On partial differential equations modified with fractional operators and integral transformations

Nonlocal and Nonlinear PDE Models

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To Dan, Hannah, and Timmy

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Abstract

We present the theory and applications of nonlocal operators in the setting of partial differential equations (PDE), with emphasis placed on the fractional Laplacian, $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$, which generalizes the Laplacian differential operator, Δ (also denoted by ∇^2). By extending the heat equation $\partial_t u - \nu\Delta u = f$ into its fractional counterpart $\partial_t u + \nu\Lambda^\alpha u = f$, we can study a whole *family* of PDE with parameter $\alpha \in [0, 2]$ all at once. This mindset naturally motivates the investigation of $\alpha's$ influence on the solution to these nonlocal PDE; it is not at all obvious what effect or meaning these colloquially named "fractional derivatives" have, especially at small values of α. We detail some decay bounds and perform numerical experiments to provide further insight into our theorems and observe how these results hold up in practice. Since our focus is on the corresponding initial value problems for these evolutionary PDE, we primarily perform 1D simulations on a periodic domain using pseudo-spectral methods. We conclude this work by incorporating advection (also known as "transport" or "convection") terms into the PDE, with both nonlinear and nonlocal modifications, e.g., integral transformations such as the Hilbert transform. These equations are more physically realistic and are often considered models for viscous incompressible fluid flow and related phenomena, particularly in the case of Burgers' equation which exhibits shock wave behavior.

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Introduction

1.1 Partial Differential Equations

Partial differential equations is one of the most diverse and ubiquitous fields of both pure and applied mathematics. A partial differential equation (PDE) is an equation involving two or more variables and some order of partial derivatives of these variables. Not entirely unlike ordinary differential equations (ODE), the vast majority of PDE do not have explicit solutions or representations. However, in contrast to ODE, PDE do not have a comprehensive framework for their analysis and solution. Strikingly similar PDE can have erratically different behavior, and with different behavior comes the need to utilize an extensive repertoire of methods. This is why in practice, to solve a PDE we typically turn to a computer and write computer codes using techniques from numerical analysis to obtain an approximate solution. Or if we desire to be solely theoretical, we deduce results about the solutions to PDE, such as existence and uniqueness of solutions, without actually writing down what the solution is. Here, mathematicians relax the *classical* sense of a smooth, continuous solution in favor of generalized *weak* solutions that are easier to work with in proofs. In either case, the theory of PDE is one of the richest and most interdisciplinary fields of mathematics.

In describing the flow of fluids, PDE such as the Navier-Stokes equation have captured the attention of mathematicians and the general science community. There is an outstanding Hilbert Millennium Prize problem for proof or disproof of existence and smoothness for the 3D Navier-Stokes equation [40]. Recently, progress has been made on the side of disproof: Buckmaster and Vicol [10] have proven that a certain (wild) class of weak solutions develop singularities in finite time, termed "finite time blow up." These extremely esoteric yet fascinating results were even communicated to a broader scientific audience via online news and magazine outlets such as WIRED and Quanta. Clearly, the *mathematical fluid dynamics* community is quite active and truly hard problems are still outstanding in this field.

In this work, we assume that all problems are **well-posed** in the sense of the definition given in [38]:

1. *A solution exists*

2. *The solution is unique*

3. *The solution has "continuous dependence" on the parameters and data of the problem*

The first two criteria are quite straightforward (and are at times proved in this work), but the meaning of the third can be up for interpretation. It is especially significant when the PDE is used to model physical phenomena that somehow constrain the allowable behavior that is expected to be exhibited by the equation. Only well-posed initial value problems arise in nature. For linear homogeneous first-order (in time) equations in one dimension, continuous dependence is satisfied by the estimate

$$\|u(\cdot,t)\| \leq C\,\|u_0(\cdot,t)\| \tag{1.1}$$

for some constant C depending on t [66]. In fact, an equation of this type satisfying inequality (1.1) is automatically well-posed.

Some common and well-known PDE are:

- Laplace equation
$$\Delta u = 0$$

- Helmholtz eigenvalue equation
$$\Delta u = \lambda u$$

- Heat equation
$$\partial_t u = \Delta u$$

- Wave equation
$$\partial_{tt} u = \Delta u$$

- Schrödinger's equation
$$i\partial_{tt} u = -\Delta u$$

- KdV equation
$$\partial_t + u\partial_x u + \partial_{xxx} u = 0$$

where $(x, t) \in \mathbb{R}^d \times (0, \infty)$, Δ is the Laplacian, and the unknown $u = u(x, t)$ is the function $u : \mathbb{R}^d \times [0, \infty) \longrightarrow \mathbb{R}$. The variable $x = (x_1, x_2, \ldots, x_d)$ is a column vector representing a point in space and t is the time variable. These conventions carry over to the rest of this work.

1.2 Motivation

Now that we have introduced an extremely basic notion of a partial differential equation, we now briefly provide some background for studying these PDE with fractional operator and integral transformation terms. Our work is largely motivated by the mathematical treatment of *diffusion*, a type of transport phenomena. We provide a historical perspective and emphasize the role it plays in numerous PDE models of the physical sciences and engineering.

Diffusion occurs in a vast variety of natural phenomena. The concept is simple: a substance that undergoes diffusion spreads out to occupy a given space. In chemistry, this may be a solvent moving from an area of high concentration to low concentration. Or in physics, the motion of free electrons and ions. We have the diffusion of heat and of a viscous fluid's momentum in the thermal/fluid sciences. The movement of financial markets in economics is also a diffusive process. Diffusion is an underlying feature of many parabolic evolution PDE that model these processes (e.g., the diffusion equation $\partial_t u = \Delta u$). However, there is an inherent probabilistic, microscopic significance to diffusion in processes that otherwise seem macroscopic in scale. Heat transfer depends on the random motion of atoms; prices are influenced by small scale fluctuations in quantities; viscous forces in a fluid flow are driven by molecular forces themselves. The small scale nature of diffusion distinguishes it from other transport phenomena that rely on bulk, large scale motion.

Scientists observed highly irregular, random motion of small particles; such motion did not damp out with time and even amplified with temperature. This was the *Brownian motion*, first noted in the late eighteenth century. After many decades, it was found that this phenomena was the result of thermodynamic action acting locally on the particles at the molecular scale. Without a deterministic way to study this behavior besides the diffusion equation— with the Laplacian operator Δ known to be responsible for Brownian motion diffusion [1]—, the mathematical development of diffusion followed an overwhelmingly statistical approach [47]. That is, probability theorists defined diffusion by Brownian motion, random walks, and related stochastic processes [74].

The renowned physicist N. Wiener proved the interesting theorem: that the trajectories of the particles undergoing Brownian motion are almost everywhere (a. e.) continuous, but nowhere differentiable. This made sense in terms of the intuitive and qualitative observations of the process, and more results followed suit in other scientific applications. A.N. Kolmogorov developed his now famous "four-thirds" scaling law for homogeneous isotropic turbulence, where the diffusivity k of the fluid varied as $k^{\frac{4}{3}}$. Here, turbulent diffusion exhibited a fast $t^{\frac{3}{2}}$ growth rate instead of the $t^{\frac{1}{2}}$ rate in normal (Gaussian) diffusion. This *superdiffusion* leads to the nonlocal Lévy motion and *anomalous diffusion*, a special case of Brownian motion and ordinary diffusion. DUBKOV, SPAGNOLO, and UCHAIKIN further describe the historical origins of Lévy motion in [36].

To describe the anomalous diffusion behavior of the Lévy motion [47], mathematicians turned to fractional operators. The fractional Laplacian is a diffusion-type operator that exhibits similar long-range diffusion behavior as that seen in the Lévy motion, and this is due to the characteristic function of what is known as a homogeneous Markovian process family acting as a (almost) Fourier transform of the solution to the generalized heat equation, which we will study in detail in Chapter 2. The fractional Laplacian is actually a generator of a stable Lévy process. Lévy processes are stochastic processes with extremely long jumps, meaning the trajectories are discontinuous [9]. This typically occurs where scale invariance takes place in a system. Some examples include diffusion in porous media flow, plasma transport, seismic behavior of earthquakes, Greenland ice core measurements, and the flight of albatrosses [71]. The interpretation from a probabilistic standpoint is that we replace Gaussian white noise in the normal diffusion case (Brownian motion) with so-called Lévy stable noise for the case of superdiffusion. The standard next-neighbor interactions of random walks and short-distance coupling of Brownian motion do not hold in these scenarios [74].

Lévy flights, these unusually long jumps in a stochastic process, have actually been observed in the field and experiments [46]. Fractional Laplacians have also been claimed to appear in a wide variety of other disciplines. In quantum mechanics, the Schrödinger equation with fractional diffusion is used to study particles in stochastic fields, again related to Lévy processes [42]. The fractional Laplacian was used in a new turbulence closure model [45] and in incompressible flow in porous media [71, 72, 73, 74] and quasi-geostrophic flows [11, 51, 57], all as examples from fluid mechanics. In chemistry and materials science, fractional Laplacians arise in stable Lévy processes related to material phase transitions, grain dislocations, and their associated free surface boundary value problems [70, 71]. In peridynamics, a nonlocal extension of continuum mechanics, PDE are modified with fractional Laplacians for modeling material fracture, crack propagation, and nonlinear elasticity [29].

One shared feature in all of these examples is the *nonlocal* nature of the associated interactions; instead of having an effect in a neighborhood, these interactions act at a distance. It is clear that PDE models for these systems could be improved with nonlocal fractional operator terms. With this mindset, we investigate the theory and numerics of both nonlocal and nonlinear PDE that model a range of behavior from anomalous heat transfer to the nonlocal velocity of vortex sheets.

1.3 Preface

It is worth noting that a significant portion of the theoretical topics to be presented in this thesis have been taken from the class lectures in [77]. While there is no claim that all results are original, deliberate effort has been placed in carefully providing proofs and interpreting both theoretical and computational studies.

The following notation is used throughout this work.

Notation 1.3.0.1. We use the acronym "PDE" to refer to both a single *partial differential equation* and multiple *partial differential equations*. The meaning should be clear from the context, and this notation is common practice in mathematics (but not so in other fields of science).

Notation 1.3.0.2. We at times use the following *Einstein notation* for convenience of writing sums of partial derivatives:

$$\partial_k \partial_k = \sum_{k=1}^{d} \partial_{x_k x_k}$$

where $(x_1, x_2, \ldots, x_k, \ldots, x_d) = x \in \mathbb{R}^d$

Notation 1.3.0.3. The variable "x" and other related symbols are used in the context of both 1-dimensional ($\mathbb{R}$) and d-dimensional ($\mathbb{R}^d$) Euclidean spaces; we do not invoke bold face text to denote vectors. Instead, the meaning should be obvious from the context. Where appropriate, the symbol $(\cdot)$ is used to denote the standard Euclidean dot (inner) product.

This thesis is organized as follows. Chapter 2 begins with the review of fundamental properties of the Fourier transform that will be instrumental in understanding and representing solutions; this naturally allows for the definition of the fractional Laplacian $\Lambda^\alpha \equiv (-\Delta)^{\frac{\alpha}{2}}$ itself. We present some theoretical results and alternate definitions. The emphasis is placed on understanding the fractional Laplacian as a generalization of the standard Laplacian Δ, and hence the simple heat equation $\partial_t u - \nu \Delta u = f$ is the first PDE modified with this fractional operator. In Chapter 3, we study decay estimates and perform numerical experiments to provide further insight into our theorems. Of great interest is $\alpha' s$ influence on the behavior of solutions to these nonlocal PDE. We incorporate advection terms into the PDE models in Chapter 4, with both nonlinear and nonlocal modifications. These equations are more physically realistic and are often considered models for fluid phenomena such as viscous incompressible flow and vortex sheet dynamics. Finally, we conclude in Chapter 5 by summarizing the ideas presented and pose ideas for future research directions.

Chapter 2

Generalized Heat Equation

2.1 Fourier Transform

We begin our study of fractional operators in PDE by examining one of the most useful tools in analysis and applied mathematics, the *Fourier transform*. The Fourier transform is a specific example of a more general idea, *integral transformations*. These are linear mappings from one function space to another of the form

$$(Tf)(\xi) = \int_{\Omega} \mathcal{K}(\xi, x) f(x) dx$$

where $\mathcal{K}$ is called the **kernel** of the transformation T and $\Omega \subseteq \mathbb{R}^d$ is the region of integration. The kernel is chosen based on the integral transformation and is designed to have certain properties for the application at hand. For example, the kernel of the Fourier transform is the function $e^{2\pi i \xi \cdot x}$. There are many other famous integral transformations, including (commonly seen in mathematical physics) the Laplace transform (4.33), Hilbert transform (4.31), Hankel transform, Mellin transform, and many others.

The fractional Laplacian operator can be defined in the terms of the Fourier transform and from this perspective we can say that the fractional Laplacian is *nonlocal*; its value at a point depends on an integral over a set typically much larger than just a neighborhood around this point. We will provide more detail on the nonlocal nature of the fractional Laplacian in the next section, where not only can it be defined using the Fourier transform but also via the principle value of a singular integral.

To even begin to rigorously understand the Fourier transform and how it can be applied to PDE, we must first return to some basic definitions from real and functional analysis. Infinite dimensional function spaces will be used extensively in the lemmas and theorems presented in this thesis. Playing such a salient role in PDE theory, we give a brief overview here.

Definition 2.1.0.1. The space L^p is the space of functions on for which the p-th power of the absolute value is *Lebesgue integrable*, that is,

$$\int_\Omega |f|^p \, d\mu < \infty \tag{2.1}$$

for a measure space (Ω, μ) and for $1 \leq p < \infty$. For a detailed review of measure theory, we refer the reader to texts on real analysis.

Definition 2.1.0.2. For $f \in L^p$, the L^p-norm is defined as

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}} \tag{2.2}$$

It is obvious that $\|f\|_{L^p} < \infty$ by the definition of L^p.

Many of the following results can be found in any standard text on Fourier analysis or graduate PDE; in particular, the book of Evans [38] forms the basis for our discussion. This material is basic setup for the study of the nonlocal and nonlinear PDE models in this work.

Definition 2.1.0.3 (Fourier Transform for L^1 functions)**.** Let $f \in L^1(\mathbb{R}^d)$, where d is the dimension of the space. Then, for $\xi \in \mathbb{R}^d$,

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx$$

is the **Fourier Transform** (FT) of f.
Conversely, $g^\vee$ denotes the **inverse Fourier Transform** (IFT) of $g \in L^1(\mathbb{R}^d)$ defined as

$$g^\vee(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} g(\xi) d\xi.$$

In particular, setting $g = \hat{f}$ recovers the original function f. These definitions can be extended for functions $f \in L^2(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d),\ p > 2$. The precise statements and proofs of these results may be found in the Appendix, see [A.1].

We can loosely think of the Fourier transform as resolving a function into its frequency components, or representing a function in a "frequency" space (e.g., signal analysis). Of course, depending on the independent variable, this interpretation does not always hold and care must always be taken in this regard.

The *convolution*, a type of "product" for functions, is defined below as it will be instrumental in solving various PDE.

Definition 2.1.0.4. The **convolution** of two functions f, g is the action of a linear operator given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy. \tag{2.3}$$

With a change of variables, we easily see that the convolution operation is commutative: $f * g = g * f$.

A useful result on convolutions is Young's inequality, provided here without proof.

Lemma 2.1.0.5 (Young's inequality)**.** *If* $1 \le p, q, r \le \infty$ *such that* $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, *then*

$$\|f * g\|_{L^p} \le \|f\|_{L^q} \|g\|_{L^r}$$

Example. Take $p = \infty, q = r = 2$. Then since $1 + 1/\infty = 1 + 0 = 1 = 1/2 + 1/2$, $\|f * g\|_{L^\infty} \le \|f\|_{L^2} \|g\|_{L^2}$.

We now present some simple properties of the Fourier Transform that will later aid in our forthcoming analysis and solution of PDE.

Proposition 2.1.0.6.

Notation. The symbol ∇ denotes the gradient operator and $|\cdot|$ is the Euclidean norm.

(1) $\widehat{\nabla^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

(2) $\widehat{x^\alpha f}\xi = \left(\frac{i}{2\pi}\right)^\alpha \nabla_\xi^\alpha \hat{f}(\xi)$

(3) $\widehat{f(ax)}(\xi) = \frac{1}{|a|^d} \hat{f}(\frac{\xi}{a}), \ \ a \neq 0$

(4) $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

(5) $\widehat{fg}(\xi) = \hat{f}(\xi) * \hat{g}(\xi)$

(6) $\left(\hat{f}(\xi)\hat{g}(\xi)\right)^\vee = f * g$

We emphasize the vector notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$ and $(2\pi i \xi)^\alpha = (2\pi i \xi_1)^{\alpha_1} \cdots (2\pi i \xi_d)^{\alpha_d}$.

Proof.

- (Prop 2.1.0.6 Property 1) By definition, $f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi$ and so formally

$$\begin{aligned}
\nabla_x^\alpha f(x) &= \nabla_x^\alpha \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi \\
&= \int_{\mathbb{R}^d} \nabla_x^\alpha e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi \\
&= \int_{\mathbb{R}^d} (2\pi i\xi)^\alpha e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi \\
&= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}((2\pi i\xi)^\alpha \hat{f}(\xi))d\xi
\end{aligned}$$

Thus by definition, $\widehat{\nabla^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi) \quad \because$

- (Prop 2.1.0.6 Property 2) Observe that

$$\begin{aligned}
\nabla_\xi^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^d} (-2\pi i x)^\alpha e^{-2\pi i x\cdot\xi} f(x)dx &= (-2\pi i)^\alpha \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}(x^\alpha f(x))dx \\
\implies \widehat{x^\alpha f}(\xi) &= \frac{1}{(-2\pi i)^\alpha}\nabla_\xi^\alpha \hat{f}(\xi) \\
&= \left(\frac{i}{2\pi}\right)^\alpha \nabla_\xi^\alpha \hat{f}(\xi) \quad \because
\end{aligned}$$

- (Prop 2.1.0.6 Property 3)

$$\begin{aligned}
\int_{\mathbb{R}^d} f(ax)e^{-2\pi i x\cdot\xi}dx &= \int_{\mathbb{R}^d} f(y)e^{-2\pi i y\cdot\frac{\xi}{a}}\frac{1}{|a|^d}dy \quad (\text{letting } y = ax \implies dx = a^{-d}dy) \\
&= \frac{1}{|a|^d}\hat{f}(\frac{\xi}{a}) \quad \because
\end{aligned}$$

The remaining three properties are easily shown using the definition of convolution (2.1.0.4). ☐

We now turn our attention to an important Fourier transform that is paramount in the forthcoming study of the heat equation and its fundamental solution.

Proposition 2.1.0.7. *Let $a > 0$. Then,*

$$\widehat{e^{-a|x|^2}} = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{-\frac{\pi^2|\xi|^2}{a}}. \tag{2.4}$$

Proof. There are two ways to proceed. 1) Using complex analysis or 2) Solving an ordinary differential equation.

Method 1: We start in 1D, the case $d = 1$. We use the basic fact from polar integration that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \implies \int_{-\infty}^{\infty} e^{-\nu x^2} dx = \sqrt{\frac{\pi}{\nu}} \quad \forall \nu > 0.$$

We take the Fourier transform to obtain

$$\begin{aligned}
\widehat{e^{-ax^2}} &= \int_{-\infty}^{\infty} e^{-2\pi i x\xi} e^{-ax^2} dx \\
&= \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x + \frac{\pi i \xi}{\sqrt{a}}\right)^2} e^{\left(\frac{\pi i \xi}{\sqrt{a}}\right)^2} dx \quad \text{(by completing the square)} \\
&= e^{-\frac{\pi^2\xi^2}{a}} \int_{-\infty}^{\infty} e^{-ax^2} dx \quad \text{(by complex contour integration to remove the ``}i\text{'' term)} \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2\xi^2}{a}}.
\end{aligned}$$

Thus in $\mathbb{R}^d$,

$$\begin{aligned}
\widehat{e^{-a|x|^2}} &= \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} e^{-a|x|^2} dx \\
&= \left(\int_{-\infty}^{\infty} e^{-2\pi i x_1\xi_1} e^{-ax_1^2} dx_1\right)\left(\int_{-\infty}^{\infty} e^{-2\pi i x_2\xi_2} e^{-ax_2^2} dx_2\right)\cdots\left(\int_{-\infty}^{\infty} e^{-2\pi i x_d\xi_d} e^{-ax_d^2} dx_d\right) \\
&= \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{-\frac{\pi^2\xi^2}{a}}. \qquad \because
\end{aligned}$$

Method 2: We now derive this result using ODE. Let $f(x) = e^{-ax^2}$. Then, $f'(x) = -2axe^{-ax^2} = -2axf(x)$ and taking the Fourier transform, $\widehat{f'(x)} = -2a\widehat{xf(x)}$. Now by using Proposition 2.1.0.6 Properties 1, 2 and solving the ODE in Fourier space, we obtain

$$\begin{aligned}
2\pi i\xi \hat{f}(\xi) &= -2a\frac{i}{2\pi}\partial_\xi \hat{f}(\xi) \\
\implies \partial_\xi \hat{f}(\xi) &= -\frac{2\pi^2}{a}\xi \hat{f}(\xi) \\
\text{or} \quad \frac{d\hat{f}}{\hat{f}} &= -\frac{2\pi^2}{a}\xi d\xi \\
\implies \hat{f}(\xi) &= \hat{f}(0)e^{-\frac{\pi^2\xi^2}{a}} \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2\xi^2}{a}}
\end{aligned}$$

where $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$. □

Often in many physical situations there arise short duration, high amplitude quantities that are concentrated at a single point, or are at least well approximated by this. Take, for example, the momentum from hitting a baseball with a bat, a hammer striking a metal beam, or a lightning strike. These situations require the use of an "impulse" function which we now define.

Definition 2.1.0.8. The **Dirac delta distribution** is a type of "generalized function" with the informal definition of

$$\delta(x-a) = \begin{cases} \infty, & x = a \\ 0, & x \neq a \end{cases}$$

for all $x, a \in \mathbb{R}^d$. More precisely, $\delta : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a linear functional acting on test functions f with the fundamental properties that

$$\int_{\mathbb{R}^d} \delta(x)dx = 1$$

and

$$\int_{\mathbb{R}^d} f(x)\delta(x-a)dx = f(a).$$

We say $\delta(x-a)$ gives *unit mass* to the point $x = a$.

Proposition 2.1.0.9. *The Fourier transform of the shifted delta function is* $e^{-2\pi i a\cdot\xi}$

Proof. The result follows from the definition of the delta function. Immediately,

$$\widehat{\delta(x-a)}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\delta(x-a)dx = e^{-2\pi i a\cdot\xi}.$$

In particular, $a = 0$ implies

$$\widehat{\delta(x)} = 1.$$

□

These properties are enough to allow us to begin our exploration of the heat equation and its generalization through the fractional Laplacian operator.

2.2 Fractional Laplacian

The Laplacian Δ is a second-order multidimensional differential operator. It is by convention taken with respect to the spatial variables $x = (x_, x_2, \dots, x_d) \in \mathbb{R}^d$. Sometimes it is denoted as the second iterate of the gradient, ∇^2, but that would be an abuse of notation we avoid in this work since the Laplacian is actually the *divergence* of the gradient, $\Delta = \nabla \cdot \nabla$.

Definition 2.2.0.1. In the standard d-dimensional Cartesian coordinate system, the **Laplacian** is the linear operator given by

$$\Delta f = \sum_{j=1}^{d} \partial_{x_j x_j} f \tag{2.5}$$

where $x \in \mathbb{R}^d$, $\partial_{x_j x_j}$ is the standard second partial derivative with respect to the component x_j, and f is a function smooth enough to admit two derivatives.

The Laplacian has many representations depending on the dimension of the space and the coordinate system used. For example, we have the following representations that come from extensive application of the chain rule [39]:

- Polar coordinates (2D)

$$\Delta f = \partial_{rr} f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_{\theta\theta} f$$

- Cylindrical coordinates (3D)

$$\Delta f = \partial_{rr} f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_{\theta\theta} f + \partial_{zz} f$$

- Spherical coordinates (3D)

$$\Delta f = \partial_{rr} f + \frac{2}{r}\partial_r f + \frac{1}{r^2}\partial_{\phi\phi} f + \frac{\cot(\phi)}{r^2}\partial_\phi f + \frac{1}{r^2 \sin^2(\phi)}\partial_{\theta\theta} f$$

The Laplacian is inherently a *local* operator; it relates the value of a function at one point to the value of the same function at its neighboring points. It is perhaps unexpected that simple non-integer (in this case, *fractional*) powers of Δ correspond to a *nonlocal* operator. We begin our discussion on the fractional Laplacian with its definition in Fourier space.

2.2.1 Fourier Definition

We now proceed to define the fractional Laplacian in the whole space $\mathbb{R}^d$.

Definition 2.2.1.1. Let $\alpha \in \mathbb{R}$. We define the **fractional Laplacian** operator $(-\Delta)^\alpha$ as

$$\widehat{(-\Delta)^\alpha f}(\xi) = (4\pi^2 |\xi|^2)^\alpha \hat{f}(\xi). \tag{2.6}$$

When $\alpha = 1$, $\widehat{(-\Delta) f} = 4\pi^2 |\xi|^2 \hat{f}$ is the standard Laplacian (in Fourier variables).

With $\alpha = \frac{1}{2}$, we can define the **Zygmund operator** $\Lambda = (-\Delta)^{\frac{1}{2}}$. Then, we may represent the fractional Laplacian in a more convenient form via powers of the Zygmund operator:

$$\widehat{(\Lambda^\alpha f)}(\xi) = (4\pi^2 |\xi|^2)^{\frac{\alpha}{2}} \hat{f}(\xi) = (2\pi |\xi|)^\alpha \hat{f}(\xi). \tag{2.7}$$

To obtain $\Lambda^\alpha f$ in physical variables, simply take the inverse Fourier transform of the right hand side above:

$$(\Lambda^\alpha f)(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} (2\pi |\xi|)^\alpha \hat{f}(\xi) d\xi. \tag{2.8}$$

The Zygmund operator simplifies the notation for these fractional operators; however, it is important to note for clarity that due to choice of notation, Λ^α has the opposite sign as the standard Laplacian Δ. For $\alpha = 2$, we recover the ordinary Laplacian $\Lambda^2 = -\Delta$. The fractional Laplacian Λ^α is clearly a nonlocal integro-differential operator due to the Fourier transform definition, and it generates long-range diffusive effects as we shall see in numerical studies.

2.2.2 Integral Definition

CÓRDOBA and CÓRDOBA have developed integral representations for the fractional Laplacian.

Lemma 2.2.2.1 (A. Córdoba and D. Córdoba [24])**.** *For* $0 < \alpha < 2$*,*

$$(\Lambda^\alpha f)(x) = \text{p. v. } C(\alpha, d) \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy \tag{2.9}$$

where p. v. *means Cauchy principle value integral and* C *is the sharp constant given by*

$$C(\alpha, d) = \frac{\Gamma(\frac{\alpha}{2} + \frac{d}{2})}{\pi^{\alpha + d/2} \Gamma(-\frac{\alpha}{2})}.$$

This is known as the Riesz potential form.

To understand this alternate definition of the fractional Laplacian, we define the principle value.

Definition 2.2.2.2 (Cauchy Principle Value)**.** The **Cauchy principle value** assigns a value to an otherwise undefined improper integral and is given by the symmetric limit

$$\text{p. v.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \text{p. v.} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_{-R}^{-\epsilon} f(x)dx + \int_{\epsilon}^{R} f(x)dx \right)$$

for all $\epsilon > 0$. The Cauchy Principle Value *must* be taken over a *symmetric interval* around the "bad" point(s).

Example 2.2.2.3.

$$\text{p. v.} \int_{-16}^{16} \frac{1}{x} dx = \lim_{\epsilon \to 0} \left(\int_{-16}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{16} \frac{1}{x} dx \right) = \lim_{\epsilon \to 0} (0) = 0$$

which is obviously true since $1/x$ is an odd function integrated over a symmetric interval on $\mathbb{R}$.

Proof of Lemma 2.2.2.1. We sketch the proof of this lemma with a classical result (STEIN [65]) from harmonic analysis. For $-2 < \beta < 0$,

$$\Lambda^\beta f(x) = C(\beta, d) \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d+\beta}} dy.$$

Thus,

$$\begin{aligned}
\Lambda^\alpha f = \Lambda^\alpha \Delta^{-1} \Delta f = \Lambda^{\alpha-2} \Delta f &= \text{p. v. } C(\alpha, d) \int_{\mathbb{R}^d} \frac{\Delta f(y)}{|x-y|^{d+\alpha-2}} dy \\
&= \text{p. v. } C(\alpha, d) \int_{\mathbb{R}^d} \frac{\Delta_y (f(y) - f(x))}{|x-y|^{d+\alpha-2}} dy \\
&= C(\alpha, d) \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \frac{\Delta_y (f(y) - f(x))}{|x-y|^{d+\alpha-2}} dy \\
&= \text{p. v. } C(\alpha, d) \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy
\end{aligned}$$

using the fact that $\Delta_y f(x) = 0$ and by integrating by parts twice to move the two derivatives in the Laplacian term to the denominator. □

The fractional Laplacian can be defined in numerous other ways both in the whole space and on bounded domains [16, 17, 27, 28]; we refer the reader to [52, 64, 73, 74] for more extensive treatments.

2.3 Heat Equation

The first partial differential equation we examine relating to the Laplacian is the *heat equation*. The heat (or diffusion) equation describes the time evolution of the density of some (potentially physical) quantity and exhibits a very nice "smoothing" behavior: For $t > 0$, the solution $u(x, t) \in C^\infty$ for all $x \in \mathbb{R}^d$. This follows from showing that all partial derivatives are bounded by the norm of the initial data of the PDE, which is easy with the Fourier transform. The heat equation dictates how a continuum spreads to occupy available space. In contrast to the wave equation ($\partial_{tt} u = c^2 \Delta u$), the heat equation propagates disturbances at an "infinite" speed and hence it is used in many applications outside of the thermal sciences.

The physical intuition for heat flow described by the diffusion equation can best be obtained through its derivation using the conservation of energy and Fourier's law of heat conduction from physics. In fact, the most common governing equations (typically PDE) in science can be traced back to fundamental principles, specifically conservation laws such as the *conservation of energy, conservation of mass, or conservation of momentum (Newton's second law).* In thermodynamics and heat transfer, the heat equation describing conduction through a solid is typically written in dimensional form as

$$\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot (\kappa \nabla T) = \dot{q}_{\text{gen}} \tag{2.10}$$

where $T(x,t)$ is the absolute temperature distribution, ρ is the density of the material, c_p is the heat capacity (specific heat), κ is the conductivity, and $\dot{q}_{\text{gen}}$ is the heat generation per unit volume. Consult texts such as [38, 39] for the full derivation.

The Cauchy problem or initial value problem (IVP) for the heat equation is

$$\begin{cases} \partial_t u - \nu \Delta u = f, \quad x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{2.11}$$

Here, the unknown is $u : \mathbb{R}^d \times [0,\infty) \longrightarrow \mathbb{R}$. No boundary conditions are specified, but the solution $u(x,t)$ is expected to approach zero sufficiently fast as $x \to \pm\infty$. Abstractly, we may formally view the system (2.11) as an ODE with unknown $u(t)$ and operator Δ. The solution represented in this way is

$$u(t) = e^{\nu \Delta t} u_0(x) + \int_0^t e^{\nu \Delta (t-\tau)} f(x,\tau) d\tau.$$

To interpret the meaning of this equation requires the language of *semigroups*, which will be detailed in section (2.4.2). See the forthcoming Lemma 2.4.1.1 for the steps to obtain $u(t)$ as above.

More familiarly, the unique solution for the homogeneous problem exists and is given by the convolution of the initial data with the special Gaussian kernel g_2 (which notationally will become clear in the next section)

$$u(x,t) = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\nu t}} * u_0 = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\nu t}} u_0(x)\ dy. \tag{2.12}$$

The solution is a weighted average of u_0 in which the high gradient, high frequency content of u_0 is damped out in time. The Laplacian facilitates this diffusive or dissipative behavior. Further, if $0 \neq u_0 \geq 0$, then clearly by observing the form of the integrand we can conclude that $u(x,t) \geq 0$ for all time; this supports our natural intuition that the flow of heat is from high to low temperatures.

The function

$$g_2(x,t) = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\nu t}}$$

is called the *heat kernel* and it is a fundamental solution (*Green's function*) of the heat equation, meaning $u = g_2$ solves

$$\begin{cases} \partial_t u - \nu \Delta u = 0, & x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = \delta(x) \end{cases} \tag{2.13}$$

where $\delta(x)$ is a Dirac delta distribution centered at $x = 0$. The solution to the non-homogeneous IVP (2.11) falls out naturally from this fundamental solution due to the principle of linear superposition [39]. The derivation of the heat kernel and its fractional generalizations are the subject of the following section.

2.4 Heat Equation with Fractional Diffusion

2.4.1 Fractional Heat Kernel

Consider the Cauchy problem for the **generalized heat equation** (or *fractional* heat equation, we use both names interchangeably when describing this PDE)

$$\begin{cases} \partial_t u + \nu \Lambda^\alpha u = f, & x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{2.14}$$

where Λ^α is the fractional Laplacian, $0 \leq \alpha \leq 2$, and $\nu > 0$ is the thermal diffusivity or viscosity parameter, and $f = f(x,t)$ is the forcing term or "body force". This system is *well-posed*; we will now prove existence and uniqueness of solutions.

Lemma 2.4.1.1 (Existence)**.** *The solution u of Eqn (2.14) exists and can be represented (formally) as the solution of an abstract ODE*

$$u(t) = e^{-\nu\Lambda^\alpha t} u_0(x) + \int_0^t e^{-\nu\Lambda^\alpha (t-\tau)} f(x,\tau) d\tau$$

which really means

$$\hat{u}(\xi,t) = e^{-\nu t (2\pi|\xi|)^\alpha} \widehat{u_0}(\xi) + \int_0^t e^{-\nu(t-\tau)(2\pi|\xi|)^\alpha} \hat{f}(\xi,\tau) d\tau$$

or by applying the inverse Fourier transform,

$$u(x,t) = g_\alpha(x,t) * u_0 + \int_0^t g_\alpha(x,t-\tau) * f(\tau) d\tau, \tag{2.15}$$

where

$$g_\alpha(x,t) = (e^{-\nu t(2\pi|\xi|)^\alpha})^\vee = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-\nu t(2\pi|\xi|)^\alpha} d\xi \tag{2.16}$$

is defined as the ***generalized heat kernel****.*

Proof of Lemma 2.4.1.1. Viewing the fractional operator in terms of the abstract ODE

$$\begin{cases} \frac{du}{dt} = -\nu\Lambda^\alpha u + f, \ t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{2.17}$$

we claim that standard results from ODE theory apply and thus the general solution for this (infinite) system of equations is given by Duhamel's Principle as

$$u(t) = e^{-\nu\Lambda^\alpha t} u_0(x) + \int_0^t e^{-\nu\Lambda^\alpha (t-\tau)} f(x,\tau) d\tau \tag{2.18}$$

We justify this claim and interpret its meaning in our later discussion of *semigroups* (2.4.2). To complete the rest of the proof, we recast the equation 2.14 into Fourier space via the Fourier transform:

$$\begin{cases} \dfrac{d\hat{u}}{dt} = -\nu(2\pi\,|\xi|)^\alpha \hat{u} + \hat{f}, \ t > 0 \\ \hat{u}(\xi,0) = \widehat{u_0}(\xi) \end{cases} \tag{2.19}$$

This is an infinite number of ODE for each $\xi \in \mathbb{R}^d$. Solving, we obtain

$$\hat{u}(\xi,t) = e^{-\nu t(2\pi|\xi|)^\alpha} \widehat{u_0}(\xi) + \int_0^t e^{-\nu(t-\tau)(2\pi|\xi|)^\alpha} \hat{f}(\xi,\tau) d\tau \tag{2.20}$$

as the solution in Fourier space. Applying the inverse Fourier transform to convert the solution back into the physical space and recalling the properties of convolution with the Fourier transform gives the desired result. □

In particular, for $\alpha = 2$ the formula (2.15) yields the solutions to the classical heat equation. We now prove uniqueness.

Lemma 2.4.1.2 (Uniqueness)**.** *The solution to (2.14) given by Equation (2.15) is unique.*

Proof. Suppose we have two solutions u_1 and u_2 that solve (2.14). Let $w(x,t) = u_1(x,t) - u_2(x,t)$. Then plugging in w into the linear PDE, we obtain

$$\begin{aligned}\partial_t w + \nu\Lambda^\alpha w &= (\partial_t u_1 + \nu\Lambda^\alpha u_1) - (\partial_t u_2 + \nu\Lambda^\alpha u_2) \\ &= f - f \\ &= 0\end{aligned}$$

Similarly, $w(x,0) = u_1(x,0) - u_2(x,0) = u_0 - u_0 = 0$. Thus, w satisfies the new homogeneous initial value problem

$$\begin{cases}\partial_t w + \nu\Lambda^\alpha w = 0, \quad x \in \mathbb{R}^d,\ t > 0 \\ w(x,0) = 0\end{cases}$$

We take the Fourier transform, yielding the ODE

$$\begin{cases}\frac{d\hat{w}}{dt} = -\nu(2\pi\,|\xi|)^\alpha \hat{w},\ t > 0 \\ \hat{w}(\xi,0) = 0\end{cases}$$

where $\xi \in \mathbb{R}^d$ is treated as a fixed parameter. The solution is trivially given by

$$\hat{w} = \hat{w}(\xi,0)e^{-\nu t(2\pi|\xi|)^\alpha} = 0$$

Thus, since $\hat{w} = 0$ implies $w = 0$, we conclude $u_1 = u_2$. □

We now present some useful properties of the generalized heat kernel.

Proposition 2.4.1.3.

(1) For $\alpha = 2,\ g_\alpha$ is the standard heat kernel

$$g_2(x,t) = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{4\nu t}}$$

(2) For $\alpha = 1,\ g_\alpha$ is the Poisson kernel

$$g_1(x,t) = C_d\frac{\nu t}{\left(|x|^2 + (\nu t)^2\right)^{\frac{d+1}{2}}}, \quad \text{where } C_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}$$

(3) For $\alpha = 0,\ g_\alpha$ is the exponential impulse

$$g_0(x,t) = e^{-\nu t}\delta(x)$$

where δ is the Dirac delta distribution defined in (2.1.0.8).

Proof.

(1) The representation for $\alpha = 2$ is a consequence of applying Prop (2.1.0.7):

$$g_2(x,t) = \left(\frac{\pi}{4\pi^2 \nu t}\right)^{\frac{d}{2}} e^{-\frac{\pi^2|x|^2}{4\nu t \pi^2}} = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\nu t}} \quad \because$$

(2) We follow the proof from Stein in [65] and the lecture notes in [77]. We first invoke the Residue Theorem from complex analysis without proof:

Theorem 2.4.1.4 (Residue)**.** *Assume $f(z)$ is analytic on $\operatorname{Im} z > 0$ (the upper half plane) and continuous on $\operatorname{Im} z \geq 0$ except at a finite number of bad points $a_1, a_2, \ldots, a_k,\ k \in \mathbb{R}$. Then, for $b > 0$,*

$$\int_{-\infty}^{\infty} e^{ibx} f(x) dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(e^{ibz} f(z), a_j). \tag{2.21}$$

We also need the next result.

Lemma 2.4.1.5. *For $\gamma > 0$,*

$$e^{-\gamma} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\gamma x} \frac{1}{1+x^2} dx.$$

Proof of Lemma. Let $f(z) = \frac{1}{1+z^2}$. We note that $z = i$ is the only bad (singular) point of f on $\operatorname{Im} z > 0$ (also note $z = i$ is a pole of order 1). By the Residue Theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\gamma x} \frac{1}{1+x^2} dx &= 2\pi i \operatorname{Res}(e^{i\gamma z} \frac{1}{1+z^2}, i) \\ &= 2\pi i \left[(z-i) e^{i\gamma z} \frac{1}{(z-i)(z+i)}\right] |_{z=i} \\ &= \pi e^{-\gamma}.\end{aligned}$$

□

We now return to the proof of (2) above. Obviously, $\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u} du$. Then,

$$\begin{aligned}e^{-\gamma} &= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\gamma x} \left(\int_0^\infty e^{-(1+x^2)u} du\right) dx \\ &= \frac{1}{\pi} \int_0^\infty e^{-u} \left(\int_{-\infty}^{\infty} e^{i\gamma x} e^{-x^2 u} dx\right) du\end{aligned}$$

by interchanging the order of integration. Completing the square yields

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\gamma x}e^{-x^2u}dx &= \int_{-\infty}^{\infty} e^{(-\sqrt{u}x-\frac{i\gamma}{2\sqrt{u}})^2}e^{-\frac{\gamma^2}{4u}}dx\\ &= e^{-\frac{\gamma^2}{4u}}\int_{-\infty}^{\infty} e^{-(\sqrt{u}x)^2}dx\\ &= \sqrt{\frac{\pi}{u}}e^{-\frac{\gamma^2}{4u}}\end{aligned}$$

where the contour integral is evaluated by using Cauchy's Integral Formula. Therefore, we can now obtain what is called *Bohler's average identity*

$$\begin{aligned}e^{-\gamma} &= \frac{1}{\pi}\int_0^{\infty} e^{-u}\sqrt{\frac{\pi}{u}}e^{-\frac{\gamma^2}{4u}}du\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} u^{-\frac{1}{2}}e^{-u-\frac{\gamma^2}{4u}}du.\end{aligned}$$

Now, for the generalized heat kernel with $\alpha = 1$ and $(2\pi\,|\xi|\,\nu t)$ acting as γ, we have

$$\begin{aligned}g_1(x,t) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}e^{-\nu t(2\pi|\xi|)}d\xi\\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\int_0^{\infty} u^{-\frac{1}{2}}e^{-u}e^{-\frac{4\pi^2\nu^2t^2|\xi|^2}{4u}}du d\xi\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} u^{-\frac{1}{2}}e^{-u}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}e^{-\frac{\pi^2\nu^2t^2|\xi|^2}{u}}d\xi du \quad \text{(switch integrals)}\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} u^{-\frac{1}{2}}e^{-u}\int_{\mathbb{R}^d} e^{-\left|\frac{\pi\nu t}{\sqrt{u}}\xi - i\frac{\sqrt{u}x}{\nu t}\right|^2}e^{-\frac{u|x|^2}{(\nu t)^2}}d\xi du \quad \text{(complete the square)}\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} u^{-\frac{1}{2}}e^{-u}e^{-\frac{u|x|^2}{(\nu t)^2}}\left(\int_{\mathbb{R}^d} e^{-\frac{\pi^2\nu^2t^2}{u}|\xi|^2}d\xi\right)du \quad \text{(by Cauchy's theorem)}\\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} u^{-\frac{1}{2}}e^{-u}e^{-\frac{u|x|^2}{(\nu t)^2}}\left(\frac{\pi}{\frac{\pi^2\nu^2t^2}{u}}\right)^{\frac{d}{2}}du\\ &= \frac{1}{\pi^{\frac{d+1}{2}}(\nu t)^d}\int_0^{\infty} u^{\frac{d-1}{2}}e^{-\left(1+\frac{|x|^2}{(\nu t)^2}\right)u}du.\end{aligned}$$

Now make the change of variables

$$y = \left(1+\frac{|x|^2}{(\nu t)^2}\right)u \implies du = \left(1+\frac{|x|^2}{(\nu t)^2}\right)^{-1}dy.$$

Then, this yields

$$\begin{aligned}
g_1(x,t) &= \frac{1}{\pi^{\frac{d+1}{2}}(\nu t)^d}\int_0^\infty y^{\frac{d-1}{2}}\left(1+\frac{|x|^2}{(\nu t)^2}\right)^{-\frac{d-1}{2}-1} e^{-y}dy \\
&= \frac{1}{\pi^{\frac{d+1}{2}}(\nu t)^d}\left(\frac{|x|^2+(\nu t)^2}{(\nu t)^2}\right)^{-\frac{d+1}{2}}\int_0^\infty e^{-y}y^{\frac{d-1}{2}}dy \\
&= \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}(\nu t)^d}\frac{(\nu t)^{d+1}}{(|x|^2+(\nu t)^2)^{\frac{d+1}{2}}} \\
&= C_d\frac{\nu t}{\left(|x|^2+(\nu t)^2\right)^{\frac{d+1}{2}}} \quad \because
\end{aligned}$$

where $\Gamma(m) = \int_0^\infty e^{-y}y^{m-1}dy$ is the gamma function. The kernel $g_1(x,t)$ solves Laplace's equation $\Delta u = 0$ in the upper half-space.

(3) For $\alpha = 0$, we have by definition and recalling Proposition 2.1.0.9,

$$\begin{aligned}
g_0(x,t) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}e^{-\nu t(2\pi|\xi|)^0}d\xi \\
&= e^{-\nu t}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}d\xi \\
&= e^{-\nu t}(1)^\vee \\
&= e^{-\nu t}\delta(x) \quad \because
\end{aligned}$$

This completes the proof. □

For general $0 < \alpha < 2$, explicit formulas for the kernels are not known. We summarize the following from above:

$$\begin{aligned}
\alpha = 0 &\implies g_0(x,t) = e^{-\nu t}\delta(x) \\
\alpha = 1 &\implies g_1(x,t) \text{ is the Poisson kernel} \\
\alpha = 2 &\implies g_2(x,t) \text{ is the Heat kernel}
\end{aligned}$$

To explore other representations of the generalized heat kernel for $0 < \alpha < 2$, we can use the formula (A.3) in Appendix A.4. Basic properties of the generalized heat kernel are now introduced. See [43, 77] for more estimates.

Lemma 2.4.1.6. *For $0 < \alpha \leq 2$, the generalized heat kernel is nonnegative:*

$$g_\alpha(x,t) \geq 0 \quad \forall\ x \in \mathbb{R}^d,\ t > 0$$

Proof. See Appendix A.4 □

Lemma 2.4.1.7. *For $0 < \alpha \leq 2$,*

$$\|g_\alpha(\cdot,t)\|_{L^1} = 1 \quad \forall t > 0$$

Proof. Since $g_\alpha \geq 0$, we must have

$$\|g_\alpha(\cdot,t)\|_{L^1} = \int_{\mathbb{R}^d} |g_\alpha(x,t)|\,dx = \int_{\mathbb{R}^d} g_\alpha(x,t)dx.$$

Also note that by definition $\widehat{g_\alpha}(\xi,t) = e^{-\nu t(2\pi|\xi|)^\alpha} = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} g_\alpha(x,t)dx$ holds for all $\xi \in \mathbb{R}^d$ and in particular $\xi = 0$. Therefore by setting $\xi = 0$, we obtain

$$\int_{\mathbb{R}^d} g_\alpha(x,t)dx = 1 = \|g_\alpha(\cdot,t)\|_{L^1}$$

as required. □

The following result is extremely important for use in forthcoming discussion involving $g_\alpha(x,t)$. It shows the timescale of heat decay for the fractional heat kernel.

Lemma 2.4.1.8 (Scaling)**.** *For $0 \leq \alpha \leq 2$,*

$$g_\alpha(x,t) = \frac{1}{t^{\frac{d}{\alpha}}} g_\alpha\left(\frac{x}{t^{\frac{1}{\alpha}}}, 1\right) \tag{2.22}$$

Proof. We prove this scaling property using a change of variables. Let $\xi = \eta t^{-\frac{1}{\alpha}}$ which implies $d\xi = t^{-\frac{d}{\alpha}} d\eta$. Then,

$$\begin{aligned} g_\alpha(x,t) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-\nu t(2\pi|\xi|)^\alpha} d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot \eta t^{-\frac{1}{\alpha}}} e^{-\nu t\left(2\pi\left|\eta t^{-\frac{1}{\alpha}}\right|\right)^\alpha} t^{-\frac{d}{\alpha}} d\eta \\ &= t^{-\frac{d}{\alpha}} \int_{\mathbb{R}^d} e^{2\pi i \frac{x}{t^{1/\alpha}}\cdot\eta} e^{-\nu(2\pi|\eta|)^\alpha} d\eta \\ &= t^{-\frac{d}{\alpha}} g_\alpha\left(\frac{x}{t^{\frac{1}{\alpha}}}, 1\right) \quad \because \end{aligned}$$

□

The following lemma extracts additional information from the fractional heat kernel and will be re-visited during our treatment of semigroups in section (2.4.2).

Lemma 2.4.1.9. *Let $\alpha \in (0,2]$. Recall $e^{-\nu t\Lambda^\alpha} f = g_\alpha(x,t) * f$ is the solution to the generalized heat equation. Then, for $p \in [1,\infty]$ and $f \in L^p$:*

(1) $\left\|e^{-\nu t\Lambda^\alpha} f\right\|_{L^p} \le \|f\|_{L^p}$

(2) For $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^d)$, $\lim_{t\to 0^+} e^{-\nu t\Lambda^\alpha} f = f$ in $L^p(\mathbb{R}^d)$ and a.e. (*almost everywhere*)

Proof. We use Young's Inequality (2.1.0.5) for convolution:

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \tag{2.23}$$

for all $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $p, d, r \in [1,\infty]$. We also use the facts that $g_\alpha(x,t) \ge 0$ and $\|g_\alpha\|_{L^1} = 1$. We prove each item separately as follows:

(1) $\left\|e^{-\nu t\Lambda^\alpha} f\right\|_{L^p} = \|g_\alpha * f\|_{L^p} \le \|g_\alpha\|_{L^1} \|f\|_{L^p} = \|f\|_{L^p} \quad \because$

(2) The proof is technical and is left to the Appendix, see A.4.

□

This lemma justifies the validity of the initial data in the IVP for the generalized heat equation

$$\begin{cases} \partial_t u + \nu\Lambda^\alpha u = 0 \\ u(x,0) = f \end{cases}$$

which is not necessarily obvious from the fundamental solution. That is, the solution $u(x,t) = g_\alpha(x,t) * f \to f$ a. e. as $t \to 0^+$ in the pointwise sense.

2.4.2 Semigroup Approach

The theory of semigroups will allow for a richer study of the fractional heat kernel as well as justify our formal solution to the heat equation and generalized heat equation.

Let X be a *Banach space* and $\mathcal{L}(X)$ be the set of bounded linear operators on X. This means X is a complete, normed vector space. One common example of such a space is $\mathbb{R}^d$ with Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2}$.

Definition 2.4.2.1 (C_0−semigroup)**.** Let $T(t) : \mathbb{R}^+ \to \mathcal{L}(X)$ be a family of bounded linear operators with parameter t. Then, we say $T(t)$ is a C_0**-semigroup** if

(1) $T(0) = I$

(2) $T(t)T(s) = T(t+s)$ for $t, s \in [0, \infty)$

(3) $T(t)f \to f$ in X as $t \to 0^+$

The third condition is called *strong continuity*. We showcase this definition with some examples.

Example 2.4.2.2. Let $X = C_0([0, \infty))$ be the set of bounded and uniformly continuous functions and $\|f\|_X = \sup_{s\in[0,\infty)} |f(s)|$. Define

$$(T(t)f)(s) = f(s+t).$$

Then, we claim $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup.

Proof. We verify the three properties from the definition of C_0-semigroup. For $f \in X$,

1. $(T(0)f)(s) = f(s+0) = f(s) \implies T(0) = I \quad \because$
2. $(T(t)T(s)f)(\tau) = T(t)(T(s)f(\tau)) = (T(t)f)(s+\tau) = f(s+\tau+t) = (T(t+s)f)(\tau) \quad \because$
3. $T(t)f(\tau) = f(t+\tau) \implies \|T(t)f(\tau) - f(\tau)\|_X = \sup_{\tau\in[0,\infty)} |f(t+\tau) - f(\tau)| \to 0$ as $t \to 0^+$ since f is uniformly continuous. Thus, $T(t)f \to f$ as $t \to 0^+$. $\quad \because$

□

The next lemma demonstrates that the fundamental solution to the 1D heat equation is in fact a semigroup.

Lemma 2.4.2.3. *Let $X = C_0((-\infty, \infty))$ be the set of bounded and uniformly continuous functions on $(-\infty, \infty)$ with $\|f\|_X = \sup_{\tau\in(-\infty,\infty)} |f(\tau)|$. We will now define the heat operator (in 1D). Let*

$$g(x,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}}, \quad t > 0.$$

Define, for $f \in X$,

$$T(t)f = \begin{cases} f(x) & \text{if } t = 0 \\ g(x,t) * f & \text{if } t > 0 \end{cases}$$

Then, $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup.

Proof.

1. For $f \in X,\ T(0)f = f$ trivially by definition $\because$

2. Let $s, t \geq 0$ and $f \in X$. We need to prove that $(T(t)T(s))f = T(t+s)f$. W.L.O.G. it suffices to consider the case $s, t > 0$. Then,

$$T(t)T(s)f = T(t)(g(x,s) * f) = g(x,t) * (g(x,s) * f) = (g(x,t) * g(x,s)) * f$$

by associativity of the convolution. Similarly,

$$T(t+s)f = g(x, t+s) * f.$$

So, we must show that $g(x,t) * g(x,s) = g(x,t+s)$, or equivalently by definition

$$\frac{1}{\sqrt{4\pi\nu t}} \frac{1}{\sqrt{4\pi\nu s}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4\nu t}} e^{-\frac{y^2}{4\nu s}} dy = \frac{1}{4\pi\nu(t+s)} e^{-\frac{x^2}{4\nu(t+s)}}.$$

The easiest way to verify this is by the Fourier transform. In fact,

$$\widehat{g(x,t) * g(x,s)} = \widehat{g(x,t)}\widehat{g(x,s)} = e^{-\nu t(2\pi|\xi|)^2} e^{-\nu s(2\pi|\xi|)^2} = e^{-\nu(t+s)(2\pi|\xi|)^2} = \widehat{g(x,t+s)}.$$

Since the Fourier transforms are the same, the kernels themselves are equal. $\because$

3. $\forall f \in X,\ \|T(t)f - f\|_X = \|f * g(x,t) - f\|_X = \sup_{x\in(-\infty,\infty)} |f * g(x,t) - f|$. Also,

$$|f * g(x,t) - f| = \left| \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f(x-y) e^{-\frac{|y|^2}{4\nu t}} dy - f(x) \right|.$$

Changing variables letting $z = \frac{y}{\sqrt{4\pi\nu t}} \implies dy = \sqrt{4\pi\nu t}dz$, we have

$$\left| \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f(x - \sqrt{4\pi\nu t}z) e^{-\pi|z|^2} dz - f(x) \right| = \left| \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} (f(x - \sqrt{4\pi\nu t}z) - f(x)) e^{-\pi|z|^2} dz \right|.$$

Since $\int_{\mathbb{R}} e^{-\pi|z|^2} dz = 1 < \infty,\ \exists M > 0$ such that $\int_{|z|\geq M} e^{-\pi|z|^2} dz < \frac{\epsilon}{2}$. Since f is uniformly continuous, $\forall \epsilon > 0,\ \exists \delta > 0$ such that for $0 < t < \delta$,

$$\left| f(x - \sqrt{4\pi\nu t}z) - f(x) \right| < \frac{\epsilon}{2}$$

for all $|z| \leq M$. So,

$$
\begin{aligned}
|f * g(x,t) - f| &= \left| \frac{1}{\sqrt{4\pi\nu t}} \int_{|z|<M} (f(x - \sqrt{4\pi\nu t} z) - f(x)) e^{-\pi|z|^2} dz \right| \\
&\quad + \left| \frac{1}{\sqrt{4\pi\nu t}} \int_{|z|\geq M} (f(x - \sqrt{4\pi\nu t} z) - f(x)) e^{-\pi|z|^2} dz \right| \\
&< \frac{\epsilon}{2} \cdot 1 + 2 \|f\|_X \cdot \frac{\epsilon}{2} \\
&= A\epsilon.
\end{aligned}
$$

Thus,

$$\|T(t)f - f\|_X = \sup_{x \in (-\infty,\infty)} |f * g(x,t) - f| < A\epsilon \quad \text{as} \quad t \to 0^+.$$

We conclude $\{T(t)\}_{t\geq 0}$ as defined is a C_0-semigroup.

This completes the proof. □

The argument above still holds for the fractional heat kernel.

Theorem 2.4.2.4. *Let $X = L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Define $T : \mathbb{R}^+ \to \mathcal{L}(X)$ by*

$$T(t)f = \begin{cases} f & \text{if } t = 0 \\ g_\alpha(x,t) * f & \text{if } t > 0 \end{cases}$$

where g_α is the fractional heat kernel (2.16)

$$g_\alpha(x,t) = (e^{-\nu t(2\pi|\xi|)^\alpha})^\vee = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-\nu t(2\pi|\xi|)^\alpha} d\xi$$

with $0 < \alpha < 2$. Then, $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on X.

Proof.

1. $T(0)f = f \quad \forall f \in X$ by definition, so $T(0) = I$. $\because$

2. $T(t)T(s)f = T(t)(g_\alpha(x,s) * f) = g_\alpha(x,t) * (g_\alpha(x,s) * f) = (g_\alpha(x,t) * g_\alpha(x,s)) * f$. On the RHS, $T(t+s)f = g_\alpha(x,t+s) * f$. So $\forall s,t > 0$, we must show that $g_\alpha(x,t) * g_\alpha(x,s) = g_\alpha(x,t+s)$. Taking the FT,

$$g_\alpha(\widehat{t) * g_\alpha}(s) = \widehat{g_\alpha(t)}\widehat{g_\alpha(s)} = e^{-\nu t(2\pi|\xi|)^\alpha} e^{-\nu s(2\pi|\xi|)^\alpha} = e^{-\nu(t+s)(2\pi|\xi|)^\alpha} = g_\alpha(\widehat{x, t+} s).$$

Since the Fourier transforms are the same, the kernels themselves are equal. $\because$

3. (Strong continuity at $t = 0$) We invoke the following lemma without proof.

Lemma 2.4.2.5 (Dominated Convergence Theorem (DCT)). *Let $\{f_n\}$ and f be measurable functions on $\mathbb{R}^d$ satisfying*

(1) $f_n(x) \to f(x)$ a. e.

(2) $|f_n(x)| \leq g(x)$ *with* $\int_{\mathbb{R}^d} |g(x)|\, dx < +\infty$ *and* $g \in L^1$

Then, $\lim\limits_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} f(x)dx$.

Now, we show $T(t)f \to f$ in $X = L^p(\mathbb{R}^d)$. We use the fact (A.4) from the Appendix: For $1 \leq p < \infty$,

$$\|f(x-y) - f(x)\|_{L^p(\mathbb{R}^d)} \to 0 \text{ as } |y| \to 0$$

The remainder of the proof follows the procedure from (A.4) in the Appendix. We have

$$\|T(t)f - f\|_{L^p} = \|g_\alpha(x,t) * f - f\|_{L^p}.$$

Then, we know from the computations in the proof (A.4) that

$$\begin{aligned} g_\alpha(x,t) * f - f &= \int_{\mathbb{R}^d} f(x-y) g_\alpha(y,t) dy - f(x) \\ &= \int_{\mathbb{R}^d} f(x-y) t^{-\frac{d}{\alpha}} g_\alpha\left(\frac{y}{t^{1/\alpha}}, 1\right) dy - f(x) \\ &= \int_{\mathbb{R}^d} f(x - t^{1/\alpha} z) g_\alpha(z,1) dz - f(x) \\ &= \int_{\mathbb{R}^d} (f(x - t^{1/\alpha} z) - f(x)) g_\alpha(z,1) dz. \end{aligned}$$

By Minkowski's inequality (A.5) and the fact that $g_\alpha(z,1) \geq 0$,

$$\|T(t)f - f\|_{L^p} \leq \int_{\mathbb{R}^d} \left\| f(x - t^{1/\alpha} z) - f(x) \right\|_{L^p_x} g_\alpha(z,1) dz.$$

For fixed z, $\left\| f(x - t^{1/\alpha} z) - f(x) \right\|_{L^p_x} \to 0$ as $t \to 0$ and also $\left\| f(x - t^{1/\alpha} z) - f(x) \right\|_{L^p_x} \leq 2\|f\|_{L^p}$ by the triangle inequality in the whole space $\mathbb{R}^d$. Therefore, by the Dominated Convergence Theorem,

$$\lim_{t\to 0^+} \|T(t)f - f\|_{L^p(\mathbb{R}^d)} = 0.$$

This concludes the proof. $\square$

The original goal of semigroup theory was to generalize the **matrix exponential** e^{At} for a matrix A to more general (possibly infinite-dimensional) operators. From ODE theory the general system

$$\begin{cases} \frac{dx}{dt} = Ax \\ x(0) = x_0 \end{cases}$$

has the solution is $x(t) = e^{At}x_0$, where $A \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^n$, and

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \ldots + \frac{1}{n!}(At)^n + \ldots$$

is defined by the Taylor series for e^x. We now make rigorous the aforementioned generalization of e^{At}.

Definition 2.4.2.6 (Infinitesimal generator)**.** Let X be a Banach space, $T(t)$ be a C_0-semigroup on X. Let $D = \{x \in X : \lim_{t\to 0} \frac{T(t)x-x}{t} \text{ exists}\}$. Then the **infinitesimal generator**, denoted by A, is defined as the mapping

$$A : D \to X$$
$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}$$

The infinitesimal generator acts as a derivative, in a sense, of the semigroup. The following lemma from [77], which we provide without proof, ensures that these objects are indeed well defined in the first place.

Lemma 2.4.2.7. *Let X be a Banach space, $T(t)$ be a C_0-semigroup on X. Then the set D defined above is dense in X.*

We illustrate the concept of infinitesimal generators with some examples ultimately of relevance to the heat equation.

Example 2.4.2.8. Let $X = C_0([0,\infty))$, the bounded uniformly continuous functions on $[0,\infty)$. Define $(T(t)x)(\tau) = x(t+\tau)$ for $x \in X$. Recall that we have shown in (2.4.2.2) that $T(t)$ is a C_0-semigroup. Then, the infinitesimal generator of T is

$$(Ax)(\tau) = \lim_{t\to 0} \frac{T(t)x - x}{t} = \lim_{t\to 0} \frac{x(t+\tau) - x(\tau)}{t} = x'(\tau).$$

So, the domain D of the infinitesimal generator A is $D = \{x \in C_0([0,\infty)) : x'(\tau) \text{ exists}\}$ and the generator is $A = \frac{d}{dx}$.

Example 2.4.2.9. Let $A \in \mathbb{R}^{d\times d}, X = \mathbb{R}^d$, and $T(t)x = e^{At}x$. We can easily show that $T(t)$ is a C_0-semigroup by the definition. But what is the generator A_{gen} of this semigroup?
For $x \in \mathbb{R}^d$,

$$\begin{aligned}
A_{gen}x = \lim_{t\to 0}\frac{T(t)x - x}{t} &= \lim_{t\to 0}\frac{e^{At}x - x}{t} \\
&= \lim_{t\to 0}\frac{(At + \frac{1}{2}(At)^2 + \ldots + \frac{1}{n!}(At)^n + \ldots)x}{t} \\
&= \lim_{t\to 0}(Ax + \frac{t}{2}A^2x + \ldots + \frac{t^{n-1}}{n!}A^n x + \ldots) \\
&= Ax.
\end{aligned}$$

Therefore, we conclude $A_{gen} = A$.

In this way, a connection can be made between the heat kernel and the infinitesimal generators of C_0-semigroups. This supports the semigroup representation of solutions to the generalized heat equation we have indicated previously and to other PDE.

Example 2.4.2.10. Let $X = C_0((-\infty,\infty))$ be the set of bounded and UC continuous functions on $(-\infty,\infty)$ with $\|f\|_X = \sup_{\tau\in(-\infty,\infty)}|f(\tau)|$ from Example 2.4.2.3. We will define the 1D heat operator as done previously,

$$g(x,t) = \frac{1}{\sqrt{4\pi\nu t}}e^{-\frac{x^2}{4\nu t}}, \quad t > 0$$

and define the corresponding semigroup, for $f \in X$, as

$$T(t)f = \begin{cases} f(x) & \text{if } t = 0 \\ g(x,t) * f & \text{if } t > 0 \end{cases}$$

We previously verified that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup. Note that clearly $T(t)f$ solves the heat equation

$$\begin{cases} u_t = \nu u_{xx} \implies \partial_t(T(t)f) = \nu\Delta(T(t)f) \\ u(0) = f \implies T(0)f = f \end{cases}$$

Now, we aim to compute the infinitesimal generator of T. We have,

$$\begin{aligned}
Af \equiv \lim_{t\to 0}\frac{T(t)f - T(0)f}{t} &= \frac{d}{dt}T(t)f|_{t=0} \\
&= \nu\Delta(T(t)f)|_{t=0} \\
&= \nu\Delta f
\end{aligned}$$

Thus, $A = \nu\Delta$, and the domain is $D(A) = \{f \in C_0((-\infty, \infty)) : \Delta f \text{ exists}\}$.

Example 2.4.2.11. Now recall Example 2.4.2.4 for the fractional heat kernel semigroup: Let $X = L^p(\mathbb{R}^d),\ 1 \leq p < \infty$. Define $T : \mathbb{R}^+ \to \mathcal{L}(X)$ by

$$T(t)f = \begin{cases} f & \text{if } t = 0 \\ g_\alpha(x,t) * f & \text{if } t > 0. \end{cases}$$

We have proved that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on X. Hence, the infinitesimal generator of T is

$$Af \equiv \lim_{t\to 0} \frac{T(t)f - f}{t} = \frac{d}{dt}T(t)f|_{t=0} = -\nu\Lambda^\alpha f.$$

using the fact that $T(t)f$ solves the generalized heat equation

$$\begin{cases} \partial_t(T(t)f) + \nu\Lambda^\alpha(T(t)f) = 0 \\ T(0)f = f \end{cases}$$

Therefore, $A = -\nu\Lambda^\alpha$. $\because$

If $A \in \mathbb{R}^{n\times n}$, then $T(t) = e^{At}$ is a semigroup on $\mathbb{R}^n$. In general, if the operator A is the infinitesimal generator of T, then we write (*represent*) its corresponding semigroup as $T(t) \equiv e^{At}$. For example, in the case of the generalized heat operator we have $A = -\nu\Lambda^\alpha$. Hence,

$$T(t)f = \begin{cases} f & \text{if } t = 0 \\ g_\alpha(x,t) * f & \text{if } t > 0 \end{cases} = e^{(-\nu\Lambda^\alpha)t}$$

This means we can *identify* the heat semigroup by its infinitesimal generator, the fractional Laplacian. The justification for the formulas like equation (2.18) is now complete and these remarkable facts conclude our mathematical analysis of the fractional heat kernel.

2.4.3 Examples of Solutions Given Specific Initial Data

We provide some examples of solutions to the fractional heat equation for some simple initial conditions. The idea is to provide some intuition through these examples on how the "fractional" derivatives compare to more standard differential operators from calculus.

Consider the ordinary heat equation with the infinite energy initial condition $u_0 = 1$:

$$\begin{cases} \partial_t u = \nu \Delta u, & x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = 1 \end{cases}$$

The solution is obviously $u(x,t) = 1$, so $\|u\|_{L^\infty} = \|u_0\|_{L^\infty}$. This means that the supremum of the solution set is equal to the supremum of the initial data; thus, we conclude that $u(x,t)$ does not decay.

In the following lemma, we claim that this is still true for the fractional Laplacian in the generalized heat equation.

Lemma 2.4.3.1. *Consider the initial value problem*

$$\begin{cases} \partial_t u + \nu \Lambda^\alpha u = 0, & x \in \mathbb{R}^d,\ t > 0,\ 0 < \alpha \leq 2 \\ u(x,0) = 1 \end{cases}$$

Then, the unique solution is $u(x,t) = u(x,0) = 1$.

Proof. Recall that the solution to the generalized heat equation is given by the convolution of the heat kernel with the initial data, $u(x,t) = g_\alpha(x,t) * u(x,0)$. So,

$$\begin{aligned} u(x,t) &= g_\alpha(x,t) * 1 \\ &= \int_{\mathbb{R}^d} g_\alpha(x-y,t) \cdot 1 \; dy \\ &= \int_{\mathbb{R}^d} g_\alpha(y,t) dy \\ &= 1 \end{aligned}$$

by Lemma 2.4.1.7. □

Even more telling, this means that the fractional Laplacian of a constant is zero, as long as $\alpha \in (0,2]$. For a more challenging example, consider the following result.

Lemma 2.4.3.2. *Consider the 1D heat equation with given initial data*

$$\begin{cases} \partial_t u = \nu \Delta u, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = x \end{cases}$$

Then, $u(x,t) = u(x,0) = x$.

Proof. Using the convolution property $f * g = g * f$, we obtain

$$\begin{aligned} u(x,t) &= g_2(x,t) * x \\ &= \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4\nu t}} y dy = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{-\frac{|y|^2}{4\nu t}} (x-y) dy \\ &= x \left(\int_{-\infty}^{\infty} e^{-\frac{|y|^2}{4\nu t}} dy \right) - \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} y e^{-\frac{|y|^2}{4\nu t}} dy \\ &= x \cdot 1 - 0 = x \end{aligned}$$

since $\|g_2\|_{L^1} = 1$ and the second term is an odd function over a symmetric interval. □

We extend this exact result to the generalized heat equation. It is not at all obvious that the solution is still $u(x,t) = x$, especially for small α in which we approach exponential decay. We now prove this fact for the 1D case.

Theorem 2.4.3.3. *Consider the initial value problem for the 1D fractional heat equation*

$$\begin{cases} \partial_t u + \nu \Lambda^\alpha u = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = x \end{cases} \tag{2.24}$$

If $0 < \alpha < 2$, *then* $u(x,t) = u(x,0) = x$.

Proof. Suppose $u(x,t) = x$ is a solution. We proceed by substituting u into the PDE. Clearly, $\partial_t x = 0$, so it remains to be shown that $\nu \Lambda^\alpha x = 0$ or equivalently $\Lambda^\alpha x = 0$.

We recall the Riesz potential integral representation of the fractional Laplacian from CÓRDOBA and CÓRDOBA [24]:

For $0 < \alpha < 2$,

$$\Lambda^\alpha f(x) = \text{p. v. } C(\alpha, d) \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy \tag{2.25}$$

where C is the sharp constant given by

$$C(\alpha, d) = \frac{\Gamma(\frac{\alpha}{2} + \frac{d}{2})}{\pi^{\alpha + d/2} \Gamma(-\frac{\alpha}{2})}$$

Thus in 1D,

$$\Lambda^\alpha f(x) = \text{p. v. } C(\alpha, 1) \int_{-\infty}^{\infty} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} dy$$

With $f(x) = x$,

$$\begin{aligned}\Lambda^\alpha x &= C(\alpha, 1)\ \text{ p. v. } \int_{-\infty}^{\infty} \frac{x-y}{|x-y|^{1+\alpha}} dy \\ &= C(\alpha, 1)\ \text{ p. v. } \int_{-\infty}^{\infty} \frac{x-y}{|x-y|} \frac{1}{|x-y|^{\alpha}} dy \\ &= C(\alpha, 1)\ \text{ p. v. } \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)}{|x-y|^{\alpha}} dy\end{aligned}$$

where $\operatorname{sgn}(x)$ is the *sign* or *signum* function, defined below.

Definition 2.4.3.4. The **signum** function is given by

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} 1,\ x > 0 \\ 0,\ x = 0 \\ -1,\ x < 0 \end{cases} \tag{2.26}$$

Continuing the proof, we make a change of variables $z = x - y \implies dz = -dy$ and assume $R > \epsilon > 0$ to yield

$$\begin{aligned}\Lambda^\alpha x &= -C(\alpha, 1)\ \text{ p. v. } \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(z)}{|z|^{\alpha}} dz \\ &= -C(\alpha, 1) \lim_{R\to\infty} \left[\lim_{\epsilon \to 0} \left(\int_{-R}^{-\epsilon} \frac{\operatorname{sgn}(z)}{|z|^{\alpha}} dz + \int_{\epsilon}^{R} \frac{\operatorname{sgn}(z)}{|z|^{\alpha}} dz \right) \right] \\ &= -C(\alpha, 1) \lim_{R\to\infty} \left[\lim_{\epsilon \to 0} \left(\int_{-R}^{-\epsilon} \frac{(-1)}{(-z)^{\alpha}} dz + \int_{\epsilon}^{R} \frac{(1)}{(z)^{\alpha}} dz \right) \right] \\ &= -C(\alpha, 1) \lim_{R\to\infty} \left[\lim_{\epsilon \to 0} \left((-1)^{1-\alpha} \int_{-R}^{-\epsilon} \frac{1}{z^{\alpha}} dz + \int_{\epsilon}^{R} \frac{1}{z^{\alpha}} dz \right) \right]\end{aligned}$$

We now consider two cases, depending on α, to compute the integrals.

1. Case 1: $\alpha \in (0,2) \setminus \{1\}$

$$\begin{aligned}\Lambda^\alpha x &= -C(\alpha,1)\lim_{R\to\infty}\left[\lim_{\epsilon\to 0}\left((-1)^{1-\alpha}\frac{z^{1-\alpha}}{1-\alpha}\bigg|_{z=-R}^{-\epsilon} + \frac{z^{1-\alpha}}{1-\alpha}\bigg|_{z=\epsilon}^{R}\right)\right]\\ &= -C(\alpha,1)\lim_{R\to\infty}\left[\lim_{\epsilon\to 0}\left((-1)^{1-\alpha}(-1)^{1-\alpha}\left(\frac{\epsilon^{1-\alpha}-R^{1-\alpha}}{1-\alpha}\right)+\frac{R^{1-\alpha}-\epsilon^{1-\alpha}}{1-\alpha}\right)\right]\\ &= -C(\alpha,1)\lim_{R\to\infty}\left[\lim_{\epsilon\to 0}\left((1)\cdot\frac{\epsilon^{1-\alpha}-R^{1-\alpha}+R^{1-\alpha}-\epsilon^{1-\alpha}}{1-\alpha}\right)\right]\\ &= 0\end{aligned}$$

since $((-1)^{1-\alpha})^2 = (-1)^{2-2\alpha} = 1\cdot((-1)^{-2})^\alpha = 1^\alpha = 1$. It remains to prove Case 1.

2. Case 2: $\alpha = 1$

$$\begin{aligned}\Lambda^1 x = \Lambda x &= -C(1,1)\ \text{p. v.}\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(z)}{|z|^\alpha}dz\\ &= -C(\alpha,1)\lim_{R\to\infty}\left[\lim_{\epsilon\to 0}\left(\int_{-R}^{-\epsilon}\frac{(-1)}{(-z)}dz+\int_{\epsilon}^{R}\frac{(1)}{(z)}dz\right)\right]\\ &= -C(\alpha,1)\lim_{R\to\infty}\left[\lim_{\epsilon\to 0}\left(-\int_{\epsilon}^{R}\frac{1}{z}dz+\int_{\epsilon}^{R}\frac{1}{z}dz\right)\right]\\ &= -C(\alpha,1)\lim_{R\to\infty}\left[\lim_{\epsilon\to 0}(0)\right]\\ &= 0\end{aligned}$$

since $\int_{-R}^{-\epsilon}\frac{1}{z}dz = \int_{R}^{\epsilon}\frac{1}{z}dz = -\int_{\epsilon}^{R}\frac{1}{z}dz$.

Therefore, we conclude that $\forall x \in \mathbb{R}$, $\Lambda^\alpha x = 0 \ \ \forall \alpha \in (0,2)$. Thus, $u(x,t) = x$ solves (2.24). □

This means that arbitrarily small "second derivatives" of the polynomial x is zero, as long as $\alpha \neq 0$ since that would just be the identity operator. We can actually prove the same result in the d-dimensional case without all of the tedious computations as above by directly exploiting the fact that the generalized heat kernel is a *radially symmetric* function, $g_\alpha(x,t) = g_\alpha(|x|,t)$ and hence is an even function.

Lemma 2.4.3.5. *Consider the initial value problem for the d-dimensional fractional heat equation*

$$\begin{cases}\partial_t u + \nu\Lambda^\alpha u = 0, & x\in\mathbb{R}^d,\ t>0\\ u(x,0) = x\end{cases} \tag{2.27}$$

If $0 < \alpha < 2$, *then* $u(x,t) = u(x,0) = x$.

Proof. Using the fact that g_α is radial, we have

$$\begin{aligned} u(x,t) = g_\alpha * x &= x * g_\alpha \\ &= \int_{\mathbb{R}^d} (x-y)\ g_\alpha(y,t)dy \\ &= x\int_{\mathbb{R}^d} g_\alpha(y,t)dy - \int_{\mathbb{R}^d} y\ g_\alpha(y,t)dy \\ &= x\cdot 1 - 0 \\ &= x \end{aligned}$$

since $g_\alpha(x,t) = g_\alpha(|x|\,,t)$ implies that $yg_\alpha(y,t)$ is odd. □

Chapter 3

Effect of Dissipation

3.1 Some Decay Bounds

We explore decay properties of the generalized heat kernel. The motivating question is the following: For the generalized heat equation, how is α related to the decay rate of the solution $u(x,t)$?

Recall that the solution to the homogeneous fractional heat equation (2.14)

$$\begin{cases} \partial_t u + \nu \Lambda^\alpha u = 0, \quad 0 \le \alpha \le 2,\ t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

in the whole space $\mathbb{R}^d$ is

$$u(x,t) = e^{-\nu t \Lambda^\alpha} u_0 = g_\alpha(x,t) * u_0$$

We have the following estimate exhibiting the decay behavior of $u(x,t)$.

Theorem 3.1.0.1. *Suppose the initial data satisfies $u_0 \in L^p(\mathbb{R}^d)$. If $0 < \alpha \le 2,\ \ 1 \le p \le q \le \infty$, and $t > 0$, then*

$$\|g_\alpha(\cdot,t) * u_0\|_{L^q(\mathbb{R}^d)} \le Ct^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)} . \tag{3.1}$$

Proof. We invoke Young's inequality (2.1.0.5). For $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$,

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} = \|g_\alpha(\cdot,t) * u_0\|_{L^q(\mathbb{R}^d)} \le \|g_\alpha(\cdot,t)\|_{L^r(\mathbb{R}^d)} \|u_0\|_{L^p(\mathbb{R}^d)} .$$

Then, it remains to be shown that

$$\|g_\alpha(\cdot,t)\|_{L^r(\mathbb{R}^d)} \le Ct^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})}.$$

By our result on scaling from Lemma 2.4.1.8, we know that

$$g_\alpha(x,t) = t^{-\frac{d}{\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}, 1\right).$$

So, we conclude

$$\begin{aligned}
\|g_\alpha(x,t)\|_{L^r(\mathbb{R}^d)} &= \left\| t^{-\frac{d}{\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}, 1\right)\right\|_{L^r(\mathbb{R}^d)} \\
&= t^{-\frac{d}{\alpha}} \left\| g_\alpha\left(\frac{x}{t^{1/\alpha}}, 1\right)\right\|_{L^r(\mathbb{R}^d)} \\
&= t^{-\frac{d}{\alpha}} \left(\int_{\mathbb{R}^d} \left| g_\alpha\left(\frac{x}{t^{1/\alpha}}, 1\right)\right|^r dx\right)^{\frac{1}{r}} \\
&= t^{-\frac{d}{\alpha}} \left(\int_{\mathbb{R}^d} \left(g_\alpha\left(\frac{x}{t^{1/\alpha}}, 1\right)\right)^r dx\right)^{\frac{1}{r}} \quad (\text{since } g_\alpha \geq 0) \\
&= t^{-\frac{d}{\alpha}} \left(t^{\frac{d}{\alpha}} \int_{\mathbb{R}^d} (g_\alpha(\eta,1))^r \, d\eta\right)^{\frac{1}{r}} \quad \left(\text{change of variables: } \eta = \frac{x}{t^{1/\alpha}} \implies d\eta = t^{-d/\alpha} dx\right) \\
&= t^{-\frac{d}{\alpha} + \frac{d}{\alpha r}} \|g_\alpha(\eta,1)\|_{L^r(\mathbb{R}^d)} \\
&= Ct^{-\frac{d}{\alpha}(1-\frac{1}{r})} \quad (\text{where } C = \|g_\alpha(\eta,1)\|_{L^r(\mathbb{R}^d)}) \\
&= Ct^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})}
\end{aligned}$$

since $\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r}$. Therefore, $\|g_\alpha(\cdot,t) * u_0\|_{L^q(\mathbb{R}^d)} \leq Ct^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}$ as desired. □

Corollary 3.1.0.2. *If* $u_0 \in L^2(\mathbb{R}^d)$, $0 < \alpha \leq 2$, *and* $t > 0$, *then*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{-\frac{d}{2\alpha}} \|u_0\|_{L^2(\mathbb{R}^d)}.$$

Proof. This follows directly from Theorem 3.1.0.1 with $p = 2,\ q = \infty$. □

Our corollary basically says that the maximum of the solution at any fixed time is bounded above algebraically. For general α, the rate is proportional to $t^{-\frac{d}{2\alpha}}$, so we would expect that a smaller value of α corresponds to slower decay in the time evolution of the solution for t small and faster decay for t large. In fact, the case $\alpha = 0$ yields exponential decay in time, $u(x,t) = e^{-\nu t} u_0(x)$, which for large t is smaller than any algebraic function $Ct^{-\frac{d}{2\alpha}}$. However, the heat equation damps the amplitude of u_0 the strongest at small times t and hence we would expect larger values of α (corresponding to more dissipation) to be most significant. We perform numerical simulations to explore the agreement between theory and computation.

3.2 Numerical Simulation

3.2.1 Spectral Method

We proceed to solve the generalized heat equation on a bounded periodic domain using *pseudo-spectral methods* [69]. Our implementation in MATLAB is based on the *Fast Fourier Transform* (FFT) algorithm. For an introduction to computing the fractional Laplacian without the use of the Fourier transform, we refer the reader to the sources [2, 35, 44, 45, 59, 68, 78], many of which are quite new in the literature.

Given some PDE, the idea behind a pseudo-spectral method is to first perform the spatial discretization using the Discrete Fourier transform (DFT), and then use a standard ODE integrator (such as fourth-order Runge-Kutta) for the time discretization. We outline this method in 1D. Let $\Omega \subset [0, 2\pi)$ be a bounded periodic grid where the point $x = 2\pi$ is identified with the point $x = 0$. The number of Fourier modes N (an even integer) is chosen such that the spacing between grid points $x_1, x_2, \ldots, x_N$ is $h = \frac{2\pi}{N}$. Due to the phenomenon of aliasing, the range of Fourier wavenumbers k resolvable on this grid is $[-\frac{\pi}{h}, \frac{\pi}{h}]$. Numerically, this amounts to the sets

$$x \in \{h, 2h, \ldots, (N-1)h, Nh = 2\pi\} \tag{3.2}$$

for the physical space domain and

$$k \in \{-\frac{N}{2} + 1, -\frac{N}{2} + 2, \ldots, \frac{N}{2}\} \tag{3.3}$$

for the Fourier wavenumber domain. In practice, the spatial domain can be shifted to any arbitrary length L using $h = \frac{L}{N}$ and

$$k \in \{-\frac{2\pi N/L}{2} + 1, -\frac{2\pi N/L}{2} + 2, \ldots, \frac{2\pi N/L}{2}\}.$$

With this notation, the well known discrete transforms on Ω are defined as follows.

Definition 3.2.1.1. The **Discrete Fourier transform** is the vector of numbers

$$\hat{v}_k = h \sum_{j=1}^{N} e^{-ikx_j} v_j$$

for $k \in \{-\frac{N}{2} + 1, -\frac{N}{2} + 1, \ldots, \frac{N}{2}\}$ and the **Inverse Discrete Fourier transform** is the vector of numbers

$$v_j = \frac{1}{Nh} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} e^{ikx_j} \hat{v}_k$$

for $j = 1, 2, \ldots, N$.

In practice, efficient algorithms have been developed to compute these transforms. One of the most used is the Fast Fourier Transform, used exclusively in this work; we denote the forward transform operation as FFT and the inverse transform operation as IFFT. Derivatives can now be computed with so-called spectral accuracy (exponential order of convergence), provided the function is sufficiently "nice" or smooth, via *spectral numerical differentiation*.

Definition 3.2.1.2. Given a function f, the **spectral derivative** is

$$f^{(m)} = \mathrm{IFFT}((ik)^m \, \mathrm{FFT}(f)) \tag{3.4}$$

where $m \in \mathbb{N}$ is the derivative order.

This formula follows from the basic properties of the Fourier transform. It is clear that numerically computing the fraction Laplacian of a suitable periodic function f is just as simple:

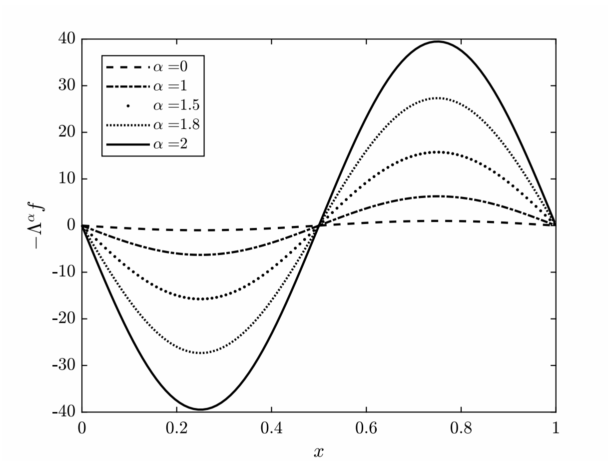
$$\Lambda^\alpha f = \mathrm{IFFT}(|k|^\alpha \, \mathrm{FFT}(f))$$

where $\alpha \in (0, 2]$ and k are the wavenumbers in (3.3). Note that this representation— particularly the Fourier multiplier $|k|^\alpha$—differs from that given in the continuous case (2.7) by a constant due to the normalization difference in the definition of the DFT.

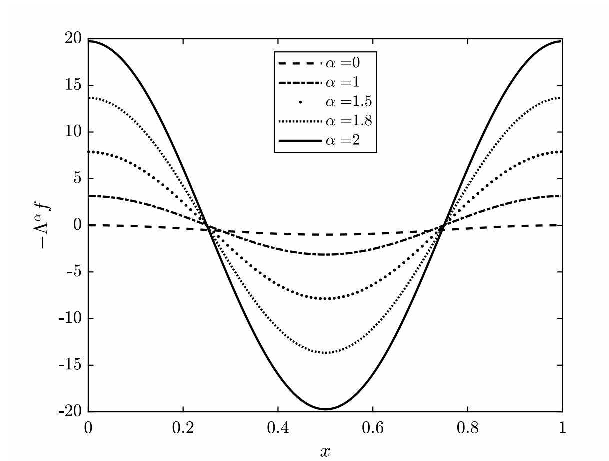
Comparisons between the 1D Laplacian $\Delta f = \partial_{xx} f$ and the (negative) fractional Laplacian $-\Lambda^\alpha f$ of a set of example functions are shown in Figure 3.1 with $N = 1024$ Fourier modes and $\alpha \in \{0, 1, 1.5, 1.8, 2\}$. The case $\alpha = 2$ corresponds to the ordinary Laplacian, $-\Lambda^2 f = \Delta f$. We observe the required behavior in the limit as $\alpha \to 2$ and $\alpha \to 0$ from the theory. When $\alpha = 0$, $-\Lambda^0 f = -f$ as expected.

For the generalized heat equation, we already have the solution in the Fourier space by directly applying the Fourier transform to equation (2.14) to obtain the system

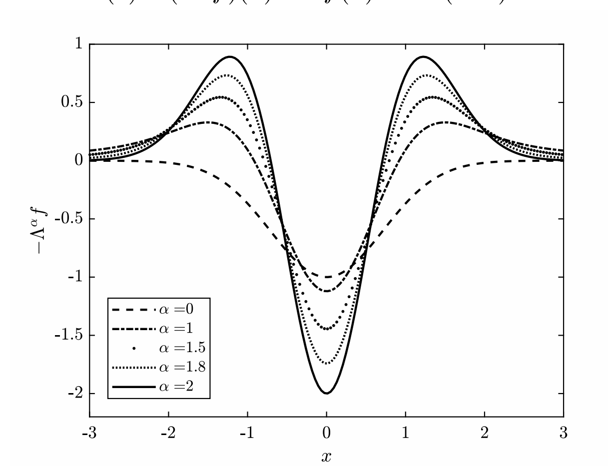
$$\begin{cases} \frac{d}{dt}\hat{u}(t) = -\nu \, |k|^\alpha \, \hat{u}(t) + \hat{f}(t), & t \in (0, T], \ \alpha \in (0, 2] \\ \hat{u}(0) = \widehat{u_0}(k) \end{cases}$$



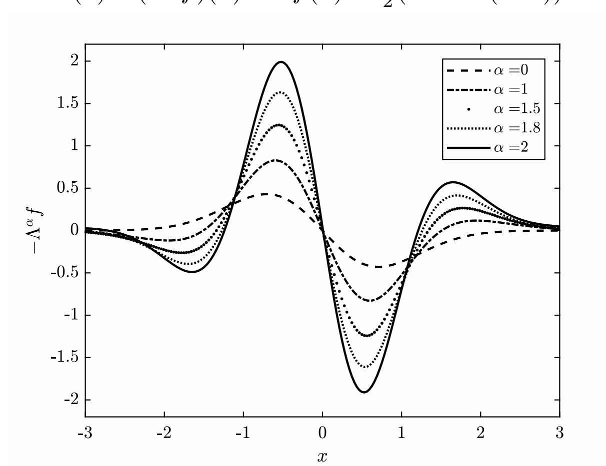
(a) $-(\Lambda^\alpha f)(x)$ for $f(x) = \sin(2\pi x)$



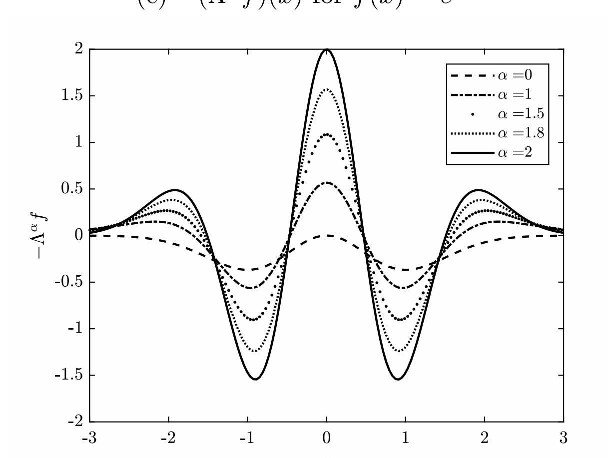
(b) $-(\Lambda^\alpha f)(x)$ for $f(x) = \frac{1}{2}(1 - \cos(2\pi x))$



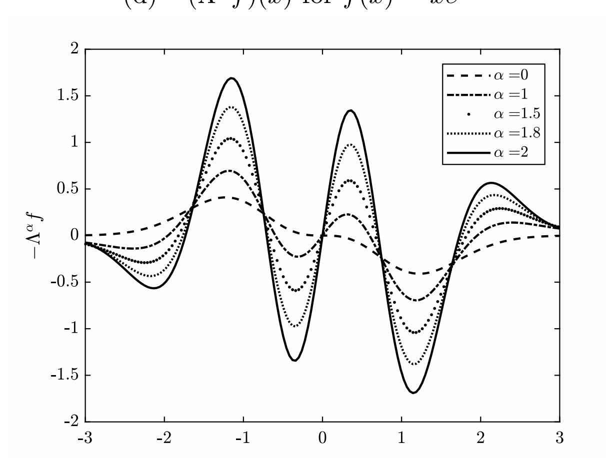
(c) $-(\Lambda^\alpha f)(x)$ for $f(x) = e^{-x^2}$



(d) $-(\Lambda^\alpha f)(x)$ for $f(x) = xe^{-x^2}$



(e) $-(\Lambda^\alpha f)(x)$ for $f(x) = x^2e^{-x^2}$



(f) $-(\Lambda^\alpha f)(x)$ for $f(x) = x^3e^{-x^2}$

Figure 3.1: Negative fractional Laplacians $-(\Lambda^\alpha f)(x)$ computed for varied α for six functions f.

where k is fixed and solving the ODE via Duhamel's principle:

$$\hat{u}(k,t) = e^{-\nu t|k|^\alpha}\widehat{u_0}(k) + \int_0^t e^{-\nu(t-\tau)|k|^\alpha}\hat{f}(k,\tau)d\tau,$$

again noting the absence of the 2π constant in the Fourier multipliers due to the definition of the DFT. We discretize time as $0 = t^0 < t^1 < \ldots < t^{N_t-1} < t^{N_t} = T$ where N_t is the number of time steps, n is the temporal index, $\Delta t = t^{i+1} - t^i$ is the time step, and T is the maximum time. Hence, using the same spatial discretization above in (3.2), the solution $u(x,t)$ to the generalized heat equation at any time t^n can be approximated by the numerical solution

$$u(x,t) \approx V(x,t) = \text{IFFT}(\hat{u}(k,t^n)).$$

Using this procedure for numerical simulation, the effect of varying α in the generalized heat equation can easily be observed.

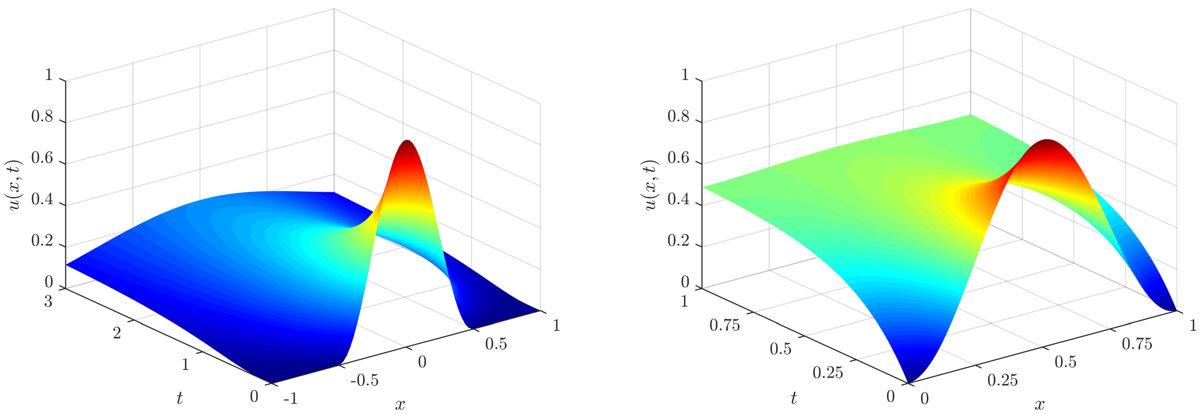
3.2.2 Results for Generalized Heat Equation

For the first set of numerical experiments (Simulation 1) using the spectral method (3.2.1), we simulate the generalized heat equation on an infinite domain by choosing L large enough such that the initial condition is supported only on a small subset $\Omega \subset [-L/2, L/2)$ of the periodic grid. For the second set of simulations (Simulation 2), a purely periodic domain is specified with $\operatorname{supp} u_0 = [-L/2, L/2)$ where L is small compared to the value of L in Simulation 1. The differences between these domains are shown in Figure 3.2; the IVP for this example is the 1D heat equation $\partial_t u = 0.1\partial_{xx}u$ with initial data $u_0(x) = \frac{1}{2}(1-\cos(2\pi x))$ defined on an interval of length 1 in both cases.

We now consider the 1D generalized heat equation

$$\begin{cases} \partial_t u + \nu\Lambda^\alpha u = f(x,t), & x \in [-L/2, L/2),\ t \in (0,T] \\ u(-L/2,t) = u(L/2,t), & t \in (0,T] \\ u(x,0) = u_0(x) \end{cases} \tag{3.5}$$

with periodic boundary conditions. Four test cases are performed each with a different initial condition. We set the forcing term $f(x,t)$ to zero in all simulations to avoid masking the effect of dissipation. Two of the initial conditions have zero mean (Case 1, Case 4) and two do not (Case 2, Case 3). Zero mean functions are preferred since the boundary conditions are periodic; the steady state solution is the mean of u_0 and thus for the zero mean initial conditions, the solution



(a) Simulated infinite domain, $L = 100$.

(b) Periodic domain $[0, L)$, $L = 1$.

Figure 3.2: Solutions to the heat equation for an infinite versus periodic numerical domain.

will damp to zero as $t \to \infty$. We use $N = 4096$ modes, $L = 50$, $T = 6$, $\nu = 1$, $\Delta t = 10^{-3}$, and $\alpha \in \{0.1, 0.5, 1, 1.5, 2\}$.

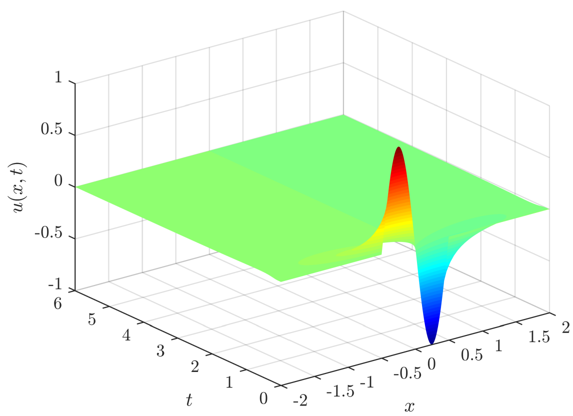
In Simulation 1, the initial conditions are:

$$\begin{aligned}
&\text{Case 1: } u_0(x) = \sin(2\pi(x + 0.5)) \\
&\text{Case 2: } u_0(x) = \frac{1}{2}(1 - \cos(2\pi(x + 0.5))) \\
&\text{Case 3: } u_0(x) = e^{-100x^2} \\
&\text{Case 4: } u_0(x) = 100xe^{-x^2}
\end{aligned}$$

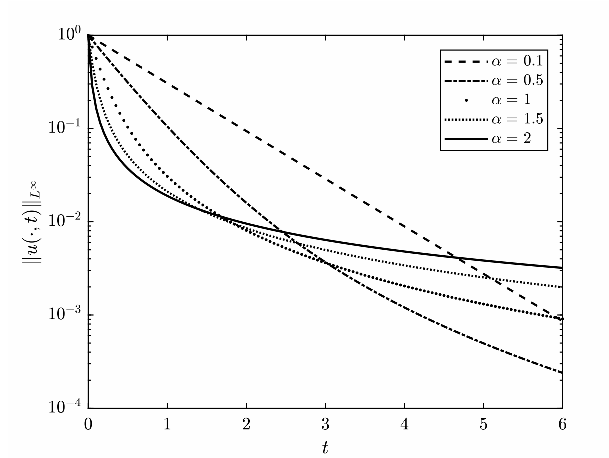
where for Case 1 and 2 we have u_0 defined as above only for $x \in [-1/2, 1/2]$ and $u_0(x) = 0$ else.

In Figures (3.3, 3.4, 3.5, 3.6), we plot the decay of the L^∞ norm $\|u(\cdot, t)\|_{L^\infty}$ as in Corollary 3.1.0.2. By the Corollary, we know that the solution is bounded above by $Ct^{-\frac{1}{2\alpha}}$ for this 1D problem. In a simple algebraic argument, we claimed in Section 3 that small values of α correspond to slow decay of $u(x, t)$ for small values of t and fast decay for large values of t. The results from the numerics support our claim about the behavior of the solution to (3.5) when α is varied.

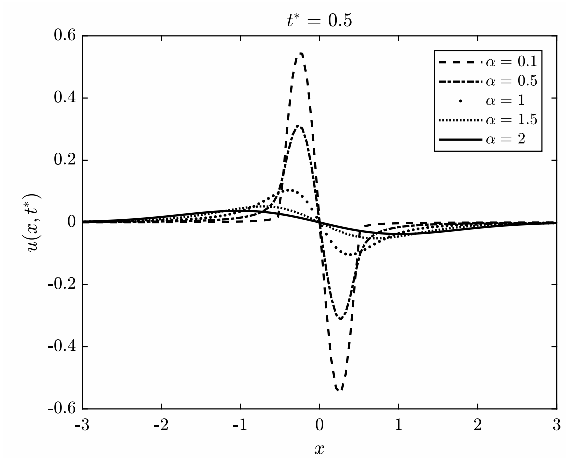
Solution profiles for small values of time are shown in (3.3c, 3.4c, 3.5c, 3.6c) for varied α and indeed depict the prediction that decreasing α corresponds to less damping in the solution. This holds in all four cases. Similarly, solution profiles for large values of time are shown in (3.3d, 3.4d, 3.5d, 3.6d) for varied α. We observe that decreasing α corresponds to more damping in the solution, but at these large times the amplitude of u is already extremely small. For real-world applications,



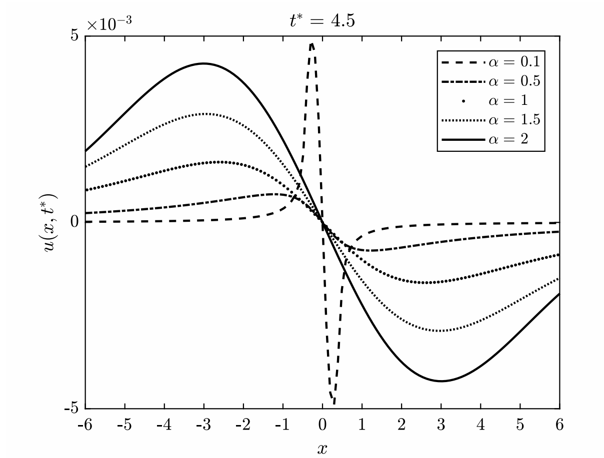
(a) Solution for $\alpha = 2$.



(b) Decay of L^∞ norm.

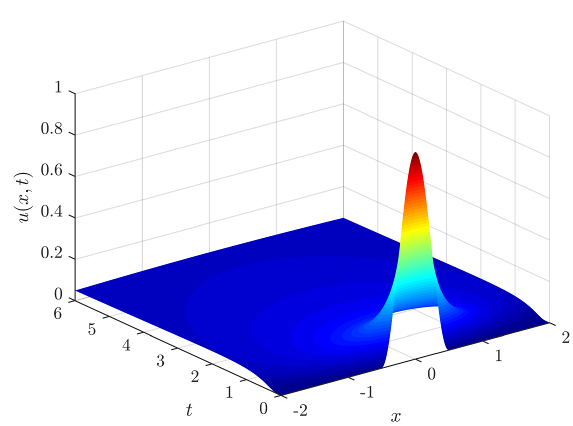


(c) Solution profiles for small time.

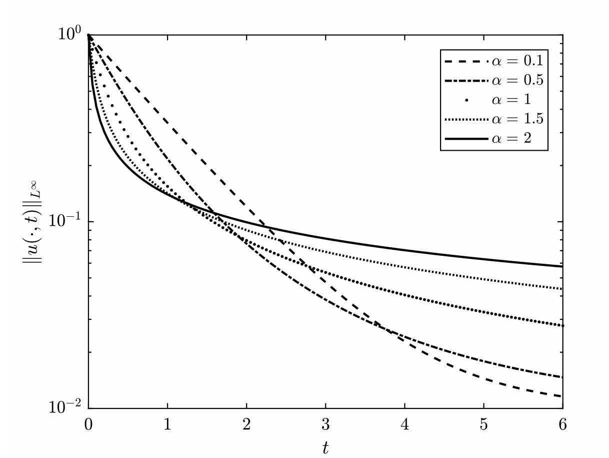


(d) Solution profiles for large time.

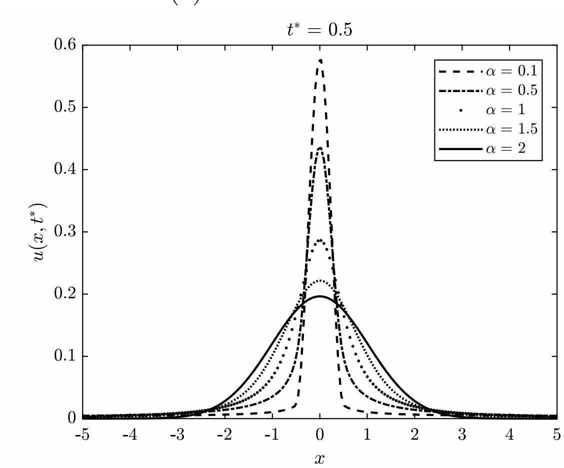
Figure 3.3: Case 1, Generalized heat equation with $u_0(x) = \sin(2\pi(x + 0.5))$ on the simulated infinite domain.



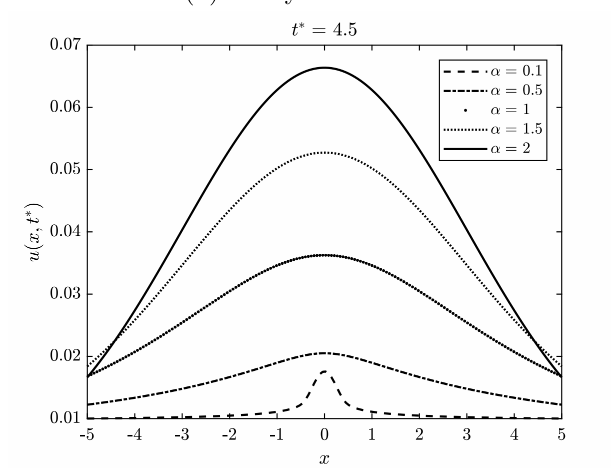
(a) Solution for $\alpha = 2$.



(b) Decay of L^∞ norm.

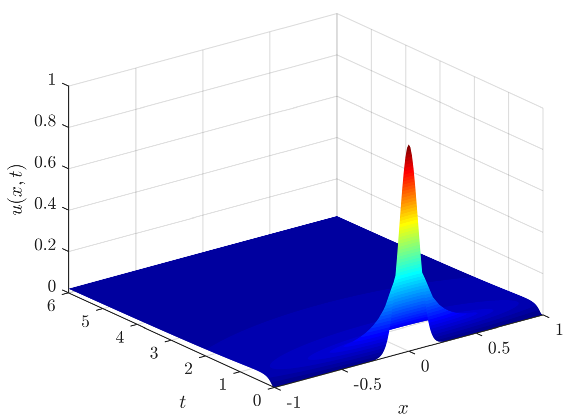


(c) Solution profiles for small time.

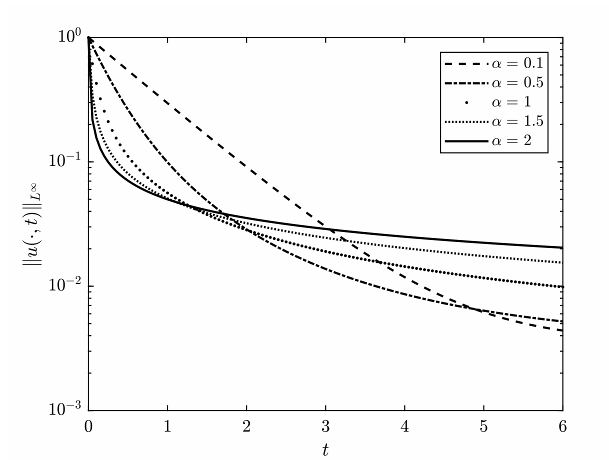


(d) Solution profiles for large time.

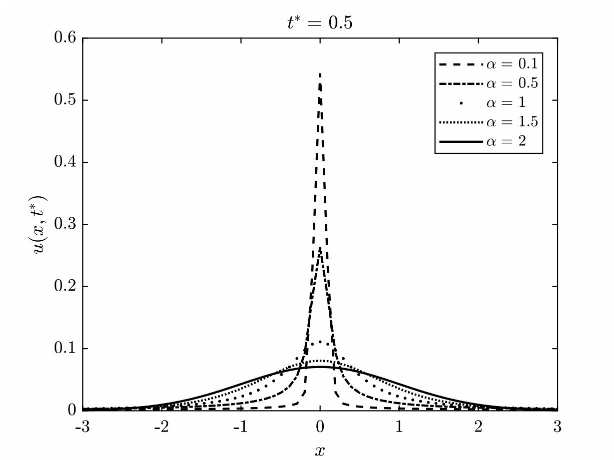
Figure 3.4: Case 2, Generalized heat equation with $u_0(x) = \frac{1}{2}(1 - \cos(2\pi(x + 0.5)))$ on the simulated infinite domain.



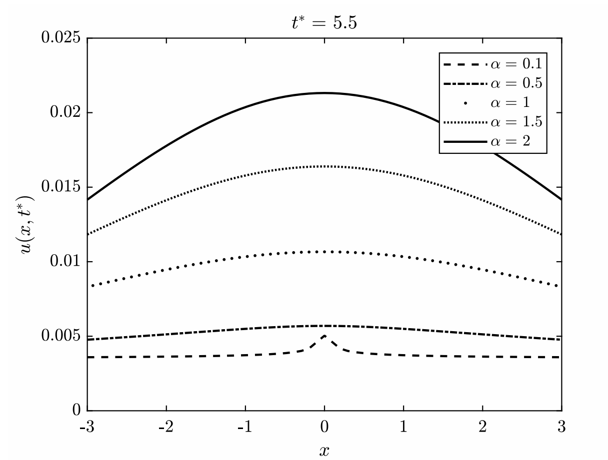
(a) Solution for $\alpha = 2$.



(b) Decay of L^∞ norm.

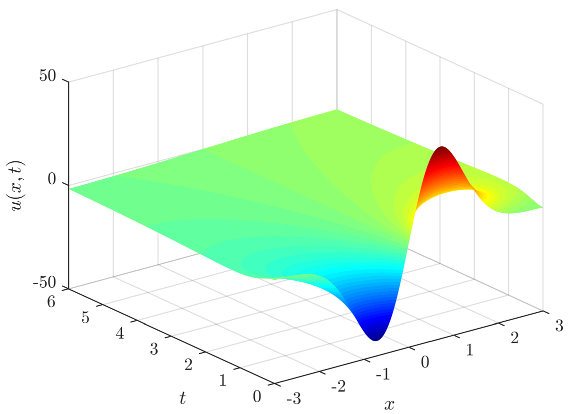


(c) Solution profiles for small time.

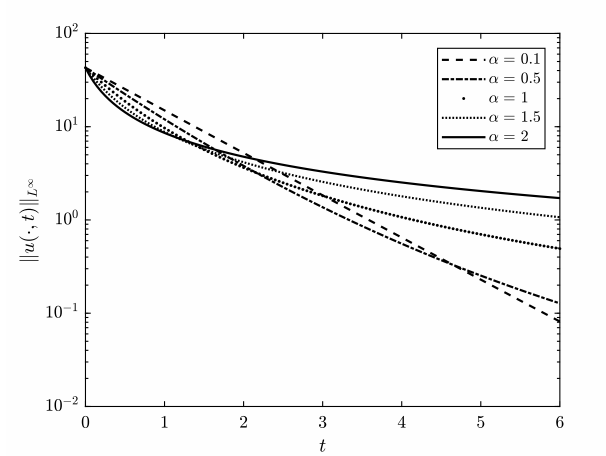


(d) Solution profiles for large time.

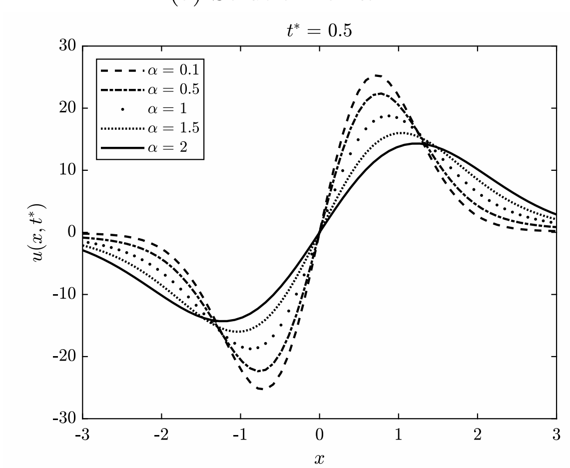
Figure 3.5: Case 3, Generalized heat equation with $u_0(x) = e^{-100x^2}$ on the simulated infinite domain.



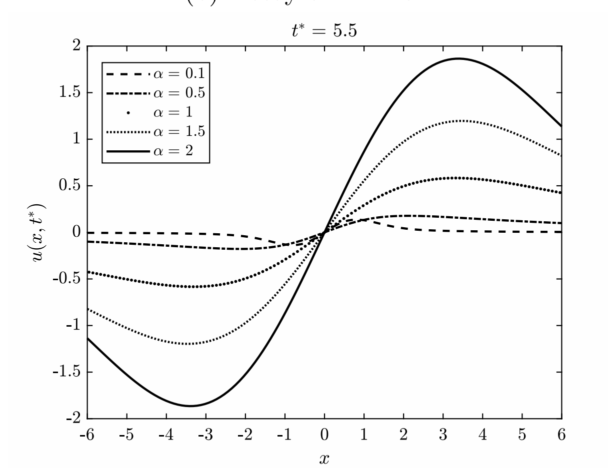
(a) Solution for $\alpha = 2$.



(b) Decay of L^∞ norm.



(c) Solution profiles for small time.



(d) Solution profiles for large time.

Figure 3.6: Case 4, Generalized heat equation with $u_0(x) = 100xe^{-x^2}$ on the simulated infinite domain.

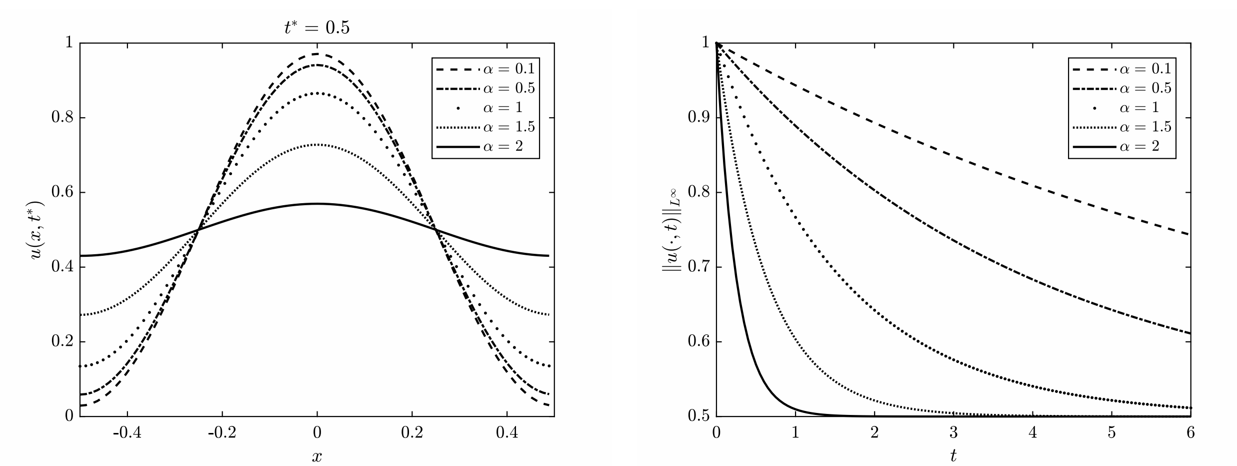
the short-time behavior is likely of more physical interest.

Finally, we examine two cases in the purely periodic domain $[-L/2, L/2)$ to investigate the influence of domain choice on the observed behavior of $u(x,t)$ as α is varied. We use $N = 4096$ modes, $L = 1$, $T = 6$, $\nu = 0.1$, $\Delta t = 10^{-3}$, and $\alpha \in \{0.1, 0.5, 1, 1.5, 2\}$.

In Simulation 2, the initial conditions are:

$$\text{Case 1: } u_0(x) = \frac{1}{2}(1 - \cos(2\pi(x + 0.5)))$$
$$\text{Case 2: } u_0(x) = 100xe^{-50x^2}$$

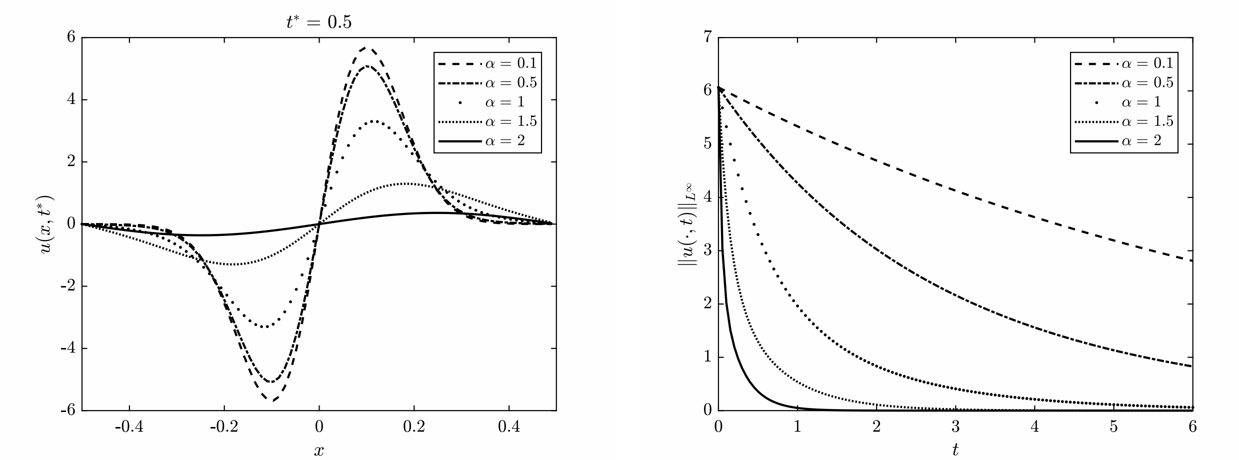
Unexpectedly, the difference in short-time versus long-time behavior is not observed on the $L = 1$ periodic domain (Figure 3.7, 3.8). In both test cases, small α corresponds to slower decay in the solution (and L^∞ norm, see 3.7b, 3.8b). For Case 1, the endpoints of the domain may be responsible for influencing the solution; the mean of u_0 in Case 1 is $1/2$ as opposed to the infinite domain approximation in which the mean was near 0. However, Case 2 in Simulation 2 also demonstrates the same trend in the solution as α decreases even thought the initial condition is zero for some distance near the boundary. Our theory—developed for the whole space $\mathbb{R}^d$—appears to not fully explain behavior observed in the periodic domain.



(a) Solution profiles for small time.

(b) Decay of L^∞ norm.

Figure 3.7: Case 1, Generalized heat equation with $u_0(x) = \frac{1}{2}(1 - \cos(2\pi(x + 0.5)))$ on the periodic domain.



(a) Solution profiles for small time.

(b) Decay of L^∞ norm.

Figure 3.8: Case 2, Generalized heat equation with $u_0(x) = 100xe^{-50x^2}$ on the periodic domain.

Chapter 4

Transport-Diffusion

4.1 Linear Transport Equation

We provide a brief overview of linear equations before diving into nonlinear transport-diffusion and Burgers' equation. The method of characteristics is where we begin our discussion. Arguably the simplest PDE, the linear transport equation (also known as the advection, convection, or one-way wave equation) is the first order PDE

$$\partial_t u + c \cdot \nabla u = 0. \tag{4.1}$$

Here, the coefficient $c \in \mathbb{R}^d$ is the vector of constant wave speeds. These equations describe advection processes such as atmospheric flow, air pollution particles, chemical dye dispersion, or even the density of car traffic.

We can immediately discern some properties of this PDE. Defining a pseudo-gradient as $\nabla_{x,t} = (\partial_t, \nabla)$, we observe that the directional derivative in the $(1, c)$ direction vanishes in the x-t plane, that is

$$\begin{pmatrix} 1 \\ c \end{pmatrix} \cdot \nabla_{x,t} u = 0.$$

Hence, the solution is constant along lines in this direction. This is the basis of what are known as the *characteristic curves* of the PDE. This idea is used to solve the initial value problem

$$\begin{cases} \partial_t u + c \cdot \nabla u = 0, \quad x \in \mathbb{R}^d,\ t > 0 \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d \end{cases} \tag{4.2}$$

where $u_0 \in C^1(\mathbb{R}^d)$. Fix a point $(\xi, 0) \in \mathbb{R}^d \times \{t = 0\}$. Then, a line in the direction $(1, c)$ containing this point is $x = \xi + ct$. We know that $u(x,t) = u(\xi + ct, t)$ for all $t \geq 0$. Hence choosing $t = 0$,

$$u(x,t) = u(\xi, 0) = u_0(x_0) = u_0(x - ct) \tag{4.3}$$

is the solution to (4.2). If $u_0 \notin C^1(\mathbb{R}^d)$, then $u_0(x-ct)$ is a weak solution. Plotting the characteristics $x = \xi + ct$ will yield parallel lines; that is, the constant coefficient advection equation propagates the initial data along lines in the x-t plane. In general, the method of characteristics converts the PDE into a system of ODE.

The modified problem [66], posed here in 1D, can be solved this way.

$$\begin{cases} \partial_t u + c\partial_x u + bu = f(x,t), & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{4.4}$$

The equation now has a zero-order term bu and inhomogeneous forcing term f. We change to the characteristic coordinates $\xi = x - ct$ and $s = t$. By the chain rule with $z(\xi, s) \equiv u(x,t)$,

$$\begin{aligned} \partial_s z(\xi, s) &= \partial_t u \partial_s t + \partial_x u \partial_s x \\ &= \partial_t u \cdot 1 + c\partial_x u \\ &= -bu + f(x,t) \\ &= -bz + f(\xi + cs, s) \end{aligned}$$

Hence, treating ξ as a parameter, we obtain the following ODE along the characteristic coordinates

$$\begin{cases} \frac{\partial z}{\partial s} = -bz + f(\xi + cs, s) \\ z(\xi, 0) = u_0(\xi) \end{cases} \tag{4.5}$$

The solution is trivial,

$$z(s) = u_0(\xi)e^{-bs} + \int_0^s f(\xi + c\tau, \tau)e^{-b(s-\tau)}d\tau.$$

Changing back to physical variables gives the desired result:

$$u(x,t) = u_0(x - ct)e^{-bt} + \int_0^t f(x,t)e^{-b(t-\tau)}d\tau. \tag{4.6}$$

We note that setting $b = 0$ recovers the general solution of the forced one-way wave equation as expected.

Combining a diffusion term with the transport equation yields the advection-diffusion equation

$$\begin{cases} \partial_t u + c \cdot \nabla u - \nu \Delta u = 0, \quad x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^d \end{cases} \tag{4.7}$$

where $c \in \mathbb{R}^d$ is constant. The solution behaves as expected, that is, u is advected with speed c while dissipated via diffusion with magnitude ν. The solution procedure relies on changing variables to characteristics coordinates $\xi = x - ct$ and defining $w(\xi, t) = u(\xi + ct, t) = u(x, t)$. Then, w solves the linear heat equation $\partial_t w - \nu \Delta_\xi w = 0$ which has an explicit solution given by Equation (2.12). Hence, the solution to the advection-diffusion equation above is simply $u(x,t) = w(x - ct, t)$.

In Appendix A.3, we prove an estimate in the sense of (1.1) for the fractional version of the advection-diffusion equation, which shows that its corresponding IVP is well-posed (at least in 1D). The next section examines nonlinear transport; structurally, the nonlinear case is similar to linear advection-diffusion. However, the behavior of solutions can be wildly different.

4.2 Burgers' Equation

The **Burgers' equation** is the simplest nonlinear PDE one can study, and it comes in two standard flavors: inviscid and viscous. The inviscid Burgers' equation is a hyperbolic conservation law (of the form $\partial_t \theta + \nabla F(\theta) = 0$ where F is a flux) exhibiting finite time singularities in its solution or gradient, making this equation difficult to simulate numerically without shock-capturing methods. The Cauchy problem is

$$\begin{cases} \partial_t u + \nabla \left(\frac{1}{2}u^2\right) = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = u_0(x). \end{cases} \tag{4.8}$$

In particular, it can be proven that shocks form if the initial data satisfies $\frac{du_0}{dx} < 0$ on some interval; specifically, the breaking time is $t_b = \frac{1}{\min_{x \in \mathbb{R}^d}\{\nabla u(x,0)\}}$.

On the other hand, the viscous Burgers' equation $\partial_t u + \nabla\left(\frac{1}{2}u^2\right) - \nu \Delta u = 0$ is parabolic and represents nonlinear advection-diffusion; the solution exhibits fierce competition between convective and diffusive phenomena [see 6, for a review]. The equation has a nice diffusion term $-\nu \Delta u$, and we generalize this term into its fractional dissipation counter part $\nu \Lambda^\alpha u$ in the IVP

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nu \Lambda^\alpha u = f,\ x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{4.9}$$

where $0 < \alpha \leq 2$. This version of the equation with fractional diffusion is well studied in the literature and is called the *fractal Burgers' equation*. For $0 < \alpha < 1$ (the supercritical case), the problem is only locally well-posed and ∇u may blow up in finite time. For the critical ($\alpha = 1$) and subcritical cases ($1 < \alpha \leq 2$), Equation (4.9) is globally well-posed with nice solutions [34]. For a thorough treatment of current results on well-posedness, blow up, and regularity, refer to [3, 7, 33, 50]. A more challenging extension of the Burgers'-type equations to 2D are the *surface quasi-geostrophic equation* (SQG) and its variants, which can be interpreted as models of geophysical fluid flows. The equation is nonlinear with nonlocal diffusion and serves as a lower dimensional model for the 3D Navier-Stokes equation due to a similar vortex stretching mechanism. A wealth of results exist for SQG, see [11, 20, 21, 22, 24, 51, 57, 77].

The family of equations (4.9), though simple, exhibit fascinating behavior that translate well into applications in science and engineering. For example, Burgers' equation is one of the first standard model problems to test new numerical methods for solving PDE, especially since we have an exact solution [6]. It shares the same quadratic nonlinearity $u \cdot \nabla u$ seen in the Euler and Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nu \Lambda^\alpha u = -\nabla p, \quad x \in \mathbb{R}^d, \ t > 0 \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{4.10}$$

with $0 < \alpha \leq 2$ and $\nu \geq 0$. While it is true that many popular fluid dynamics solvers are designed for complex 3D flows governed by the Navier-Stokes equations or its many coupled extensions (e.g.: radiative heat transfer, chemically reacting flows, particle problems, magnetohydrodynamics), engineers test their numerical frameworks on the Burgers' equation first because if the code fails for this simple case, it will certainly fail for more challenging PDE. We first gleam some insight into the structure of solutions by solving (4.8) and (4.9) with $\alpha = 2$ analytically. Then, we perform a numerical study on Equation (4.9) with other values of $0 < \alpha \leq 2$, since this corresponds to physically observed phenomena (e.g., combustion models [58]).

4.2.1 Analytical Solution

We can solve the inviscid Burgers' equation (4.8) analytically using the method of characteristics. We note that shock front formation is a result of the characteristic curves intersecting; similarly, an expansion or rarefaction wave is generated by the characteristics diverging from one another. Using a similar argument as in (4.2) but now considering the curve $x = \xi + u(x,t)t$ with variable speed depending on the solution itself, an implicit solution for the inviscid Burgers' equation is found to be

$$u(x,t) = u_0(x - u(x,t)t). \tag{4.11}$$

There is no closed form solution in general, but using this relationship to plot the characteristics in the x-t plane can provide insight on how the curves interact.

The viscous Burgers' equation (with $\alpha = 2$ only, in this case) can also be solved exactly for the whole space. We use a powerful technique thats converts the nonlinear PDE into an easy linear PDE, the **Cole-Hopf transformation** [38]. Many mathematicians perform the *ansatz* setting $w = e^{\frac{-cu}{\nu}}$ and changing variables in (4.9) to obtain the heat equation. We will take this approach, but first justify where the substitution came from. Consider the Cauchy problem for the parabolic PDE with a quadratic nonlinearity in $\mathbb{R}^d$:

$$\begin{cases} \partial_t u - \nu \Delta u + c\,|\nabla u|^2 = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{4.12}$$

where $\nu > 0$. We assume that u has sufficient regularity and assign the unknown function

$$\phi : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{smooth}$$

with $w := \phi(u)$. The goal here is to find a smooth function ϕ such that we force w to solve a linear PDE. Computing derivatives with the chain rule yields

$$\partial_t w = \phi'(u)\partial_t u$$

for time and

$$\Delta w = \phi'(u)\Delta u + \phi''(u)\,|\nabla u|^2$$

for space. But by our carefully selected system (4.12) above, we see that

$$\begin{aligned} \partial_t w &= \phi'(u)\partial_t u = \phi'(u)(\nu\Delta u - c\,|\nabla u|^2) \\ &= \nu\phi'(u)\Delta u - c\phi'(u)\,|\nabla u|^2 \\ &= \nu(\Delta w - \phi''(u)\,|\nabla u|^2) - c\phi'(u)\,|\nabla u|^2 \\ &= \nu\Delta w - (\nu\phi''(u) + c\phi'(u))\,|\nabla u|^2 \end{aligned}$$

Clearly, we obtain the linear heat equation if ϕ solves the ODE $\nu\phi'' + c\phi' = 0$. Thus, set

$$\phi(y) = e^{-\frac{cy}{\nu}} \tag{4.13}$$

as the *Cole-Hopf transform*. More specifically, if u solves the nonlinear equation (4.12), then $w = e^{-\frac{cu}{\nu}}$ solves the diffusion equation IVP

$$\begin{cases} w_t - \nu\Delta w = 0, \quad x \in \mathbb{R}^d,\ t > 0 \\ w(x,0) = e^{\frac{-cu_0(x)}{\nu}}, \quad x \in \mathbb{R}^d. \end{cases} \tag{4.14}$$

We recall that the unique solution for this problem is given by the convolution with the heat kernel

$$w(x,t) = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\nu t}} e^{-\frac{cu_0(x)}{\nu}}\ dy$$

and hence by the Cole-Hopf transform we may recover u by

$$u(x,t) = -\frac{\nu}{c}\log(w(x,t)) = -\frac{\nu}{c}\log\left(\frac{1}{(4\pi\nu t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\nu t}} e^{-\frac{cu_0(x)}{\nu}}\ dy\right). \tag{4.15}$$

We can now proceed to use the Cole-Hopf transform to solve the 1D viscous Burgers' equation on the whole real line. Our system of interest is

$$\begin{cases} \partial_t u + u\partial_x u - \nu\partial_{xx}u = 0,\ x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{4.16}$$

since $-\partial_{xx} = \Lambda^2$ in 1D. Our goal is to solve this equation using the above procedure. The first step is to re-write equation (4.16) into the form of the parabolic PDE (4.12). We begin with the not so obvious change of variables

$$h(x,t) \equiv \int_{-\infty}^{x} u(y,t)dy. \tag{4.17}$$

Applying this transformation is straightforward for the linear terms of (4.16). For the nonlinear term $u\partial_x u$, we re-write it in conservation form as $\partial_x(u^2/2)$ so that

$$\int_{-\infty}^{x} \partial_x(u(y,t)^2/2)dy = \frac{d}{dx}\int_{-\infty}^{x}(u(y,t)^2/2)dy = u(x,t)^2/2 = \frac{1}{2}(\partial_x h)^2$$

where $u(x,t) = \partial_x h$ is obvious from (4.17) and the fundamental theorem of calculus. Then, the transformed PDE is

$$\begin{cases} \partial_t h + \frac{1}{2}(\partial_x h)^2 - \nu \partial_{xx} h = 0, \ x \in \mathbb{R}, \ t > 0 \\ h(x,0) = \int_{-\infty}^{x} u_0(\eta) \ d\eta \end{cases} \tag{4.18}$$

which indeed is of the form (4.12) with $c = 1/2$. Using the formula (4.15), the solution for the auxiliary variable h is

$$h(x,t) = -2\nu \log\left(\frac{1}{(4\pi\nu t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4\nu t}} e^{-\frac{\int_{-\infty}^{y} u_0(\eta) \ d\eta}{2\nu}} \ dy \right).$$

Finally, taking a spatial derivative and canceling terms yields the exact solution to (4.16)

$$u(x,t) = \frac{\displaystyle\int_{\mathbb{R}} \frac{x-y}{t} \, e^{\frac{-|x-y|^2}{4\nu t} - \frac{\int_{-\infty}^{y} u_0(\eta) d\eta}{2\nu}} \ dy}{\displaystyle\int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4\nu t} - \frac{\int_{-\infty}^{y} u_0(\eta) d\eta}{2\nu}} \ dy} \quad \forall x \in \mathbb{R}, \ t > 0. \tag{4.19}$$

4.2.2 Numerical Study of Burgers' Equation

We begin a numerical study with a simple approach by implementing the (1,2) accurate forward-time central-space (FTCS) finite difference scheme to numerically solve the 1D viscous Burgers' equation (4.16) in conservative flux form

$$\begin{cases} u_t + \partial_x\left(\frac{1}{2}u^2\right) + \nu \Lambda^\alpha u = f(x,t), \ x \in [0,L), \ t \in (0,T], \ \alpha \in (0,2] \\ u(0,t) = u(L,t), \ t \in (0,T] \\ u(x,0) = u_0(x) \end{cases} \tag{4.20}$$

but with periodic boundary conditions and external forcing. Since $x \in \mathbb{R}$, the fractional Laplacian is really $\Lambda^\alpha = (-\partial_{xx})^{\frac{\alpha}{2}}$. These baseline finite difference results for the case $\alpha = 2$ will be used to ensure that the eventual pseudo-spectral method is implemented correctly.

We begin by discretizing the space $(0,L) \times (0,T)$ into a grid of equally spaced points (x_j, t^n), $0 = x_0 < x_1 < \ldots < x_{N_x - 1} < x_{N_x} = 1$ and $0 = t^0 < t^1 < \ldots < t^{N_t - 1} < t^{N_t} = T$ where N_x is the number of points in space, N_t is the number of time steps, j is the spatial index, and n is the temporal index (time level). It is well known from the *von Neumann stability analysis* that the time step for FTCS for parabolic problems in d-dimensions is limited by the CFL-type stability condition [60]

$$\nu \frac{k}{h^2} \leq \frac{1}{2d}$$

where $k = t^{i+1} - t^i$ is the time step and $h = x_{i+1} - x_i$ is the mesh size of the discretization. As a buffer, we choose the time step

$$k = 0.95\left(\frac{h^2}{2\nu}\right) \tag{4.21}$$

to avoid numerical blow up of the computed approximation to the solution. In the FTCS scheme, the time derivative is discretized in the Forward-Euler sense,

$$\frac{\partial u}{\partial t}(x_j, t^n) \approx \frac{\partial V}{\partial t}(x_j, t^n) = \frac{V_j^{n+1} - V_j^n}{k} \tag{4.22}$$

where $V(x_j, t^n) = V_j^n \approx u(x_j, t^n)$ is the numerical solution to the discretized difference equations at the grid point (x_j, t^n). Forward-Euler converges as $\mathcal{O}(k)$. Similarly,

$$\frac{\partial F}{\partial x}(x_j, t^n) \approx \frac{F_{j+1}^n - F_{j-1}^n}{2h} \tag{4.23}$$

where $F(u) = \frac{1}{2}u^2$ is the nonlinear flux and

$$\frac{\partial^2 u}{\partial x^2}(x_j, t^n) \approx \frac{\partial^2 V}{\partial x^2}(x_j, t^n) = \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{h^2} \tag{4.24}$$

are the central difference approximations to the spatial derivatives; both are second-order accurate $(\mathcal{O}(h^2))$ [60, 66]. Plugging into the PDE, we obtain

$$\frac{V_j^{n+1} - V_j^n}{k} + \frac{F_{j+1}^n - F_{j-1}^n}{2h} - \nu\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{h^2} = f_j^n$$

where $f_j^n = f(x_j, t^n)$ is the forcing. We define the numbers $\lambda = \frac{k}{h}$ and $\mu = \frac{k}{h^2}$ and rearrange to obtain the explicit form

$$V_j^{n+1} = V_j^n - \frac{\lambda}{2}(F_{j+1}^n - F_{j-1}^n) + \nu\mu(V_{j+1}^n - 2V_j^n + V_{j-1}^n) + kf_j^n. \tag{4.25}$$

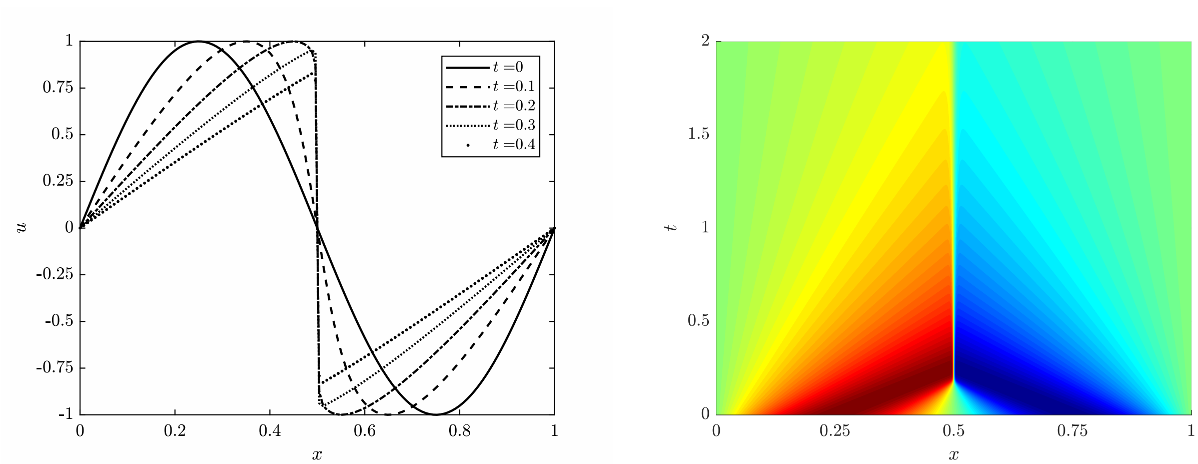
To treat the periodic boundary conditions, a similar procedure is applied. For the left "boundary" $x = 0$, $j = 0$ and we apply the FTCS scheme at $j = 0$ and make use of the periodicity $u(x_j, t^n) = u(x_{N_x+j}, t^n)$ for all j:

$$V_0^{n+1} = V_0^n - \frac{\lambda}{2}(F_1^n - F_{N_x-1}^n) + \nu\mu(V_1^n - 2V_0^n + V_{N_x-1}^n) + kf_0^n. \tag{4.26}$$

In summary, the numerical solution to can be obtained by solving the linear system of difference equations (DE)

$$\begin{cases} \text{DE:} \quad V_j^{n+1} = V_j^n - \frac{\lambda}{2}(F_{j+1}^n - F_{j-1}^n) + \nu\mu(V_{j+1}^n - 2V_j^n + V_{j-1}^n) + kf_j^n \\ \text{BC:} \quad V_0^{n+1} = V_0^n - \frac{\lambda}{2}(F_1^n - F_{N_x-1}^n) + \nu\mu(V_1^n - 2V_0^n + V_{N_x-1}^n) + kf_0^n \\ \text{IC:} \quad V_j^0 = u_0(x_j). \end{cases} \tag{4.27}$$

For example, prescribing the two mode sine wave $u_0(x) = \sin(2\pi x)$ in the Burgers' system (4.20) leads to the well-known shock behavior at $x = 0.5$ (Figure 4.1a). The advection term causes $u(x,t)$ to progressively steepen with time. The viscosity term provides enough dissipation to prevent a singularity here, as we know that the solution is smooth for all time. For this computation, we set $f(x,t) = 0, L = 1, T = 2, \nu = 10^{-3}$, and $h = \frac{1}{500}$. To investigate the behavior of the solution for



(a) Time evolution of $u(x,t)$ into a shock.

(b) Contours of characteristic curves in x-t plane.

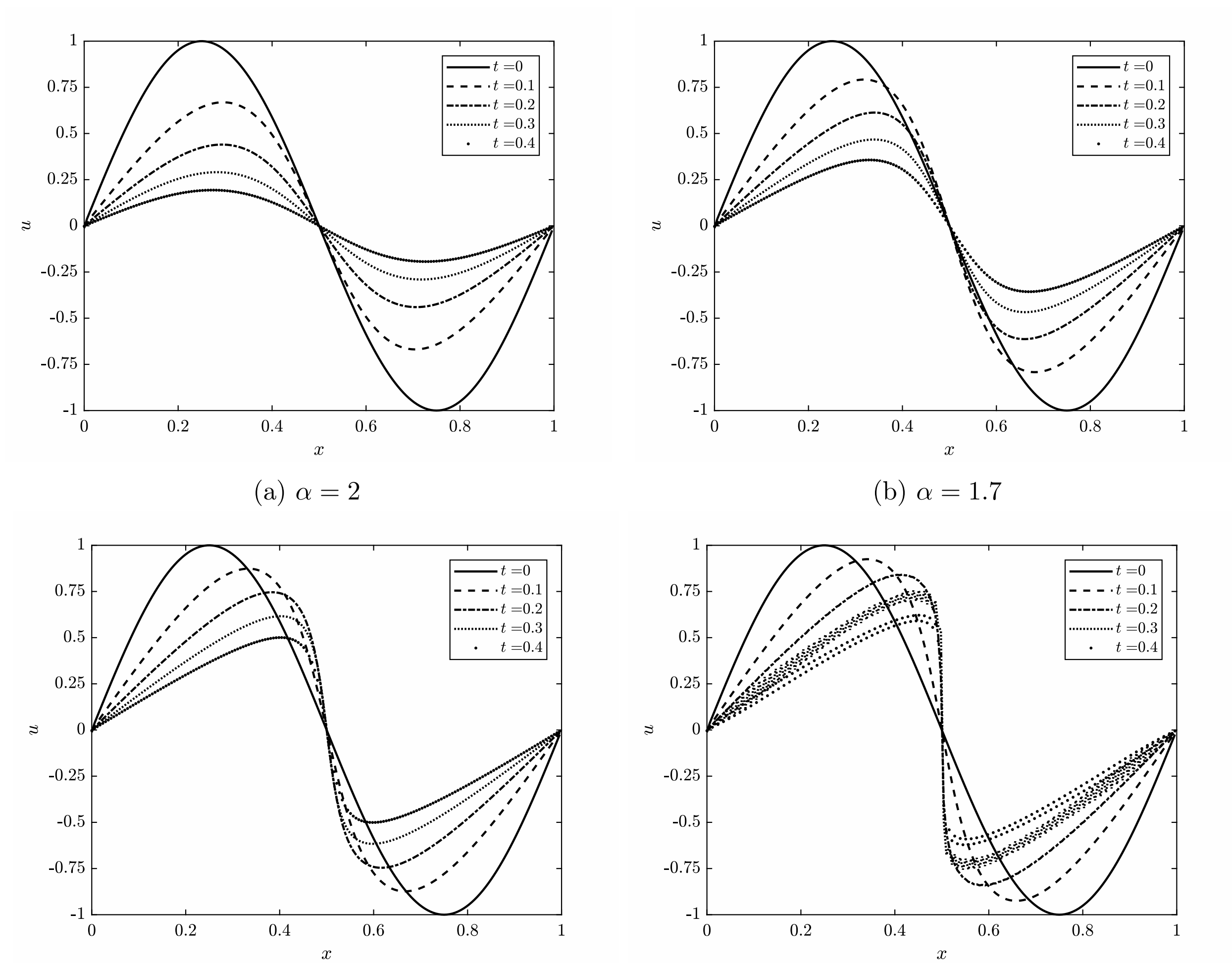
Figure 4.1: FTCS solution of Burgers' equation with $u_0(x) = \sin(2\pi x)$.

other $\alpha \neq 2$, we must resort to the pseudo-spectral method since finite differences alone cannot account for the nonlocal operator Λ^α.

Our implementation of the spectral method follows exactly the description in subsection 3.2.1. However, since the Burgers' equation is nonlinear, this time we do not have an exact solution in Fourier space. We apply the Discrete Fourier Transform to the Burgers' problem (4.20) with N Fourier modes and $f(x,t) = 0$ to obtain the system of N ODE for each wavenumber k:

$$\begin{cases} \frac{d}{dt}\hat{u}(t) = -ik\widehat{F(u)}(t) - \nu\,|k|^\alpha\,\hat{u}(t), \quad t \in (0,1],\ \alpha \in (0,2] \\ \hat{u}(0) = \widehat{u_0}(k) \end{cases} \tag{4.28}$$

where $\hat{u}(t) = \hat{u}(k,t)$ means $\text{FFT}(u(x,t))$ and $F(u) = \frac{1}{2}u^2$ is the flux. This system is solved with the well known `ode45` (fourth-order with adaptive time-stepping) integrator in Matlab. We recover the solution $u(x,t)$ in the physical space by taking the inverse Discrete Fourier Transform of the complex-valued matrix $\hat{u}(k,t)$ along the wavenumber coordinate. The 1D pseudo-spectral method allows us to obtain the same numerical solution as in Figure 4.1 by setting $\alpha = 2$, which confirms that the implementation is correct.

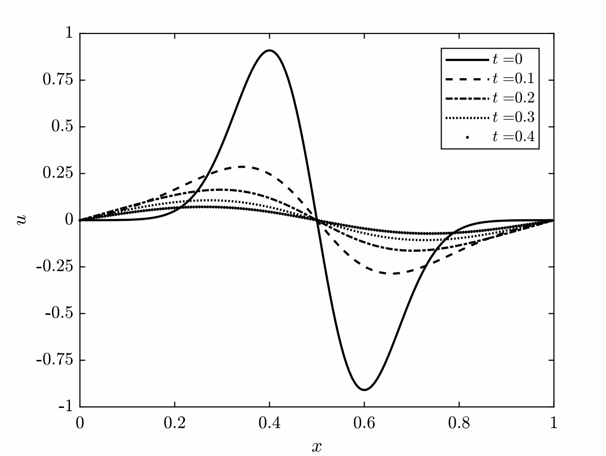


(a) $\alpha = 2$ (b) $\alpha = 1.7$

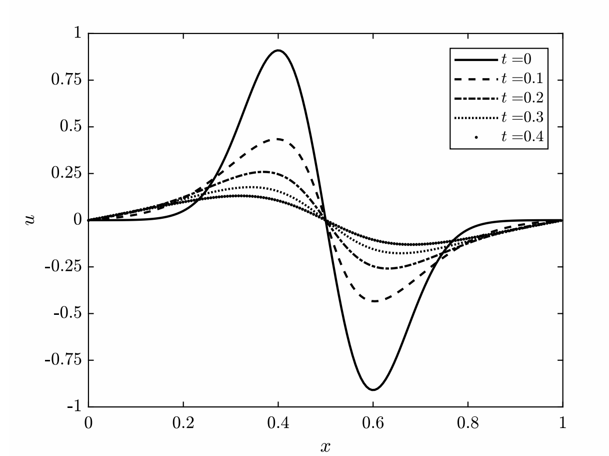
(c) $\alpha = 1.4$ (d) $\alpha = 1.1$

Figure 4.2: Case 1, Solution profiles of fractal Burgers' equation with $u_0(x) = \sin(2\pi(x))$ and varied α.

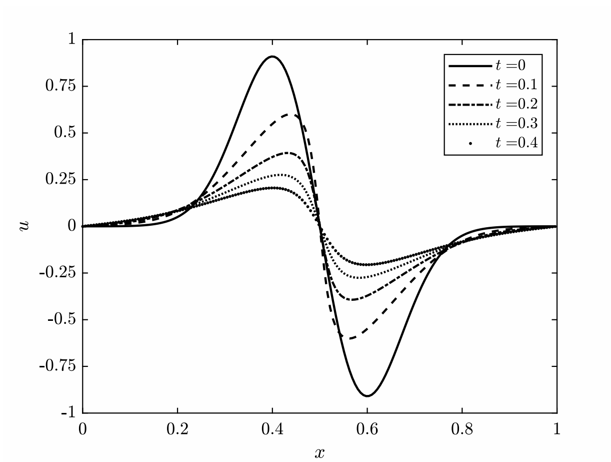
The goal now is to describe the behavior of the solution $u(x,t)$ as the parameter α is decreased. It is well known that values of α below the critical dissipation threshold $\alpha = 1$ may induce blow up depending on the initial condition. Hence, we choose ν large ($\nu = 10^{-1}$) and decrease α in the



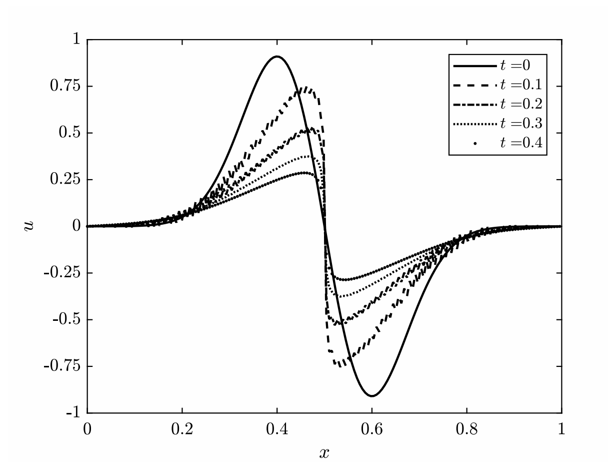
(a) $\alpha = 2$



(b) $\alpha = 1.7$

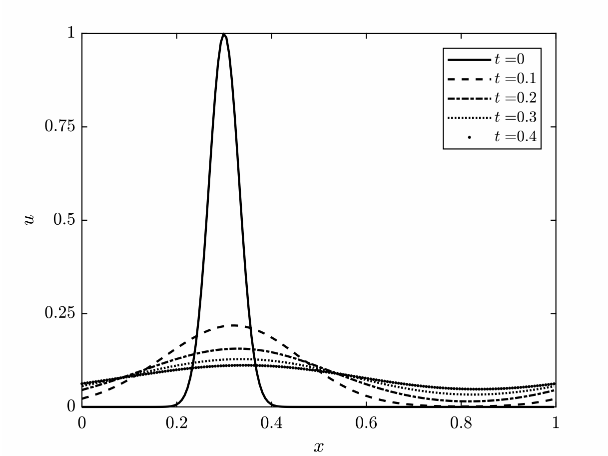


(c) $\alpha = 1.4$

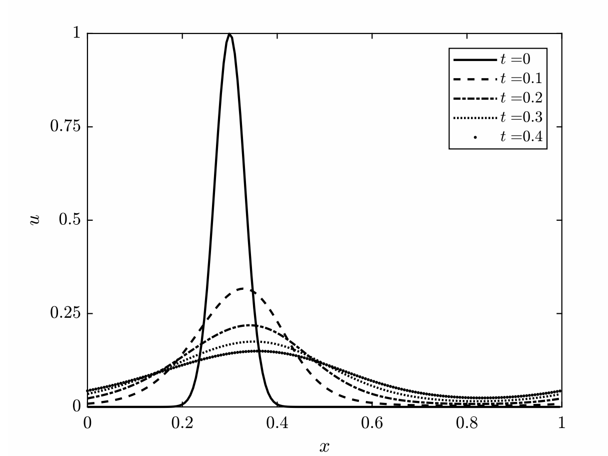


(d) $\alpha = 1.1$

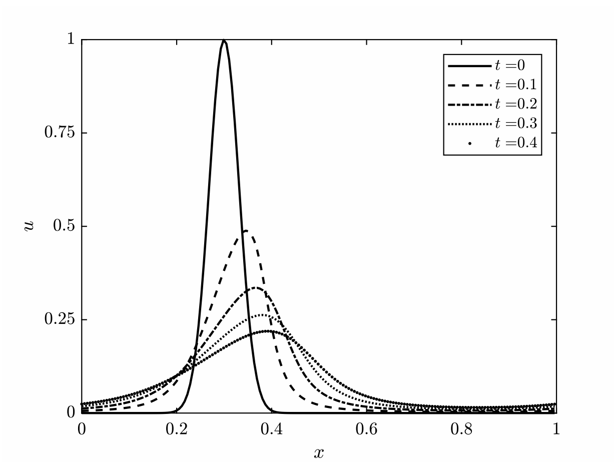
Figure 4.3: Case 2, Solution profiles of fractal Burgers' equation with $u_0(x) = -15(x-0.5)e^{-50(x-0.5)^2}$ and varied α.



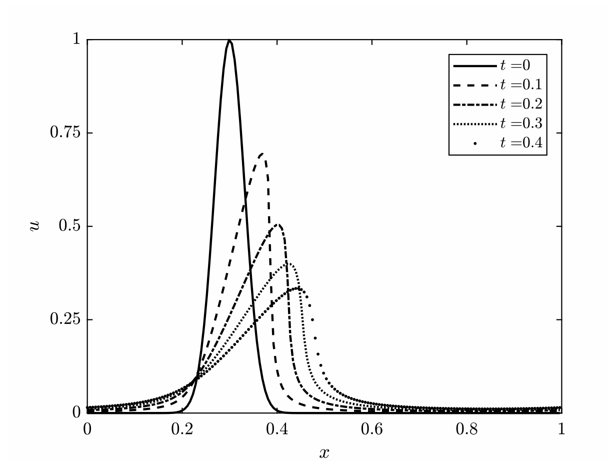
(a) $\alpha = 2$



(b) $\alpha = 1.7$



(c) $\alpha = 1.4$



(d) $\alpha = 1.1$

Figure 4.4: Case 3, Solution profiles of fractal Burgers' equation with $u_0(x) = e^{-500(x-0.3)^2}$ and varied α.

following numerical simulations.

The code is set up with $N = 512$ modes, spatial domain $\Omega = [0, L)$, $L = 1$, $\nu = 10^{-1}$, $T = 1$, and $\Delta t = 10^{-3}$. The dissipation exponent of the fractional Laplacian is varied as $\alpha \in \{2, 1.7, 1.4, 1.1\}$. The initial conditions for three numerical experiments are:

$$\begin{aligned}
&\text{Case 1: } u_0(x) = \sin(2\pi(x)/L)\\
&\text{Case 2: } u_0(x) = -15(x-0.5)e^{-50(x-0.5)^2}\\
&\text{Case 3: } u_0(x) = e^{-500(x-0.3)^2}
\end{aligned}$$

In the results presented in Figures 4.2, 4.3, 4.4, we observe the influence of convection competing with diffusion. At early times, the solutions steepen rapidly and develop sharp gradients. After some time, diffusion begins to win back and regularizes these sharp interfaces with a smoothing effect. In all three cases, the effect of decreasing α appears to be twofold. First, the amount of dissipation decreases as indicated by the decreased damping of the initial condition. This is especially evident in Figure 4.4. Second, less regularization leads to sharper interfaces that develop in the solution due to the nonlinear convection term in the PDE. This natural tendency to form shocks in the absence of diffusion is enhanced as α decreases.

Setting α near one from above for both Case 1 and Case 2, such as $0 < \alpha - 1 < 0.1$, causes the numerical solution to blow up or develop unphysical high amplitude oscillations. Of course, $\alpha < 1$ facilitates the same behavior. We begin to observe such numerical artifacts in the plots 4.2d and 4.3d with $\alpha = 1.1$. These results suggest that while values of $\alpha < 2$ may be more physical for some problems, larger α leads to nicer solutions. One would expect values of $\alpha > 2$ to damp solutions at a faster rate. Numerically, this could not be tested using the current implementation of the spectral method for the Burgers' equation because of issues arising in computing the Discrete Fourier Transforms. Relatedly, in the case of the generalized heat equation it is known that no maximum principle holds for α outside of the range $(0, 2]$. For completeness, we note from some short trial runs that Case 3 can actually handle $\alpha = 1$, but not values of α below $\alpha = 0.9$.

4.3 Nonlocal Burgers'-type PDE

We investigate a nonlinear advection-diffusion equation of the Burgers' type that admits a *nonlocal velocity* (wave speed) in the advection term given by the Hilbert transform. The equation exhibits a strong tendency to form singularities or at least sharp cusps, perhaps due to the nature of the singular kernel in the Hilbert transform itself.

Singular kernels have applications outside of the mathematical theory. A well-known use of such a construct is in the realms of electromagnetism and fluid dynamics, where we have the *Bio-Savart law*. For example, consider the 2D fractional Navier-Stokes vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = 0, \quad 0 < \alpha < 2 \\ u = K_2 * \omega \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{4.29}$$

where $x \in \mathbb{R}^2$, $\omega = \nabla \times u$ is the vorticity, and K_2 is the *Bio-Savart kernel* given by

$$K_2(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2} = \frac{1}{2\pi} \frac{(-z_2, z_1)}{|z|^2}. \tag{4.30}$$

This kernel essentially inverts the curl operator. Thus, we can recover the velocity field of the fluid flow via the 2D version of the Bio-Savart law by convolving equation (4.30) with the vorticity

$$u = K_2 * \omega = \int_{\mathbb{R}^2} K_2(x-y)\omega(y)dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(-x_2 - y_2, x_1 - y_1)}{|x-y|^2} \omega(y)dy.$$

This is just one of many examples of physically significant singular kernels; another is in [70], for instance.

4.3.1 Hilbert Transform

We begin this section with the study of the Hilbert transform, a famous nonlocal operator in one dimension.

Definition 4.3.1.1. The **Hilbert transform** is a 1D *singular integral operator* acting on "nice enough" functions f [see 67, for details]. We represent this nonlocal linear operator by the convolution

$$Hf(x) \equiv \frac{1}{\pi t} * f(t) = -\frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt = \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt \tag{4.31}$$

We derive the Fourier symbol for H, which will be used in subsequent numerical methods for solving the nonlocal advection-diffusion PDE modified with a Hilbert transform term.

Lemma 4.3.1.2. *Let $f \in \mathcal{S}(\mathbb{R})$, the Schwartz space [see 67] on $\mathbb{R}$. Then,*

$$\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi) \tag{4.32}$$

is the Fourier transform of $Hf(x)$.

Proof. By properties of the Fourier transform,

$$\widehat{Hf}(\xi) = \widehat{\frac{1}{\pi x}}\hat{f}.$$

So, we must show that

$$\widehat{\frac{1}{\pi x}} = -i\,\mathrm{sgn}(\xi).$$

For simplicity in the computations that follow, for this proof only we define the forward Fourier transform as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx.$$

Then, in the principle value sense when necessary,

$$\begin{aligned}\widehat{\frac{1}{\pi x}}(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{1}{x}dx \\ &= \frac{1}{\pi}\left(\int_{-\infty}^{\infty}\frac{\cos(x\xi)}{x}dx - i\int_{-\infty}^{\infty}\frac{\sin(x\xi)}{x}dx\right) \\ &= -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin(x\xi)}{x}dx\end{aligned}$$

since $\frac{\cos(x\xi)}{x}$ is odd. We proceed with a change of variables, letting $t = \xi x \implies dt = \xi dx$. So for $\xi > 0$,

$$\int_{-\infty}^{\infty}\frac{\sin(x\xi)}{x}dx = \int_{-\infty}^{\infty}\frac{\sin(t)}{t}dt.$$

We evaluate this integral over the half-range $\mathbb{R}^+$ (in which it is commonly called the *Dirichlet integral*), using the Laplace transform.

Definition 4.3.1.3. The forward **Laplace transform** is defined as the linear operator

$$\mathcal{L}f(s) = \int_0^{\infty} e^{-st} f(t)dt \tag{4.33}$$

for $f \in L^1(0,\infty),\ s \geq 0$.

Using a result found in any standard Laplace transform table, we have

$$\begin{aligned}\mathcal{L}\left(\frac{\sin(t)}{t}\right)(s) &= \int_s^\infty \mathcal{L}(\sin(t))(\tau)d\tau \\ &= \int_s^\infty \frac{1}{1+\tau^2}d\tau \\ &= \lim_{R\to\infty} \arctan(\tau)|_{\tau=s}^R \\ &= \frac{\pi}{2} - \arctan(s).\end{aligned}$$

So,

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right)(s) = \int_0^\infty e^{-st}\frac{\sin(t)}{t}dt = \frac{\pi}{2} - \arctan(s)$$

and by setting $s = 0$, we obtain

$$\int_0^\infty \frac{\sin(t)}{t}dt = \frac{\pi}{2}. \tag{4.34}$$

Since $\frac{\sin(t)}{t}$ is even, we have (undoing the change of variables)

$$\int_0^\infty \frac{\sin(t)}{t}dt = \int_0^\infty \frac{\sin(\xi x)}{x}dx = \int_{-\infty}^0 \frac{\sin(\xi x)}{x}dx.$$

Define $w(\xi) := \int_0^\infty \frac{\sin(\xi x)}{x}dx$. We consider three cases for $\xi \in \mathbb{R}$.

1. $\xi > 0$: We have

$$w(\xi) = \int_0^\infty \frac{\sin(\xi x)}{x}dx = \frac{\pi}{2}$$

as shown above.

2. $\xi = 0$: Clearly, $w(0) = 0$.

3. $\xi < 0$: We consider

$$w(\xi) = -w(-\xi) = -\int_0^\infty \frac{\sin(-\xi x)}{x}dx = -\frac{\pi}{2}.$$

Hence $\forall \xi \in \mathbb{R}$,

$$\int_0^\infty \frac{\sin(\xi x)}{x}dx = \frac{\pi}{2}\operatorname{sgn}(\xi) \tag{4.35}$$

so we must have

$$\int_{\mathbb{R}} \frac{\sin(\xi x)}{x}dx = 2\int_0^\infty \frac{\sin(\xi x)}{x}dx = \pi \operatorname{sgn}(\xi). \tag{4.36}$$

Thus,

$$\widehat{\frac{1}{\pi x}}(\xi) = -\frac{i}{\pi}(\pi \operatorname{sgn}(\xi)) = -i \operatorname{sgn}(\xi)$$

as required. □

4.3.2 Hilbert-modified Burgers' Equation

The Hilbert transform has been used in simple 1D models to form an analogy with higher dimensional vortex sheet motion in incompressible fluid flows [30]. See [48] for a review and [14, 63] for another detailed example. The original origin for the outburst of these basic models arise from the seminal paper of Constantin, Lax, and Majda [18], where the equation $\partial_t \omega + (-H\omega)\omega = 0$—aptly named the *Constantin-Lax-Majda equation*—served as a analog in many respects to the full 3D Euler vorticity equation (e.g., vortex stretching, singularity formation and breakdown). We explore the 1D model of nonlocal and nonlinear advection first studied in [5] by Baker, Li, and Morlet, but with fractional dissipation:

$$\begin{cases} \partial_t \theta + u \partial_x \theta + \nu \Lambda^\alpha \theta = 0, \;\; x \in \mathbb{R}, \;\; t > 0, \;\; \alpha \in [0, 2] \\ u = -H\theta \\ \theta(x, 0) = \theta_0(x) \end{cases} \tag{4.37}$$

where $\nu > 0$ and H is the Hilbert transform defined in Equation (4.31). We refer to this equation as the **Hilbert-modified Burgers' Equation**. The velocity of the transported quantity $\theta(x, t)$ is nonlocal, given by the negative Hilbert transform $-H\theta$, and nonlinear, depending on θ itself. This 1D model is known to share salient characteristics to higher dimensional fluid PDE such as equations of the quasi-geostrophic type, the 3D Euler vorticity equation, and the Birkhoff-Rott vortex sheet equation [14, 25].

In [25], Córdoba, Córdoba, and Fontelos showed that solutions exist globally in time for $1 < \alpha \leq 2$, whereas the case $\nu = 0$ exhibited singularity formation in finite time. Later, Li and Rodrigo proved in [55] that solutions blow up in finite time for $0 \leq \alpha < \frac{1}{4}$. The global well-posedness results require the use of a nonlocal maximum principle similar to that in [24]; Dong and Li prove this for the case $\alpha = 2$, where the nonlocal approach does not apply since the operator Λ^α becomes local (and thus is no longer represented as an integral). Baker, Li, and Morlet prove existence and uniqueness for the nonlocal model (4.37) but with periodic boundary conditions. A wealth of other results exist in the literature for this and other Burgers'-type equations modified with varying Hilbert transform fluxes, e.g.: [25, 32, 62].

Remark 4.3.2.1. Some care is needed when browsing the literature on these models, as the sign of the term $-H\theta\partial_x\theta$ depends on the definition of H. That is, some authors define the Hilbert transform with the opposite sign of (4.31). Thus, Equation (4.37) is sometimes instead seen as $\partial_t\theta + (H\theta)\partial_x\theta + \nu\Lambda^\alpha\theta = 0$. It is evident that this just changes the direction of the wave-like propagation of the solution (since it is still a transport-diffusion equation). For clarity, the change in sign simply maps $\theta \longrightarrow -\theta$. We choose the negative sign variant $-H\theta\partial_x\theta$ because it produces solutions that evolve more physically (e.g., damping to smaller amplitudes instead of damping to higher amplitudes) [see 25].

4.3.3 Numerical Study of Nonlocal Burgers' Equation

The numerical simulation of the Hilbert-modified Burgers' Equation has been studied by a few authors for some simple initial conditions and $\alpha = 2$. See [5, 61, 62] for detailed simulation parameters and corresponding plots.

Here, we continue along this vein by solving Equation (4.37) on a periodic domain with the pseudo-spectral methods of subsection 3.2.1, but now with specific values of α to examine the influence of fractional dissipation. In particular, we would like to determine the extent of viscous regularization due to α and explore results even for $2 \leq \alpha \leq 3$ which represents enhanced dissipation. The periodic problem reads

$$\begin{cases} \theta_t + (-H\theta)\theta_x + \nu\Lambda^\alpha\theta = 0,\ x \in [0, L),\ t \in (0, T],\ \alpha \in [0, 3] \\ u(0, t) = u(L, t),\ t \in (0, T] \\ u(x, 0) = u_0(x) \end{cases} \tag{4.38}$$

where $\nu > 0$ and the Hilbert transform H is given by

$$H\theta = \frac{1}{\pi}\,\text{p. v.} \int_{-\infty}^{\infty} \frac{\theta(y)}{x - y} dy.$$

Again employing the 1D pseudo-spectral method (3.2.1), the Discrete Fourier Transform is applied to the nonlocal, nonlinear problem (4.38) with $N = 512$ Fourier modes. However, this system cannot be integrated in Fourier space due to the divergence form product of $H\theta$ and θ_x. Hence, we convert each term into the Fourier space via the DFT, compute derivatives and the Hilbert transform using corresponding Fourier symbols, and then convert back into physical space with the inverse DFT before solving the ODE. We obtain the system of N nonlinear ODE:

$$\begin{cases} \frac{d}{dt}\theta(t) = (H\theta)^*\theta_x^* - \nu(\Lambda^\alpha\theta)^*, \quad t \in (0, T],\ \alpha \in [0, 3] \\ \theta(0) = \theta_0 \end{cases} \tag{4.39}$$

where $(H\theta)^* = \text{IFFT}(-i\,\text{sgn}(k)\,\text{FFT}(\theta))$ and $(\Lambda^\alpha\theta)^* = \text{IFFT}(|k|^\alpha\,\text{FFT}(\theta))$. This system is solved with the well known `ode15s` integrator (a variable-order method for highly singular, stiff problems) in Matlab.

Three initial conditions are tested,

$$\text{Case 1: } u_0(x) = (1-(2x-1)^2)^2$$

$$\text{Case 2: } u_0(x) = -2(1+0.2\cos(2\pi x)+0.02\cos(10\pi x)+0.002\cos(18\pi x))+2.444$$

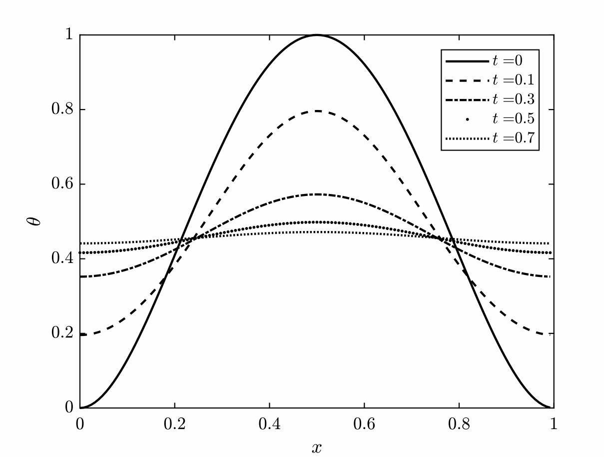
$$\text{Case 3: } u_0(x) = \begin{cases} 1-x, & x\in[0.25,0.75] \\ 0, & \text{else} \end{cases},$$

and numerical experiment parameters are prescribed as $L=1$, $T=2$, $\Delta t = 10^{-3}$, and $\nu = 0.05$.

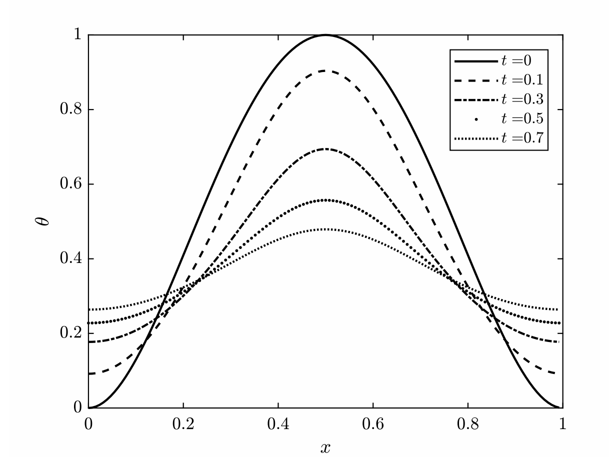
In Figures 4.5, 4.7, and 4.9, we observe that the effect of decreasing α on the solution is to form a cusp-like singularity feature at the maximum of θ. This behavior confirms the numerical simulations of other work [see 5, 61]. In the absence of viscosity ν (here, $\nu = 0.05$), the solution would blow up in finite time at this cusp location. The viscous regularization due to the fractional Laplacian term, even with small α, still serves to provide enough dissipation to smear out the sharp gradients that form. The contour plots 4.6, 4.8, and 4.10 visually illustrate this fact. Case 3 with $\alpha = 0.9$ (Figure 4.10d) is especially drastic, likely as a result of the piecewise initial condition that was prescribed.

In Figure 4.11, the gradients $\partial_x\theta$ are plotted at various times t for the most singular case $\alpha = 0.9$. It is clear that at the cusp locations, the derivative becomes extremely discontinuous with high magnitudes. In a way, this is reflective of the singular kernel defining the Hilbert transform itself. We would expect a vertical line in the limit $\alpha \to 0$ or $\nu \to 0$, in which $\partial_x\theta$ reaches infinity.

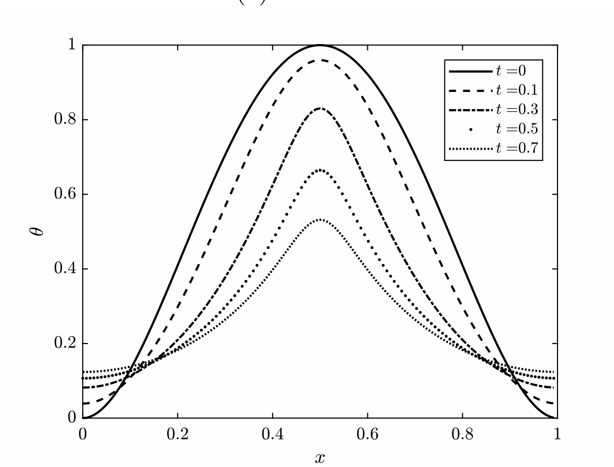
The trend of decreasing α is the same as that observed in the fractal Burgers' equation (see subsection 4.2.2); small α corresponds to less smoothing and greater potential for shock-like interfaces to develop. As expected, *increasing* α above 2 to 2.5 provided even an greater dissipative influence than did the ordinary Laplacian with $\alpha = 2$. Even in this nonlocal, nonlinear model, the fractional dissipation conforms to the theory presented in Chapter 2.



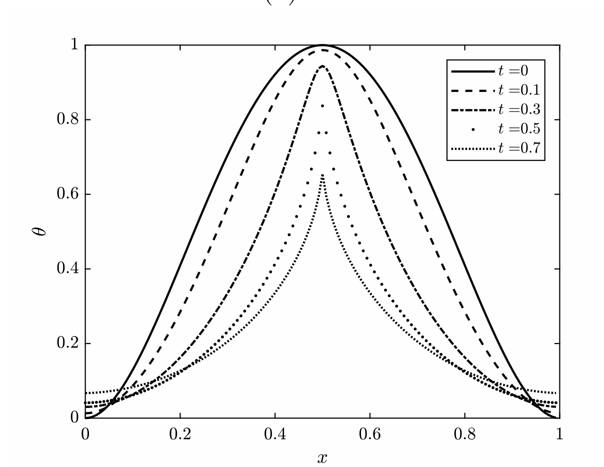
(a) $\alpha = 2.5$



(b) $\alpha = 2$

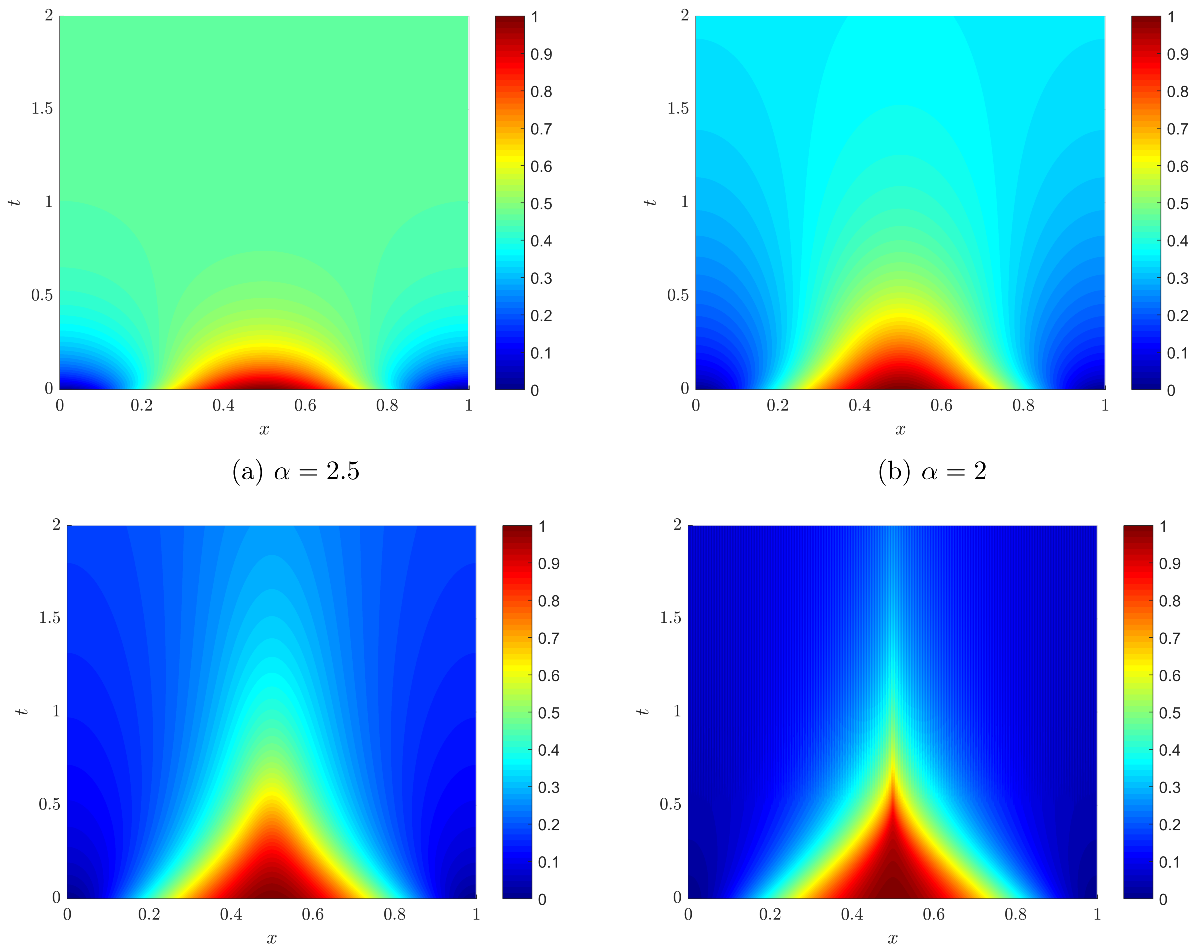


(c) $\alpha = 1.5$



(d) $\alpha = 0.9$

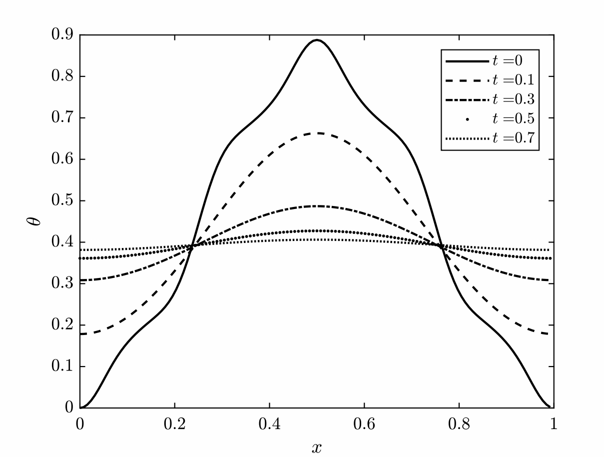
Figure 4.5: Case 1, Solution profiles of Hilbert-modified Burgers' equation with $u_0(x) = (1-(2x-1)^2)^2$ and varied α.



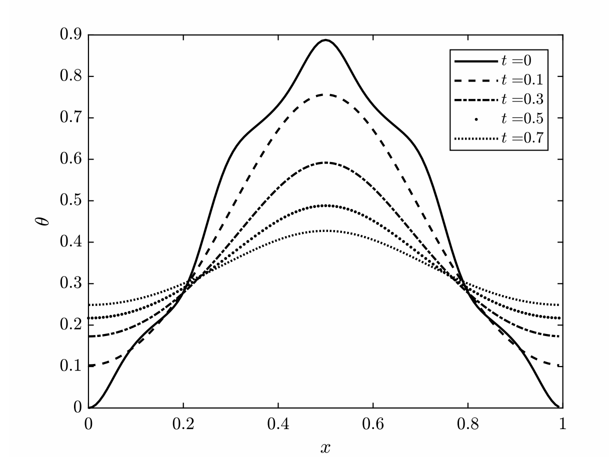
(a) $\alpha = 2.5$ (b) $\alpha = 2$

(c) $\alpha = 1.5$ (d) $\alpha = 0.9$

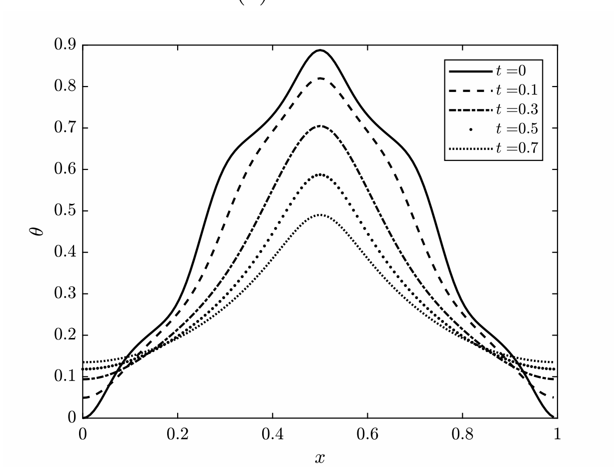
Figure 4.6: Case 1, Contours of Hilbert-modified Burgers' equation with $u_0(x) = (1 - (2x - 1)^2)^2$ and varied α.



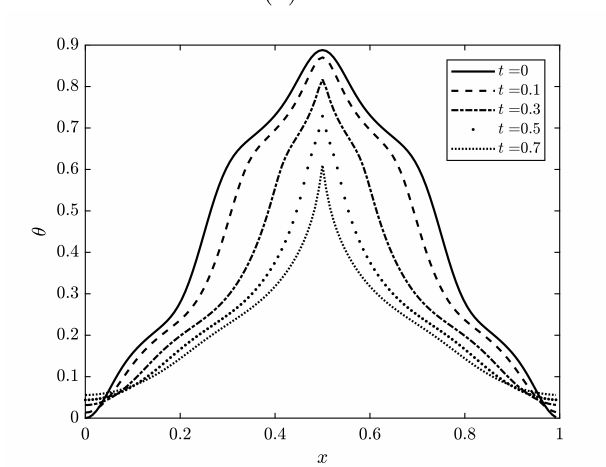
(a) $\alpha = 2.5$



(b) $\alpha = 2$

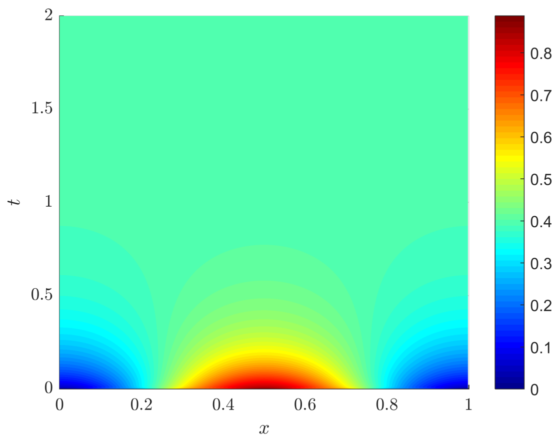


(c) $\alpha = 1.5$

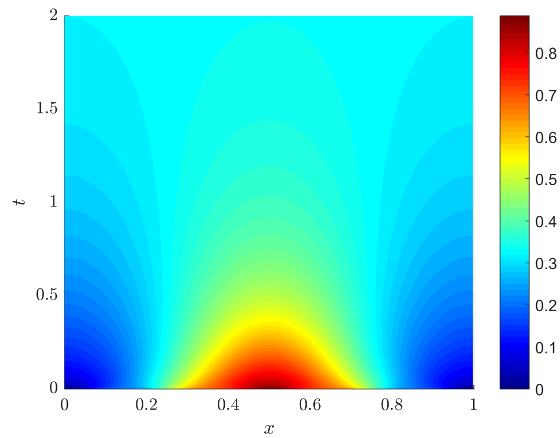


(d) $\alpha = 0.9$

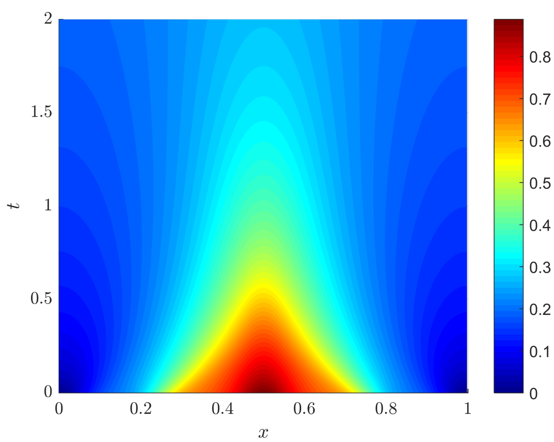
Figure 4.7: Case 2, Solution profiles of Hilbert-modified Burgers' equation with $u_0(x) = -2(1 + 0.2\cos(2\pi x) + 0.02\cos(10\pi x) + 0.002\cos(18\pi x)) + 2.444$ and varied α.



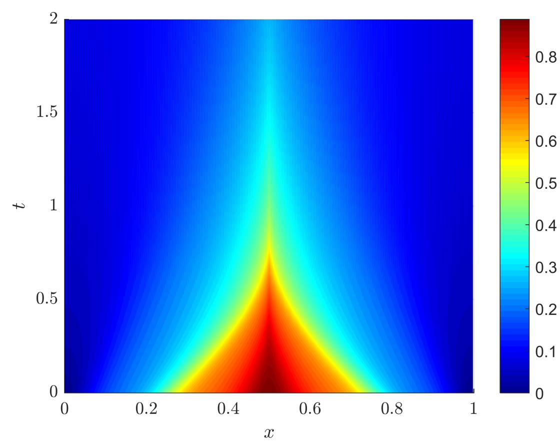
(a) $\alpha = 2.5$



(b) $\alpha = 2$

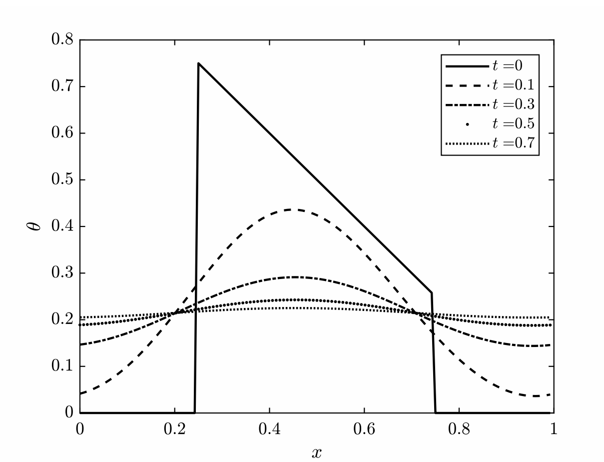


(c) $\alpha = 1.5$

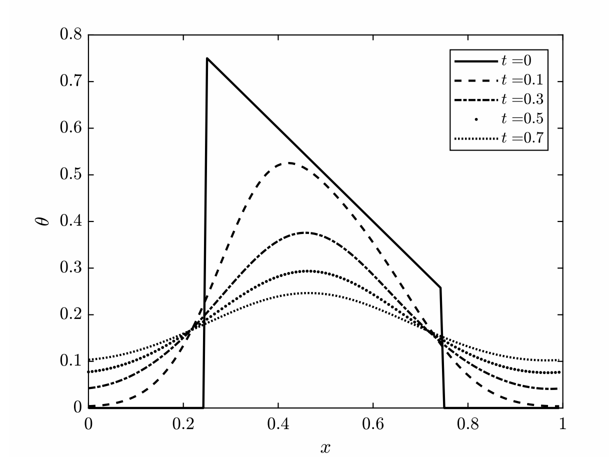


(d) $\alpha = 0.9$

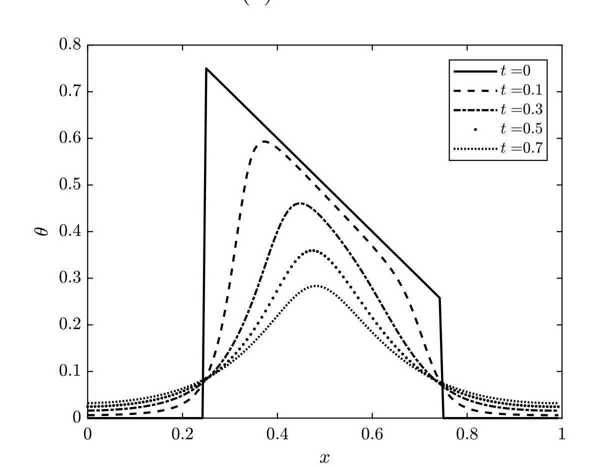
Figure 4.8: Case 2, Contours of Hilbert-modified Burgers' equation with $u_0(x) = -2(1+0.2\cos(2\pi x)+0.02\cos(10\pi x)+0.002\cos(18\pi x))+2.444$ and varied α.



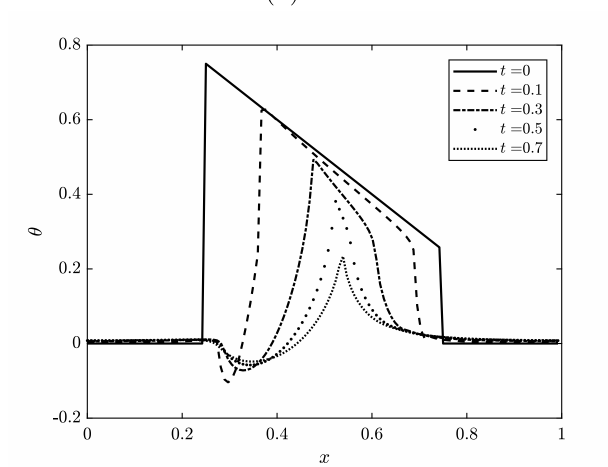
(a) $\alpha = 2.5$



(b) $\alpha = 2$

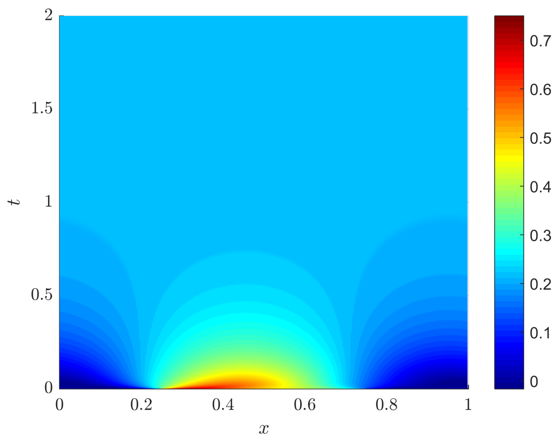


(c) $\alpha = 1.5$

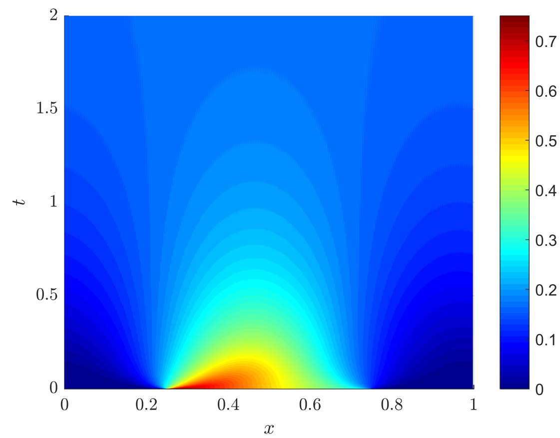


(d) $\alpha = 0.9$

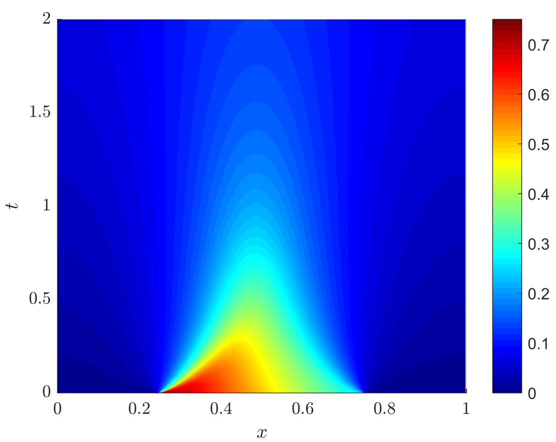
Figure 4.9: Case 3, Solution profiles of Hilbert-modified Burgers' equation with $u_0(x)$ defined piecewise linear and varied α.



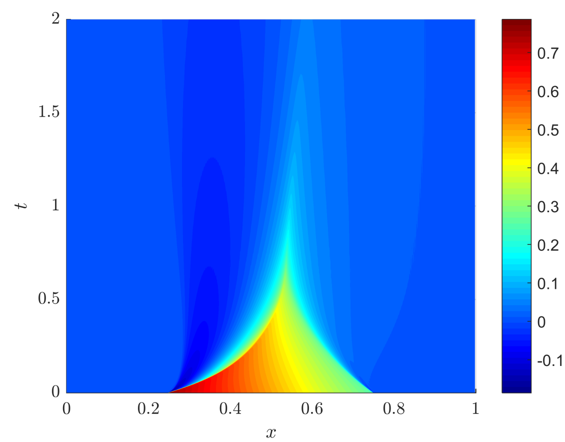
(a) $\alpha = 2.5$



(b) $\alpha = 2$

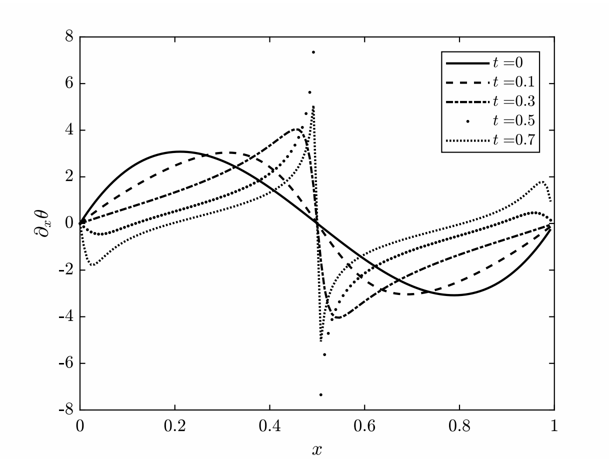


(c) $\alpha = 1.5$

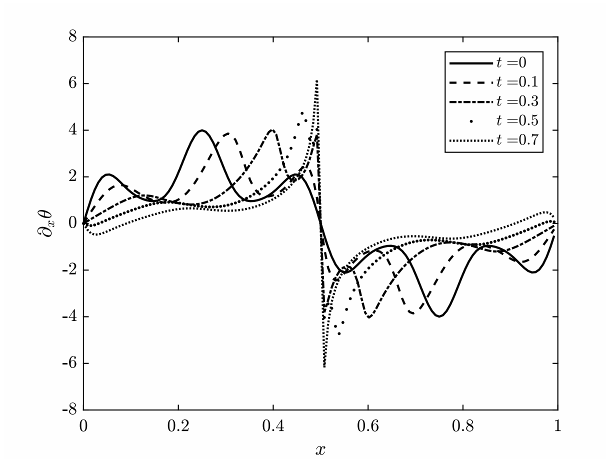


(d) $\alpha = 0.9$

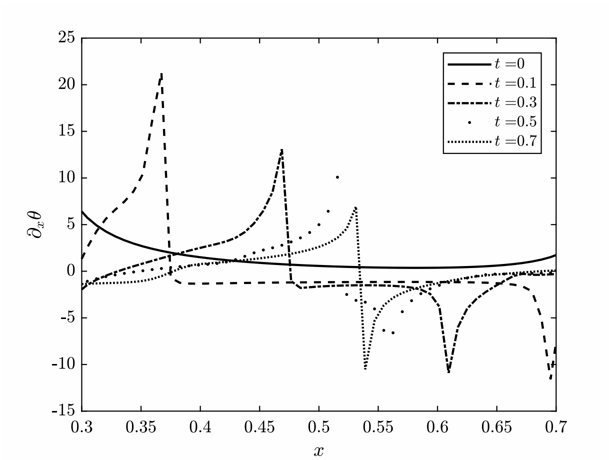
Figure 4.10: Case 3, Contours of Hilbert-modified Burgers' equation with $u_0(x)$ defined piecewise linearly and varied α.



(a) Case 1, $\theta_x(x,t)$



(b) Case 2, $\theta_x(x,t)$



(c) Case 3, $\theta_x(x,t)$

Figure 4.11: Comparison of gradient $\theta_x(x,t)$ at $\alpha = 0.9$ for the three test cases of the Hilbert-modified Burgers' equation.

Chapter 5

Conclusion

In summary, we have presented an overview of the theory of the fractional Laplacian operator in partial differential equations and numerically investigated the behavior of solutions to three important PDE modified with fractional dissipation. The fractional Laplacian $\Lambda^\alpha \equiv (-\Delta)^\alpha$, defined either through the Fourier transform or principal value integral in this work, induces a nonlocal dissipative effect that affects the regularity and structural characteristics of solutions. We posed the generalized heat equation $\partial_t u + \nu\Lambda^\alpha u = f$ and analyzed its fundamental solution, the fractional heat kernel, with both standard estimates and the semigroup approach. Several solutions for the Cauchy problem with simple initial data were discussed. We then compared the results from numerical simulations of the generalized heat equation to a decay estimate. This suggested that as α is decreased, the most salient diffusion and damping behavior of the solution occurs more slowly.

Turning to more complicated PDE involving transport terms, we studied the fractal Burgers' equation $\partial_t u + u\cdot\nabla u + \nu\Lambda^\alpha u = f$ and the Hilbert-modified Burgers' equation $\partial_t\theta + (-H\theta)\partial_x\theta + \nu\Lambda^\alpha\theta = 0$. These nonlinear PDE are prototypical examples of models exhibiting similar structure and behavior to more difficult equations that arise in the study of incompressible fluid flow and other physical systems. We performed numerical experiments with the pseudo-spectral method to solve these equations on a 1D periodic interval. Our results further suggest the influence of α's role in viscous regularization of solutions, namely, that smaller α imparts less dissipation and hence relinquishes some control to allow solutions to evolve in less desirable ways (e.g., development of shocks or sharper flow structures).

This thesis has only touched on a small subset of the vast theory of nonlocal operators and integral transformations in partial differential equations; there are certainly many other avenues for future research in the field of nonlocal and nonlinear PDE. On the one hand, development of highly accurate and reliable numerical methods for solving nonlocal equations would enrich the current state of applied mathematics. In particular, methods for fractional Laplacians on bounded domains would be of great interest to computational scientists in the various fields that anomalous diffusion and nonlocal interactions arise in frequently. On the other hand, the theory itself is not even fully developed for bounded domains, especially for challenging boundary conditions. Of interest in mathematical fluid dynamics, the Navier-slip boundary conditions continue to pose a challenge for fractional Laplacians and nonlocal equations. Other areas requiring more work are proofs of global existence, regularity, and blow up for PDE in the critical and supercritical regimes of the fractional exponent ($0 < \alpha \leq 1$).

Regardless of application area, recasting familiar or classical equations into their fractional counterparts is philosophically a valuable action to take. Studying a whole family of PDE with parameter α in the diffusion term $\Lambda^\alpha(\cdot)$ may unlock behavior or deep results that were previously masked by the assumption of locality. For instance, another equally important use for fractional Laplacians other than the topics presented in this thesis is in the global existence and regularity Millennium Prize Problem for the 3D Navier-Stokes equation (NSE) [40]. While proof has eluded many of the most eminent mathematicians of our time, progress has been made by generalizing the Navier-Stokes equations into a fractional form, similar to our discussion of the heat equation in Chapter 2. The Millennium problem asks the following: If the initial data is in a suitable Schwartz class of functions with finite L^2 norm, then does the 3D NSE (4.10) admit a global-in-time smooth solution with finite energy? The problem is still open, but it is known that the 3D NSE is *supercritical* in the sense that dissipation from the term $\nu\Delta u$ is not sufficient for global regularity and existence. However, if we consider the 3D fractional NSE

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^\alpha u \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{5.1}$$

where $\alpha > 0$, then the system (5.1) does indeed have a unique global-in-time solution for any $u_0 \in H^1(\mathbb{R}^3)$ provided that $\alpha \geq \frac{5}{4}$. See [15, 53, 76] for the work of the work of CIORANESCU and LIONS, LADYZHENSKAYA and SEREGIN, and WU. In essence, this recasting of the problem implies that a proof showing that $\alpha = 1$ is a sufficient exponent will solve the Millennium Prize Problem.

By incorporating fractional Laplacians into previously well studied models, perhaps similar breakthroughs will occur. This field is still in its infancy, at least on a mathematical timescale, and new results continue to appear on the arXiv and in high impact journals each year. Nonlocal operators undoubtedly have utility outside of pure mathematical theory, and it would be encouraging to see fractional Laplacians applied to more engineering problems and scientific studies in the near future.

Appendix A

A.1 Fourier Transform for L^p Functions

We need the following lemma before defining $\hat{f}$ for $f \in L^p(\mathbb{R}^d)$ or even the case $f \in L^2(\mathbb{R}^d)$:

Lemma A.1.0.1 (Weak version of Plancherel's Theorem). *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then*

$$\|\hat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$$

Notation A.1.0.2. We drop the $(\mathbb{R}^d)$ for the rest of this work; that is, $L^p = L^p(\mathbb{R}^d)$.

Definition A.1.0.3 (FT for L^2 functions). Let $f \in L^2$. Define $f_n(x) \in L^1 \cap L^2$ such that $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$. One choice would be to choose $f_n(x) = f(x) \cdot \chi_{B(0,n)}$, where χ is the indicator function and $B(0,n)$ is the ball centered at the origin **0** with radius n.

Claim A.1.0.4. $f_n \in L^1$ and $f_n \in L^2$

Proof.

$$\begin{aligned}
\int |f_n(x)|\, dx &= \int_{B(0,n)} |f(x)|\, dx \\
&\leq \left(\int_{B(0,n)} |f(x)|^2\, dx \right)^{\frac{1}{2}} \left(\int_{B(0,n)} 1 dx \right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2} |B(0,n)|^{\frac{1}{2}} \\
&< +\infty
\end{aligned}$$

Further, $f_n \in L^2$ trivially. □

Now, observe

$$\|f - f_n\|_{L^2}^2 = \int_{\mathbb{R}^d \setminus B(0,n)} |f(x)|^2\, dx \to 0 \text{ as } n \to \infty.$$

So, $\{f_n\}$ is a Cauchy sequence in L^2 (a sequence in a *Banach space* converges iff it is a Cauchy sequence), that is, $\|f_n - f_m\|_{L^2} \to 0$ as $n, m \to \infty \quad \forall n, m \in \mathbb{N}$. By Plancherel's Theorem,

$$\|f_n - f_m\|_{L^2} = \|\hat{f}_n - \hat{f}_m\|_{L^2},$$

which implies that $\hat{f}_n$ is Cauchy in the space L^2. Then, $\hat{f}_n$ has a limit in L^2, denoted by $\hat{f}$, because L^2 is a complete normed vector space (a Banach space). We define the **Fourier Transform** of $f \in L^2$ by the limit $\hat{f}$. The **Inverse Fourier Transform** of $f \in L^2$ follows similarly.

Remark A.1.0.5. To define the Fourier Transform of $f \in L^p \quad \forall p \in \mathbb{R}$ such that $1 < p < 2$, we use the **Hausdorff-Young Inequality**:

Lemma A.1.0.6 (Hausdorff-Young)**.** *Let $1 \le p \le 2$. Assume $f \in L^1 \cap L^p$. Let q be the dual exponent of p, $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\|\hat{f}\|_{L^q} \le C(d,p)\, \|f\|_{L^p}\,,$$

where (for $d = 2$), $C(2,p) = p^{\frac{1}{p}} q^{-\frac{1}{q}}$.

Definition A.1.0.7 (FT for L^p functions, where $1 < p < 2$)**.** For $f \in L^p$, define $f_n \in L^1 \cap L^p$, say $f_n = f \cdot \chi_{B(0,n)}$ and $f_n \to f$ in L^p. Then, f_n is Cauchy in L^p. By Hausdorff-Young (Lemma A.1.0.6),

$$\|\hat{f}_n - \hat{f}_m\|_{L^q} \le C(d,p)\, \|f_n - f_m\|_{L^p}\,.$$

Thus, $\hat{f}_n$ is Cauchy in L^q and has a limit in L^q, denoted by $\hat{f}$. So, we define $\hat{f}$ as the **Fourier Transform** of $f \in L^p$.

It is natural to think about the case $f \in L^p$ with $p > 2$. Here, f is actually treated as a *tempered distribution*, and then $\hat{f}$ is defined in the distributional sense. Such discussion is outside of the scope of this thesis, however.

A.2 More on Semigroups

There are more exotic and general semigroups than just the heat kernel or its fractional variant.

Definition A.2.0.1 (symmetric diffusion semigroup)**.** Let $X = L^2(\mathbb{R}^d)$. Then $T(t)$ is a **symmetric diffusion C_0-semigroup** if $T(t)$ satisfies

(1) $T(t)$ is a C_0-semigroup.

(2) For all $f, g \in L^2$, $\langle T(t)f, g\rangle = \langle f, T(t)g\rangle$ (symmetric) where $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)dx$.

(3) For all $f \in L^2$, $T(t)f = g * f$ where $g \geq 0$ and $\int_{\mathbb{R}^d} g(x)dx = 1$ (g is a probability measure).

Remark. The generalized heat kernel is a special case of a symmetric diffusion operator.

Let L be the infinitesimal generator of $T(t)$ on L^2. We write $T(t) = e^{Lt}$.

Lemma A.2.0.2. *Let $T(t)$ be a symmetric diffusion C_0-semigroup on $L^2(\mathbb{R}^d)$ and let L be its infinitesimal generator. Then for any convex function ϕ,*

$$\phi'(f)Lf \leq L(\phi(f)).$$

Example A.2.0.3. Let $L = \nu\Delta$, $\phi(f) = f^2$. We have $\phi'(f) = 2f$ and by the claim in the lemma, $2f\nu\Delta f \leq \nu\Delta(f^2)$.

Proof. Using Einstein notation, we have

$$\begin{aligned} \Delta(f^2) = \partial_k\partial_k(f^2) = \partial_k(2f\partial_k f) &= 2\partial_k f\partial_k f + 2f\partial_k\partial_k f \\ &= 2\left|\nabla f\right|^2 + 2f\Delta f \\ &\geq 2f\Delta f. \end{aligned}$$

Therefore, we have verified $2f\nu\Delta f \leq \nu\Delta(f^2)$ directly. □

Example A.2.0.4. Let $L = -\nu\Lambda^\alpha$, $\phi(f) = f^2$. Then, by the lemma we obtain

$$2f(-\nu\Lambda^\alpha f) \leq -\nu\lambda^\alpha(f^2) \implies f\Lambda^\alpha f \geq \Lambda^\alpha(\frac{1}{2}f^2)$$

which is a fact we will prove again in Lemma A.4.0.5.

Example A.2.0.5. Let $\phi(f) = |f|$, $L = -\nu\Lambda^\alpha$. Then $\phi'(f) = \frac{f}{|f|} = \operatorname{sgn}(f)$. Applying the lemma,

$$\frac{f}{|f|}(-\nu\Lambda^\alpha f) \leq -\nu\Lambda^\alpha |f| \implies f\Lambda^\alpha f \geq |f|\,\Lambda^\alpha |f|. \tag{A.1}$$

For example, this means $f\Delta f \leq |f|\,\Delta|f|$ if $\Delta|f|$ exists.

A.3 An Estimate for Linear Transport-Diffusion

We now present a result bounding the solution of the homogeneous linear transport equation with fractional dissipation

$$\partial_t\theta + u\cdot\nabla\theta + \nu\Lambda^\alpha\theta = 0$$

by its initial data θ_0. By (1.1), the problem in 1D is well-posed. Additionally, we prove that the gradient of the solution is bounded above by its initial data. This important result implies that $\nabla\theta$ and θ do not admit finite time blowup; singularities do not form in these solutions. In the lemma, we denote the norm $\|\theta(\cdot,t)\| = \|\theta(t)\|$.

Lemma A.3.0.1. *Let $u \in C^{0,1}(\mathbb{R}^d)$. Consider the fractional advection-diffusion equation:*

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu\Lambda^\alpha\theta = 0, \quad 0<\alpha<2 \\ \nabla\cdot u = 0 \\ \theta(x,0) = \theta_0(x) \end{cases} \tag{A.2}$$

Then,

(1) For any $1 \le q < \infty$,

$$\|\theta(t)\|_{L^q}^q + C(q)\int_0^t \|\theta(\tau)\|^q_{L^{\frac{qd}{d-\alpha}}}\,d\tau \le \|\theta_0\|^q_{L^q}$$

and

$$\|\theta\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$$

(2) For any $1 \le q < \infty$,

$$\|\nabla\theta(t)\|_{L^q} \le C(t)\,\|\nabla\theta_0\|_{L^q}$$

Proof. Let $q \ge 2$, multiplying by $\theta\,|\theta|^{q-2}$ and integrating over $\mathbb{R}^d$ yields

$$\frac{1}{q}\frac{d}{dt}\|\theta(t)\|^2_{L^q} + \int_{\mathbb{R}^d} u\cdot\nabla\left(\frac{1}{q}|\theta|^q\right)dx + \nu\int_{\mathbb{R}^d}\theta\,|\theta|^{q-2}\,\Lambda^\alpha\theta dx = 0$$

where

$$\int_{\mathbb{R}^d} u\cdot\nabla\left(\frac{1}{q}|\theta|^q\right)dx = \int_{\mathbb{R}^d}\nabla\cdot\left(u\frac{1}{q}|\theta|^q\right)dx = 0.$$

Using the fact that $f\Lambda^\alpha f \ge \Lambda^\alpha(\frac{1}{2}f^2)$ by the Lemma A.4.0.5 in the next section of Appendix A, we have

$$\int_{\mathbb{R}^d}\theta\,|\theta|^{q-2}\,\Lambda^\alpha\theta dx \ge C(q)\int_{\mathbb{R}^d}|\theta|^{\frac{q}{2}}\,\Lambda^\alpha\,|\theta|^{\frac{q}{2}}\,dx = C(q)\int_{\mathbb{R}^d}\left|\Lambda^{\frac{\alpha}{2}}\,|\theta|^{\frac{\alpha}{2}}\right|^2 dx.$$

By Sobolev embedding,

$$\begin{aligned}\left\|\Lambda^{\frac{\alpha}{2}}|\theta|^{\frac{\alpha}{2}}\right\|_{L^2} &\geq C\left\||\theta|^{\frac{\alpha}{2}}\right\|_{L^r}\\ &= C\left(\int_{\mathbb{R}^d}|\theta|^{\frac{q}{2}\cdot r}\,dx\right)^{\frac{1}{r}}\\ &= C\left(\int_{\mathbb{R}^d}|\theta|^{\frac{q}{2}\cdot\frac{2d}{d-\alpha}}\,dx\right)^{\frac{d-\alpha}{2d}}\\ &= C\left(\int_{\mathbb{R}^d}|\theta|^{\frac{qd}{d-\alpha}}\,dx\right)^{\frac{d-\alpha}{qd}\frac{qd}{2d}}\\ &= C\left\|\theta\right\|^{\frac{q}{2}}_{L^{\frac{qd}{d-\alpha}}}.\end{aligned}$$

Putting these estimates together yields

$$\frac{1}{q}\frac{d}{dt}\left\|\theta(t)\right\|^q_{L^q}+C(q)\left\|\theta\right\|^q_{L^{\frac{qd}{d-\alpha}}}\leq 0.$$

Integrating in time, we have $\forall t>0$,

$$\left\|\theta(t)\right\|^q_{L^q}+C(q)\int_0^t\left\|\theta(\tau)\right\|^q_{L^{\frac{qd}{d-\alpha}}}\,d\tau\leq\left\|\theta_0\right\|^q_{L^q}.$$

This result holds for $2\leq q<\infty$. Clearly,

$$\left\|\theta(t)\right\|_{L^q}\leq\left\|\theta_0\right\|_{L^q}.$$

Letting $q\to\infty$, we obtain

$$\left\|\theta(t)\right\|_{L^\infty}\leq\left\|\theta_0\right\|_{L^\infty}\quad\because$$

Now we prove the second part of the lemma. Applying the gradient to the equation in θ and multiplying by $\nabla\theta\left|\nabla\theta\right|^{q-2}$ yields

$$\frac{1}{q}\frac{d}{dt}\left\|\nabla\theta\right\|^q_{L^q}+\int u\cdot\left(\frac{1}{q}\left|\nabla\theta\right|^q\right)dx+\nu\int\left|\nabla\theta\right|^{q-2}\nabla\theta\cdot\Lambda^\alpha(\nabla\theta)dx=-\int\partial_k\theta\partial_k u_j\partial_j\theta\left|\nabla\theta\right|^{q-2}dx$$

where we use the Einstein summation notation. Due to the incompressibility constraint $\nabla\cdot u=0$, we have

$$\frac{1}{q}\frac{d}{dt}\left\|\nabla\theta\right\|^q_{L^q}+\nu C(q)\int(\Lambda^{\frac{\alpha}{2}}\left|\nabla\theta\right|^{\frac{q}{2}})^2dx\leq C\left\|\nabla u\right\|_{L^\infty}\left\|\nabla\theta\right\|^q_{L^q}$$

where C is independent of q. By Sobolev embedding,

$$\frac{1}{q}\frac{d}{dt}\|\nabla\theta\|_{L^q}^q + C(q)\|\nabla\theta\|_{L^{\frac{qd}{d-\alpha}}}^q \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q .$$

Since $C(q) \geq 0$, it follows that

$$\frac{1}{q}\frac{d}{dt}\|\nabla\theta\|_{L^q}^q \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q .$$

Treating $\|\nabla\theta\|_{L^q}$ as a function of q, we have by the chain rule

$$\frac{1}{q}q\|\nabla\theta\|_{L^q}^{q-1}\frac{d}{dt}\|\nabla\theta\|_{L^q} \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q$$

which implies

$$\frac{d}{dt}\|\nabla\theta\|_{L^q} \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q} .$$

We require the next lemma, Gronwall's inequality for ODE, which we provide here without proof.

Lemma A.3.0.2 (Gronwall's Inequality). *If $\frac{d}{dt}f(t) \leq a(t)f(t)$ and $f(0) = f_0$, then*

$$f(t) \leq f_0\, e^{\int_0^t a(\tau)d\tau}.$$

We finish the proof by using Gronwall's inequality on $\frac{d}{dt}\|\nabla\theta\|_{L^q} \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}$ to obtain

$$\begin{aligned}\|\nabla\theta(t)\|_{L^q} &\leq \|\nabla\theta_0\|_{L^q}\, e^{C\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau}\\ &\leq C(t)\|\nabla\theta_0\|_{L^q}\end{aligned}$$

as required. □

A.4 Miscellaneous Results and Proofs

- The following theorem is fundamental to the study of the standard $\alpha = 2$ heat equation.

  Theorem A.4.0.1 (Maximum Principle, from EVANS [38]). *Assume $u = u(x,t) \in C_1^2(\mathbb{R}^d \times (0,T]) \cap C(\mathbb{R}^d \times [0,T])$ (namely that u is twice continuously differentiable in space and continuously differentiable in time) solves the heat equation*

$$\begin{cases}\partial_t u = \nu\Delta u, & x \in \mathbb{R}^d,\ t > 0\\ u(x,0) = g(x)\end{cases}$$

and $|u(x,t)| \le Ae^{a|x|^2} \quad \forall x \in \mathbb{R}^d$*, where* $A > 0$ *and* $a > 0$*. Then,*

$$\sup_{x\in\mathbb{R}^d, t\in[0,T]} |u(x,t)| = \sup_{x\in\mathbb{R}^d} |g(x)| .$$

- We now justify the claim that $g_\alpha \geq 0$.

Proof of Lemma 2.4.1.6. We use the following fact from FELLER [41]:

$$e^{-\lambda^\alpha} = \int_0^\infty e^{-s\lambda^2} d\mu_\alpha(s) \tag{A.3}$$

where $\mu_\alpha(s)$ is a *Borel probability measure*, which averages $e^{-\lambda^\alpha}$ in a sense and satisfies $\int_0^\infty d\mu_\alpha(s) = 1$. Then,

$$\begin{aligned} g_\alpha(x,t) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-\nu t(2\pi|\xi|)^\alpha} d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \int_0^\infty e^{-s((\nu t)^{\frac{1}{\alpha}} 2\pi|\xi|)^2} d\mu_\alpha(s) d\xi \\ &= \int_0^\infty \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-s(\nu t)^{\frac{2}{\alpha}} 4\pi^2|\xi|^2} d\xi d\mu_\alpha(s) \\ &= \int_0^\infty \left(\frac{\pi}{s(\nu t)^{\frac{2}{\alpha}} 4\pi^2} \right)^{\frac{d}{2}} e^{-\frac{|x|^2}{4s(\nu t)^{\frac{2}{\alpha}}}} d\mu_\alpha(s) \\ &= \frac{1}{(4\pi(\nu t)^{\frac{2}{\alpha}})^{\frac{d}{2}}} \int_0^\infty s^{\frac{d}{2}} e^{-\frac{|x|^2}{4s(\nu t)^{\frac{2}{\alpha}}}} d\mu_\alpha(s) \\ &\geq 0. \end{aligned}$$

□

- We now prove Property (2) of Lemma 2.4.1.9.

Proof. We remark on a basic fact from real analysis .

Remark A.4.0.2. For all $f \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$, f can be approximated by a continuous function with compact support (dense in L^p). More precisely, $\forall \epsilon > 0, \exists f_1 \ni f = f_1 + f_2$, where f_1 is continuous with $\operatorname{supp} f_1$ compact and $\|f - f_2\|_{L^p} \le \frac{\epsilon}{2} < \epsilon$.

Our goal is to prove that if $f \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$, then

$$e^{-\nu t\Lambda^\alpha} f \to f \quad \text{in } L^p \text{ as } t \to 0^+$$

which means

$$\|g_\alpha(x,t) * f - f\|_{L^p} \to 0 \quad \text{as } t \to 0^+.$$

To continue the proof of (2), we invoke the remark. For $f \in L^p$, $\epsilon > 0$, and $0 < |x-y| < \delta$, we write $f = f_1 + f_2$ with f_1 continuous such that $\operatorname{supp} f_1$ is compact (in the sense that f_1 approximates f) and $\|f_2\|_{L^p} < \frac{\epsilon}{2}$. We show

$$\|f(x-y) - f(x)\|_{L^p(\mathbb{R}^d)} \to 0 \text{ as } |y| \to 0 \tag{A.4}$$

as the first step. This is true for f_1:

$$\|f_1(x-y) - f(x)\|_{L^p} \to 0 \text{ as } |y| \to 0$$

since f_1 is continuous. To clarify explicitly,

$$\begin{aligned}\int_{\mathbb{R}^d} |f_1(x-y) - f_1(x)|^p \, dx &= \int_{|x|\le M} |f_1(x-y) - f_1(x)|^p \, dx \quad \text{(by compact support)}\\ &\le \frac{\epsilon^p}{M} \cdot M = \epsilon^p\end{aligned}$$

for $|y| < \delta$. Then,

$$\begin{aligned}\|f(x-y) - f(x)\|_{L^p} &\le \|f_1(x-y) - f_1(x)\|_{L^p} + \|f_2(x-y) - f_2(x)\|_{L^p}\\ &\le \epsilon + (\|f_2(x-y)\|_{L^p} + \|f_2(x)\|_{L^p})\\ &\le \epsilon + \epsilon/2 + \epsilon/2\\ &= \int_{\mathbb{R}^d} |f_2(z)|^p \, dz = \|f_2\|^p_{L^p})\\ &= 2\epsilon\end{aligned}$$

because $\|f_2(x-y)\|_{L^p}^p = \int_{\mathbb{R}^d} |f_2(x-y)|^p \, dx$ and by *translation invariance* of L^p spaces. Now,

$$\|g_\alpha(x,t) * f - f\|_{L^p} = \left\| \int_{\mathbb{R}^d} f(x-y) g_\alpha(y,t) dy - f(x) \right\|_{L^p_x}$$

$$= \left\| \int_{\mathbb{R}^d} f(x-y) \frac{1}{t^{\frac{d}{\alpha}}} g_\alpha(\frac{y}{t^{\frac{1}{\alpha}}}, 1) dy - f(x) \right\|_{L^p_x} \qquad \text{(by scaling)}$$

(Change of variables in $\mathbb{R}^d$: Let $z = \frac{y}{t^{\frac{1}{\alpha}}} \implies dy = t^{\frac{d}{\alpha}} dz$)

$$= \left\| \int_{\mathbb{R}^d} f(x - t^{\frac{1}{\alpha}} z) g_\alpha(z,1) - f(x) \right\|_{L^p_x}$$

$$= \left\| \int_{\mathbb{R}^d} f(x - t^{\frac{1}{\alpha}} z) g_\alpha(z,1) - \int_{\mathbb{R}^d} g_\alpha(z,1) f(x) dz \right\|_{L^p_x}$$

$\left(\int_{\mathbb{R}^d} g_\alpha(z,1) dz = 1 \quad \text{since} \quad \|g_\alpha\|_{L^1} \right)$

$$\le \left\| \int_{\mathbb{R}^d} \left| f(x - t^{\frac{1}{\alpha}} z) - f(x) \right| g_\alpha(z,1) dz \right\|_{L^p_x}$$

$$\le \int_{\mathbb{R}^d} \left\| f(x - t^{\frac{1}{\alpha}} z) - f(x) \right\|_{L^p_x} g_\alpha(z,1) dz$$

by the following important lemma:

Lemma A.4.0.3 (Minkowski's Inequality)**.** *If $q \ge p$, then*

$$\left\| \|f(x,y)\|_{L^p_x} \right\|_{L^q_x} \le \left\| \|f(x,y)\|_{L^q_x} \right\|_{L^p_x}. \tag{A.5}$$

To continue the proof, we split up the integral into two pieces:

$$\|g_\alpha(x,t) * f - f\|_{L^p} \le \int_{\left|t^{\frac{1}{\alpha}} z\right| < \delta} \left\| f(x - t^{\frac{1}{\alpha}} z) - f(x) \right\|_{L^p_x} g_\alpha(z,1) dz$$

$$+ \int_{\left|t^{\frac{1}{\alpha}} z\right| \ge \delta} \left\| f(x - t^{\frac{1}{\alpha}} z) - f(x) \right\|_{L^p_x} g_\alpha(z,1) dz$$

$$\le \int_{\left|t^{\frac{1}{\alpha}} z\right| < \delta} \epsilon \cdot g_\alpha(z,1) dz + \int_{\left|t^{\frac{1}{\alpha}} z\right| \ge \delta} 2 \|f\|_{L^p_x} g_\alpha(z,1) dz$$

$$\le \epsilon \cdot 1 + 2 \|f\|_{L^p_x} \int_{\left|t^{\frac{1}{\alpha}} z\right| \ge \delta} g_\alpha(z,1) dz.$$

We require the next lemma to finish the proof.

Lemma A.4.0.4. *If $g \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$, then $\int_{|x|\ge R} |g|^p\,dx \to 0 \quad as \quad R \to \infty$.*

Thus,

$$\begin{aligned}
\|g_\alpha(x,t) * f - f\|_{L^p} &\le \epsilon \cdot 1 + 2\,\|f\|_{L^p_x} \int_{\left|t^{\frac{1}{\alpha}} z\right| \ge \delta} g_\alpha(z,1) dz \\
&\le \epsilon + 2\,\|f\|_{L^p_x} \cdot \epsilon \\
&< C\epsilon
\end{aligned}$$

as required. □

- We now provide a useful pointwise estimate arising from the integral representation for the fractional Laplacian (2.9). This inequality is used extensively in the study of the surface quasi-geostrophic equation [19, 24, 77] and other fluid PDE models, and was first introduced by Córdoba and Córdoba [23, 24].

Lemma A.4.0.5. *For $\alpha \in (0,2)$, $f(x)\Lambda^\alpha f(x) \ge \Lambda^\alpha \left(\frac{1}{2}\,|f(x)|^2\right)$.*

Corollary A.4.0.6. *For $\alpha \in (0,2)$, $f(x)\Lambda^\alpha f(x) = \Lambda^\alpha \left(\frac{1}{2}\,|f(x)|^2\right) + D_\alpha(f)$, where*

$$D_\alpha(f) = \text{p.v. } C(\alpha,d) \int_{\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} dy \ge 0.$$

Proof.

$$\begin{aligned}
f(x)\Lambda^\alpha f(x) &= f(x)\,\text{p.v. } C(\alpha,d) \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}} dy \\
&= \text{p.v. } C(\alpha,d) \int_{\mathbb{R}^d} \frac{f^2(x) - f(x)f(y)}{|x-y|^{d+\alpha}} dy \\
&= \text{p.v. } C(\alpha,d) \int_{\mathbb{R}^d} \frac{\frac{1}{2}f^2(x) - \frac{1}{2}f^2(y)}{|x-y|^{d+\alpha}} dy + \text{p.v. } C(\alpha,d) \int_{\mathbb{R}^d} \frac{\frac{1}{2}f^2(x) - f(x)f(y) + \frac{1}{2}f^2(y)}{|x-y|^{d+\alpha}} dy \\
&= \Lambda^\alpha(\frac{1}{2}\,|f(x)|^2) + \text{p.v. } C_{new}(\alpha,d) \int_{\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} dy \\
&= \Lambda^\alpha(\frac{1}{2}\,|f(x)|^2) + D_\alpha(f) \\
&\ge \Lambda^\alpha(\frac{1}{2}\,|f(x)|^2).
\end{aligned}$$

□

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