MOMENTS AND THE CHARACTERISTIC FUNCTIONAL IN HILBERT SPACE AND SOME CHARACTERIZATIONS IN PROBABILITY THEORY

By

ROBERT MICHAEL NORTON

Bachelor of Science
Bachelor of Science in Education
Northeast Missouri State University
Kirksville, Missouri
1968

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1971

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
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Thesis Approved:

K. Marski
Thesis Adviser

W. Warde

L. A. Johnson

Bennett Basore

Paul Duvall

D. D. Durham
Dean of the Graduate College

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CHAPTER I

PRELIMINARY CONCEPTS

The purpose of this chapter is to present some of the background and notions which underlie the material in the remaining chapters.

In Chapter II analogs of several well known theorems relating moments and derivatives of characteristic functions are proved in the setting of a real separable Hilbert space. It should be noted that according to Grenander (see [5], p. 28) not much work has been done with moments of order greater than one in abstract spaces. An appropriate notion of moment function is introduced, and the Gateaux derivative is used in order that the theorems may be extended from the finite dimensional case.

The following theorem due to Carleman is also extended to the setting under consideration.

Theorem 1.1 (Carleman) Let the k-dimensional moment problem corresponding to the moments $m_{j_1,j_2,\ldots,j_k} = \int_{R^k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}\, dP(x_1,x_2,\ldots,x_k)$, $j_1,j_2,\ldots,j_k = 0, 1, 2, \ldots,$ have a solution. Let $\lambda_{2n} = m_{2n,0,\ldots,0} + m_{0,2n,0,\ldots,0} + \cdots + m_{0,0,\ldots,2n}$. A sufficient condition for the moment problem to be determined is that

$$\sum_{n=1}^{\infty} \lambda_{2n}^{-1/(2n)} = \infty.$$

This theorem is proved in a distinguished monograph on the moment problem by J. A. Shohat and J. D. Tamarkin [12].

The theorems in Chapter III are related to binomial destruction in nature, a problem considered by C. R. Rao [9]. In the setting under consideration a non-negative integer valued random variable X assumes a value which may then be reduced ("partially damaged" or "destroyed") by some random destructive mechanism S, to yield $S(X)$. As an example, let X denote the number of auto accidents in a town during the month of May and $S(X)$ the number of accidents reported for that month. Given that one knows how the destructive mechanism acts (the distribution of $S(X)$ given X) it is natural to ask what relationships hold between X and $S(X)$, or if certain relationships are known to exist between X and $S(X)$, what the destructive mechanism is.

Rao and H. Rubin [10] proved the following theorem.

<u>Theorem 1.2</u> Let X be a discrete random variable taking the values $0, 1, \ldots,$ and let $P[S(X) = r \mid X = n] = \binom{n}{r} p^r (1-p)^{n-r}$, $r = 0, 1, \ldots, n$. Then

$$P[S(X) = r] = P[S(X) = r \mid S(X) < X] = P[S(X) = r \mid S(X) = X],$$

$r = 0, 1, \ldots,$ if, and only if, X has a Poisson distribution.

Chapter III considers the following setting. Suppose that X is acted upon by a random destructive mechanism yielding the variable S_1, which is acted upon by some destructive mechanism yielding the variable S_2, ... yielding the variable S_t, for some positive integer t. A result similar to the above theorem is proved.

The second part of Chapter III is devoted to a proposition of R. C. Srivastava and A. B. L. Srivastava [13].

<u>Conjecture 1.1</u> Let $\vec{X} = (X_1, X_2)$ be a discrete random vector where X_1, X_2 take on the values $0, 1, \ldots$, and let S be a process acting on $\vec{X}$ satisfying for some $p_1, p_2 \in (0,1)$

$$P[S(\vec{X}) = (r,s) | \vec{X} = (i,j)] = \binom{i}{r} p_1^{\,r} (1-p_1)^{i-r} \binom{j}{s} p_2^{\,s} (1-p_2)^{j-s}$$

for all non-negative integers i and j and all $r = 0, 1, \ldots, i$ and all $s = 0, 1, \ldots, j$. Then

$$\begin{aligned} P[S(\vec{X}) = (r,s)] &= P[S(\vec{X}) = (r,s) | S(\vec{X}) \neq \vec{X}] \\ &= P[S(\vec{X}) = (r,s) | S(\vec{X}) = \vec{X}] \end{aligned} \tag{1.1}$$

for all $r = 0, 1, \ldots$ and all $s = 0, 1, \ldots$ only if $\vec{X}$ obeys a bivariate Poisson law possessing probability generating function

$$G(x_1, x_2) = \exp\{\alpha_1(x_1 - 1) + \alpha_2(x_2 - 1)\} \qquad (|x_i| \leq 1)$$

for some positive real numbers α_1, α_2.

The authors could not prove this conjecture, but noted that the probability generating function associated with $\vec{X}$ satisfies

$$G(p_1 x_1, p_2 x_2) = G(p_1, p_2)\, G(p_1 x_1 + (1-p_1), p_2 x_2 + (1-p_2)) \tag{1.2}$$

for all $x_i \in [-\frac{1}{p_i}, 1]$, $i = 1, 2$.

This is not difficult to see since taking, say, $P[S(\vec{X}) = (r,s)] = P[S(\vec{X}) = (r,s) | S(\vec{X}) = \vec{X}]$, then

$$\sum_{\substack{i \geq r \\ j \geq s}} P[S(\vec{X}) = (r,s), \vec{X} = (i,j)] = \frac{P[S(\vec{X}) = (r,s), \vec{X} = (r,s)]}{\sum_{\substack{i \geq 0 \\ j \geq 0}} P[S(\vec{X}) = \vec{X}, \vec{X} = (i,j)]},$$

whence

$$\sum_{\substack{i\geq r\\ j\geq s}} \binom{i}{r} p_1^{\,r}(1-p_1)^{i-r}\binom{j}{s} p_2^{\,s}(1-p_2)^{j-s} P[\vec{X}=(i,j)]$$

$$= \frac{p_1^{\,r}\, p_2^{\,s}\, P[\vec{X}=(r,s)]}{\sum_{\substack{i\geq 0\\ j\geq 0}} p_1^{\,i}\, p_2^{\,j}\, P[\vec{X}=(i,j)]} .$$

The denominator of the right member of the preceding equation is equal to $G(p_1, p_2)$. Multiplying both sides of the equation by $x_1^{\,r} x_2^{\,s}$ and summing over $r, s \geq 0$ yields (1.2). By equating the first two members of (1.1) or the last two members of (1.1), (1.2) follows similarly.

Now (1.2) is equivalent to

$$G(x,y) = G(p_1,p_2)\, G(x+q_1, y+q_2), \qquad -1 \leq x \leq p_1, \quad -1 \leq y \leq p_2,$$

where $q_i = 1-p_i$, $i=1,2$. It is shown in Chapter III that an even stronger property holds, namely

$$G(x,y) = G^r(p_1,p_2)\, G(x+r\,q_1, y+r\,q_2)$$

for all real numbers r and all (x,y) in the strip $-q_1\, p_2 < q_2\, x - q_1\, y < q_2\, p_1$ satisfying $x, y, x+r\,q_1, y+r\,q_2 \;\varepsilon\; [-1, 1]$.

Chapter IV presents the following characterization of the arc sine law as suggested by I. I. Kotlarski.

<u>Theorem 1.3</u> Let X_1, X_2 be independent identically distributed random variables with common density

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{\left(\frac{2}{b}\right)^2 - x^2}} & |x| < \left|\frac{2}{b}\right| \\ 0 & |x| \geq \left|\frac{2}{b}\right| \quad (b \neq 0). \end{cases} \tag{1.3}$$

Then $Y = \frac{X_1 + X_2}{b}$ and $Z = X_1 \cdot X_2$ are identically distributed.

<u>Theorem 1.4</u> Let X_1, X_2 be independent identically distributed random variables with common symmetric non-degenerate distribution function. Suppose that all moments $a_k = E[X_i^k]$, $k = 1, 2, \ldots$ exist. Let $b \neq 0$ be a real number. If

$$Y = \frac{X_1 + X_2}{b} \quad \text{and} \quad Z = X_1 \cdot X_2$$

are identically distributed, then X_i are distributed according to density (1.3).

The proofs of these theorems are based on the moments of the random variables, and Theorem 1.1 is an important tool in the proof of Theorem 1.4. An example is given to illustrate that the symmetry requirement in Theorem 1.4 may not be deleted.

CHAPTER II

ON MOMENTS AND THE CHARACTERISTIC FUNCTIONAL IN HILBERT SPACE

Terminology

The purpose of this chapter is to generalize to the Hilbert space setting several well known theorems relating moments, derivatives of characteristic functions, and Carleman's condition.

Let $\mathcal{H}$ be a real separable Hilbert space and $\mathcal{B}$ denote the sigma algebra of Borel sets of $\mathcal{H}$. Let P be a probability measure (p.m.) on $\mathcal{B}$, and R denote the set of real numbers. The characteristic functional (c.f.) $g(\cdot)$ of P is defined by

$$g(y) = \int_{\mathcal{H}} e^{\,i\,(x,y)}dP(x), \qquad y \in \mathcal{H},$$

where $(\cdot,\cdot)$ denotes the inner product. The Gateaux derivative is the perfect type of derivative needed to achieve the generalizations of this chapter. The following notation is used for Gateaux derivatives of the complex valued function g.

$$\delta^1 g(y;u) \equiv \delta^1{}_u g(y) \equiv \lim_{r\to 0} \frac{g(y+r\,u) - g(y)}{r} \qquad r \in R,$$

for $u \in \mathcal{H}$ if this limit exists. Inductively, for $u_1, u_2, \ldots, u_k \in \mathcal{H}$, derivatives of order k are defined by

$$\delta^k g(y;u_1,u_2,\ldots,u_k) = \delta^1{}_{u_k}\delta^{k-1}g(y;u_1,u_2,\ldots,u_{k-1}), \quad k = 2, 3, \ldots .$$

For convenience, define $\delta^2 g(y;u^2) \equiv \delta^2 g(y;u,u)$, and inductively, for $a_1, a_2, \ldots, a_m$ positive integers with $\sum_{n=1}^{m} a_n = k$,

$$\delta^k g(y;u_1^{a_1},u_2^{a_2},\ldots,u_m^{a_m}) = \begin{cases} \delta^1_{u_m} \delta^{k-1} g(y;u_1^{a_1},u_2^{a_2},\ldots,u_{m-1}^{a_{m-1}},u_m^{a_m-1}), & \text{if } a_m > 1 \\ \\ \delta^1_{u_m} \delta^{k-1} g(y;u_1^{a_1},u_2^{a_2},\ldots,u_{m-1}^{a_{m-1}}), & \text{if } a_m = 1. \end{cases}$$

The reader is cautioned not to view the a_j's above as exponents of the u_j's, even if $\mathcal{H} = R$. Set $A = \{(a_1,a_2,\ldots)$: a_n non-negative integers, $0 < \sum_{n=1}^{\infty} a_n < \infty\}$. If $(a_1,a_2,\ldots) \in A$ and $\sum_{n=1}^{\infty} a_n = j$, then define

$$\delta^j g(y;u_1^{a_1},u_2^{a_2},\ldots) = \delta^j g(y;u_{1'}^{a_{1'}},u_{2'}^{a_{2'}},\ldots,u_{m'}^{a_{m'}})$$

where $a_{1'}$ is the first nonzero a_n, $a_{2'}$ is the second nonzero a_n, $\ldots$, $a_{m'}$ is the last nonzero a_n.

The definition of moment to be introduced now will easily indicate the analogy between the theorems of finite dimensional space and the theorems in this chapter. Hereafter we have the convention that $(\cdot,\cdot)^0 = 1$.

<u>Definition 2.1</u> Let $\mathcal{H}$ be a real separable Hilbert space and $\mathcal{B}$ denote the sigma algebra of Borel sets of $\mathcal{H}$ and P a p.m. on $\mathcal{B}$. For $u \in \mathcal{H}$ and n a non-negative integer the n th moment of P with respect to u is

$$m_n(u^n) = \int_{\mathcal{H}} (x,u)^n dP(x)$$

provided that this integral exists. Inductively, for $u_1,u_2, \ldots, u_k \in \mathcal{H}$

and non-negative integers $a_1, a_2, \ldots, a_k$, with $\sum_{j=1}^{k} a_j = n$,

$$m_n(u_1^{a_1}, u_2^{a_2}, \ldots, u_k^{a_k}) = \int_{\mathcal{H}} \prod_{j=1}^{k} (x,u_j)^{a_j} dP(x).$$

Finally, if $u_1, u_2, \ldots, \varepsilon \mathcal{H}$ and $a_1, a_2, \ldots,$ are non-negative integers satisfying $\sum_{j=1}^{\infty} a_j = n < \infty$, then

$$m_n(u_1^{a_1}, u_2^{a_2}, \ldots) = \int_{\mathcal{H}} \prod_{j=1}^{\infty} (x,u_j)^{a_j} dP(x).$$

Now in a separable Hilbert space every orthonormal system is countable, and there is a complete orthonormal system $\{\alpha_n\}$ ($\{\alpha_n\}$ is complete if $(x,\alpha_n) = 0$ for every $n = 1, 2, \ldots,$ implies that $x = \theta$) (see [11], p. 212). Further, for any $x \varepsilon \mathcal{H}$,

$$x = \sum_{n=1}^{\infty} (x,\alpha_n)\alpha_n \quad \text{and} \quad \| x \|^2 = \sum_{n=1}^{\infty} (x,\alpha_n)^2,$$

where $\| \cdot \|$ denotes the norm of $\mathcal{H}$.

The real and imaginary parts of a complex valued function are denoted by R and I as usual.

The Theorems

<u>Theorem 2.1</u> If $\int_{\mathcal{H}} \| x \|^k dP(x) < \infty$ for some positive integer k, and g denotes the c.f. of P, then for all integers j, $1 \le j \le k$, and all $y \varepsilon \mathcal{H}$, all j th order Gateaux derivatives of g exist and are given by

$$\delta^j g(y;u_1,\ldots,u_j) = \int_{\mathcal{H}} i^j \prod_{m=1}^{j} (x,u_m) e^{i(x,y)} dP(x). \tag{2.1}$$

All jth order moments exist and are given by

$$i^j m_j(u_1,\ldots,u_j) = \delta^j g(\theta;u_1,\ldots,u_j).$$

Theorem 2.2 Let $\{\alpha_n\}_{n=1}^{\infty}$ be a complete orthonormal system in $\mathcal{H}$, and k be a positive integer. If, for each positive integer n, $\delta^{2k}g(\theta;\alpha_n^{2k})$ exists and $\sum_{n=1}^{\infty}\{R[\delta^{2k}g(\theta;\alpha_n^{2k})]\}^{1/k}$ exists, then $\int_{\mathcal{H}} \| x \|^{2k} dP(x) < \infty$. The theorem holds if $R[\delta^{2k}g(\theta;\alpha_n^{2k})]$ is replaced by $m_{2k}(\alpha_n^{2k})$.

Theorem 2.3 Let $\{\alpha_n\}_{n=1}^{\infty}$ be a complete orthonormal system in $\mathcal{H}$, and k be a positive integer. If $\int_{\mathcal{H}} \| x \|^{k} dP(x) < \infty$, then as $\| y \| \to 0$,

$$g(y) = 1 + \sum_{j=1}^{k} \frac{i^j}{j!}\left[\sum_{\substack{(a_1,a_2,\ldots)\in A \\ \sum_{m=1}^{\infty} a_m = j}} \frac{j!\prod_{m=1}^{\infty}(y,\alpha_m)^{a_m}}{\prod_{m=1}^{\infty}(a_m!)} m_j(\alpha_1^{a_1},\alpha_2^{a_2},\ldots)\right] + o(\|y\|^k).$$

Theorem 2.4 (Carleman) Let $\mathcal{H}$ be a real separable Hilbert space, $u_1 = (1,0,0,\ldots)$, $u_2 = (0,1,0,0,\ldots),\ldots$, be the standard complete orthonormal system, P be a p.m. on $\mathcal{B}$, the Borel sets of $\mathcal{H}$, and $m_n(u_j^n)$ exist for every $j = 1,2,\ldots$ and $n = 0,1,\ldots$. Denote $\lambda_{2n} = \sum_{j=1}^{\infty} m_{2n}(u_j^{2n})$, where, here, "$\infty$" may be replaced by the dimension of $\mathcal{H}$, if it is finite. If $\sum_{n=1}^{\infty}\lambda_{2n}^{-1/(2n)} = \infty$, then the moment problem is determined.

Lemma 2.1 If $r, c \in R$ with $r \neq 0$, then $\left|\frac{e^{icr}-1}{r}\right| \le |c|$.

Proof Set $h(r) = \left|\frac{e^{icr}-1}{r}\right|$, $r \neq 0$. Then $h(r) = \frac{\sqrt{2}(1-\cos cr)^{1/2}}{|r|}$.

We bound $|h(r)|$ by bounding $h^2(r) = \frac{2(1 - \cos cr)}{r^2}$. Without loss of generality we may assume that $c, r > 0$.

$1 - \cos cr$ increases from 0 to 2 as r increases from 0 to $\frac{\pi}{c}$, which leads to two cases.

Case I. $$\sup_{r \geq \frac{\pi}{c}} h^2(r) = h^2(\tfrac{\pi}{c}) = \frac{4c^2}{\pi^2}.$$

Case II. Since $\cos t = 1 - \frac{t^2}{2!} + \frac{(\sin s)t^3}{3!}$ for some $s \in (0,t)$, then for $r \in (0,\frac{\pi}{c})$,

$$h^2(r) = \frac{2}{r^2}\left(1 - \left(1 - \frac{r^2c^2}{2!} + (\sin s)\frac{r^3c^3}{3!}\right)\right),$$

for some $s \in (0, rc) \subseteq (0,\pi)$. Therefore $h^2(r) \leq c^2$. Thus, $|h(r)| \leq |c|$.

<u>Proof of Theorem 2.1</u> Note that $\int_{\mathcal{H}} \| x \|^j dP(x) < \infty$, $0 \leq j \leq k$. First let $j = k = 1$.

$$\frac{g(y + r\,u_1) - g(y)}{r} = \int_{\mathcal{H}} \frac{e^{i(x,y + r\,u_1)} - e^{i(x,y)}}{r}\, dP(x)$$

$$= \int_{\mathcal{H}} \frac{e^{i(x,y)}\{e^{ir(x,u_1)} - 1\}}{r}\, dP(x). \qquad (2.2)$$

Denoting the integrand in (2.2) by $f_r(x;u_1,y)$, we have $|f_r(x;u_1,y)| \leq |(x,u_1)| \leq \| x \| \cdot \| u_1 \|$, from which $\lim_{r \to 0} \int_{\mathcal{H}} f_r(x;u_1,y)dP(x) = \int_{\mathcal{H}} i\,(x,u_1)\,e^{i\,(x,y)}dP(x)$. To see this, assume the opposite, that there is a sequence $\{r_k\}$ of nonzero real numbers tending to zero for which it is not true that $\lim_{k\to\infty} \int_{\mathcal{H}} f_{r_k}(x;u_1,y)dP(x) = \int_{\mathcal{H}} i\,(x,u_1)\,e^{i\,(x,y)}dP(x)$. Since $\lim_{k\to\infty} f_{r_k}(x;u_1,y) = i\,(x,u_1)\,e^{i\,(x,y)}$ for all x, we have a counterexample to the dominated convergence theorem. Inductively,

$$\frac{1}{r}\,[\delta^{j-1}g(y + r\,u_j;u_1,\ldots,u_{j-1}) - \delta^{j-1}g(y;u_1,\ldots,u_{j-1})]$$

$$= \frac{1}{r}\int_{\mathcal{H}} i^{j-1}\prod_{m=1}^{j-1}(x,u_m)\,e^{i\,(x,y)}\{e^{ir(x,u_j)} - 1\}\,dP(x).$$

By the same reasoning as above we have (2.1).

<u>Corollary 2.1</u> If $\int_{\mathcal{H}} |(x,u)|^k dP(x)$ exists for some positive integer k, and $u \in \mathcal{H}$, then, for all integers j, $1 \le j \le k$, $\delta^j g(r\,u; u^j)$ exists and is equal to $\int_{\mathcal{H}} i^j (x,u)^j e^{i(x,\,r\,u)} dP(x)$, $r \in R$.

<u>Lemma 2.2</u> If, for some positive integer k and some $u \in \mathcal{H}$, $\delta^{2k}g(\theta;u^{2k})$ exists where $g(\cdot)$ is the c.f. of a p.m. P, then $m_{2k}(u^{2k}) < \infty$.

<u>Proof</u> First suppose that $k = 1$. Then $\delta^1 g(r\,u; u)$ exists for every $r \in (-a,a)$, some open interval which contains 0. Now we look at $g(r\,u)$, $r \in R$, as a complex valued function of r with

$$R[\delta^1 g(r\,u;u)] = [Rg(r\,u)]' \quad\text{and}\quad R[\delta^2 g(r\,u;u^2)] = [Rg(r\,u)]''$$

(and similar equalities for I and for higher order derivatives).

Letting "$-$" denote the complex conjugate function we have

$$\lim_{r\to 0}\frac{R\,g(r\,u) - 1}{r} = R[\delta^1 g(\theta;u)] = \lim_{r\to 0}\frac{R\,g(r\,u) - 1}{-r}$$

since $g(-x) = \bar{g}(x)$, $x \in \mathcal{H}$. Therefore $R[\delta^1 g(\theta;u)] = 0$.

$$\int_{\mathcal{H}}(x,u)^2 dP(x) = 2\int_{\mathcal{H}}\lim_{r\to 0}\frac{1 - \cos[r(x,u)]}{r^2}\,dP(x)$$

$$\le \varliminf_{r\to 0}\int_{\mathcal{H}}\frac{-[e^{ir(x,u)} - 2 + e^{-ir(x,u)}]}{r^2}\,dP(x)$$

$$= \lim_{r\to 0} \frac{-[g(ru) - 2 + g(-ru)]}{r^2} = \lim_{r\to 0} \frac{-2[Rg(ru) - 1]}{r^2}$$

$$= \lim_{r\to 0} \frac{-[Rg(ru)]'}{r} = \lim_{r\to 0} - \frac{R[\delta^1 g(ru;u)] - R[\delta^1 g(\theta;u)]}{r}$$

$$= - R[\delta^2 g(\theta;u^2)].$$

Inductively, suppose that $\delta^{2k} g(\theta;u^{2k})$ exists and that $\int_{\mathcal{H}} (x,u)^{2k-2} dP(x)$ exists. Set $G(C) = \int_C (x,u)^{2k-2} dP(x)$, $C \in \mathcal{B}$. Assuming momentarily that $G(\mathcal{H}) > 0$, then $\frac{G(\cdot)}{G(\mathcal{H})}$ is a p.m. and has c.f. $f(\cdot)$ satisfying

$$f(ru) = \frac{1}{G(\mathcal{H})} \int_{\mathcal{H}} (x,u)^{2k-2} e^{i(x,ru)} dP(x), \qquad r \in R.$$

By Corollary 2.1

$$f(ru) = \frac{(-1)^{k-1}}{G(\mathcal{H})} \delta^{2k-2} g(ru;u^{2k-2}).$$

Therefore $\delta^2 f(\theta;u^2)$ exists, and, by the induction hypothesis,

$$\frac{1}{G(\mathcal{H})} \int_{\mathcal{H}} (x,u)^2 (x,u)^{2k-2} dP(x) \le -R[\delta^2 f(\theta;u^2)] = -R\, \frac{(-1)^{k-1}}{G(\mathcal{H})} \delta^{2k} g(\theta;u^{2k}).$$

Or

$$\int_{\mathcal{H}} (x,u)^{2k} dP(x) \le (-1)^k R[\delta^{2k} g(\theta;u^{2k})]. \tag{2.3}$$

If $G(\mathcal{H}) = 0$ then $P(\{x: (x,u) = 0\}) = 1$, and by Corollary 2.1 this lemma and (2.3) remain true.

<u>Proof of Theorem 2.2</u> By Lemma 2.2, $\int_{\mathcal{H}} (x,\alpha_n)^{2k} dP(x) \le (-1)^k R[\delta^{2k} g(\theta;\alpha_n^{2k})]$ for $n = 1, 2, \ldots$. Let m be a positive integer. By repeated use of

Minkowski's inequality

$$\int_{\mathcal{H}} \left(\sum_{n=1}^{m} (x,\alpha_n)^2 \right)^k dP(x) \le \left\{ \sum_{n=1}^{m} \left(\int_{\mathcal{H}} (x,\alpha_n)^{2k} dP(x) \right)^{1/k} \right\}^k$$

$$\le \left\{ \sum_{n=1}^{\infty} [(-1)^k \, R[\delta^{2k} g(\theta;\alpha_n{}^{2k})]]^{1/k} \right\}^k .$$

The theorem is proved since

$$\int_{\mathcal{H}} \| x \|^{2k} dP(x) = \int_{\mathcal{H}} \lim_{m\to\infty} \left[\sum_{n=1}^{m} (x,\alpha_n)^2 \right]^k dP(x) \le \underline{\lim}_{m\to\infty} \int_{\mathcal{H}} \left[\sum_{n=1}^{m} (x,\alpha_n)^2 \right]^k dP(x).$$

In the following, if $x \in \mathcal{H} - \{\theta\}$, then x' will denote $\frac{x}{\| x \|}$.

<u>Proof of Theorem 2.3</u> If $y = \theta$ the theorem is clear. Suppose $y \neq \theta$.

$$g(y) = Rg(y) + i\,Ig(y) = Rg(\| y \| \cdot y') + i\,Ig(\| y \| \cdot y').$$

By Taylor's theorem, as $\| y \| \to 0$,

$$g(y) = 1 + \sum_{j=1}^{k} \frac{R\delta^j g(\theta;y'^j)}{j!} \|y\|^j + o(\|y\|^k) + i \sum_{j=1}^{k} \frac{I\delta^j g(\theta;y'^j)}{j!} \|y\|^j + o(\|y\|^k)$$

$$= 1 + \sum_{j=1}^{k} \frac{\delta^j g(\theta;y'^j)}{j!} \|y\|^j + o(\|y\|^k)$$

$$= 1 + \sum_{j=1}^{k} \frac{1}{j!} \left[\int_{\mathcal{H}} i^j (x,y')^j dP(x) \right] \|y\|^j + o(\|y\|^k)$$

$$= 1 + \sum_{j=1}^{k} \frac{1}{j!} \left[\int_{\mathcal{H}} i^j (x,y)^j dP(x) \right] + o(\|y\|^k)$$

$$= 1 + \sum_{j=1}^{k} \frac{1}{j!}\left[\int_{\mathcal{H}} i^j \left\{\sum_{n=1}^{\infty} (x,\alpha_n)(y,\alpha_n)\right\}^j dP(x)\right] + o(\|y\|^k).$$

Now $0 \leq (x \pm y,\alpha_n)^2 = (x,\alpha_n)^2 \pm 2(x,\alpha_n)(y,\alpha_n) + (y,\alpha_n)^2$. From this $|(x,\alpha_n)(y,\alpha_n)| \leq \frac{1}{2}\left[(x,\alpha_n)^2 + (y,\alpha_n)^2\right]$, so that

$$\begin{aligned}
\left|\sum_{n=1}^{m} (x,\alpha_n)(y,\alpha_n)\right| &= \left|\sum_{n=1}^{m} (x',\alpha_n)(y',\alpha_n)\right| \|x\|\,\|y\| \\
&\leq \left\{\sum_{n=1}^{m} |(x',\alpha_n)(y',\alpha_n)|\right\} \|x\|\,\|y\| \\
&\leq \left\{\sum_{n=1}^{\infty} \frac{1}{2}\left[(x',\alpha_n)^2 + (y',\alpha_n)^2\right]\right\}\|x\|\,\|y\| \\
&= \|x\|\,\|y\|.
\end{aligned}$$

By the dominated convergence theorem and the fact that $\int_{\mathcal{H}} \|x\|^k dP(x) < \infty$,

$$g(y) = 1 + \sum_{j=1}^{k}\frac{1}{j!}\left[\sum_{\substack{(a_1,a_2,\ldots)\in A \\ \sum_{m=1}^{\infty} a_m = j}} \frac{j!}{\prod_{m=1}^{\infty}(a_m!)}\int_{\mathcal{H}} i^j \prod_{m=1}^{\infty}(x,\alpha_m)^{a_m}\prod_{n=1}^{\infty}(y,\alpha_n)^{a_n} dP(x)\right] +$$

$$o(\|y\|^k)$$

$$= 1 + \sum_{j=1}^{k}\frac{1}{j!}\left[\sum_{\substack{(a_1,a_2,\ldots)\in A \\ \sum_{m=1}^{\infty} a_m = j}} \frac{j!\prod_{n=1}^{\infty}(y,\alpha_n)^{a_n}}{\prod_{m=1}^{\infty}(a_m!)}\, \delta^j g(\theta;\alpha_1^{a_1},\alpha_2^{a_2},\ldots)\right] + o(\|y\|^k).$$

Proof of Theorem 2.4 For $\mathcal{H}$ finite dimensional Euclidean space the theorem is known (see [12], p. 21), so assume otherwise. Let Q be a p.m. on $\mathcal{B}$ having the given moments, $\Pi_k : \mathcal{H} \to R^k$ be the projection map, $\Pi_k(x) = ((x,u_1),(x,u_2),\ldots,(x,u_k))$, for k a positive integer, and $\Pi_k(\mathcal{B})$ be the sigma algebra of subsets of R^k induced by $\mathcal{B}$. $\Pi_k(\mathcal{B})$ is just the usual sigma algebra of Borel sets of R^k since $\Pi_k(\mathcal{B})$ is generated by $\|\cdot\|$ when restricted to R^k, and all norms on R^k generate the same topology. Note that $\Pi_k(\cdot)$ is a measurable function, hence it makes sense to define the p.m. $Q_k : \Pi_k(\mathcal{B}) \to [0,1]$ by $Q_k(C) = Q(\Pi_k^{-1}(C))$. For integers j, k with $1 \leq j \leq k$, and denoting $u_j' = \Pi_k(u_j)$, then

$$m_{2n}(u_j^{2n}) = \int_{\mathcal{H}} (x,u_j)^{2n} dQ(x) = \int_{R^k} (x,u_j')^{2n} dQ_k(x).$$

Now since

$$\sum_{n=1}^{\infty} \{m_{2n}(u_1{}^{2n}) + \ldots + m_{2n}(u_k{}^{2n})\}^{-1/(2n)} = \infty,$$

then by Carleman's condition for finite dimensional spaces Q_k is uniquely determined $(k = 1, 2, \ldots)$. Let $y \in \mathcal{H}$ and $y_k = \sum_{j=1}^{k}(y,u_j)u_j$. Then as $k \to \infty$, $\| y_k - y \| \to 0$, and if we let $h(\cdot)$ denote the c.f. of Q, then $h(y_k) \to h(y)$. But

$$h(y_k) = \int_{\mathcal{H}} \exp\{i(x,y_k)\} dQ(x) = \int_{R^k} \exp\{i(x,\Pi_k(y_k))\} dQ_k(x)$$

is a uniquely determined complex number since $Q_k(\cdot)$ is determined. Hence $h(y)$ is a uniquely determined number. Thus, the c.f.'s of Q and P are equal, whence P = Q (see [8], p. 152).

Examples and Comments

It is easy to show that the convergence condition given in Theorem 2.2 is necessary if $k = 1$. Unfortunately this is not the case for larger k, as is seen in the following. Let $\mathcal{H} = \ell^2$ and $a_n = \sqrt{n}\,\alpha_n$ where $\alpha_1 = (1,0,0,\ldots)$, $\alpha_2 = (0,1,0,0,\ldots)$, $\ldots$. Let $p_n = P(\{a_n\}) = \frac{c}{n^{k+s}}$ where s is a constant, $1 < s < 2$, and c is a norming constant defined by $\sum_{n=1}^{\infty} p_n = 1$. Then $\int_{\mathcal{H}} \|x\|^{2k} dP(x) < \infty$ while $\int_{\mathcal{H}} i^{2k}(x,\alpha_n)^{2k} dP(x) = \frac{(-1)^k c}{n^s}$. $\sum_{n=1}^{\infty} \frac{1}{n^{s/k}}$ diverges for $k = 2, 3, \ldots$. However, if $\int_{\mathcal{H}} \|x\|^{2k} dP(x) < \infty$, we can conclude at least that $\sum_{n=1}^{\infty} (-1)^k \delta^{2k} g(\theta; \alpha_n^{2k}) < \infty$ since

$$\int_{\mathcal{H}} \sum_{n=1}^{\infty} (x,\alpha_n)^{2k} dP(x) \leq \int_{\mathcal{H}} \left(\sum_{n=1}^{\infty} (x,\alpha_n)^2 \right)^k dP(x).$$

An easy example which illustrates the use of Theorem 2.4 is obtained by letting $\mathcal{H}$ and $\{\alpha_j\}$ be as given above and setting $P(\{\alpha_j\}) = \frac{1}{2^j}$. Then $m_n(\alpha_j^{\,n}) = \frac{1}{2^j}$, and $\lambda_{2n} = 1$, $n = 1, 2, \ldots$. Thus, $\sum_{n=1}^{\infty} \lambda_{2n}^{-1/(2n)} = \infty$, whence P is uniquely determined by $\{m_n(\alpha_j^{\,n}) : j = 1,2,\ldots, n = 0,1,\ldots\}$.

This chapter contains a package of theorems relating moments and derivatives of characteristic functions. Theorem 2.1 can be shown to hold in an arbitrary real Banach space, Ω, say, with the c.f. defined by $\int_{\Omega} e^{i\,x^*(x)} dP(x)$, x^* a continuous linear functional. In order to obtain the package in an optimal setting, a suitable notion of moment must be determined, which would give rise to "the moment problem" in an abstract space, an appealing problem.

CHAPTER III

ON A CHARACTERIZATION OF THE POISSON DISTRIBUTION

Sequential Damage

C. R. Rao [9] considered the following situation. A random variable X is produced by nature where X takes on only non-negative integer values. X is then acted upon by a process S to yield $S(X)$ where S reduces or destroys X, that is, if i is the value of X then $S(X) = r$ for some integer r, $0 \leq r \leq i$. If we know the distribution of $S(X)$ given X then it is natural to ask what distribution(s) X must have in order that

$$P[S(X) = r] = P[S(X) = r|\text{damaged}] = P[S(X) = r|\text{undamaged}]$$

or equivalently

$$P[S(X) = r] = P[S(X) = r|S(X) < X] = P[S(X) = r|S(X) = X].$$

Rao and H. Rubin proved the following theorem (see [10]):

<u>Theorem 3.1</u> Let X be a discrete random variable taking the values $0, 1, \ldots,$ and let $P[S(X) = r|X = n] = \binom{n}{r}p^r(1-p)^{n-r}$, $r = 0, 1, \ldots, n$. Then

$$P[S(X) = r] = P[S(X) = r|S(X) < X] = P[S(X) = r|S(X) = X], \quad r = 0, 1, \ldots,$$

if, and only if, X has a Poisson distribution.

Now let us consider the following situation. A random variable X is as above, and X is reduced to $S_1(X)$ by a process S_1. Suppose further that $S_1(X)$ is reduced to $S_2(S_1(X))$ by some process S_2. This leads to the following.

Theorem 3.2 Let t be a positive integer and $X \equiv S_0$ and $S_1, S_2, \ldots, S_t$ be random variables taking on the values $0, 1, \ldots,$ and let $p_1, p_2, \ldots, p_t \in (0,1)$. Suppose for all $k = 1, 2, \ldots, t$ and for all integers $j_0, j_1, \ldots, j_t$ satisfying $0 \leq j_t \leq j_{t-1} \leq \cdots \leq j_0$ that

$$P[S_k = j_k | S_{k-1} = j_{k-1}, S_{k-2} = j_{k-2}, \ldots, S_0 = j_0] = P[S_k = j_k | S_{k-1} = j_{k-1}]$$
$$= \binom{j_{k-1}}{j_k} p_k^{\,j_k} (1 - p_k)^{j_{k-1} - j_k}. \tag{3.1}$$

Then

$$P[S_t = r] = P[S_t = r | S_t < X] = P[S_t = r | S_t = X], \qquad r = 0, 1, \ldots,$$

if, and only if, X has a Poisson distribution.

Proof We first make an observation. Let j_0, j_t be integers such that $P[X = j_0] > 0$, and $0 \leq j_t \leq j_0$.

$$P[S_t = j_t | X = j_0] = \sum_{j_t \leq j_{t-1} \leq \cdots \leq j_0} P[S_t = j_t, S_{t-1} = j_{t-1}, \ldots, S_1 = j_1 | X = j_0]. \tag{3.2}$$

Note that the summand is equal to

$$P[S_t = j_t, S_{t-1} = j_{t-1}, \ldots, S_1 = j_1, X = j_0] / P[X = j_0]$$

$$= P[S_t = j_t | S_{t-1} = j_{t-1}, S_{t-2} = j_{t-2}, \ldots, X = j_0] \cdot$$

$$P[S_{t-1} = j_{t-1}, \ldots, X = j_0] / P[X = j_0]$$

$$= P[S_t = j_t | S_{t-1} = j_{t-1}]\ P[S_{t-1} = j_{t-1}, \ldots, X = j_0]/P[X = j_0]$$

$$\vdots$$

$$= \prod_{m=1}^{t} P[S_m = j_m | S_{m-1} = j_{m-1}]. \tag{3.3}$$

Substituting (3.3) in (3.2), multiplying each side by x^{j_t} then summing each side from $j_t = 0$ to $j_t = j_0$, we obtain the probability generating function, G_{t,j_0}, of S_t given $X = j_0$:

$$G_{t,j_0}(x) = \sum_{j_t=0}^{j_0} x^{j_t} \left\{ \sum_{j_t \le j_{t-1} \le \cdots \le j_0} \prod_{m=1}^{t} \binom{j_{m-1}}{j_m} p_m^{\,j_m} (1-p_m)^{j_{m-1} - j_m} \right\}$$

$$= \sum_{j_t=0}^{j_0} \sum_{j_t \le \cdots \le j_0} \left\{ \left[\prod_{m=1}^{t-1} \binom{j_{m-1}}{j_m} p_m^{\,j_m} (1-p_m)^{j_{m-1} - j_m} \right] \cdot \right.$$

$$\left. \binom{j_{t-1}}{j_t} (p_t\, x)^{j_t} (1-p_t)^{j_{t-1} - j_t} \right\}$$

$$= \sum_{0 \le j_{t-1} \le \cdots \le j_0} \left\{ \prod_{m=1}^{t-1} \binom{j_{m-1}}{j_m} p_m^{\,j_m} (1-p_m)^{j_{m-1} - j_m} \right\} [p_t\, x + (1-p_t)]^{j_{t-1}}$$

$$= \sum_{0 \le j_{t-1} \le \cdots \le j_0} \left\{ \prod_{m=1}^{t-2} \binom{j_{m-1}}{j_m} p_m^{\,j_m} (1-p_m)^{j_{m-1} - j_m} \right\} \cdot$$

$$\binom{j_{t-2}}{j_{t-1}} [p_{t-1}[p_t\, x + (1-p_t)]]^{j_{t-1}} (1-p_{t-1})^{j_{t-2} - j_{t-1}}$$

$$= \sum_{0 \leq j_{t-2} \leq \cdots \leq j_0} \left\{ \prod_{m=1}^{t-2} \binom{j_{m-1}}{j_m} p_m^{j_m} (1-p_m)^{j_{m-1} - j_m} \right\} \cdot$$

$$[p_{t-1} p_t x + (1 - p_{t-1} p_t)]^{j_{t-2}}$$

$$\vdots$$

$$= \sum_{0 \leq j_1 \leq j_0} \left\{ \binom{j_0}{j_1} p_1^{j_1} (1-p_1)^{j_0 - j_1} \right\} [p_2 \cdots p_t x + (1 - p_2 \cdots p_t)]^{j_1}$$

$$= \{p_1 p_2 \cdots p_t x + (1 - p_1 p_2 \cdots p_t)\}^{j_0}.$$

Thus the distribution of S_t given $X = j_0$ is binomial with parameters j_0 and $p_1 p_2 \cdots p_t$. (3.4)

Both parts of the proof follow from Theorem 3.1 and observation (3.4).

It should be noted that Theorem 3.2 is easily related to the language of Markov processes, where, for instance, X could represent the size of a population, S_k the size of the k th generation, and (3.1) describes the one step transition probabilities.

Corollary 3.1 Let X be a random variable possessing a binomial distribution with parameters n and p_0, where n is a positive integer and $p_0 \in [0,1]$. Let t be a positive integer, $p_1, p_2, \ldots, p_t \in [0,1]$, and $S_1, S_2, \ldots, S_t$ with $S_0 \equiv X$ be random variables taking on only the values $0, 1, \ldots,$ such that for all $k = 1, 2, \ldots, t$ and for all integers $j_0, j_1, \ldots, j_t$ satisfying $0 \leq j_t \leq j_{t-1} \leq \cdots \leq j_0 \leq n$

$$P[S_k = j_k | S_{k-1} = j_{k-1}, S_{k-2} = j_{k-2}, \ldots, S_0 = j_0] = \binom{j_{k-1}}{j_k} p_k^{j_k} (1-p_k)^{j_{k-1} - j_k}.$$

Then S_t has a binomial distribution with parameters n and $p_0 p_1 \cdots p_t$.

Proof Just as in the proof of Theorem 3.2 the probability generating function of S_t given $X = j$ is $\{p_1 p_2 \cdots p_t x + (1 - p_1 p_2 \cdots p_t)\}^j$ for $j = 0, 1, \ldots, n$. Thus for $k = 1, 2, \ldots, n$

$$P(S_t = k) = \sum_{j=k}^{n} P(S_t = k, X = j) = \sum_{j=k}^{n} P(S_t = k | X = j) P(X = j)$$

$$= \sum_{j=k}^{n} \binom{j}{k} (p_1 p_2 \cdots p_t)^k (1 - p_1 p_2 \cdots p_t)^{j-k} \binom{n}{j} p_0^j (1 - p_0)^{n-j}.$$

Thus the probability generating function of S_t is

$$\sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} (p_1 p_2 \cdots p_t)^k (1 - p_1 p_2 \cdots p_t)^{j-k} \binom{n}{j} p_0^j (1 - p_0)^{n-j} x^k$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{j}{k} (p_1 p_2 \cdots p_t x)^k (1 - p_1 p_2 \cdots p_t)^{j-k} \binom{n}{j} p_0^j (1 - p_0)^{n-j}$$

$$= \sum_{j=0}^{n} \{p_1 p_2 \cdots p_t x + (1 - p_1 p_2 \cdots p_t)\}^j \binom{n}{j} p_0^j (1 - p_0)^{n-j}$$

$$= [p_0 p_1 \cdots p_t x + (1 - p_0 p_1 \cdots p_t)]^n.$$

The Bivariate Case

R. C. Srivastava and A. B. L. Srivastava [13] considered the two dimensional analog of Theorem 3.1 and conjectured as follows.

Conjecture 3.1 Let $\vec{X} = (X_1, X_2)$ be a discrete random vector where X_1, X_2 take on the values $0, 1, \ldots,$ and let S be a process acting on $\vec{X}$ satisfying for some $p_1, p_2 \in (0,1)$

$$P[S(\vec{X}) = (r,s) \mid \vec{X} = (i,j)] = \binom{i}{r} p_1^{\,r}(1-p_1)^{i-r}\binom{j}{s} p_2^{\,s}(1-p_2)^{j-s}$$

for all non-negative integers i and j and all $r = 0, 1, \ldots, i$ and all $s = 0, 1, \ldots, j$. Then

$$P[S(\vec{X}) = (r,s)] = P[S(\vec{X}) = (r,s) \mid S(\vec{X}) \neq \vec{X}] = P[S(\vec{X}) = (r,s) \mid S(\vec{X}) = \vec{X}] \qquad (3.5)$$

for all $r = 0, 1, \ldots$ and all $s = 0, 1, \ldots$ only if $\vec{X}$ obeys a bivariate Poisson law possessing probability generating function

$$G(x_1, x_2) = \exp\{\alpha_1(x_1 - 1) + \alpha_2(x_2 - 1)\} \qquad (|x_i| \leq 1)$$

for some positive real numbers α_1, α_2.

In [13] the authors verified the converse, leaving the conjecture open, with the observation that (3.5) implies

$$G(p_1 x_1, p_2 x_2) = G(p_1, p_2) G(p_1 x_1 + (1-p_1), p_2 x_2 + (1-p_2)) \qquad (3.6)$$

for all $x_i \in [-\frac{1}{p_i}, 1]$, $i = 1, 2$.

Aczél [2] solved (3.6) under the assumption that (3.6) holds for all $x_1, x_2 \in (-1,1)$ and all $p_1, p_2 \in (0,1)$. The solution of Theorem 3.1 did not require that p be a variable, and it is natural to wonder if the conjecture is true for p_1, p_2 fixed. This author, using a technique which appeared in the paper of Rao and Rubin, has arrived at the following extension of (3.6), where $p_1, p_2 \in (0,1)$ are assumed fixed.

In the following let Z denote the set of integers, R the set of real numbers, and $q_i = 1 - p_i$, $i = 1, 2$.

<u>Theorem 3.3</u> Under the hypothesis of the preceding conjecture the probability generating function G of $\vec{X}$ satisfies

$$G(x,y) = G^r(p_1,p_2)\,G(x+r\,q_1,\, y+r\,q_2)$$

for all $r \varepsilon R$ and all (x,y) in the strip $-q_1\,p_2 < q_2\,x - q_1\,y < q_2\,p_1$ satisfying $x,y,x+r\,q_1,y+r\,q_2 \;\; \varepsilon \; [-1,1]$.

Proof Replacing $p_1\,x_1$ by x and $p_2\,x_2$ by y, (3.6) becomes

$$G(x,y) = G(p_1,p_2)G(x+q_1,y+q_2), \qquad -1 \leq x \leq p_1,\; -1 \leq y \leq p_2. \tag{3.7}$$

Using (3.7) we can define a function $F(x,y)$ such that F and G agree on the Cartesian product $(0,1)\times(0,1)$ and F is defined on the open domain D bounded by the two polygonal lines L_1, L_2:

$$L_1 = \{(k\,q_1+x,1+k\,q_2),(k\,q_1,p_2+k\,q_2+y): k\,\varepsilon\,Z,\; 0\leq x\leq q_1,\; 0\leq y\leq q_2\}$$

$$L_2 = \{(1+(k-1)q_1+x,k\,q_2),(1+k\,q_1,k\,q_2+y): k\,\varepsilon\,Z,\; 0\leq x\leq q_1,\; 0\leq y\leq q_2\}$$

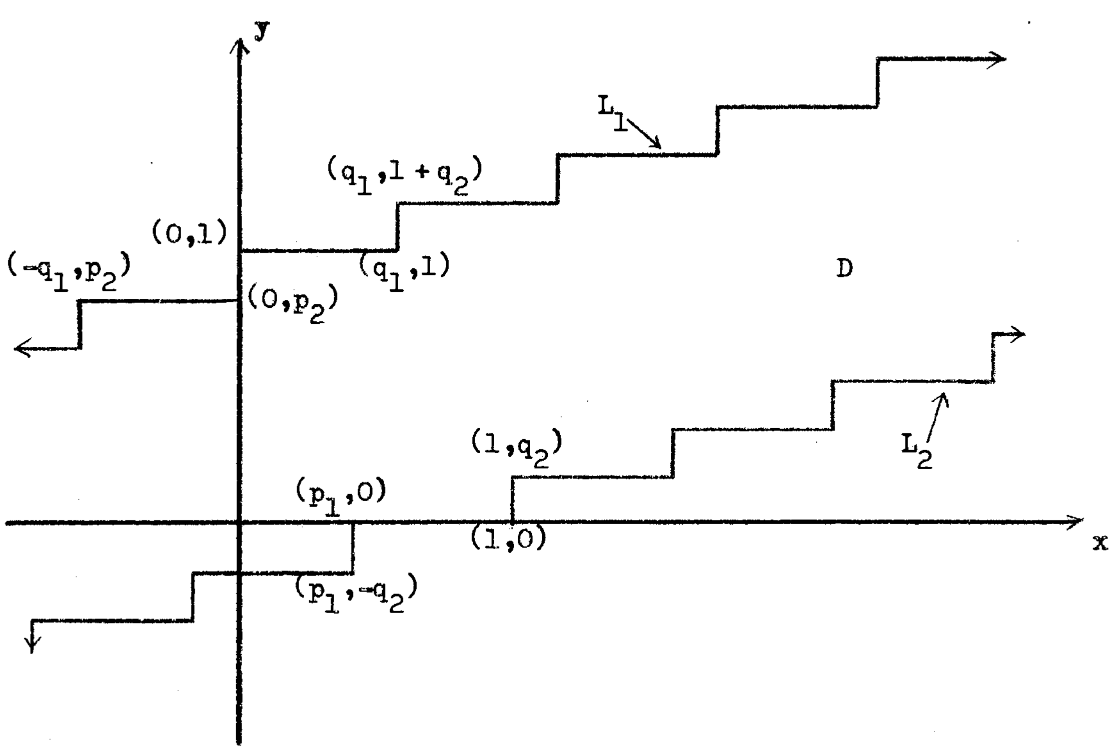

Figure 1. D and Its Boundaries

(see Figure 1) and

$$F(x,y) = F(p_1,p_2)\,F(x+q_1,y+q_2), \qquad (x,y)\in D. \tag{3.8}$$

It is easily shown that each line lying wholly in D has equation

$$x = \frac{q_1}{q_2}(y-p_2)+c \quad \text{for some} \quad c, \quad 0<c<p_1+\frac{q_1}{q_2}p_2. \tag{3.9}$$

Let the strip in R^2 determined by all lines (3.9) be denoted D_1. Pick c from the interval given in (3.9). Inductively (3.8) becomes (on the line determined by c)

$$F\left(\frac{q_1}{q_2}(y-p_2)+c,\,y\right) = F^k(p_1,p_2)\,F\left(\frac{q_1}{q_2}(y-p_2)+c+kq_1,\,y+kq_2\right),$$

$k\in Z$, $y\in R$. Set

$$S(y) \equiv F\left(\frac{q_1}{q_2}(y-p_2)+c,\,y\right), \qquad y\in R.$$

Then for $k\in Z$,

$$\begin{aligned} S(y+kq_2) &= F\left(\frac{q_1}{q_2}(y+kq_2-p_2)+c,\,y+kq_2\right) \\ &= F\left(\frac{q_1}{q_2}(y-p_2)+c+kq_1,\,y+kq_2\right). \end{aligned}$$

Hence

$$S(y) = F^k(p_1,p_2)\,S(y+kq_2), \qquad k\in Z,\ y\in R. \tag{3.10}$$

Let

$$H(y) = S(y)\,e^{y(\log F(p_1,p_2))/q_2}, \qquad y\in R. \tag{3.11}$$

Then for $k \varepsilon Z$,

$$H(y+kq_2) = S(y+kq_2)\,e^{y(\log F(p_1,p_2))/q_2}\,e^{k\log F(p_1,p_2)} \tag{3.12}$$

$$= S(y)\,e^{y(\log F(p_1,p_2))/q_2} = H(y),$$

and in particular $H(kq_2) = H(0) = S(0) \neq 0$ for every $k \varepsilon Z$ ($S(0) \neq 0$ follows from (3.10) and the fact that G is a probability generating function).

The line of D_1 determined by c intersects $(0,1)\times(0,1)$ at points whose y coordinates belong to an interval (l_c, u_c) the length of which is greater than q_2. That S is absolutely monotonic on R will follow from (3.10) if it can be shown that S is absolutely monotonic on (l_c, u_c).

Let $y_0 \varepsilon (l_c, u_c)$. Then $\left(\frac{q_1}{q_2}(y_0 - p_2) + c, y_0\right) \varepsilon (0,1)\times(0,1)$ and

$$S(y_0) = F\left(\frac{q_1}{q_2}(y_0-p_2)+c, y_0\right) = G\left(\frac{q_1}{q_2}(y_0-p_2)+c, y_0\right)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} p_{k,j}\left(\frac{q_1}{q_2}(y_0-p_2)+c\right)^k y_0^j \tag{3.13}$$

where $p_{k,j} \equiv P[\vec{X} = (k,j)]$. For a given k and j the function

$$h_{k,j}(y) = p_{k,j}\left(\frac{q_1}{q_2}(y-p_2) + c\right)^k y^j$$

is absolutely monotonic on (l_c, u_c) since $h_{k,j}^{(n)}(y) \geq 0$ for every $n = 0, 1, \ldots$, and for every $y \varepsilon (l_c, u_c)$. Since (3.13) converges, $l_c < y_0 < u_c$, then $S(y)$ is absolutely monotonic on (l_c, u_c) (see [14], p. 151), and S is absolutely monotonic on R, and in particular on $(-\infty, 0)$. Hence for $-\infty < y < 0$, $S(y) = S(0) \int_0^\infty e^{yt} d\alpha(t)$ for some distribution function α (see [14], p. 162 - Bernstein's Theorem). Therefore

$$H(y) = e^{y(\log F(p_1, p_2))/q_2} S(0) \int_0^\infty e^{yt} d\alpha(t). \tag{3.14}$$

Recalling from (3.12) that $H(y)$ is periodic(q_2) then

$$H(-q_2) = \frac{1}{F(p_1,p_2)} S(0) \int_0^\infty e^{-q_2 t} d\alpha(t)$$

$$= H(-2q_2) = \frac{S(0)}{F^2(p_1,p_2)} \int_0^\infty e^{-2q_2 t} d\alpha(t);$$

or

$$F(p_1,p_2) = \int_0^\infty e^{-q_2 t} d\alpha(t) \quad \text{and} \quad F^2(p_1,p_2) = \int_0^\infty e^{-2q_2 t} d\alpha(t).$$

Thus $\left[E\left(e^{-q_2 t}\right)\right]^2 = E\left(e^{-q_2 t}\right)^2$ which implies that $e^{-q_2 t} = F(p_1,p_2)$ a.e.(α). But $\exp(-q_2 t)$ is a strictly decreasing function of t. Therefore $\alpha(t)$ is a distribution with point mass 1 at $t = \frac{\log F(p_1,p_2)}{-q_2}$. So from (3.14), for $y < 0$

$$H(y) = e^{y(\log F(p_1, p_2))/q_2} S(0)\, e^{-y(\log F(p_1, p_2))/q_2} = S(0),$$

a constant. This and (3.11) with H periodic on R implies

$$S(y) = S(0)\, e^{-y(\log F(p_1, p_2))/q_2},$$

or

$$F\left(\frac{q_1}{q_2}(y - p_2) + c, y\right) = F\left(\frac{q_1}{q_2}(-p_2) + c, 0\right) e^{\rho y q_1}, \qquad y \in R,$$

where

$$\rho = -\frac{\log G(p_1, p_2)}{q_1 q_2}.$$

In rectangular coordinates with $x = \frac{q_1}{q_2}(y - p_2) + c$ this becomes

$$F(x,y) = F\left(x - \frac{q_1}{q_2} y, 0\right) e^{\rho y q_1}, \qquad (x,y) \varepsilon D_1. \tag{3.15}$$

Symmetrically it can be shown that

$$F(x,y) = F\left(0, y - \frac{q_2}{q_1} x\right) e^{\rho x q_2}, \qquad (x,y) \varepsilon D_1.$$

From (3.15), for $r \varepsilon R$, $(x,y) \varepsilon D_1$, we have $F(x + r q_1, y + r q_2) = F(x,y) F^{-r}(p_1, p_2)$. The domain of the hypothesis follows by intersecting $[-1,1] \times [-1,1]$ with D_1.

CHAPTER IV

ON PROPERTIES OF THE ARC SINE LAW

Two properties of the arc sine law appear in this chapter. In particular, let X_1, X_2 be independent identically distributed random variables satisfying $X_1 + X_2 \sim X_1 \cdot X_2$ or $\frac{X_1 + X_2}{2} \sim X_1 \cdot X_2$, where "$\sim$" denotes "is distributed as". Theorem 4.2 exhibits a non-degenerate distribution for the random variables in each case.

<u>Theorem 4.1</u> Let X_1, X_2 be independent identically distributed random variables with common density

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{\left(\frac{2}{b}\right)^2 - x^2}} & |x| < \left|\frac{2}{b}\right| \\ 0 & |x| \geq \left|\frac{2}{b}\right| \qquad (b \neq 0). \end{cases} \tag{4.1}$$

Then $Y = \dfrac{X_1 + X_2}{b}$ and $Z = X_1 \cdot X_2$ are identically distributed.

<u>Proof</u> The support of X_i is in a finite interval hence so are the supports of Y, Z and their moments characterize their distribution functions (see [3], p. 178). Let k be a positive integer. If k is odd then since the distribution corresponding to $f(x)$ is symmetric then $m_k = E[X_i^k] = 0 = E[X_1^k \cdot X_2^k] = E[Z^k]$. Examining the other case

$$m_{2k} = \frac{1}{\pi}\int_{-\left|\frac{2}{b}\right|}^{\left|\frac{2}{b}\right|} \frac{x^{2k}}{\sqrt{\left(\frac{2}{b}\right)^2 - x^2}}\,dx = \frac{2}{\pi}\int_0^{\left|\frac{2}{b}\right|} \frac{x^{2k}}{\sqrt{\left(\frac{2}{b}\right)^2 - x^2}}\,dx$$

$$= \frac{2}{\pi}\int_0^1 \frac{\left(\frac{2}{b}\right)^{2k} y^{2k}}{\sqrt{1-y^2}}\,dy$$

$$= \left(\frac{2}{b}\right)^{2k} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)} = \left(\frac{2}{b}\right)^{2k} \frac{(2k)!}{(k!)^2\, 2^{2k}}$$

$$= b^{-2k}\binom{2k}{k}.$$

Hence $E[Z^{2k}] = b^{-4k}\binom{2k}{k}^2$ while $E[Z^{2k-1}] = 0.$

$$E[Y^k] = b^{-k}\, E\left\{\sum_{j=0}^{k}\binom{k}{j} X_1^j X_2^{k-j}\right\} = b^{-k}\sum_{j=0}^{k}\binom{k}{j} m_j\, m_{k-j}.$$

Note that if k is odd then $E[Y^k]$ like $E[Z^k]$ equals zero. On the other hand

$$E[Y^{2k}] = b^{-2k}\sum_{j=0}^{2k}\binom{2k}{j} m_j\, m_{2k-j} = b^{-2k}\sum_{j=0}^{k}\binom{2k}{2j} m_{2j}\, m_{2k-2j}$$

$$= b^{-2k}\sum_{j=0}^{k}\binom{2k}{2j}\, b^{-2j}\binom{2j}{j} b^{2j-2k}\binom{2k-2j}{k-j} = b^{-4k}\sum_{j=0}^{k}\binom{2k}{2j}\binom{2j}{j}\binom{2k-2j}{k-j}$$

$$= b^{-4k}\,(2k)!\sum_{j=0}^{k}\frac{1}{(j!)^2[(k-j)!]^2}$$

$$= b^{-4k}\,\frac{(2k)!}{k!\;k!}\sum_{j=0}^{k}\binom{k}{j}\binom{k}{k-j}\;. \tag{4.2}$$

This last summation may be looked upon as the coefficient of x^k in the product $(1+x)^k(1+x)^k$. Therefore (4.2) becomes $b^{-4k}\binom{2k}{k}^2$. Thus $E[Y^k] = E[Z^k]$, and Y and Z are identically distributed.

Theorem 4.2 Let X_1, X_2 be independent identically distributed random variables with common symmetric non-degenerate distribution function F. Suppose that all moments $a_k = E[X_i^k]$, $k=1, 2, \ldots$, exist. Let $b \neq 0$ be a real number. If

$$Y = \frac{X_1 + X_2}{b} \quad \text{and} \quad Z = X_1 \cdot X_2$$

are identically distributed, then X_i are distributed according to density (4.1).

Proof Now $E[Z^k]$ and $E[Y^k]$ exist since $E[X_i^k]$ exist, and the hypothesis implies that $E[Z^k] = E[Y^k]$, $k=1, 2, \ldots$. If k is odd then $a_k = 0$ since F is symmetric. Hence $E[Z^k] = E[X_1 \cdot X_2]^k = E[X_1^k]\,E[X_2^k] = 0$. For even subscripts

$$\begin{aligned} a_{2k}^2 &= E[X_1^{2k}]\,E[X_2^{2k}] = E\left(\frac{X_1+X_2}{b}\right)^{2k} \\ &= b^{-2k}\,E\left\{\sum_{j=0}^{2k}\binom{2k}{j}X_1^{\,j}\,X_2^{\,2k-j}\right\} \\ &= b^{-2k}\sum_{j=0}^{2k}\binom{2k}{j}a_j\,a_{2k-j} \\ &= b^{-2k}\sum_{j=0}^{k}\binom{2k}{2j}a_{2j}\,a_{2k-2j}. \end{aligned} \tag{4.3}$$

Letting $k=1$ this becomes $a_2^2 = b^{-2}\left[\binom{2}{0}a_2 + \binom{2}{2}a_2\right]$ or $b^2 a_2^2 - 2a_2 = 0$, whence $a_2 = 0$ or $a_2 = \frac{2}{b^2}$. But since F is non-degenerate, no moment of even order can be zero. Therefore $a_2 = \frac{2}{b^2} = b^{-2}\binom{2}{1}$. An inductive argument used on (4.3) shows that $a_{2k} = b^{-2k}\binom{2k}{k}$, $k = 1, 2, \dots$.

From Theorem 1.1 it is known that a distribution function is completely determined by the sequence $\{\alpha_k\}$ of its moments if the sum

$$\sum_{k=1}^{\infty} \alpha_{2k}^{-1/(2k)} \tag{4.4}$$

is divergent. Since $a_{2k} = \binom{2k}{k} b^{-2k}$ then $a_{2k}^{-1/(2k)} = |\,b\,|\binom{2k}{k}^{-1/(2k)}$.

We now compare $\binom{2k}{k}^{-1/(2k)}$ to $\frac{1}{k}$, the general term of the harmonic series. We note the inequality

$$k^{2k} > \frac{(2k)!}{k!\;k!} \qquad (k > 1). \tag{4.5}$$

To see this, the right member of (4.5) may be looked upon as the number of 2k-long binary sequences in which exactly as many zeros as ones appear. The left member is the number of 2k-long k-ary sequences (i. e., with a choice of k digits for each position). Hence (4.5) is clear.

Therefore

$$\binom{2k}{k}^{-1} > k^{-2k}$$

$$\binom{2k}{k}^{-1/(2k)} > k^{-1}$$

and (4.4) diverges if α is replaced by a.

It was shown in Theorem 4.1 that the moments associated with density (4.1) are $m_{2k} = \binom{2k}{k} b^{-2k}$, and $m_{2k-1} = 0$. Since the moments $\{a_k : k = 1, 2, \dots\}$ characterize F and $a_k = m_k$, then X_i are distributed according to density (4.1).

It should be noted that the assumption of symmetry cannot be removed from the hypothesis of Theorem 4.2 as is seen by letting X_i have the common distribution

$$P[X_i = -\tfrac{1}{2}] = \tfrac{2}{3} \quad \text{and} \quad P[X_i = 1] = \tfrac{1}{3}, \qquad i = 1, 2,$$

and $b = 2$.

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A SELECTED BIBLIOGRAPHY

[1] Aczél, J. Lectures on Functional Equations and their Applications. New York: Academic Press, 1966.

[2] Aczél, J. "On a Characterization of Poisson Distributions." Journal of Applied Probability, Vol. 9 (1972), 852-856.

[3] Chung, K. L. A Course in Probability Theory. New York: Harcourt, Brace and World, 1968.

[4] Feller, W. An Introduction to Probability Theory and Its Applications. 2nd ed. New York: John Wiley and Sons, 1968.

[5] Grenander, U. Probabilities on Algebraic Structures. Stockholm: Almqvist and Wiksell, 1963.

[6] Liu, C. L. Introduction to Combinatorial Mathematics. New York: McGraw-Hill, 1968.

[7] Lukacs, E. Characteristic Functions. 2nd ed. London: Griffin, 1970.

[8] Parthasarathy, K. R. Probability Measures on Metric Spaces. New York: Academic Press, 1967.

[9] Rao, C. R. "On Discrete Distributions Arising Out of Methods of Ascertainment." Classical and Contagious Discrete Distributions. Ed. G. P. Patil. Calcutta: Statistical Publishing Society, 1965.

[10] Rao, C. R. and H. Rubin. "On a Characterization of the Poisson Distribution." Sankhyā, Series A, Vol. 26 (1964), 295-298.

[11] Royden, H. L. Real Analysis. New York: Macmillan, 1968.

[12] Shohat J. A. and J. D. Tamarkin. The Problem of Moments. New York: American Mathematical Society, 1943.

[13] Srivastava R. C. and A. B. L. Srivastava. "On a Characterization of the Poisson Distribution." Journal of Applied Probability, Vol. 7, No. 2 (1970), 497-501.

[14] Widder, D. V. The Laplace Transform. Princeton: Princeton University Press, 1946.

VITA

Robert Michael Norton

Candidate for the Degree of

Doctor of Philosophy

Thesis: MOMENTS AND THE CHARACTERISTIC FUNCTIONAL IN HILBERT SPACE AND SOME CHARACTERIZATIONS IN PROBABILITY THEORY

Major Field: Mathematics

Biographical:

Personal Data: Born in Richmond Heights, Missouri, July 1, 1946, the son of Robert and Eunice L. Norton.

Education: Graduated from Webster Groves High School, Webster Groves, Missouri, in June, 1964; received Bachelor of Science degree in Mathematics and Bachelor of Science in Education degree from Northeast Missouri State University, Kirksville, Missouri, in May, 1968; received Master of Science in Mathematics from Oklahoma State University, Stillwater, Oklahoma, in July, 1971; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in July, 1974.

Professional Experience: National Science Foundation trainee, Oklahoma State University, fall semester of 1968, 1970-72; Graduate Assistant, Oklahoma State University, fall semester of 1971, 1972-74; Member of the Mathematical Association of America, the American Mathematical Society, and the National Council of Teachers of Mathematics.