölder's inequality and the assumption $1<p<2$ show that

$$\begin{aligned}
|I\bar{I}_1| &\le p\left(\int\limits_M \left|\tfrac{(|u|^2+\delta)^{\frac{p-2}{2}} u\nabla r}{r^a}\right|^q dv\right)^{\frac{1}{q}} \left(\int\limits_M \left|\tfrac{\nabla u}{r^b}\right|^p dv\right)^{\frac{1}{p}} \\
&= p\left(\int\limits_M \tfrac{(|u|^2+\delta)^{\frac{(p-2)q}{2}}|u|^q}{r^{aq}} dv\right)^{\frac{1}{q}} \left(\int\limits_M \tfrac{|\nabla u|^p}{r^{bp}} dv\right)^{\frac{1}{p}} \\
&\le p\left(\int\limits_M \tfrac{|u|^p}{r^{aq}} dv\right)^{\frac{1}{q}} \left(\int\limits_M \tfrac{|\nabla u|^p}{r^{bp}} dv\right)^{\frac{1}{p}}
\end{aligned} \tag{3.61}$$

Combine (3.60) and (3.61), one obtains the desired (3.50).

In particular, if $\mathrm{Ric}^M \ge 0$ then by the Laplacian comparison theorem $r\Delta r \le n-1$. If $a+b+1\ge n$, then $a+b-r\Delta r \ge a+b+1-n \ge 0$. Hence we obtain

$$\frac{1}{p}\int\limits_M |u|^p \frac{a+b-r\Delta r}{r^{a+b+1}} dv \ge \frac{a+b+1-n}{p}\int\limits_M \frac{|u|^p}{r^{a+b+1}} dv$$

and

$$\begin{aligned}
\frac{1}{p}\int\limits_M (|u|^2+\delta)^{\frac{p}{2}} \frac{a+b-r\Delta r}{r^{a+b+1}} dv &\ge \frac{a+b+1-n}{p}\int\limits_M \frac{(|u|^2+\delta)^{\frac{p}{2}}}{r^{a+b+1}} dv \\
&\ge \frac{a+b+1-n}{p}\int\limits_M \frac{|u|^p}{r^{a+b+1}} dv
\end{aligned}$$

Combine the above inequalities and (3.49)-(3.50), we obtain the desired (3.51).

$\square$

Theorem 3.12. *[50] Let M be an n-dimensional Cartan-Hadamard manifold. Then for every $x_0 \in M$, every $u \in C_0^\infty(M\setminus\{x_0\})$, and every $a, b \in \mathbb{R}$, with $a+b+1 \le n$, the following inequality holds:*

$$\frac{n-(a+b+1)}{p}\int_M \frac{|u|^p}{r^{a+b+1}}dv \le \left(\int_M \frac{|u|^p}{r^{aq}}dv\right)^{\frac{1}{q}} \left(\int_M \frac{|\nabla u|^p}{r^{bp}}dv\right)^{\frac{1}{p}} \tag{3.62}$$

where dv is the volume element of M, r is the distance to x_0, and p, q satisfy $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Since M is a Cartan-Hadamard manifold, then every point in M is a pole. Thus, given a fixed point $x_0 \in M$, ∇r is well defined in $M\setminus\{x_0\}$. By the Hessian comparison theorem, $r\Delta r \ge n-1$.

If $p \ge 2$, from (3.55), one obtains

$$\begin{aligned} -\bar{I}I &= \tfrac{1}{1-(a+b)}\int_M |u|^p \left(\tfrac{1-(a+b)}{r^{a+b}}\Delta r + (1-(a+b))(-(a+b))\tfrac{|\nabla r|^2}{r^{a+b+1}}\right)dv \\ &\ge \tfrac{1}{1-(a+b)}\int_M |u|^p \tfrac{(1-(a+b))(n-(a+b+1))}{r^{a+b+1}}dv \\ &= (n-(a+b+1))\int_M \tfrac{|u|^p}{r^{a+b+1}}dv \end{aligned} \tag{3.63}$$

Combine (3.63) and (3.56), one obtains the desired (3.62).

If $1 < p < 2$, from (3.60), one obtains

$$\begin{aligned}
-\bar{II}_1 &= \tfrac{1}{1-(a+b)} \int\limits_M (|u|^2+\delta)^{\frac{p}{2}} \left(\tfrac{1-(a+b)}{r^{a+b}} \Delta r + (1-(a+b))(-(a+b)) \tfrac{|\nabla r|^2}{r^{a+b+1}} \right) dv \\
&\geq \tfrac{1}{1-(a+b)} \int\limits_M (|u|^2+\delta)^{\frac{p}{2}} \tfrac{(1-(a+b))(n-(a+b+1))}{r^{a+b+1}} dv \\
&= (n-(a+b+1)) \int\limits_M \tfrac{(|u|^2+\delta)^{\frac{p}{2}}}{r^{a+b+1}} dv \\
&\geq (n-(a+b+1)) \int\limits_M \tfrac{|u|^p}{r^{a+b+1}} dv
\end{aligned} \tag{3.64}$$

Combine (3.64) and (3.61), one obtains the desired (3.62).

□

Using the same technique and the Hessian comparison theorem (Theorem 2.3), we obtain the following:

Theorem 3.13. *Let M be a complete n-dimensional manifold with a pole of radial curvature $0 \leq K \leq \frac{c(1-c)}{r^2}$, where $c \in [0, 1]$. Then for every $u \in C_0^\infty(M)$ and every $a, b \in \mathbb{R}$ with $c(n-1)-(a+b) \geq 0$, the following inequality holds:*

$$\frac{cn-(a+b+c)}{p} \int\limits_M \frac{|u|^p}{r^{a+b+1}} dv \leq \left(\int\limits_M \frac{|u|^p}{r^{aq}} dv \right)^{\frac{1}{q}} \left(\int\limits_M \frac{|\nabla u|^p}{r^{bp}} dv \right)^{\frac{1}{p}} \tag{3.65}$$

where dv is the volume element of M, r is the distance to x_0, and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

In the Euclidean spaces $\mathbb{R}^n$, Costa gave a short proof of (3.62) for the case $p = 2$ in [9] using divergence theorem and completing the square technique. Later, Catrina and Costa (cf. [8]) showed the constants are sharp when $p = 2$ and they found the functions that achieve them. However, for $p \neq 2$, the sharpness of the constants is still unknown.

Chapter 4

Application to p-harmonic Geometry

We use Hardy type inequalities and techniques and results of S.-C. Chang, J.-T. Chen and S.W. Wei (cf. [10]), to study Liouville theorems of p-harmonic functions, p-harmonic morphisms, and weakly conformal maps, with assumption only on curvature and q-energy growth. As further applications we obtain Picard type theorems in p-harmonic geometry.

4.1 Preliminaries

First of all, let us recall some related basic facts, notations, definitions, and formulas.

Definition 4.1. A C^2 function $u : M \to \mathbb{R}$ is said to be p-harmonic (resp. p-superharmonic, and p-subharmonic) in a storng sense if its p-Laplacian $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (resp. ≤ 0 , and ≥ 0). A function $u : M \to \mathbb{R}$ is said to be p-harmonic (resp. p-superharmonic, and p-subharmonic) in a weak sense if its p-Laplacian $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (resp. ≤ 0 , and ≥ 0) in the sense of distributions.

Definition 4.2. Let (M, g) be an n-dimensional Riemannian manifold with local orthonormal frame $\{e_i\}_{i=1}^n$, where $n \geq 2$. The q-energy functional E_q , $q > 1$ of smooth map $u : M \to \mathbb{R}$ is given by

$$E_q\left(u\right) = \frac{1}{q}\int_M |du|^p \, dv$$

where $|du| = \sum_{i=1}^{m} \langle du\,(e_i)\,, du\,(e_i)\rangle$ is the Hilbert-Schmidt norm of the differential du of u , and dv is the volume element of M .

Definition 4.3. A map $u : M \to N$ is said to be horizontally weakly conformal if for any $x \in M$ such that the differential $du_x \neq 0$, the restriction of du_x to the orthogonal complement of the Kernel of du_x is conformal and surjective.

Definition 4.4. Let M, N be differentiable manifolds. A differentiable mapping $\phi : M \to N$ is said to be an immersion if $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective for all $p \in M$. if, in addition, ϕ ia a homeomorphism onto $\phi(M) \subset N$, where $\phi(M)$ has the subspace topology induced from N, we say that ϕ is an embedding. If $M \subset N$ and the inclusion $i : M \to N$ is an embedding, we say that M is a submanifold of N.

Definition 4.5. An immersion f of an m-dimensional manifold M with boundary ∂M (possibly empty) into a Riemannian manifold N is called minimal if the mean curvature vector field H of M with respect to the induced Riemannian metric vanishes identically. Then M is called a minimal submanifold of N.

Definition 4.6. A minimal submanifold M is called stable if for every compact region on M all the second variations of the volume are positive.

Definition 4.7. If M and N are differentiable manifolds. $dimN - dimM = 1$, and if an immersion $f : M \to N$ has been defined, then $f(M)$ is a hypersurface in M.

Definition 4.8. A C^2 map $u : M \to N$ is called a p-harmonic morphism if for any p-harmonic function f defined on an open set V of N, the composition $f \circ u$ is p-harmonic on $u^{-1}(V)$.

In [33], Roger Moser introduced the following linearized operator $\mathcal{L}$:

$$\mathcal{L}(\Psi) = \operatorname{div}\left(f^{p-2} A(\nabla \Psi)\right)$$

where

$$A := \mathrm{id} + (p-2)\frac{\nabla u \otimes \nabla u}{f^2}, \quad \text{and } f = |\nabla u|.$$

In [10], Chang-Chen-Wei introduced an operator $\mathcal{L}_{s,\varepsilon}$ by

$$\mathcal{L}_{s,\varepsilon}(\Psi) = \operatorname{div}\left(f_\varepsilon^s A_\varepsilon(\nabla \Psi)\right),$$

for $\Psi \in C^2(M)$, where $s \in \mathbb{R}$, $p > 1$, $\varepsilon > 0$, $f_\varepsilon = \sqrt{f^2+\varepsilon}$ and

$$A_\varepsilon := \mathrm{id} + (p-2)\tfrac{\nabla u \otimes \nabla u}{f_\varepsilon^2}.$$

$\mathcal{L}_{s,\varepsilon}$ is a linearized operator of the nonlinear p-harmonic equation, and $\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2)(x)$ is well define for all $x \in M$ since $f_\varepsilon > 0$ and $f_\varepsilon^2 \in C^2(M)$.

They further derive

Theorem 4.9 (a generalized Bochner formula for a p-harmonic function, $p > 1$)**.** *[10] Let $u \in C^3(M)$ be a p-harmonic function, $f = |\nabla u|$ and $f_\varepsilon = \sqrt{f^2+\varepsilon}$. Then for any $s \in \mathbb{R}$, and $\varepsilon > 0$, the following formula*

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2) &= \tfrac{s}{4} f_\varepsilon^{s-2}\left|\nabla f_\varepsilon^2\right|^2 + f_\varepsilon^s \textstyle\sum_{i,j=1}^n \left(u_{ij}^2 + R_{ij}u_i u_j\right) \\ &\quad + \tfrac{(p-2)(s-p+2)}{4} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle^2 \\ &\quad + \varepsilon\left(f_\varepsilon^{s-2}\langle \nabla u, \nabla \Delta u\rangle + \tfrac{p-4}{2} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u\right) \end{aligned} \tag{4.1}$$

holds at every point in M, where u_{ij} is the Hessian of u, and R_{ij} is the Ricci *curvature tensor of M . In particular, if $p = 2$, then*

$$\frac{1}{2}\mathcal{L}_{s,\varepsilon}\left(f_\varepsilon^2\right) = \frac{s}{4} f_\varepsilon^{s-2} \left|\nabla f_\varepsilon^2\right|^2 + f_\varepsilon^s \sum_{i,j=1}^n \left(u_{ij}^2 + R_{ij} u_i u_j\right) \tag{4.2}$$

holds on all of M and for all $s \in \mathbb{R}$.

and derive

Theorem 4.10 (a sharp Kato's inequality for a p-harmonic function, $p > 1$)**.** *[10] Let $u \in C^2(M)$ be a p-harmonic function on a complete manifold M^n, $p > 1$ and $\kappa = \min\left\{\frac{(p-1)^2}{n-1}, 1\right\}$. Then at any $x \in M$ with $du(x) \neq 0$,*

$$|\nabla (du)|^2 \geq (1 + \kappa) |\nabla |du||^2, \tag{4.3}$$

and "=" holds if and only if

$$\begin{cases} u_{\alpha\beta} = 0 \text{ and } u_{11} = -\frac{n-1}{p-1} u_{\alpha\alpha}, & \text{for } (p-1)^2 = n-1, \\ u_{\alpha\beta} = 0, \ u_{1\alpha} = 0 \text{ and } u_{11} = -\frac{n-1}{p-1} u_{\alpha\alpha}, & \text{for } (p-1)^2 < n-1, \\ u_{\alpha\beta} = 0 \text{ and } u_{ii} = 0, & \text{for } (p-1)^2 > n-1, \end{cases}$$

for all $\alpha, \beta = 2, \ldots, n$, $\alpha \neq \beta$ and $i = 1, \ldots, n$.

4.2 Liouville Theorem for p-harmonic functions on manifolds

Using a generalized Bochner formula and sharp Kato's inequality, S.-C. Chang, J.-T. Chen and S.W. Wei prove the following Liouville type

Theorem C (Liouville Theorem for p-harmonic functions, $p > 1$)**.** *[10] Let M be a complete noncompact Riemannian n-manifold that supports a weighted*

Poincaré inequality

$$\int_M \rho(x)\,\Psi^2(x)\,dv \le \int_M |\nabla\Psi(x)|^2\,dv. \tag{4.4}$$

for every smooth function Ψ with compact support on M, where $\rho(x)$ is a positive function a.e.. Let Ricci curvature $Ric^M \ge -\tau\rho$, where τ is a constant satisfying

$$\tau < \frac{4(q-1+\kappa+b)}{q^2}, \text{ in which } \kappa = \min\{\frac{(p-1)^2}{n-1}, 1\} \text{ and } b = \min\{0, (p-2)(q-p)\}. \tag{4.5}$$

Let $u \in C^3(M)$ be a p-harmonic function in a weak sense for $p \in \{2\} \cup [4,\infty)$, and in a strong sense for $p \in (1,2) \cup (2,4)$, with finite q-energy $E_q(u) = \int_M |du|^q dv$, for p and q satisfying one of the following:

(1) $p = 2$ *and* $q > \frac{n-2}{n-1}$,

(2) $p = 4$, $q > 1$ *and* $q - 1 + \kappa + b > 0$,

(3) $p > 2$, $p \ne 4$, *and either* $\max\left\{1, p-1-\frac{\kappa}{p-1}\right\} < q \le p - \frac{(p-4)^2 n}{4(p-2)}$, *or both* $q > 2$ *and* $q - 1 + \kappa + b > 0$.

Then u is constant. If p and q satisfy

(4) $1 < p < 2$ *and* $q > 2$,

then u does not exist.

The following Liouville theorem in p-harmonic geometry follows from the above theorem and Theorem 3.3 in which we choose $p = 2$, M supports a weighted Poincaré inequality with $\rho(x) = \frac{(n-2)^2}{4r(x)^2}$.

Theorem 4.11 (Liouville Theorem for p-harmonic functions)**.** *[11] Let M be a complete noncompact Riemannian n-manifold with non-positive sectional curvature. Suppose that $Ric^M \ge -\tau\frac{(n-2)^2}{4r^2}$ a.e., where τ is as in (4.5). Let*

$u \in C^3(M)$ be a p-harmonic function with finite q energy, for p and q as in Theorem C. Then the same conclusion as in Theorem C holds.

For completeness, we sketch the proof as follows:

Proof: Following [10], giving a fixed point $x_0 \in M$, let $0 \le \eta \le 1$ be a smooth cut-off function satisfying $\eta \equiv 1$ in $\overline{B_R(x_0)}$, $\eta \equiv 0$ off $B_{2R}(x_0)$, and $|\nabla \eta| \le \frac{C}{R}$ in $B_{2R}(x_0)\backslash\overline{B_R(x_0)}$.

For the case $p \ne 2$, in view of the divergence theorem and the Cauchy-Schwarz inequality, one obtains:

$$\frac{1}{2}\int_M \eta^2 \mathcal{L}_{s,\varepsilon}\left(f_\varepsilon^2\right) dv \le \varepsilon_1 \int_M \eta^2 f_\varepsilon^s \left|\nabla f_\varepsilon\right|^2 dv + \frac{(1+|p-2|)^2}{\varepsilon_1}\int_M \left|\nabla \eta\right|^2 f_\varepsilon^{s+2} dv, \tag{4.6}$$

where ε_1 is a positive constant.

On the other hand, combining Theorems 4.9 and 4.10, one obtains

$$\begin{aligned} \frac{1}{2}\mathcal{L}_{s,\varepsilon}\left(f_\varepsilon^2\right) \ \ge\ & (s+1+\kappa) f_\varepsilon^s \left|\nabla f_\varepsilon\right|^2 + f_\varepsilon^s \sum_{i,j=1}^n R_{ij} u_i u_j \\ & + \frac{(p-2)(s-p+2)}{4} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle^2 \\ & + \varepsilon\left(f_\varepsilon^{s-2} u_{ij}^2 + f_\varepsilon^{s-2}\left\langle \nabla u, \nabla \Delta u\right\rangle + \frac{p-4}{2} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u\right), \end{aligned} \tag{4.7}$$

for $p > 1$ and $p \ne 2$.

Let $b = \min\{0, (p-2)(s-p+2)\}$. Then via the Cauchy-Schwarz inequality

$$\begin{aligned} & \frac{(p-2)(s-p+2)}{4}\int_M \eta^2 f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle^2 dv \\ \ge\ & b\int_M \eta^2 f_\varepsilon^s \left|\nabla f_\varepsilon\right|^2 dv - b\varepsilon \int_M \eta^2 f_\varepsilon^{s-2}\left|\nabla f_\varepsilon\right|^2 dv \end{aligned} \tag{4.8}$$

Combining (4.6)-(4.8), one obtains

$$\begin{aligned} & A_1 \int_M \eta^2 f_\varepsilon^s \left|\nabla f_\varepsilon\right|^2 dv + \int_M \eta^2 f_\varepsilon^s \sum_{i,j=1}^n R_{ij} u_i u_j dv + \varepsilon B \\ \le\ & \frac{(1+|p-2|)^2}{\varepsilon_1}\int_M \left|\nabla \eta\right|^2 f_\varepsilon^{s+2} dv, \end{aligned} \tag{4.9}$$

where $A_1 = s + 1 + \kappa + b - \varepsilon_1$ and

$$\begin{aligned} B &= \int_M \eta^2 \Bigg(f_\varepsilon^{s-2} \sum_{i,j=1}^n u_{ij}^2 + f_\varepsilon^{s-2} \langle \nabla u, \nabla \Delta u \rangle + \tfrac{p-4}{2} f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2 \rangle \Delta u \\ & \quad - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \Bigg) dv. \end{aligned}$$

Let $q = s + 2$, then the first term on the left hand side of (4.9) becomes

$$\begin{aligned} & A_1 \int_M \eta^2 f_\varepsilon^s |\nabla f_\varepsilon|^2 dv \\ = & \tfrac{4A_1}{q^2} \int_M \eta^2 \left| \nabla f_\varepsilon^{q/2} \right|^2 dv \\ \geq & \tfrac{4A_1(1-\varepsilon_2)}{q^2} \int_M \left| \nabla \left(\eta f_\varepsilon^{q/2} \right) \right|^2 + \tfrac{4A_1\left(1 - \frac{1}{\varepsilon_2}\right)}{q^2} \int_M |\nabla \eta|^2 f_\varepsilon^q dv. \end{aligned}$$

where ε_2 is a positive constant satisfying $\varepsilon_2 < 1$. Thus, (4.9) implies

$$\begin{aligned} & \tfrac{4(1-\varepsilon_2)A_1}{q^2} \int_M \left| \nabla \left(\eta f_\varepsilon^{q/2} \right) \right|^2 dv + \int_M \eta^2 f_\varepsilon^{q-2} \sum_{i,j=1}^n R_{ij} u_i u_j \, dv + \varepsilon B \\ \leq & \left(\tfrac{(1+|p-2|)^2}{\varepsilon_1} + \tfrac{4\left(\frac{1}{\varepsilon_2} - 1\right) A_1}{q^2} \right) \int_M |\nabla \eta|^2 f_\varepsilon^q dv. \end{aligned} \tag{4.10}$$

By assumption and Theorem 3.3 (in which we select $p = 2$), for every $u \in W_0^{1,2}(M)$,

$$\int_M \frac{|n-2|^2}{4r^2} |u|^2 \, dv \leq \int_M |\nabla u|^2 \, dv. \tag{4.11}$$

To simplify (4.10), we apply the generalized sharp Hardy inequality (4.11) to the first term on the left hand side of (4.10) in which $u = \eta f_\varepsilon^{\frac{q}{2}}$. Then with the assumption $q - 1 + \kappa + b > 0$, one obtains

$$\int_{B_R} A_2 f_\varepsilon^{q-2} dv + \varepsilon B \leq \tfrac{C^2 B_1}{R^2} \int_{B_{2R} \setminus B_R} f_\varepsilon^q dv, \tag{4.12}$$

for all fixed $R > 0$, where

$$A_2 = \tfrac{(1-\varepsilon_2)(q-1+\kappa+b-\varepsilon_1)(n-2)^2}{q^2r^2} f_\varepsilon^2 - \textstyle\sum_{i,j=1}^n R_{ij}u_iu_j$$

and

$$B_1 = \tfrac{(1+|p-2|)^2}{\varepsilon_1} + \tfrac{4\left(\frac{1}{\varepsilon_2}-1\right)(q-1+\kappa+b-\varepsilon_1)}{q^2}.$$

By the Ricci curvature assumption, there exists a constant $0 < \delta < 1$ such that

$$\mathrm{Ric}^M \geq -\frac{(q-1+\kappa+b)(n-2)^2\delta}{q^2r^2}.$$

Since

(i) If $s > 0$, then $\varepsilon B \to 0$ as $\varepsilon \to 0$.

(ii) If $b \leq -\frac{(p-4)^2n}{4}$ and $s > -1$, then $\varepsilon B \geq -\varepsilon \int_M \eta^2 f_\varepsilon^{s-2} f\, |\nabla \Delta u| \to 0$ as $\varepsilon \to 0$.

(iii) In particular, if $p = 4$ and $s > -1$, then $\varepsilon B \geq -\varepsilon \int_M \eta^2 f_\varepsilon^{s-2} f\, |\nabla \Delta u| \to 0$ as $\varepsilon \to 0$.

Then let $\varepsilon \to 0$ in (4.12), and $q = s + 2$, we have

$$\textstyle\int_{B_R} A_3 f^q dv \leq \frac{C^2B_1}{R^2} \int_{B_{2R}\setminus B_R} f^q dv, \tag{4.13}$$

where

$$A_3 = \left(\tfrac{(1-\varepsilon_2)(q-1+\kappa+b-\varepsilon_1)}{q^2} - \tfrac{(q-1+\kappa+b)\delta}{q^2}\right)\tfrac{(n-2)^2}{r^2}.$$

We note $A_3 > 0$ for sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Since $f \in L^q(M)$, the assertion follows by letting $R \to \infty$ in (4.13).

For the case $p = 2$, use the same method as above, the assertion follows.

For the case $1 < p < 2$ and $q > 2$, by letting $R \to \infty$ in (4.13), u is constant. But a constant function is not a p-harmonic function in a strong sense for $1 < p < 2$. The nonexistence result follows.

$\square$

This idea can be extended to a large class of manifolds or submanifolds, such as stable minimal hypersurfaces:

Theorem 4.12. *[11] Let N be a Riemannian $(n+1)$-manifold, M be a stable minimal hypersurface in N, and ν be a unit normal vector to M, such that the length $|A|$ of the second fundamental form of M in N satisfying $|A|^2 + Ric^N(\nu) > 0$ a.e.. Suppose $Ric^M \geq -\tau(|A|^2 + Ric^N(\nu))$ where τ is as in (4.5). Let $u \in C^3(M)$ be a p-harmonic function with finite q-energy, for p and q as in Theorem C. Then the same conclusion as in Theorem C holds.*

Proof: Since $M \subset N$ is a stable minimal hypersurface in N, then for every smooth function Ψ with compact support on M the following inequality holds:

$$\int_M \left(|A|^2 + \mathrm{Ric}^N(\nu)\right) \Psi^2(x)\, dv \leq \int_M |\nabla \Psi(x)|^2\, dv. \tag{4.14}$$

Precede as in the proof of Theorem 4.11, the assertion follows. $\square$

There are examples of stable minimal hypersurfaces M in N that satisfy the conditions in Theorem 4.12. These include a counter-example to Bernstein conjecture, i.e. a nonlinear entire minimal graph in $\mathbb{R}^9$ that was found by Bombieri-de Giorgi-Giusti [2] satisfying the assumption $|A|^2 + \mathrm{Ric}^N(\nu) > 0$ a.e.. For appropriate p and q, such a minimal hypersurface M satisfies the assumption $Ric^M \geq -\tau|A|^2 + \mathrm{Ric}^N(\nu)$, since $0 \geq Ric^M$ and $-|A|^2 = Scal^M$, where $Scal^M$ is the scalar curvature of M (see e.g. [28]).

4.3 Applications to p-harmonic morphisms and Weakly Conformal Maps

Lemma 4.13. *[47]Let M, N and K be manifolds of dimension n, k, and ℓ respectively, and $u : M \to N$, and $w : N \to K$ be C^2. If u is horizontally weak conformal, then $|d(w \circ u)|^{p-2} = (\frac{1}{k})^{\frac{p-2}{2}} |dw|^{p-2} |du|^{p-2}$.*

Theorem 4.14 (Liouville Theorem for p-harmonic morphisms)**.** *[11] Let M be as in Theorem 4.11 or in Theorem 4.12. If $u \in C^3(M)$ is a p-harmonic morphism $u : M \to \mathbb{R}^k$, with finite q-energy, for p and q as in Theorem C. Then the same conclusion as in Theorem C holds.*

Proof: Let $u^i = \pi_i \circ u$, where $\pi_i : \mathbb{R}^k \to \mathbb{R}$ is the i-th projection. Then the linear function π_i is a p-harmonic function (cf. 2.2 in [46]). Hence u^i, a composition of a p-harmonic morphism and a p-harmonic function, is p-harmonic. Since u is horizontally weak conformal, it follows from Lemma 4.13 that $E_p(u) < \infty$ implies $E_p(u^i) < \infty$. Now apply u^i to Theorem C, the assertion follows. □

Our previous result can be applied to weakly conformal maps between equal dimensional manifolds based on the following:

Theorem D. [*34*] *$u : M \to N$ is an n-harmonic morphism, if and only if u is weakly conformal, where $n = \dim M = \dim N$.*

Theorem 4.15 (Liouville Theorem for weakly conformal maps)**.** *[11] Let M be as in Theorem 4.11 or in Theorem 4.12, in which $p = n$ in* (4.5)*. If $u : M \to \mathbb{R}^n$ is a weakly conformal map with finite q-energy, for n and q satisfying one of the following:*

(1) *$n = 2$ and $q > 0$,*

(2) *$n = 4$, $q > 1$ and $q + b > 0$,*

(3) $n > 2$, $n \neq 4$, *and either* $\frac{n(n-2)}{n-1} < q \leq n - \frac{(n-4)^2 n}{4(n-2)}$, *or both* $q > 2$ *and* $q + b > 0$,

then u *is a constant.*

Proof: By Theorem D, u is an n-harmonic morphism. Now the result follows immediately from Theorem 4.14 in which $p = n$. □

4.4 Further Applications: Picard Theorems

Theorem 4.16 (Picard Theorem for p-harmonic morphisms)**.** *[11] Let* M *be as in Theorem 4.11 or Theorem 4.12. Suppose that* $u \in C^3(M)$ *is a* p*-harmonic morphism* $u : M \to \mathbb{R}^k \backslash \{y_0\}$*, and the function* $x \mapsto |u(x) - y_0|^{\frac{p-n}{p-1}}$ *has finite* q*-energy where* $p \neq n$*, for* p *and* q *satisfying one of the following:* (1), (2), *and* (3) *as in Theorem C. Then* u *is constant. For* p *and* q *satisfying* (4) *as in Theorem C, then* u *does not exist.*

Proof: Since $x \mapsto |x|^{\frac{p-n}{n-1}}$ is a p-harmonic function on $\mathbb{R}^n$, and $|u(x) - y_0|^{\frac{p-n}{n-1}} : M \to \mathbb{R}$ is a p-harmonic function with finite q-energy. By Theorem 4.11 or Theorem 4.12, when $p \neq n$, $|u(x) - y_0|^{\frac{p-n}{n-1}}$ is constant. This implies that on M , rank $du < n$. Since a p-harmonic morphism is a horizontally weakly conformal map, u is constant. □

Theorem 4.17 (Picard Theorem for weakly conformal maps)**.** *[11] Let* M *be as in Theorem 4.11 or in Theorem 4.12, in which* $p = n$ *in* (4.5)*. Suppose that* $u : M \to \mathbb{R}^n \backslash \{y_0\}$ *is a weakly conformal map and the function* $x \mapsto \log |u(x) - y_0|$ *has finite* q*-energy, for* n *and* q *satisfying one of the following:* (1), (2), *and* (3) *as in Theorem 4.15. Then* u *is constant.*

Proof: Since $x \mapsto \log |x|$ is an n-harmonic function, and $\log |u(x) - y_0| : M \to \mathbb{R}$ is an n-harmonic function with finite q-energy. Now the result follows from

Theorem 4.11 or Theorem 4.12, when $p = n$, and u is a a weakly conformal map. $\square$

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