INVESTIGATING THE ENERGY OF MUSICAL CHORDS

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Abstract. In western music, each octave is divided into 12 semitones. The number of semitones between notes played at the same time largely determines whether a chord sounds pleasing or not. In a mathematical sense, a relative energy can be defined for chords. In this sense, consonant (pleasant) chords should be of lower energy than dissonant ones. A function of frequency ratios has been developed to calculate relative energies of chords. This project aims to modify this function to improve its accuracy with respect to expectations from music theory.

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1. Introduction

Music is essential to the evolutionary story of *homo sapiens*, having originated tens of thousands of years in the past. It has been hypothesized that music, in its most primitive form, served as a method of "communicat[ing] coalition quality" between groups of humans. (Hagen and Bryant, 2003). If this or similar hypotheses hold any water, the implications are an intrinsic connection of music to human communication and interaction.

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In addition to this, music has been observed in some form or another in various other animal species, such as birds and whales. The biological universality and relevance of music to human thought and communication motivate a deeper understanding of its fundamental mechanisms, an issue upon which this paper attempts to shed light.

We begin with a brief overview of the structure of human hearing and contemporary music theory. We then incorporate mathematical concepts as a connection between musical constructions and psychological preferences for certain sounds over others. In this way we relate the frequency of sound waves with human perception, in terms of the prevalent twelve-tone system of music.

2. Background

2.1. **How do humans hear?** Why are some sounds more pleasing than others? At a very basic level, this is based on the nerve fibers found in the inner ear. It has been shown that humans show a preference for harmonic sounds. That is, that pitches played at the same time, and whose harmonic series overlap or share a base frequency, activate nerve fibers of their harmonically related pitches, which results in a pleasing sound (Tramo et al, 2015). Conversely, combined pitches sounding unpleasant do not activate nerve fibers correlated with harmonically related pitches. We naturally turn to examine the most common relationships between musical notes - intervals.

2.2. **Some Music Theory as a Framework.** We will be in interested the consonance and dissonance of musical intervals and chords. *Consonance* in this context is a measure of how pleasing something sounds to the human ear, which we presume is *based on the ratio of the frequency of the sound waves played at the same time.* An *interval* is the distance between two notes, and a *chord* is multiple notes played at the same time. For the purposes of this paper, we refer to an interval when considering two notes played at the same time, and a chord for three or more (and in terms of the intervals between those notes).

Many systems of music have been developed in different cultures throughout history. The most prevalent of these in Western Music, developed during the Middle Ages, divides the octave into 12 semitones. The number of semitones between two notes determines their interval. An octave is separated by 12 semitones. From 1 to 11 semitones the intervals are named as follows: *Major and minor Second, Major and minor Third, Perfect Fourth, Tritone, Perfect Fifth, Major and minor Sixth, and Major and minor Seventh.* Classifying the intervals as consonant or dissonant has historically been a point of debate, and is the motivating reason for this paper. According to Fux (1725) and the classical method of counterpoint, the consonant intervals were the octave, fifth, major and minor third, and major and minor sixth. The fourth, seconds, and sevenths, and especially the tritone were not used in counterpoint composition because of their apparent dissonance.

It seems apparent that the intervals defined as consonant are not equal in their consonance, and the same for dissonant. An ordering seems natural, but any objective ordering is based on the tuning system used (Hindemith, 1942).

2.2.1. *Tuning Systems.* How the ratios of the frequencies of intervals are defined depends on the tuning system used. The two primary tuning systems considered in this paper are Pythagorean and equal-temperament. **Pythagorean Tuning** is a tuning system that uses strictly the octave and perfect fifth to construct the scale. The intervals and their corresponding frequency ratios are shown in Figure 1.

Interval	Frequency Ratio	Interval	Frequency Ratio
Perfect Unison	1 : 1	Perfect Fifth	3 : 2
Minor Second	16 : 15	Minor Sixth	8 : 5
Major Second	9 : 8	Major Sixth	5 : 3
Minor Third	6 : 5	Minor Seventh	16 : 9
Major Third	5 : 4	Major Seventh	15 : 8
Perfect Fourth	4 : 3	Octave	2 : 1
Tritone	32 : 27		

FIGURE 1. Every interval in the twelve-tone system with its associated Pythagorean frequency ratio.

For example, the Major second is obtained by applying two successive fifths (producing a ratio of $9 : 4$) followed by moving down an octave (cutting the frequency in half, resulting in the $9 : 8$ ratio presented in the table). If we start from C, adding fifths gives us G, D, A, E, B, and F♯. Working in the opposite direction produces F, B♭, E♭, A♭, D♭, and G♭. Now F♯ and G♭. F♯ and G♭ should be equivalent up to octave, but when this method of tuning is used, the notes are almost a quarter of a semitone apart. This error is known as the **Pythagorean Comma**, and after some more advanced mathematics became known, the equally-tempered scale was constructed.

Twelve-Tone Equal Temperament Tuning spaces the error from the Pythagorean Comma equally between each interval, so that no one interval is particularly out of tune, but each interval is slightly out of tune. The idea of equal-temperament is to split the octave into twelve equal intervals. Since the octave has a ratio of $2 : 1$, this means, for example, each half-step (or the minor Second interval) has a ratio of $2^{\frac{1}{12}}$, and the Perfect Fifth has a ratio of $2^{\frac{7}{12}} \approx 1.498$.

We move towards classifying intervals as consonant or dissonant based on the complexity of their frequency ratio.

3. The Mathematics Used

Using mathematics to describe musical constructs in itself is not a new idea. When studying the motion and direction of a musical composition, dynamical systems can be useful (Pinto, 2012). Tymoczko (2006) has done extensive work modeling musical chords and transitions between them using non-Euclidean geometry. We use applications of number theory to describe the complexity of interval frequency ratios, which in turn we use to describe the relative consonance of the interval.

3.1. **Logarithmic Height.** When considering the complexity of a ratio, we begin with the notion of **logarithmic height** defined as

$$h(\frac{a}{b}) = \ln(\max\{|a|, |b|\}) \tag{1}$$

This height would work well within the Pythagorean tuning system, yielding a lower height for the octave and fourth and fifth, and a higher height for sevenths, seconds, and the tritone. But simply using this height will not work in equal-tempered tuning. How do we measure the height of an irrational frequency ratio? The answer lies in the ratio's continued fraction expansion.

3.2. **Continued Fractions.**

Definition 1. A regular **continued fraction** is a fraction of the form

$$c = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2}$$

where each a_i is a positive integer.

The continued fraction can be written in its expanded form, as a sequence of a_is, or as a sequence of its convergents. The k-th **convergent**, $\frac{p_k}{q_k}$ is the segment of the ratio defined by its first k terms:

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots \cfrac{1}{a_k}}}}} = [a_0, a_1, a_2, \cdots, a_k] \tag{3}$$

Theorem 2. *The continued fraction expansion for $c = [a_0, a_1, a_2, \cdots]$ terminates if and only if $c \in \mathbb{Q}$.*

Example 3. $\frac{7}{5} = [1, 2, 2] = \{1, \frac{3}{2}, \frac{7}{5}\}$

Example 4. $\frac{27}{16} = [1, 1, 2, 5] = \{1, 2, \frac{5}{3}, \frac{27}{16}\}$

Example 5. $\sqrt{2} = [1, 2, 2, 2, 2, \cdots] = \{1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \cdots\}$

Example 6. $\pi = [3, 7, 15, 1, 292, \cdots] = \{3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103{,}993}{33{,}102}, \cdots\}$

Now we can calculate the height for each of the convergents of a ratio, rational or not, but if we simply sum the heights for our energy, it will surely diverge if c is irrational. To ensure convergence, we choose a k and sum the first k heights. To further represent the complexity of the ratio, we also add to this energy calculation a weight that decreases according to how accurate the convergent is.

3.3. **Weight.** Means (2015) used a simple exponential weight,

$$E(\frac{F_2}{F_1}) = \sum_{i=1}^{k} h(\frac{p_i}{q_i}) \cdot e^{-(i-1)} \tag{4}$$

but, as will be discussed later, the energies for the intervals did not particularly match expectations from music theory. When considering the weight presented here, it is first important to consider the Fibonacci sequence.

Theorem 7. *The denominator of a continued fraction convergent, q_i is defined recursively by the following relation*

$$q_i = a_i \cdot q_{i-1} + q_{i-2} \tag{5}$$

The larger the i-th denominator, the better an approximation that term is, so we consider now the "worst case" where each a_i is 1. This is the case where the denominators are numbers from the Fibonacci sequence (i.e. $q_i = f_i$, where $\{f_1, f_2, f_3, \cdots\}$, is the Fibonacci sequence), and the ratio described is the golden ratio: $\varphi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \{1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \cdots\}$

Since each a_i in a continued fraction expansion is a positive integer, for any ratio we have the following inequality

$$q_i \geq f_i. \tag{6}$$

Now, how much greater q_i is than f_i, or equivalently, how large $\frac{q_i}{f_i}$ is, is a measure of the accuracy of the preceding term with respect to the actual ratio.

Theorem 8 (Khinchin, 1935, Theorem 9)**.**

$$|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}, \tag{7}$$

where $\frac{p_k}{q_k}$ is the k-th convergent of α.

In Example 6. above, for example, $\frac{22}{7}$ is a more accurate approximation for π than $\frac{3}{2}$ is for $\sqrt{2}$, because, looking at the third terms of each sequence, $\frac{106}{2}$ is much larger than $\frac{5}{2}$.

Definition 9 (Weight)**.**

$$w_i = \frac{q_i}{f_i}, \tag{8}$$

where q_i is the denominator of the i-th convergent, and f_i is the i-th Fibonacci number.

4. The Energy Function

The energy function for an interval, whose notes are of frequencies F_1, F_2 in increasing order, and $\frac{p_i}{q_i}$ are the partial convergents of $\frac{F_2}{F_1}$, with w_i defined above:

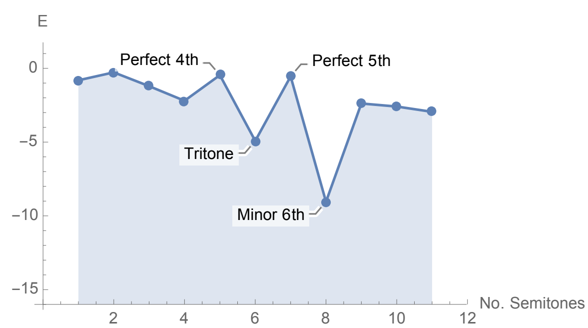
$$E(\frac{F_2}{F_1}) = \sum_{i=1}^{k} h(\frac{p_i}{q_i})/w_i \tag{9}$$

To calculate the energy of a chord, we simply add the energies of every possible frequency ratio within the chord:

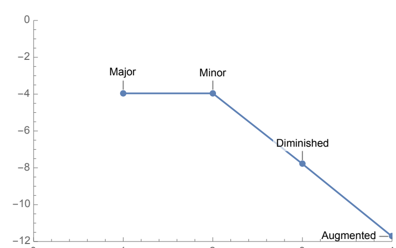
$$E(c) = \sum_{i=1}^{k}\sum_{j=1}^{i-1} E(\frac{F_i}{F_j}) \tag{10}$$

4.1. **Calculations.** We used this function to calculate the energy for each interval from minor second to major seventh, as well as the major, minor, diminished, and augmented triads. We performed these calculations within the Pythagorean and equal-temperament tuning systems.

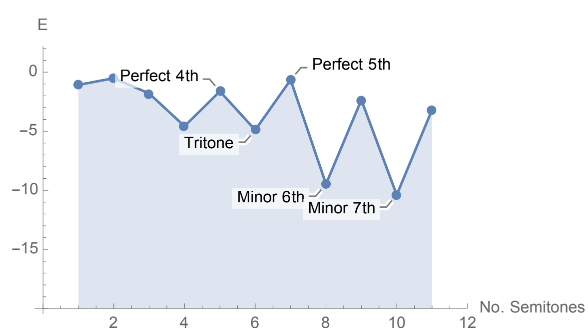
For intervals, the x-axis represents the number of semitones between notes in the interval, and the y-axis is the inverted energy, so *higher energy, dissonant intervals are at the bottom, and consonant ones are at the top.*



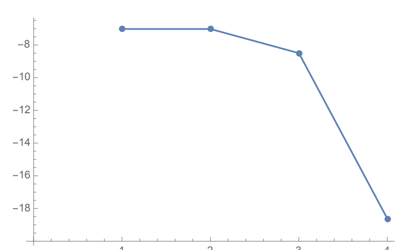
a. Energy for intervals in Pythagorean tuning



b. Energy for major, minor, diminished, and augmented triads in Pythagorean tuning



c. Energy for intervals in twelve-tone equal-temperament tuning



d. Energy for triads in twelve-tone equal-temperament tuning

FIGURE 2. Energy calculations for Pythagorean and equal-temperament tuning. The summation was capped at k = 42.

5. Discussion

The following two graphics are from social science experiments in which participants rated the relative pleasantness of intervals and triads. These figures serve as a representation of more-or-less what we expect from music theory to be mirrored by our calculations.

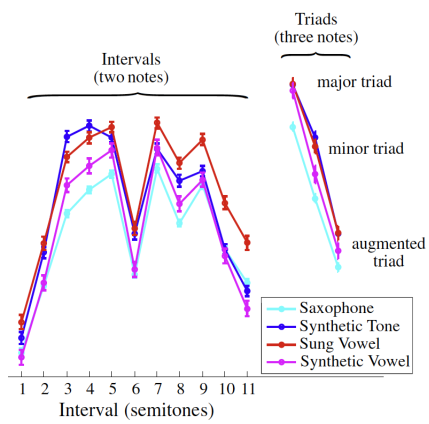


FIGURE 3. Participants rated the pleasantness of each interval and the major, minor, and augmented triads (N=265) (McDermott et al, 2010)

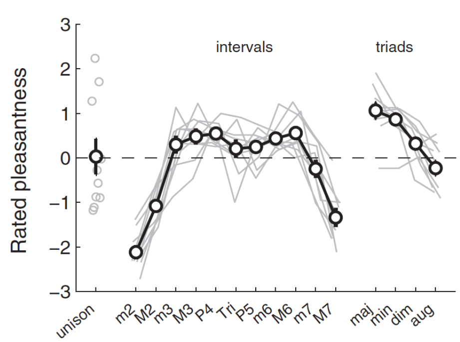


FIGURE 4. Participants in this study also rated the diminished triad, although there were fewer participants (N=10) (Cousineau et al, 2012)

Figure 5 shows energy calculations for the previous energy function, using the exponential as the weight.

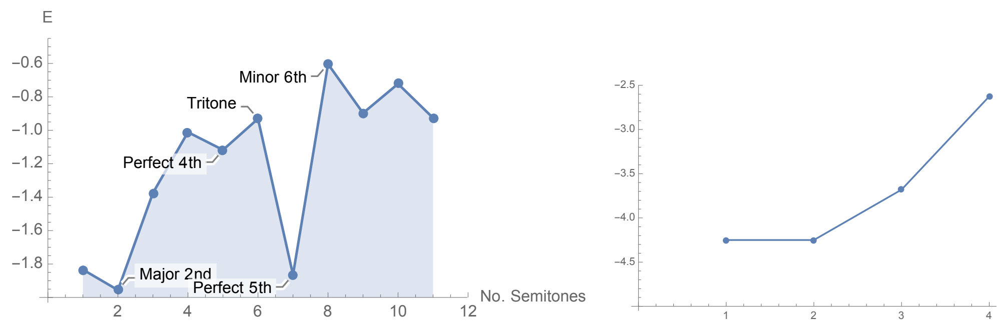


Figure 5. Energy calculations for intervals and triads from previous energy function, which utilized an exponential weight (Means, 2015).

5.1. **Improvements.** Comparing figure 5 to McDermott and Cousineau (figures 3 and 4), the trend on for the seconds and sevenths is better displayed with the previous energy, but the perfect fifth (7 semitones) has the second highest energy, which is not expected. Additionally, the trend in the triads is reversed.

The expected trend for the triads is observed after applying our new weight (figures 2b and 2d). For the Pythagorean tuning (figure 2a), the fourth (5 semitones) and fifth are nicely consonant, and the tritone between them has a higher energy. The peculiar calculation was that of the minor sixth (8 semitones).

Within the equal-tempered tuning, the trend is less clear. The fourth and fifth are again low in energy, but here the minor sixth, as well as the minor seventh, have the highest energies.

5.2. **Limitations.** The energy calculated for major and minor triads was equal in every case because the energy function presented here sums over every interval in the chord. A major triad includes the major third from root to third, the perfect fifth from root to fifth, and the minor third from third to fifth. The minor triad contains the same intervals in a different order, resulting in the same energy calculation.

6. Future Directions

The timeline of the project did not allow for analysis of actual sound waves. To do this, we would break the waves into their integer frequencies, and weight them according to their Fourier coefficients.

Because of octave equivalence, musical intervals are translation invariant. That is, they are the same regardless of between which notes in which octave they are played. The present energy is translation invariant, because it depends solely upon the ratios of the frequencies between notes and not the frequencies themselves. Future work should use the more applicable model mentioned above and investigate methods of unifying the two ideas.

Also outside the scope of this project was a consideration of melody and harmonic progressions. It is clear that some melodies are of higher energy than others, as well as harmonic progressions, but a canonical representation of these ideas may be less intuitive than the one presented here.

References

[1] A. Y. Khinchin, *Continued Fractions.* Dover Publications, Inc., Mineola, NY, Russian edition (1997).

[2] M. Cousineau, J. H. McDermott, and I. Peretz. *The basis of musical consonance as revealed by congenital amusia.* PNAS. 109 (2012), no. 48, 19858–19863.

[3] J. H. McDermott, A. J. Lehr, and A. J. Oxenham. *Individual Differences Reveal the Basis of Consonance.* Current Biology. 20 (2010), no. 11, 1035–1041.

[4] M. J. Tramo, P. A. Cariani, B. Delgutte, and L. D. Braida. *Neurobiological Foundations for the Theory of Harmony in Western Tonal Music.* Ann. N. Y. Acad. Sci. 930 (2001), 92-116.

[5] E. H. Hagen, G. A. Bryant. *Music and Dance as a Coalition Signaling System.* Human Nature. 14 (2003), no. 1, 21–51.

[6] A. Pinto. *A Dynamical Approach to Music Analysis.* Journal of Mathematics and Music. 6 (2012), no. 3, 207–219.

[7] D. Tymoczko. *The Geometry of Musical Chords.* Science. 313 (2006), 72-74.

[8] A. Means. Computing the Relative Energy of Musical Intervals. Thesis Creative Component submitted to Mathematics Department. OSU. (2015).

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