THE USE OF PRIME IDEALS IN THE CHARACTERIZATIONS OF COMMUTATIVE RINGS

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in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
May, 1974

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OF COMMUTATIVE RINGS

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ACKNOWLEDGMENTS

To my thesis adviser, Dr. Craig A. Wood, I am deeply indebted. He contributed immense time and effort toward the completion of this thesis and his advice in other areas has been excellent. The other members of my committee, Professor Jeanne Agnew, Professor Dennis Bertholf, and Professor Doug Aichele have been most helpful and cooperative. Also, I wish to thank Professor Gerald Goff for his friendship and for sharing his years of educational and classroom experience with me during my stay at Oklahoma State University.

To my family, I owe the greatest gratitude. My parents have given me support and encouragement for many years. My brother and sister, Hal and Margaret, have also contributed in many ways toward the completion of this degree. For her patience and encouragement, my sincerest thanks go to my wife, Linda.

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CHAPTER I

INTRODUCTION

This thesis is concerned with a part of the subject in commutative ring theory usually referred to as multiplicative ideal theory. In his book, entitled Multiplicative Ideal Theory [9], Robert Gilmer makes the following statement:

> Multiplicative ideal theory has its roots in the works of Richard Dedekind, in algebraic number theory, and ultimately in Fermat's Last Theorem. But much credit must be given to Emmy Noether and Wolfgang Krull for the origin and application of the axiomatic method in the subject.

Other eminent mathematicians have contributed much to the advancement of knowledge in this area as well. As we discuss some highlights of each succeeding chapter in this thesis, we include the names of some of the major contributors to this field of study.

All classes of rings we consider in this dissertation have at least one property in common. Each has one characterization in terms of its set of prime ideals. The purpose of this thesis is to make these characterizations and many other facts readily accessible to the student interested in this part of multiplicative ideal theory. We have also tried to demonstrate the close relationships which exist among these classes of rings and to include several examples which illustrate the distinctions among them. Included in Chapters III and IV is a study of two types of rings without identity. These are not considered in such standard references as [9], [13], or [22].

The reader of this work should have a background in commutative algebra roughly equivalent to the material contained in Chapters 1, 3, and 4 of Zariski and Samuel's book [22]. Also, a limited acquaintance with field theory, Chapter 2 of [22], will help. However, every attempt has been made to make this work self contained. In Chapter II, we have included several results which are used in later chapters. As most of these facts are contained in [22] or [23], they are simply listed and no proofs are given. A few things are referenced individually, as they may be a bit more difficult to find. We use the material contained in Chapter II as the need arises.

Chapter III begins the main body of the thesis. This chapter considers integral domains D which satisfy what we refer to as property C: Every ideal of D has a representation as a finite product of prime ideals of D. In the first part of this chapter we give a brief historical account of the basic ideas which were present in algebraic number theory in the mid-nineteenth century that inspired Dedekind and others to develop the tools necessary to study rings of algebraic integers more thoroughly. Eventually this led to the concept of a Dedekind domain, which is an integral domain D with identity such that D satisfies property C. We develop some of the basic properties possessed by Dedekind domains in this chapter. In fact, Theorem 3.7 contains a list of several equivalent conditions on an integral domain with identity, each of which is equivalent to D being a Dedekind domain. Many authors have contributed to the advancement of the ideal theory of Dedekind domains. Among these are W. Krull, E. Noether, M. Sono, Keizi Kubo, Kameo Matusita, Noburu Nakano, and I. S. Cohen.

The remainder of Chapter III is concerned with integral domains without identity which satisfy property C. A characterization of these domains, due to Gilmer [11], is developed. Theorem 3.17 shows such a domain J is the maximal ideal of a rank one discrete valuation ring V having the property that V is generated over J by the identity element e of V. Several examples are included which are intended to illustrate some of the basic ideas of the chapter.

In a more general setting a commutative ring R is called a general Z.P.I.-ring if R satisfies property C. This class of rings is the topic of study in Chapter IV. General Z.P.I.-rings arise in a natural way, as Examples 4.1 and 4.2 show. Some properties of the ideals of a general Z.P.I.-ring R are discussed. A proof that R is Noetherian, due to Craig Wood in [21], is given. Shinziro Mori was the first to prove this result in [16], but the proof by Wood is shorter and more straightforward. Several relationships are established as we develop the tools necessary to prove the direct sum decomposition theorem for general Z.P.I.-rings. We show that a commutative ring R is a general Z.P.I.-ring if and only if R has the following structure:

(1) If $R = R^2$, then $R = R_1 \oplus \cdots \oplus R_n$ where each R_i is a Dedekind domain or a special primary ring.

(2) If $R \neq R^2$, then $R = F \oplus T$ or $R = T$ where F is a field and T is a ring without identity such that each nonzero ideal of T is a power of T. This theorem was also originally proved by Mori [16], but, in this revised form, (1) appears in [1] by Keizo Asano and (2) appears in Wood [21].

In [15] Kathleen Levitz gives the following characterization of a general Z.P.I.-ring: A commutative ring R is a general Z.P.I.-ring if and only if every ideal of R generated by at most two elements is the finite product of prime ideals of R. The remainder of Chapter IV is primarily concerned with establishing this result. Some examples illustrating various properties of general Z.P.I.-rings are also given.

Chapter V is concerned with another generalization of Dedekind domains. This is the class of integral domains D with identity having the property that every quotient ring D_M with respect to a maximal ideal M of D is a Dedekind domain. Gilmer [12] calls such domains almost Dedekind domains. Several characterizations of almost Dedekind domains, which are results of Gilmer [12] and H. S. Butts and R. C. Phillips in [4], are given in this chapter. As the name suggests, a Dedekind domain is an almost Dedekind domain. We also show that a Noetherian almost Dedekind domain is Dedekind and offer an example of an almost Dedekind domain which is not Dedekind.

A commutative ring R with identity is said to be a multiplication ring if whenever A and B are ideals of R such that $A \subseteq B$, then there is an ideal C of R such that $A = BC$. The term multiplication ring appears in the literature as early as 1925 and much of the early theory was developed by W. Krull, S. Mori, and Yasuo Akizuki. In Chapter VI we classify all integral domains and Noetherian rings which are multiplication rings. These are shown to be the classes of Dedekind domains and general Z.P.I.-rings, respectively, which we studied in Chapters III and IV. An example of a non-Noetherian multiplication ring is also given. In the remainder of the chapter, a characterization of a multiplication ring in terms of its prime ideal structure is given. This result of Joe Mott is given in [17] and states that a commutative ring R with identity is a multiplication ring if and only if whenever P is a prime ideal of R containing an ideal A, then there is an ideal C such that $A = PC$.

In the last chapter we summarize the interrelationships which exist among several classes of rings considered in this work. Included in this summary are several examples and results from the body of this thesis which indicate the similarities and differences which exist among these classes of rings. The overall picture is displayed in a chart.

CHAPTER II

PRELIMINARY RESULTS

This chapter is devoted to listing some definitions and results that will be used in the succeeding chapters. It is not intended to be an exhaustive study of any topic nor is it a complete list of all of the facts which will be used in later chapters. Instead, it is intended to be a collection of those results which play important roles in what follows. We simply list these in the form of propositions and no proofs are included. This chapter is divided into several sections with a brief description at the beginning of most sections of the types of things to be found within that section. The section on notation and terminology should be read before proceeding into the main body of the thesis. The remaining sections may be omitted by those who feel well-acquainted with these areas. For the reader interested in seeing the proofs of these results, most can be found in either [22] or [23].

Notation and Terminology

Since we are only concerned in this thesis with commutative rings, "ring" will always mean "commutative ring". The notation and terminology of this thesis is that of [22] with two exceptions: (1) $\subset$ will denote proper containment while $\subseteq$ means contained in or equal to and (2) we do not assume a Noetherian ring contains an identity. Since it is imperative that the reader distinguish between proper and genuine ideals, we include these definitions here.

<u>2.1 Definitions</u>: Let R be a ring and let A be an ideal of R.

(a) A is said to be genuine if $(0) \subseteq A \subset R$.

(b) A is said to be proper if $(0) \subset A \subset R$.

General Remarks

<u>2.2 Definitions</u>: Let A be an ideal of a ring R.

(a) A is said to be <u>regular</u> if A contains a regular element of R.

(b) A is said to be <u>irreducible</u> if A is not a finite intersection of ideals properly containing A.

<u>2.3 Definition</u>: Let R be a ring with identity and let $R[[x]] = \{\sum_{i=0}^{\infty} r_i x^i \mid r_i \in R \text{ for each } i\}$. Define $\sum_{i=0}^{\infty} r_i x^i + \sum_{i=0}^{\infty} s_i x^i = \sum_{i=0}^{\infty} (r_i + s_i)x^i$ and $(\sum_{i=0}^{\infty} r_i x^i)\cdot(\sum_{i=0}^{\infty} s_i x^i) = \sum_{i=0}^{\infty} t_i x^i$ where $t_k = \sum r_i s_j$. Then R[[x]] is called the formal power series ring in

prime ideals and include here some facts which possibly are not so well-known.

2.5 Proposition: Let A be an ideal of a ring R and let $\{P_i\}_{i=1}^{n}$ be a finite collection of ideals of R such that at most two of the P_i are not prime. If $A \subseteq \bigcup_{i=1}^{n} P_i$, then $A \subseteq P_j$ for some j, j = 1, ⋯, n. [2; Proposition 2, p. 52].

2.6 Definition: Let A be an ideal of a ring R. The prime ideal P of R is called a minimal prime divisor of A (minimal prime of A) if

(a) $A \subseteq P$ and

(b) If P_1 is a prime ideal such that $A \subseteq P_1 \subseteq P$, then $P_1 = P$.

2.7 Proposition: Let R be a ring with identity. If an ideal A of R is contained in a prime ideal P of R, then P contains a minimal prime of A.

2.8 Definition: Let R be a ring. If $P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of n + 1 genuine prime ideals of R, we say this chain has length n. If R has a chain of genuine primes of length n but no such chain of length n + 1, we say that R has dimension n or R is n-dimensional. If contains no genuine prime ideal, we say that R has dimension -1. The dimension of a ring R is sometimes denoted dim R.

Pairwise Comaximal Ideals

These results offer a way of expressing a ring as a finite direct sum of rings if the ideal (0) can be represented as a finite intersection of ideals which are pairwise comaximal. We use this method in Chapter IV.

2.9 Definition: Let R be a ring and let $A_1, \cdots, A_n$ be a collection of ideals of R. If $A_i + A_j = R$ for each i and j, $i \neq j$, then the collection is said to be pairwise comaximal.

2.10 Proposition: Let R be a ring with identity and let $A_1, \cdots, A_n$ be a collection of ideals of R.

(a) The A_i are pairwise comaximal if and only if their radicals are.

(b) $A_1 \cap \cdots \cap A_n = A_1 \cdots A_n$ if the A_i are pairwise comaximal.

(c) $R/(A_1 \cap \cdots \cap A_n) \cong R/A_1 \oplus \cdots \oplus R/A_n$ if the A_i are pairwise comaximal.

Ideals of Finite Direct Sums

Included here are several properties possessed by ideals of a ring which is a finite direct sum of rings. Throughout the remainder of this section, we will assume that $R = R_1 \oplus \cdots \oplus R_n$ where at most one of the rings R_i does not have an identity.

2.11 Proposition: If A is an ideal of R, then $A = A_1 \oplus \cdots \oplus A_n$ where, for $i = 1, \cdots, n$, A_i is an ideal of R_i. In particular, $A_i = \{a_i \in R_i \mid (a_1, \cdots, a_i, \cdots, a_n) \in A \text{ for some } a_j \in R_j, j \neq i\}$.

2.12 Proposition: If $A = (a_1, \cdots, a_m)$ is an ideal of R where a_j is the n-tuple $(r_{1j}, \cdots, r_{nj})$, then the ideal A_i of Proposition 2.11 is $(r_{i1}, r_{i2}, \cdots, r_{im})$.

2.13 Proposition: An ideal P of R is prime if and only if $P = A_1 \oplus \cdots \oplus A_n$ where $A_i = R_i$ for each $i \neq j$ and A_j is a prime ideal of R_j.

Finitely Generated Ideals and Noetherian Rings

The ideal theory of Noetherian rings, which are rings in which every ideal is finitely generated, is basic to what we do in later chapters. The classes of rings which constitute much of our study, Chapters III and IV, are Noetherian rings. The first two propositions we include in this section, though true in Noetherian rings, are stated in a more general setting.

2.14 Proposition: Let A be a finitely generated ideal of the ring R and suppose A = AB for some ideal B of R.

(a) Then there exists an element b of B such that a = ab for each a in A.

(b) If A is a regular ideal, then R has an identity and B = R. Therefore, if $A = A^2$, then A = R.

2.15 Proposition: Let A and B be ideals of the ring R such that $A \subseteq B$ and A is finitely generated. If B/A is finitely generated in R/A, then B is finitely generated in R.

2.16 Proposition: Let R be a ring. The following statements are equivalent.

(a) R is Noetherian.

(b) There is no infinite strictly ascending chain of ideals of R.

(c) Each prime ideal of R is finitely generated. [5; Theorem 2, p. 29].

2.17 Proposition: Let A be an ideal of a Noetherian ring R. If $B = \bigcap_{i=1}^{\infty} A^i$, then AB = B. [21; Lemma 4, p. 843].

2.18 Proposition: If A is a genuine ideal of a Noetherian domain D, then $\bigcap_{i=1}^{\infty} A^i = (0)$. [21; Lemma 5, p. 843].

2.19 Definition: Let A be an ideal of a ring R. A representation $A = Q_1 \cap \cdots \cap Q_n$, where Q_i is P_i-primary for i = 1, ⋯, n, is called an irredundant (shortest) representation if it satisfies the following conditions:

(a) No Q_j contains the intersection of the other Q_i's.

(b) $P_i \neq P_j$ for $i \neq j$.

2.20 Proposition: In a Noetherian ring R, every ideal has an irredundant representation.

Extension and Contraction of Ideals

2.21 Definition: Let R and S be rings with identities and let f be a homomorphism of R into S such that $f(1_R) = 1_S$. If B is an ideal of S, the ideal $B^c = f^{-1}(B)$ is called the contraction of B. If A is an ideal of R, the ideal $A^e = f(A)S$ generated by f(A) in S is called the extension of A.

2.22 Proposition: Let R, S, and f be as in Definition 2.21 and let A and B be ideals of R and S, respectively. Then $A^{ec} \supseteq A$ and $B^{ce} \subseteq B$.

2.23 Remark: When R is a subring of S such that the identity of S is in R, then the identity map is a homomorphism of R into S. In this case, $B^c = B$ R and $A^e = AS$ and we use the notation $B \cap R$ and AS.

In most instances we are concerned with the extension and contraction of ideals in the following setting: R is a ring with identity and $S = R_M = \{\frac{a}{m} \mid a \in R, m \in M$ where R_M is the quotient ring of R with respect to the regular multiplicative system M of R.

2.24 Proposition: Let R be a ring with identity and let M be a regular multiplicative system in R.

(a) For an ideal A of R, $AR_M \cap R = A$ if and only if $A:(m) = A$ for each $m \in M$.

(b) For each ideal B of R_M, $(B \cap R)R_M = B$.

2.25 Remark: Let R be a ring with identity and let P be a proper prime ideal of R. Whenever $M = R - P$ is a regular multiplicative system in R, we can consider the quotient ring R_M of R with respect to M. We denote this ring by R_P and say that it is the quotient ring of R with respect to the prime ideal P.

2.26 Definition: Let R be a ring and let T be the total quotient ring of R. If R' is any ring between R and T, R' is called an overring of R.

The Ring*

Let R and S be rings such that $R \subseteq S$ and S has an identity. If in addition R has an identity, then we also insist that the identities be the same. In this section we consider some relationships between R and R* which we define below.

2.27 Definition: Let R and S be as above. The smallest subring of S containing R and the identity of S is denoted by R*.

2.28 Proposition: If A is an ideal of R*, then A is an ideal of R if and only if $A \subseteq R$.

2.29 Proposition: If R is Noetherian, then R* is Noetherian.

2.30 Proposition: Let D be an integral domain and let K be the quotient field of D. If D* is the smallest subring of K containing D and the identity of K, then K is also the quotient field of D*.

Fractional and Invertible Ideals

These concepts are usually introduced within the setting of an integral domain and its quotient field. In this section we consider a more general setting. Throughout this discussion, R will denote a ring containing a regular element and having total quoteint ring T. Proofs of the results which follow may be found in [20].

2.31 Definition: An R-module M contained in T is called a fractional ideal of R if there exists a regular element a in R such that $aM = \{am \mid m \in M\} \subseteq R$. F(R) denotes the collection of fractional ideals of R.

Let R* be the smallest subring of T containing R and the identity of T. We note that every ideal of R is a fractional ideal of R.

2.32 Definition: Let $F_1 \in F(R)$. F_1 is said to be invertible if there exists an $F_2 \in F(R)$ such that $F_1F_2 = R^*$. We denote F_2 by F_1^{-1} and call F_1^{-1} the inverse of F_1.

2.33 Proposition: Let A be a nonzero ideal of R. If there exists an ideal B of R such that AB = (r) where r is regular in R, then A is invertible.

2.34 Proposition: If A is an invertible ideal of R, then A^{-1} is unique and $A^{-1} = [R^*:A]_T = \{t \in T \mid tA \subseteq R^*\}$.

2.35 Proposition: If A is an invertible ideal of R, then A is a finitely generated ideal of R.

2.36 Proposition: If r is a regular element of R, then (r) is invertible.

2.37 Proposition: Let $F_1, \cdots, F_n \in F(R)$. Then $F_1 \cdots F_n$ is invertible if and only if each F_i is invertible.

2.38 Proposition: If $A_1, \cdots, A_n$ are ideals of R such that $A_1 \cdots A_n = (r)$ where r is regular in R, then each A_i is invertible.

2.39 Proposition: If A is an invertible ideal of R and if B is an ideal such that $B \subset R$, then $AB \subset AR$.

2.40 Proposition: Let F_1, F_2, and F_3 be fractional ideals of R such that F_1 is invertible. If $F_1F_2 = F_1F_3$, then $F_2 = F_3$.

Valuation Rings

The class of valuation rings is an important class of rings in the study of multiplicative ideal theory. We need only a few facts about these rings for our purposes.

2.41 Definition: Let V be an integral domain with identity. If $A \subseteq B$ or $B \subseteq A$ for every pair of ideals A and B of V, then V is called a *valuation ring*.

2.42 Proposition: An integral domain V with identity is a valuation ring if and only if for each x in the quotient field of V, either $x \in V$ or $x^{-1} \in V$.

<u>2.43 Definition</u>: A valuation ring V is <u>discrete</u> if primary ideals of V are prime powers.

<u>2.44 Definition</u>: Let V be a valuation ring. Then the ordinal type of the set of proper prime ideals of V is called the <u>rank</u> of V.

<u>2.45 Proposition</u>: Let V be a valuation ring. Then V is Noetherian if and only if V is rank one and discrete.

<u>2.46 Proposition</u>: If V is a rank one valuation ring and M is the unique maximal ideal of V, then $(0) = \bigcap_{i=1}^{\infty} M^i$.

CHAPTER III

INTEGRAL DOMAINS SATISFYING PROPERTY C

In this chapter, our overall objective is to investigate integral domains D satisfying property C: Every ideal of D can be represented as a finite product of prime ideals of D. To do this, it will be necessary to consider domains with identity separately from domains without identity.

An integral domain with identity satisfying property C is called a Dedekind domain. Included in our study of Dedekind domains is a brief description of some basic ideas that led a few mathematicians into this area of study. Also, we investigate some of the properties and characterizations of Dedekind domains. These include a proof of the Unique Factorization Theorem: Each proper ideal of a Dedekind domain has a unique representation as a finite product of proper prime ideals. There are many ideal-theoretic characterizations of Dedekind domains and a summary of these is given.

Our last objective of this chapter is to classify an integral domain J without identity having property C. This result is due to Robert Gilmer [11]. He proved that J is the maximal ideal of a rank one discrete valuation ring V such that V is generated over J by the identity element e of V, and conversely. We prove this and some other results related to domains without identity having property C.

The topics we consider throughout this chapter and in succeeding chapters come under that part of commutative ring theory called multiplicative ideal theory. Ultimately, multiplicative ideal theory has its roots in the works of Kummer and Dedekind in algebraic number theory. It grew out of a mistake made by Kummer in the nineteenth century when he was trying to solve Fermat's Last Theorem, a famous problem in number theory. He assumed the rings of algebraic integers were unique factorization domains (UFD's). That this actually was a mistake can be shown by the following example.

<u>3.1 Example</u>: Let $F = Q(\sqrt{10}) = \{a + b\sqrt{10} \mid a, b \in Q\}$ where Q denotes the field of rational numbers. Then F is a finite algebraic extension of Q. Let $D = Z(\sqrt{10}) = \{t \in F \mid t$ is an algebraic integer in $F\}$ $= \{t \in F \mid t$ is a root of a polynomial of the form $x^2 + cx + d$ for some c and d in $Z\}$ where Z denotes the ring of integers. In the terminology of the algebraic number theorist, the integral domain D is a ring of algebraic integers when considered as a subring of the algebraic number field F. Some of the properties possessed by D and F are summarized below. We define $N(x + y\sqrt{10}) = x^2 - 10y^2$ where $x, y \in Q$.

(a) An element $t = a + b\sqrt{10}$ of F is an algebraic integer if and only if a and b are integers. Thus $Z(\sqrt{10}) = \{a + b\sqrt{10} \mid a, b \in Z\}$.

(b) An element $u = a + b\sqrt{10}$ of D is a unit if and only if $N(u) = a^2 - 10b^2$ is equal to ± 1.

(c) Let $t \in D$, $t \neq 0$ and not a unit. Then t is prime in D if and only if whenever $t = t_1 t_2$ where t_1 and t_2 are in D, then either t_1 or t_2 is a unit of D.

(d) $Z(\sqrt{10})$ is the integral closure of Z in $Q(\sqrt{10})$.

(e) If $t \in Z(\sqrt{10})$, the $N(t)$ is an integer.

Now consider the element 6 of D. We have $6 = 2 \cdot 3 = (4 + \sqrt{10})(4 - \sqrt{10})$. We show that 2, 3, $4 + \sqrt{10}$, and $4 - \sqrt{10}$ are all prime in D and neither of 2 nor 3 is an associate of $4 + \sqrt{10}$ or $4 - \sqrt{10}$.

Suppose that for some nonunits t_1 and t_2 of D, $2 = t_1 t_2$ where $t_1 = a_1 + b_1\sqrt{10}$ and $t_2 = a_2 + b_2\sqrt{10}$ for some integers a_1, a_2, b_1, b_2. Then $N(2) = 4 = N(t_1)N(t_2) = (a_1^2 - 10b_1^2)(a_2^2 - 10b_2^2)$. Since neither t_1 nor t_2 is a unit, we must have either $a_1^2 - 10b_1^2 = a_2^2 - 10b_2^2 = 2$ or $a_1^2 - 10b_1^2 = a_2^2 - 10b_2^2 = -2$. We claim there are no integers x and y such that

$$x^2 - 10y^2 = 2 \tag{2.1.1}$$

or

$$x^2 - 10y^2 = -2 \tag{2.1.2}$$

If there are such integers, we may assume x to be a natural number, since $x^2 = (-x)^2$. For some integers q and r, $0 \leq r \leq 9$, $x = 10q + r$. Then $x^2 = 100q^2 + 20qr + r^2$ and substitution into 2.1.1 and 2.1.2 yields $r^2 - 2 = 10(y^2 - 10q^2 - 2qr)$ and $r^2 + 2 = 10(y^2 - 10q^2 - 2qr)$. Since r can only take integer values from 0 to 9, we have the following table.

r	0	1	2	3	4	5	6	7	8	9
r^2-2	-2	-1	2	7	14	23	34	47	62	79
r^2+2	2	3	6	11	18	27	38	51	66	83

Thus, neither $r^2 - 2$ nor $r^2 + 2$ is divisible by 10 for any of the values of r. It follows that neither $x^2 - 10y^2 = 2$ nor $x^2 - 10y^2 = -2$ has a solution in integers. We may conclude if $2 = t_1 t_2$, then at least one of

t_1 or t_2 is a unit in D. Therefore, 2 is prime in D.

The proof that 3 is prime in D is very similar to the above argument. The assumption that $3 = t_1t_2$, where neither t_1 nor t_2 is a unit, leads to the consideration of the solutions of the equations $x^2 - 10y^2 = 3$ and $x^2 - 10y^2 = -3$ for some integers x and y. Setting $x = 10q + r$ as before, we have $r^2 - 3 = 10(y^2 - 10q^2 - 2qr)$ or $r^2 + 3 = 10(y^2 - 10q^2 - 2qr)$ for some r, $0 \leq r \leq 9$. Thus, $r^2 - 3$ or $r^2 + 3$ would have to be divisible by 10. However, the table below shows no such r exists.

r	0	1	2	3	4	5	6	7	8	9
r^2-3	-3	-2	1	6	13	22	33	46	61	78
r^2+3	3	4	7	12	19	28	39	52	67	84

Now consider $4 + \sqrt{10}$. If for some nonunits t_1 and t_2 of D, $4 + \sqrt{10} = t_1t_2 = (a_1 + b_1\sqrt{10})(a_2 + b_2\sqrt{10})$, then we must have $6 = (a_1^2 - 10b_1^2)(a_2^2 - 10b_2^2)$. Therefore, $a_1^2 - 10b_1^2$ must be either ± 2 or ± 3 and we have already seen that none of these four possibilities can occur. This proves $4 + \sqrt{10}$ is prime in D. Similarly, $4 - \sqrt{10}$ is prime in D.

The only thing left to show is that neither 2 nor 3 is an associate of $4 + \sqrt{10}$ or $4 - \sqrt{10}$. Suppose $4 + \sqrt{10} = 2y$ where $y \in Z(\sqrt{10})$. Then $N(4 + \sqrt{10}) = N(2)N(y)$ implies that $6 = 4N(y)$ and hence $N(y) = 3/2$. This shows $y \notin Z(\sqrt{10})$ since $N(y) \notin Z$. Therefore, 2 and $4 + \sqrt{10}$ are not associates. Similarly, 2 is not an associate of $4 - \sqrt{10}$ and 3 is not an associate of $4 + \sqrt{10}$ or $4 - \sqrt{10}$. Hence, we have 6 represented as a product of prime elements of D in two distinct ways. This proves D is not a UFD. ▲

The concept of an ideal, which was introduced by Dedekind, was eventually used in the study of algebraic number theory. The notion of a prime ideal can be used to introduce a unique factorization property into the rings of algebraic integers. This is accomplished by representing ideals as finite products of prime ideals. In particular Remark 3.8 shows every ideal of $Z(\sqrt{10})$ has a unique representation as a finite product of prime ideals of $Z(\sqrt{10})$. The use of prime ideals in characterizing rings has become an extensive area of study.

In the definition of a Dedekind domain, it is not required that the representation of ideals as finite products of prime ideals be unique. To prove that this factorization is unique for certain ideals of a Dedekind domain, Theorem 3.5, is our next objective. The proof of this result is a direct consequence of the three lemmas which immediately follow this discussion.

<u>3.2 Lemma</u>: Let D be an integral domain with identity. If an ideal A of D has a representation as a finite product of invertible proper prime ideals of D, then the representation is unique to within the order of the factors.

<u>Proof</u>: Suppose $A = P_1 \cdots P_k = Q_1 \cdots Q_t$ where P_i and Q_j are invertible proper prime ideals of D for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, t$. Assume P_1 is minimal among $P_1, \cdots, P_k$. Since $Q_1 \cdots Q_t \subseteq P_1$, some $Q_j \subseteq P_1$, say $Q_1 \subseteq P_1$. Since $P_1 \cdots P_k \subseteq Q_1$, some P_i is contained in Q_1. The minimality of P_1 implies that $i = 1$, and so $Q_1 = P_1$. By the cancellation property for invertible ideals, Proposition 2.40, $P_2 \cdots P_k = Q_2 \cdots Q_t$, and we can repeat the argument. Since each P_i and each Q_j is a proper ideal of D, we must have $k = t$. ▲

3.3 Lemma: Let D be a Dedekind domain. Then every invertible proper prime ideal of D is a maximal ideal.

Proof: Let P be an invertible proper prime ideal of D and let $a \in D - P$. Consider the ideals $P + (a)$ and $P + (a^2)$ of D. Then $P + (a) = \prod_{i=1}^{n} P_i$ and $P + (a^2) = \prod_{j=1}^{m} Q_j$, where the P_i and the Q_j are prime ideals of D. Let $\overline{R} = R/P$ and $\overline{a} = a + P$. We have $(\overline{a}) = \prod_{i=1}^{n} (P_i/P)$, $(\overline{a}^2) = \prod_{j=1}^{m} (Q_j/P)$ where the ideals P_i/P and Q_j/P are prime. Proposition 2.38 implies these prime ideals are invertible. Since $(\overline{a}^2) = \prod_{i=1}^{n} (P_i/P)^2$, Lemma 3.2 shows the ideals $Q_1/P, \cdots, Q_m/P$ are the ideals $P_1/P, \cdots, P_n/P$, each repeated twice. Consequently, we have $m = 2n$, and we can renumber the Q_j's in such a way that $Q_{2i}/P = Q_{2i-1}/P = P_i/P$. Thus, $Q_{2i} = Q_{2i-1} = P_i$ and $P + (a^2) = [P + (a)]^2$. Hence, $P \subseteq [P + (a)]^2 \subseteq P^2 + (a)$. Thus, if $x \in P$, $x = y + da$ where $y \in P^2$ and $d \in D$. Therefore, $da \in P$ and since $a \notin P$, $d \in P$; that is, $P \subseteq P^2 + P(a)$. As the inclusion $P^2 + P(a) \subseteq P$ always holds, we have $P = P^2 + P(a)$. Since P is invertible, it follows that $D = P + (a)$. Therefore, P is a maximal ideal of D. ▲

3.4 Lemma: Every nonzero prime ideal of a Dedekind domain D is invertible.

Proof: Let P be a nonzero prime ideal of D. If $P = D$, then P is invertible. We can assume that $P \neq D$. Let $a \in P$, $a \neq 0$. Then $(a) = P_1 \cdots P_k$, where each P_i is a proper prime ideal of D. Since (a) is invertible, each P_i is invertible by Proposition 2.37 and maximal by Lemma 3.3. Since $P_1 \cdots P_k \subseteq P$, $P_i \subseteq P$ for some i. Hence, $P_i = P$ and

P is invertible. ▲

<u>3.5 Theorem</u>: Let D be a Dedekind domain. Then every proper ideal of D has a unique representation as a finite product of proper prime ideals.

<u>Proof</u>: If A is a proper ideal of D, then A has a representation as a finite product of proper prime ideals. By Lemma 3.4 these primes are invertible. It follows from Lemma 3.3 that the representation for A is unique. ▲

This establishes the unique factorization property for proper ideals of a Dedekind domain. Three more well known properties of Dedekind domains are given in the next theorem.

<u>3.6 Theorem</u>: If D is a Dedekind domain, then (1) D is Noetherian, (2) the dimension of D is less than two, and (3) every nonzero ideal of D is invertible.

<u>Proof</u>: Proposition 2.37 states that the product of invertible ideals is invertible and Proposition 2.35 shows invertible ideals are finitely generated. Thus, the first and third properties hold. Lemmas 3.3 and 3.4 imply the dimension of D is less than two. ▲

As we mentioned earlier, there are many ideal-theoretic characterizations of Dedekind domains. We present here a list of some of these without proofs. For the reader who wishes to see these proofs or to study Dedekind domains in depth, several books are available. For example, [22], [13], and [9] all contain one or more chapters on Dedekind domains.

<u>3.7 Theorem</u>: Let D be an integral domain with identity, $\{M_r\}$ the set of maximal ideals of D, and F(D) the set of nonzero fractional ideals of D. The following are equivalent.

(1) D is a Dedekind domain.

(2) Each proper ideal of D is uniquely expressible as a finite product of proper prime ideals of D.

(3) D is Noetherian, integrally closed, and has dimension less than two.

(4) F(D) is a group with respect to multiplication.

(5) Each proper ideal of D is invertible.

(6) Each proper prime ideal of D is invertible.

(7) Each ideal of D has a basis of at most two elements.

(8) Each proper homomorphic image of D is a principal ideal ring.

(9) If A and C are ideals of D and if $A \subseteq C$, then there is an ideal B of D such that A = BC.

For (10) - (24) let D be a Noetherian integral domain with identity.

(10) Each nonzero ideal of D generated by two elements is invertible.

(11) Whenever AB = AC for ideals A, B, and C of D with $A \neq (0)$, then B = C.

(12) $A(B \cap C) = AB \cap AC$ for all ideals A, B, and C of D.

(13) $(A + B)(A \cap B) = AB$ for all ideals A and B of D.

(14) $(A + B):C = (A:C) + (B:C)$ for all ideals A, B, and C of D.

(15) $C:(A \cap B) = (C:A) + (C:B)$ for all ideals A, B, and C of D.

(16) $A + (B \cap C) = (A + B) \cap (A + C)$ for all ideals A, B, and C of D.

(17) $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all ideals A, B and C of D.

(18) If $a, b \in D$, then $[(a):(b)] + [(b):(a)] = D$.

(19) D_P is a valuation ring for each proper prime ideal P of D.

(20) D_M is a valuation ring for each maximal ideal M of D.

(21) For each M_r, there are no ideals properly between M_r and M_r^2.

(22) For each M_r, the set of M_r-primary ideals of D is linearly ordered under $\subseteq$.

(23) For each M_r, M_r-primary ideals are powers of M_r.

(24) For each M_r, M_r-primary ideals are irreducible.

(25) For each M_r, M_r^2 is irreducible.

(26) Primary ideals of D are prime powers.

(27) Primary ideals of D are irreducible.

In example 3.1 we saw that the ring of algebraic integers $Z(\sqrt{10})$ of the algebraic number field $Q(\sqrt{10})$ was not a UFD. However, $Z(\sqrt{10})$ is a Dedekind domain. We show this and exhibit the prime ideal factorization of the ideal (6) in the remark which follows.

3.8 Remark: Let D be a Dedekind domain and let L be a finite algebraic extension of its quotient field. Then the integral closure of D in L is also a Dedekind domain. A proof of this result may be found in [22; Theorem 19, p. 281]. In particular since the ring of integers Z is a Dedekind domain and since $Z(\sqrt{10})$ is the integral closure of Z in $Q(\sqrt{10})$, we may conclude $Z(\sqrt{10})$ is a Dedekind domain. Previously, we observed that $Z(\sqrt{10})$ is not a UFD. We proved the element 6 of $Z(\sqrt{10})$ has at least two distinct factorizations into prime elements. Now we investigate the prime ideal representation of the ideal (6) in $Z(\sqrt{10})$.

Consider the three ideals $P_1 = (2, \sqrt{10})$, $P_2 = (3, 4 + \sqrt{10})$ and $P_3 = (3, 4 - \sqrt{10})$ of $Z(\sqrt{10})$. We show $(6) = P_1^2P_2P_3$ and P_1, P_2 and P_3 are all prime ideals of $Z(\sqrt{10})$.

We first observe the equality $P_1^2 = (4, 2\sqrt{10}), 10)$. We claim $P_1^2 = (2)$. Since $2|4$, $2|2\sqrt{10}$, and $2|10$, $P_1^2 \subseteq (2)$. But $2 = 10 - 2\cdot 4$ and so $(2) \subseteq P_1^2$. Hence, $P_1^2 = (2)$.

Next, we show $P_2 \cdot P_3 = (3)$. We have $P_2 \cdot P_3 = (3, 4 + \sqrt{10})(3, 4 - \sqrt{10}) = (9, 12 - 3\sqrt{10}, 12 + 3\sqrt{10}, 6)$. Since 3 divides all of the elements of the basis of $P_2 \cdot P_3$, it follows that $P_2 \cdot P_3 \subseteq (3)$. Also, $3 = 9 - 6$ and so $(3) \subseteq P_2 \cdot P_3$. Hence, $P_2 \cdot P_3 = (3)$ and $P_1^2 P_2 P_3 = (2)(3) = (6)$.

We now show the ideals P_1, P_2 and P_3 are prime. Since $P_1 = (2, \sqrt{10})$, any number $t = 2a + \sqrt{10}\, b$ where $a, b \in Z$ belongs to P_1. We claim every element of P_1 can be written in this form. For if $t \in P_1$, then $t = 2r_1 + \sqrt{10}\, r_2$ where $r_1, r_2 \in Z(\sqrt{10})$. Then $r_1 = c_1 + \sqrt{10}\, d_1$ and $r_2 = c_2 + \sqrt{10}\, d_2$ where $c_1, c_2, d_1, d_2 \in Z$. Substituting these expressions for r_1 and r_2 into $t = 2r_1 + \sqrt{10}\, r_2$, we obtain $t = 2(c_1 + \sqrt{10}\, d_1) + \sqrt{10}\,(c_2 + \sqrt{10}\, d_2) = 2c_1 + 10d_2 + \sqrt{10}\,(2d_1 + c_2)$. Since $2c_1 + 10d_2$ is an even integer, $P_1 = \{t \in Z(\sqrt{10}) \mid t = 2a + \sqrt{10}\, b \text{ where } a, b \in Z\}$. Let $t_1 = a_1 + \sqrt{10}\, b_1$ and $t_2 = a_2 + \sqrt{10}\, b_2$ and suppose $t_1 \notin P_1$ and $t_2 \notin P_1$. Then a_1 and a_2 are odd integers as is $a_1a_2 + 10b_1b_2$. However, $t_1t_2 = (a_1a_2 + 10b_1b_2) + \sqrt{10}\,(a_1b_2 + b_1a_2)$ and we conclude $t_1t_2 \notin P_1$. Thus, P_1 is prime.

Now consider the ideal $P_3 = (3, 4 - \sqrt{10})$, and let $t = a + \sqrt{10}\, b$ be any element of $Z(\sqrt{10})$. We show $t \in P_3$ if and only if $a + b$ is divisible by 3.

First, suppose $3 \mid a + b$. Then $a + b = 3k$ for some $k \in Z$. Thus, $t = a + \sqrt{10}\, b = 3k - b + \sqrt{10}\, b = 3(k + b) - (4 - \sqrt{10})b$ and we can conclude that $t \in P_3$.

Now suppose $t \in P_3$. Then $k = 3r_1 + (4 - \sqrt{10})r_2$ where $r_1 = a_1 + \sqrt{10}\, b_1$, $r_2 = a_2 + \sqrt{10}\, b_2$ for some $a_1, a_2, b_1, b_2 \in Z$. Hence, $t = (3a_1 + 4a_2 - 10b_2) + \sqrt{10}\,(3b_1 - a_2 + 4b_2)$. Comparing this with the expression $t = a + \sqrt{10}\, b$, we see that $a + b = 3(a_1 + b_1 + a_2 - 2b_2)$.

Hence, $3 \mid a + b$ and $P_3 = \{t \in Z(\sqrt{10}) \mid t = a + \sqrt{10}\, b$ where $a, b \in Z$ and $3 \mid a + b\}$.

Now consider the product $t_1t_2 = (a_1a_2 + 10b_1b_2) + \sqrt{10}\,(a_1b_2 + b_1a_2)$ of the elements $t_1 = a_1 + \sqrt{10}\, b_1$ and $t_2 = a_2 + \sqrt{10}\, b_2$ where $a_1, a_2, b_1, b_2 \in Z$. Letting $t_1t_2 = a + \sqrt{10}\, b$ where $a, b \in Z$, we see that $a = a_1a_2 + 10b_1b_2$, $b = a_1b_2 + b_1a_2$, and $a + b = a_1a_2 + 10b_1b_2 + a_1b_2 + b_1a_2 = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 + 9b_1b_2 = (a_1 + b_1)(a_2 + b_2) + 9b_1b_2$. Then $a + b$ is divisible by 3 if and only if $3 \mid (a_1 + b_1)(a_2 + b_2)$. If neither t_1 nor t_2 belongs to P_3, then neither $a_1 + b_1$ nor $a_2 + b_2$ is divisible by 3. Therefore, $(a_1 + b_1)(a_2 + b_2)$ and hence $a + b$ are not divisible by 3. We conclude that $t_1t_2 \notin P_3$ which shows P_3 is prime.

A similar argument proves that P_2 is prime. One can show $P_2 = \{t \in Z(\sqrt{10}) \mid t = a + b\sqrt{10}$ where $a, b \in Z$ and $3 \mid a - b\}$, and $t_1t_2 \in P_2$ if and only if $3 \mid (a_1 - b_1)(a_2 - b_2)$ where $t_1 = a_1 + \sqrt{10}\, b_1$ and $t_2 = a_2 + \sqrt{10}\, b_2$ for some $a_1, a_2, b_1, b_2 \in Z$. If neither t_1 nor t_2 belongs to P_3, then neither $a_1 - b_1$ nor $a_2 - b_2$ is divisible by 3. Thus, if $t_1 \notin P_3$ and $t_2 \notin P_3$, then $t_1t_2 \notin P_3$ and P_3 is prime. Therefore, the equality $(6) = P_1^2P_2P_3$ represents the factorization of the ideal (6) into prime ideals. ▲

When studying ring (integral domain, field) theory the student usually encounters the idea of a polynomial ring in one indeterminate over a ring (integral domain, field). It is well known that if D is an integral domain with identity, then D[x] is an integral domain with identity. From this observation a question arises: If in addition D is a Dedekind domain, then is D[x] a Dedekind domain? The answer is no, unless D is a field, and the next remark proves this.

3.9 Remark: If D is a Dedekind domain which is not a field, then D[x] is not a Dedekind domain. Let P be a proper prime ideal of D. We show the dimension of D[x] is greater than one. By Proposition 2.4, $D[x]/PD[x] \cong (D/P)[x]$. Since $(D/P)[x]$ is an integral domain, $D[x]/PD[x]$ is an integral domain. Thus, $PD[x]$ is prime in D[x].

From part (7) of Theorem 3.7, P = (a,b) for some elements a, b ε D. Consider the ideal (a,b,x) of D[x] and the mapping $f: D[x] \to D/(a,b)$ where $f(\sum_{i=0}^{n} a_i x^i) = \overline{a}_0$. It is straightforward to show that f is an epimorphism. Also, $\ker f = \{\sum_{i=0}^{n} a_i x^i \in D[x] \mid a_0 \in (a,b)\} = (a,b,x)$. Therefore, $D[x]/(a,b,x) \cong D/(a,b)$. Since (a,b) is maximal in D, D/(a,b) is a field. Thus, D[x]/(a,b,x) is a field and so (a,b,x) is a maximal prime of D[x]. Hence, $(0) \subset PD[x] \subset (a,b,x)$ is a chain of three genuine primes of D[x]. This proves that the dimension of D[x] is greater than one and, consesequently, is not a Dedekind domain.

However, if D is a field, then D[x] is a principal ideal domain with identity and hence a Dedekind domain. ▲

In the definition of a Dedekind domain D, we require that D have an identity. Gilmer, in [11], has investigated integral domains J without identity satisfying property C. In the remainder of this chapter, we consider Gilmer's results. Henceforth in this chapter, J denotes an integral domain without identity satisfying property C. Also, if D is a domain with quotient field K, then D* denotes the subring of K generated by D and the identity element e of K.

It seems that Gilmer originally set out to prove an analog to characterization (3) of Theorem 3.7: An integral domain with identity is a

Dedekind domain if and only if D is Noetherian, integrally closed, and has dimension less than two. This particular result is essentially due to E. Noether [18]. Gilmer did prove that J satisfies Noether's three conditions. However, an integral domain without identity satisfying Noether's three conditions need not have property C. The even integers E is an example of such a domain and this is proved in Example 3.10. In [11] Gilmer states that while an analog to Noether's result is obtainable, the following characterization of J seems more desirable: J is the maximal ideal of a rank one discrete valuation ring V such that V is generated over J by the identity e of V, and conversely. After we establish some basic properties of J, we prove this characterization.

<u>3.10 Example</u>: The domain of even integers E is Noetherian, integrally closed, and has dimension less than two but does not have property C. The fact that E is Noetherian is clear. The set of proper prime ideals of E is $\mathcal{A} = \{(2p) \mid p \text{ is an odd prime in } Z\}$. Thus, no chain $P_1 \subset P_2 \subset P_3$ of three genuine prime ideals exists in E and hence the dimension of E is less than two. To see that E is integrally closed, we must prove every rational number x which satisfies $x^{n+1} + d_n x^n + \cdots + d_1 x + d_0 = 0$, where $d_0, \cdots, d_n$ are in E and n is a nonnegative integer, must be an even integer. If $x = p/q$ where p and q are integers then $p \mid d_0$ and $q \mid 1$. Therefore, x is an integer. In addition $d_n x^n + \cdots + d_0$ is an even integer and so x^{n+1} must be an even integer. This proves that $x \in E$ and hence E is integrally closed.

Next we show E does not have property C. Consider the ideal (18) in E. It is not prime since $6 \cdot 6 \in (18)$ but $6 \notin (18)$. If we assume that $(18) = P_1 \cdots P_n$, where P_i is a proper prime for $i = 1, 2, \cdots, n$ and $n \geq 2$, then the characterization of the proper prime ideals of E shows that $4 \mid 18$. Thus, (18) is not the finite product of proper prime ideals of E. Therefore, E does not have property C. ▲

The proof of the following lemma, which generalizes the unique factorization property we proved in Lemma 3.2, is very similar to the proof of Lemma 3.2. For this reason, the proof is omitted.

3.11 Lemma: Suppose A is a proper ideal of the domain D such that A can be expressed as a product of invertible prime ideals of D. This representation is unique if $D \subset D^*$ and unique to within factors of D if $D = D^*$.

The maximality and invertibility of the proper prime ideals of J is established next. Putting these two properties together, in conjunction with Lemma 3.11, gives us the unique factorization property for proper ideals of J. A slight modification of the proof of Lemma 3.3, the analogous result for Dedekind domains, is used to prove Theorem 3.12. The proof of Theorem 3.13, with "proper" replacing "nonzero", is identical to the proof of Lemma 3.4.

3.12 Theorem: Every invertible proper prime ideal of J is a maximal ideal.

Proof: Let P be an invertible proper prime ideal of J and choose $a \in J - P$. We express $P + (a)$ and $P + (a^2)$ as products of prime ideals: $P + (a) = J^k P_1 \cdots P_r$, $P + (a^2) = J^* Q_1 \cdots Q_s$ where each P_i and each Q_j is a proper ideal of J. In $\overline{J} = J/P$ we have, $(\overline{a}) = \overline{J}^k\ \overline{P}_1 \cdots \overline{P}_r$ and $(\overline{a})^2 = \overline{J}^*\ \overline{Q}_1 \cdots \overline{Q}_s$. By Lemma 3.11, $s = 2r$ and by proper labeling $P_i = Q_{2i-1} = Q_{2i}$ for $i = 1, \cdots, r$. If $\overline{J}$ does not contain an identity, Lemma 3.11 implies that $t = 2k$ and thus $P + (a^2) = [P + (a)]^2$. Next assume $\bar{J}$ does contain an identity. If $(\bar{a}) = \bar{J}$, then $(\bar{a})^2 = \bar{J}$ and $[P + (a)]^2 = P + (a)^2$. If $(\bar{a}) \subset \bar{J}$, then r is positive, $(\bar{a}) = \overline{P_1 \cdots P_r}$, $(\bar{a})^2 = \overline{Q_1 \cdots Q_s}$, and it follows that $[P + (a)]^2 = P_1^2 \cdots P_r^2 = P + (a^2)$. Therefore, in each case we have $P + (a^2) = [P + (a)]^2$. As in the proof of Lemma 3.3, we can show that $P = P^2 + P(a) = P[P + (a)]$. Multiplying both sides of this expression by JP^{-1} gives us $J = J[P + (a)] \subseteq P + (a)$. Therefore, $J = P + (a)$ and P is a maximal ideal of J. ▲

3.13 Theorem: Every proper prime ideal of J is invertible.

Theorems 3.12 and 3.13 imply J satisfies the second of Noether's three conditions; that is, the dimension of J is less than two. Theorem 3.14, together with Cohen's Theorem (Proposition 2.16(c)), show J is Noetherian.

3.14 Theorem: Every prime ideal of J is finitely generated. Thus, J is Noetherian.

Proof: Since proper prime ideals of J are invertible, they are finitely generated by Proposition 2.35. Thus, we need only show J is finitely generated. If J contains a proper prime ideal P, then $P = (p_1, \cdots, p_s)$ is maximal by Theorem 3.12. Hence, if $d \in J - P$, then $J = (p_1, \cdots, p_s, d)$. If J does not contain a proper prime ideal, then given $d \in J - (0)$, $(d) = J^k$ for some integer $k \geq 1$. Since J is an integral domain, Proposition 2.36 shows that (d) is invertible. Therefore, J is invertible by Proposition 2.38 and thus finitely generated. Since every prime ideal of J is finitely generated, Cohen's Theorem implies J is Noetherian. ▲

We have seen that if D is a Dedekind domain then every ideal of D is generated by at most two elements. We can now establish a similar result for J.

3.15 Theorem: Every nonzero ideal of J is a power of J and, in fact, J is a principal ideal domain (PID).

Proof: Since J is Noetherian and $J \subset J^*$, $J^2 \subset J$ by Proposition 2.14(b). Choose $x \in J - J^2$. Then $(x) = P_1 \cdots P_n$ where P_i is a prime ideal of J for $i = 1, 2, \cdots, n$. Since $x \notin J^2$, we must have $n = 1$. Therefore, (x) is a prime ideal of J. We will show that (x) = J. We suppose $(x) \subset J$. Since (x) is invertible and $J \subset J^*$, $(x) \supset (x)J \supset (x^2)$. If A is any ideal such that $(x) \supset A \supset (x^2)$ and if P is a prime factor of A, then $P \supseteq (x)$. Theorem 3.12 implies P = (x) or P = J. Because $(x) \supset A \supset (x^2)$, $A = (x)J^k$ for some $k \geq 1$. But $x \notin J^2$ so that $x^2 \notin (x)J^k$ for $k \geq 2$. Therefore, k = 1 and (x)J is the unique ideal properly between (x) and (x^2).

We next show (x^2) is a primary ideal. Suppose $a, b \in J$ such that $ab \in (x^2)$ and $a \notin (x)$. Now $b \in (x)$ so that $(x^2) \subseteq (x^2, b) \subseteq (x)$. Since (x) is maximal and prime in J, J/(x) contains an identity element $\overline{u}$. Because $a \notin (x)$, $ua \notin (x)$ so that $uax \notin (x^2)$. Thus, $ux \notin (x^2, b)$ since $(x^2, ab) = (x^2)$. Hence, $(x^2, b) \not\supseteq (x)J$ and we conclude $(x^2, b) = (x^2)$ by the preceding paragraph. Therefore, $b \in (x^2)$ and (x^2) is primary.

Now $ua - a \in (x)$ so that $(ua - a)^3 \in (x^2)$. If $z \in J$, then $z(ua - a)^3 = a^3(tz - z) \in (x^2)$ where t is a fixed element of J independent of z. Since $a^3 \notin (x)$ and (x^2) is (x)-primary, $tz - z \in (x^2)$ for each $z \in J$; that is, $J/(x^2)$ contains an identity. This means, however, that $V = (x)/(x^2)$ is a vector space of the field J/(x). There is a one-to-one correspondence between the collection of subspaces of V and the set of ideals of J between (x) and (x^2). Thus, V has exactly one nonzero proper subspace, which is impossible. We conclude that J = (x).

Let P be a proper prime ideal of J. We have seen that P is not properly contained between (x^2) and (x). If $P \not\subseteq (x^2)$ then there exists an element $y \in P$ such that $y \notin (x^2) = J^2$. The argument above shows $(y) = J \subseteq P$ which contradicts the fact that P is a proper ideal of J. Thus, we must have $P \subseteq J^2 = (x^2)$. Since (x^2) is an invertible ideal of J*, there exists an ideal B of J* such that $P = (x^2)B = (x)[(x)B] = (x)A$ where A = (x)B. Since $(x) \not\subseteq P$ we must have $A \subseteq P$ and thus A = P. Now $(x) = J \subset J^*$ so that P is not invertible and hence P = (0). Consequently, J is the only nonzero prime ideal of J. Therefore, if A is a nonzero ideal of J, $A = J^k = (x^k)$ for some positive integer k. ▲

Having established these preliminaries, we are now ready to prove Gilmer's characterization of J.

<u>3.16 Theorem</u>: [11; Theorem 4, p. 581]. J* is a rank one discrete valuation ring and J is the maximal ideal of J*. Conversely, if D is a rank one discrete valuation ring with maximal ideal M and if D = M*, then M is a domain without identity having property C.

<u>Proof</u>: Since J is Noetherian, J* is Noetherian by Proposition 2.29. Since a Noetherian valuation ring is necessarily discrete and of rank one (Proposition 2.45), we need only show that J* is a valuation ring. Let $y \in K$, the quotient field of J*. We need to show either y or y^{-1} is in J*. By Proposition 2.30, K is also the quotient field of J. Thus, y = a/b for some elements a and b of J. By Theorem 3.15 the ideals (a) and (b) of J compare; that is, $(a) \subseteq (b) \subseteq J \subset J^*$ or $(b) \subseteq (a) \subseteq J \subset J^*$. Therefore, either $a/b \in J^*$ or $b/a \in J^*$. This proves that J* is a

Noetherian valuation ring and hence discrete and of rank one.

If M is the maximal ideal of J*, then $J = M^r$ for some positive integer r. Since $M \subset J^*$ and M is invertible, $M^{r+1} \subset M^r = J$. Theorem 3.15 asserts that $M^{r+1} = J^s = (M^r)^s$ for some integer s. Consequently, r + 1 = rs. Hence, r = 1 and J = M.

To prove the converse, we recall that a subset B of the maximal ideal M of D is an ideal of M* = D if and only if B is an ideal of M (Proposition 2.28). Since D is a rank one discrete valuation ring, every proper ideal of D is a power of M. Therefore, if A is an ideal of M, $A = M^k$ for some integer k. This proves M is a domain without identity having property C. ▲

I. S. Cohen in [6] has classified all rank one discrete valuation rings D with maximal ideal M such that D = M*. Indeed, he proved $D \cong \pi_p[[x]]$ and M = (x) where $\pi_p[[x]]$ is the formal power series ring over a prime field of characteristic p for some prime integer p.

<u>3.17 Remark</u>: Gilmer states that the proof of the converse of Theorem 3.16 is an immediate consequence of the relationship between the ideals of M and M* = D and the fact that a rank one discrete valuation ring is a Dedekind domain. This observation is certainly true but could be misleading if the reader interprets the statement incorrectly. It seems to suggest, at first glance, that if H is a Dedekind domain and N is a maximal ideal of H such that H = N*, then N is an integral domain without identity having property C. The maximal ideal (2) of the Dedekind domain Z has the property that (2)* = Z but does not have property C (Example 3.10). Referring to the proof of the converse of Theorem 3.17, we see the proof depends on the fact that M is the <u>unique</u> maximal ideal

of D so that every proper ideal of D is a power of M. However, an ideal of N may not be a power of N if N is any maximal ideal of the Dedekind domain H. ▲

CHAPTER IV

GENERAL Z.P.I.-RINGS

A natural question arises from the study of Dedekind domains and domains without identity satisfying property C. What properties are possessed by rings satisfying property C? Such a ring is called a <u>general</u> <u>Z.P.I.-ring</u>.

We begin this chapter by considering two examples. These examples show general Z.P.I.-rings are very natural generalizations of integral domains satisfying property C.

<u>4.1 Example:</u> Let $R = Z \oplus Z$ where Z is the ring of integers. If A is an ideal of R, then Proposition 2.11 shows $A = B \oplus C$ for some ideals B and C of Z. Since Z is a Dedekind domain, there exist prime ideals $P_1, \cdots, P_n$ and $Q_1, \cdots, Q_m$ such that $B = P_1 \cdots P_n$ and $C = Q_1 \cdots Q_m$. Thus, $A = P_1 \cdots P_n \oplus Q_1 \cdots Q_m = (P_1 \oplus Z) \cdots (P_n \oplus Z)\cdot(Z \oplus Q_1) \cdots (Z \oplus Q_m)$ and Proposition 2.13 shows each of $P_i \oplus Z$ and $Z \oplus Q_j$, $1 \leq i \leq n$ and $1 \leq j \leq m$, is a prime ideal of R. Therefore, R is a general Z.P.I.-ring with identity. ▲

<u>4.2 Example:</u> Let $R = Q \oplus (x)$ where Q is the field of rational numbers and (x) is the maximal ideal of the rank one discrete valuation ring $Z_p[[x]]$, where Z_p represents the integers modulo p for a prime integer p. If A is an ideal of R, then $A = B \oplus C$ where B and C are ideals of Q and (x), respectively. Then $B = Q$ or $B = (0)$ and $C = (x)^k$

for some positive integer k. Thus, $A = (B \oplus x)\cdot(Q \oplus (x)^k) = (B \oplus (x)) \cdot R^k$ where $B \oplus (x)$ and R are prime ideals of R. Hence, R is a general Z.P.I.-ring without identity. ▲

The above two examples are chosen out of the two main classes of general Z.P.I.-rings. The first, $Z \oplus Z$, has an identity while $Q \oplus (x)$ does not have an identity.

In this chapter we are concerned with the properties and characterizations of general Z.P.I.-rings. Our first main objective is to prove a general Z.P.I.-ring is Noetherian. Together with this and several other results, we can prove the direct sum decomposition theorem for general Z.P.I.-rings. In fact, we prove a general Z.P.I.-ring R with identity has the following structure: R is the finite direct sum of Dedekind domains and special primary rings. For the structure of a general Z.P.I.-ring S without identity, we show that $S = F \oplus T$ or $S = T$ where F is a field and T is a ring without identity having the property that each nonzero ideal of T is a power of T. These results are originally due to Mori [16], but appear in the above revised forms in papers by Asano [1] and Wood [21]. Henceforth, whenever we refer to the structure theorem for general Z.P.I.-rings, we are alluding to the direct sum decomposition of general Z.P.I.-rings. With this structure theorem at hand, we can develop several ideal-theoretic characterizations of general Z.P.I.-rings. Our last main objective is to focus on one of these characterizations. We show R is a general Z.P.I.-ring if and only if every ideal of R generated by at most two elements is the finite product of prime ideals of R. This characterization is due to Levitz [15].

Mori was the first to prove that a general Z.P.I.-ring R is Noetherian in [16]. However, the proof given here is due to Wood [21] since it is shorter and more straightforward than the proof given by Mori. Through a series of lemmas concerning the minimal prime ideals of R, we eventually prove every prime ideal of R is finitely generated. The application of Cohen's Theorem, Proposition 2.16(c), then gives us our first main result.

4.3 Lemma: If R is a general Z.P.I.-ring, R contains only finitely many minimal prime ideals and dim R $\leq$ 1.

Proof: If R contains no proper prime ideal, then the lemma is clearly true. Therefore, we assume R contains a proper prime ideal P and we show R contains a minimal prime ideal. If P is not a minimal prime of R, there exists a prime ideal P_1 such that $P_1 \subset P \subset R$. It follows that R/P_1 is a domain satisfying property C and containing a proper prime ideal P/P_1. We conclude from Theorem 3.15 that R/P_1 is a Dedekind domain. Since the dimension of a Dedekind domain is one, P_1 is a minimal prime ideal of R. Therefore, dim R $\leq$ 1.

Since R is a general Z.P.I.-ring, there exist prime ideals $Q_1, \cdots, Q_n$ in R and positive integers $e_1, \cdots, e_n$ such that $(0) = Q_1^{e_1} \cdots Q_n^{e_n}$. If M is any minimal prime ideal of R, then we have $(0) = Q_1^{e_1} \cdots Q_n^{e_n} \subseteq M$. This implies $Q_i \subseteq M$ for some i since M is prime. Hence, $M = Q_i$ since M is minimal and it follows that the collection $\{Q_1, \cdots, Q_n\}$ contains all the minimal prime ideals of R. Therefore, R contains only finitely many minimal prime ideals. ▲

Next we prove every minimal prime ideal P of R is finitely generated by showing how to select a finite number of elements in P which generate P.

4.4 Lemma: If R is a general Z.P.I.-ring containing a genuine prime ideal, then each minimal prime ideal of R is finitely generated.

Proof: Let P be a minimal prime ideal of R and let $\{P_1, \cdots, P_n\}$ be the collection of minimal primes of R distinct from P. If P = (0), we are done. Assume $(0) \subset P$. We divide the proof into three cases.

Case 1. $P = P^2$. Then $P = P^2 \subseteq RP \subseteq P$ implies $P = RP$. Since $P \not\subseteq P_i$ for $1 \leq i \leq n$, $P \not\subseteq \bigcup_{i=1}^{n} P_i$ by Proposition 2.5. So let $x_1 \in P - (\bigcup_{i=1}^{n} P_i)$. Then there exist prime ideals $M_1, \cdots, M_s$, positive integers $e_0, e_1, \cdots, e_s$ and a nonnegative integer e_{s+1} such that

$(x_1) = P^{e_0} M_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} = PM_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} = PM_1^{e_1} \cdots M_s^{e_s}$ since $P = RP$. Let $q = \sum_{i=1}^{s} e_i$. If $P = (x_1)$, we are done. If $(x_1) \subset P$, then by the choice of x_1 each M_i is a maximal prime ideal of R. By Proposition 2.5, $P \not\subseteq \{(x_1) \cup (\bigcup_{i=1}^{n} P_i)\}$. Choose $x_2 \in P - \{(x_1) \cup (\bigcup_{i=1}^{n} P_i)\}$. Then $(x_2) = PM_1^{f_1} \cdots M_s^{f_s} R^{f_{s+1}} Q_1^{g_1} \cdots Q_t^{g_t} = PM_1^{f_1} \cdots M_s^{f_s} Q_1^{g_1} \cdots Q_t^{g_t}$ where, for $1 \leq j \leq t$, Q_j is a maximal prime ideal of R, g_j is a positive integer and for $1 \leq i \leq s + 1$, f_i is a nonnegative integer. Since $(x_2) \not\subseteq (x_1)$ we have $e_{i_0} > f_{i_0}$ for some i_0, $1 \leq i_0 \leq s$. Thus,

$(x_1, x_2) = PM_1^{e_1} \cdots M_s^{e_s} + PM_1^{f_1} \cdots M_s^{f_s} Q_1^{g_1} \cdots Q_t^{g_t}$

$= PM_1^{m_1} \cdots M_s^{m_s}(M_1^{e_1-m_1} \cdots M_s^{e_s-m_s} + M_1^{f_1-m_1} \cdots M_s^{f_s-m_s} Q_1^{g_1} \cdots Q_t^{g_t})$ where $m_i = \min\{e_i, f_i\}$ for $1 \leq i \leq s$. We note that if $e_i - m_i \neq 0$ then $f_i - m_i = 0$, and if $f_i - m_i \neq 0$, then $e_i - m_i = 0$. Also, $e_{i_0} - m_{i_0} \neq 0$. Let $A = M_1^{e_1-m_1} \cdots M_s^{e_s-m_s}$ and

$B = M_1^{f_1-m_1} \cdots M_s^{f_s-m_s} Q_1^{g_1} \cdots Q_t^{g_t}$. We will show A + B is contained in no maximal prime ideal of R. If M is a maximal prime ideal of R containing A, then M a maximal prime implies there exists a k, $1 \leq k \leq s$, such that $e_k - m_k \neq 0$ and $M_k \subseteq M$. Thus, $M = M_k$. Since $e_k - m_k \neq 0$, $f_k - m_k = 0$ and it follows that $B \nsubseteq M_k = M$. Therefore, if M is a maximal prime ideal of R containing A, M does not contain B and, thus, cannot contain A + B. We conclude $A + B = R^{\ell}$ for some positive integer ℓ and

$(x_1, x_2) = PM_1^{m_1} \cdots M_s^{m_s}(A + B) = PM_1^{m_1} \cdots M_s^{m_s}R^{\ell} = PM_1^{m_1} \cdots M_s^{m_s}$. By our choice of m_i, $e_i \geq m_i$ for $1 \leq i \leq s$. But $e_{i_0} > f_{i_0}$ implies $q - 1 \geq \sum_{i=1}^{s} m_i \geq 0$.

Assume we have chosen, as described above, $x_1, x_2, \cdots, x_u$ in P such that $(x_1, \cdots, x_u) = PM_1^{v_1} \cdots M_s^{v_s}$ and $q - (u - 1) \geq \sum_{i=1}^{s} v_i \geq 0$. Then by the above method, either $P = (x_1, \cdots, x_u)$ or there exists an $x_{u+1} \in P - \{(x_1, \cdots, x_u) \cup (\bigcup_{i=1}^{n} P_i)\}$ such that $(x_1, \cdots, x_{u+1}) = PM_1^{w_1} \cdots M_s^{w_s}$ where each w_i is a nonnegative integer and $q - (u + 1 - 1) \geq \sum_{i=1}^{s} w_i \geq 0$. Since q is a finite positive number, there exists a positive integer b and $x_1, \cdots, x_b \in P$ such that $P = (x_1, \cdots, x_b)$.

Case 2. $P^2 \subset P$ and $P = RP$. Again, by Proposition 2.5, $P \nsubseteq \{P^2 \cup (\bigcup_{i=1}^{n} P_i)\}$ so let $x_1 \in P - \{P^2 \cup (\bigcup_{i=1}^{n} P_i)\}$. Then there exist prime ideals $M_1, \cdots, M_s$ of R, positive integers $e_1, \cdots, e_s$ and a nonnegative integer e_{s+1} such that $(x_1) = PM_1^{e_1} \cdots M_s^{e_s}R^{e_{s+1}} = PM_1^{e_1} \cdots M_s^{e_s}$ since $P = RP$. If $P = (x_1)$ we are done. If $(x_1) \subset P$, then we can choose $x_2 \in P - \{(x_1) \cup P^2 \cup (\bigcup_{i=1}^{n} P_i)\}$. We now consider (x_1, x_2) and the remainder of the proof of Case 2 is the same as the proof of Case 1. Therefore, P is a finitely generated ideal of R.

Case 3. $P^2 \subset P$ and $RP \subset P$. Let $x \in P - RP$. Then there are prime ideals $M_1, \cdots, M_s$ of R and nonnegative integers $e_1, \cdots, e_{s+1}$ such that $(x) = PM_1^{e_1} \cdots M_s^{e_s}R^{e_{s+1}} \not\subseteq RP$. Thus, $e_i = 0$ for $1 \leq i \leq s + 1$ and so $P = (x)$. ▲

<u>4.5 Lemma</u>: Each prime ideal of a general Z.P.I.-ring is finitely generated.

<u>Proof</u>: Let R be a general Z.P.I.-ring. We consider two cases.

Case 1. R contains no proper prime ideals. If $R = R^2$, let $x \in R - (0)$. Since R is a general Z.P.I.-ring, there exists a positive integer n such that $(x) = R^n = R$. If $R^2 \subset R$, let $x \in R - R^2$. It then follows that $(x) = R$.

Case 2. R contains a proper prime ideal M. If M is a minimal prime ideal of R, M is finitely generated by the previous lemma. If M is not a minimal prime, the proof of Lemma 3.2 implies there exists a minimal prime ideal P of R such that $P \subset M$. Thus, R/P is Noetherian which implies M/P is finitely generated in R/P. Since P is a minimal prime ideal of R, P is finitely generated in R. It follows from Proposition 2.15 that M is a finitely generated ideal of R. ▲

4.6 Theorem: A general Z.P.I.-ring is Noetherian.

Proof: This is an immediate consequence of Lemma 4.5 and Cohen's Theorem, Proposition 2.16(c). ▲

4.7 Remark: We could have proved Theorem 4.6 without using Cohen's Theorem. For if A is an ideal of a general Z.P.I.-ring R, then A is a finite product of prime ideals of R. Each of these prime ideals is finitely generated by Lemma 4.5. Since a finite product of finitely generated ideals is finitely generated, A is finitely generated. This shows R is Noetherian. ▲

As mentioned earlier, our next main objective is to prove the structure theorem for general Z.P.I.-rings. To accomplish this we need to develop several more properties of general Z.P.I.-rings and to establish various relationships among these properties. Before doing this, we look more closely at the approach we intend to use to prove the structure theorem.

One method which is sometimes used to prove that a ring R with identity is the finite direct sum of rings is outlined below.

Step 1. Show that $(0) = A_1 \cap \cdots \cap A_k$ where $A_1, \cdots, A_k$ are ideals of R.

Step 2: Show that $A_1, \cdots, A_k$ are pairwise comaximal ideals of R.

Step 3: Use the fact that $R/(A_1 \cap \cdots \cap A_k) \cong R/A_1 \oplus \cdots \oplus R/A_k$.

This is the method we use when we eventually prove the structure theorem for general Z.P.I.-rings with identity. The first step is already accessible to us since a general Z.P.I.-ring is Noetherian. Thus, we need only develop the tools necessary to apply Step 2.

For rings without identity, we use the following result of Butts and Gilmer. The proof is not included here but it should be pointed out that they use modified versions of the three steps outlined above in their proof of this fact.

<u>4.8 Proposition</u>: If R is a ring such that $R \neq R^2$ and if every ideal of R is an intersection of a finite number of prime power ideals, then $R = F_1 \oplus \cdots \oplus F_k \oplus T$ where each F_i is a field and T is a nonzero ring without identiy in which every nonzero ideal is a power of T [3; Theorem 14, p. 1195].

We precede our quest for the necessary results to use the ideas outlined above with several definitions. These definitions single out some of the more important properties that are related to general Z.P.I.-rings.

<u>4.9 Definitions</u>: Let R be a ring.

(a) We say R has <u>property (α)</u> [3] if each primary ideal of R is a power of its (prime) radical.

(b) If each ideal of R is an intersection of a finite number of prime power ideals, we say R has <u>property (δ)</u> [3].

(c) If R is a ring without identity such that each nonzero ideal of R is a power of R, we say R satisfies <u>property (#)</u> [21].

(d) A ring R with identity having a unique maximal ideal M such that each genuine ideal of R is a power of M is called a <u>special primary ring</u>.

(e) Let A be an ideal of a ring R. We say A is <u>simple</u> if there exist no ideals properly between A and A^2. To avoid conflicts with other definitions of a simple ring, we say, in case A = R, that R satisfies <u>property S</u>.

Relationships between properties (α) and (δ) defined above have been considered by Butts and Gilmer in [3]. We need some of these to obtain the structure theorem for general Z.P.I.-rings and we consider them next.

4.10 Theorem: If R is a ring satisfying property (δ), then (α) also holds in R.

Proof: Suppose Q is primary for the prime ideal P. Let $Q = \bigcap_{i=1}^{n} P_i^{e_i}$ be a representation of Q as an intersection of powers of distinct prime ideals. We have $P = \sqrt{Q} = \sqrt{(\bigcap_{i=1}^{n} P_i^{e_i})} = \bigcap_{i=1}^{n} \sqrt{(P_i^{e_i})}$ $= \bigcap_{i=1}^{n} P_i \subseteq P_j$ for each j, $1 \leq j \leq n$. Since P is prime and $P \supseteq \bigcap_{i=1}^{n} P_i^{e_i} \supseteq \prod_{i=1}^{n} P_i^{e_i}$, we must have $P \supseteq P_j$ for some j and therefore, say, $P = P_1$. Then $P_1 \subset P_i$ for $i \geq 2$, so $\bigcap_{i=2}^{n} P_i^{e_i}$ is not contained in P_1. Since $P_1^{e_1}(\bigcap_{i=2}^{n} P_i^{e_i}) \subseteq Q$ and Q is P_1-primary, $P_1^{e_1} \subseteq Q$. It follows that $Q = P_1^{e_1}$ and Q is a prime power. ▲

4.11 Theorem: Let R be a Noetherian ring satisfying property (α). Then R has property (δ).

Proof: Let A be an ideal of R. Since R is Noetherian, A has a shortest representation in R, say, $A = Q_1 \cap \cdots \cap Q_n$ where Q_i is P_i-primary. Then $Q_i = P_i^{e_i}$ since (α) holds in R. Thus, $A = \bigcap_{i=1}^{n} P_i^{e_i}$ and (δ) holds in R. ▲

4.12 Theorem: Let R be a ring with identity. If R satisfies property (α), then each maximal ideal of R is simple.

Proof: Let M be a maximal ideal of R and A an ideal between M^2 and M. Then $\sqrt{A} = M$ implies that A is M-primary. Since R has property (α), $A = M^k$. But $M^2 \subseteq A \subseteq M$ implies $k = 1$ or $k = 2$. Therefore, each maximal ideal of R is simple. ▲

The next result is a well known fact concerning simple ideals. For the reader interested in seeing a proof, we refer you to [20; Lemma 2.2.7, p. 38].

4.13 Proposition: Let A be a simple ideal of a ring R. Then for each positive integer i there are no ideals properly between A^i and A^{i+1}. Further for each positive integer n, the only ideals between A and A^n are $A, A^2, \cdots, A^n$. ▲

The preceding discussion gives some interesting relationships among properties (α) and (δ) and simple ideals. The next theorem allows us to use these results when we are working with general Z.P.I.-rings.

4.14 Theorem: If Q is a P-primary ideal of a general Z.P.I.-ring R, then Q is a power of P; that is, R has property (α).

Proof: There exist distinct prime ideals $P_1, \cdots, P_n$ and positive integers $e_1, \cdots, e_n$ such that $Q = P_1^{e_1} \cdots P_n^{e_n}$. Since $Q = P_1^{e_1} \cdots P_n^{e_n} \subseteq P$ and P is prime, $P_i \subseteq P$ for some i, say i = 1. Now,

$$P = \sqrt{Q} = \sqrt{(P_1^{e_1} \cdots P_n^{e_n})} = \sqrt{(P_1^{e_1})} \cap \cdots \cap \sqrt{(P_n^{e_n})} = P_1 \cap \cdots \cap P_n$$

$\subseteq P_i$ for each i. Therefore, $P = P_1$ and we have $Q = P^{e_1} P_2^{e_2} \cdots P_n^{e_n}$ where

P P_i for $2 \leq i \leq n$. Since $Q = P^{e_1}(P_2^{e_2} \cdots P_n^{e_n}) \subseteq Q$ and $P_2^{e_2} \cdots P_n^{e_n} \nsubseteq P$, it follows that $P^{e_1} \subseteq Q$ since Q is primary. Hence, $Q = P^{e_1}$. ▲

We have now established that a general Z.P.I.-ring without identity satisfies the hypotheses of Theorem 4.8. This gives us some information concerning the finite direct sum decomposition of such a ring. The next three results are used when we consider general Z.P.I.-rings with identity. Lemma 4.15 allows us to conclude that the ideals which constitute an irredundant representation of (0) are pairwise comaximal ideals. The other two results pertain to the summands of our eventual decomposition of a general Z.P.I.-ring with identity.

<u>4.15 Lemma</u>: Let A be a proper simple ideal of a Noetherian ring R. If there exists a prime ideal P of R such that $(0) \subset P \subset A \subset R$, P is unique and $P = \bigcap_{i=1}^{\infty} A^i$. Also, if Q is a P-primary ideal of R, $Q = P$.

<u>Proof</u>: We first show by an inductive argument that $P \subset A^i$ for each positive integer i. By hypothesis $P \subset A$. Assume $P \subset A^k$ for some positive integer k. Since A/P is a proper ideal of R/P, a Noetherian integral domain, $A^k/P \supset (A^k/P)(A/P) = (A^{k+1} + P)/P \supset P/P$ by Proposition 2.14(b). This implies $A^k \supset A^{k+1} + P \supseteq A^{k+1}$. Therefore, $A^{k+1} + P = A^{k+1}$ since A is a simple ideal. Since $A^{k+1} + P \supset P$, it follows that $P \subset A^{k+1}$. Thus, $P \subset A^i$ for each positive integer i.

We now show $P = \bigcap_{i=1}^{\infty} A^i$. Since A/P is a proper ideal of a Noetherian domain, $P/P = \bigcap_{i=1}^{\infty} (A/P)^i$ by Proposition 2.18. Also, since $\bigcap_{i=1}^{\infty} (A/P)^i = \bigcap_{i=1}^{\infty} ((A^i + P)/P) = \bigcap_{i=1}^{\infty} (A^i/P) = (\bigcap_{i=1}^{\infty} A^i)/P$, it follows that

$P = \bigcap_{i=1}^{\infty} A^i$.

Finally, we show that if Q is a P-primary ideal of R, then Q = P. By Proposition 2.17 $P = A(\bigcap_{i=1}^{\infty} A^i) = AP$. Thus, there exists an $a \in A$ such that $ap = p$ for each $p \in P$ by Proposition 2.14(a). If $x \in R - A$, then $p(ax - x) = apx - px = 0 \in Q$ for each $p \in P$. Since $x \notin A$, $ax - x \notin A \supset P$. Thus, $p \in Q$ for each $p \in P$ since $p(ax - x) \in Q$, $ax - x \notin P$, and Q is a p-primary ideal of P. Thus, $P \subseteq Q$ which shows P = Q. ▲

<u>4.16 Lemma</u>: If a ring R is a finite direct sum of general Z.P.I.-rings with identity, then R is a general Z.P.I.-ring with identity.

<u>Proof</u>: Let $R = R_1 \oplus \cdots \oplus R_n$ where each R_i is a general Z.P.I.-ring with identity. Then R has an identity since each summand has an identity. Let A be an ideal of R. Then by Proposition 2.11, $A = A_1 \oplus \cdots \oplus A_n$ where, for $1 \leq i \leq n$, A_i is an ideal of R_i. Since each summand is a general Z.P.I.-ring, we have $A_i = P_{i1} \cdots P_{im_i}$ where P_{ij} is a prime ideal of R_i for $1 \leq j \leq m_i$. Thus, $A = P_{11} \cdots P_{1m_1} \oplus \cdots \oplus P_{n1} \cdots P_{nm_n}$ $= (P_{11} \oplus R_2 \oplus \cdots \oplus R_n)\cdot(P_{12} \oplus R_2 \oplus \cdots \oplus R_n) \cdots (P_{1m_1} \oplus R_2 \oplus \cdots \oplus R_n)$ $\cdots (R_1 \oplus \cdots \oplus R_{n-1} \oplus P_{n1}) \cdots (R_1 \oplus \cdots \oplus R_{n-1} \oplus P_{nm_n})$ and for each i and j, $1 \leq i \leq n$ and $1 \leq j \leq m_i$, $R_1 \oplus \cdots \oplus P_{ij} \oplus \cdots \oplus R_n$ is a prime ideal of R by Proposition 2.13. Hence, R is a general Z.P.I.-ring with identity. ▲

<u>4.17 Remark</u>: If R is a ring with identity and M is a simple maximal ideal of R, then R/M^n is a special primary ring for each positive

integer n. For if A/M^n is a genuine ideal of R/M^n, then $M^n \subseteq A \subseteq M$. Proposition 4.13 shows that $A = M^k$ for some integer k, $1 \leq k \leq n$. Thus, $A/M^n = M^k/M^n = (M/M^n)^k$ and R/M^n is a special primary ring. It is clear that a special primary ring is a general Z.P.I.-ring. ▲

<u>4.18 Theorem:</u> [Structure Theorem of a General Z.P.I.-Ring.] A ring R is a general Z.P.I.-ring if and only if R has the following structure:

(a) If $R = R^2$, then $R = R_1 \oplus \cdots \oplus R_n$ where R_i is either a Dedekind domain or a special primary ring for $1 \leq i \leq n$.

(b) If $R \neq R^2$, then either $R = F \oplus T$ or $R = T$ where F is a field and T is a ring satisfying property (#).

<u>Proof:</u> (→) If R is a general Z.P.I.-ring, then R is a Noetherian ring having property (α) by Theorem 4.6 and Theorem 4.14, respectively. Thus, R has property (δ) by Theorem 4.11. Also, dim R $\leq$ 1 by Lemma 4.3. We consider two cases.

Case 1. $R = R^2$. Then by Proposition 2.14, R has an identity. Theorem 4.12 implies each maximal ideal of R is simple. We consider an irredundant representation of (0), $(0) = Q_1 \cap \cdots \cap Q_n$, where Q_i is P_i-primary for each i. Theorem 4.14 shows there exist positive integers r and s such that $Q_i = P_i^r$ and $Q_j = P_j^s$. Since $Q_1 \cap \cdots \cap Q_n$ is an irredundant representation of (0), $P_i \neq P_j$ for $i \neq j$. We claim $P_1, \cdots, P_n$ are pairwise comaximal. For if $P_i + P_j$ R then $P_i + P_j \subseteq M$ R for some maximal ideal M of R. Lemma 4.15 shows that

$P_i = \bigcap_{i=1}^{\infty} M^i = P_j$, a contradiction. Thus, $P_1, \cdots, P_n$ are pairwise

comaximal and Proposition 2.10(a) states that $Q_1, \cdots, Q_n$ are pairwise comaximal. By Proposition 2.10(c) $R \cong R/(0) \cong R/(Q_1 \cap \cdots \cap Q_n)$ $\cong R/Q_1 \oplus \cdots \oplus R/Q_n$. Either P_i is maximal, in which case Q_i is a power of P_i since (α) holds in R, or P_i is nonmaximal, in which case $Q_i = P_i$ by Lemma 4.15. Remark 4.17 shows that R/Q_i is a special primary ring if P_i is maximal, and if P_i is nonmaximal, R/Q_i is a Dedekind domain.

Case 2. $R = R^2$. Since R has property (δ), Theorem 4.8 implies $R = F_1 \oplus \cdots \oplus F_u \oplus T$ where each F_i is a field and T is a nonzero ring satisfying property (#). Using a contrapositive argument, we show $u \not\geq 2$.

Assume $u \geq 2$. We now show R is not a general Z.P.I.-ring. Since $u \geq 2$, T is an ideal of R that is not prime. The prime ideals of R containing T are R and $P_i = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_u \oplus T$ for $1 \leq i \leq u$ where $T \subset P_i$ for each i. Now $P_iP_j = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0)$ $\oplus F_{i+1} \oplus \cdots \oplus F_{j-1} \oplus (0) F_{j+1} \oplus \cdots \oplus F_u \oplus T^2$, $RP_i = F_1 \oplus \cdots \oplus F_{i-1}$ $\oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_u \oplus T^2$, and $R^2 = F_1 \oplus \cdots \oplus F_u \oplus T^2$. Since $T^2 \subset T$, it follows that $T \not\subseteq P_iP_j$, $T \not\subseteq RP_i$, and $T \not\subseteq R^2$ for $1 \leq i, j \leq u$. Thus, T cannot be represented as a finite product of prime ideals of R; that is, R is not a general Z.P.I.-ring. Therefore, if R is a general Z.P.I.-ring, $u \not\geq 2$; that is, $R = F_1 \oplus T$ or $R = T$ where F_1 is a field and T is a ring satisfying property (#).

(←) If R is a direct sum of finitely many Dedekind domains and special primary rings, R is a general Z.P.I.-ring by Lemma 4.16. If R = T where T is a ring satisfying property (#), then R is clearly a general Z.P.I.-ring. If $R = F \oplus T$ where F is a field and T is a ring satisfying property (#), then $\{F \oplus T^i, T^i, F, (0)$: where i is any positive integer$\}$ is the collection of ideals of R. It follows that each ideal of R is a finite product of prime ideals. Therefore, if R satisfies either (a) or (b), R is a general Z.P.I.-ring. ▲

At times it may be difficult to use the definitions or the structure theorem to prove that a ring is a general Z.P.I.-ring. The next two theorems give us some equivalent ideal-theoretic conditions which at times may be easier to apply. For the first of these results, we concern ourselves with rings with identity and use the method of proof outlined previously. The second theorem handles the case when R is a ring without identity.

<u>4.19 Theorem</u>: Let R be a ring with identity. The following conditions are equivalent in R.

(1) R is a general Z.P.I.-ring.

(2) R is Noetherian and property (α) holds in R.

(3) R is Noetherian and each maximal ideal of R is simple.

<u>Proof</u>: (1) → (2) This is Theorems 4.6 and 4.14.

(2) → (3) This is Theorem 4.12.

(3) → (1) We make the following observations:

(i) dim R $\leq$ 1 For if M is a maximal ideal properly containing the prime ideal P, then Lemma 4.15 shows that $P = \bigcap_{i=1}^{\infty} M^i$.

(ii) If Q is P-primary for the nonmaximal prime ideal P of R, then Q = P by Lemma 4.15.

(iii) The set of M-primary ideals for the maximal ideal M is $\{M^i\}_{i=1}^{\infty}$. For each M^i is M-primary since M is maximal and R has an identity. Further, if Q is M-primary, then $M^k \subseteq Q$ for some positive integer k since R is Noetherian. Proposition 4.13 shows that $Q = M^j$ for some positive integer j.

Having made these observations, we consider a shortest representation of (0), $(0) = Q_1 \cap \cdots \cap Q_n$, where Q_i is P_i-primary. The rest of the proof is identical to the proof of Theorem 4.18(a) which shows that $R = R_1 \oplus \cdots \oplus R_n$ where R_i is either a Dedekind domain or a special primary ring. Therefore, R is a general Z.P.I.-ring. ▲

4.20 Theorem: Let R be a ring without identity.

(a) If R contains a proper prime ideal, then R is a general Z.P.I.-ring if and only if R satisfies the following four conditions:

(1) R is Noetherian.

(2) R satisfies property S.

(3) Each maximal prime ideal of R is simple.

(4) $\bigcap_{i=1}^{\infty} R^i$ is a field.

(b) If R does not contain a proper prime ideal, then R is a general Z.P.I.-ring if and only if R satisfies the following two conditions:

(1) R is Noetherian.

(2) R satisfies property S.

Proof of (a): (→) Assume R is a general Z.P.I.-ring. Then R is Noetherian by Theorem 4.6. Since R contains a proper prime ideal, Theorem 4.18 shows $R = F \oplus T$ where F is a field and T is a ring satisfying property (#). Hence, R satisfies property S since $R^2 = F \oplus T^2$ and there are no ideals of T between T and T^2. If T is a domain, then F and T are the maximal prime ideals of R. If T is not a domain, then T is the maximal prime ideal of R. It follows that each maximal prime ideal of R is simple. Finally, $\bigcap_{i=1}^{\infty} R^i = \bigcap_{i=1}^{\infty} (F \oplus T)^i = F$, a field.

(←) Assume conditions (1), (2), (3), and (4) hold. Let Q be a P-primary ideal of R. If P = R or if P is a maximal prime ideal of R, there exists an integer n such that $P^n \subseteq Q$ since R is Noetherian. Thus, Proposition 4.13 implies $Q = P^k$ for some integer k. If P is a proper nonmaximal prime ideal of R, Lemma 4.15 shows that Q = P. Thus, R is Noetherian and satisfies property (α) which shows property (δ) holds in R by Theorem 4.11. Therefore, by Theorem 4.8, $R = F_1 \oplus \cdots \oplus F_m \oplus T$ where each F_i is a field and T is a ring satisfying property (#). Since R contains a proper prime ideal, $m \geq 1$. Condition (4) implies that $m \ngtr 1$. Thus $R = F_1 \oplus T$ and by Theorem 4.18, R is a general Z.P.I.-ring.

<u>Proof of (b)</u>: (→) If R is a general Z.P.I.-ring containing no proper prime ideal, then R = T where T is a ring satisfying property (#). Thus, R is Noetherian and satisfies property S.

(←) Assume conditions (1) and (2) hold. Since R is Noetherian and R is the only nonzero prime ideal of R, R has property (α). Thus, (δ) holds and Theorem 4.8 shows that $R = F_1 \oplus \cdots \oplus F_n \oplus T$ where F_i is a field and T is a ring satisfying property (#). If $n \geq 1$, then $P = (0) \oplus F_2 \oplus \cdots \oplus F_n \oplus T$ is a proper prime ideal of R. Thus, R = T and R is a general Z.P.I.-ring. ▲

We are now ready to take on our last main objective of this chapter, the characterization of general Z.P.I.-rings due to Levitz [15]. In proving this result, we arrive at a similar characterization of Dedekind domains. Because they are so similar and since the proofs overlap, both are presented here. However, before attempting these proofs we need some knowledge of the properties possessed by Krull domains and π(1)-domains.

These two concepts are defined immediately following this discussion. Some relationships among Krull domains, Dedekind domains, and $\pi(1)$-domains will also be needed. Probably the most important of these, for our purposes, is the equivalence of Dedekind domains and one-dimensional Krull domains. To include the proofs of this fact and many of the others would require a lengthy deviation to areas outside of our primary topic of discussion. For this reason, only references are given. A detailed study of Krull domains may be found in [9] while $\pi(1)$-domains are considered in [20].

4.21 Definition: An integral domain D with identity is a Krull domain if there is a set of rank one discrete valuation rings $\{V_\alpha\}$ such that $D = \bigcap_\alpha V_\alpha$ and such that each nonzero element of D is a non-unit in only finitely many of the V_α.

4.22 Definition: Let R be a ring and n a positive integer. If each ideal of R generated by n or fewer elements can be expressed as a finite product of prime ideals of R, R is called a $\pi(n)$-ring. If, in addition, R is an integral domain, we say R is a $\pi(n)$-domain.

4.23 Example: Let D be a UFD and let $\{p_\alpha\}_{\alpha\varepsilon A}$ be a complete set of nonassociate, nonunit prime elements of D. Then D is a Krull domain. In fact, the set of quotient rings $\{D_{(p_\alpha)}\}$ is a set of rank one discrete valuation rings satisfying the conditions of Definition 4.21. A proof of this can be found in [9, Proposition 43.2, p. 525]. ▲

From the definition of R being a $\pi(n)$-ring, it is clear that R is a $\pi(k)$-ring for $k \leq n$. $Z[x]$ is an example of a $\pi(1)$-domain which is not a $\pi(2)$-domain and this is proved in Example 4.40. Very soon we will see there are no $\pi(2)$-rings which are not $\pi(k)$-rings for $k > 2$.

4.24 Proposition: Let D be an integral domain with identity. Then D is a Dedekind domain if and only if D is a one-dimensional Krull domain [9; Theorem 43.16, p. 536].

4.25 Proposition: Let D be a Krull domain. Then each nonzero prime ideal P of D contains a minimal prime ideal of D [9; Corollary 43.10, p. 529].

4.26 Proposition: Let D be an integral domain with identity. D is a $\pi(1)$-domain if and only if D is a Krull domain in which each minimal prime ideal is invertible [14; Theorem 1.2, p. 377].

4.27 Proposition: If D is a $\pi(1)$-domain, then each proper principal ideal of D can be represented as a finite product of minimal prime ideals of D [20; Theorem 1.1.8, p. 18].

4.28 Proposition: Let D be a $\pi(2)$-domain. Then each minimal prime ideal of D is a principal ideal. In addition, if D has a unique minimal prime ideal (p), then R = (p) [15; Theorem 2.4, p. 149].

4.29 Proposition: Let D be a $\pi(2)$-domain with identity, P_1 and Q distinct minimal prime ideals of D, and $a \in P_1 - Q$. Suppose $b \in Q - \bigcup_{i=1}^{n} P_i$, where $(a) = P_1^{e_1} \cdots P_n^{e_n}$ and each P_i is a minimal prime ideal of R. If $bt \in (a)$ for some $t \in R$, then $t \in (a)$ [15; Lemma 2.2, p. 148].

As noted earlier, the definition of a Dedekind domain does not require that the representation of ideals as finite products of prime ideals be unique. However, it was shown that the representation was unique. Example 4.37 shows the factorization is not necessarily unique in a general Z.P.I.-ring. In order that the representation be unique, it is sufficient for the ideal to satisfy the conditions of the following proposition.

4.30 Proposition: Let B be a proper, finitely generated, regular ideal of a ring R such that B is a finite product of proper prime ideals of R. Then the representation of B is unique [8; Theorem 2, p. 72].

Now we are ready to prove a lemma which leads to our characterization of Dedekind domains.

4.31 Lemma: Let R be a $\pi(2)$-domain with identity. Then R is a Krull domain in which each minimal prime ideal is invertible. Moreover, the minimal prime ideals are pairwise comaximal.

Proof: If R is a $\pi(2)$-domain, R is a $\pi(1)$-domain. It follows from Proposition 4.26 that R is a Krull domain in which each minimal prime ideal is invertible. Let P_1 and Q be distinct minimal prime ideals of R and let $a \in P_1 - Q$. Then $(a) = P_1^{e_1} \cdots P_n^{e_n}$ where, for each i, $e_i \geq 1$, $P_i \neq Q$, and P_i is a minimal prime ideal by Proposition 4.27. Let $b \in Q - \bigcup_{i=1}^{n} P_i$. Then $(a, b) = \prod_{j=1}^{m} A_j$ and $(a, b^2) = \prod_{k=1}^{s} B_k$ where, for each j and k, A_j and B_k are prime ideals of R.

Consider $\overline{R} = R/(a)$ and $\overline{b}$, the image of b in $\overline{R}$. Suppose $\overline{b}\,\overline{t} = \overline{0}$ for some $\overline{t} \in \overline{R}$. Then $bt \in (a)$ and by Proposition 4.29, $t \in (a)$. Thus, $\overline{b}$ is a regular element in $\overline{R}$. We have in $\overline{R}$, $(\overline{b}) = \prod_{j=1}^{m} (A_j/(a))$ and

$(\bar{b}^2) = \prod_{k=1}^{s} (B_k/(a))$. Also, $(\bar{b}^2) = \prod_{j=1}^{m} (A_j/(a))^2$. By Proposition 4.30 the factorization of the ideal $(\bar{b}^2)$ is unique up to factors of $\bar{R}$. It follows that $s = 2m$, and we can index the ideals B_k, $1 \leq k \leq s$, so that $A_j = B_{2j-1} = B_{2j}$. Hence, $(a, b^2) = \prod_{j=1}^{m} (A_j)^2 = (a, b)^2$. Thus, $(a) \subset (a, b^2) = (a, b)^2 \subset (a^2, b)$. Now if $x \in (a) \subset (a^2, b)$, then $x = ra^2 + sb$ where $r, s \in R$. This implies $sb \in (a)$, and, consequently, $s \in (a)$ by Proposition 4.29. Thus, we can conclude $(a) \subseteq (a)(a, b)$. But we always have $(a) \supseteq (a)(a, b)$ and so $(a) = (a)(a, b)$. Since $a \neq 0$ and R is an integral domain, (a) is invertible. Therefore, by the cancellation property for invertible ideals $R = (a, b) \subseteq P_1 + Q \subseteq R$. Hence, $R = P_1 + Q$ and the minimal prime ideals of R are pairwise comaximal. ▲

<u>4.32 Theorem</u>: Let R be an integral domain with identity. Then R is a Dedekind domain if and only if R is a $\pi(2)$-domain.

<u>Proof</u>: (→) By the definition of a Dedekind domain, R is a $\pi(2)$-domain.

(←) By Lemma 4.31 R is a Krull domain in which the minimal prime ideals are invertible. To conclude that R is a Dedekind domain, it suffices by Proposition 4.24 to show dim R = 1; that is, proper prime ideals are maximal. First we note that every non-unit of R is contained in some minimal prime ideal of R. For if a is a non-unit in R, then $(a) = \prod_{i=1}^{n} P_i^{e_i} \subseteq P_i$ where, for each i, P_i is a minimal prime of R by Proposition 4.27. We now consider two cases.

Case 1. R has a unique minimal prime ideal P. Then P is also the unique maximal ideal of R. For if there exists a maximal ideal M of R such that $M \nsubseteq P$, then for any $d \in M$, $(d) = P^k \subseteq P$ for some positive integer k. This follows from Proposition 4.27. Therefore, $d \in P$ and $M \subseteq P$, which implies $M = P$. We conclude that P is the unique maximal ideal of R and hence dim R = 1.

Case 2. R has more than one minimal prime ideal. Let P be any proper prime ideal of R. By Proposition 4.25, there exists a minimal prime ideal Q of R such that $Q \subseteq P$. If $Q \nsubseteq P$, there exists $b \in P - Q$. Then $(b) = \prod_{i=1}^{r} B_i$ where, for each i, B_i is a minimal prime ideal of R and $B_i \nsubseteq Q$ for any i, $1 \leq i \leq r$. Since $b \in P$, $\prod_{i=1}^{r} B_i \subseteq P$. Thus $B_j \subseteq P$ for some j, $1 \leq j \leq r$, since P is prime. By Lemma 4.31 the minimal prime ideals of R are pairwise comaximal. But this implies $R = Q + B_j \subseteq P$, contradicting the choice of P. Hence Q = P and each proper prime ideal of R is maximal. Thus, dim R = 1. ▲

The following theorem completes the characteriztion of $\pi(2)$-domains.

<u>4.33 Theorem</u>: Let R be a $\pi(2)$-domain without identity. Then R is a general Z.P.I.-ring.

<u>Proof</u>: By Proposition 4.28 each minimal prime ideal of R is a principal ideal and if R has a unique minimal prime ideal (p), then R = (p). We assume R contains two distinct minimal prime ideals, (p) and (q). Using the same argument we did in Lemma 4.31, we can show that (p) = (p)(p, q). Since (p) is a regular ideal, Proposition 2.14(b) implies R must have an identity. Since R has no identity, it must be the case that R is the only nonzero prime ideal of itself.

Let A be a nonzero ideal of R such that $R^2 \subseteq A \subseteq R$. Suppose $R^2 \subset A$ and choose $a \in A - R^2$. Since R is a $\pi(2)$-domain and R = (p) is the only nonzero prime ideal of itself, $(a) = R^k$ for some positive integer k. Since $a \notin R^2$, we must have k = 1. Thus, (a) = R and R has property S.

Now let B be any nonzero ideal of R. For $b \in B$, $R^j = (b) \subseteq B$ for some positive integer j. Since R has property S, Proposition 4.13 shows that $B = R^n$ for some positive integer n. We may conclude each ideal of R is a power of R. By Theorem 4.18, R is a general Z.P.I.-ring. ▲

Having characterized $\pi(2)$-domains, we are now ready to consider $\pi(2)$-rings. With the structure theorem for general Z.P.I.-rings available to us, we would hope to be able to prove something pertaining to a direct sum decomposition of $\pi(2)$-rings. The next two results give us two ways of considering the decomposition of $\pi(2)$-rings. Proposition 4.34 is due to Mori and a proof may be found in [15; Theorem 3.2, p. 150].

<u>4.34 Proposition</u>: Let R be a $\pi(2)$-ring.

(a) If R has an identity, then R is a direct sum of finitely many $\pi(1)$-domains and special primary rings.

(b) If R does not have an identity, then $R = F \oplus T$ or R = T where F is a field and T is a ring satisfying property (#).

<u>4.35 Lemma</u>: Suppose R is a ring with identity such that R is a finite direct sum of rings, $R = R_1 \oplus \cdots \oplus R_k$. Then R is a $\pi(n)$-ring if and only if each summand R_i is also a $\pi(n)$-ring. Thus, R is a $\pi(2)$-ring if and only if each summand R_i is a $\pi(2)$-ring.

<u>Proof:</u> ($\rightarrow$) Suppose R is a $\pi(n)$-ring and let R_j be a direct summand of R. Let $A_j = (a_{1j}, a_{2j}, \cdots, a_{nj})$ be an ideal of R_j generated by n

elements. Let e_i denote the identity of the direct summand R_i, $1 \leq i \leq k$. Then if A is the ideal of R generated by the n elements

$(\sum_{i=j} e_i) + a_{1j}$, $(\sum_{i=j} e_i) + a_{2j}$, $\cdots$, $(\sum_{i=j} e_i) + a_{nj}$, $A = \prod_{r=1}^{t} P_r$ where for each r, $1 \leq r \leq t$, P_r is a prime ideal of R. Then

$A_j = AR_j = (\prod_{r=1}^{t} P_r)R_j = \prod_{r=1}^{t} (P_rR_j)$. Since for each r, P_rR_j is a prime ideal of R_j, A_j can be expressed as a finite product of prime ideals. Therefore, R_j is a $\pi(n)$-ring.

(←) Suppose each summand R_j is a $\pi(n)$-ring and let A be an ideal of R generated by n elements. Then for each i, $1 \leq i \leq k$, AR_i is an ideal of R_i and $A = AR_1 \oplus \cdots \oplus AR_k$ by Proposition 2.11. In addition Proposition 2.12 states that AR_i is generated by n elements. Thus, AR_i has a representation as a finite product of prime ideals of R_i. The remainder of the proof is very similar to the proof of Lemma 4.16. Hence, R is a $\pi(n)$-ring. ▲

<u>4.36 Theorem</u>: Let R be a ring. Then R is a general Z.P.I.-ring if and only if R is a $\pi(2)$-ring [15; Theorem 3.2, p. 150].

<u>Proof</u>: (→) By the definition of a general Z.P.I.-ring, R is a $\pi(2)$-ring.

(←) If R has an identity, then by Proposition 4.34, R is a finite direct sum of $\pi(1)$-domains with identity and special primary rings. Using Theorem 4.18, we can conclude R is a general Z.P.I.-ring if any summand of R which is a domain is a Dedekind domain. From Lemma 4.35 it follows that each summand of R is a $\pi(2)$-ring. Hence, if the summand is a domain, it is a Dedekind domain by Theorem 4.32. Thus, a $\pi(2)$-ring with identity is a general Z.P.I.-ring.

If R is a ring without identity, but with zero-divisors, then Proposition 4.34 shows $R = F \oplus T$ or $R = T$ where F is a field and T is a ring with property (#). Therefore, R is a general Z.P.I.-ring by Theorem 4.18(b). Δ

Levitz [15] comments that this result is surprising. She notes that Mori characterized $\pi(1)$-domains and that Wood [20] generalized these results. It is then easy to construct a $\pi(1)$-domain which is not a $\pi(2)$-domain. It is shown at the end of this chapter that Z[x] is a $\pi(1)$-domain but is not a $\pi(2)$-domain. However, in another sense, this result is not so surprising. Recall one particular characterization of a Dedekind domain: An integral domain D with identity is Dedekind if and only if every ideal of D has a basis consisting of two elements. In this way it would seem natural that one need only consider ideals generated by two elements.

We conclude this chapter with four examples which illustrate some of the concepts we have discussed. The first of these shows that in a general Z.P.I.-ring R, the factorization of an ideal into a finite product of prime ideals may not be unique. Property (#) and property (α) are considered in the next two examples. The final example shows that it is not difficult to construct a $\pi(1)$-domain which is not a $\pi(2)$-domain as mentioned in the previous paragraph.

<u>4.37 Example</u>: This example shows the representation of ideals of a general Z.P.I.-ring as a finite product of prime ideals is not necessarily unique. Let $R = Z \oplus Z$. Then by Theorem 4.18, R is a general Z.P.I.-ring. If $P = Z \oplus (0)$, then P is a prime ideal of R by Proposition 2.13. But

$P = P^n$ for each positive integer n and so the representation of P is not unique. ▲

4.38 Example: This is an example of a ring with property (#). Let R be a ring with identity and let M be a simple maximal ideal of R. Then M/M^n is a ring with property (#) for each positive integer n. For if A/M^n is an ideal of M/M^n, then $M^n \subseteq A \subseteq M$. Since M is simple, $A = M^k$ for some k, $1 \leq k \leq n$. Thus, $A/M^n = M^k/M^n = (M/M^n)^k$. Hence, M/M^n is a ring without identity such that each ideal of M/M^n is a power of M/M^n. For a specific example, let R = Z and M = (p) where p is a prime integer. ▲

4.39 Example: This is an example of a ring which does not satisfy property (α). Let E be the ring of even integers. Then $\mathcal{A} = \{(2p)$: p is an odd prime in Z$\}$ is the set of proper prime ideals of E. Consider the ideal (18) in E. We claim (18) is (6)-primary but is not a power of (6). $\sqrt{(18)} = \bigcap_\alpha P_\alpha$ where $\{P_\alpha\}_\alpha$ is the set of prime ideals of E containing (18). Thus, $\sqrt{(18)} = (2) \cap (6) = (6)$. Now we need to show that if x, y ε E, $x \cdot y \;\varepsilon\; (18)$ and $y \notin (6)$, then x ε (18). Since $x \cdot y \;\varepsilon\; (18)$, xy = 18m + 18j where m ε E, j ε Z. Thus xy = 18n for some n ε Z. Since x, y ε E, x = 2t and y = 2s for some t, s ε Z. Since $y \notin (6)$ and y ε E, y = 6k + 2 or y = 6k + 4 for some integer k. If y = 6k + 2, then xy = x(6k + 2) = 4t(3k + 1) = 18n. Now, $3^2 | 4t(3k + 1)$ and (3,4(3k + 1)) = 1. Since Z is a UFD, we conclude that $3^2 | t$. Hence, t = 9u for some u ε Z. Therefore, x = 2t = 18u; that is, x ε (18). If y = 6k + 4, a similar argument shows that x ε (18). Consequently, (18) is (6)-primary and is not a power of (6).

Also, we note $(36) = (6)^2$ is not (6)-primary since $2 \cdot 18 \;\varepsilon\; (36)$, $18 \notin (36)$ and $2^k \notin (6)$ for any positive integer k. Thus, powers of prime or maximal ideals are not necessarily primary. ▲

4.40 Example: This is an example of a $\pi(1)$-domain which is not a $\pi(2)$-domain. Consider Z[x] and let (a) be a principal ideal of Z[x]. Since Z[x] is a UFD, a can be written as a finite product of prime elements of Z[x], $a = p_1^{e_1} \cdots p_n^{e_n}$. Thus $(a) = (p_1)^{e_1} \cdots (p_n)^{e_n}$ and each (p_i), $1 \leq i \leq n$, is a prime ideal of Z[x]. To see that Z[x] is not a $\pi(2)$-domain, we note that Z[x] is not a Dedekind domain since $\dim Z[x] > 1$. In particular, $(0) \subset (3) \subset (3, x)$ is a chain of three genuine prime ideals of Z[x]. Thus, Z[x] is a $\pi(1)$-domain but is not a $\pi(2)$-domain. ▲

CHAPTER V

ALMOST DEDEKIND DOMAINS

Let D be an integral domain with identity and let P be a proper prime ideal of D. In Chapter II we introduced the quotient ring D_P of D with respect to the prime ideal P of D. Some authors choose to call D_P the localization of D at P. If one is interested in some aspect of the ideal theory of D, then in some instances it is possible to gain insight into this by considering the localization of D and P for some prime ideal P of D. Due to the "nice" relationships between the P-primary ideals of D and the PD_P-primary ideals of D_P (Proposition 5.1), this method of study is especially useful if the prime or primary ideals of D or D_P play an important role in the topic under consideration. This is one reason it is important to study the structure of the quotient rings of commutative rings.

In this chapter we consider integral domains D with identity which have the following property: D_M is a Dedekind domain for each maximal ideal M of D. Such a domain is called an <u>almost Dedekind domain</u> [12]. We prove several characterizations of almost Dedekind domains, making heavy use of the ideas discussed in the previous paragraph. These provide us with the tools necessary to classify the almost Dedekind domains among the classes of Dedekind domains and Prüfer domains. Most of these results are due to Gilmer in [12] and Butts and Phillips in [4]. We also present an outline of an example due to Nakano and given in [9; Example

42.6, p. 516] of an almost Dedekind domain which is not a Dedekind domain (Example 5.4). Finally, we consider overrings of Dedekind domains, almost Dedekind domains, and general Z.P.I.-rings. Before proceeding with these things, we state the important relationships we alluded to in the previous paragraph and we give references where proofs of these well known facts can be found.

5.1 Proposition: Let D be an integral domain with identity and let P be a proper prime ideal of D.

(1) PD_P is the unique maximal ideal of D_P [22; Theorem 19, p. 228].

(2) $QD_P \cap D = Q$ for each P-primary ideal Q of D [22; Theorem 19, p. 228].

(3) If $\mathcal{S}$ = {P-primary ideals of D} and τ = {PD_P-primary ideals of D_P}, then there is a one-to-one correspondence between $\mathcal{S}$ and τ [22; Theorem 19, p. 228].

(4) If B and C are ideals of D, then B = C if and only if $BD_M = CD_M$ for each maximal ideal M of D [13; Proposition 3.13, p. 70].

(5) If D_P is a valuation ring, then $\bigcap_{n=1}^{\infty} P^nD_P$ is a prime ideal of D_P which contains every prime ideal of D_P which is properly contained in PD_P [13; Theorem 5.10, p. 105].

The following lemma provides us with a useful result pertaining to the cancellation law for ideals. We use this in proving Theorem 5.3 and, together with condition (5) of Theorem 5.3, this lemma gives us another characterization of almost Dedekind domains.

5.2 Lemma: Let D be an integral domain with identity having quotient field K and let A, B, and C be ideals of D with $A \neq (0)$. The following are equivalent.

(1) The cancellation law for ideals of D.

(2) If $AB \subseteq AC$, then $B \subseteq C$.

Proof: (2) → (1) This implication is clear.

(1) → (2) Assume $AB \subseteq AC$. It is straightforward to show that $A[CA:A]_K = AC$ where $[CA:A]_K = \{k \in K \mid kA \subseteq CA\}$. Since (1) holds in D, we may conclude $[CA:A]_K = C$. By assumption $AB \subseteq AC$, so we must have $B \subseteq [CA:A]_K$. Thus, $B \subseteq C$ and (2) holds in D. ▲

We are now prepared to proceed with our discussion of almost Dedekind domains. Theorem 5.3 gives us six conditions on an integral domain D with identity each of which is equivalent to D being almost Dedekind. After the statement of this theorem and before its proof, we include some general remarks about these equivalent conditions.

5.3 Theorem: Let D be an integral domain with identity which is not a field. The following statements are equivalent.

(1) D is an almost Dedekind domain.

(2) D_M is a Noetherian valuation ring for each maximal ideal M of D; that is, D_M is a rank one discrete valuation ring (Proposition 2.45).

(3) D is one-dimensional and primary ideals of D are prime powers; that is, D is a one-dimensional domain satisfying property (α).

(4) Each ideal of D which has prime radical is a power of its radical.

(5) The cancellation law for ideals of D holds.

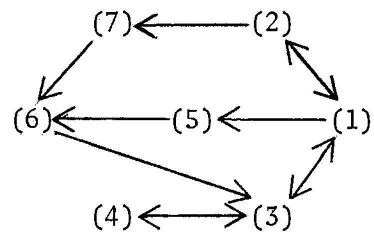
(6) D is a one-dimensional Prüfer domain and D contains no idempotent maximal ideals.

(7) D is a Prufer domain and $\bigcap_{n=1}^{\infty} A^n = (0)$ for each genuine ideal A of D.

Condition (2) offers a characterization of an almost Dedekind domain D in terms of the localization at a maximal ideal of D. The next three characterizations are related to the ideal structure of D. These also give us another result that classifies the Dedekind domains within the class of almost Dedekind domains. Referring to Theorem 3.7 (11), we see that in a Noetherian domain D condition (5) is equivalent to D being Dedekind. Thus, a Noetherian almost Dedekind domain is a Dedekind domain.

The final two conditions classify the almost Dedekind domains within the class of Prüfer domains. A <u>Prüfer domain</u> is an integral domain D with identity such that D_M is a valuation ring for each maximal ideal M of D. Prüfer domains occupy a central role in the study of multiplicative ideal theory. There are several characterizations of Prüfer domains. In fact, many of these are contained in Theorem 3.7. Without the Noetherian assumption, conditions (10), (12), (13), (14), (15), (16), (17), and (19) on an integral domain D with identity are all equivalent to D being Prüfer. Therefore, a Noetherian Prüfer domain is a Dedekind domain and the above theorem shows that the class of almost Dedekind domains lies between Prüfer domains and Dedekind domains. A detailed study of Prüfer domains may be found in [9; Chapter IV] or [13; Chapter VIII].

To show the conditions of Theorem 5.3 are equivalent, we prove the following implications.



Proof of Theorem 5.3: (1) → (2). Suppose D is almost Dedekind and let M be a maximal ideal of D. Since D_M is Dedekind, Theorem 3.6(1) shows D_M is Noetherian. If A and B are proper ideals of D_M, then the radical of both A and B is MD_M since D_M is one-dimensional. Thus, A and B are both MD_M-primary and hence are powers of MD_M by Theorem 3.7(26). Thus, either $A \subseteq B$ or $B \subseteq A$ which proves D_M is a valuation ring. Therefore, D_M is a Noetherian valuation ring.

(2) → (1) Let M be a maximal ideal of D. By assumption D_M is a Noetherian valuation ring and hence a rank one discrete valuation ring by Proposition 2.45. Thus, D_M is a PID and a PID with identity is a Dedekind domain. This proves that (1) holds.

(1) → (3) Assume D is almost Dedekind. Then D_M is a Dedekind domain for each maximal ideal M of D. Since a Dedekind domain is one-dimensional, D_M is one-dimensional. If the dimension of D were greater than one, then we could find a chain $P_1 \subset P_2 \subset M$ of genuine prime ideals of D such that M is maximal. This would imply the existence of the chain $P_1D_M \subset P_2D_M \subset MD_M$ of genuine primes of D_M, a contradiction. Consequently, D is one-dimensional.

Now suppose Q is primary for the prime ideal P of D. Then QD_P is PD_P-primary in the Dedekind domain D_P and is therefore a power of PD_P; that is, $QD_P = (PD_P)^n = P^nD_P$ for some integer n. Since P is maximal in D, P^n is P-primary. Consequently, $Q = QD_P \cap D = P^nD_P \cap D = P^n$ by Proposition 5.1(2), and (3) holds.

(3) → (1) Let M be a maximal ideal of D and let Q be a proper ideal of D_M. Since D is one-dimensional D_M is also and this implies the radical of Q is MD_M. Hence, Q is MD_M-primary and it follows that $Q \cap D$ is M-primary. By hypothesis $Q \cap D = M^n$ for some integer n and thus $Q = M^n D_M$. Therefore, the proper ideals of D_M are powers of MD_M and this proves D_M is Dedekind.

(3) → (4) Let A be an ideal of D and suppose the radical of A is P for some prime ideal P of D. Since D is one-dimensional, P is maximal and hence A is P-primary. Consequently, $A = P^n$ for some integer n and (4) holds.

(4) → (3) Let P be a proper prime ideal of D. We want to prove P is maximal. We first assume P is a minimal prime of a principal ideal (p). In the domain D_P PD_P is maximal and there are no prime ideals properly between $(p)D_P$ and PD_P since P is a minimal prime of (p). Hence, the radical of $(p)D_P$ is the maximal ideal PD_P and it follows that $(p)D_P$ is PD_P-primary. Consequently, $(p)D_P \cap D$ is P-primary in D. By hypothesis $(p)D_P \cap D = P^n$ for some integer n and $(p)D_P = P^nD_P = (PD_P)^n$. Since the ideal $(p)D_P$ is invertible in D_P, PD_P is invertible. Therefore, Proposition 2.14(b) implies that $P^2D_P \subset PD_P$ and thus $P \supset P^2D_P \cap D \supseteq P^2$ by Proposition 2.22. We conclude that $P^2D_P \cap D$ has radical P and so is a power of P; that is, $P^2D_P \cap D = P^2$. Since P^2D_P D is P-primary, P^2 is P-primary. We next choose $x \in P - P^2$, and let y be any element in D - P. Then $P \supseteq P^2 + (xy) \supseteq P^2$ and so $P^2 + (xy)$ has radical P. By hypothesis $P^2 + (xy)$ is a power of P and hence is equal either to P or P^2. However, $xy \notin P^2$, for P^2 is P-primary and $x \notin P^2$, $y \notin P$. Therefore, $P^2 + (xy) = P$ and $x \in P^2 + (xy)$. Thus, $x = t + dxy$ for some $t \in P^2$ and $d \in D$. Then $x(1 - dy) \in P^2$ and $x \notin P^2$ implies that $1 - dy \in P$ and so $1 \in P + (y)$.

It follows that P is maximal in D if P is a minimal prime of a principal ideal.

In the general case, we let x be a non-zero element of P. Then P contains a minimal prime P_1 of (x) by Proposition 2.7, and we have just shown P_1 is maximal in D. Therefore, $P_1 = P$, P is maximal, and D is one-dimensional. By hypothesis, primary ideals of D are prime powers and (3) holds.

(2) → (7) If (2) holds in D, then it is clear that D is a Prüfer domain. Let A be any genuine ideal of D and let M be a maximal ideal of D containing A. Since D_M is a rank one discrete valuation ring, the powers of MD_M descend to (0). Thus,

$\bigcap_{n=1}^{\infty} A^n \subseteq \bigcap_{n=1}^{\infty} M^n = \bigcap_{n=1}^{\infty} (M^nD_M \cap D) = \bigcap_{n=1}^{\infty} (M^nD_M) \cap D = (0)$ and (7) is established.

(7) → (6) Let M be a maximal ideal of D. Then D_M is a valuation ring since D is Prüfer. By hypothesis

$(0) = \bigcap_{n=1}^{\infty} M^n = \bigcap_{n=1}^{\infty} (M^nD_M \cap D) = \bigcap_{n=1}^{\infty} (M^nD_M) \cap D.$ Thus, $\bigcap_{n=1}^{\infty} M^nD_M = (0)$ and by Proposition 5.1(6) we may conclude there is no prime ideal properly between (0) and MD_M. This implies D_M is one-dimensional. Hence, by a previous result we may conclude D is one-dimensional. Next, assuming M is an idempotent maximal ideal of D, it follows that

$MD_M = \bigcap_{n=1}^{\infty} M^nD_M$ which contradicts our assumption. Therefore, $M^2 \neq M$ and (6) is proved.

(1) → (5) Let A, B and C be ideals of D such that $A \neq (0)$ and $AB = AC$. If M is a maximal ideal of D, then $(AD_M)(BD_M) = (AD_M)(CD_M)$ by

well known properties of extended ideals. Since D_M is a Dedekind domain, the cancellation law for ideals holds in D_M. Thus, $BD_M = CD_M$ and since this is true for every maximal ideal M of D, Proposition 5.1(5) shows that B = C.

(5) → (6) The cancellation law for ideals of D implies D is Prüfer since the cancellation law for finitely generated ideals of D is equivalent to D being a Prufer domain [9; Theorem 24.3, p. 299]. Also, if we assume $M^2 = M = MD$ for some maximal ideal M of D, then (5) states that M = D. Thus, there are no idempotent maximal ideals of D. It remains to be shown that D is one-dimensional. Let P be a proper prime ideal of D and choose $x \in D - P$. Since $[P + (x)]^3 = [P^2 + (x^2)][P + (x)]$ and since (5) holds, $[P + (x)]^2 = P^2 + P(x) + (x^2) = P^2 + (x^2)$. Hence, $P(x) \subseteq P^2 + (x^2)$ and if $t \in P$, then there exists $q \in P^2$, $d \in D$ such that $tx = q + dx^2$. Thus, $dx^2 \in P$ and it follows that $d \in P$ since P is prime and $x^2 \notin P$. Therefore, $P(x) \subseteq P^2 + (x^2) \subseteq P^2 + P(x^2) = P[P + (x^2)]$. It follows from condition (5) and Lemma 5.2 that $(x) \subseteq P + (x^2)$ and so $x = p + rx^2$ for some $p \in P$ and $r \in D$. This implies $x(1 - rx) \in P$ and since $x \notin P$, $1 - rx \in P$. Therefore, $1 \in P + (x)$ and P is a maximal ideal of D. This proves that (5) → (6).

(6) → (3) Let Q be a P-primary ideal of D. If P = (0), then Q = (0) and hence Q is a power of P. Thus, we may assume Q and P are proper ideals of D. Since D is one-dimensional, D_P is one-dimensional. Since D is Prüfer and D_P is one dimensional, D_P is a rank one valuation ring. By Proposition 2.46 $(0) = \bigcap_{n=1}^{\infty} (PD_P)^n$. Since QD_P is PD_P-primary QD_P is not contained in every power of PD_P. Therefore, there exists a positive integer m such that $P^{m+1}D_P \subseteq QD_P$ and $P^mD_P \supset QD_P$ since D_P is a

valuation ring. Let $x \in P^mD_P - QD_P$. Then $(x) \supseteq QD_P$ since D_P is a valuation ring. Proposition 2.36 implies the ideal (x) is invertible and we have $QD_P = (x)((x)^{-1}QD_P) = (x)A$. Since $(x)^{-1}QD_P \subseteq (x)^{-1}(x) = D$, $A = (x)^{-1}QD_P$ is an ideal of D_P. Furthermore, since QD_P is PD_P-primary, $x \notin QD_P$, and $(x)A = QD_P$, we have $A \subseteq PD_P$. Therefore, $QD_P = (x)A \subseteq (x)PD_P \subseteq P^{m+1}D_P$. We conclude that $QD_P = P^{m+1}D_P$ and hence $Q = P^{m+1}$ by Proposition 5.1(2). This proves (3) holds in D.

This completes the proof of Theorem 5.2 ▲

The following example of an almost Dedekind domain Z' which is not Dedekind is due to Nakano. We present a brief outline of the construction of Z' [4] and some indication as to how one can prove Z' has these properties. For the reader interested in seeing a detailed discussion of this example, we refer you to [9; Example 42.6, p. 516].

5.4 Example: Let K be the field obtained by the adjunction, to the field of rational numbers, of the pth roots of unity for every prime p in the set of integers Z. Let Z' be the integral closure of Z in K. Nakano proved Z' has no idempotent proper prime ideals. In view of [4; Corollary 1.4, p. 271], this is sufficient to imply that Z' is an almost Dedekind domain. He also proved Z' contains a proper prime ideal which is not finitely generated. Hence, Z' is not Noetherian and so cannot be Dedekind. ▲

In this chapter we have investigated some special overrings of an integral domain with identity. In some instances the algebraist studies overrings of integral domains in a more general setting than what we have done. One question which is usually of interest is the following: Given an integral domain D with certain properties, which of these properties are inherited by an overring D' of D? Before we consider this question for the three main classes of rings and domains we have studied so far we consider a specific example.

5.5 Example: Let Z be the ring of integers and let Z' be any overring of Z. We will show Z' is a Dedekind domain. First we prove Z' is a quotient ring of Z. Let $M = \{b \in Z \mid \frac{a}{b} \in Z' \text{ for some } a \in Z, (a, b) = 1\}$. It is clear that M is a multiplicative system of Z and $Z' \subseteq Z_M$. If $\frac{x}{y}$ is any element of Z_M, then there exists an integer a such that $\frac{a}{y} \in Z'$ and $(a, y) = 1$. Since a and y are relatively prime we can find integers n and m such that $na + my = 1$. Multiplying this equality by $\frac{x}{y}$ we get $nx\frac{a}{y} + mx = \frac{x}{y}$. But, $nx\frac{a}{y} + mx \in Z'$ and hence $\frac{x}{y} \in Z'$. Thus, $Z_M \subseteq Z'$ and it follows that $Z_M = Z'$.

Using the above, we show each ideal of Z_M can be represented as a finite product of prime ideals. If B is an ideal of Z_M, then $B = AZ_M$ for some ideal A of Z by Proposition 2.24(b). Then $A = \prod_{i=1}^{n} P_i^{e_i}$ for prime ideals $P_1, \cdots, P_n$ of Z. Therefore, $B = AZ_M = \prod_{i=1}^{n} (P_iZ_M)^{e_i}$ where each P_iZ_M is a prime ideal of Z_M.

This proves Z' is a Dedekind domain. ▲

The above example shows every overring of Z is also a Dedekind domain. In fact, every overring of a Dedekind domain is a Dedekind domain [12; Theorem 4, p. 815]. If we replace "Dedekind domain" with "almost Dedekind domain", then the statement is still true [12; Theorem 4, p. 815]. A similar result holds for general Z.P.I.-rings with identity; that is, every overring of a general Z.P.I.-ring with identity is a general Z.P.I.-ring [9; p. 489].

CHAPTER VI

MULTIPLICATION RINGS

Theorem 3.7 of Chapter III contains several equivalent conditions on an integral domain D with identity, each of which is equivalent to D being Dedekind. Several of these conditions have been studied in the more general setting of a ring with identity. In this chapter we study condition (9) of Theorem 3.7 in this manner.

Let A and B be ideals of a ring R with identity such that $A \subseteq B$. If there exists an ideal C of R such that A = BC, then B is said to be a _multiplication ideal_. If every ideal of R is a multiplication ideal, then R is called a _multiplication ring_. Multiplication rings have been studied extensively by W. Krull, S. Mori, as well as other noted mathematicians.

We consider multiplication rings in three settings. First, we prove that in an integral domain with identity the concepts of multiplication ring and Dedekind domain are equivalent. We then consider Noetherian multiplication rings. We prove these are precisely the general Z.P.I.-rings which we studied in Chapter IV. The infinite direct sum of the integers modulo two provides us with an example of a non-Noetherian multiplication ring. Thus, this is also an example of a multiplication ring which is not a general Z.P.I.-ring.

Our final objective is to prove that a multiplication ring R can be characterized in terms of the prime ideals of R. Specifically, we show

show that R is a multiplication ring if and only if every prime ideal of R is a multiplication ideal. This is accomplished through a sequence of steps which also establishes another characterization of multiplication rings. These final two results are due to Mott in [17].

We begin our study by considering integral domains which are multiplication rings.

6.1 Theorem: Let D be an integral domain with identity. Then D is a multiplication ring if and only if D is a Dedekind domain.

Proof: (→) Assume D is a multiplication ring and let A be a non-zero ideal of D. We prove that A is invertible. Choose $a \in A$, $a \neq 0$. Since $(a) \subseteq A$, there exists an ideal B such that $(a) = AB$. It follows that A is invertible since (a) is invertible. Therefore, D is a Dedekind domain by Theorem 3.7(5).

(←) Suppose D is a Dedekind domain and let A and B be ideals of D such that $A \subseteq B$. Since D is Dedekind, B is invertible and we have $A = B(B^{-1}A)$. Since $B^{-1}A \subseteq B^{-1}B = D$, $B^{-1}A$ is an ideal of D. Therefore, D is a multiplication ring. ▲

The next result classifies a multiplication ring which is a finite direct sum of rings. Together with Lemma 6.3 and the Structure Theorem for General Z.P.I.-Rings, we have the tools necessary to characterize all Noetherian multiplication rings.

6.2 Lemma: Let R be a ring with identity. If R is a finite direct sum of rings, then R is a multiplication ring if and only if each summand is a multiplication ring.

Proof: (→) Suppose R is a multiplication ring such that

$R = R_1 \oplus \cdots \oplus R_n$ where each R_i is a ring with identity. Let A_i and B_i be ideals of R_i such that $A_i \subseteq B_i$. Consider the ideals

$A = R_1 \oplus \cdots \oplus R_{i-1} \oplus A_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ and

$B = R_1 \oplus \cdots \oplus R_{i-1} \oplus B_i \oplus R_{i+1} \oplus \cdots \oplus R_n$. It is clear that $A \subseteq B$ and since R is a multiplication ring there is an ideal C of R such that $A = BC$. By the choice of A and B, we must have

$C = R_1 \oplus \cdots \oplus R_{i-1} \oplus C_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ for some ideal C_i of R_i. It follows that $A_i = B_iC_i$ and R_i is a multiplication ring.

($\leftarrow$) Now let A and B be ideals of $R = R_1 \oplus \cdots \oplus R_n$ such that $A \subseteq B$. Then $A = A_1 \oplus \cdots \oplus A_n$ and $B = B_1 \oplus \cdots \oplus B_n$ where, for each i, A_i and B_i are ideals of R_i. Then $A_i \subseteq B_i$ for each i and since each R_i is a multiplication ring, there exists an ideal C_i of R_i such that $A_i = B_iC_i$. Letting $C = C_1 \oplus \cdots \oplus C_n$ it follows that $A = BC$. Thus, R is a multiplication ring. ▲

<u>6.3 Lemma</u>: If R is a special primary ring, then R is a multiplication ring.

<u>Proof</u>: Let A and B be ideals of R with $A \subseteq B$. If A or B is not a proper ideal of R, then it is clear that $A = BC$ for some ideal C of R. If A and B are both proper ideals, then $A = M^i$ and $B = M^j$ for some positive integers i and j where M is the unique maximal ideal of R and $i \geq j$. It follows that $A = BM^{i-j}$ and hence R is a multiplication ring.

▲

<u>6.4 Theorem</u>: Let R be a ring with identity. Then R is a general Z.P.I.-ring if and only if R is a Noetherian multiplication ring.

<u>Proof</u>: ($\rightarrow$) Let R be a general Z.P.I.-ring. Then R is Noetherian

by Theorem 4.6 and R is a finite direct sum of Dedekind domains and special primary rings by Theorem 4.18(a). Since each summand of R is a multiplication ring, Lemma 6.2 shows R is also a multiplication ring.

($\leftarrow$) Let R be a Noetherian multiplication ring. We show R is a general Z.P.I.-ring by showing each maximal ideal of R is simple. Let M be a maximal ideal of R and let A be an ideal such that $M^2 \subseteq A \subseteq M$. By hypothesis, there exists an ideal B such that A = MB. If $B \subseteq M$, then $A = MB \subseteq M^2$, and $A = M^2$. If $B \not\subseteq M$, then $R = M + B$ and $M = RM = (M + B)M = M^2 + MB = A$ since $M^2 \subseteq MB = A$. Therefore, A = M or $A = M^2$ and by Theorem 4.19(3) R is a general Z.P.I.-ring. ▲

The previous theorem implies we must consider rings which are not finite direct sums of rings if we are to find a non-Noetherian multiplication ring. It is not difficult to prove that an infinite direct sum of rings is not Noetherian. Thus, it seems natural to look in such a setting to find a non-Noetherian multiplication ring.

<u>6.5 Example</u>: Let $R = \sum_{i=1}^{\infty} F_i$ where each $F_i = Z_2$, the integers modulo two. Then $F_1 \oplus \sum_{i=2}^{\infty} (0) \subset F_1 \oplus F_2 \oplus \sum_{i=3}^{\infty} (0) \subset \cdots \subset F_1 \oplus \cdots \oplus F_k \oplus \sum_{i=k+1}^{\infty} (0) \subset \cdots$ is an infinite strictly ascending chain of ideals of R. Thus, R is not Noetherian by Proposition 2.16(b).

Every element of R is idempotent. Therefore, if A is an ideal of R, $A = A^2$. Let A and B be ideals of R such that $A \subseteq B$. Then $A = A^2 \subseteq AB = A$. Hence, A = AB and R is a multiplication ring.

This example is a special case of a result of Krull as stated in [4]. He proved every multiplication ring is a subring of a direct sum of Dedekind domains and special primary rings. ▲

The classes of rings we have studied so far have at least one characteristic in common; namely, various conditions on their sets of prime ideals characterize them. Mott [17] uses the prime ideals of a ring with identity to characterize multiplication rings. Our next objective is to prove this equivalence. This result is contained in the statement of the following theorem.

<u>6.6 Theorem</u>: Let R be a ring with identity. The following statements are equivalent.

(1) R is a multiplication ring.

(2) Each prime ideal of R is a multiplication ideal.

(3) R is a ring in which the following three conditions are valid:

(a) Each ideal of R is equal to its kernel.

(b) Each primary ideal of R is a power of its radical.

(c) If P is a minimal prime ideal of an ideal B and n is the least positive integer such that P^n is an isolated primary component of B and if $P^n \neq P^{n+1}$, then P does not contain the intersection of the remaining isolated primary components of B.

It is clear condition (1) implies condition (2). We proceed to show condition (2) implies condition (3). This is accomplished through a sequence of steps as we establish several characteristics of a ring which satisfies condition (2). Temporarily, we call a ring with identity which satisfies Theorem 6.6(2) a <u>weak multiplication ring</u>. We now define the concepts we need as we pursue the goal outlined above.

<u>6.7 Definition</u>: If A is an ideal of a ring R with identity and P is a minimal prime ideal of A, then the intersection of all P-primary ideals containing A is called an <u>isolated P-primary component</u> of A. The

intersection of all isolated primary components of A is called the <u>kernel</u> of A.

Several of the properties of rings which have appeared in the previous chapters are also possessed by weak multiplication rings. The next lemma gives some of these.

<u>6.8 Lemma</u>: Let R be a weak multiplication ring.

(1) R/A is a weak multiplication ring for each ideal A of R.

(2) If R is an integral domain, then R is a Dedekind domain.

(3) There is no prime ideal chain $P_1 \subset P_2 \subset P_3 \subset R$ in R.

(4) Each maximal ideal of R is simple.

<u>Proof</u>: (1) This follows directly from properties of the ideals of R/A.

(2) Let P be a nonzero prime ideal of R and let $p \in P$. Then $(p) \subseteq P$ and hence there is an ideal B such that $(p) = PB$. Since (p) is invertible, P is invertible. By Theorem 3.7(6) R is a Dedekind domain.

(3) If a chain $P_1 \subset P_2 \subset P_3 \subset R$ of prime ideals exists in R, then in the Dedekind domain R/P_1 we have the chain $P_1/P_1 \subset P_2/P_1 \subset P_3/P_1$ of genuine prime ideals of R/P_1. However, a Dedekind domain is one-dimensional and hence no such chain exists.

(4) The second half of the proof of Theorem 6.4 also proves this result. ▲

The next result is useful when we prove a weak multiplication ring satisfies condition 3(a) of Theorem 6.6. It establishes a relationship between the intersection of the powers of a maximal ideal M and the prime ideals contained in M.

<u>6.9 Lemma</u>: Let R be a weak multiplication ring. If M is a maximal ideal of R which properly contains a prime ideal P, then

$$P = \bigcap_{n=1}^{\infty} M^n \text{ and } P = MP.$$

<u>Proof</u>: Consider the Dedekind domain $\overline{R} = R/P$. Then $(0) = \overline{P} = \bigcap_{n=1}^{\infty} \overline{M}^n$ and, consequently, $P \supseteq \bigcap_{n=1}^{\infty} M^n$. Since $P \subseteq M$, there is an ideal C such that $P = MC \subseteq C$. Using the fact that P is a prime ideal and since $M \not\subseteq P$, it follows that $C \subseteq P$. Therefore, $C = P$ and $P = MP = M^2P = \cdots$. Thus, $P \subseteq \bigcap_{n=1}^{\infty} M^n$ and we have the desired equality, $P = \bigcap_{n=1}^{\infty} M^n$. ▲

<u>6.10 Theorem</u>: If R is a weak multiplication ring, then every ideal is equal to its kernel.

<u>Proof</u>: Let A be an ideal of R and suppose $A \neq A^*$ where A^* denotes the kernel of A. Let $a \in A^* - A$, and consider the ideal $A' = A:(a)$. Let M be a minimal prime ideal of A'. By a known result of Krull and stated in [17; (vi), p. 430], A' properly contains a minimal prime ideal P of A. Thus, Lemma 6.8(3) implies M is a maximal ideal. By Lemma 6.9, $P = \bigcap_{n=1}^{\infty} M^n$ and $P = MP$. Since $A' \subseteq M$, there is an ideal C such that $A' = MC$. If $C \subseteq A'$, then $A' = MC \subseteq MA' \subseteq A'$ and hence $A' = MA' = M^2A' = \cdots$. Thus, $A' \subseteq \bigcap_{n=1}^{\infty} M^n = P$ and this implies M is not a minimal prime ideal of A'. Therefore, $C \not\subseteq A'$ and hence $(a)\cdot C \not\subseteq A$. However, $(a)\cdot C \subseteq (a) \subseteq P$ since $a \in A^*$. Consequently, there is an ideal B such that $(a)\cdot C = PB = MPB = M(a)C = (a)A' \subseteq A$. This is a contradiction and proves $A = A^*$. ▲

Before we consider condition 3(b) of Theorem 6.6, we need two results about the powers of the maximal ideals of a weak multiplication ring. Lemma 6.11 shows the powers of proper maximal ideals are multiplication ideals while Lemma 6.12 shows the intersection of the powers of a maximal ideal is a prime ideal under certain restrictions on the maximal ideals of a weak multiplication ring.

<u>6.11 Lemma</u>: If M is a proper maximal ideal of a weak multiplication ring R and if A is an ideal contained in M^n for some integer n, then there is an ideal C such that $A = M^nC$. Furthermore, if $A \not\subseteq M^{n+1}$, then $C \not\subseteq M$.

<u>Proof</u>: The proof is by induction. It is clear the statement is true for n = 1. Suppose $A \subseteq M^k$ implies $A = M^kC$. Then if $A \subseteq M^{k+1} \subseteq M^k$, $A = M^kC$. If $M^{k+1} = M^k$, then $A = M^{k+1}C$. Suppose $M^{k+1} \neq M^k$. Since M is maximal, M^{k+1} is an M-primary ideal containing $A = M^kC$. Since $M^k \not\subseteq M^{k+1}$, it follows that $C \subseteq M$. Hence, $C = MC'$ and $A = M^k(MC') = M^{k+1}C'$. This completes the proof of the first part of the lemma.

If $A \subseteq M^n$ and $A \not\subseteq M^{n+1}$, then $A = M^nC$ by the above. However, $C \not\subseteq M$ because if $C \subseteq M$, then $C = MB$ and this would imply that $A = M^nC = M^{n+1}B \subseteq M^{n+1}$. Since this can't occur, we conclude that $C \not\subseteq M$ and the proof is complete. ▲

<u>6.12 Lemma</u>: Let R be a weak multiplication ring. If M is a maximal ideal and $M^n \neq M^{n+1}$ for each positive integer n, then

$$P = \bigcap_{n=1}^{\infty} M^n \text{ is a prime ideal.}$$

<u>Proof</u>: Suppose $x \notin P$ and $y \notin P$. Then there are positive integers k and n such that $x \in M^k$ and $y \in M^n$, but $x \notin M^{k+1}$ and $y \notin M^{n+1}$. Thus,

there exist ideals B and C, not contained in M, such that $(x) = M^kB$ and $(y) = M^nC$. Therefore, $(xy) = M^{n+k}BC$ and $BC \nsubseteq M$ since $B \nsubseteq M$ and $C \nsubseteq M$. Then $(xy) \nsubseteq M^{n+k+1}$, for if it were this would imply that $M^{n+k} \subseteq M^{n+k+1}$ since $BC \nsubseteq M$ and M^{n+k+1} is M-primary. Consequently, $xy \notin P$ and P is a prime ideal. ▲

The following useful result of S. Mori will shorten our work in proving the next theorem: If P is a non-maximal prime ideal and Q is P-primary in a ring with identity in which every ideal is equal to its kernel, then $P = P^2$ and $Q = P$ [17; (ix), p. 431]. Thus, to establish condition 3(b) of Theorem 6.6, we need only consider the maximal ideals of a weak multiplication ring.

<u>6.13 Theorem</u>: If Q is P-primary in the weak multiplication ring R, then Q is a power of P.

<u>Proof</u>: By the result of Mori mentioned above, we need only consider maximal ideals of R. Assume P is a maximal ideal of R. The following two cases will be considered: (a) $P^n \neq P^{n+1}$ for every positive integer n and (b) $P^n = P^{n+1}$ for some positive integer n.

(a) By Lemma 6.12, $P' = \bigcap_{n=1}^{\infty} P^n$ is a prime ideal of R. Since Q is P-primary and $P' \subset P$, Q is not contained in P'. Therefore, there is an integer k such that $Q \subseteq P^k$ but $Q \nsubseteq P^{k+1}$. By Lemma 6.11, $Q = P^kC$ and since $Q \nsubseteq P^{k+1}$ we may conclude that $C \nsubseteq P$. We now show $C = R$. If C is a proper ideal, any proper prime P_1 containing C must contain Q and hence must contain the maximal ideal P. This would imply that $P = P_1$ and therefore $C \subseteq P$. This contradiction shows $C = R$ and $Q = P^k$.

(b) If $P^n = P^{n+1}$ for some positive integer n, then let k be the least positive integer such that $P^k = P^{k+1}$. We consider the following two possibilities separately: (i) $Q \subseteq P^k$ and (ii) $Q \nsubseteq P^k$.

(i) If $Q \subseteq P^k = P^{2k}$, then for each $a \in P^k$ there is an ideal C such that $(a) = P^kC = P^{2k}C = P^k(a)$. Since (a) is finitely generated, Proposition 2.14(a) implies there exists an element $p \in P^k \subseteq P$ such that $a = pa = p^2a = \cdots$. Since the radical of Q is P, $p^s \in Q$ for some integer s. Consequently, $a \in Q$, $P^k \subseteq Q$ and hence $Q = P^k$.

(ii) If $Q \nsubseteq P^k$, then the radical of $Q + P^k$ is P and hence $Q + P^k$ is a P-primary ideal which properly contains P^k. Since P is maximal, P is simple by Lemma 6.8(4). By Proposition 4.13 the only ideals between P and P^k are powers of P. Therefore, $Q + P^k = P^t$ for some integer $t < k$. Since $Q \subseteq P^t$ and $Q \nsubseteq P^k$, there is a positive integer m such that $t < m < k$ and $Q \subseteq P^m$ but $Q \nsubseteq P^{m+1}$. As before, there is an ideal C such that $Q = P^mC$ and $C \nsubseteq P$. Repeating the argument used in (a), we see that $C = R$ and $Q = P^m$.

Thus, each primary ideal of R is a power of its radical. ▲

To complete the proof that theorem 6.6(2) implies Theorem 6.6(3), we need only establish condition 3(c). This is the content of the next theorem as we again use Lemma 6.11 and Theorem 6.10.

<u>6.14 Theorem</u>: Let R be a weak multiplication ring. If P is a minimal prime ideal of an ideal B and n is the least positive integer such that P^n is an isolated primary component of B and if $P^n \neq P^{n+1}$, then P does not contain the intersection of the remaining isolated primary components of B.

<u>Proof</u>: Since R has property (α), the set of isolated primary components of B is a set of prime power ideals. Let $B' = \bigcap_r P_r^{n_r}$ where

$\{P_r^{n_r}\}$ is the set of isolated primary components of B except P^n. Since B is equal to its kernel, $B = B' \cap P^n$. Also, since $P^n \neq P^{n+1}$, $P \neq P^2$. By the result to Mori cited earlier, we may conclude that P is a maximal ideal since $P \neq P^2$. The definition of an isolated primary component of B and the fact that P-primary ideals are powers of P, implies n is the largest integer such that $B \subseteq P^n$. Since P is maximal, P^{n+1} is P-primary and we conclude $B \nsubseteq P^{n+1}$. Lemma 3.11 implies there exists an ideal C such that $B = P^nC$ and $C \nsubseteq P$. Since $B \subseteq B'$, P^nC is contained in the P_r-primary ideal $P_r^{n_r}$ for each r. Since P is maximal, $P^n \nsubseteq P_r$ and we conclude that $C \subseteq P_r$ for each r. Therefore, $C \subseteq B'$ and $B' \nsubseteq P$ since $C \nsubseteq P$. This completes the proof. ▲

Summarizing our progress toward proving Theorem 6.6, we observe that we have shown condition (1) implies condition (2) and condition (2) implies condition (3). It remains to be shown that condition (3) implies condition (1). We will prove this by showing how to select an ideal C such that A = BC whenever $A \subseteq B$. The desired equality will follow by showing the kernel of A equals the kernel of BC.

<u>6.15 Theorem:</u> Let R be a ring with identity satisfying condition (3) of Theorem 6.6. Then R is a multiplication ring.

<u>Proof:</u> Let A and B be ideals of R such that $A \subset B$. Let $\{P_r\}$ be the set of prime ideals of R which are minimal primes of A and B, $\{P'_s\}$ be the set of minimal primes of B but not A, and $\{P''_t\}$ be the set of minimal primes of A but not B. Since primary ideals of R are prime powers, the isolated primary components of A and B are intersections of powers of the prime ideals of these three sets. Since every ideal is equal to its kernel we have

$$B = (\bigcap_r P_r^{n_r}) \cap (\bigcap_s P_s'^{\ell_s})$$

and

$$A = (\bigcap_r P_r^{m_r}) \cap (\bigcap_t P_t''^{k_t})$$

where the exponents n_r, m_r, ℓ_s, and k_t denote the least positive integers such that $P_r^{n_r}$, $P_s'^{\ell_s}$ are isolated primary components of B and $P_r^{m_r}$, $P_t''^{k_t}$ are isolated primary components of A. Since $A \subset B$, $P_r^{m_r} \subseteq P_r^{n_r}$ and hence $n_r \leq m_r$ for each r. Let $C = (\bigcap_r P_r^{m_r - n_r}) \cap (\bigcap_t P_t''^{k_t})$.

It is clear that $A \subseteq C$. Next we show $BC \subseteq A$. For $x \in BC$, $x = \sum_{i=1}^{n} b_i c_i$, where $b_i \in B$ and $c_i \in C$ for each i. Then $b_i \in P_r^{n_r}$, $c_i \in P_r^{m_r - n_r}$, and $c_i \in P_t''^{k_t}$ for each r, t, and i. Consequently, $b_i c_i \in P_r^{m_r}$ and $b_i c_i \in P_t''^{k_t}$ for each r, t, and i. We conclude that $x \in A$, and, as a result, $BC \subseteq A$.

We now proced to show every minimal prime of BC is a minimal prime of A. Let P be any minimal prime of BC. Then P contains B or C. If $B \subseteq P$, then P is a minimal prime of B since $BC \subseteq B$. By assumption, $A \subseteq B$ and hence $BC \subseteq A \subseteq P$. Thus, P is a minimal prime of A. Therefore, P is a minimal prime of A and B and hence $P = P_r$ for some r. If $B \nsubseteq P$, then $C \subseteq P$ and since $BC \subseteq A \subseteq C$, P is a minimal prime of A and C. In this case $P = P_t''$ for some t. In particular, any minimal prime ideal of BC must be a minimal prime ideal of A. Therefore, let

$$BC = (\bigcap_r P_r^{m_r'}) \cap (\bigcap_t P_t''^{k_t'})$$

be the kernel of BC where m'_r and k_t are the minimal exponents such that $P_r^{m'}$ and $P_t''^{k'_t}$ are isolated primary components of BC.

To finish the proof we need to show that $m'_r = m_r$ and $k'_t = k_t$ for each r and t. Since BC $\subseteq$ A we must have $m_r \leq m'_r$ and $k_t \leq k'_t$ for each r and t. Furthermore, for each t, $P_t''^{k_t}$ is an isolated primary component of C, since $A \subseteq C \subseteq P_t''^{k_t}$ and $P_t''^{k_t}$ is an isolated primary component of A. Since $B \not\subseteq P''_t$ and $BC \subseteq P_t''^{k'_t}$, it follows that $C \subseteq P_t''^{k'_t}$ since $P_t''^{k'_t}$ is P''_t-primary. Thus, $P_t''^{k_t} \subseteq P_t''^{k'_t}$ and we may conclude $k_t \geq k'_t$ for each t. This being the case, one concludes $k_t = k'_t$ for each t.

For a particular isolated primary component of A, say $P_{r_0}^{m_{r_0}}$, we consider the two possibilities: (i) $P_{r_0}^{m_{r_0}} = P_{r_0}^{m_{r_0}+1}$ and (ii) $P_{r_0}^{m_{r_0}} \neq P_{r_0}^{m_{r_0}+1}$. If (i) holds, then $P_{r_0}^{m_{r_0}} = P_{r_0}^{j_{r_0}}$ for each $j_{r_0} \geq m_{r_0}$. Since $m'_{r_0} \geq m_{r_0}$ we have $P_{r_0}^{m_{r_0}} = P_{r_0}^{m'_{r_0}}$. Since m'_{r_0} is the least positive integer such that $P_{r_0}^{m'_{r_0}}$ is an isolated primary component of BC and since $m_{r_0} \leq m'_{r_0}$, we may conclude $m_0^{r_0} = m_0^{r'_0}$. Now assume (ii) holds for the ideals $P_{r_0}^{m_{r_0}}$ and $P_{r_0}^{m_{r_0}+1}$. Since every ideal is equal to its kernel, every non-maximal prime ideal is idempotent by Mori's result cited earlier. Since (ii) holds, $P_{r_0} \neq P_{r_0}^2$ and, consequently, P_{r_0} is a maximal ideal. Let

$$C' = (\bigcap_{r \neq r_0} P_r^{m_r - n_r}) \cap (\bigcap_t P''^{\ell_t}) \quad ,$$

$$B' = (\bigcap_{r \neq r_0} P_r^{n_r}) \cap (\bigcap_s P'^{\ell_s}) \quad ,$$

and

$$A' = (\bigcap_{r \neq r_0} P_r^{m_r}) \cap (\bigcap_t P_t''^{k_t}) \quad .$$

Then by condition 3(c) of Theorem 6.6, $P_{r_0} \not\supseteq A'$, and since P_{r_0} is maximal, $P_{r_0} + A' = R$. Since $P_{r_0}^{m_{r_0}}$ is P_{r_0}-primary, Proposition 2.10(a) shows that $P_{r_0}^{m_{r_0}} + A' = R$. Thus, $A = P_{r_0}^{m_{r_0}} \cap A' = P_{r_0}^{m_{r_0}} \cdot A'$ by Proposition 2.10(b). Similarly, $B' \not\subseteq P_{r_0}$, $P_{r_0}^{n_{r_0}} + B' = R$, and $B = P_{r_0}^{n_{r_0}} \cdot B'$. Also, $C' \not\subseteq P_{r_0}$ since $A' \subseteq C'$ and $A' \not\subseteq P_{r_0}$. Therefore, $P_{r_0}^{m_{r_0} - n_{r_0}} + C' = R$ and $P_{r_0}^{m_{r_0} - n_{r_0}} C' = C$. As a consequence, $BC = P_{r_0}^{m_{r_0}} B'C'$ where $B'C' \not\subseteq P_{r_0}$. Thus, $P_{r_0}^{m_{r_0}}$ is an isolated primary component of BC. Since $m_0^{r_0'}$ is the least positive integer such that $P_{r_0}^{m'_{r_0}}$ is an isolated primary component of BC, we may conclude $m_{r_0}^{r_0'} \leq m_{r_0}^{r_0}$. This implies $m_{r_0} = m'_{r_0}$ and consequently, $m_r = m'_r$ for each r. We have shown the kernels of BC and A are equal and hence by condition 3(a) of Theorem 6.6, BC = A. Therefore, R is a multiplication ring. ▲

The results of Theorems 6.10, 6.13, 6.14, and 6.15 imply the three conditions of Theorem 6.6 are equivalent. Hence, we have a characterization of a multiplication ring R in terms of the prime ideals of R.

CHAPTER VII

SUMMARY

This thesis has centered around the discussion of several closely related classes of rings. This chapter presents a summary which will clarify the interrelationships among these classes of rings and, in addition, the relation of these to PID's and UFD's. As a means of studying these similarities and differences more effectively, the reader is encouraged to use the chart which appears at the end of this chapter in conjunction with the discussion below. All rings and integral domains considered in this chapter are assumed to contain an identity.

It is well known that a PID is a UFD. However, $Z[x]$ is an example of a UFD which is not a PID. Since each element of a PID has a representation as a finite product of prime elements and since these prime elements generate prime ideals, a PID is a Dedekind domain. The reverse implication is not true as the domain $Z(\sqrt{10})$ illustrates.

The class of Dedekind domains is closely related, both historically and conceptually, to the class of UFD's. However, they are distinct entities. $Z[x]$ is a UFD which is not a Dedekind domain (Remark 3.9, page 27) and $Z(\sqrt{10})$ is a Dedekind domain which is not a UFD (Example 3.1, page 17 and Remark 3.8, page 24).

As a natural generalization of Dedekind domains, we study the class of general Z.P.I.-rings. From the definitions it is clear that a general Z.P.I.-ring which is also an integral domain is a Dedekind domain while

$Z \oplus Z$ is an example of a general Z.P.I.-ring with proper zero-divisors (Example 4.1, page 35). Simiarly, the classes of multiplication rings and Dedekind domains coincide in an integral domain (Theorem 6.1, page 73).

The Structure Theorem for General Z.P.I.-Rings (Theorem 4.18, page 47) shows that a general Z.P.I.-ring is a finite direct sum of Dedekind domains and special primary rings. Since these two classes of rings are multiplication rings (Theorem 6.1, page 73 and Lemma 6.3, page 74), Lemma 6.2 shows that a general Z.P.I.-ring is a multiplication ring. In a Noetherian ring these two classes of rings are equivalent (Theorem 6.4, page 74) while $\sum_{i=1}^{\infty} Z_2$ is an example of a non-Noetherian multiplication ring which is not a general Z.P.I.-ring (Example 6.5, page 75).

Another generalization of Dedekind domains is the class of almost Dedekind domains. As the name suggests a Dedekind domain is an almost Dedekind domain while a Noetherian almost Dedekind domain is Dedekind (Theorem 5.3(5), page 64). If $T = \{t \mid t$ is a pth root of unity for some prime integer $p\}$, then the integral closure Z' of Z in the field Q(T) is an example of an almost Dedekind domain which is not Dedekind (Example 5.4, page 70).

The following chart illustrates the interrelationships alluded to above. A solid arrow between two classes of rings A and B ($A \rightarrow B$) indicates that $A \subseteq B$. A broken arrow of the form C--- EX---> D implies that $C \nsubseteq D$ and EX is an example of a ring which is in class C and not in class D. A curved arc connecting two classes of rings E and F ($E \xrightarrow{\text{CONDITION}} F$) indicates $E \subseteq F$ if the condition written on the arc is assumed to hold for rings in class E.

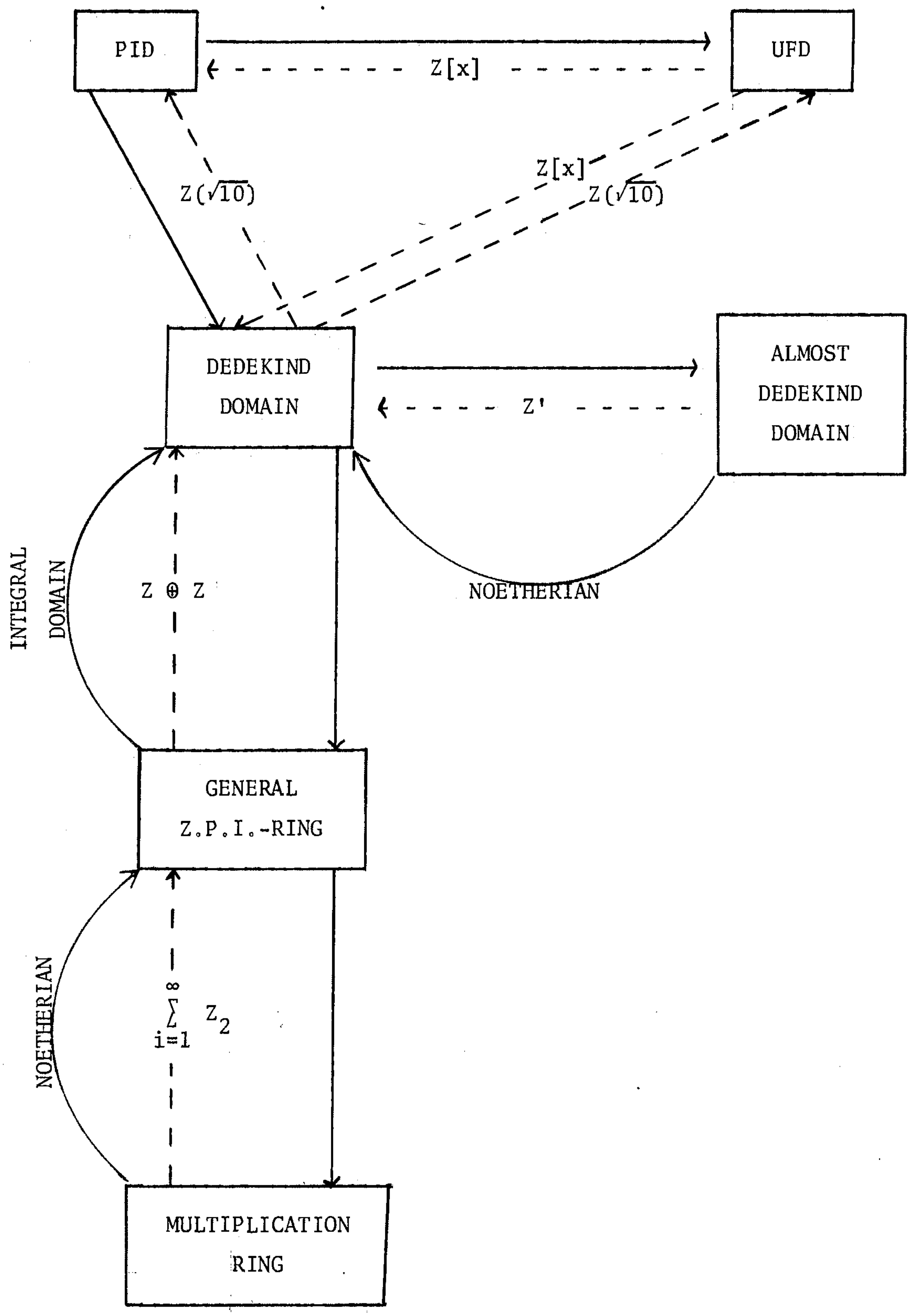


Figure 1. Summary of Interrelationships

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