SOME NONPARAMETRIC STATISTICAL TESTS

By

LOY ELBERT PUFFINBARGER

Bachelor of Science
Northwestern State College
Alva, Oklahoma
1959

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1964

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
May, 1974

SOME NONPARAMETRIC STATISTICAL TESTS

Thesis Approved:

P. L. Claypool
Thesis Adviser

E. K. McFadden

Robert L Brown

D B Aichele

D D Durham
Dean of the Graduate College

ACKNOWLEDGMENTS

To Professor P. L. Claypool, my thesis adviser, my gratitude for the many considerations, time and patience extended to me during the preparation of this thesis.

I also wish to thank Professor E. K. McLachlan for both serving as chairman of my advisory committee and for guidance throughout my years of graduate study.

I also wish to express my thanks to the other members of my advisory committee: Professor R. S. Brown and Professor Douglas B. Aichele.

Additionally, I am deeply aware of and grateful for the many sacrifices which my wife, Anne, and son, William Ray, have made in order that I might have the opportunity to prepare this dissertation.

Finally, I wish to express my appreciation to Mary Bonner for the typing of this manuscript.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

INTRODUCTION

This dissertation deals with some nonparametric statistical tests which broaden the scope of applications of nonparametric methods to include some of the experimental designs which are usually presented in elementary statistical textbooks dealing with parametric methods of analysis but are not presented in elementary statistical textbooks which deal with nonparametric methods of analysis. In addition, multivariate nonparametric methods are presented for some of these design structures.

Most of the theory for nonparametric tests has been developed since 1940, with a tremendous increase in the speed of development since 1950. These tests were viewed with skepticism for many years since they appear to use only a part of the data (usually the order relation between observations). However, the relative efficiencies of these tests, relative to the standard parametric tests, have proven to be quite satisfactory. In many cases the nonparametric tests are superior even when the (often questioned) assumption of normality is met.

Until the latter part of the last decade most of the nonparametric tests proposed in the literature were applicable to only the most elementary design structures. Within the past five years a number of tests have appeared which broaden this scope of application. Due to the newness of these tests as well as the level of mathematical sophistication required to develop them, they have not yet appeared in a form which is accessible to most applied researchers.

The purposes of this dissertation are to present some of these tests in a format analogous to that used by Conover [8] which is more accessible to applied research workers and to present the theory for some of these tests at a level such that individuals with a minimal background in mathematics and statistics may understand the general nature of the tests. Persons interested in the complete theory of these tests will find that most theories are developed and that complete references are given by Puri and Sen [17].

Several nonparametric tests are available for most of the situations discussed in this dissertation. The tests illustrated are those based on the function of the ranks $a(r_i) = \frac{r_i}{n+1}$, (in most cases) which is a Wilcoxon type test. This choice was made because of the ease of computing the test statistic and because most people are familiar with the Wilcoxon statistic. Several examples are included to illustrate the use of the tests presented and, for each example the critical level $\hat{\alpha}$ of the test statistic is reported. The definition of critical level as given by Conover [8, p. 81] is "The critical level $\hat{\alpha}$ is the smallest significance level at which the null hypothesis would be rejected for the given observations." Therefore, if $\hat{\alpha} \leq \alpha$, H_0 is rejected at the α level of significance. The terms "observed significance level" and "associated probability" are used by various subject matter areas instead of critical level. In the tests illustrated, midranks have been assigned as ranks in examples where ties have occurred in the data.

In most examples the exact distribution of the test statistic is laborious to tabulate; however, all the test statistics presented in this dissertation (except the univariate Wilcoxon Signed Rank statistic) are asymptotically chi-square random variables. From the examples illustrated by this dissertation and the various authors referenced in this dissertation, it appears that the limiting distribution is a "satisfactory" approximation of the distribution of the test statistic, in most cases, even when the sample size is quite small. For this reason, the decision rule will be given in terms of the chi-square random variable with appropriate degrees of freedom.

To be specific, Chapter II presents a bivariate sign test and the "basic permutation principle" which is used in many multivariate nonparametric statistical tests. Chapter II serves as a background for the tests presented throughout the remainder of the dissertation. Chapters III, IV and V present tests which appear to be the basic tests for extending the use of nonparametric statistics to more complicated designs and Chapter VI discusses the use of nonparametric statistical tests for interaction in a factorial experiment with a randomized complete block design of the experimental units.

CHAPTER II

DISTRIBUTION-FREE MULTIVARIATE RANK STATISTICS

Nonparametric univariate rank tests are usually based on some function G of the ranks assigned to the data where the distribution of G does not depend on the distribution function of the sampled population when the null hypothesis is true. In considering multivariate data, each observation in the univariate case is replaced by a vector; however, when each rank going into G is replaced by the corresponding vector of ranks it occurs that the distribution of G now depends on the unknown distribution function of the sampled population even when the null hypothesis is true. It is usually the case that distribution-free multivariate rank tests may be developed by considering some type of conditional distribution involving the multivariate function G or a similar function [17]. To see how this may be accomplished consider, a one sample bivariate sign test and a general procedure which can be used to develop some multivariate permutation tests which are based on the ranks assigned to the data.

The Bivariate Sign Test

Let us first consider the bivariate sign test. This test is applicable to a bivariate randomized block structure; that is, observations on $p = 2$ variables under $t = 2$ treatment conditions within each of b blocks. Let Z_{ij} denote the bivariate observation in block i (= 1, 2, ... b) receiving treatment j (= 1, 2) having elements Z_{ij1} and Z_{ij2}; that is, $Z_{ij} = (Z_{ij1}, Z_{ij2})$. Then for each block define the bivariate vector of differences X_i as $X_i = Z_{i1} - Z_{i2} = (X_{i1}, X_{i2})$ where the element $X_{is} = Z_{i1s} - Z_{i2s}$ for $s = 1, 2$. The vectors Z_{ij} and X_i are defined here as row vectors instead of column vectors simply to facilitate the geometric argument which follows. Suppose the b vectors of differences are stochastically independent and the vector X_i has a continuous distribution function $F_i(X)$ for $i = 1, 2, \ldots b$ where X is an element of the two-dimensional Euclidean vector space R^2. We wish to test the hypothesis that the marginal medians of X are both zero, so we have $H_0: F_i(0, \infty) = F_i(\infty, 0) = \frac{1}{2}$ for all $i = 1, 2, \ldots, b$, where the F_i are otherwise arbitrary. We are then testing the hypothesis that the two treatments have the same location parameters within variables.

Define the quadrants of the plane R^2 as usual denoting them as Q_1, Q_2, Q_3 and Q_4. Let $\alpha_i = P[X_i \in Q_1 \cup Q_3]$ and assume $0 < \alpha_i < 1$. If we let $\beta_i = P[X_i \in Q_3 | X_i \in Q_1 \cup Q_3]$ and $\gamma_i = P[X_i \in Q_2 | X_i \in Q_2 \cup Q_4]$, then H_0 may be written as $H_0: \beta_i = \gamma_i = \frac{1}{2}$ for each i. This is true since we are saying that each component of the vector X_i is as likely to be positive as to be negative. From the sample X_i with $i = 1, 2, \ldots, b$, let Y_q be the number of X's in quadrant q for $q = 1, 2, 3, 4$. Note that $0 \leq Y_q \leq b$ for each q and $\sum_{q=1}^{4} Y_q = b$. When $H_0: \beta_i = \gamma_i = \frac{1}{2}$ is true, Y_1 and Y_3 are identically distributed and Y_1 and Y_3 have different distributions if H_0 is false. The same statement is true for Y_2 and Y_4. The statement that Y_1 and Y_3 are identically distributed follows from the definition of β_i and the null hypothesis, since we are saying that an X has the same probability of being in Q_3 and Q_1 given it is in $Q_1 \cup Q_3$. A similar situation occurs with Y_2 and Y_4. This suggests basing a test on $Y_1 - Y_3$ and $Y_2 - Y_4$; however the joint distribution of (Y_1, Y_2, Y_3, Y_4) depends on the unknown values of $F_i(0,0)$ for $i = 1, 2, \ldots, b$. This problem can be resolved by considering the conditional distribution of (Y_1, Y_2, Y_3, Y_4) given N the number of X_i's in $Q_1 \cup Q_3$ and, hence, $b - N$ the number in $Q_2 \cup Q_4$. In order to have notation to use in considering the problem let $n(0 \leq n \leq b)$ be an integer and let $(i_1, i_2, \ldots, i_n)$, $(i_{n+1}, i_{n+2}, \ldots, i_b)$ be a two part partition of the integers $1, 2, 3, \ldots, b$ with $i_1 < i_2 < \ldots < i_n$ and $i_{n+1} < i_{n+2} < \ldots < i_b$ where if $n = 0$ or $n = b$ then one subset of the partition is empty. Also let $E_{i_1, i_2, \ldots, i_n}$ be the event that $X_i \epsilon Q_1 \cup Q_3$ for $i = i_1, i_2, \ldots, i_n$ and $X_i \epsilon Q_2 \cup Q_4$ otherwise. Using this notation we have

$$P(E_{i_1, i_2, \ldots, i_n}) = (\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n})(1 - \alpha_{i_{n+1}})(1 - \alpha_{i_{n+2}}) \ldots (1 - \alpha_{i_b})$$

and

$$P[Y_3 = r_1, Y_2 = r_2 | E_{i_1, i_2, \ldots, i_n}, H_0] = P[Y_3 = r_1, Y_2 = r_2 | N = n, H_0]$$
$$= \binom{n}{r_1}\left(\frac{1}{2}\right)^n \binom{b-n}{r_2}\left(\frac{1}{2}\right)^{b-n}$$

where $0 \leq r_1 \leq n$, $0 \leq r_2 \leq b-n$. But this shows that Y_3 and Y_2 are conditionally independent binomial variables with parameters $(n, \frac{1}{2})$ and $(b-n, \frac{1}{2})$ respectively; therefore, a test can be based on

$$T = \frac{4}{N}\left(Y_3 - \frac{N}{2}\right)^2 + \frac{4}{b-N}\left(Y_2 - \frac{b-N}{2}\right)^2 .$$

The distribution of probabilities for the statistic T, given N and H_0, may be tabulated from the knowledge that Y_3 and Y_2 are conditionally independent binomial variables and it may be approximated by the distribution of the chi-square random variable with two degrees of freedom for large b and N.

The more general multivariate sign test will be discussed in Chapter IV as a special case of the multivariate Friedman statistic.

The Multivariate Signed Rank Test

A general procedure which is used in several multivariate rank tests will be illustrated by considering the multivariate signed rank test. The multivariate signed rank test is a multivariate extension of the Wilcoxon signed rank test for matched pairs. The design structure to which it applies may be viewed as a p-variate randomized complete block design of 2 treatments and b blocks; that is, each sample point consists of a p-tuple of observed values. Also, the observations on the two treatments within each block represent repeated measures.

Data:

The data consists of b p-variate vectors of observations on each of 2 treatments; that is, $Z_{ij} = (Z_{ij1}, Z_{ij2}, \ldots, Z_{ijp})'$ for $i = 1, 2, \ldots, b$ and $j = 1, 2$ is the p-variate observation from the i^{th} block and j^{th} treatment. Let $X_i = Z_{i1} - Z_{i2}$ be the p-variate vector of differences between the two treatment vectors in block i.

Assumptions:

(1) The X_i are independent p-variate random vectors.

(2) The X_i have continuous cumulative distribution functions $F_i(X, \Omega)$ with $X \epsilon R^p$, $\Omega \epsilon R^p$ where Ω is a vector of location parameters.

(3) $F_i(X, \Omega)$ is diagonally symmetric about Ω; that is, ($X - \Omega$ and $\Omega - X$ have the same distribution).

(4) The scale of measurement is at least ordinal.

Suppose we want to test the hypothesis $H_0 : \Omega = \phi$ against the alternative $H_1 : \Omega \neq \phi$, then as before the joint distribution of the ranks assigned within the set $\{|X_{ij}|, i = 1, 2, \ldots, b\}$ for each $j = 1, 2, \ldots, p$ depends on the distribution functions $F_1, F_2, \ldots, F_b$ of the sampled populations even when H_0 is true -- unless the variables $X_{i1}, X_{i2}, \ldots, X_{ip}$ are mutually independent. To obtain a conditionally distribution-free test, let D_x be the $b \times p$ matrix of differences whose i^{th} row is X_i. Consider the group $\mathcal{G}$ of transformations $\{f_b\}$ given by

$$f_b(D_x) = \{(-1)^{r_1} X_1, (-1)^{r_2} X_2, \ldots, (-1)^{r_b} X_b\}$$

for $r_i = 0, 1$ and $i = 1, 2, \ldots, b$. For any D_x (fixed), $\mathcal{G}$ has 2^b distinct points. Under H_0, the points in $\mathcal{G}$ are equally likely because of the diagonal symmetry of each F_i. Denote the set of 2^b distinct points in $\mathcal{G}$ by $S(D_x)$, then the distribution of D_x given $D_x \epsilon S(D_x)$ is uniform on the 2^b points. A test function of $\varphi(D_x)$ is selected by considering the particular alternative we want to test and such that the probability of type I error is α. The test statistic is developed as follows.

The matrix of difference was defined as

$$D_x = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & & & X_{2p} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ X_{b1} & & \cdots & X_{bp} \end{bmatrix}$$

where one can think about the i^{th} row as the p treatment differences observed in block i. Define

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & & & r_{2p} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ r_{b1} & & \cdots & r_{bp} \end{bmatrix}$$

as the $b \times p$ matrix of ranks obtained by ranking the absolute values of the elements within columns of the sample matrix D_x in ascending order; that is, r_{ij} is the rank of $|X_{ij}|$ among the $|X_{1j}|, |X_{2j}|, \ldots, |X_{bj}|$ for each fixed $j = 1, 2, \ldots, p$. Let

$$T^j = \frac{1}{b+1} \sum_{i=1}^{b} r_{ij} c_{ij} \qquad \text{for } j = 1, 2, \ldots, p$$

where

$$c_{ij} = \begin{cases} 1 & \text{if } X_{ij} > 0 \\ -1 & \text{if } X_{ij} < 0 \end{cases}$$

be $\frac{1}{b+1}$ times the sum of the signed ranks assigned to variable j. Also let $T = (T^1, T^2, \ldots, T^p)'$ and let C be the $b \times p$ matrix whose (i,j) element is c_{ij}. The covariance matrix V of the vector T whose (i,j) element is v_{ij} is obtained by noting that $E[T^j | D_x \epsilon S(D_x), H_0] = 0$ for each j and

$$E[T^i T^j | D_x \epsilon S(D_x), H_0] = \frac{1}{(b+1)^2} \sum_{k=1}^{b} r_{ki} r_{kj} c_{ki} c_{kj} = b v_{ij} .$$

Thus

$$v_{ij} = \frac{1}{b(b+1)^2} \sum_{i=1}^{b} r_{ki} r_{kj} c_{ki} c_{kj} .$$

If $i = j$, then v_{ij} becomes

$$v_{ii} = \sum_{k=1}^{b} \frac{(r_{ki})^2}{(b+1)^2} = \frac{b(2b+1)}{6(b+1)} .$$

The test statistic for testing H_0 against H_1 is now defined as

$$S = \frac{1}{b} (T' V^{-1} T) .$$

A problem arises in that V may be singular; however, this may be overcome by using the highest order non-singular minor matrix of V and the corresponding components of T.

The conditional distribution of probabilities for the statistic S, given that $D_x \epsilon S(D_x)$ and H_0 is true, can be calculated from the discrete uniform distribution with 2^b points. The amount of arithmetic required to do this calculation is prohibitive if b is large; however, the limiting distribution of S is the distribution of the chi-square random variable with p degrees of freedom [17]. The following numerical example is given to illustrate the calculation of the exact conditional distribution and the test statistic for a particular case.

Example:

Four students were given a form A test and a form B test which were designed to measure both verbal usage and quantitative skills. Scores are recorded in Table I with the verbal usage score first and the quantitative skill score second.

We wish to test the hypothesis that the two test forms are equivalent. This can be accomplished by defining

$$D_x = \begin{pmatrix} -5 & 1 \\ -1 & -5 \\ 6 & 3 \\ -4 & 8 \end{pmatrix},$$

where D_x is obtained from Table I by subtracting the test scores on form B from the corresponding test scores on form A and testing

$$H_0 : \Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \phi \; ; \qquad H_1 : \Omega \neq \phi .$$

TABLE I

TEST SCORES FROM TWO TEST FORMS MEASURING VERBAL USAGE (VU) AND QUANTITATIVE SKILLS (QS)

Student	Variable	Form A	Form B
1	VU	70	75
	QS	89	90
2	VU	64	65
	QS	45	50
3	VU	73	67
	QS	64	61
4	VU	89	93
	QS	87	79

Now, under H_0, X is symmetric about ϕ and we may use the test outlined. This gives

$$R = \begin{pmatrix} 3 & 1 \\ 1 & 3 \\ 4 & 2 \\ 2 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & -1 \\ -1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$V = .06 \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, \qquad V^{-1} = \frac{25}{36} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix},$$

$$T' = (-4, .4) \qquad \text{and} \qquad S = .3333 .$$

From the distribution of probabilities for S developed in the discussion which follows, $\hat{\alpha} = P(S \geq .3333) = 1$ which implies that our data has supplied the least possible support for H_1; hence, H_0 would not be rejected.

Theory:

The distribution of the ranks assigned to the data depends on the distribution functions of the populations sampled, so it is necessary to use a conditional distribution or a limiting distribution to tabulate the distribution of probabilities for S. Tables of probabilities for the conditional distribution of S are not readily available, nor easily calculated, so it is convenient to use the limiting distribution. Even so, we will use the data in the previous example to illustrate the computations involved in tabulating the conditional distribution of S. The 2^4 points in $S(D_x)$ and the corresponding values of C, T, and S are given in Table II.

The values of S are found by considering the group of transformations

$$\{f_4(D_x)\} = \left\{ (-1)^{j_1}\begin{pmatrix}-5\\-1\end{pmatrix}', (-1)^{j_2}\begin{pmatrix}-1\\-5\end{pmatrix}', (-1)^{j_3}\begin{pmatrix}6\\3\end{pmatrix}', (-1)^{j_4}\begin{pmatrix}-4\\8\end{pmatrix}' \right\}$$

where $j_i = 0, 1$ and $i = 1, 2, 3, 4$. As Table II shows we have 16 points in $S(D_x)$ with 8 distinct values of S; since the points in $S(D_x)$ are equally likely under H_0 and each value of S appears exactly twice, the 8 distinct values of S are equally likely, each with probability $.125$. We see that $P[S \geq .3333] = 1$ which does not give any support for the alternate hypothesis. We also note

TABLE II

THE EXACT DISTRIBUTION OF S

Points of $S(D_x)$	C'	5 T'	S
$\begin{pmatrix} -5 & -1 & 6 & -4 \\ -1 & -5 & 3 & 8 \end{pmatrix}'$	$\begin{pmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$	(-2, 2)	.3333
$\begin{pmatrix} -5 & 1 & 6 & -4 \\ -1 & 5 & 3 & 8 \end{pmatrix}'$	$\begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$	(0, 8)	2.2222
$\begin{pmatrix} -5 & 1 & -6 & -4 \\ -1 & 5 & -3 & 8 \end{pmatrix}'$	$\begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$	(-8, 4)	3.2222
$\begin{pmatrix} -5 & 1 & -6 & 4 \\ -1 & 5 & -3 & -8 \end{pmatrix}'$	$\begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}$	(-4, 4)	.8888
$\begin{pmatrix} -5 & -1 & -6 & -4 \\ -1 & -5 & -3 & 8 \end{pmatrix}'$	$\begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$	(-10, -2)	3.6111
$\begin{pmatrix} -5 & -1 & -6 & 4 \\ 5 & -5 & -3 & -8 \end{pmatrix}'$	$\begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$	(-6, -10)	3.8888
$\begin{pmatrix} -5 & -1 & 6 & 4 \\ -1 & -5 & 3 & -8 \end{pmatrix}'$	$\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}$	(2, -6)	1.5555
$\begin{pmatrix} -5 & 1 & 6 & 4 \\ -1 & 5 & 3 & -8 \end{pmatrix}'$	$\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$	(4, 0)	.5555
$\begin{pmatrix} 5 & -1 & 6 & -4 \\ 1 & -5 & 3 & 8 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$	(4, 4)	.8888
$\begin{pmatrix} 5 & 1 & 6 & -4 \\ 1 & 5 & 3 & 8 \end{pmatrix}'$	$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	(6, 10)	3.8888
$\begin{pmatrix} 5 & 1 & -6 & -4 \\ 1 & 5 & -3 & 8 \end{pmatrix}'$	$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$	(-2, 6)	1.5555
$\begin{pmatrix} 5 & 1 & -6 & 4 \\ 1 & 5 & -3 & -8 \end{pmatrix}'$	$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$	(2, -2)	.3333
$\begin{pmatrix} 5 & -1 & -6 & -4 \\ 1 & -5 & -3 & 8 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	(-4, 0)	.5555
$\begin{pmatrix} 5 & -1 & -6 & 4 \\ 1 & -5 & -3 & -8 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$	(0, -8)	2.2222
$\begin{pmatrix} 5 & -1 & 6 & 4 \\ 1 & -5 & 3 & -8 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	(8, -4)	3.2222
$\begin{pmatrix} 5 & 1 & 6 & 4 \\ 1 & 5 & 3 & -8 \end{pmatrix}'$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$	(10, 2)	3.6111

from this distribution that for a .125 level of significance the critical value is 3.8888 and that a randomized test is necessary for any smaller level of significance.

CHAPTER III

THE MULTIVARIATE KRUSKAL-WALLIS TEST FOR THE COMPLETELY RANDOMIZED DESIGN

The Multivariate Kruskal-Wallis test is a generalization of the univariate Kruskal-Wallis test and is used for the multivariate multi-sample problem. It is applicable in the case of t independent random samples of size n_i, $i = 1, 2, \ldots, t$, from a p-variate random vector in a completely randomized design or a one-way classification. In the case that $t = 2$, the test statistic is $L = \frac{N-1}{N-2} T^2$ where $N = \sum_{i=1}^{t} n_i$ and T^2 is the Hotelling's T^2 statistic except that the ranks assigned are used to calculate the statistic T^2 instead of the observed data.

Data:

The data consists of a random sample from each of t treatment populations where the i^{th} sample is a p-variate sample of size n_i for $i = 1, 2, \ldots, t$ and $p \geq 1$. The data from the i^{th} sample may be displayed as the $p \times n_i$ matrix X_i where

$$X_i = \begin{bmatrix} X_{i11} & X_{i21} & \cdots & X_{in_i1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ X_{i1p} & X_{i2p} & & X_{in_ip} \end{bmatrix}$$

and X_{ijs} denotes observation number j on variable number s under treatment i (or within sample i). Let $F_i(X, \Omega_i)$ for $X \epsilon R^p$ and Ω_i in R^p denote the distribution function for the i^{th} treatment, where Ω_i is a vector of location parameters. Let $X_N = [X_1, X_2, \ldots, X_t]$ be the $p \times N$ matrix of observations. Assign the rank r_{ijs} to X_{ijs} where the X_{ijs}'s are ranked in ascending algebraic order among the N elements $\{X_{ijs}\}$ for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n_i$ for each fixed variable s; that is, the observations taken within each variable are ranked as in the univariate Kruskal-Wallis test. Let R_i denote the $p \times n_i$ matrix of ranks assigned to the data in sample i where the (s, j) element is r_{ijs}. Also define the $p \times N$ matrix $R = [R_1, R_2, \ldots, R_t]$; that is, rows in R contains the ranks assigned within variable s.

Assumptions:

(1) $F_i(X, \Omega_i)$ is continuous for each i.

(2) The N p-variate vectors of observations are independent.

(3) The scale of measurement is at least ordinal (within each variable).

Hypothesis:

$$H_0: \Omega_1 = \Omega_2 = , \ldots, \Omega_t$$

H_1: Some two Ω's are not equal.

Note that H_0 involves vector valued parameters $\Omega_i = (\omega_{i1}, \omega_{i2}, \ldots, \omega_{ip})'$ and implies that $\omega_{is} = \omega_{ms}$ for all i, m and s, but does not imply that $\omega_{is} = \omega_{ik}$ for any i, s or k with $s \neq k$.

Test statistic:

Compute the $p \times p$ covariance matrix $V = RR' - \frac{1}{4}N(N+1)^2 J_p^p$ where J_c^r is an $r \times c$ matrix with all elements 1. Compute the t p-variate vectors $\bar{R}_i = (\bar{r}_{i1}, \bar{r}_{i2}, \ldots, \bar{r}_{ip})'$ whose components are the means of the ranks assigned to the p-variables for the i^{th} treatment; that is, for fixed i define $\bar{r}_{is} = \frac{1}{n_i} \sum_{j=1}^{n_i} r_{ijs}$ for $s = 1, 2, \ldots, p$. We also need

$$\bar{u}_i = \bar{R}_i - \frac{N+1}{2} J_1^p$$

which is the p-variate vector of mean deviations for each treatment. The test statistic is

$$L = (N-1) \sum_{i=1}^{t} n_i \bar{u}_i' V^{-1} \bar{u}_i .$$

Decision rule:

The exact conditional distribution of the statistic L is laborious to calculate so the limiting distribution of L is used. According to Puri and Sen [17], the limiting distribution of L is the distribution of the chi-square random variable with $p(t-1)$ degrees of freedom. Thus, H_0 is rejected at the α level of significance if $L > \chi^2[p(t-1), \alpha]$ where $\chi^2[p(t-1), \alpha]$ is the $1-\alpha$ quantile from the distribution of the chi-square random variable with $p(t-1)$ degrees of freedom.

Example:

Jerome L. Meyers [14] gives the data (Table III) from a completely randomized two-factor experiment. For a plausible context, suppose the levels of A represent two varieties of corn and the levels of B represent three fertilizers with the scores being yields in bushels for one-third acre plots. The Multivariate Kruskal-Wallis may be used to test the "main effect" of Treatment A.

H_0: There is no difference in varieties.

H_1: There is a difference in varieties.

This gives $t = 2$ treatments and $p = 3$ variates (levels of B) each with $n_1 = n_2 = 8$ observations per treatment so $N = 16$. Ranks assigned over the two levels of A for each fixed level of B are given in Table III.

TABLE III

CORN YIELDS PER ONE-THIRD ACRE AND THE ASSIGNMENT OF RANKS

A_1						A_2					
B_1		B_2		B_3		B_1		B_2		B_3	
Obs'n	Rank	Obs'n	Rank	Obs'n	Rank	Obs'n	Rank	Obs'n	Rank	Obs'n	Rank
7	2	6	3.5	9	9	42	16	28	15	13	13
33	15	11	6.5	12	12.5	25	9	6	3.5	18	14
26	10	11	6.5	6	6.5	8	3	1	1	23	15
27	11	18	12.5	24	16	28	12	15	10.5	1	1.5
21	7	14	8.5	7	8	30	13	9	5	3	4
6	1	18	12.5	10	10.5	22	8	15	10.5	4	5
14	4	19	14	1	1.5	17	5	2	2	6	6.5
19	6	14	8.5	10	10.5	32	14	37	6	2	3

From Table III we calculate

$$V = \begin{bmatrix} 340 & 93.50 & -18.0 \\ 93.50 & 337.5 & -86.5 \\ -18.0 & -86.5 & 338.5 \end{bmatrix}$$

$$\bar{u}_1 = \left(-\frac{12}{8}, \frac{4.5}{8}, \frac{6}{8}\right)' \qquad \text{and} \qquad \bar{u}_2 = \left(\frac{12}{8}, -\frac{4.5}{8}, -\frac{6}{8}\right)'$$

thus

$$L = 8 \cdot 2 \cdot 15\left(-\frac{12}{8}, \frac{4.5}{8}, \frac{6}{8}\right)\begin{pmatrix} .003185 & -.000898 & -.00060 \\ -.000898 & .003424 & .000827 \\ -.00060 & .000827 & .003162 \end{pmatrix}\begin{pmatrix} -\frac{12}{8} \\ \frac{4.5}{8} \\ \frac{6}{8} \end{pmatrix}$$

$$= 3.261949 .$$

Using the chi-square approximation (3 degrees of freedom) gives $.25 < \hat{\alpha} < .50$. An analysis of variance for the factorial experiment using the F test gives $\hat{\alpha} > .25$ for the main effect of factor A.

Theory:

The distribution of the ranks assigned to the data depends on the sampled distribution functions (even when H_0 is true). The statistic L is conditionally distribution-free under H_0 and may be calculated from the uniform distribution over N! points. For this reason tables of probabilities for L are not readily available so it is convenient and apparently satisfactory to use the limiting distribution to approximate the distribution of L. Puri and Sen [17] show that the conditional limiting distribution of L is the distribution of the chi-square random variable with $p(t-1)$ degrees of freedom.

CHAPTER IV

SOME TESTS FOR TWO-WAY CLASSIFICATIONS

The Ranking After Alignment Procedure for the Friedman Test

One of the basic designs in a two-way layout is the randomized complete block design. Conover [8] illustrates the use of the well known "Friedman test" for this design. He gives the usual assumptions as:

(1) The observations in different blocks are independent.

(2) The observations within each block may be arranged in increasing order according to some criterion of interest.

Since the Friedman test makes use only of the intrablock ranks, the efficiency of this test may be improved in certain examples by a procedure Puri and Sen [17] call "ranking after alignment." The "ranking after alignment" procedure uses an intrablock transformation then a ranking procedure which ignores the blocks and thus makes use of interblock information as well as intrablock information.

Data:

Let X_{ij} be the observation in the i^{th} of $b(b \geq 2)$ blocks receiving the j^{th} of t treatments. Let $\overline{X}_{i.}$ be the mean of the observations in block i. Align the data by subtracting $\overline{X}_{i.}$ from each observation in block i; that is, let $Y_{ij} = X_{ij} - \overline{X}_{i.}$ for each i and j and call Y_{ij} the aligned observations. Next assign ranks r_{ij} to the $N = bt$ aligned observations with rank 1 assigned to the smallest, rank 2 assigned to the next smallest and continuing in this manner with rank N assigned to the largest aligned observation.

Assumptions:

To simplify the communication consider the model $X_{ij} = \mu + \beta_i + \tau_j + \epsilon_{ij}$ for $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, t$ where as usual μ is the mean effect, β_i the effect of the i^{th} block, τ_j the effect of the j^{th} treatment and the ϵ_{ij}'s the residual error components. Using this notation we may write the assumptions as follows:

(1) $\epsilon_i = (\epsilon_{i1}, \epsilon_{i2}, \ldots, \epsilon_{it})$ for $i = 1, 2, \ldots, b$ are independent t-variate random vectors.

(2) $G_i(X_1, X_2, \ldots, X_t)$, the joint commulative distribution function of the elements of ϵ_{ij}, is continuous and symmetric in its t arguments for each $i = 1, 2, \ldots, b$.

(3) The measurement scale is at least interval.

(4) $\sum_{i=1}^{t} \tau_i = 0$.

Hypothesis:

$$H_0: \tau = \tau_1, \tau_2, \ldots, \tau_t)' = \phi$$

$$H_1: \tau \neq \phi .$$

Test statistic:

A few preliminary definitions and calculations are needed in the computation of the test statistic. Let

$$T_j = \frac{1}{b} \sum_{i=1}^{b} \frac{r_{ij}}{N+1} , \qquad T = (T_1, T_2, \ldots, T_t) ,$$

$$\sigma^2 = \frac{1}{(N+1)^2 b(t-1)} \sum_{i=1}^{b} \sum_{j=1}^{t} \left(r_{ij} - \frac{1}{t} \sum_{k=1}^{t} r_{ik}\right)^2 \quad \text{and}$$

$$E_i = \frac{1}{t} \sum_{k=1}^{t} \frac{r_{ik}}{N+1} \qquad \text{then} \qquad \overline{E} = \frac{1}{b} \sum_{i=1}^{b} E_i = \frac{1}{2} .$$

The test statistic is defined as

$$S_N = \frac{b}{\sigma^2} \sum_{j=1}^{t} (T_j - \frac{1}{2})^2 .$$

Decision Rule:

We can reject H_0 at the α level of significance if S_N exceeds the $1 - \alpha$ quantile of the conditional distribution of S_N under H_0. Again the conditional distribution of S_N is laborious to tabulate if b is large so the limiting distribution is used. Thus, the decision rule is to reject H_0 at the α level of significance if S_N exceeds the

$1 - \alpha$ quantile of the chi-square random variable with $(t - 1)$ degrees of freedom, since the limiting distribution of S_N is the distribution of the chi-square random variable with $t - 1$ degrees of freedom [17].

Example:

Bing [1] compared the effect of several herbicides on the spike weight of gladiolus. The average weight per spike in ounces is recorded in Table IV.

TABLE IV

AVERAGE SPIKE WEIGHT IN OUNCES

Blocks \ Treatment	2.4-D TCA	Check	DN/cr	Sesin	$\overline{X}_{i.}$
1	2.05	1.25	1.95	1.75	1.75
2	1.56	1.73	2.00	1.93	1.80
3	1.68	1.82	1.83	1.70	1.76
4	1.69	1.31	1.81	1.59	1.60

From Table V we can calculate $T = (.49, .25, .78, .49)$ and $S_N = 5.9956$. Comparing S_N to the chi-square random variable with 3 degrees of freedom gives $.10 < \hat{\alpha} < .25$.

TABLE V

RANKS OF THE ALIGNED OBSERVATIONS

Blocks \ Treatments	2.4-D TCA	Check	DN/cr	Sesin	E_i
1	15	1	14	8	.5588
2	3	5	13	12	.4853
3	4	9	10	6	.4265
4	11	2	16	7	.5294
T_j	.4853	.25	.7794	.4853	

Theory:

The Friedman statistic is distribution-free under H_0, but the aligned variables within each block are usually dependent and it is necessary to use a conditionally distribution-free statistic in the ranking after alignment procedure. The alignment procedure subtracts out the block effect and under H_0 leaves interchangable random variables. So the test is a test of interchangability of the aligned values $Y_{i1}, Y_{i2}, \ldots, Y_{it}$ for each i. The joint commulative distribution of $(Y_{i1}, Y_{i2}, \ldots, Y_{it})$ is invariant under the $t!$ permutations of the coordinates among themselves, for each $i = 1, 2, \ldots, b$, so there are $(t!)^b$ equally likely points in the group of transformations. For large b or t this makes computation of the distribution function difficult and for this reason it is again convenient to use the limiting distribution of S_N which is the distribution of the chi-square random variable with $(t - 1)$ degrees of freedom [17]. Puri and Sen

[17] also show that the efficiency of the Friedman statistic relative to the variance ratio test for normal alternatives (parametric analysis of variances) is less than $3/\pi$ and that the efficiency of S_N relative to the variance ratio test is $3/\pi$ when $t = 2$ and strictly greater than $3/\pi$ when $t > 2$. Such an increase in efficiency is not surprising, since the statistic S_N assumed an interval scale of measurement (as does the variance ratio test) while the Friedman statistic assumes only an ordinal scale of measurement within blocks.

The Randomized Complete Block Design With Several Observations per Cell

A "ranking after alignment" procedure for the randomized complete block design with one observation per cell has been discussed in the previous section. This section will extend the consideration to the case of several observations per cell [17].

Data:

The data consists of m_j observations on the j^{th} of t treatments within each of b blocks. The total number of observations is then $N = b \sum_{j=1}^{t} m_j = b M$, where $M = \sum_{j=1}^{t} m_j$. As before we align the observations by subtracting the block mean from each observation within that block. Next we assign ranks to the N aligned observations in ascending algebraic order (ignoring treatments) and let r_{ijk} be the rank assigned to the k^{th} observation under the j^{th} treatment in block i. We also denote the "average" of the ranks assigned to the $b m_j$ observations on treatment j by $(N+1)\, T_{(N,j)}$ where

$$T_{(N,j)} = \frac{1}{b\,m_j} \sum_{i=1}^{b} \sum_{k=1}^{m_j} \frac{r_{ijk}}{N+1}$$

and define

$$\sigma^2 = \frac{1}{b(M-1)} \sum_{i=1}^{b} \sum_{j=1}^{t} \sum_{k=1}^{m_j} \frac{\left(r_{ijk} - \frac{1}{b} \sum_{i=1}^{b} r_{ijk}\right)^2}{(N+1)^2}$$

as the (pooled) within block mean square of the rank scores.

Assumptions:

The assumptions, hypotheses and theory are the same as when we have one observation per cell except that the number of points in the transformation space, used to calculate the conditional distribution of the test statistic, is larger.

Test statistic:

The test statistic is $S_N = \frac{b}{\sigma^2} \sum_{j=1}^{t} m_j \, [T_{(N,j)} - \frac{1}{2}]^2$.

Decision rule:

Reject H_0 at the α level of significance if S_N exceeds the $1 - \alpha$ quantile from the chi-square distribution with $t - 1$ degrees of freedom. The chi-square approximation is used because the exact conditional permutation distribution of S_N is laborious to compute.

Example:

Five fertilizers were tested for possible different effects on yields of oats. The design is the randomized complete block design with six blocks. The experimenter selected 3 sample quadrats, each three feet square, as experimental plots and determined the yield of each of the 90 quadrats. The coded yields from Ostle [15] are given in Table VI.

TABLE VI

OATS YIELDS

Blocks \ Fertilizers	1	2	3	4	5	Block Mean
1	57 46 28	67 72 66	95 90 89	102 88 109	123 101 113	83.06667
2	26 38 20	44 68 64	92 89 106	96 89 106	93 110 115	77.06667
3	39 39 43	57 61 61	91 82 98	102 93 98	112 104 112	79.46667
4	23 36 18	74 47 69	105 85 85	103 90 105	120 101 111	78.13333
5	48 35 48	61 60 75	78 89 95	99 87 113	113 109 111	81.40
6	50 37 19	68 65 61	85 74 80	117 93 107	124 102 118	80.00

TABLE VII

RANKS OF THE ALIGNED OBSERVATIONS ON OATS YIELDS

Blocks \ Fertilizers	1	2	3	4	5
1	20 13 4	28 32 27	52 47 44	63.5 41 70	88 60 78
2	5.5 12 3	17 34 30	57 52 67	63.5 52 75	58 84 86
3	10.5 10.5 14	21 25.5 25.5	49 40 61.5	66 55 61.5	81.5 68 81.5
4	4 9 2	37 18 33	71.5 45.5 45.5	69 50 71.5	89 67 83
5	15.5 7 15.5	22 23 35	38 48 56	59 43 79.5	79.5 74 77
6	19 8 1	31 29 24	42 36 39	85 54 73	90 65 87
$91\, T_j$	9.6389	27.3333	50	62.8611	77.5833
$(T_j - \frac{1}{2})^2$	.1553	.0398	.00244	.0364	.1243

Using Tables VI and VII we can calculate $\sigma^2 = .08728$ and $S_N = \frac{(6)(3)}{.08728} (.358294) = 73.8920$. Then using the chi-square approximation (4 degrees of freedom) we see that $\hat{\alpha}$ is less than

.001 and we have strong support for the alternative that the effects of the five fertilizers are unequal.

Gerig's Multivariate Extension of the Friedman Statistic

The univariate randomized complete block design has been discussed in the previous sections. Gerig [9] developed the Multivariate Friedman test which is a generalization of the Friedman test to the case where the observation in each cell of the randomized complete block design is a p-variate observation. Puri and Sen [17] also suggest a "ranking after alignment" procedure which will improve the efficiency in the case of additive block effects. Both of the procedures will be discussed in the following pages.

Data:

The data consists of a p-variate observation from each of the $N = bt$ cells with $X_{ij} = (X_{ij}^1, X_{ij}^2, \ldots, X_{ij}^p)'$ denoting the p-variate observation in the i^{th} of b blocks which received the j^{th} of t treatments. Let r_{ij}^s denote the rank assigned to the observation on variable s in block i which received treatment j where the observations on each variable are ranked from 1 to t across the treatments within each block. For block i let R_i denote the $p \times t$ matrix of ranks whose (s, j) element is r_{ij}^s ; that is,

$$R_i = \begin{bmatrix} r_{i1}^{1} & r_{i2}^{1} & \cdots & r_{it}^{1} \\ & & & r_{it}^{2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ r_{i1}^{p} & & & r_{it}^{p} \end{bmatrix}$$

and let

$$V_i = \frac{1}{t-1} [R_i R_i' - \frac{1}{4} t(t+1)^2 J_p^p] .$$

Assumptions:

(1) $F_{ij}(X)$, the distribution function of X_{ij}, is continuous for each i and j.

(2) $F_{ij}(X) = F_i(X - \Omega_j)$ for each $i = 1, 2, \ldots, b$ where Ω_j is a vector of location parameters; that is, $\Omega_j = (\omega_j^1, \omega_j^2, \ldots, \omega_j^p)'$.

(3) The blocks are independent; however X_{ij} and $X_{ij'}$ for $j \neq j'$ may not be independent.

(4) The measurement scale is at least ordinal within each block.

Hypotheses:

H_0: $\Omega_j = \Omega_k$ for $j = 1, 2, \ldots, t$ and $k = 1, 2, \ldots, t$.

H_1: $\Omega_j \neq \Omega_k$ for at least two values $j \neq k$.

Please note again that H_0 involves vector valued parameters and does not imply equality of components within the vectors Ω_j, but implies equality between corresponding components of Ω_j and Ω_k for $j \neq k$.

Test statistic:

Some additional notation is needed to define the test statistic, so let

$$V = \frac{1}{b} \sum_{k=1}^{b} V_i , \qquad T_j^s = \frac{1}{b} \sum_{i=1}^{b} r_{ij}^s - \frac{1}{2}(t+1)$$

and

$$T_j = (T_j^1, T_j^2, T_j^3 \ldots T_j^p)' ,$$

then the test statistic

$$Q = b \sum_{j=1}^{t} T_j' V^{-1} T_j .$$

Since V^{-1} does not depend on the index of summation j, Q may also be written as

$$Q = b \sum_{s=1}^{p} \sum_{s'=1}^{p} v^{-1}(s,s') \, w(s,s')$$

where $v^{-1}(s,s')$ and $w(s,s')$ are the (s,s') elements in the matrices V^{-1} and $W = T'T$, respectively, and T is the $t \times p$ matrix whose (s,j) element is T_j^s.

Decision rule:

We can reject H_0 at the α level of significance if Q exceeds the $1-\alpha$ quantile of the conditional distribution of Q under H_0. Again the conditional distribution of Q is laborious to construct if b and t are not small so the limiting distribution is used. Then our decision rule is "reject H_0 at the α level of significance if Q exceeds the $1-\alpha$ quantile of the distribution of the chi-square random variable with $p(t-1)$ degrees of freedom" since the limiting distribution of Q is the distribution of the chi-square random variable with $p(t-1)$ degrees of freedom [9].

Koch [11] has indicated that the multivariate Friedman test is a valid procedure to use in a randomized block experiment with a factorial arrangement of treatments to test the hypothesis that the main effect of one of the factors and all its interactions are zero. This is accomplished by letting the levels of one factor be the treatments and the combinations of levels of the other factors be the variables.

Example:

Pearce [16] gives a 3×3 factorial experiment in four blocks which will be used as an example. The experiment was conducted to determine the effect of growth substances upon peas. Doses of either 1, 10, or 100 micrograms of Gibberellic acid were applied to either the 3rd, 6th or 9th node of the plant and a measurement of the thickness of the leaf was taken at node 10. The data is given in Table VIII with factor A representing the doses and factor B representing the node of application. Let a_1, a_2 and a_3 denote the treatment levels of 1, 10, and 100 microgram doses, respectively, and let b_1, b_2, and b_3 denote the variables corresponding to an application of the dose to the 3rd, 6th and 9th nodes, respectively.

TABLE VIII

THICKNESS OF THE PEA LEAVES AT NODE TEN AND THE ASSIGNMENT OF RANKS

Block	Variable	Treatments					
		a_1		a_2		a_3	
		Obs'n	Rank	Obs'n	Rank	Obs'n	Rank
1	b_1	9.0	3	6.6	1	6.7	2
	b_2	7.6	3	6.0	2	5.9	1
	b_3	7.1	1	8.7	2	9.1	3
2	b_1	8.9	3	6.5	1	8.8	2
	b_2	8.1	3	5.6	1	5.8	2
	b_3	8.3	3	9.0	3	7.8	1
3	b_1	9.1	2	9.2	3	6.5	1
	b_2	9.3	3	7.0	2	6.4	1
	b_3	8.3	1	8.5	2	9.0	3
4	b_1	9.0	3	8.9	2	7.0	1
	b_2	7.2	3	6.3	2	5.9	1
	b_3	8.0	2	8.3	3	7.0	1

If we wish to test that the main effect of A and the AB interaction are zero, we can use the theory in the previous pages when the levels of B are the variables within the vectors, thus we have $p = 3$. To test this hypothesis, let the levels of A be the treatments then assign ranks within the blocks across the levels of A for each variable (levels of B). The ranks assigned are also given in Table VIII. From Table VIII we calculate

$$R_1 = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1.00 & .75 & -.25 \\ .75 & 1.00 & -.5 \\ -.25 & -.5 & 1.00 \end{bmatrix}.$$

Using the corrected mean vectors $T_1 = (.75, 1, -.5)'$, $T_2 = (-.25, -.25, .5)'$, and $T_3 = (-5, -.75, 0)$ for the three levels of A, the value of the test statistic becomes $Q = 8.30$. Using the chi-square approximation with 6 degrees of freedom shows $.20 < \alpha < .25$. Thus the null hypothesis would not be rejected at any of the commonly used significance levels.

Theory:

As in tests discussed previously, the distribution of the ranks assigned to the data depends on the sampled distribution functions even when H_0 is true. For this reason it is necessary to use a conditional permutation distribution as in other multivariate tests. Gerig shows the limiting distribution of Q to be the chi-square distribution with p(t-1) degrees of freedom [9]. Gerig also discusses the efficiency of the statistic Q relative to the univariate likelihood ratio test and gives a table of the efficiency when p is 2 and $t \geq 2$ for some different correlation coefficients.

Special Cases of Gerig's Statistic

In the case where $t = 2$ the Multivariate Friedman Statistic may be calculated by considering the signs of the entries in the p-variate vector d_i which is obtained by subtracting one column of the $p \times 2$ matrix R_i from the other and thus it is a multivariate sign test. To see how this is accomplished let $d_i^s = r_{i1}^s - r_{i2}^s$ be the s^{th} component of the p-variate vector d_i. Note that $d_i^s = \pm 1$ for each i and s, since $r_{ij}^s = 1$ or 2 for all i, j and s. Let k(s, s') be the number of blocks in which the signs of d_i^s and $d_i^{s'}$ are the same. Now $2b\,V(s,s') = \sum_{i=1}^{b} d_i^s d_i^{s'} = 2k(s,s') - b$ for $s \neq s'$ and $2b\,V(s,s') = b$ for $s = s'$. Thus, $V(s,s') = \frac{2k(s,s') - b}{2b}$ for $s \neq s'$ and $V(s,s') = \frac{1}{2}$ for $s = s'$. Now if we let P_s be the number of blocks in which the s^{th} component of d_i is positive, then

$$T_1^s = \frac{1}{b}\sum_{i=1}^{b} r_{i1}^s - \frac{3}{2} = \frac{1}{b}\sum_{i=1}^{b} r_{i1}^s - \frac{1}{b}\sum_{i=1}^{b}\left(\frac{r_{i1}^s + r_{i2}^s}{2}\right)$$

$$= \frac{1}{b}\sum_{i=1}^{b}\left(\frac{r_{i1}^s - r_{i2}^s}{2}\right) = \frac{1}{b}\sum_{i=1}^{b}\frac{d_i^s}{2}$$

$$= \frac{1}{2b}\,P_s - (b - P_s) = \frac{1}{2b}(2P_s - b)$$

$$= \frac{P_s}{b} - \frac{1}{2} .$$

From this it follows that $T_1^s = -T_2^s$ for all s. The test statistic Q may now be written as

$$Q = b \sum_{i=1}^{2} T_i' V^{-1} T_i = 2b(T_1' V^{-1} T_1)$$

where both T_1 and V may be calculated by using the signs of the components of d_i as indicated.

One can also show that if the number of treatments is 2 and the number of variables is 2, the statistic Q is identical to the statistic T of the bivariate sign test discussed in Chapter II. So that the Multivariate Friedman is a generalization of the bivariate sign test.

As an example, consider the example discussed in the previous section with the last two levels of A (deleting a_1) as treatments and test the hypothesis that the effects of a_2 and a_3 are the same using the data in Table VIII. After ranking across the levels a_2 and a_3 for each of the levels of B, we have

$$R_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and}$$

$$R_4 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{thus} \quad d_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

$$d_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad d_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the calculation of T_1 and V we have $P_1 = 2$, $P_2 = 3$, $P_3 = 2$, $k(1,2) = 3$, $k(1,3) = 2$, and $k(2,3) = 1$ which gives

$$T_1 = \begin{pmatrix} \frac{2}{4} - \frac{1}{2} \\ \frac{3}{4} - \frac{1}{2} \\ \frac{2}{4} - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \quad \text{and} \quad V = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} .$$

Further calculations show

$$V^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

and

$$Q = 2 \cdot 4 \begin{bmatrix} 0, \frac{1}{4}, 0 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{4} \\ 0 \end{bmatrix} = 8(4) \frac{1}{16} = 2 .$$

Using the chi-square approximation (3 degrees of freedom) shows $\hat{\alpha} > .5$, so we cannot say that the two levels of A appear to be different.

The "Ranking After Alignment" Procedure for the Multivariate Friedman Test

As in the univariate case an alignment procedure can be introduced, if we add two additional assumptions. The assumptions needed are:

(1) block effects are additive

(2) The scale of measurement is at least interval.

Using $X_{ij} = (X_{ij}^1, X_{ij}^2, \ldots, X_{ij}^p)'$ to represent the p-variate observation in the i^{th} block receiving the j^{th} treatment, then the p-variate aligned observation Y_{ij} is $Y_{ij} = X_{ij} - \overline{X}_{i\cdot}$ where $Y_{ij} = (Y_{ij}^1, Y_{ij}^2, \ldots, Y_{ij}^p)'$ and

$$Y_{ij}^s = X_{ij}^s - \frac{1}{t} \sum_{j=1}^{t} X_{ij}^s \quad \text{for} \quad s = 1, 2, \ldots, p .$$

Next, ranks are assigned for each variable as in the univariate case; that is, r_{ij}^s is the rank assigned to Y_{ij}^s among the $N = bt$ observations Y_{ij}^s with $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, t$ for each fixed $s = 1, 2, \ldots, p$.

Test statistic:

Let $T_j^s = \frac{1}{b(N+1)} \sum_{i=1}^{b} r_{ij}^s$ denote the weighted mean over the blocks for the s^{th} variable and the j^{th} treatment and let V be the $p \times p$ matrix whose (s, s') element is

$$v(s, s') = \frac{1}{b(t-1)(N+1)^2} \sum_{i=1}^{b} \sum_{j=1}^{t} \left(r_{ij}^s - \frac{1}{t} \sum_{j=1}^{t} r_{ij}^s \right) \left(r_{ij}^{s'} - \frac{1}{t} \sum_{j=1}^{t} r_{ij}^{s'} \right) .$$

Also denote the (s, s') element of V^{-1} by $v^{-1}(s, s')$ then the test statistic is

$$S = b \sum_{s=1}^{p} \sum_{s'=1}^{p} v^{-1}(s, s') \sum_{j=1}^{t} \left(T_j^s - \frac{1}{2} \right) \left(T_j^{s'} - \frac{1}{2} \right) .$$

Decision rule:

Reject H_0 at the α level of significance if S exceed S_α where S_α is the $1-\alpha$ quantile of the conditional distribution of S when b is small and S_α is the $1-\alpha$ quantile of the chi-square random variable with $p(t-1)$ degrees of freedom when b is large [17].

Example:

Again look at a test for the main effect of A in the 3×3 factorial experiment from Pearce [16] which was discussed in the previous two sections. Table IX gives the aligned observations and the ranks assigned to the data.

Computations from Table IX give $T_1^1 = .730769$, $T_2^2 = .423077$, $T_2^3 = .634615$, $T_3^1 = .346154$, $T_3^2 = .269231$, $T_1^2 = .807692$, $T_1^3 = .394231$, $T_2^1 = .423077$, $T_3^3 = .471154$ and

$$V = \frac{1}{(4)(2)(13)^2} \begin{bmatrix} 118.6667 & 92.6665 & -61.1662 \\ 92.6665 & 127.9983 & -27.4995 \\ -61.1662 & -27.4995 & 117.3329 \end{bmatrix}.$$

Further computation gives

$$\frac{1}{(4)(2)(13)^2} V^{-1} = \begin{bmatrix} .027549 & -1.017775 & .010202 \\ -.017775 & .019667 & -.004646 \\ .010202 & -.004646 & .012753 \end{bmatrix}$$

and $S = 7.6415$.

TABLE IX

ALIGNED OBSERVATIONS ON THE PEA LEAVES

Block	Variable	Treatments					
		a_1		a_2		a_3	
		Obs'n	Rank	Obs'n	Rank	Obs'n	Rank
1	b_1	1.5889	12	- .8111	3	- .7111	4
	b_2	.1889	10	-1.4111	6	-1.5111	5
	b_3	-.3111	2	1.2889	10	1.6889	12
2	b_1	1.2556	9	-1.1444	2	1.1556	8
	b_2	.4556	11	-2.0444	1	-1.8444	2
	b_3	1.1112	9	1.3556	11	.1556	3.5
3	b_1	.9556	6	-1.0556	7	-1.6444	1
	b_2	1.1556	12	-1.1444	8	-1.7444	3
	b_3	.1556	3.5	.3556	5	.8556	8
4	b_1	1.4889	11	1.3889	10	- .5111	5
	b_2	-.3111	9	-1.2111	7	-1.6111	4
	b_3	.4889	6	.7889	7	- .5111	1

The chi-square approximation with 6 degrees of freedom gives $\hat{\alpha} > .25$ so we would not reject H_0 at the levels of significance usually quoted.

Theory:

The theory for the multivariate case is similar to the univariate case. It differs in that the conditional distribution involves $(t!)^b$ points instead of $t!$ points which of course increase the amount of computation needed to calculate the exact conditional distribution of S. The limiting distribution of S is again the practical distribution to use and is the distribution of the chi-square random variable with $p(t-1)$ degrees of freedom [17].

CHAPTER V

SOME STATISTICS FOR THE MIXED MODEL

Most disciplines have many experiments which involve subjects being treated by several distinct treatments. Quite often we may consider the treatments fixed and the subjects random. The analysis of this type experiment leads to four cases depending on whether or not we have the following two assumptions [12].

A_1 : The 'additivity' of subject effects.

A_2 : The 'compound symmetry' of the error vectors.

Statistics which may be used in an analysis of two of these cases have been discussed in Chapter IV. Statistics which may be used in the other two cases will be discussed here.

Data:

The data consists of a sample of size n from a p-variate random vector. Denote the n p-variate sample points by X_i for $i = 1, 2, \ldots, n$ where $X_i = (X_{i1}, X_{i2}, \ldots, X_{ip})'$. Let $F_i(X, \Omega_i)$ denote the distribution function of X_i and suppose $\Omega_i = (\omega_{i1}, \omega_{i2}, \ldots, \omega_{ip})'$ is a vector of location parameters with $\omega_{ij} = \beta_i + \tau_j$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p$ where β_i is the effect of the i^{th} subject and τ_j is the effect of the j^{th} treatment.

Assumptions:

(1) The $F_i(X)$ are continuous for each i,

(2) The X_i are independent random vectors,

(3) The joint distribution of any linearly independent set of contrasts among the observations on any particular subject is diagonally symmetric.

(4) The scale of measurement is at least ordinal.

Hypothesis:

H_0: $\tau_1 = \tau_2 = , \ldots , = \tau_p = 0$

H_1: Some τ is not equal to zero.

The test statistic and the decision rule depend on whether or not the additional assumptions A_1 and A_2 are valid. For this reason, test statistics will be given separately for the four cases.

Case I: Assume A_2 holds and A_1 does not hold. The test statistic is the Friedman Statistic which (with a decision rule) is discussed in Conover [8].

Case II: Assume A_2 and A_1 both hold. The "ranking after alignment procedure" is a proper procedure to use and the test statistic S (with a decision rule) was discussed in Chapter IV, section I.

Case III: Assume that neither A_2 nor A_1 hold. Assign the rank r_{ij} to the observation X_{ij} where the ranks are assigned over the variables within the subjects; that is, r_{ij} denotes the rank assigned to X_{ij} among the observations $X_{i1}, X_{i2}, \ldots, X_{ip}$ for each fixed i. Let R be the $n \times p$ matrix of ranks whose (i,j) element is r_{ij} and let C be a $p-1 \times p$ matrix of constants such that $C J_1^p = \phi$. Let $T_j = \frac{1}{n} \sum_{i=1}^{n} r_{ij}$ denote the average rank for the j^{th} treatment with $T = (T_1, T_2, \ldots, T_p)'$. Also let

$$v_{jk} = \frac{1}{n^2} \sum_{i=1}^{n} \left(r_{ij} - \frac{p+1}{2}\right)\left(r_{ik} - \frac{p+1}{2}\right)$$

be the (j,k) element of the $p \times p$ matrix V. The test statistic is

$$W = T'C'(CVC')^{-1} CT .$$

It should be noted that $n^2 CVC' = (RC')' RC' = CR'RC'$. This relationship will frequently facilitate computations. Similarly, $nCT = CR' J_1^n$.

Decision rule:

For Case III as in previous examples the conditional distribution for W has 2^n (not necessarily distinct) realizations and is laborious to calculate. For this reason the asymptotic theory is used and H_0 is rejected at the α level of significance if W exceeds the $1-\alpha$ quantile of the distribution of the chi-square random variable with $p-1$ degrees of freedom [12].

Case IV: Assume A_1 holds but A_2 does not hold. The test procedure is a generalization of the Wilcoxon signed rank test where we use certain contrasts of the variables as aligned variables. Calculate all possible within subject differences $u_{ijk} = X_{ij} - X_{ik}$ for

$i = 1, 2, \ldots, n$ and $j, k = 1, 2, \ldots, p$. Then assign the rank r_{ijk} to u_{ijk} where ranks are assigned to $|u_{ijk}|$ among $|u_{1jk}|, |u_{2jk}|, \ldots, |u_{njk}|$ for $j \neq k = 1, 2, \ldots, p$. Let

$$s_{ij} = \sum_{k=1}^{p} r_{ijk} Z_{ijk}$$

be the (i, j) element of the $n \times p$ matrix S of scores assigned where

$$Z_{ijk} = \begin{cases} 1 & \text{if } u_{ijk} > 0 \\ 0 & \text{if } u_{ijk} = 0 \\ -1 & \text{if } u_{ijk} < 0 . \end{cases}$$

Also let $T_j = \frac{1}{n} \sum_{i=1}^{n} s_{ij}$, $T = (T_1, T_2, \ldots, T_p)'$ and let

$$v^*(j, k) = \frac{1}{n^2} \sum_{i=1}^{n} s_{ij} s_{ik}$$

be the (j, k) element of the $p \times p$ matrix V^*. One should note that it is necessary to calculate u_{ijk} only for $i = 1, 2, \ldots, n$ and $1 \leq j < k \leq p$ since $r_{ijk} = r_{ikj}$ and $z_{ijk} = -z_{ikj}$. The test statistic is

$$W^* = T'C'(C V^* C')^{-1} CT$$

where C is defined as in Case III. Note that W^* is a quadratic form in Wilcoxon signed rank statistics. As will be illustrated shortly by example the relationships $n^2 C V^* C' = C S' S C'$ and $nCT = C S' J_1^n$ will usually facilitate computations.

Decision rule for Case IV: Again the limiting distribution is used and H_0 is rejected at the α level of significance if W^* exceeds the $1 - \alpha$ quantile of the distribution of the chi-square random variable with $p - 1$ degrees of freedom [12].

Example:

Clark and Schkade [4] suggest an experiment to study the rate of arrivals of automobiles at 4 particular toll stations. The number of automobiles arriving at the four toll stations in a 4 hour time period (8:00 a.m. - 12:00 p.m.) for each of six days is recorded in Table X.

TABLE X

NUMBER OF ARRIVALS AT THE GATES AND THE RANKS ASSIGNED

Days	Gates							
	1		2		3		4	
	Obs'n	Rank	Obs'n	Rank	Obs'n	Rank	Obs'n	Rank
1	490	2	525	3	475	1	527	4
2	450	1	506	3	460	2	507	4
3	510	3	473	1	525	4	492	2
4	478	2	526	4	420	1	505	3
5	504	3	502	2	499	1	530	4
6	482	2	505	3	472	1	555	4

TABLE XI

DIFFERENCES OF THE RANKS ASSIGNED TO THE NUMBER OF ARRIVALS AT THE GATES

Days	$G_1 - G_2$	$G_1 - G_3$	$G_1 - G_4$
1	-1	1	-2
2	-2	-1	-3
3	2	-1	1
4	-2	1	-1
5	1	2	-1
6	-1	1	-2

Computation of W (the test statistic for Case III) will be illustrated for this example.

The entries under "Rank" in Table X give the 6×4 matrix R of ranks assigned to the data where ranks are assigned within each day. The column headings in Table XI result if and only if the matrix C is defined as

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

and the body of the table viewed as a matrix gives the 6×3 matrix RC' of differences in ranks. With RC' as defined in Table XI

$$36\ C\,V\,C' = C\,R'R\,C' = \begin{bmatrix} 15 & -2 & 13 \\ -2 & 9 & -5 \\ 13 & -5 & 20 \end{bmatrix}$$

and from the column totals in Table XI $6(C\,T)' = J_6^1\,R\,C' = (-3, 3, -8)$. From these we obtain

$$\frac{1}{36}\,(C\,V\,C')^{-1} = \begin{bmatrix} .15752 & -.02541 & -.10874 \\ -.02541 & .13313 & .04980 \\ -.10874 & .04980 & .13313 \end{bmatrix}$$

and $W = 3.9837$. Note that it is not necessary to calculate T or V explicitly. Using the chi-square approximation with 3 degrees of freedom indicates that $\hat{\alpha} > .25$. This gives little evidence that the rate of arrivals at the four gates are different.

Example:

To illustrate the computations for the statistic W^* (Case IV) an experiment discussed by Winer [21] to study the effect of four drugs upon reaction time to a series of standardized tasks will be used. The scores are mean reaction time on the series of tasks and are given in Table XII.

To compute the statistic W^* the within subject differences are calculated and signed ranks assigned to the data by assigning ranks across subjects within each difference; that is, assign ranks within each column of differences in the same manner as one assigns ranks in the Wilcoxon Signed Rank Test. The differences are given in Table XIII and the signed ranks are given in Table XIV.

TABLE XII

MEAN REACTION TIME ON THE SERIES OF TASKS

Persons \ Drugs	1	2	3	4
1	30	28	16	34
2	14	18	10	22
3	24	20	18	30
4	38	34	20	44
5	26	28	14	30

TABLE XIII

DIFFERENCES IN THE MEAN REACTION TIMES

Persons	$D_1 - D_2$	$D_1 - D_3$	$D_1 - D_4$	$D_2 - D_3$	$D_2 - D_4$	$D_3 - D_4$
1	2	14	-4	12	-6	-18
2	-4	4	-8	8	-4	-12
3	4	6	-6	2	-10	-12
4	4	18	-6	14	-10	-24
5	-2	12	-4	14	-2	-16

TABLE XIV

SIGNED RANKS

Persons	$D_1 - D_2$	$D_1 - D_3$	$D_1 - D_4$	$D_2 - D_3$	$D_2 - D_4$	$D_3 - D_4$
1	1.5	4	-1.5	3	-3	-4
2	-4	1	-5	2	-2	-1.5
3	+4	2	-3.5	1	-4.5	-1.5
4	4	5	-3.5	4.5	-4.5	-5
5	-1.5	3	-1.5	4.5	-1	-3

A score s_{ij} (Table XV) is now assigned for each person and each drug from Table XIV by adding the ranks assigned where the drug is the minuend and subtracting the ranks assigned where the drug is the subtrahend in the columns of Table XIV. For example, for person 1, $D_1 = 1.5 + 4 - 1.5 = 4$, $D_2 = -1.5 + 3 - 3 = -1.5$, $D_3 = -4 - 3 - 4 = -11$, and $D_4 = -(-1.5) - (-3) - (-4) = 8.5$. The matrix S of scores is given by Table XV.

The statistic W^* is now calculated in the same manner as W in Case III where the scores assigned in Table XV constitute the 5×4 matrix S which now plays the same role as the matrix R played in Case III. This indicates that we need to calculate the matrix SC' of differences of the scores assigned to each drug for each person. This 5×3 matrix is displayed as the body of Table XVI where C is the same as in the previous example (note column headings in Table XVI).

TABLE XV

SCORES FOR THE DRUGS

Persons	D_1	D_2	D_3	D_4
1	4.0	-1.5	-11.0	8.5
2	-8.0	4.0	- 4.5	8.5
3	2.5	-7.5	- 4.5	9.5
4	5.5	-4.0	-14.5	13.0
5	0.0	5.0	-10.5	5.5

TABLE XVI

DIFFERENCES OF THE SCORES

Persons	$D_1 - D_2$	$D_1 - D_3$	$D_1 - D_4$
1	5.5	15.0	- 4.5
2	-12.0	- 3.5	-16.5
3	10.0	7.0	- 7.0
4	9.5	20.0	- 7.5
5	- 5.0	10.5	- 5.5

From Table XVI we can calculate directly

$$25\, C V^* C' = C S' S C' = \begin{bmatrix} 389.50 & 332.00 & 59.50 \\ 332.00 & 795.50 & -266.50 \\ 59.50 & -266.50 & 428.00 \end{bmatrix}$$

and $5(T\,C)' = J_5^1 S C' = (8, 49, -41)$ which gives

$$\frac{1}{25}\,(CV^* C')^{-1} = \begin{bmatrix} .00605 & -.00354 & -.00305 \\ -.00354 & .00366 & .00277 \\ -.00305 & .00277 & .00449 \end{bmatrix}$$

and $W^* = 3.81777$. If we again use the chi-square approximation with 3 degrees of freedom, W^* gives $\hat{\alpha} > .25$ which again gives little evidence against the null hypothesis. From this we would conclude that the effects of the four drugs are the same.

CHAPTER VI

FACTORIAL EXPERIMENTS

2^m Factorial Experiment in b Blocks

The general setting is the 2^m factorial arrangement of treatments replicated in $b \geq 2$ complete blocks.

Data:

For the sake of communication let X_{iL} denote the observation in block i under treatment combination L where $L = (\ell_1, \ell_2, \ldots, \ell_m)$ and $\ell_j = 1, 2$ denotes the level of factor j for $j = 1, 2, \ldots, m$. Also let τ_t denote the main effect or interaction effect given by the vector $t = (t_1, t_2, \ldots, t_m)$ where $t_j = 0, 1$ denotes the absence or presence, respectively, of factor j for $j = 1, 2, \ldots, m$ with the understanding that $\tau_t = 0$ for $t = \phi$. For example if $m = 3$, treatment combination $L = (2, 1, 2)$ would indicate presence of the high level of the first factor, low level of the second factor and high level of the third factor. Similarly $\tau_{(0,1,0)}$ would represent the main effect of the second factor and $\tau_{(1,0,1)}$ would represent the interaction effect of the first and third factors. One can also remember that $\tau_{(0,0,0)} = 0$. In addition, let S denote the set containing the 2^m vectors which are the possible values of the vector t and let β_i denote the effect of the i^{th} block for $i = 1, 2, \ldots, b$. The response of the plot in the i^{th} block receiving treatment combination L may be represented by

$X_{iL} = \beta_i + \frac{1}{2} \Sigma_S (-1)^{Lt'} \tau_t + \epsilon_{iL}$ where ϵ_{iL} is an error variable for treatment combination L in the i^{th} block. The various statistics used are based on the aligned observations [18]

$$Y_{i,t} = 2^{-(m-1)} \sum_W (-1)^{Lt'} X_{iL} \quad \text{for} \quad i = 1, 2, \ldots, b ,$$

where W is the set containing the 2^m vectors which are the possible treatment combinations; that is, the possible values of L.

Assumptions:

(1) The ϵ_{iL} for all $L \in W$ have jointly a continuous cumulative distribution function G_i for each $i = 1, 2, \ldots, b$.

(2) G_i is symmetric in its 2^m arguments.

(3) The b sets of 2^m within block errors are independent.

(4) The treatment effects are additive.

(5) The scale of measurement is at least interval.

Hypotheses:

Several different hypotheses may be tested, however with some additional notation they can be written as one statement. If we let $S^* = \{t : t \in S, \; t \neq \emptyset\}$ be the set of non-zero elements in S and let P be any non-empty subset of S^* then the hypotheses may be written as:

$H_{0,P}: \tau_t = 0$ for all $t \in P$

$H_{1,P}: \tau_t \neq 0$ for some $t \in P$.

Test statistic:

The test statistic will be given using different cases which will depend on the nature of the set P.

Case I: Suppose P contains a single element t.

Test statistic:

For small b the Wilcoxon signed rank statistic is used; that is, we assign the rank $r_{(i,t)}$ to $|Y_{i,t}|$, ranking the $|Y_{i,t}|$ among the magnitudes $|Y_{1,t}|, |Y_{2,t}|, \ldots, |Y_{b,t}|$ for each $i = 1, 2, \ldots, b$. Then the test statistic is

$$T(t) = \sum_{i=1}^{b} r_{(i,t)} \, c_{(i,t)}$$

where

$$c_{(i,t)} = \begin{cases} 1 & \text{if } Y_{i,t} > 0 \\ 0 & \text{if } Y_{i,t} \leq 0 . \end{cases}$$

For larger b the test statistic is

$$Z(t) = \frac{T(t) - \frac{b(b+1)}{4}}{\sqrt{\frac{b(b+1)(2b+1)}{24}}} .$$

Decision rule:

For small b, reject H_0 at the α level of significance if $T > T_{1-(\alpha/2)}$ or if $T < T_{(\alpha/2)}$ where $T_{1-(\alpha/2)}$ and $T_{(\alpha/2)}$ are the $1-(\alpha/2)$ and $\alpha/2$ quantiles, respectively, from the distribution for the Wilcoxon signed rank statistic. For large b, reject H_0 at the α level of significance if $|Z(t)|$ exceeds $Z_{(\alpha/2)}$ where $Z_{(\alpha/2)}$ is the $1-(\alpha/2)$ quantile from the standard normal distribution.

Case II: P contains $n \geq 2$ distinct points of S^*; that is, $P = \{t_1, t_2, \ldots, t_n\}$. If P contains more than 2 distinct points the additional assumption that the joint distribution of the errors $\epsilon_{i,L}$, $L \epsilon W$, is not only symmetric in the 2^m arguments but is also diagonally symmetric about zero is needed. Let

$$Q(t) = \frac{1}{b(b+1)} \sum_{i=1}^{b} r_{(i,t)} \left(c_{(i,t)} - \frac{1}{2}\right) = \frac{1}{b(b+1)} T(t) - \frac{1}{4},$$

then for small b, the test statistic is the multivariate signed rank statistic $W(P) = b\,\overline{Q(t)}'\, V^* \,\overline{Q(t)}$ where $\overline{Q(t)} = \left[Q(t_1), Q(t_2), \ldots, Q(t_n)\right]'$ and V^* is the generalized inverse of the $n \times n$ matrix V whose (j,k) element

$$v(j,k) = \frac{1}{b(b+1)^2} \sum_{i=1}^{b} r_{(i,t_j)}\, r_{(i,t_k)} \left(c_{ij} - \frac{1}{2}\right)\left(c_{ik} - \frac{1}{2}\right)$$

for $j,k = 1,2,\ldots,n$. For large b, the test statistic is

$$W^*(P) = \frac{24\,b(b+1)}{2b+1} \sum_{j=1}^{n} [Q(t_j)]^2$$

which is easier to compute.

Decision rule:

For small b, it is necessary to use the conditional distribution discussed in Chapter II. For large b, reject H_0 at the α level of significance if $W^*(P)$ exceeds the $1-\alpha$ quantile of a chi-square random variable with n degrees of freedom.

Example:

Snedecor [19] reports the data from an unpublished randomized block experiment which was used to learn the effect of two supplements to a corn ration for feeding pigs. The treatments had three factors; Lysine at 2 levels, Soybean meal at two levels and sex of pigs at two levels. The data is reported in Table XVII. The number 1 is used to indicate the low level of a factor and a 2 to indicate the high level of a factor. Ordered three-tuples then give the treatment combinations. First we will consider the situation where P has a single element and test

$$H_{0P}: \tau_t = 0$$

$$H_{1P}: \tau_t \neq 0$$

for each of the seven possible singleton sets P. The necessary calculations are given in the tables below.

TABLE XVII

AVERAGE DAILY GAINS OF SWINE

Blocks	Treatments							
	(1, 1, 1)	(1, 1, 2)	(1, 2, 1)	(1, 2, 2)	(2, 1, 1)	(2, 1, 2)	(2, 2, 1)	(2, 2, 2)
1	1.11	1.03	1.52	1.48	1.22	.87	1.38	1.09
2	.97	.97	1.45	1.22	1.13	1.00	1.08	1.09
3	1.09	.99	1.27	1.53	1.34	1.16	1.40	1.47
4	.99	.99	1.22	1.19	1.41	1.29	1.21	1.43
5	.85	.99	1.67	1.16	1.34	1.00	1.46	1.24
6	1.21	1.21	1.24	1.57	1.19	1.14	1.39	1.17
7	1.29	1.19	1.34	1.13	1.25	1.36	1.17	1.01
8	.96	1.24	1.32	1.43	1.32	1.32	1.21	1.13

TABLE XVIII

CALCULATIONS FOR TESTING SINGLETON SETS P

P / Blocks	{(1, 0, 0)}			{(1, 0, 1)}			{(1, 1, 0)}			{(1, 1, 1)}			{(0, 1, 0)}			{(0, 1, 1)}			{(0, 0, 1)}		
	$4T_{it}$	c_{it}	r_{it}	$4T_{it}$	c_{it}	r_{it}	$4T_{it}$	c_{it}	r_{it}	$4T_{it}$	c_{it}	r_{it}	$4T_{it}$	c_{it}	r_{it}	$4T_{it}$	c_{it}	r_{it}	$4T_{it}$	c_{it}	r_{it}
1	.58	1	7	-.52	0	7	-.48	0	4	.02	1	1	1.24	1	7	.10	1	2	-.76	0	7
2	.31	1	3	.11	1	1	-.69	0	7	.37	1	5.5	.77	1	5	-.09	0	1	-.35	0	5
3	-.49	0	6	-.27	0	5	-.35	0	2	-.11	0	3	1.09	1	6	.61	1	8	.05	1	1
4	-.95	0	8	.13	1	2	-.49	0	5	.37	1	5.5	.37	1	2	.31	1	5	.07	1	3
5	-.37	0	5	-.19	0	3	-.63	0	6	.77	1	8	1.35	1	8	-.53	0	7	-.93	0	8
6	.34	1	4	-.60	0	8	-.16	0	1	-.50	0	7	.62	1	4	.16	1	3	.06	1	2
7	.16	1	2	.26	1	4	-.42	0	3	-.16	0	4	-.44	0	3	-.38	0	6	-.36	0	6
8	-.03	0	1	-.47	0	6	-.85	0	8	.09	1	2	.25	1	1	-.25	0	4	.31	1	4
T(r)	16			7			0			22			33			18			10		
Ω(r)	.02778			-.15278			-.25			.05556			.20833			0			-.11111		
$\hat{\alpha}$	.8438			.1484			.0078			.6406			.0390			1.00			.3126		

Entries in Table XVIII are calculated using Table XVII. From the observed significance levels in Table XVIII we see that the interaction of Lysine and soybean meal is significant at the .01 level and that the main effect of soybean meal is significant at the .05 level.

Suppose we want to test the hypothesis that the effect soybeans was zero; that is,

$$H_0: \tau_{(0,1,0)} = \tau_{(1,1,0)} = \tau_{(1,1,1)} = \tau_{(0,1,1)} = 0$$

against

$$H_1: \text{not } H_0.$$

To test this hypothesis we have $P = \{(0,1,0), (1,1,0), (1,1,1), (0,1,1)\}$ and we calculate

$$W^*(P) = \frac{24(8)(9)}{17} \sum_{j=1}^{4} [Q(t_j)]^2 = 11.078 .$$

The chi-square approximation $(df = 4)$ gives $.025 < \alpha < .05$ so the hypothesis would be rejected at the .05 level of significance but not at the .01 level.

Theory:

The distribution of the statistic T has been tabulated quite extensively by McCormack [13], Wilcoxon, Katti and Wilcox [20] and others which are referenced by these two publications. Approximations to this distribution of probabilities are given by Claypool [5] and Claypool and Holbert [6]. The asymptotic relative efficiency of $W(t)$ is the same as that of the Wilcoxon signed-rank test with respect to the Student t-test and is discussed at some length by Conover [8]. The conditional distribution of $W(P)$ has been discussed in the section on the Multivariate signed-rank statistic in Chapter II. Sen [18] states that both the conditional distribution of W(P) and the conditional distribution of $W^*(P)$ are asymptotically the distribution of the chi-square random variable with n degrees of freedom and that the asymptotic relative efficiency of each relative to the parametric variance ratio test is the same as the asymptotic relative efficiency of W(t) mentioned above. At the same time the two statistics W and W^* are quite different for small b; however, a meaningful comparison of them would be expensive to conduct due to the large number of points involved in their conditional distributions.

Sen [18] discusses using the statistics based on T to estimate main effects and interactions and also discusses extensions of the tests to cover confounded or partially confounded designs. It is also of interest to note that the parametric procedures suggested by Cochran and Cox [7] are based on the aligned observations used in this procedure.

Testing for Interaction in Factorial Experiments

The general setting considered is the 2-factor factorial arrangement of treatments in a randomized complete block design with one observation per cell. Denote the response in the i^{th} block receiving the j^{th} level of the first factor A and the k^{th} level of the second factor B as X_{ijk} and assume the model

$$X_{ijk} = \mu + \beta_i + \gamma_j + \delta_k + \pi_{jk} + \epsilon_{ijk}$$

for $1 \leq i \leq n$, $1 \leq j \leq a$, $1 \leq k \leq b$ and $N = nab$; where μ, the β_i, γ_j, δ_k, π_{jk}, and ϵ_{ijk} represent the overall mean, the effect of block i, the effect of the j^{th} level of factor A, the effect of the k^{th} level of factor B, the effect of the AB interaction due to the j^{th} level of factor A and the k^{th} level of factor B and the residual error, respectively. Also assume

$$\sum_{i=1}^{n} \beta_i = \sum_{j=1}^{a} \gamma_j = \sum_{j=1}^{b} \delta_k = \sum_{j=1}^{a} \pi_{jk} = \sum_{k=1}^{b} \pi_{jk} = 0 .$$

The general idea of the tests is to align the observations to eliminate all effects except π_{jk} and then use a statistic based on the ranks assigned to the aligned observations to test $H_0 : \pi_{jk} = 0$ for each $j = 1, 2, \ldots, a$, $k = 1, 2, \ldots, b$ against the alternative $H_1 : \pi_{jk} \neq 0$ for some j and k.

Data:

The data consists of n independent random vectors $(X_{i11}, X_{i12}, \ldots, X_{iab})$ with $i = 1, 2, \ldots, n$. In the following we will let a subscript $\cdot$ indicate the sum over the variable replaced by $\cdot$; that is, $X_{i \cdot k} = \sum_{j=1}^{a} X_{ijk}$ and let

$$X_i = \begin{bmatrix} X_{i11} & X_{i12} & \cdots & X_{i1b} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ X_{ia1} & & \cdots & X_{iab} \end{bmatrix} .$$

Also let Z_i be the $a \times b$ matrix of aligned observations,

$$Z_i = \left(I_a - \frac{1}{a} J_a^a\right) X_i \left(I_b - \frac{1}{b} J_b^b\right),$$

with the (j, k) element denoted by Z_{ijk} where J_c^r is an $r \times c$ matrix with each element having the value 1. One can note that the elements of Z_i are the contrasts used to calculate the sum of squares for interaction in the univariate parametric analysis of variance for a 2-factor experiment [21], although they are not visible in the usual formulas used in the computation.

Assumptions:

(1) $\epsilon_i = (\epsilon_{i11}, \epsilon_{i12}, \ldots, \epsilon_{iab})$ for $i = 1, 2, \ldots, n$ are n independent random vectors.

(2) The distribution function $G_i(X)$ of ϵ_i is continuous for each $i = 1, 2, \ldots, n$.

(3) G_i is symmetric in its arguments for each $i = 1, 2, \ldots, n$.

(4) The scale of measurement is at least interval.

Hypotheses:

H_0: $\pi_{jk} = 0$ for all $j = 1, 2, \ldots, a$, and all $k = 1, 2, \ldots, b$.

H_1: $\pi_{jk} \neq 0$ for some i and j.

Test statistic:

The test statistic is given by cases for three different cases.

Case I: $a = b = 2$ when $a = b = 2$ we have

$Z_{i11} = -Z_{i21} = -Z_{i12} = Z_{i22}$ for each $i = 1, 2, \ldots, n$ so we may use a one-sample location test based on Z_{i11} for $i = 1, 2, \ldots, n$. The test statistic is the univariate Wilcoxon signed-rank statistic discussed by Conover [8].

Case II: $a > 2$, $b = 2$

When $a > 2$, $b = 2$ we have $Z_{ij1} = Z_{ij2}$ for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, a$, and our test can be based on the vectors $(Z_{i11}, Z_{i21}, Z_{i31} \ldots Z_{ia1})$ for $i = 1, 2, \ldots, n$. The test statistic is S_N which was discussed in Chapter IV. It is based on the na aligned observations Z_{ij1} where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, a$.

Example (Case II):

For a hypothetical context suppose it is desired to test the effect of two fertilizers (B) on three varieties (A) of wheat. One acre plots were harvested and results were recorded in bushels per acre (Table XIX).

From Table XIX we have

$$X_1 = \begin{pmatrix} 35 & 39 \\ 29 & 39 \\ 33 & 43 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 36 & 40 \\ 31 & 39 \\ 41 & 49 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 46 & 50 \\ 55 & 62 \\ 57 & 69 \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad 3Z_2 = \begin{pmatrix} 4 & -4 \\ 3 & -3 \\ -2 & 2 \end{pmatrix},$$

and

$$6Z_3 = \begin{pmatrix} 11 & -11 \\ 1 & -1 \\ -2 & 2 \end{pmatrix}$$

so $30T = (17, 14, 14)$, $\sigma^2 = \frac{13}{900}$ and $S_N = 1.384$. The chi-square approximation $(df = 1)$ gives $.10 < \hat{\alpha} < .25$ which shows little evidence of interaction between varieties and fertilizers.

TABLE XIX

WHEAT YIELDS IN BUSHELS PER ACRE

Blocks \ Varieties	a_0		a_1		a_2	
1	35	39	29	39	33	43
2	36	40	31	39	41	40
3	46	50	55	52	57	69
Fertilizers	b_0	b_1	b_0	b_1	b_0	b_1

Case III: $a \geq 3$, $b \geq 3$.

Test statistic:

Assign the rank r_{ijk} to the aligned observation Z_{ijk} in the combined ranking of all N aligned observations. Let

$$\bar{r}_{\cdot jk} = \frac{1}{n} \sum_{i=1}^{n} r_{ijk} ,$$

$$R_{jk} = \frac{1}{(N+1)n} \sum_{i=1}^{n} \left(r_{ijk} - \bar{r}_{i \cdot k} - \bar{r}_{ij\cdot} + \bar{r}_{i\cdot\cdot} \right) ,$$

$$T = (R_{11}\ R_{12}, \ldots, R_{ab})$$

and

$$\sigma^2 = \frac{(N+1)^2}{n(a-1)(b-1)} \sum_{i=1}^{n} \sum_{j=1}^{a} \sum_{k=1}^{b} \left(r_{ijk} - \bar{r}_{i \cdot k} - \bar{r}_{ij\cdot} + \bar{r}_{i\cdot\cdot} \right)^2 .$$

The test statistic is

$$L = \frac{n}{\sigma^2} \sum_{j=1}^{a} \sum_{k=1}^{b} (R_{jk})^2 .$$

Rejection rule:

Reject H_0 at the α level of significance if L exceeds the $1-\alpha$ quantile of the distribution of the chi-square random variable with $(a-1)(b-1)$ degrees of freedom. The distribution of the chi-square random variable is used since it is the limiting distribution of L and the exact permutation distribution of L is laborious to compute since it involves the discrete uniform distribution on $(a!b!)^n$ points [17].

Example:

The example by Pearce (Table VIII) which was used to illustrate the use of the Multivariate Friedman test (a test for main effects) will be used to illustrate the test for interaction. Corresponding to the notation used in this section we have

$$X_1 = \begin{pmatrix} 9.0 & 7.6 & 7.1 \\ 6.6 & 6.0 & 8.7 \\ 6.7 & 5.9 & 9.1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 8.9 & 8.1 & 8.3 \\ 6.5 & 5.6 & 9.0 \\ 8.8 & 5.8 & 7.8 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 9.1 & 9.3 & 8.3 \\ 9.2 & 7.0 & 8.5 \\ 6.5 & 6.4 & 9.0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 9.0 & 7.2 & 8.0 \\ 8.9 & 6.3 & 8.3 \\ 7.0 & 5.9 & 7.0 \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1.0778 & 0.6111 & -1.6889 \\ -.5222 & -.1889 & .7111 \\ -.5556 & -.4222 & .9778 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} .0444 & .8111 & -.8556 \\ -.9556 & -.2889 & 1.2444 \\ .9111 & -.5222 & -.3889 \end{pmatrix},$$

$$Z_3 = \begin{pmatrix} .0778 & .9778 & -1.0556 \\ .8444 & -.6556 & -.1889 \\ -.9222 & -.3222 & 1.2444 \end{pmatrix},$$

and

$$Z_4 = \begin{pmatrix} .1444 & .1778 & -.3222 \\ .2778 & -.4889 & .2111 \\ -.4222 & .3111 & .1111 \end{pmatrix}.$$

In addition, let R_i be the $a \times b$ matrix of rank whose (j, k) element is r_{ijk} then

$$R_1 = \begin{pmatrix} 34.0 & 27.0 & 1.0 \\ 8.5 & 17.5 & 28.0 \\ 7.0 & 11.5 & 32.5 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 19.0 & 29.0 & 5.0 \\ 3.0 & 16.0 & 35.5 \\ 31.0 & 8.5 & 13.0 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 20.0 & 32.5 & 2.0 \\ 30.0 & 6.0 & 17.5 \\ 4.0 & 14.5 & 32.5 \end{pmatrix}, \qquad R_4 = \begin{pmatrix} 22.0 & 23.0 & 14.5 \\ 25.0 & 10.0 & 24.0 \\ 11.5 & 26.0 & 21.0 \end{pmatrix},$$

$(4 \cdot 37)T = (21, 35.3333, 56.3333, -4.8333, -24, 28.8333, -16.6666, -11.3333, 27.5)$, $\sigma^2 = .174179$ and $L = 7.8172$. The chi-square approximation $(df = 4)$ shows $.05 < \hat{\alpha} < .10$, so H_0 would not be rejected at the .05 level of significance.

Theory:

Let $E_i = (I_a - \frac{1}{a} J_a^a)\ \epsilon_i (I_b - \frac{1}{b} J_b^b)$ where ϵ_i is the a by b matrix with (j, k) element ϵ_{ijk}. Let Γ be the a by b matrix with (j, k) element π_{jk}, then considering the aligned observations we have $Z_i = \Gamma + E_i$. The conditional distribution of the test statistic L may be computed by considering a group of transformations on the matrices E_i. This group of transformations has $(a!b!)^n$ points and leads to a discrete uniform distribution on $(a!b!)^n$ points for the conditional distribution of $(Z_1\ Z_2, \ldots, Z_n)$. This makes the distribution of L difficult to compute when a, b, or n is large, but the limiting distribution of L is the distribution of the chi-square random variable with $(a-1)(b-1)$ degrees of freedom [17].

BIBLIOGRAPHY

[1] Bing, A. "Gladiolus Control Experiments." The Gladiolus 1954. Boston: New England Gladiolus Society, Inc., (1954), pp. 14-19.

[2] Chatterjee, S.K. "A Bivariate Sign Test for Location." Annals of Mathematical Statistics, 37 (1966), pp. 1771-1782.

[3] Chatterjee, S.K. and P.K. Sen. "Nonparametric Tests for the Bivariate Location Problem." Calcutta Statistical Association Bulletin, 13 (1964), pp. 18-58.

[4] Clark, Charles T. and L.L. Schkade. Statistical Methods for Business Decisions. Cincinnatti, Ohio: South-Western Publishing Co., 1969.

[5] Claypool, P.L. "Linear Interpolation within McCornack's Table of the Wilcoxon Matched Pair Signed Rank Statistic," Journal of the American Statistical Association, 65 (1970), pp. 974-975.

[6] Claypool, P.L. and Donald Holbert. "Accuracy of Normal and Edgeworth Approximations To the Distribution of the Wilcoxon Signed Rank Statistic." Journal of The American Statistical Association, 69 (1974), in press.

[7] Cochran, W.G. and G.M. Cox. Experimental Design. New York: John Wiley and Sons, Inc., 1957.

[8] Conover, W.J. Practical NonParametric Statistics. New York: John Wiley and Sons, Inc., 1971.

[9] Gerig, Thomas M. "A Multivariate Extension of Friedman's χ^2-test." Journal of the American Statistical Association, 64 (1969), pp. 1595-1608.

[10] Koch, Gary G. "Some Aspects of the Statistical Analysis of Split Plot Experiments in Completely Randomized Layouts." Journal of the American Statistical Association, 64 (1969), pp. 485-505.

[11] Koch, Gary G. "The Use of Nonparametric Methods in the Statistical Analysis of a Complex Split Plot Experiment." Biometrics, 26 (1970), pp. 105-128.

[12] Koch, Gary G. and P.K. Sen. "Some Aspects of the Statistical Analysis of the 'Mixed Model'." Biometrics, 24 (1968), pp. 27-48.

[13] McCornack, R. L. "Extended Tables of the Wilcoxon Matched Pair Signed Rank Statistics." Journal of the American Statistical Association, 60 (1965), pp. 864-874.

[14] Myers, Jerome L. Fundamentals of Experimental Design. Boston: Allan and Bacon, Inc., 1972.

[15] Ostle, Bernard. Statistics in Research. Ames, Iowa: Iowa State College Press, 1954.

[16] Pearce, S. C. Biological Statistic "An Introduction." New York: McGraw-Hill, 1965.

[17] Puri, M. L. and P.K. Sen. Nonparametric Methods in Multivariate Analysis. New York: John Wiley and Sons, Inc., 1971.

[18] Sen, P.K. "Nonparametric Inference in n Replicated 2^m Factorial Experiments." Annals of the Institute of Statistical Mathematics, 22 (1970), pp. 281-294.

[19] Snedecor, George W. and W. G. Cocran. Statistical Methods. Ames, Iowa: Iowa State Press, 1967.

[20] Wilcoxon, F., S.K. Katti, and R.A. Wilcox. "Critical Values and Probability Levels for the Wilcoxon Rank Sum Test and the Wilcoxon Signed Rank Test." Selected Tables in Mathematical Statistics. 1 (Edited by H. L. Harter and D.B. Owens). Chicago: Markum Publishing Col, (1970), pp. 177-237.

[21] Winer, B.J. Statistical Principles in Experimental Design. New York: McGraw-Hill, 1965.

VITA

Loy Elbert Puffinbarger

Candidate for the Degree of

Doctor of Education

Thesis: SOME NONPARAMETRIC STATISTICAL TESTS

Major Field: Higher Education

Biographical:

Personal Data: Born in Byron, Oklahoma, June 17, 1937, the son of Elmer R. and Jessie Florence Puffinbarger.

Education: Attended Byron Grade school and graduated from Byron-Driftwood High School, Cherokee, Oklahoma in 1955; received the Associate degree from the Pratt Junior College, Pratt, Kansas, 1957; received the Bachelor of Science degree from Northwestern State College, Alva, Oklahoma, 1959; received the Master of Science degree from Oklahoma State University, Stillwater, Oklahoma 1964; completed requirements for the Doctor of Education degree from Oklahoma State University, May, 1974.

Professional Experience: Taught Junior High School mathematics at Woodward Junior High, Woodward, Oklahoma 1959-1964; Instructor and assistant professor of mathematics at Central State University, Edmond, Oklahoma from 1964 to date.