THE FUNCTIONAL EQUATION OF THE TWISTED L-FUNCTION ASSOCIATED WITH AN AUTOMORPHIC FORM ON $GL(3, \mathbf{R})$

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Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the Requirements for
the Degree of
DOCTOR OF PHILOSOPHY
May, 1993

OKLAHOMA STATE UNIVERSITY

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ACKNOWLEDGEMENTS

It would be impossible for me to acknowledge everyone who has helped me obtain the understanding of mathematics necessary to complete this project. However, I would like to acknowledge those who have directly assisted with this work.

First of all, I wish to thank God for giving me the ability to work at this level. My parents, Cecil and Mary Ann Flood, have shown me by their example that anything is attainable through hard work and patience. A special thanks goes to my wife, Chris. She has encouraged me throughout this work and, by her sacrifices, has allowed me the time to complete it. I would also like to thank my son, Aaron, for just being himself.

I am also indebted to David Farmer for introducing me to mathematical research and for the suggestions he has given me concerning this manuscript. Finally, I wish to express my appreciation to Dr. David Wright, Dr. Alan Adolphson, Dr. Mark McConnell, Dr. Arlene Fulton, and Dr. Brian Conrey for serving on my graduate committee. Their suggestions have been very helpful in preparing this work. Most of all I would like to thank my adviser, Dr. Brian Conrey. Without his constant encouragement and guidance this work would never have been completed.

TABLE OF CONTENTS

CHAPTER I

BACKGROUND

Introduction

Automorphic forms on $GL(3,\mathbf{R})$ can be thought of as a generalization of the more classical and familiar automorphic forms on $GL(2,\mathbf{R})$. The aspects which generalize to $GL(3,\mathbf{R})$ are outlined in Chapter 1 of [B1] and are reproduced here for comparison to the $GL(3,\mathbf{R})$ case. Automorphic forms on $GL(2,\mathbf{R})$ are functions on the upper half plane H. In particular, there are two general types of automorphic forms on $GL(2,\mathbf{R})$; namely, holomorphic or modular forms and Maass forms. It is the Maass forms which generalize to $GL(3,\mathbf{R})$.

In order to generalize Maass forms to automorphic forms on $GL(3,\mathbf{R})$ we consider H as $GL(2,\mathbf{R})/ZK$ where Z is the center of $GL(2,\mathbf{R})$ and K is the subgroup of orthogonal matrices. Thus we have

$$H \cong \left\{ \begin{pmatrix} y & x \\ & 1 \end{pmatrix} : x, y \in \mathbf{R},\ y > 0 \right\}.$$

In this way the natural action of $SL(2,\mathbf{Z})$ on H is given by matrix multiplication. With this action we define the automorphic forms of Maass. A complex-valued function f on H is a *Maass form* if

(a) $f(gz) = f(z)$ for all $g \in SL(2,\mathbf{Z})$ and $z \in H$,

(b) f is an eigenfunction of the G-invariant differential operators on H, and

(c) there exists an n such that $f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) y^n$ is bounded for $y > 1$.

We say f is a *cusp form*, if in addition

$$\int_0^1 f\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) dx = 0 \quad \text{for all } y > 0.$$

Condition (b) in this definition can be made more precise. That is, the space of G-invariant differential operators on $\mathcal{H}$ is generated by

$$\Delta = -y^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right).$$

Therefore, condition (b) implies $\Delta f = \lambda f$ for some eigenvalue $\lambda \in \mathbf{C}$. Thus associated to f is a complex number λ; however, a more natural parameter to associate with f is ν where $\lambda = \nu(1-\nu)$. In this situation we say f is a *Maass form* (or respectively *cusp form*) *of type* ν. In the $GL(3,\mathbf{R})$ case it is known that the space of differential operators is generated by two elements. Thus the type is given by two complex numbers.

Condition (a) in the definition of a Maass form implies that

$$f\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right)$$

is periodic in x. This, along with the other conditions, gives a Fourier expansion of the form

$$f\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{n\in\mathbf{Z}} a_n W^\nu\left(\begin{pmatrix} ny & nx \\ & 1 \end{pmatrix}\right)$$

where

$$W^\nu\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = 2\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi y)\mathrm{e}(x)$$

and $K_\nu(z)$ is the standard K-Bessel function. If f is a cusp form then $a_0 = 0$, and we can write

$$f\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{n\neq 0} a_n W^\nu\left(\begin{pmatrix} ny & nx \\ & 1 \end{pmatrix}\right).$$

Thus associated to a cusp form is a sequence $\{a_n\}_{n\neq 0}$. This sequence is used to define the L-function associated with f. For f a cusp form, the *L-function associated with f* is

$$L(s,f) = \sum_{n\neq 0} \frac{a_n}{n^s}.$$

It is known that this series is absolutely convergent in a right half pane, extends to an entire function of s, and satisfies a functional equation of the form

$$\pi^{-s}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{1+s-\nu}{2}\right)L(s,f)=\pi^{s-1}\Gamma\left(\frac{1-s+\nu}{2}\right)\Gamma\left(\frac{2-s-\nu}{2}\right)L(1-s,f).$$

A similar functional equation for the L-function associated with a cusp form on $GL(3,\mathbf{R})$ was first proven by Godement and Jacquet ([GJ]). A more direct proof using the machinery of representation theory was given by Jacquet, Piatetski-Shapiro, and Shalika ([JPS]); however the gamma factors were not explicitly evaluated. Later, following methods of [JPS], a classical proof was given by Bump ([B1]). In this work the gamma factors were specifically evaluated, but the method relied on the introduction of Eisenstein series. Finally, by introducing an auxiliary variable, a classical and direct method was presented by Hoffstein and Murty ([HM1]).

Jacquet, Piatetski-Shapiro, and Shalika ([JPS]) go much further and prove the converse theorem in the more general setting of automorphic forms on the Adele group. The $GL(2,\mathbf{R})$ analog had previously been proven by Weil ([W]) (See the appendix for remarks on Weil's converse theorem). The converse theorem states that a function is a cusp form if and only if all twists by characters of the L-function associated with the form have an Euler product, are entire and bounded in every vertical strip, and satisfy a similar functional equation.

In the following work, the exact form of the functional equation for the L-function associated with a cusp form on $GL(3,\mathbf{R})$ which has been twisted by a primitive Dirichlet character will be established. The methods of [B1] will be used while incorporating the methods of [HM1]. We will also be relying on the handwritten notes of Hoffstein and Murty ([HM2]), which were provided by Hoffstein. In these notes they work out the functional equation for the L-function twisted by an additive character of prime modulus. Their results were easily extended to the case of a primitive Dirichlet character of prime modulus and by a modification of their argument the result was obtained for even primitive Dirichlet characters with any modulus. Finally, by introducing an auxiliary variable, the result was obtained for all primitive Dirichlet characters. It is hoped that by interpreting the results of [JPS] into classical language more insight may be obtained regarding the behavior of the cusp forms themselves.

$GL(3,\mathbf{R})$ Preliminaries

We can now define an automorphic form on $G = GL(3,\mathbf{R})$. Let K denote the subgroup of orthogonal matrices in G, let Z denote the center of G, and let $\mathcal{H} = G/ZK$. It is this space $\mathcal{H}$ which plays the role of the upper half plane. We note, by the Iwasawa decomposition, that each coset in $\mathcal{H}$ has a unique representative of the form

$$\tau = \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \quad \text{where } y_1, y_2 > 0 \text{ and } x_1, x_2, x_3 \in \mathbf{R}.$$

We also introduce an auxiliary coordinate x_4 given by the relation

$$x_1x_2 = x_3 + x_4$$

which will greatly simplify some of the formulas. Finally, we let $\Gamma = GL(3,\mathbf{Z})$. A $GL(3,\mathbf{R})$ *automorphic form* is a complex-valued function F on $\mathcal{H}$ such that

(a) $F(g\tau) = F(\tau)$ for all $g \in \Gamma$ and $\tau \in \mathcal{H}$,

(b) F is an eigenfunction of the G-invariant differential operators on $\mathcal{H}$, and

(c) there exist constants n_1, n_2 such that $F\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) y_1^{n_1} y_2^{n_2}$ is bounded on the subset of $\mathcal{H}$ determined by $y_1, y_2 > 1$.

We say F is a *cusp form* if in addition

$$\int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) dx_1dx_3 = 0$$
$$\int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & \\ & & 1 \end{pmatrix}\tau\right) dx_2dx_3 = 0 \qquad \text{for all } \tau \in \mathcal{H}.$$

As in the $GL(2,\mathbf{R})$ case, condition (b) can be made more precise. The space of G-invariant differential operators on $\mathcal{H}$ is generated by two elements ([B1] 2.33 and 2.37); namely,

$$\Delta_1 = y_1^2\frac{\partial^2}{\partial y_1^2} + y_2^2\frac{\partial^2}{\partial y_2^2} - y_1y_2\frac{\partial^2}{\partial y_1\partial y_2} + y_1^2(x_2^2+y_2^2)\frac{\partial^2}{\partial x_3^2} + y_1^2\frac{\partial^2}{\partial x_1^2} + y_2^2\frac{\partial^2}{\partial x_2^2} + 2y_1^2x_2\frac{\partial^2}{\partial x_1\partial x_3}$$

and

$$\begin{aligned}\Delta_2 = {} & -y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2} + y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2} - y_1^3y_2^2\frac{\partial^3}{\partial x_3^2\partial y_1} + y_1y_2^2\frac{\partial^3}{\partial x_2^2\partial y_1} - 2y_1^2y_2x_2\frac{\partial^3}{\partial x_1\partial x_3\partial y_2}\\ & + y_1^2y_2(y_2^2-x_2^2)\frac{\partial^3}{\partial x_3^2\partial y_2} - y_1^2y_2\frac{\partial^3}{\partial x_1^2\partial y_2} + 2y_1^2y_2^2\frac{\partial^3}{\partial x_1\partial x_2\partial x_3} + 2y_1^2y_2x_2\frac{\partial^3}{\partial x_2\partial x_3^2}\\ & + y_1^2\frac{\partial^2}{\partial y_1^2} - y_2^2\frac{\partial^2}{\partial y_2^2} + 2y_1^2x_2\frac{\partial^2}{\partial x_1\partial x_3} + y_1^2(x_2^2+y_2^2)\frac{\partial^2}{\partial x_3^2} + y_1^2\frac{\partial^2}{\partial x_1^2} - y_2^2\frac{\partial^2}{\partial x_2^2}.\end{aligned}$$

Therefore, condition (b) implies $\Delta_1F = \lambda F$ and $\Delta_2F = \mu F$ for some eigenvalues $\lambda,\mu \in \mathbf{C}$. However, more natural parameters to associate with F are $\nu_1,\nu_2 \in \mathbf{C}$ which are given by the relations ([B1] p. 33)

$$\lambda = 3(\nu_1^2+\nu_1\nu_2+\nu_2^2-\nu_1-\nu_2)$$

and ([B1] p. 34)

$$\mu = -2\nu_1^3 - 3\nu_1^2\nu_2 + 3\nu_1\nu_2^2 + 2\nu_2^3 + 3\nu_1^2 - 3\nu_2^2 - \nu_1 + \nu_2.$$

In this situation we say F is an *automorphic form* (or respectively *cusp form*) *of type* (ν_1,ν_2).

Also of particular interest in the theory of automorphic forms for $GL(3,\mathbf{R})$ is the involution

$$^{\iota}\tau = w_1\,{}^t\tau^{-1}w_1 \quad \text{where} \quad w_1 = \begin{pmatrix} & & -1\\ & -1 & \\ -1 & & \end{pmatrix}.$$

We use this involution to define the *dual* $\widetilde{f}$ of any function f on $\mathcal{H}$, which is given by

$$\widetilde{f}(\tau) = f(^{\iota}\tau).$$

If F is an automorphic form of type (ν_1,ν_2) then $\widetilde{F}$ is an automorphic form of type (ν_2,ν_1) ([B1] p. 71).

We now recall the expansion of an automorphic form in terms of Whittaker functions ([B1] Chapter 4). This will give the coefficients used to define the L-function for $GL(3,\mathbf{R})$. We first must define several subgroups of $\Gamma = GL(3,\mathbf{Z})$. We let

$$\Gamma_\infty = \left\{ \tau \in \Gamma : \tau = \begin{pmatrix} 1 & A & B \\ & 1 & C \\ & & 1 \end{pmatrix} \right\}$$

$$\Gamma^2 = \left\{ \tau \in \Gamma : \tau = \begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix} \right\}$$

$$\Gamma_1^2 = \{ \tau \in \Gamma^2 : \det(\tau) = 1 \}$$

$$\Gamma_\infty^2 = \Gamma^2 \cap \Gamma_\infty.$$

We now let F be an automorphic form of type (ν_1, ν_2). Since F is invariant under $\begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$ we have the Fourier expansion

$$F(\tau) = \sum_{n_1, n_3 \in \mathbf{Z}} F_{n_1}^{n_3}(\tau)$$

where

$$F_{n_1}^{n_3}(\tau) = \int_0^1 \int_0^1 F\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \tau \right) \mathrm{e}(-n_1 x_1 - n_3 x_3) dx_1 dx_3.$$

We also note ([B1] 4.5) that for $A, B, C, D, m \in \mathbf{Z}$ with $AD - BC = 1$ and $m > 0$ we have

$$F_{mD}^{mC}(\tau) = F_m^0\left(\begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix} \tau \right).$$

Thus we have that

$$F(\tau) = F_0^0(\tau) + \sum_{g \in \Gamma_\infty^2 \backslash \Gamma_1^2} \sum_{m=1}^{\infty} F_m^0(g\tau).$$

Noting that F_m^0 is invariant under $\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$ we see that we have the Fourier expansion

$$F_m^0(\tau) = \sum_{n \in \mathbf{Z}} F_{m,n}(\tau)$$

where

$$F_{n_1,n_2} = \int_0^1 \int_0^1 \int_0^1 F\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \tau \right) \mathrm{e}(-n_1 x_1 - n_2 x_2) dx_1 dx_2 dx_3.$$

Hence,

$$F(\tau) = \sum_{n_2 \in \mathbf{Z}} F_{0,n_2}(\tau) \; + \sum_{g \in \Gamma^2_\infty \backslash \Gamma^2_1} \sum_{n_1=1}^{\infty} \sum_{n_2 \in \mathbf{Z}} F_{n_1,n_2}(g\tau).$$

We observe that

$$F_{n_1,n_2}\left(\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tau\right) = F_{n_1,-n_2}(\tau)$$

and if we assume that F is a cusp form we see that $F_{0,n_2} = 0$ and $F_{n_1,0} = 0$, whence

$$F(\tau) = \sum_{g \in \Gamma^2_\infty \backslash \Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} F_{n_1,n_2}(g\tau).$$

There exist a_{n_1,n_2} ([B1] 4.12) such that

$$F_{n_1,n_2}(\tau) = \frac{a_{n_1,n_2}}{|n_1 n_2|} W^{\nu_1,\nu_2}_{1,1}\left(\begin{pmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{pmatrix} \tau\right)$$

where

$$\begin{aligned} &W^{\nu_1,\nu_2}_{n_1,n_2}(\tau) \\ &= \pi^{\frac{1}{2}-3\nu_1-3\nu_2}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \mathrm{e}(n_1 x_1 + n_2 x_2) \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi_3^2 + \xi_2^2 y_1^2 + y_1^2 y_2^2)^{-\frac{3\nu_1}{2}} (\xi_4^2 + \xi_1^2 y_2^2 + y_1^2 y_2^2)^{-\frac{3\nu_2}{2}} \mathrm{e}(-n_1\xi_1 - n_2\xi_2) d\xi_1 d\xi_2 d\xi_3 \end{aligned}$$

with τ written in the standard coordinates and $\xi_1\xi_2 = \xi_3 + \xi_4$. With this we have established

Lemma 1.1. [B1] *If F is a cusp form of type (ν_1, ν_2) then there exist coefficients a_{n_1,n_2} such that*

$$F(\tau) = \sum_{g \in \Gamma^2_\infty \backslash \Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{a_{n_1,n_2}}{n_1 n_2} W^{\nu_1,\nu_2}_{1,1}\left(\begin{pmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{pmatrix} g\tau\right).$$

The array a_{n_1,n_2} is called the *matrix of Fourier coefficients* of F and we have

$$a_{n_1,n_2} = a_{|n_1|,|n_2|}.$$

It is this array which will be used to define the L-functions. For convenience of notation, if a_{n_1,n_2} is the matrix of coefficients for F then we let $\widetilde{a}_{n_1,n_2}$ be the matrix of coefficients for the dual $\widetilde{F}$. We have ([B1] 4.15) that

$$\widetilde{a}_{n_1,n_2} = a_{n_2,n_1}.$$

It is now possible to define the L-function for $GL(3,\mathbf{R})$. Let F be an automorphic form of type (ν_1,ν_2) and a_{n_1,n_2} be its matrix of Fourier coefficients. The *L-function associated with* F is given by

$$L(w,F)=\sum_{n=1}^{\infty}\frac{a_{1,n}}{n^w}.$$

We see for the dual form $\widetilde{F}$ that

$$L(w,\widetilde{F})=\sum_{n=1}^{\infty}\frac{\widetilde{a}_{1,n}}{n^w}=\sum_{n=1}^{\infty}\frac{a_{n,1}}{n^w}.$$

We have $a_{n_1,n_2}=O(|n_1n_2|)$ ([B1] 8.4); thus the L-functions converge absolutely for $\mathrm{Re}(w)>2$. Also, for χ a Dirichlet character we define the *twisted L-function associated with* F by

$$L_\chi(w,F)=\sum_{n=1}^{\infty}\frac{a_{1,n}\chi(n)}{n^w},$$

which is also absolutely convergent for $\mathrm{Re}(w)>2$.

It is the L-function for which Bump [B1] established a functional equation. Later, Hoffstein and Murty [HM1] gave a more direct proof of this functional equation. It will be their method which will be used in this paper to develop a functional equation for the twisted L-function. We now, very briefly, state Hoffstein and Murty's results. They begin by introducing an auxiliary variable s and considering the two Mellin transforms

$$\widetilde{\Phi}(s,w)=\int_0^\infty\int_0^\infty\widetilde{W}\left(\begin{pmatrix}t&&\\&v&\\&&1\end{pmatrix}\right)t^wv^s\frac{dt}{t}\frac{dv}{v}$$

and

$$\Phi(s,w)=\int_0^\infty\int_0^\infty\int_{-\infty}^\infty W\left(\begin{pmatrix}tv&&\\yv&v&\\&&1\end{pmatrix}\right)t^wv^s dy\frac{dt}{t}\frac{dv}{v}$$

where

$$\widetilde{W}(\tau)=W_{1,1}^{\nu_2,\nu_1}(\tau)$$

and

$$W(\tau)=W_{1,1}^{\nu_1,\nu_2}(\tau).$$

It is now possible to show

$$L(w,\widetilde{F})\widetilde{\Phi}(s-1,w-1)=\sum_{n=1}^{\infty}\frac{\widetilde{a}_{1,n}}{n^{w}}\int_0^\infty\int_0^\infty \widetilde{W}\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$
$$=\frac{1}{2}\int_0^\infty\int_0^\infty\sum_{n\in\mathbf{Z}}\widetilde{F}_{1,n}\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$
$$=\frac{1}{2}\int_0^\infty\int_0^\infty \widetilde{F}_0^1\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}.$$

Exploiting the fact that

$$\widetilde{F}_0^1\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right)=\int_{-\infty}^{\infty}F_0^1\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)dy$$

the above equals

$$\frac{1}{2}\int_0^\infty\int_0^\infty\int_{-\infty}^{\infty}F_0^1\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$
$$=\frac{1}{2}\int_0^\infty\int_0^\infty\int_{-\infty}^{\infty}\sum_{n\in\mathbf{Z}}F_{1,n}\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$
$$=\sum_{n=1}^{\infty}\frac{a_{1,n}}{n^{1-w}}\int_0^\infty\int_0^\infty\int_{-\infty}^{\infty}W\left(\begin{pmatrix} tv & & \\ yv & v & \\ & & 1\end{pmatrix}\right)t^{-w}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$
$$=L(1-w,F)\Phi(s-1,-w).$$

So we have by meromorphic continuation that

$$L(w,\widetilde{F})\widetilde{\Phi}(s-1,w-1)=L(1-w,F)\Phi(s-1,-w).$$

Evaluating the Mellin transforms we obtain

$$\widetilde{\Phi}(s-1,w-1)=\frac{\Gamma\left(\frac{w+\alpha}{2}\right)\Gamma\left(\frac{w+\beta}{2}\right)\Gamma\left(\frac{w+\gamma}{2}\right)\Gamma\left(\frac{w+s-\alpha-1}{2}\right)\Gamma\left(\frac{w+s-\beta-1}{2}\right)\Gamma\left(\frac{w+s-\gamma-1}{2}\right)}{4\pi^{2w+s-1}\Gamma\left(\frac{2w+s-1}{2}\right)}$$

and

$$\Phi(s-1,-w)=\frac{\Gamma\left(\frac{1-w-\alpha}{2}\right)\Gamma\left(\frac{1-w-\beta}{2}\right)\Gamma\left(\frac{1-w-\gamma}{2}\right)\Gamma\left(\frac{w+s-\alpha-1}{2}\right)\Gamma\left(\frac{w+s-\beta-1}{2}\right)\Gamma\left(\frac{w+s-\gamma-1}{2}\right)}{4\pi^{s-w+\frac{1}{2}}\Gamma\left(\frac{s+2w-1}{2}\right)}$$

where

$$\alpha = -\nu_1 - 2\nu_2 + 1$$

$$\beta = -\nu_1 + \nu_2$$

$$\gamma = 2\nu_1 + \nu_2 - 1.$$

Combining these facts and noting that

$$\int_0^\infty \int_0^\infty \widetilde{F}_0^1 \left(\begin{pmatrix} t & & \\ & v & \\ & & 1 \end{pmatrix} \right) t^{w-1} v^{s-1} \frac{dt}{t} \frac{dv}{v}$$

converges for $\mathrm{Re}(s+w)$ sufficiently large we obtain

Theorem 1.2. [B1] *The L-function of an automorphic form F converges for large values of w. If F is a cusp form then $L(w, F)$ has an analytic continuation to all values of w and satisfies*

$$\pi^{\frac{-3w}{2}} L(w, \widetilde{F}) \Gamma\left(\frac{w+\alpha}{2}\right) \Gamma\left(\frac{w+\beta}{2}\right) \Gamma\left(\frac{w+\gamma}{2}\right)$$
$$= \pi^{\frac{-3(1-w)}{2}} L(1-w, F) \Gamma\left(\frac{1-w-\alpha}{2}\right) \Gamma\left(\frac{1-w-\beta}{2}\right) \Gamma\left(\frac{1-w-\gamma}{2}\right).$$

CHAPTER II

SOME BASIC LEMMAS

In this chapter we will present some basic lemmas which will be useful in the development of the functional equation of the twisted L-function. Throughout this chapter we let F be an automorphic form of type (ν_1, ν_2). We will use the notation of Chapter 1, specifically for F_n^m, W, Φ, and $\widetilde{\Phi}$. We first recall ([B1] 4.4) that if $n_2 \in \mathbf{Z}$ then

$$F_{n_1}^{n_3}\left(\begin{pmatrix} 1 & n_2 & \\ & 1 & \\ & & 1 \end{pmatrix}\tau\right) = F_{n_1+n_2n_3}^{n_3}(\tau).$$

We also need a similar result.

Lemma 2.1. *If $n_3 \in \mathbf{Z}$ then*

$$F_{n_1}^{n_3}\left(\begin{pmatrix} 1 & & \\ n_2 & 1 & \\ & & 1 \end{pmatrix}\tau\right) = F_{n_1}^{n_3+n_1n_2}(\tau).$$

Proof: We have by the definition of $F_{n_1}^{n_3}$ that

$$\begin{aligned}
&F_{n_1}^{n_3}\left(\begin{pmatrix} 1 & & \\ n_2 & 1 & \\ & & 1 \end{pmatrix}\tau\right) \\
&\quad = \int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ n_2 & 1 & \\ & & 1 \end{pmatrix}\tau\right) \mathrm{e}(-n_1x_1 - n_3x_3)dx_1dx_2 \\
&\quad = \int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & & x_3 \\ n_2 & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) \mathrm{e}(-n_1x_1 - n_3x_3)dx_1dx_2 \\
&\quad = \int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & & \\ n_2 & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 - n_2x_3 \\ & & 1 \end{pmatrix}\tau\right) \mathrm{e}(-n_1x_1 - n_3x_3)dx_1dx_2.
\end{aligned}$$

By a change of variables and the fact that F is invariant on the left by Γ, the above equals

$$\int_0^1\int_{-n_2x_3}^{1-n_2x_3} F\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) \mathrm{e}(-n_1(x_1 + n_2x_3) - n_3x_3)dx_1dx_3$$

which by periodicity equals

$$F_{n_1}^{n_3+n_1n_2}(\tau).$$

∎

We now need a simple result about an integral of a Whittaker function.

Lemma 2.2.

$$\int_{-\infty}^{\infty} W\left(\begin{pmatrix} -\frac{n}{t} & & \\ x & 1 & \\ & & 1 \end{pmatrix}\right) x^k dx = (-1)^k \int_{-\infty}^{\infty} W\left(\begin{pmatrix} \frac{n}{t} & & \\ x & 1 & \\ & & 1 \end{pmatrix}\right) x^k dx$$

Proof: Since W is invariant on the right under ZK we have

$$\begin{aligned}\int_{-\infty}^{\infty} W\left(\begin{pmatrix} -\frac{n}{t} & & \\ x & 1 & \\ & & 1 \end{pmatrix}\right) x^k dx &= \int_{-\infty}^{\infty} W\left(\begin{pmatrix} -\frac{n}{t} & & \\ x & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right) x^k dx \\ &= \int_{-\infty}^{\infty} W\left(\begin{pmatrix} \frac{n}{t} & & \\ -x & 1 & \\ & & 1 \end{pmatrix}\right) x^k dx.\end{aligned}$$

Now by a change of variables the above equals

$$\int_{-\infty}^{\infty} W\left(\begin{pmatrix} \frac{n}{t} & & \\ x & 1 & \\ & & 1 \end{pmatrix}\right) (-x)^k dx.$$

∎

Next we compute the coordinates of a certain matrix in order to calculate a certain partial derivative of W.

Lemma 2.3. *For $a, b, c, d \in \mathbf{R}$*

$$\begin{pmatrix} a & & \\ b & 1 & \\ c & & d \end{pmatrix} \text{ and } \begin{pmatrix} \frac{ad}{\sqrt{c^2+d^2}\sqrt{b^2d^2+c^2+d^2}} & \frac{abd^2}{(c^2+d^2)\sqrt{b^2d^2+c^2+d^2}} & \frac{ac}{c^2+d^2} \\ & \frac{\sqrt{b^2d^2+c^2+d^2}}{c^2+d^2} & \frac{bc}{c^2+d^2} \\ & & 1 \end{pmatrix}$$

differ by multiplication on the right by an element of ZK and hence have the coordinates

$$y_1 = \frac{\sqrt{b^2d^2+c^2+d^2}}{c^2+d^2} \qquad x_1 = \frac{bc}{c^2+d^2}$$

$$y_2 = \frac{ad\sqrt{c^2+d^2}}{b^2d^2+c^2+d^2} \qquad x_2 = \frac{abd^2}{b^2d^2+c^2+d^2}$$

$$x_3 = \frac{ac}{c^2+d^2}.$$

Proof: We first note

$$\begin{pmatrix} a & & \\ b & 1 & \\ c & & d \end{pmatrix}\begin{pmatrix} d & & c \\ & \sqrt{c^2+d^2} & \\ -c & & d \end{pmatrix}\begin{pmatrix} \sqrt{c^2+d^2} & bd & \\ -bd & \sqrt{c^2+d^2} & \\ & & \sqrt{b^2d^2+c^2+d^2} \end{pmatrix}$$

$$= \begin{pmatrix} ad & & ac \\ bd & \sqrt{c^2+d^2} & bc \\ & & c^2+d^2 \end{pmatrix}\begin{pmatrix} \sqrt{c^2+d^2} & bd & \\ -bd & \sqrt{c^2+d^2} & \\ & & \sqrt{b^2d^2+c^2+d^2} \end{pmatrix}$$

$$= \begin{pmatrix} ad\sqrt{c^2+d^2} & abd^2 & ac\sqrt{b^2d^2+c^2+d^2} \\ & b^2d^2+c^2+d^2 & bc\sqrt{b^2d^2+c^2+d^2} \\ & & (c^2+d^2)\sqrt{b^2d^2+c^2+d^2} \end{pmatrix}.$$

Multiplying by $\frac{1}{(c^2+d^2)\sqrt{b^2d^2+c^2+d^2}}I$, where I is the identity matrix, we obtain

$$\begin{pmatrix} \frac{ad}{\sqrt{c^2+d^2}\sqrt{b^2d^2+c^2+d^2}} & \frac{abd^2}{(c^2+d^2)\sqrt{b^2d^2+c^2+d^2}} & \frac{ac}{c^2+d^2} \\ & \frac{\sqrt{b^2d^2+c^2+d^2}}{c^2+d^2} & \frac{bc}{c^2+d^2} \\ & & 1 \end{pmatrix}$$

and comparing this to $\begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix}$ we have the coordinates as stated. ∎

We use this to obtain:

Lemma 2.4. *For $a,b,c,d \in \mathbf{R}$ with $d>0$ we have*

$$\frac{\partial}{\partial c}\left(W\left(\begin{pmatrix} a & & \\ b & 1 & \\ c & & d \end{pmatrix}\right)\right)\Bigg|_{c=0} = \frac{2\pi i b}{d^2}W\left(\begin{pmatrix} a & & \\ b & 1 & \\ & & d \end{pmatrix}\right).$$

Proof: From the definition of W, the previous lemma, and letting

$$k(\nu_1,\nu_2) = \pi^{\frac{1}{2}-3\nu_1-3\nu_2}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)$$

we see that

$$W\left(\begin{pmatrix} a & & \\ b & 1 & \\ c & & d \end{pmatrix}\right)$$

$$= k(\nu_1,\nu_2)\left(\frac{\sqrt{b^2d^2+c^2+d^2}}{c^2+d^2}\right)^{2\nu_1+\nu_2}\left(\frac{ad\sqrt{c^2+d^2}}{b^2d^2+c^2+d^2}\right)^{\nu_1+2\nu_2}$$

$$\times\, \mathrm{e}\left(\frac{bc}{c^2+d^2}+\frac{abd^2}{b^2d^2+c^2+d^2}\right)$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\xi_3^2+\xi_2^2\left(\frac{b^2d^2+c^2+d^2}{(c^2+d^2)^2}\right)+\frac{a^2d^2}{(b^2d^2+c^2+d^2)(c^2+d^2)}\right)^{-\frac{3\nu_1}{2}}$$

$$\times\left(\xi_4^2+\xi_1^2\left(\frac{a^2d^2(c^2+d^2)}{(b^2d^2+c^2+d^2)^2}\right)+\frac{a^2d^2}{(b^2d^2+c^2+d^2)(c^2+d^2)}\right)^{-\frac{3\nu_2}{2}}$$

$$\times\, \mathrm{e}(-\xi_1-\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

We also note

$$\frac{\partial}{\partial c}\left(\left(\frac{\sqrt{b^2d^2+c^2+d^2}}{c^2+d^2}\right)^{2\nu_1+\nu_2}\right)\Bigg|_{c=0}$$

$$= \left((2\nu_1+\nu_2)\left(\frac{\sqrt{b^2d^2+c^2+d^2}}{c^2+d^2}\right)^{2\nu_1+\nu_2-1}\right.$$

$$\left.\times\frac{c(c^2+d^2)(b^2d^2+c^2+d^2)^{-\frac{1}{2}}-2c(b^2d^2+c^2+d^2)^{\frac{1}{2}}}{(c^2+d^2)^2}\right)\Bigg|_{c=0}$$

$$=0$$

$$\frac{\partial}{\partial c}\left(\left(\frac{ad\sqrt{c^2+d^2}}{b^2d^2+c^2+d^2}\right)^{\nu_1+2\nu_2}\right)\Bigg|_{c=0}$$

$$= \left((\nu_1+2\nu_2)\left(\frac{ad\sqrt{c^2+d^2}}{b^2d^2+c^2+d^2}\right)^{\nu_1+2\nu_2-1}\right.$$

$$\left.\times\frac{acd(b^2d^2+c^2+d^2)(c^2+d^2)^{-\frac{1}{2}}-2cad(c^2+d^2)^{\frac{1}{2}}}{(b^2d^2+c^2+d^2)^2}\right)\Bigg|_{c=0}$$

$$=0$$

$$\frac{\partial}{\partial c}\left(\mathrm{e}\left(\frac{abd^2}{b^2d^2+c^2+d^2}+\frac{bc}{c^2+d^2}\right)\right)\Bigg|_{c=0}$$

$$=\left(2\pi i\left(\frac{-2abcd^2}{(b^2d^2+c^2+d^2)^2}+\frac{b(c^2+d^2)-2bc^2}{(c^2+d^2)^2}\right)\mathrm{e}\left(\frac{abd^2}{b^2d^2+c^2+d^2}+\frac{bc}{c^2+d^2}\right)\right)\Bigg|_{c=0}$$

$$=\frac{2\pi ib}{d^2}\mathrm{e}\left(\frac{ab}{b^2+1}\right)$$

$$\frac{\partial}{\partial c}\left(\left(\xi_3^2+\xi_2^2\left(\frac{b^2d^2+c^2+d^2}{(c^2+d^2)^2}\right)+\frac{a^2d^2}{(b^2d^2+c^2+d^2)(c^2+d^2)}\right)^{-\frac{3\nu_1}{2}}\right)\Bigg|_{c=0}$$

$$=\left(-\frac{3\nu_1}{2}\left(\xi_3^2+\xi_2^2\left(\frac{b^2d^2+c^2+d^2}{(c^2+d^2)^2}\right)+\frac{a^2d^2}{(b^2d^2+c^2+d^2)(c^2+d^2)}\right)^{-\frac{3\nu_1}{2}-1}\right.$$

$$\left.\times\left(\left(\frac{2c(c^2+d^2)-4c(b^2d^2+c^2+d^2)}{(c^2+d^2)^3}\right)\xi_2^2+\frac{-2a^2d^2c(b^2d^2+2c^2+2d^2)}{(b^2d^2+c^2+d^2)^2(c^2+d^2)^2}\right)\right)\Bigg|_{c=0}$$

$$=0$$

$$
\begin{aligned}
&\frac{\partial}{\partial c}\left(\xi_4^2+\xi_1^2\frac{a^2d^2(c^2+d^2)}{(b^2d^2+c^2+d^2)^2}+\frac{a^2d^2}{(b^2d^2+c^2+d^2)(c^2+d^2)}\right)^{-\frac{3\nu_2}{2}}\Bigg|_{c=0}\\
&=\left(-\frac{3\nu_2}{2}\left(\xi_4^2+\xi_1^2\frac{a^2d^2(c^2+d^2)}{(b^2d^2+c^2+d^2)^2}+\frac{a^2d^2}{(b^2d^2+c^2+d^2)(c^2+d^2)}\right)^{-\frac{3\nu_2}{2}-1}\right.\\
&\quad\times\left.\left(\left(\frac{2a^2d^2c(b^2d^2+c^2+d^2)-4a^2c(c^2+d^2)}{(b^2d^2+c^2+d^2)^3}\right)\xi_1^2+\frac{-2a^2d^2c(b^2d^2+2c^2+2d^2)}{(b^2d^2+c^2+d^2)^2(c^2+d^2)^2}\right)\right)\Bigg|_{c=0}\\
&=0.
\end{aligned}
$$

Thus, by the product rule

$$
\begin{aligned}
&\frac{\partial}{\partial c}\left(W\left(\begin{pmatrix} a & & \\ b & 1 & \\ c & & d\end{pmatrix}\right)\right)\Bigg|_{c=0}\\
&\qquad=k(\nu_1,\nu_2)\left(\frac{\sqrt{b^2d^2+d^2}}{d^2}\right)^{2\nu_1+\nu_2}\left(\frac{a}{b^2+1}\right)^{\nu_1+2\nu_2}\frac{2\pi ib}{d^2}e\left(\frac{ab}{b^2+1}\right)\\
&\qquad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\xi_3^2+\xi_2^2\left(\frac{b^2+1}{d^2}\right)+\frac{a^2}{b^2d^2+d^2}\right)^{-\frac{3\nu_1}{2}}\\
&\qquad\qquad\times\left(\xi_4^2+\xi_1^2\left(\frac{a^2}{(b^2+1)^2}\right)+\frac{a^2}{b^2d^2+d^2}\right)^{-\frac{3\nu_2}{2}}e(-\xi_1-\xi_2)d\xi_1d\xi_2d\xi_3.
\end{aligned}
$$

Now using the previous lemma with $c=0$ and the definition of W, the above equals

$$
\frac{2\pi ib}{d^2}W\left(\begin{pmatrix} a & & \\ b & 1 & \\ & & d\end{pmatrix}\right).
$$

∎

In the next chapter we will be interested in twisting the L-function of an automorphic form F by a Dirichlet character mod q. In the following work we will follow the structure of [HM2]. We define a function

$$
G(\tau)=F\left(\begin{pmatrix} 1 & \frac{u}{q} & \\ & 1 & \\ & & 1\end{pmatrix}\tau\right)
$$

which depends on q and another integer u. This function will be used to simplify some of the formulas in the next chapter. Several relationships involving G will be useful.

We first note that for $n_1, n_3 \in \mathbf{Z}$ we have

$$\begin{aligned}
G\left(\begin{pmatrix}1 & & n_3\\ & 1 & n_1 q\\ & & 1\end{pmatrix}\tau\right) &= F\left(\begin{pmatrix}1 & \frac{u}{q} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & n_3\\ & 1 & n_1 q\\ & & 1\end{pmatrix}\tau\right)\\
&= F\left(\begin{pmatrix}1 & \frac{u}{q} & n_3 + u n_1\\ & 1 & n_1 q\\ & & 1\end{pmatrix}\tau\right)\\
&= F\left(\begin{pmatrix}1 & & n_3 + u n_1\\ & 1 & n_1 q\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \frac{u}{q} & \\ & 1 & \\ & & 1\end{pmatrix}\tau\right)\\
&= F\left(\begin{pmatrix}1 & \frac{u}{q} & \\ & 1 & \\ & & 1\end{pmatrix}\tau\right)\\
&= G(\tau)
\end{aligned}$$

since $\begin{pmatrix}1 & & n_3 + u n_1\\ & 1 & n_1 q\\ & & 1\end{pmatrix} \in \Gamma$. From this fact we have the Fourier expansion

$$G(\tau) = \sum_{n_1, n_3 \in \mathbf{Z}} G_{n_1}^{n_3}(\tau)$$

where

$$G_{n_1}^{n_3}(\tau) = \frac{1}{q}\int_0^1\int_0^q G\left(\begin{pmatrix}1 & & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}\tau\right) \mathrm{e}\left(-\frac{n_1 x_1}{q} - n_3 x_3\right) dx_1 dx_3.$$

With this we have the following lemmas.

Lemma 2.5. [HM2] *For $\xi_1, \xi_3 \in \mathbf{R}$ we have*

$$G_q^m\left(\begin{pmatrix}1 & & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right) = \mathrm{e}(\xi_1 + m\xi_3) G_q^m(\tau).$$

Proof: By the definition of G_q^m we have

$$\begin{aligned}
&G_q^m\left(\begin{pmatrix}1 & & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right)\\
&\qquad = \frac{1}{q}\int_0^1\int_0^q G\left(\begin{pmatrix}1 & & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right) \mathrm{e}\left(-\frac{qx_1}{q} - m x_3\right) dx_1 dx_3\\
&\qquad = \frac{1}{q}\int_0^1\int_0^q G\left(\begin{pmatrix}1 & & x_3 + \xi_3\\ & 1 & x_1 + \xi_1\\ & & 1\end{pmatrix}\tau\right) \mathrm{e}(-x_1 - m x_3) dx_1 dx_3\\
&\qquad = \frac{1}{q}\int_{\xi_3}^{1+\xi_3}\int_{\xi_1}^{q+\xi_1} G\left(\begin{pmatrix}1 & & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}\tau\right) \mathrm{e}\left(-(x_1 - \xi_1) - m(x_3 - \xi_3)\right) dx_1 dx_3
\end{aligned}$$

which by periodicity equals

$$\frac{e(\xi_1 + m\xi_3)}{q} \int_0^1 \int_0^q G\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \tau\right) e(-x_1 - mx_3)dx_1dx_3.$$

∎

Lemma 2.6. [HM2]

$$\sum_{n_3 \in \mathbf{Z}} G_q^{n_3}(\tau) = \frac{1}{q} \int_0^q G\left(\begin{pmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{pmatrix} \tau\right) e(-y)dy.$$

Proof: We have by the Fourier expansion

$$\begin{aligned}
&\frac{1}{q} \int_0^q G\left(\begin{pmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{pmatrix} \tau\right) e(-y)dy \\
&\quad = \frac{1}{q} \int_0^q \sum_{n_1, n_3 \in \mathbf{Z}} G_{n_1}^{n_3}\left(\begin{pmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{pmatrix} \tau\right) e(-y)dy. \\
&\quad = \frac{1}{q^2} \int_0^q \sum_{n_1, n_3 \in \mathbf{Z}} \int_0^1 \int_0^q G\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{pmatrix} \tau\right) \\
&\quad \times e(-y)e\left(-\frac{n_1x_1}{q} - n_3x_3\right) dx_1dx_2dy \\
&\quad = \frac{1}{q^2} \int_0^q \sum_{n_1, n_3 \in \mathbf{Z}} \int_0^1 \int_0^q G\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & y + x_1 \\ & & 1 \end{pmatrix} \tau\right) e\left(-y - \frac{n_1x_1}{q} - n_3x_3\right) dx_1dx_2dy.
\end{aligned}$$

Now by a change of variables and periodicity, the above equals

$$\frac{1}{q^2} \int_0^q \sum_{n_1, n_3 \in \mathbf{Z}} \int_0^1 \int_0^q G\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \tau\right) e\left(-y - \frac{n_1(x_1 - y)}{q} - n_3x_3\right) dx_1dx_2dy.$$

We observe that

$$\int_0^q e\left(\frac{n_1y}{q} - y\right) dy = \begin{cases} q & \text{if } n_1 = q \\ 0 & \text{if } n_1 \neq q; \end{cases}$$

thus the above equals

$$\frac{1}{q} \sum_{n_3 \in \mathbf{Z}} \int_0^1 \int_0^q G\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \tau\right) e\left(-\frac{qx_1}{q} - n_3x_3\right) dx_1dx_2,$$

and by the definition, the above equals $\sum_{n_3 \in \mathbf{Z}} G_q^{n_3}(\tau)$. ∎

Lemma 2.7. *If* $(u,q)=1$ *then*

$$G_q^{qm}(\tau) = F_q^{m-\bar{u}}\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}\tau\right).$$

Proof: Since we have assumed that $(u,q)=1$, there exist $\bar{u}, a \in \mathbf{Z}$ such that $u\bar{u}+aq=1$.

We observe

$$\begin{pmatrix} q & -u & \\ \bar{u} & a & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{u}{q} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & qx_3 \\ & 1 & \bar{u}x_3+\frac{x_1}{q} \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}.$$

We observe from the definition of G_q^{qm} that

$$G_q^{qm}(\tau) = \frac{1}{q}\int_0^1\int_0^q F\left(\begin{pmatrix} 1 & \frac{u}{q} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right)\mathrm{e}(-x_1-qmx_3)dx_1dx_3.$$

Noting that $\begin{pmatrix} q & -u & \\ \bar{u} & a & \\ & & 1 \end{pmatrix} \in \Gamma$ and using the matrix fact above we obtain

$$\frac{1}{q}\int_0^1\int_0^q F\left(\begin{pmatrix} 1 & & qx_3 \\ & 1 & \bar{u}x_3+\frac{x_1}{q} \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}\tau\right)\mathrm{e}(-x_1-qmx_3)dx_1dx_3$$

which by a change of variables equals

$$\int_0^1\int_{\bar{u}x_3}^{1+\bar{u}x_3} F\left(\begin{pmatrix} 1 & & qx_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}\tau\right)\mathrm{e}(-qx_1+q\bar{u}x_3-qmx_3)dx_1dx_3.$$

By periodicity this equals

$$\int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & & qx_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}\tau\right)\mathrm{e}(-qx_1+q\bar{u}x_3-qmx_3)dx_1dx_3.$$

Now, by another change of variables we obtain

$$\frac{1}{q}\int_0^q\int_0^1 F\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}\tau\right)\mathrm{e}(-qx_1-(m-\bar{u})x_3)dx_1dx_3$$

which by periodicity equals

$$\int_0^1\int_0^1 F\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}\tau\right)\mathrm{e}(-qx_1-(m-\bar{u})x_3)dx_1dx_3,$$

and by the definition this equals $F_q^{m-\bar{u}}\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\tau\right)$. ∎

Before we can proceed to develop the functional equation for the twisted L-function, we must evaluate the two Mellin transforms introduced in Chapter 1, along with another related Mellin transform. Before we do that, we will give a proof of equation 10.1 of [B1], for which Bump had admittedly not worked out all of the details, following the methods of [BF]. We begin with a rather technical result.

Lemma 2.8. *For* $\mathrm{Re}(\nu_1) > \frac{1}{3}$, $\mathrm{Re}(\nu_2) > \frac{1}{3}$, $-1 < \mathrm{Re}(s_1 - \nu_1 - 2\nu_2) < 0$, *and* $-1 < \mathrm{Re}(s_2 - 2\nu_1 - \nu_2) < 0$ *we have*

$$\int_0^\infty\int_0^\infty W\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= \frac{\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\Gamma\left(\frac{s_1-\nu_1-2\nu_2+1}{2}\right)\Gamma\left(\frac{s_2-2\nu_1-\nu_2+1}{2}\right)}{\pi^{s_1+s_2+\frac{1}{2}}\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right)\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)}$$
$$\times\int_0^\infty\int_0^\infty\int_{-\infty}^\infty(\xi_3^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi_3)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

Proof: Let

$$k(\nu_1,\nu_2) = \pi^{\frac{1}{2}-3\nu_1-3\nu_2}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right).$$

So

$$W\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) = k(\nu_1,\nu_2)y_1^{2\nu_1+\nu_2}y_2^{\nu_1+2\nu_2}$$
$$\times\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}(\xi_4^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}e(-\xi_1-\xi_2)d\xi_1d\xi_2d\xi_3,$$

whence

$$\int_0^\infty\int_0^\infty W\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= k(\nu_1,\nu_2)\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}(\xi_4^2+\xi_1^2y_1^2+y_2^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, e(-\xi_1-\xi_2)y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_1d\xi_2d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

Replacing ξ_4 by $\xi_1\xi_2 - \xi_3$ we obtain

$$k(\nu_1,\nu_2)\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times \mathrm{e}(-\xi_1-\xi_2)y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_1 d\xi_2 d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

We consider

$$\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty=\int_{-\infty}^\infty\int_{-\infty}^0\int_{-\infty}^0+\int_{-\infty}^\infty\int_{-\infty}^0\int_0^\infty+\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^0+\int_{-\infty}^\infty\int_0^\infty\int_0^\infty$$

and introduce a family of integrals that depend on θ by replacing

$$\begin{array}{cccc}
\xi_1\to -e^{i\theta}\xi_1 & \xi_1\to e^{-i\theta}\xi_1 & \xi_1\to -e^{i\theta}\xi_1 & \xi_1\to e^{-i\theta}\xi_1\\
\xi_2\to -e^{i\theta}\xi_2 & \xi_2\to -e^{i\theta}\xi_2 & \xi_2\to e^{-i\theta}\xi_2 & \xi_2\to e^{-i\theta}\xi_2\\
\xi_3\to e^{2i\theta}\xi_3 & \xi_3\to -\xi_3 & \xi_3\to -\xi_3 & \xi_3\to e^{-2i\theta}\xi_3\\
y_1\to e^{i\theta}y_1 & y_1\to e^{-i\theta}y_1 & y_1\to e^{i\theta}y_1 & y_1\to e^{-i\theta}y_1\\
y_2\to e^{i\theta}y_2 & y_2\to e^{i\theta}y_2 & y_2\to e^{-i\theta}y_2 & y_2\to e^{-i\theta}y_2
\end{array}$$

in each of the summands, respectively. We let I_θ equal

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_0^\infty$$
$$(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}e^{-6\nu_1 i\theta}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}e^{-6\nu_2 i\theta}$$
$$\times\mathrm{e}(e^{i\theta}\xi_1+e^{i\theta}\xi_2)y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}e^{(3\nu_1+3\nu_2+s_1+s_2+2)i\theta}$$
$$+(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_3-\xi_1\xi_2)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\mathrm{e}(-e^{-i\theta}\xi_1+e^{i\theta}\xi_2)y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}e^{(-\nu_1+\nu_2-s_1+s_2)i\theta}$$
$$+(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_3-\xi_1\xi_2)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\mathrm{e}(e^{i\theta}\xi_1-e^{-i\theta}\xi_2)y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}e^{(\nu_1-\nu_2+s_1-s_2)i\theta}$$
$$+(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}e^{6\nu_1 i\theta}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}e^{6\nu_2 i\theta}$$
$$\times\mathrm{e}(-e^{-i\theta}\xi_1-e^{-i\theta}\xi_2)y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}e^{(-3\nu_1-3\nu_2-s_1-s_2-2)i\theta}$$
$$d\xi_1 d\xi_2 d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

which in turn equals

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times \Big(\mathrm{e}(e^{i\theta}\xi_1+e^{i\theta}\xi_2)e^{(-3\nu_1-3\nu_2+s_1+s_2+2)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_1+e^{i\theta}\xi_2)e^{(-\nu_1+\nu_2-s_1+s_2)i\theta}$$
$$+\mathrm{e}(e^{i\theta}\xi_1-e^{-i\theta}\xi_2)e^{(\nu_1-\nu_2+s_1-s_2)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_1-e^{-i\theta}\xi_2)e^{(3\nu_1+3\nu_2-s_1-s_2-2)i\theta}\Big)$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_1d\xi_2d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2},$$

which equals

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times \Big(\mathrm{e}(e^{i\theta}\xi_1)e^{(-\nu_1-2\nu_2+s_1+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_1)e^{(\nu_1+2\nu_2-s_1-1)i\theta}\Big)$$
$$\times \Big(\mathrm{e}(e^{i\theta}\xi_2)e^{(-2\nu_1-\nu_2+s_2+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_2)e^{(2\nu_1+\nu_2-s_2-1)i\theta}\Big)$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_1d\xi_2d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

We see from the definition of I_θ that

$$\int_0^\infty \int_0^\infty W\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2}=k(\nu_1,\nu_2)I_0,$$

and since

$$I_\theta=\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times \Big(\mathrm{e}(e^{i\theta}\xi_1)e^{(-\nu_1-2\nu_2+s_1+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_1)e^{(\nu_1+2\nu_2-s_1-1)i\theta}\Big)$$
$$\times \Big(\mathrm{e}(e^{i\theta}\xi_2)e^{(-2\nu_1-\nu_2+s_2+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_2)e^{(2\nu_1+\nu_2-s_2-1)i\theta}\Big)$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_1d\xi_2d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

we see I_θ is absolutely convergent for $0<\theta<\frac{\pi}{2}$.

We now want to integrate along the contours given by first integrating each variable along the real axis out to a value M, then integrating along $Me^{i\phi}$ for $\phi\in[0,\theta]$, and finally going back to the origin along $xe^{i\theta}$. Cauchy's theorem gives that the integral along these contours is zero. We first let

$$f(\xi_1,\xi_2,\xi_3,y_1,y_2,\phi)=$$

$$(\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}$$
$$\left(\mathrm{e}(e^{i\phi}\xi_1)e^{(-\nu_1-2\nu_2+s_1+1)i\phi}+\mathrm{e}(-e^{-i\phi}\xi_1)e^{(\nu_1+2\nu_2-s_1-1)i\phi}\right)$$
$$\left(\mathrm{e}(e^{i\phi}\xi_2)e^{(-2\nu_1-\nu_2+s_2+1)i\phi}+\mathrm{e}(-e^{-i\phi}\xi_2)e^{(2\nu_1+\nu_2-s_2-1)i\phi}\right)$$

and observe that for $0\le\phi\le\pi$ and as $M\to\infty$ we have

$$\begin{aligned}
|Mf(M,\xi_2,\xi_3,y_1,y_2,\phi)| &\ll \left|M\left((M\xi_2-\xi_3)^2+M^2y_2^2+y_1^2y_2^2\right)^{-\frac{3\nu_2}{2}}\right| \\
&\quad\times\left|\mathrm{e}(e^{i\phi}M)e^{(-\nu_1-2\nu_2+s_1+1)i\phi}\mathrm{e}(-e^{-i\phi}M)e^{(\nu_1+2\nu_2-s_1-1)i\phi}\right| \\
&\ll M^{1-3\mathrm{Re}(\nu_2)}\left(\left|e^{2\pi i e^{i\phi}M}\right|+\left|e^{-2\pi i e^{-i\phi}M}\right|\right) \\
&\ll M^{1-3\mathrm{Re}(\nu_2)}e^{-2\pi M\sin\phi} \\
&\ll M^{1-3\mathrm{Re}(\nu_2)},
\end{aligned}$$

$$\begin{aligned}
|Mf(\xi_1,M,\xi_3,y_1,y_2,\phi)| &\ll \left|M\left(\xi_3+M^2y_1^2+y_1^2y_2^2\right)^{-\frac{3\nu_1}{2}}\left((\xi_1M-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2\right)^{-\frac{3\nu_2}{2}}\right| \\
&\quad\times\left|\mathrm{e}(e^{i\phi}M)e^{(-2\nu_1-\nu_2+s_2+1)i\phi}+\mathrm{e}(-e^{-i\phi}M)e^{(2\nu_1+\nu_2-s_2-1)i\phi}\right| \\
&\ll M^{1-3\mathrm{Re}(\nu_1+\nu_2)}\left(\left|e^{2\pi i e^{i\phi}M}\right|+\left|e^{-2\pi i e^{-i\phi}M}\right|\right) \\
&\ll M^{1-3\mathrm{Re}(\nu_1+\nu_2)}e^{-2\pi M\sin\phi} \\
&\ll M^{1-3\mathrm{Re}(\nu_1+\nu_2)},
\end{aligned}$$

$$\begin{aligned}
|Mf(\xi_1,\xi_2,M,y_1,y_2,\phi)| &\ll \left|M\left(M^2+\xi_2^2y_1^2+y_1^2y_2^2\right)^{-\frac{3\nu_1}{2}}\left((\xi_1\xi_2-M)^2+\xi_1^2y_2^2+y_1^2y_2^2\right)^{-\frac{3\nu_2}{2}}\right| \\
&\ll M^{1-3\mathrm{Re}(\nu_1+\nu_2)}
\end{aligned}$$

$$\begin{aligned}
|Mf(\xi_1,\xi_2,\xi_3,M,y_2,\phi)| &\ll \left|M\left(\xi_3^2+\xi_2^2M^2+M^2y_2^2\right)^{-\frac{3\nu_1}{2}}\right| \\
&\quad\times\left|\left((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+M^2y_2^2\right)^{-\frac{3\nu_2}{2}}M^{2\nu_1+\nu_2+s_1-1}\right| \\
&\ll M^{1+\mathrm{Re}(-\nu_1-2\nu_2+s_1-1)} \\
&= M^{\mathrm{Re}(s_1-\nu_1-2\nu_2)},
\end{aligned}$$

and

$$|Mf(\xi_1,\xi_2,\xi_3,y_1,M,\phi)| \ll \left|M\left(\xi_3^2+\xi_2^2y_1^2+y_1^2M^2\right)^{-\frac{3\nu_1}{2}}\right|$$
$$\times\left|\left((\xi_1\xi_2-\xi_3)^2+\xi_1^2M^2+y_1^2M^2\right)^{-\frac{3\nu_2}{2}}M^{\nu_1+2\nu_2+s_2-1}\right|$$
$$\ll M^{1+\mathrm{Re}(-2\nu_1-\nu_2+s_2-1)}$$
$$= M^{\mathrm{Re}(s_2-2\nu_1-\nu_2)}.$$

Now since $\mathrm{Re}(\nu_1), \mathrm{Re}(\nu_2) > \frac{1}{3}$ and $\mathrm{Re}(s_1-\nu_1-2\nu_2), \mathrm{Re}(s_2-2\nu_1-\nu_2) < 0$, we observe that as $M \to \infty$ the integral along the path $Me^{i\phi}$ for $\phi \in [0,\theta]$ goes to zero. Thus by Cauchy's theorem we have

$$I_\theta = I_0 \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{2},$$

or

$$\int_0^\infty\int_0^\infty W\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2} = k(\nu_1,\nu_2)I_\theta \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{2}.$$

We now must evaluate I_θ. Since I_θ is absolutely convergent for $0 < \theta < \frac{\pi}{2}$ we are free to interchange the order of integration as necessary, so

$$I_\theta = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((\xi_1\xi_2-\xi_3)^2+\xi_1^2y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\left(\mathrm{e}(e^{i\theta}\xi_1)e^{(-\nu_1-2\nu_2+s_1+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_1)e^{(\nu_1+2\nu_2-s_1-1)i\theta}\right)$$
$$\times\left(\mathrm{e}(e^{i\theta}\xi_2)e^{(-2\nu_1-\nu_2+s_2+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_2)e^{(2\nu_1+\nu_2-s_2-1)i\theta}\right)$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}\,d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}d\xi_1d\xi_2.$$

We now make the change of variables $y_1 \to y_1\xi_1$, $y_2 \to y_2\xi_2$, and $\xi_3 \to \xi_1\xi_2\xi_3$ and obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi_3^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi_3)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}(\xi_1\xi_2)^{-3\nu_1-3\nu_2}$$
$$\times\left(\mathrm{e}(e^{i\theta}\xi_1)e^{(-\nu_1-2\nu_2+s_1+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_1)e^{(\nu_1+2\nu_2-s_1-1)i\theta}\right)$$
$$\times\left(\mathrm{e}(e^{i\theta}\xi_2)e^{(-2\nu_1-\nu_2+s_2+1)i\theta}+\mathrm{e}(-e^{-i\theta}\xi_2)e^{(2\nu_1+\nu_2-s_2-1)i\theta}\right)$$
$$\times(\xi_1y_1)^{2\nu_1+\nu_2+s_1-1}(\xi_2y_2)^{\nu_1+2\nu_2+s_2-1}\xi_1\xi_2\,d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}d\xi_1d\xi_2$$

which simplifies to

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty (\xi_3^2 + y_1^2 + y_1^2 y_2^2)^{-\frac{3\nu_1}{2}} ((1-\xi_3)^2 + y_2^2 + y_1^2 y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times \left(\mathrm{e}(e^{i\theta}\xi_1) e^{(-\nu_1-2\nu_2+s_1+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_1) e^{(\nu_1+2\nu_2-s_1-1)i\theta} \right)$$
$$\times \left(\mathrm{e}(e^{i\theta}\xi_2) e^{(-2\nu_1-\nu_2+s_2+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_2) e^{(2\nu_1+\nu_2-s_2-1)i\theta} \right)$$
$$\times \xi_1^{s_1-\nu_1-2\nu_2} \xi_2^{s_2-2\nu_1-\nu_2} y_1^{2\nu_1+\nu_2+s_1-1} y_2^{\nu_1+2\nu_2+s_2-1} d\xi_3 \frac{dy_1}{y_1} \frac{dy_2}{y_2} d\xi_1 d\xi_2.$$

We now interchange the order of integration again and split up the integrals to obtain

$$\int_0^\infty \left(\mathrm{e}(e^{i\theta}\xi_1) e^{(-\nu_1-2\nu_2+s_1+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_1) e^{(\nu_1+2\nu_2-s_1-1)i\theta} \right) \xi_1^{s_1-\nu_1-2\nu_2} d\xi_1$$
$$\times \int_0^\infty \left(\mathrm{e}(e^{i\theta}\xi_2) e^{(-2\nu_1-\nu_2+s_2+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_2) e^{(2\nu_1+\nu_2-s_2-1)i\theta} \right) \xi_2^{s_2-2\nu_1-\nu_2} d\xi_2$$
$$\times \int_0^\infty \int_0^\infty \int_{-\infty}^\infty (\xi_3^2 + y_1^2 + y_1^2 y_2^2)^{-\frac{3\nu_1}{2}} ((1-\xi_3)^2 + y_2^2 + y_1^2 y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1} y_2^{\nu_1+2\nu_2+s_2-1} d\xi_3 \frac{dy_1}{y_1} \frac{dy_2}{y_2}.$$

Thus for $0 < \theta < \frac{\pi}{2}$ we have

$$\int_0^\infty \int_0^\infty W\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \right) y_1^{s_1-1} y_2^{s_2-1} \frac{dy_1}{y_1} \frac{dy_2}{y_2}$$
$$= k(\nu_1, \nu_2) \int_0^\infty \left(\mathrm{e}(e^{i\theta}\xi_1) e^{(-\nu_1-2\nu_2+s_1+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_1) e^{(\nu_1+2\nu_2-s_1-1)i\theta} \right) \xi_1^{s_1-\nu_1-2\nu_2} d\xi_1$$
$$\times \int_0^\infty \left(\mathrm{e}(e^{i\theta}\xi_2) e^{(-2\nu_1-\nu_2+s_2+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_2) e^{(2\nu_1+\nu_2-s_2-1)i\theta} \right) \xi_2^{s_2-2\nu_1-\nu_2} d\xi_2$$
$$\times \int_0^\infty \int_0^\infty \int_{-\infty}^\infty (\xi_3^2 + y_1^2 + y_1^2 y_2^2)^{-\frac{3\nu_1}{2}} ((1-\xi_3)^2 + y_2^2 + y_1^2 y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1} y_2^{\nu_1+2\nu_2+s_2-1} d\xi_3 \frac{dy_1}{y_1} \frac{dy_2}{y_2}.$$

We will now evaluate these first two integrals by letting $\theta \to \frac{\pi}{2}$ and observing they are both of the form

$$\int_0^\infty \left(\mathrm{e}(it) i^a + \mathrm{e}(it) i^{-a} \right) t^{a-1} dt$$

or

$$(i^a + i^{-a}) \int_0^\infty \mathrm{e}(it) t^a \frac{dt}{t}.$$

Now noting that

$$\begin{aligned} i^a + i^{-a} &= e^{\frac{a\pi i}{2}} + e^{-\frac{a\pi i}{2}} \\ &= \mathrm{e}\left(\frac{a}{4}\right) + \mathrm{e}\left(-\frac{a}{4}\right) \end{aligned}$$

and recalling that $\mathrm{e}(t) + \mathrm{e}(-t) = 2\cos(2\pi t)$, we have

$$i^a + i^{-a} = 2\cos\left(\frac{a\pi}{2}\right).$$

We also observe that for $\mathrm{Re}(a) > 0$ we have

$$\begin{aligned} \int_0^\infty \mathrm{e}(it)t^a \frac{dt}{t} &= \int_0^\infty e^{-2\pi t} t^a \frac{dt}{t} \\ &= \int_0^\infty e^{-t}\left(\frac{t}{2\pi}\right)^a \frac{dt}{t} \\ &= (2\pi)^{-a}\Gamma(a); \end{aligned}$$

thus

$$\int_0^\infty \left(\mathrm{e}(it)i^a + \mathrm{e}(it)i^{-a}\right) t^{a-1} dt = \frac{2}{(2\pi)^a}\cos\left(\frac{a\pi}{2}\right)\Gamma(a) \quad \text{for} \quad \mathrm{Re}(a) > 0.$$

We note ([GR] 8.334.2) that

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \frac{\pi}{\cos(\pi x)};$$

thus we have

$$\frac{2}{(2\pi)^a}\cos\left(\frac{a\pi}{2}\right)\Gamma(a) = \frac{\Gamma(a)}{(2\pi)^{a-1}\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1-a}{2}\right)}.$$

We also recall the doubling formula ([GR] 8.335.1)

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$$

so

$$\Gamma(a) = \frac{2^{a-1}}{\sqrt{\pi}}\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)$$

or

$$\frac{\Gamma(a)}{\Gamma\left(\frac{1+a}{2}\right)} = \frac{2^{a-1}\Gamma\left(\frac{a}{2}\right)}{\sqrt{\pi}}$$

which then gives

$$\int_0^\infty \left(\mathrm{e}(it)i^a + \mathrm{e}(it)i^{-a}\right) t^{a-1}dt = \frac{\Gamma\left(\frac{a}{2}\right)}{\pi^{a-\frac{1}{2}}\Gamma\left(\frac{1-a}{2}\right)} \quad \text{for} \quad \mathrm{Re}(a) > 0.$$

Hence for $\mathrm{Re}(s_1 - \nu_1 - 2\nu_2 + 1) > 0$ we have

$$\int_0^\infty \left(\mathrm{e}(e^{i\theta}\xi_1)e^{(-\nu_1-2\nu_2+s_1+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_1)e^{(\nu_1+2\nu_2-s_1-1)i\theta}\right)\xi_1^{s_1-\nu_1-2\nu_2}d\xi_1$$
$$= \frac{\Gamma\left(\frac{s_1-\nu_1-2\nu_2+1}{2}\right)}{\pi^{s_1-\nu_1-2\nu_2+\frac{1}{2}}\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right)}$$

and for $\mathrm{Re}(s_2 - 2\nu_1 - \nu_2 + 1) > 0$ we have

$$\int_0^\infty \left(\mathrm{e}(e^{i\theta}\xi_2)e^{(-2\nu_1-\nu_2+s_2+1)i\theta} + \mathrm{e}(-e^{-i\theta}\xi_2)e^{(2\nu_1+\nu_2-s_2-1)i\theta}\right)\xi_2^{s_2-2\nu_1-\nu_2}d\xi_2$$
$$= \frac{\Gamma\left(\frac{s_2-2\nu_1-\nu_2+1}{2}\right)}{\pi^{s_2-2\nu_1-\nu_2+\frac{1}{2}}\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)}.$$

Thus for $\mathrm{Re}(\nu_1) > \frac{1}{3}$, $\mathrm{Re}(\nu_2) > \frac{1}{3}$, $-1 < \mathrm{Re}(s_1 - \nu_1 - 2\nu_2) < 0$, and $-1 < \mathrm{Re}(s_2 - 2\nu_1 - \nu_2) < 0$ we have that

$$\int_0^\infty\int_0^\infty W\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= k(\nu_1,\nu_2)\frac{\Gamma\left(\frac{s_1-\nu_1-2\nu_2+1}{2}\right)\Gamma\left(\frac{s_2-2\nu_1-\nu_2+1}{2}\right)}{\pi^{s_1+s_2-3\nu_1-3\nu_2+1}\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right)\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)}$$
$$\times\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi_3^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi_3)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= \frac{\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\Gamma\left(\frac{s_1-\nu_1-2\nu_2+1}{2}\right)\Gamma\left(\frac{s_2-2\nu_1-\nu_2+1}{2}\right)}{\pi^{s_1+s_2+\frac{1}{2}}\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right)\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)}$$
$$\times\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi_3^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi_3)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}. \qquad \blacksquare$$

We now want to concern ourselves with the integral

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi_3^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi_3)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi_3\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

For this integral we have the following lemma:

Lemma 2.9. *For* $\mathrm{Re}(\nu_1) > \frac{2}{3}$, $\mathrm{Re}(\nu_2) > \frac{2}{3}$, $-1 < \mathrm{Re}(s_1 - \nu_1 - 2\nu_2) < 0$, *and* $-1 < \mathrm{Re}(s_2 - 2\nu_1 - \nu_2) < 0$ *we have*

$$\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{\nu_2-\nu_1+s_1}{2}\right)\Gamma\left(\frac{\nu_1-\nu_2+s_2}{2}\right)\Gamma\left(\frac{2\nu_1+\nu_2+s_1-1}{2}\right)\Gamma\left(\frac{\nu_1+2\nu_2+s_2-1}{2}\right)}{4\Gamma\left(\frac{s_1+s_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)}$$
$$\times\,\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right).$$

Proof: We observe that by the definition of the gamma function,

$$\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$=\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi\frac{dy_1}{y_1}\frac{dy_2}{y_2}\int_0^\infty t_1^{\frac{3\nu_1}{2}}e^{-t_1}\frac{dt_1}{t_1}\int_0^\infty t_2^{\frac{3\nu_2}{2}}e^{-t_2}\frac{dt_2}{t_2}$$

which in turn equals

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_0^\infty\left(\frac{t_1}{\xi^2+y_1^2+y_1^2y_2^2}\right)^{\frac{3\nu_1}{2}}\left(\frac{t_2}{(1-\xi)^2+y_2^2+y_1^2y_2^2}\right)^{\frac{3\nu_2}{2}}$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}e^{-t_1-t_2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}d\xi\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

We make the change of variables

$$t_1 \to \frac{t_1(\xi^2+y_1^2+y_1^2y_2^2)}{y_1^2}$$
$$t_2 \to \frac{t_2((1-\xi)^2+y_2^2+y_1^2y_2^2)}{y_2^2}$$

and obtain

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_0^\infty\left(\frac{t_1}{y_1^2}\right)^{\frac{3\nu_1}{2}}\left(\frac{t_2}{y_2^2}\right)^{\frac{3\nu_2}{2}}y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}$$
$$\times e^{-\frac{t_1(\xi^2+y_1^2+y_1^2y_2^2)}{y_1^2}-\frac{t_2((1-\xi)^2+y_2^2+y_1^2y_2^2)}{y_2^2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}d\xi\frac{dy_1}{y_1}\frac{dy_2}{y_2},$$

which equals

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty t_1^{\frac{3\nu_1}{2}} t_2^{\frac{3\nu_2}{2}} y_1^{\nu_2-\nu_1+s_1-1} y_2^{\nu_1-\nu_2+s_2-1} \times e^{-\frac{t_1\xi^2}{y_1^2}-\frac{t_2(1-\xi)^2}{y_2^2}} e^{-t_1(1+y_2^2)-t_2(1+y_1^2)} \frac{dt_1}{t_1}\frac{dt_2}{t_2} d\xi \frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

Since we have assumed that $\mathrm{Re}(\nu_1) > \frac{2}{3}$ and $\mathrm{Re}(\nu_2) > \frac{2}{3}$, the inner integrals are absolutely convergent, so we can interchange the order of integration and proceed to do the ξ integral. We observe

$$\begin{aligned}\int_{-\infty}^\infty e^{-Ax^2-B(1-x)^2} dx &= \int_{-\infty}^\infty e^{-(A+B)\left(x^2-\frac{2Bx}{A+B}+\frac{B}{A+B}\right)} dx \\ &= \int_{-\infty}^\infty e^{-(A+B)\left(\left(x-\frac{B}{A+B}\right)^2+\frac{B}{A+B}-\frac{B^2}{(A+B)^2}\right)} dx \\ &= e^{\frac{-AB}{A+B}} \int_{-\infty}^\infty e^{-(A+B)\left(x-\frac{B}{A+B}\right)^2} dx \\ &= e^{\frac{-AB}{A+B}} \int_{-\infty}^\infty e^{-(A+B)x^2} dx \\ &= e^{\frac{-AB}{A+B}} \sqrt{\frac{\pi}{A+B}}.\end{aligned}$$

Thus

$$\begin{aligned}\int_{-\infty}^\infty e^{-\frac{t_1\xi^2}{y_1^2}-\frac{t_2(1-\xi)^2}{y_2^2}} d\xi &= e^{\frac{-t_1t_2}{y_1^2y_2^2\left(\frac{t_1}{y_1^2}+\frac{t_2}{y_2^2}\right)}} \sqrt{\frac{\pi}{\frac{t_1}{y_1^2}+\frac{t_2}{y_2^2}}} \\ &= e^{\frac{-t_1t_2}{t_1y_2^2+t_2y_1^2}} \sqrt{\frac{\pi y_1^2y_2^2}{t_1y_2^2+t_2y_1^2}}.\end{aligned}$$

So, the above equals

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty t_1^{\frac{3\nu_1}{2}} t_2^{\frac{3\nu_2}{2}} y_1^{\nu_2-\nu_1+s_1-1} y_2^{\nu_1-\nu_2+s_2-1} \times \sqrt{\frac{\pi y_1^2y_2^2}{t_1y_2^2+t_2y_1^2}} e^{\frac{-t_1t_2}{t_1y_2^2+t_2y_1^2}} e^{-t_1(1+y_2^2)-t_2(1+y_1^2)} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

which equals

$$\sqrt{\pi}\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty t_1^{\frac{3\nu_1}{2}} t_2^{\frac{3\nu_2}{2}} y_1^{\nu_2-\nu_1+s_1} y_2^{\nu_1-\nu_2+s_2} \times \frac{e^{\frac{-t_1t_2}{t_1y_2^2+t_2y_1^2}} e^{-t_1(1+y_2^2)-t_2(1+y_1^2)}}{\sqrt{t_1y_2^2+t_2y_1^2}} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

We now observe $\mathrm{Re}(\nu_2-\nu_1+s_1)=\mathrm{Re}(s_1-\nu_1-2\nu_2)+3\mathrm{Re}(\nu_2)>-1+2=1$ and similarly $\mathrm{Re}(\nu_1-\nu_2+s_2)>1$, so the above integral is absolutely convergent. We now interchange the order of integration, make the change of variables $y_1\to\sqrt{\frac{y_1}{t_2}}$, $y_2\to\sqrt{\frac{y_2}{t_1}}$, and obtain

$$\frac{\sqrt{\pi}}{4}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty t_1^{\frac{3\nu_1}{2}}t_2^{\frac{3\nu_2}{2}}\left(\frac{y_1}{t_2}\right)^{\frac{\nu_2-\nu_1+s_1}{2}}\left(\frac{y_2}{t_1}\right)^{\frac{\nu_1-\nu_2+s_2}{2}}$$
$$\times\frac{e^{\frac{-t_1t_2}{y_2+y_1}}e^{-t_1-y_2-t_2-y_1}}{\sqrt{y_2+y_1}}\frac{dy_1}{y_1}\frac{dy_2}{y_2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

which equals

$$\frac{\sqrt{\pi}}{4}\int_0^\infty\int_0^\infty t_1^{\frac{2\nu_1+\nu_2-s_2}{2}}t_2^{\frac{2\nu_2+\nu_1-s_1}{2}}e^{-t_1-t_2}$$
$$\times\int_0^\infty\int_0^\infty\frac{y_1^{\frac{\nu_2-\nu_1+s_1}{2}}y_2^{\frac{\nu_1-\nu_2+s_2}{2}}e^{\frac{-t_1t_2}{y_2+y_1}-y_2-y_1}}{\sqrt{y_2+y_1}}\frac{dy_1}{y_1}\frac{dy_2}{y_2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

So we need to evaluate

$$\int_0^\infty\int_0^\infty\frac{y_1^Ay_2^Be^{\frac{-C}{y_2+y_1}-y_2-y_1}}{\sqrt{y_2+y_1}}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

for $\mathrm{Re}(A)>\frac{1}{2}$, $\mathrm{Re}(B)>\frac{1}{2}$, and C any positive real number. We note that in this region the integrals are absolutely convergent. Under the change of variables $y_1\to y_1-y_2$ the above equals

$$\int_0^\infty\int_{y_2}^\infty\frac{(y_1-y_2)^{A-1}y_2^{B-1}e^{\frac{-C}{y_1}-y_1}}{\sqrt{y_1}}dy_1dy_2.$$

Since we are in the region of absolute convergence we can interchange the order of integration to obtain

$$\int_0^\infty\frac{e^{\frac{-C}{y_1}-y_1}}{\sqrt{y_1}}\int_0^{y_1}(y_1-y_2)^{A-1}y_2^{B-1}dy_2dy_1.$$

Letting $y_2\to y_1y_2$ this becomes

$$\int_0^\infty\frac{e^{\frac{-C}{y_1}-y_1}}{\sqrt{y_1}}\int_0^1(y_1-y_1y_2)^{A-1}(y_1y_2)^{B-1}y_1dy_2dy_1,$$

which in turn equals

$$\int_0^\infty y_1^{B+A-\frac{3}{2}} e^{\frac{-C}{y_1}-y_1} \int_0^1 (1-y_2)^{A-1}(y_2)^{B-1} dy_2 dy_1.$$

Recalling ([GR] 3.191.3) that

$$\int_0^1 t^{z-1}(1-t)^{w-1}dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \quad \text{for} \quad \mathrm{Re}(z), \mathrm{Re}(w) > 0$$

the above equals

$$\frac{\Gamma(A)\Gamma(B)}{\Gamma(A+B)} \int_0^\infty y_1^{B+A-\frac{3}{2}} e^{\frac{-C}{y_1}-y_1} dy_1.$$

Thus using the fact ([GR] 3.471.12) that

$$\int_0^\infty x^{\nu-1} e^{-x-\frac{\mu^2}{4x}} dx = 2\left(\frac{\mu}{2}\right)^\nu K_\nu(\mu) \quad \text{for} \quad |\arg(\mu)| < \frac{\pi}{2} \text{ and } \mathrm{Re}(\mu^2) > 0$$

we obtain

$$\int_0^\infty \int_0^\infty \frac{y_1^A y_2^B e^{\frac{-C}{y_2+y_1}-y_2-y_1}}{\sqrt{y_2+y_1}} \frac{dy_1}{y_1}\frac{dy_2}{y_2} = 2C^{\frac{B}{2}+\frac{A}{2}-\frac{1}{4}} K_{B+A-\frac{1}{2}}\left(2\sqrt{C}\right)\frac{\Gamma(A)\Gamma(B)}{\Gamma(A+B)}$$

for $\mathrm{Re}(A) > \frac{1}{2}$, $\mathrm{Re}(B) > \frac{1}{2}$, and C any positive real number. So we have

$$\int_0^\infty \int_0^\infty \frac{y_1^{\frac{\nu_2-\nu_1+s_1}{2}} y_2^{\frac{\nu_1-\nu_2+s_2}{2}} e^{\frac{-t_1t_2}{y_2+y_1}-y_2-y_1}}{\sqrt{y_2+y_1}} \frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= 2(t_1t_2)^{\frac{s_1+s_2-1}{4}} K_{\frac{s_1+s_2-1}{2}}\left(2\sqrt{t_1t_2}\right) \frac{\Gamma\left(\frac{\nu_2-\nu_1+s_1}{2}\right)\Gamma\left(\frac{\nu_1-\nu_2+s_2}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)},$$

whence

$$\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\int_0^\infty\int_0^\infty\int_{-\infty}^\infty (\xi^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times y_1^{2\nu_1+\nu_2+s_1-1} y_2^{\nu_1+2\nu_2+s_2-1} d\xi \frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$= \frac{\sqrt{\pi}\Gamma\left(\frac{\nu_2-\nu_1+s_1}{2}\right)\Gamma\left(\frac{\nu_1-\nu_2+s_2}{2}\right)}{2\Gamma\left(\frac{s_1+s_2}{2}\right)}$$
$$\times \int_0^\infty\int_0^\infty t_1^{\nu_1+\frac{\nu_2}{2}+\frac{s_1}{4}-\frac{s_2}{4}-\frac{1}{4}} t_2^{\nu_2+\frac{\nu_1}{2}+\frac{s_2}{4}-\frac{s_1}{4}-\frac{1}{4}} e^{-t_1-t_2} K_{\frac{s_1+s_2-1}{2}}\left(2\sqrt{t_1t_2}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Under the change of variables $t_1 \to \dfrac{t_1^2}{t_2}$ we have

$$\begin{aligned}\int_0^\infty \int_0^\infty & t_1^{\nu_1+\frac{\nu_2}{2}+\frac{s_1}{4}-\frac{s_2}{4}-\frac{1}{4}} t_2^{\nu_2+\frac{\nu_1}{2}+\frac{s_2}{4}-\frac{s_1}{4}-\frac{1}{4}} e^{-t_1-t_2} K_{\frac{s_1+s_2-1}{2}}\left(2\sqrt{t_1 t_2}\right) \frac{dt_1}{t_1}\frac{dt_2}{t_2} \\ &= 2\int_0^\infty \int_0^\infty t_1^{2\nu_1+\nu_2+\frac{s_1}{2}-\frac{s_2}{2}-\frac{1}{2}} t_2^{\frac{\nu_2}{2}-\frac{\nu_1}{2}+\frac{s_2}{2}-\frac{s_1}{2}} e^{-\frac{t_1^2}{t_2}-t_2} K_{\frac{s_1+s_2-1}{2}}(2t_1) \frac{dt_1}{t_1}\frac{dt_2}{t_2}.\end{aligned}$$

Since we have the exponential function and

$$\begin{aligned}\mathrm{Re}&\left(2\nu_1+\nu_2+\frac{s_1}{2}-\frac{s_2}{2}-\frac{1}{2}\right) \\ &= \frac{1}{2}\Big(\mathrm{Re}(s_1-\nu_1-\nu_2)+\mathrm{Re}(2\nu_1+\nu_2-s_2)+\mathrm{Re}(3\nu_1+3\nu_2-1)\Big) \\ &> \frac{-1+0+2+2-1}{2} = 1\end{aligned}$$

both of the integrals are absolutely convergent. So we can interchange the order of integration and let $t_2 \to \dfrac{t_1}{t_2}$ to obtain

$$2\int_0^\infty \int_0^\infty t_1^{\frac{3\nu_1+3\nu_2-1}{2}} t_2^{\frac{\nu_1-\nu_2+s_1-s_2}{2}} e^{-t_1 t_2-\frac{t_1}{t_2}} K_{\frac{s_1+s_2-1}{2}}(2t_1) \frac{dt_2}{t_2}\frac{dt_1}{t_1}.$$

Using the fact that

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-\frac{z}{2}(w+\frac{1}{w})} w^\nu \frac{dw}{w}$$

the above becomes

$$4\int_0^\infty t_1^{\frac{3\nu_1+3\nu_2-1}{2}} K_{\frac{\nu_1-\nu_2+s_1-s_2}{2}}(2t_1) K_{\frac{s_1+s_2-1}{2}}(2t_1) \frac{dt_1}{t_1}.$$

Now, we recall ([GR] 6.576.4) that for $\mathrm{Re}(a+b)>0$ and $\mathrm{Re}(\lambda)+|\mathrm{Re}(\mu)|+|\mathrm{Re}(\nu)|<1$ we have

$$\begin{aligned}\int_0^\infty & x^{-\lambda} K_\mu(ax) K_\nu(bx)\,dx \\ &= 2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda+\mu-\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right) \\ &\times \frac{\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)}{\Gamma(1-\lambda)}\end{aligned}$$

where F is the hypergeometric function. We observe

$$\mathrm{Re}\left(\frac{3-3\nu_1-3\nu_2}{2}\right)+\left|\mathrm{Re}\left(\frac{\nu_1-\nu_2+s_1-s_2}{2}\right)\right|+\left|\mathrm{Re}\left(\frac{s_1+s_2-1}{2}\right)\right|$$
$$=\mathrm{Re}\left(\frac{3-3\nu_1-3\nu_2}{2}\right)+\frac{1}{2}\Big|\mathrm{Re}(s_1-\nu_1-2\nu_2)+\mathrm{Re}(2\nu_1+\nu_2-s_2)\Big|$$
$$+\frac{1}{2}\Big|\mathrm{Re}(s_1-\nu_1-2\nu_2)+\mathrm{Re}(s_2-2\nu_1-\nu_2)+3\mathrm{Re}(\nu_1)+3\mathrm{Re}(\nu_2)-1\Big|.$$

Since

$$\mathrm{Re}(s_1-\nu_1-2\nu_2)+\mathrm{Re}(s_2-2\nu_1-\nu_2)+3\mathrm{Re}(\nu_1)+3\mathrm{Re}(\nu_2)-1>1$$

the above equals

$$\frac{1}{2}\Big|\mathrm{Re}(s_1-\nu_1-2\nu_2)+\mathrm{Re}(2\nu_1+\nu_2-s_2)\Big|+\frac{1}{2}\Big(\mathrm{Re}(s_1-\nu_1-2\nu_2)+\mathrm{Re}(s_2-2\nu_1-\nu_2)+2\Big)$$

which by our hypotheses is less than 1. Thus we can apply the above identity, and using the fact that $F(a,b;c;0)=1$ we obtain

$$4\int_0^\infty t_1^{\frac{3\nu_1+3\nu_2-1}{2}}K_{\frac{\nu_1-\nu_2+s_1-s_2}{2}}(2t_1)K_{\frac{s_1+s_2-1}{2}}(2t_1)\frac{dt_1}{t_1}$$
$$=\frac{\Gamma\left(\frac{2\nu_1+\nu_2+s_1-1}{2}\right)\Gamma\left(\frac{\nu_1+2\nu_2+s_2-1}{2}\right)\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right)}{2\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)}.$$

Thus

$$\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right)\int_0^\infty\int_0^\infty\int_{-\infty}^\infty(\xi^2+y_1^2+y_1^2y_2^2)^{-\frac{3\nu_1}{2}}((1-\xi)^2+y_2^2+y_1^2y_2^2)^{-\frac{3\nu_2}{2}}$$
$$\times\, y_1^{2\nu_1+\nu_2+s_1-1}y_2^{\nu_1+2\nu_2+s_2-1}d\xi\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$=\frac{\sqrt{\pi}\Gamma\left(\frac{\nu_2-\nu_1+s_1}{2}\right)\Gamma\left(\frac{\nu_1-\nu_2+s_2}{2}\right)\Gamma\left(\frac{2\nu_1+\nu_2+s_1-1}{2}\right)\Gamma\left(\frac{\nu_1+2\nu_2+s_2-1}{2}\right)}{4\Gamma\left(\frac{s_1+s_2}{2}\right)\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)}$$
$$\times\,\Gamma\left(\frac{2\nu_1+\nu_2-s_2}{2}\right)\Gamma\left(\frac{\nu_1+2\nu_2-s_1}{2}\right).\qquad\blacksquare$$

Combining Lemmas 2.8 and 2.9 we obtain:

Corollary 2.10. *For* $\mathrm{Re}(\nu_1)>\frac{2}{3}$, $\mathrm{Re}(\nu_2)>\frac{2}{3}$, $-1<\mathrm{Re}(s_1-\nu_1-2\nu_2)<0$, *and* $-1<\mathrm{Re}(s_2-2\nu_1-\nu_2)<0$ *we have*

$$\int_0^\infty\int_0^\infty W\left(\begin{pmatrix}y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right)y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$=\frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{4\pi^{s_1+s_2}\Gamma\left(\frac{s_1+s_2}{2}\right)}$$

where, as before,

$$\alpha = -\nu_1 - 2\nu_2 + 1$$

$$\beta = -\nu_1 + \nu_2$$

$$\gamma = 2\nu_1 + \nu_2 - 1.$$

We now recall Theorem 2.1 of [B1].

Theorem 2.11. *There exist $N_1(\nu_1, \nu_2)$ and $N_2(\nu_1, \nu_2)$ which depend on ν_1 and ν_2 in a continuous fashion such that if $n_1 > N_1(\nu_1, \nu_2)$ and $n_2 > N_2(\nu_1, \nu_2)$ then*

$$W\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) y_1^{n_1} y_2^{n_2}$$

is bounded on $\mathcal{H}$.

By analytic continuation we obtain equation 10.1 of [B1], namely

Corollary 2.12. *For $\mathrm{Re}(s_1 - 1) > N_1(\nu_1, \nu_2)$ and $\mathrm{Re}(s_2 - 1) > N_2(\nu_1, \nu_2)$*

$$\int_0^\infty \int_0^\infty W\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) y_1^{s_1 - 1} y_2^{s_2 - 1} \frac{dy_1}{y_1} \frac{dy_2}{y_2}$$
$$= \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right) \Gamma\left(\frac{s_1+\beta}{2}\right) \Gamma\left(\frac{s_1+\gamma}{2}\right) \Gamma\left(\frac{s_2-\alpha}{2}\right) \Gamma\left(\frac{s_2-\beta}{2}\right) \Gamma\left(\frac{s_2-\gamma}{2}\right)}{4\pi^{s_1+s_2} \Gamma\left(\frac{s_1+s_2}{2}\right)}$$

and the integral is absolutely convergent.

By the Mellin inversion formula we have a correction ([BF] p. 208) to equation 10.2 of [B1] which should read

Corollary 2.13. *For $\mathrm{Re}(s_1 - 1) > N_1(\nu_1, \nu_2)$ and $\mathrm{Re}(s_2 - 1) > N_2(\nu_1, \nu_2)$*

$$W\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right)$$
$$= \frac{1}{4\pi^2 (2\pi i)^2} \int_{\sigma - i\infty}^{\sigma + i\infty} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right) \Gamma\left(\frac{s_1+\beta}{2}\right) \Gamma\left(\frac{s_1+\gamma}{2}\right) \Gamma\left(\frac{s_2-\alpha}{2}\right) \Gamma\left(\frac{s_2-\beta}{2}\right) \Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}$$
$$\times (\pi y_1)^{1-s_1} (\pi y_2)^{1-s_2} \, ds_1 ds_2$$

where σ is sufficiently large.

We can now evaluate the Mellin transforms occurring in Chapter 1. We recall that if W is of type (ν_1, ν_2) then $\widetilde{W}$ is of type (ν_2, ν_1). Thus

$$\widetilde{\alpha} = -\nu_2 - 2\nu_1 + 1$$

$$\widetilde{\beta} = -\nu_2 + \nu_1$$

$$\widetilde{\gamma} = 2\nu_2 + \nu_1 - 1.$$

So we see

$$\widetilde{\alpha} = -\gamma, \quad \widetilde{\beta} = -\beta, \quad \text{and} \quad \widetilde{\gamma} = -\alpha.$$

With this we now have

Lemma 2.14. [HM1] *$\widetilde{\Phi}(s,w)$ is absolutely convergent for $\mathrm{Re}(w+s) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) > \widetilde{N}_2(\nu_1,\nu_2)$ and*

$$\widetilde{\Phi}(s,w) = \frac{\Gamma\left(\frac{w+s+1-\alpha}{2}\right)\Gamma\left(\frac{w+s+1-\beta}{2}\right)\Gamma\left(\frac{w+s+1-\gamma}{2}\right)\Gamma\left(\frac{w+1+\alpha}{2}\right)\Gamma\left(\frac{w+1+\beta}{2}\right)\Gamma\left(\frac{w+1+\gamma}{2}\right)}{4\pi^{2w+s+2}\Gamma\left(\frac{2w+s+2}{2}\right)}.$$

Proof: We have by a change of variables

$$\begin{aligned}\widetilde{\Phi}(s,w) &= \int_0^\infty\int_0^\infty \widetilde{W}\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right) t^w v^s \frac{dt}{t}\frac{dv}{v} \\ &= \int_0^\infty\int_0^\infty \widetilde{W}\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1\end{pmatrix}\right) t^w v^{w+s} \frac{dt}{t}\frac{dv}{v}.\end{aligned}$$

Now applying Corollary 2.12, we obtain

$$\frac{\Gamma\left(\frac{w+s+1+\widetilde{\alpha}}{2}\right)\Gamma\left(\frac{w+s+1+\widetilde{\beta}}{2}\right)\Gamma\left(\frac{w+s+1+\widetilde{\gamma}}{2}\right)\Gamma\left(\frac{w+1-\widetilde{\alpha}}{2}\right)\Gamma\left(\frac{w+1-\widetilde{\beta}}{2}\right)\Gamma\left(\frac{w+1-\widetilde{\gamma}}{2}\right)}{4\pi^{2w+s+2}\Gamma\left(\frac{2w+s+2}{2}\right)}$$

which in turn equals

$$\frac{\Gamma\left(\frac{w+s+1-\alpha}{2}\right)\Gamma\left(\frac{w+s+1-\beta}{2}\right)\Gamma\left(\frac{w+s+1-\gamma}{2}\right)\Gamma\left(\frac{w+1+\alpha}{2}\right)\Gamma\left(\frac{w+1+\beta}{2}\right)\Gamma\left(\frac{w+1+\gamma}{2}\right)}{4\pi^{2w+s+2}\Gamma\left(\frac{2w+s+2}{2}\right)}.$$

∎

For the other Mellin transform of Chapter 1 we have

Lemma 2.15. [HM1] $\Phi(s,w)$ *is absolutely convergent for* $\mathrm{Re}(s) > N_1(\nu_1,\nu_2)$ *and* $\mathrm{Re}(w) > N_2(\nu_1,\nu_2)$ *and*

$$\Phi(s,w) = \frac{\Gamma\left(\frac{w-\alpha+1}{2}\right)\Gamma\left(\frac{w-\beta+1}{2}\right)\Gamma\left(\frac{w-\gamma+1}{2}\right)\Gamma\left(\frac{s-w-\alpha}{2}\right)\Gamma\left(\frac{s-w-\beta}{2}\right)\Gamma\left(\frac{s-w-\gamma}{2}\right)}{4\pi^{s+w+\frac{3}{2}}\Gamma\left(\frac{s-2w}{2}\right)}.$$

Proof: We first note

$$\begin{pmatrix} tv & & \\ yv & v & \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+y^2}} & \frac{y}{\sqrt{1+y^2}} & \\ \frac{-y}{\sqrt{1+y^2}} & \frac{1}{\sqrt{1+y^2}} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{ty}{y^2+1} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{tv}{\sqrt{1+y^2}} & & \\ & v\sqrt{1+y^2} & \\ & & 1 \end{pmatrix}.$$

Thus, by invariance under ZK and using the fact that

$$W\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) = \mathrm{e}(x_1+x_2)W(\tau)$$

we have

$$\begin{aligned}\Phi(s,w) &= \int_0^\infty\int_0^\infty\int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ yv & v & \\ & & 1 \end{pmatrix}\right)t^w v^s dy\frac{dt}{t}\frac{dv}{v} \\ &= \int_0^\infty\int_0^\infty\int_{-\infty}^\infty W\left(\begin{pmatrix} \frac{tv}{\sqrt{1+y^2}} & & \\ & v\sqrt{1+y^2} & \\ & & 1 \end{pmatrix}\right)\mathrm{e}\left(\frac{ty}{y^2+1}\right)t^w v^s dy\frac{dt}{t}\frac{dv}{v}.\end{aligned}$$

By Theorem 2.11, for $n_1 > N_1(\nu_1,\nu_2)$ and $n_2 > N_2(\nu_1,\nu_2)$ we have that

$$\left|W\left(\begin{pmatrix} \frac{tv}{\sqrt{1+y^2}} & & \\ & v\sqrt{1+y^2} & \\ & & 1 \end{pmatrix}\right)\left(v\sqrt{1+y^2}\right)^{n_1}\left(\frac{t}{1+y^2}\right)^{n_2}\right|$$

is bounded. Thus

$$W\left(\begin{pmatrix} \frac{tv}{\sqrt{1+y^2}} & & \\ & v\sqrt{1+y^2} & \\ & & 1 \end{pmatrix}\right)$$

is bounded as $y \to 0$. Taking n_1 large we see this function is rapidly decreasing as $y \to \pm\infty$ and $v \to \infty$, and taking n_2 large it is rapidly decreasing as $t \to \infty$. Thus $\Phi(s,w)$ is absolutely convergent for $\mathrm{Re}(s) > N_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) > N_2(\nu_1,\nu_2)$.

Now by a change of variables we have

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty W\left(\begin{pmatrix} \frac{tv}{\sqrt{1+y^2}} & & \\ & v\sqrt{1+y^2} & \\ & & 1 \end{pmatrix}\right) \mathrm{e}\left(\frac{ty}{y^2+1}\right) t^w v^s dy \frac{dt}{t}\frac{dv}{v}$$

$$= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1 \end{pmatrix}\right) \frac{\mathrm{e}(ty)}{(1+y^2)^{\frac{s-2w}{2}}} t^w v^s dy \frac{dt}{t}\frac{dv}{v}.$$

We recall the fact that, for the K-Bessel function, we have

$$\int_{-\infty}^\infty \frac{\mathrm{e}(ty)}{(1+y^2)^\nu} dy = \frac{2\pi^\nu}{\Gamma(\nu)} t^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}(2\pi t).$$

So the above equals

$$\frac{2\pi^{\frac{s-2w}{2}}}{\Gamma\left(\frac{s-2w}{2}\right)} \int_0^\infty \int_0^\infty W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1 \end{pmatrix}\right) K_{\frac{s-2w-1}{2}}(2\pi t) t^{\frac{s-1}{2}} v^s dy \frac{dt}{t}\frac{dv}{v}.$$

As in [B2] we let

$$K_{\alpha,\beta,\gamma}(y_1, y_2)$$
$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}$$
$$\times \left(\frac{y_1}{2}\right)^{-s_1} \left(\frac{y_2}{2}\right)^{-s_2} ds_1 ds_2.$$

By Corollary 2.13 we have

$$W\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right)$$
$$= \frac{1}{4\pi^2(2\pi i)^2} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}$$
$$\times (\pi y_1)^{1-s_1} (\pi y_2)^{1-s_2} ds_1 ds_2$$

which in turn equals

$$\frac{y_1 y_2}{4(2\pi i)^2} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}$$
$$\times (\pi y_1)^{-s_1} (\pi y_2)^{-s_2} ds_1 ds_2.$$

Thus we have

$$W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1 \end{pmatrix}\right) = \frac{tv}{4} K_{\alpha,\beta,\gamma}(2\pi v, 2\pi t),$$

and so

$$\frac{2\pi^{\frac{s-2w}{2}}}{\Gamma\left(\frac{s-2w}{2}\right)}\int_0^\infty\int_0^\infty W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1\end{pmatrix}\right)K_{\frac{s-2w-1}{2}}(2\pi t)t^{\frac{s-1}{2}}v^s dy\frac{dt}{t}\frac{dv}{v}$$
$$=\frac{\pi^{\frac{s-2w}{2}}}{2\Gamma\left(\frac{s-2w}{2}\right)}\int_0^\infty\int_0^\infty K_{\alpha,\beta,\gamma}(2\pi v,2\pi t)K_{\frac{s-2w-1}{2}}(2\pi t)t^{\frac{s+1}{2}}v^{s+1}\frac{dt}{t}\frac{dv}{v}.$$

By a change of variables this equals

$$\frac{\pi^{\frac{s-2w}{2}}}{2(2\pi)^{\frac{3s+3}{2}}\Gamma\left(\frac{s-2w}{2}\right)}\int_0^\infty\int_0^\infty K_{\alpha,\beta,\gamma}(v,t)K_{\frac{s-2w-1}{2}}(t)t^{\frac{s+1}{2}}v^{s+1}\frac{dt}{t}\frac{dv}{v}.$$

By examining [B2] we see equation 1.2 should read

$$\int_0^\infty\int_0^\infty K_{\alpha,\beta,\gamma}(y_1,y_2)K_\nu(y_2)(y_1^2y_2)^s\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$
$$=2^{3s-1}\Gamma\left(\frac{s-\alpha-\nu}{2}\right)\Gamma\left(\frac{s-\beta-\nu}{2}\right)\Gamma\left(\frac{s-\gamma-\nu}{2}\right)$$
$$\times\Gamma\left(\frac{s-\alpha+\nu}{2}\right)\Gamma\left(\frac{s-\beta+\nu}{2}\right)\Gamma\left(\frac{s-\gamma+\nu}{2}\right).$$

So the above double integral equals

$$\frac{\pi^{\frac{s-2w}{2}}2^{\frac{3s+1}{2}}\Gamma\left(\frac{w-\alpha+1}{2}\right)\Gamma\left(\frac{w-\beta+1}{2}\right)\Gamma\left(\frac{w-\gamma+1}{2}\right)\Gamma\left(\frac{s-w-\alpha}{2}\right)\Gamma\left(\frac{s-w-\beta}{2}\right)\Gamma\left(\frac{s-w-\gamma}{2}\right)}{2(2\pi)^{\frac{3s+3}{2}}\Gamma\left(\frac{s-2w}{2}\right)}.$$

∎

We now must introduce one more Mellin transform which will appear in the functional equation for the twisted L-function. We let

$$\widehat{\Phi}(s,w)=\int_0^\infty\int_0^\infty\int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ yv & v & \\ & & 1\end{pmatrix}\right)yt^wv^s dy\frac{dt}{t}\frac{dv}{v}.$$

For this function we have

Lemma 2.16. *$\widehat{\Phi}(s,w)$ is absolutely convergent for* $\mathrm{Re}(s)>N_1(\nu_1,\nu_2)$ *and* $\mathrm{Re}(w)>N_2(\nu_1,\nu_2)$ *and*

$$\widehat{\Phi}(s,w)=\frac{\Gamma\left(\frac{w-\alpha+2}{2}\right)\Gamma\left(\frac{w-\beta+2}{2}\right)\Gamma\left(\frac{w-\gamma+2}{2}\right)\Gamma\left(\frac{s-w-\alpha-1}{2}\right)\Gamma\left(\frac{s-w-\beta-1}{2}\right)\Gamma\left(\frac{s-w-\gamma-1}{2}\right)}{-4i\pi^{s+w+\frac{3}{2}}\Gamma(\frac{s-2w}{2})}.$$

Proof: As in the proof of Lemma 2.15 we have

$$\widehat{\Phi}(s,w) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ yv & v & \\ & & 1 \end{pmatrix}\right) y t^w v^s dy \frac{dt}{t}\frac{dv}{v}$$

$$= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1 \end{pmatrix}\right) \frac{\mathrm{e}\,(ty)}{(1+y^2)^{\frac{s-2w}{2}}} y t^w v^s dy \frac{dt}{t}\frac{dv}{v}$$

which is absolutely convergent for $\mathrm{Re}(s) > N_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) > N_2(\nu_1,\nu_2)$. So we need to evaluate

$$\int_{-\infty}^\infty \frac{y\mathrm{e}\,(ty)}{(1+y^2)^\nu} dy.$$

We observe that differentiating the identity in Lemma 2.15 yields

$$2\pi i \int_{-\infty}^\infty \frac{y\mathrm{e}\,(ty)}{(1+y^2)^\nu} dy = \frac{2\pi^\nu}{\Gamma(\nu)}\frac{d}{dt}\left(t^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}(2\pi t)\right).$$

We see from equation 8.486.12 of [GR] that

$$z\frac{d}{dz}K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z),$$

so

$$\frac{d}{dt}\left(t^\nu K_\nu(2\pi t)\right) = -2\pi t^\nu K_{\nu-1}(2\pi t).$$

Thus

$$\int_{-\infty}^\infty \frac{y\mathrm{e}\,(ty)}{(1+y^2)^\nu} dy = \frac{-2\pi^\nu}{i\Gamma(\nu)} t^{\nu-\frac{1}{2}} K_{\nu-\frac{3}{2}}(2\pi t).$$

So we have

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1 \end{pmatrix}\right) \frac{\mathrm{e}\,(ty)}{(1+y^2)^{\frac{s-2w}{2}}} y t^w v^s dy \frac{dt}{t}\frac{dv}{v}$$

$$= \frac{-2\pi^{\frac{s-2w}{2}}}{i\Gamma(\frac{s-2w}{2})} \int_0^\infty \int_0^\infty W\left(\begin{pmatrix} tv & & \\ & v & \\ & & 1 \end{pmatrix}\right) K_{\frac{s-2w-3}{2}}(2\pi t) t^{\frac{s-2w-1}{2}} t^w v^s dy \frac{dt}{t}\frac{dv}{v}.$$

As in the proof of Lemma 2.15 this equals

$$\frac{-\pi^{\frac{s-2w}{2}}}{2i(2\pi)^{\frac{3s+3}{2}}\Gamma(\frac{s-2w}{2})} \int_0^\infty \int_0^\infty K_{\alpha,\beta,\gamma}(v,t) K_{\frac{s-2w-3}{2}}(t) t^{\frac{s+1}{2}} v^{s+1} \frac{dt}{t}\frac{dv}{v}.$$

Now applying the corrected version of equation 1.2 of [B2] we obtain

$$\frac{-\pi^{\frac{s-2w}{2}} 2^{\frac{3s+1}{2}} \Gamma\left(\frac{w-\alpha+2}{2}\right)\Gamma\left(\frac{w-\beta+2}{2}\right)\Gamma\left(\frac{w-\gamma+2}{2}\right)\Gamma\left(\frac{s-w-\alpha-1}{2}\right)\Gamma\left(\frac{s-w-\beta-1}{2}\right)\Gamma\left(\frac{s-w-\gamma-1}{2}\right)}{2i(2\pi)^{\frac{3s+3}{2}}\Gamma(\frac{s-2w}{2})}. \blacksquare$$

CHAPTER III

FUNCTIONAL EQUATION

Before we can begin to derive the functional equation for the twisted L-function we must prove a result about characters. For χ a Dirichlet character, we let $\overline{\chi}(n)$ be the complex conjugate of $\chi(n)$. Since $|\chi(n)|$ is 0 or 1 we have $\chi(n)\overline{\chi}(n) = 1$ if $\chi(n) \neq 0$. In particular, we have $\chi(-1) = \pm 1$. If $\chi(-1) = 1$ we say χ is *even* and if $\chi(-1) = -1$ we say χ is *odd*. For any Dirichlet character χ mod q, the *Gauss sum associated with* χ is

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) \mathrm{e}\Big(\frac{n}{q}\Big).$$

Finally a Dirichlet character mod q is *primitive* if for each positive integer $d|q$, there exists $a \equiv 1 \bmod d$, $(a,q) = 1$ such that $\chi(a) \neq 1$. With this we have

Lemma 3.1. *If χ is a primitive character mod q then*

$$\chi(n) = \frac{\tau(\chi)\chi(-1)}{q} \sum_{a=1}^{q} \overline{\chi}(a) \mathrm{e}\Big(\frac{an}{q}\Big)$$

where $\tau(\chi)$ is the Gauss sum.

Proof: By finite Fourier expansion we have

$$\chi(n) = \sum_{a=1}^{q} f(a) \mathrm{e}\Big(\frac{an}{q}\Big) \quad \text{where} \quad f(a) = \frac{1}{q} \sum_{m=1}^{q} \chi(m) \mathrm{e}\Big(\frac{-am}{q}\Big).$$

We now must consider the sum in the expression for $f(a)$ in two separate cases.

If $(a,q)=1$ then we have

$$\begin{aligned}\sum_{m=1}^{q}\chi(m)\mathrm{e}\Big(\frac{-am}{q}\Big) &= \sum_{n=1}^{q}\chi(-\overline{a}n)e\Big(\frac{n}{q}\Big)\\ &= \chi(\overline{a})\chi(-1)\sum_{n=1}^{q}\chi(n)\mathrm{e}\Big(\frac{n}{q}\Big)\\ &= \overline{\chi}(a)\chi(-1)\tau(\chi)\end{aligned}$$

where $a\overline{a}\equiv 1 \bmod q$. Otherwise we have $(a,q)>1$. In this case we let $d_a=(a,q)$ and $kd_a=q$. Now for any $b\in\mathbf{Z}$ such that $(b,q)=1$ and $b\equiv 1 \bmod k$ we have

$$\begin{aligned}\sum_{m=1}^{q}\chi(m)\mathrm{e}\Big(\frac{-am}{q}\Big) &= \sum_{n=1}^{q}\chi(bn)\mathrm{e}\Big(\frac{-abn}{q}\Big)\\ &= \chi(b)\sum_{n=1}^{q}\chi(n)\mathrm{e}\Big(\frac{-abn}{q}\Big).\end{aligned}$$

Choosing $z\in\mathbf{Z}$ such that $b=kz+1$, we have $\dfrac{abn}{q}=\dfrac{an}{q}+\dfrac{anz}{d}$. But $d|a$, so $\dfrac{anz}{d}\in\mathbf{Z}$, whence $\mathrm{e}\Big(\dfrac{-abn}{q}\Big)=\mathrm{e}\Big(\dfrac{-an}{q}\Big)$. Thus

$$\sum_{m=1}^{q}\chi(m)\mathrm{e}\Big(\frac{-am}{q}\Big)=\chi(b)\sum_{n=1}^{q}\chi(n)\mathrm{e}\Big(\frac{-an}{q}\Big).$$

If we assume $\displaystyle\sum_{m=1}^{q}\chi(m)\mathrm{e}\Big(\frac{-am}{q}\Big)\neq 0$ then $\chi(b)=1$, so we have $k|q$. Since $d>1$ we have a $k<q$ such that for all $b\equiv 1 \bmod k$ with $(b,q)=1$ we have $\chi(b)=1$. This contradicts the fact that χ is primitive, whence

$$\sum_{m=1}^{q}\chi(m)\mathrm{e}\Big(\frac{-am}{q}\Big)=0.$$

Since $(a,q)>1$ we have $\overline{\chi}(a)=0$, so

$$\sum_{m=1}^{q}\chi(m)\mathrm{e}\Big(\frac{-am}{q}\Big)=\overline{\chi}(a)\chi(-1)\tau(\chi).$$

Thus in either case

$$\chi(n)=\frac{\tau(\chi)\chi(-1)}{q}\sum_{a=1}^{q}\overline{\chi}(a)\mathrm{e}\Big(\frac{an}{q}\Big). \qquad \blacksquare$$

We can now proceed to derive the functional equation for the twisted L-function. As in the untwisted case (Chapter 1) we would like to begin with

$$L_{\overline{\chi}}(w,\widetilde{F})\widetilde{\Phi}(s-1,w-1)$$

and express the L-function as a sum over both the positive and negative integers instead of just the positive integers. However, recalling that $a_{n_1,n_2} = a_{|n_1|,|n_2|}$, we have

$$\begin{aligned}\sum_{n\neq 0} \frac{\widetilde{a}_{1,n}\overline{\chi}(n)}{|n|^w} &= \sum_{n=1}^{\infty} \frac{\widetilde{a}_{1,n}\,(\overline{\chi}(n)+\overline{\chi}(-n))}{n^w} \\ &= \begin{cases} 2L_{\overline{\chi}}(w,\widetilde{F}) & \text{if } \chi \text{ is even} \\ 0 & \text{if } \chi \text{ is odd.}\end{cases}\end{aligned}$$

We take our idea from the derivation of the functional equation for

$$L_{\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

which is worked out in Chapter 9 of [D] and consider

$$\begin{aligned}\sum_{n\neq 0} \frac{n\widetilde{a}_{1,n}\overline{\chi}(n)}{|n|^w} &= \sum_{n=1}^{\infty} \frac{\widetilde{a}_{1,n}\,(\overline{\chi}(n)-\overline{\chi}(-n))}{n^{w-1}} \\ &= \begin{cases} 0 & \text{if } \chi \text{ is even} \\ 2L_{\overline{\chi}}(w-1,\widetilde{F}) & \text{if } \chi \text{ is odd.}\end{cases}\end{aligned}$$

Introducing the parameter $\delta_n^{\chi} = \begin{cases} 1 & \text{if } \chi \text{ is even} \\ n & \text{if } \chi \text{ is odd}\end{cases}$ we obtain

$$\sum_{n\neq 0} \frac{\widetilde{a}_{1,n}\overline{\chi}(n)\delta_n^{\chi}}{|n|^w} = \begin{cases} 2L_{\overline{\chi}}(w,\widetilde{F}) & \text{if } \chi \text{ is even} \\ 2L_{\overline{\chi}}(w-1,\widetilde{F}) & \text{if } \chi \text{ is odd.}\end{cases}$$

Thus we will take as our starting point

$$\widetilde{\Phi}(s-1,w-1)\sum_{n\neq 0} \frac{\widetilde{a}_{1,n}\overline{\chi}(n)\delta_n^{\chi}}{|n|^w}.$$

For this we observe:

Lemma 3.2. *Let F be a cusp form and χ a character mod q with $(a,q) = d_a$ and $aa' \equiv -d_a \bmod q$. If χ is even, we have, for* $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ *and* $\mathrm{Re}(w) >$

$\max\left(\widetilde{N}_2(\nu_1,\nu_2)+1,2\right)$,

$$\frac{q^{3w}\chi(-1)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)\sum_{n\neq 0}\frac{\widetilde{a}_{1,n}\overline{\chi}(n)\delta_n^{\chi}}{|n|^w}$$
$$=\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1&z&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}d_a&\frac{a'}{q}&\\&\frac{1}{d_a}&\frac{a\bar{u}}{qd_a}\\&&1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3}&&\\&v&\\&&1\end{pmatrix}\right)\Bigg|_{z=0}$$
$$\times\, t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

and in this region all of the integrals and sums are absolutely convergent. If χ is odd, we have, for $\mathrm{Re}(s+w-2)>\widetilde{N}_1(\nu_1,\nu_2)$ *and* $\mathrm{Re}(w)>\max\left(\widetilde{N}_2(\nu_1,\nu_2)+1,3\right)$,

$$\frac{q^{3w}\chi(-1)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)\sum_{n\neq 0}\frac{\widetilde{a}_{1,n}\overline{\chi}(n)\delta_n^{\chi}}{|n|^w}$$
$$=\frac{1}{2\pi i}\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty$$
$$\times\frac{\partial}{\partial z}\left(\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1&z&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}d_a&\frac{a'}{q}&\\&\frac{1}{d_a}&\frac{a\bar{u}}{qd_a}\\&&1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3}&&\\&v&\\&&1\end{pmatrix}\right)\right)\Bigg|_{z=0}t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

and in this region all of the sums and integrals are absolutely convergent.

Proof: We first observe, that by equations 4.7 and 4.8 of [B1] we have

$$\widetilde{F}^0_d\left(\begin{pmatrix}1&z&\\&1&\\&&1\end{pmatrix}\tau\right)=\sum_{n\in\mathbf{Z}}\widetilde{F}_{d,n}\left(\begin{pmatrix}1&z&\\&1&\\&&1\end{pmatrix}\tau\right)$$
$$=\sum_{n\in\mathbf{Z}}\mathrm{e}(nz)\widetilde{F}_{d,n}(\tau).$$

Thus,

$$\frac{\partial}{\partial z}\left(\widetilde{F}^0_d\left(\begin{pmatrix}1&z&\\&1&\\&&1\end{pmatrix}\tau\right)\right)\Bigg|_{z=0}=\frac{\partial}{\partial z}\left(\sum_{n\in\mathbf{Z}}\mathrm{e}(nz)\widetilde{F}_{d,n}(\tau)\right)\Bigg|_{z=0}$$
$$=2\pi i\sum_{n\in\mathbf{Z}}n\widetilde{F}_{d,n}(\tau)$$

and

$$\widetilde{F}^0_d\left(\begin{pmatrix}1&z&\\&1&\\&&1\end{pmatrix}\tau\right)\Bigg|_{z=0}=\sum_{n\in\mathbf{Z}}\widetilde{F}_{d,n}(\tau).$$

Since $\widetilde{F}$ is a cusp form, we have that $\widetilde{F}_{d,0}(\tau) = 0$, so

$$\sum_{n\neq 0} \delta_n^{\chi} \widetilde{F}_{d,n}(\tau) = \begin{cases} \widetilde{F}_d^0\left(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}\tau\right)\Big|_{z=0} & \text{if } \chi \text{ is even} \\ \dfrac{1}{2\pi i}\dfrac{\partial}{\partial z}\left(\widetilde{F}_d^0\left(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}\tau\right)\right)\Big|_{z=0} & \text{if } \chi \text{ is odd.} \end{cases}$$

Thus, in either case, we are interested in

$$\int_0^\infty \int_0^\infty \sum_{n\neq 0} \delta_n^{\chi} \widetilde{F}_{d_a,n}\left(\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) t^{w-1} v^{s-1} \frac{dt}{t}\frac{dv}{v}$$

which by a change of variables equals

$$\int_0^\infty \int_0^\infty \sum_{n\neq 0} \delta_n^{\chi} \widetilde{F}_{d_a,n}\left(\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} t & & \\ & v & \\ & & 1 \end{pmatrix}\right) (tq^3)^{w-1} v^{s-1} \frac{dt}{t}\frac{dv}{v}$$

and in turn equals

$$q^{3w-3}\int_0^\infty \int_0^\infty \sum_{n\neq 0} \delta_n^{\chi} \widetilde{F}_{d_a,n}\left(\begin{pmatrix} 1 & \frac{a'd_a}{q} & \\ & 1 & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} td_a & & \\ & \frac{v}{d_a} & \\ & & 1 \end{pmatrix}\right) t^{w-1} v^{s-1} \frac{dt}{t}\frac{dv}{v}.$$

Now applying equations 4.8 and 4.12 of [B1] we obtain

$$q^{3w-3}\int_0^\infty \int_0^\infty \sum_{n\neq 0} \frac{\widetilde{a}_{d_a,n}\delta_n^{\chi}}{|d_a n|} \widetilde{W}\left(\begin{pmatrix} d_a n & & \\ & d_a & \\ & & 1 \end{pmatrix}\begin{pmatrix} td_a & & \\ & \frac{v}{d_a} & \\ & & 1 \end{pmatrix}\right) \times \mathrm{e}\left(\frac{a\bar{u}}{q} + \frac{na'd_a}{q}\right) t^{w-1} v^{s-1} \frac{dt}{t}\frac{dv}{v}$$

which by interchanging the order of integration and summation equals

$$q^{3w-3}\sum_{n\neq 0}\int_0^\infty \int_0^\infty \frac{\widetilde{a}_{d_a,n}\delta_n^{\chi}}{d_a|n|} \widetilde{W}\left(\begin{pmatrix} tnd_a^2 & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}\left(\frac{a\bar{u}}{q} + \frac{na'd_a}{q}\right) t^{w-1} v^{s-1} \frac{dt}{t}\frac{dv}{v}.$$

We observe

$$\begin{aligned} \widetilde{W}\left(\begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix}\right) &= \widetilde{W}\left(\begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix}\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right) \\ &= \widetilde{W}\left(\begin{pmatrix} -A & & \\ & B & \\ & & C \end{pmatrix}\right). \end{aligned}$$

Thus the above equals

$$q^{3w-3}\sum_{n\neq 0}\int_0^\infty\int_0^\infty \frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a|n|}\widetilde{W}\left(\begin{pmatrix} t|n|d_a^2 & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{a\bar{u}}{q}+\frac{na'd_a}{q}\right)t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}.$$

With a change of variables this equals

$$q^{3w-3}\sum_{n\neq 0}\int_0^\infty\int_0^\infty \frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a|n|}\widetilde{W}\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{a\bar{u}}{q}+\frac{na'd_a}{q}\right)\left(\frac{t}{|n|d_a^2}\right)^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

which in turn equals

$$q^{3w-3}\sum_{n\neq 0}\frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a^{2w-1}|n|^w}\mathrm{e}\left(\frac{a\bar{u}}{q}+\frac{na'd_a}{q}\right)\int_0^\infty\int_0^\infty \widetilde{W}\left(\begin{pmatrix} t & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}.$$

By Lemma 2.14 this double integral is absolutely convergent for $\mathrm{Re}(s+w-2)>\widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w-1)>\widetilde{N}_2(\nu_1,\nu_2)$, and since $\widetilde{a}_{n_1,n_2}=O(|n_1n_2|)$ the sum is absolutely convergent for $\mathrm{Re}(w)>2$ if χ is even and for $\mathrm{Re}(w)>3$ if χ is odd.

Thus we have shown

$$\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty\sum_{n\neq 0}\delta_n^\chi\widetilde{F}_{d_a,n}\left(\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2)>\widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w)>\max\left(\widetilde{N}_2(\nu_1,\nu_2)+1,2\right)$ if χ is even, and for $\mathrm{Re}(s+w-2)>\widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w)>\max\left(\widetilde{N}_2(\nu_1,\nu_2)+1,3\right)$ if χ is odd. In either case it equals

$$q^{3w-3}\widetilde{\Phi}(s-1,w-1)\sum_{u=1}^{q}\overline{\chi}(u)\sum_{n\neq 0}\frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a^{2w-1}|n|^w}\mathrm{e}\left(\frac{a\bar{u}}{q}+\frac{na'd_a}{q}\right)$$

which is absolutely convergent in this region.

Now by applying Lemma 3.1 to the sum over u we have

$$\begin{aligned}\sum_{u=1}^{q}\overline{\chi}(u)\sum_{a=1}^{q^2}\sum_{n\neq 0}\frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a^{2w-1}|n|^w}\mathrm{e}\left(\frac{a\bar{u}}{q}+\frac{na'd_a}{q}\right)&\\ &=\sum_{a=1}^{q^2}\sum_{n\neq 0}\frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a^{2w-1}|n|^w}\mathrm{e}\left(\frac{na'd_a}{q}\right)\sum_{u=1}^{q}\overline{\chi}(u)\mathrm{e}\left(\frac{a\bar{u}}{q}\right)\\ &=\frac{q}{\tau(\overline{\chi})\overline{\chi}(-1)}\sum_{a=1}^{q^2}\sum_{n\neq 0}\frac{\widetilde{a}_{d_a,n}\delta_n^\chi}{d_a^{2w-1}|n|^w}\mathrm{e}\left(\frac{na'd_a}{q}\right)\overline{\chi}(a).\end{aligned}$$

Now since $\overline{\chi}(a)=0$ unless $(a,q)=1$, in which case $d_a=1$ and $aa'\equiv -1 \bmod q$, this equals

$$\frac{q}{\tau(\overline{\chi})\overline{\chi}(-1)}\sum_{\substack{a=1\\(a,q)=1}}^{q^2}\sum_{n\neq 0}\frac{\widetilde{a}_{1,n}\delta_n^{\chi}}{|n|^w}\mathrm{e}\left(-\frac{n\overline{a}}{q}\right)\overline{\chi}(a),$$

which by interchanging the order of summation and by periodicity equals

$$\frac{q^2}{\tau(\overline{\chi})\overline{\chi}(-1)}\sum_{n\neq 0}\frac{\widetilde{a}_{1,n}\delta_n^{\chi}}{|n|^w}\sum_{\substack{a=1\\(a,q)=1}}^{q}\mathrm{e}\left(-\frac{n\overline{a}}{q}\right)\overline{\chi}(a).$$

Finally, applying Lemma 3.1, we obtain

$$\frac{q^3\chi(-1)}{(\tau(\overline{\chi}))^2}\sum_{n\neq 0}\frac{\widetilde{a}_{1,n}\overline{\chi}(n)\delta_n^{\chi}}{|n|^w}. \qquad \blacksquare$$

Hence, from Lemma 3.2 we are interested in expressing

$$\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\overline{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$

in terms of F. If χ is even we will simply evaluate this function at $z=0$. Whereas if χ is odd we will first differentiate this function with respect to z and then evaluate it at $z=0$. In either case, for this function we have

Lemma 3.3. *For $t,v,z\in\mathbf{R}$, $(u,q)=1$, $d_a=(a,q)$, $u\overline{u}\equiv 1 \bmod q$, $aa'\equiv -d_a \bmod q$, and $d_k=(k,q)$ we have*

$$q\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\overline{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$=\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_q^b\left(\begin{pmatrix}q & & \\ \overline{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\overline{u})}{q}\right)dy.$$

Proof: We first observe, by equation 4.2 of [B1], that

$$\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\overline{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$=\int_0^1\int_0^1\widetilde{F}\left(\begin{pmatrix}1 & & y \\ & 1 & x \\ & & 1\end{pmatrix}\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\overline{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}(-xd_a)\,dx\,dy$$
$$=\int_0^1\int_0^1\widetilde{F}\left(\begin{pmatrix}1 & z & y \\ & 1 & x \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\overline{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}(-xd_a)\,dx\,dy.$$

We have $aa' \equiv -d_a \bmod q$, so there exists $k \in \mathbf{Z}$ such that $kq - aa' = d_a$. Thus by a change of variables, the above equals

$$\int_{\frac{a\bar{u}z}{qd_a}-\frac{k\bar{u}}{q}}^{1+\frac{a\bar{u}z}{qd_a}-\frac{k\bar{u}}{q}} \int_0^1 \widetilde{F}\left(\begin{pmatrix} 1 & z & y+\frac{k\bar{u}}{q}-\frac{a\bar{u}z}{qd_a} \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \times \mathrm{e}(-xd_a)dxdy$$

which by periodicity equals

$$\int_0^1 \int_0^1 \widetilde{F}\left(\begin{pmatrix} 1 & z & y+\frac{k\bar{u}}{q}-\frac{a\bar{u}z}{qd_a} \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-xd_a)dxdy.$$

Integrating over the region

$$\left\{(x,y) : 0 \le x \le \frac{q}{d_a}, \frac{xd_aa'}{q} \le y \le \frac{xd_aa'}{q} + \frac{d_a}{q}\right\}$$

we obtain

$$\int_0^{\frac{q}{d_a}} \int_{\frac{xd_aa'}{q}}^{\frac{d_a}{q}+\frac{xd_aa'}{q}} \widetilde{F}\left(\begin{pmatrix} 1 & z & y+\frac{k\bar{u}}{q}-\frac{a\bar{u}z}{qd_a} \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \times \mathrm{e}(-xd_a)dydx.$$

By a change of variables, the above equals

$$\begin{aligned} &\int_0^{\frac{d_a}{q}} \int_0^{\frac{q}{d_a}} \widetilde{F}\left(\begin{pmatrix} 1 & z & y+\frac{k\bar{u}}{q}+\frac{xd_aa'}{q}-\frac{a\bar{u}z}{qd_a} \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \\ &\quad \times \mathrm{e}(-xd_a)dxdy \\ &\quad = \int_0^{\frac{d_a}{q}} \int_0^{\frac{q}{d_a}} \widetilde{F}\left(\begin{pmatrix} d_a & \frac{a'}{q}+\frac{z}{d_a} & y+\frac{ku}{q}+\frac{xd_aa'}{q} \\ & \frac{1}{d_a} & x+\frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-xd_a)dxdy \\ &\quad = \int_0^{\frac{d_a}{q}} \int_0^{\frac{q}{d_a}} \widetilde{F}\left(\begin{pmatrix} d_a & \frac{a'}{q} & \frac{k\bar{u}}{q} \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{z}{d_a^2} & \frac{y}{d_a} \\ & 1 & xd_a \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-xd_a)dxdy. \end{aligned}$$

Again, by a change of variables, the above equals

$$\frac{1}{q^3}\int_0^{q^2} \int_0^{q} \widetilde{F}\left(\begin{pmatrix} d_a & \frac{a'}{q} & \frac{k\bar{u}}{q} \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{z}{d_a^2} & \frac{y}{q^3} \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-x)dxdy.$$

We note that

$$\begin{pmatrix} d_a & \frac{a'}{q} & \frac{k\bar{u}}{q} \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{z}{d_a^2} & \frac{y}{q^3} \\ & 1 & x \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{d_a}{q^3} & \frac{a'}{q} & \frac{k\bar{u}}{q} \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix} \begin{pmatrix} t & \frac{zq^3v}{d_a^2} & y \\ & v & x \\ & & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{q}{d_a} & -a' & \\ -\frac{a}{d_a} & k & \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{d_a}{q^3} & \frac{a'}{q} & \frac{k\bar{u}}{q} \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{q^2} & & \frac{\bar{u}}{q} \\ -\frac{a}{q^3} & \frac{1}{q} & \\ & & 1 \end{pmatrix}.$$

Now, since $\begin{pmatrix} \frac{q}{d_a} & -a' & \\ -\frac{a}{d_a} & k & \\ & & 1 \end{pmatrix} \in \Gamma$, the above integral equals

$$\frac{1}{q^3} \int_0^{q^2} \int_0^q \widetilde{F} \left(\begin{pmatrix} \frac{1}{q^2} & & \frac{\bar{u}}{q} \\ -\frac{a}{q^3} & \frac{1}{q} & \\ & & 1 \end{pmatrix} \begin{pmatrix} t & \frac{zq^3v}{d_a^2} & y \\ & v & x \\ & & 1 \end{pmatrix} \right) e(-x)dxdy.$$

We are in the situation that $u\bar{u} \equiv 1 \bmod q$, so there exists m such that $u\bar{u} - mq = 1$, and we note that

$$\begin{pmatrix} & \bar{u} & m \\ -1 & & \\ & q & u \end{pmatrix} \begin{pmatrix} 1 & & -\frac{a}{q^2} \\ & 1 & -\frac{u}{q} \\ & & 1 \end{pmatrix} \begin{pmatrix} & -\frac{1}{q} & \\ & & \frac{1}{q} \\ -\frac{1}{q} & & \end{pmatrix} = \begin{pmatrix} \frac{1}{q^2} & & \frac{\bar{u}}{q} \\ -\frac{a}{q^3} & \frac{1}{q} & \\ & & 1 \end{pmatrix}.$$

Since $\begin{pmatrix} & \bar{u} & m \\ -1 & & \\ & q & u \end{pmatrix} \in \Gamma$, and after multiplying by qI, the above integral becomes

$$\frac{1}{q^3} \int_0^{q^2} \int_0^q \widetilde{F} \left(\begin{pmatrix} 1 & & -\frac{a}{q^2} \\ & 1 & -\frac{u}{q} \\ & & 1 \end{pmatrix} \begin{pmatrix} & -1 & \\ & & 1 \\ -1 & & \end{pmatrix} \begin{pmatrix} t & \frac{zq^3v}{d_a^2} & y \\ & v & x \\ & & 1 \end{pmatrix} \right) e(-x)dxdy.$$

We note that

$$\widetilde{F} \left(\begin{pmatrix} 1 & & -\frac{a}{q^2} \\ & 1 & -\frac{u}{q} \\ & & 1 \end{pmatrix} \begin{pmatrix} & -1 & \\ & & 1 \\ -1 & & \end{pmatrix} \begin{pmatrix} t & \frac{zq^3v}{d_a^2} & y \\ & v & x \\ & & 1 \end{pmatrix} \right)$$
$$= F \left({}^{\iota}\begin{pmatrix} 1 & & -\frac{a}{q^2} \\ & 1 & -\frac{u}{q} \\ & & 1 \end{pmatrix} {}^{\iota}\begin{pmatrix} & -1 & \\ & & 1 \\ -1 & & \end{pmatrix} {}^{\iota}\begin{pmatrix} t & \frac{zq^3v}{d_a^2} & y \\ & v & x \\ & & 1 \end{pmatrix} \right),$$

$${}^{\iota}\begin{pmatrix} 1 & & -\frac{a}{q^2} \\ & 1 & -\frac{u}{q} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{u}{q} & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix},$$

$${}^{\iota}\begin{pmatrix} & -1 & \\ & & 1 \\ -1 & & \end{pmatrix} = \begin{pmatrix} & & -1 \\ 1 & & \\ & -1 & \end{pmatrix},$$

and

$$^{\iota}\begin{pmatrix} t & \frac{zq^3v}{d_a^2} & y \\ & v & x \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{x}{v} & \frac{xq^3z}{td_a^2} - \frac{y}{t} \\ & \frac{1}{v} & -\frac{q^3z}{td_a^2} \\ & & \frac{1}{t} \end{pmatrix}$$

so the above integral equals

$$\frac{1}{q^3}\int_0^{q^2}\int_0^{q} F\left(\begin{pmatrix} 1 & \frac{u}{q} & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} & & -1 \\ 1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix} 1 & -\frac{x}{v} & \frac{xq^3z}{td_a^2} - \frac{y}{t} \\ & \frac{1}{v} & -\frac{q^3z}{td_a^2} \\ & & \frac{1}{t} \end{pmatrix}\right) \mathrm{e}(-x)\,dx\,dy.$$

Noting

$$\begin{pmatrix} & & -1 \\ 1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix} 1 & -\frac{x}{v} & \frac{xq^3z}{td_a^2} - \frac{y}{t} \\ & \frac{1}{v} & -\frac{q^3z}{td_a^2} \\ & & \frac{1}{t} \end{pmatrix}\begin{pmatrix} & v & \\ & & -v \\ -v & & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ y - \frac{xq^3z}{d_a^2} & 1 & x \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}$$

and since $\begin{pmatrix} & v & \\ & & -v \\ -v & & \end{pmatrix} \in ZK$, the integral equals

$$\frac{1}{q^3}\int_0^{q^2}\int_0^{q} F\left(\begin{pmatrix} 1 & \frac{u}{q} & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y - \frac{xq^3z}{d_a^2} & 1 & x \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-x)\,dx\,dy.$$

Thus we have shown

$$q^3\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right)$$

$$= \sum_{a=1}^{q^2}\int_0^{q^2}\int_0^{q} F\left(\begin{pmatrix} 1 & \frac{u}{q} & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y - \frac{xq^3z}{d_a^2} & 1 & x \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-x)\,dx\,dy.$$

In order to simplify the computations, we recall from Chapter 2 the function $G(\tau) = F\left(\begin{pmatrix} 1 & \frac{u}{q} & \\ & 1 & \\ & & 1 \end{pmatrix}\tau\right)$. We first observe

$$\begin{pmatrix} 1 & \frac{u}{q} & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y - \frac{xq^3z}{d_a^2} & 1 & x \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{u}{q} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix},$$

whence

$$\sum_{a=1}^{q^2}\int_0^{q^2}\int_0^{q} F\left(\begin{pmatrix} 1 & \frac{u}{q} & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y-\frac{xq^3z}{d_a^2} & 1 & x \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-x)dxdy$$

$$=\sum_{a=1}^{q^2}\int_0^{q^2}\int_0^{q} G\left(\begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}(-x)dxdy.$$

Applying Lemma 2.6 we obtain

$$q\sum_{a=1}^{q^2}\int_0^{q^2}\sum_{m\in\mathbf{Z}} G_q^m\left(\begin{pmatrix} 1 & & \frac{a}{q^2} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) dy$$

and with Lemma 2.5 we have

$$q\sum_{a=1}^{q^2}\int_0^{q^2}\sum_{m\in\mathbf{Z}} G_q^m\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_a^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}\left(\frac{ma}{q^2}\right) dy.$$

Writing $a = hq + k$ and so $d_a = (a, q) = (hq + k, q) = (k, q) = d_k$, we obtain

$$q\sum_{h=0}^{q-1}\sum_{k=1}^{q}\int_0^{q^2}\sum_{m\in\mathbf{Z}} G_q^m\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}\left(\frac{m(hq+k)}{q^2}\right) dy.$$

We can now do the sum on h and obtain

$$\begin{aligned}\sum_{h=0}^{q-1}\mathrm{e}\left(\frac{m(hq+k)}{q^2}\right) &= \mathrm{e}\left(\frac{mk}{q^2}\right)\sum_{h=0}^{q-1}\mathrm{e}\left(\frac{mh}{q}\right) \\ &= \begin{cases} q\mathrm{e}\left(\frac{mk}{q^2}\right) & \text{if } q|m \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

So we are only interested in the case where $q|m$. In this case we replace m by qm, whence the above sum equals

$$q^2\sum_{k=1}^{q}\int_0^{q^2}\sum_{m\in\mathbf{Z}} G_q^{qm}\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}\left(\frac{mk}{q}\right) dy.$$

Thus we have

$$q\sum_{a=1}^{q^2}\widetilde{F}_{d_a}^0\left(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right)$$

$$=\sum_{k=1}^{q}\int_0^{q^2}\sum_{m\in\mathbf{Z}} G_q^{qm}\left(\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right) \mathrm{e}\left(\frac{mk}{q}\right) dy.$$

Since $(u,q)=1$ we can apply Lemma 2.7 and obtain

$$\sum_{k=1}^{q}\int_0^{q^2}\sum_{m\in\mathbf{Z}} G_q^{qm}\left(\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{mk}{q}\right)dy$$

$$=\sum_{k=1}^{q}\int_0^{q^2}\sum_{m\in\mathbf{Z}} F_q^{m-\bar{u}}\left(\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{mk}{q}\right)dy.$$

Now replacing m by $m+\bar{u}$ we obtain

$$\sum_{k=1}^{q}\int_0^{q^2}\sum_{m\in\mathbf{Z}} F_q^{m}\left(\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(m+\bar{u})}{q}\right)dy.$$

We now let $m=aq+b$ so the above equals

$$\sum_{k=1}^{q}\int_0^{q^2}\sum_{a\in\mathbf{Z}}\sum_{b=0}^{q-1} F_q^{aq+b}\left(\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$\times\mathrm{e}\left(\frac{k(aq+b+\bar{u})}{q}\right)dy.$$

Applying Lemma 2.1 the above equals

$$\sum_{k=1}^{q}\int_0^{q^2}\sum_{a\in\mathbf{Z}}\sum_{b=0}^{q-1} F_q^{b}\left(\begin{pmatrix}1 & & \\ a & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$\times\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

We now note that

$$\begin{pmatrix}1 & & \\ a & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}=\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y+aq^2 & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix};$$

thus the above equals

$$\sum_{k=1}^{q}\int_0^{q^2}\sum_{a\in\mathbf{Z}}\sum_{b=0}^{q-1} F_q^{b}\left(\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y+aq^2 & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

Interchanging the sum and integral and with a change of variables the above equals

$$\sum_{k=1}^{q}\sum_{a\in\mathbf{Z}}\int_{aq^2}^{q^2+aq^2}\sum_{b=0}^{q-1} F_q^{b}\left(\begin{pmatrix}q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix}\frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy$$

which equals

$$\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_q^b\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy. \quad \blacksquare$$

In the region of absolute convergence for Lemma 3.2 we can interchange the order of integration and summation, and applying Lemma 3.3 we see that we are interested in writing

$$\sum_{u=1}^{q}\overline{\chi}(u)\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_q^b\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy$$

in terms of W, which we do in the following lemma.

Lemma 3.4. *For χ a primitive character mod q, F a cusp form, and $t, v > 0$ we have*

$$\sum_{u=1}^{q}\overline{\chi}(u)\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_q^b\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy$$
$$= \frac{q^2}{\tau(\chi)}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3 zv}{t} & & 1\end{pmatrix}\right)dy$$

where $d_k = (k,q)$. Both integrals and the sum are absolutely convergent.

Proof: We begin by letting $d_b = (b,q)$, $C_b = \dfrac{b}{d_b}$, and $D_b = \dfrac{q}{d_b}$. Since $(C_b, D_b) = 1$ there exist integers A_b, B_b such that $A_bD_b - B_bC_b = 1$. Thus by equation 4.5 of [B1] we have

$$\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_q^b\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy$$
$$=\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_{d_b}^0\left(\begin{pmatrix} A_b & B_b & \\ C_b & D_b & \\ & & 1\end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$\times\,\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

Expanding this we have

$$\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}\sum_{n\in\mathbf{Z}}F_{d_b,n}\left(\begin{pmatrix} A_b & B_b & \\ C_b & D_b & \\ & & 1\end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3 z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$\times\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

We note that

$$\begin{pmatrix} 1 & -\frac{B_b}{D_b} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} A_b & B_b & \\ C_b & D_b & \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{D_b} & & \\ C_b & D_b & \\ & & 1 \end{pmatrix}\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} d_b & & \\ C_bq + D_b\bar{u} & \frac{1}{d_b} & \\ & & 1 \end{pmatrix}.$$

Thus, by equation 4.8 of [B1] and noting that since F is a cusp form we have $F_{d_b,0} = 0$, the above sum equals

$$\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}\sum_{n\neq 0} F_{d_b,n}\left(\begin{pmatrix} d_b & & \\ C_bq + D_b\bar{u} & \frac{1}{d_b} & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right)$$
$$\times \mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

Writing this in terms of W we obtain

$$\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}$$
$$\times W\left(\begin{pmatrix} d_bn & & \\ & d_b & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_b & & \\ C_bq + D_b\bar{u} & \frac{1}{d_b} & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right)$$
$$\times \mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

Observing

$$\begin{pmatrix} d_bn & & \\ & d_b & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_b & & \\ C_bq + D_b\bar{u} & \frac{1}{d_b} & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix} = \begin{pmatrix} d_b^2n & & \\ y + C_bqd_b + D_b\bar{u}d_b & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}$$

the above equals

$$\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}W\left(\begin{pmatrix} d_b^2n & & \\ y + C_bqd_b + D_b\bar{u}d_b & 1 & \\ -\frac{q^3z}{d_k^2} & & 1 \end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1 \end{pmatrix}\right)$$
$$\times \mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

Now, interchanging the integral with the finite sum over b and with a change of variables we obtain

$$\sum_{k=1}^{q}\sum_{b=0}^{q-1}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}W\left(\begin{pmatrix} \frac{d_b^2nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{td_k^2} & & 1 \end{pmatrix}\right)\mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

Thus we have shown

$$\sum_{u=1}^{q}\overline{\chi}(u)\sum_{k=1}^{q}\int_{-\infty}^{\infty}\sum_{b=0}^{q-1}F_q^b\left(\begin{pmatrix} q & & \\ \bar{u} & \frac{1}{q} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ y & 1 & \\ -\frac{q^3z}{d_k^2} & & 1\end{pmatrix}\begin{pmatrix} \frac{v}{t} & & \\ & v & \\ & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy$$

$$=\sum_{u=1}^{q}\overline{\chi}(u)\sum_{k=1}^{q}\sum_{b=0}^{q-1}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}W\left(\begin{pmatrix} \frac{d_b^2nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{td_k^2} & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{k(b+\bar{u})}{q}\right)dy.$$

We observe by Lemma 3.1 that

$$\sum_{u=1}^{q}\overline{\chi}(u)\mathrm{e}\left(\frac{k\bar{u}}{q}\right)=\frac{q\overline{\chi}(k)}{\tau(\overline{\chi})\overline{\chi}(-1)};$$

thus the above equals

$$\frac{q}{\tau(\overline{\chi})\overline{\chi}(-1)}\sum_{k=1}^{q}\overline{\chi}(k)\sum_{b=0}^{q-1}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}W\left(\begin{pmatrix} \frac{d_b^2nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{td_k^2} & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{kb}{q}\right)dy.$$

Since $\overline{\chi}(k)=0$ unless $(k,q)=1$ we can take $d_k=1$ in the above and obtain

$$\frac{q}{\tau(\overline{\chi})\overline{\chi}(-1)}\sum_{\substack{k=1\\(k,q)=1}}^{q}\overline{\chi}(k)\sum_{b=0}^{q-1}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}W\left(\begin{pmatrix} \frac{d_b^2nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{B_bn}{D_b}\right)\mathrm{e}\left(\frac{kb}{q}\right)dy.$$

Again, by Lemma 3.1 we have

$$\sum_{\substack{k=1\\(k,q)=1}}^{q}\overline{\chi}(k)\mathrm{e}\left(\frac{kb}{q}\right)=\frac{q\chi(b)}{\tau(\chi)\chi(-1)},$$

so using the facts that $\chi\overline{\chi}=1$ and $\tau(\chi)\tau(\overline{\chi})=q$ the above equals

$$q\sum_{b=0}^{q-1}\chi(b)\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{d_b,n}}{|nd_b|}W\left(\begin{pmatrix} \frac{d_b^2nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{B_bn}{D_b}\right)dy.$$

Again, $\chi(b)=0$ unless $(b,q)=1$, in which case $D_b=q$ and $C_b=b$. So $A_bq-B_bb=1$ or $B_bb\equiv -1 \bmod q$, thus we have

$$q\sum_{\substack{b=0\\(b,q)=1}}^{q-1}\chi(b)\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}}{|n|}W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)\mathrm{e}\left(\frac{-\bar{b}n}{q}\right)dy.$$

Finally

$$\sum_{\substack{b=0\\(b,q)=1}}^{q-1} \chi(b)\mathrm{e}\left(\frac{-\bar{b}n}{q}\right) = \frac{q\chi(-n)}{\tau(\chi)\chi(-1)},$$

so using the facts that $(\chi(-1))^2 = 1$ and $\chi(-1)\chi(-n) = \chi(n)$ we have

$$\frac{q^2}{\tau(\chi)} \int_{-\infty}^{\infty} \sum_{n\neq 0} \frac{a_{1,n}\chi(n)}{|n|} W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1 \end{pmatrix}\right) dy.$$

We observe

$$\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{v} & & \\ & \frac{1}{v} & \\ & & \frac{1}{v} \end{pmatrix} = \begin{pmatrix} \frac{n}{t} & & \\ \frac{y}{t} & 1 & \\ -\frac{q^3z}{t} & & \frac{1}{v} \end{pmatrix}.$$

By Lemma 2.3 the coordinates of this matrix are

$$y_1 = \frac{\sqrt{\frac{y^2}{t^2v^2} + \frac{q^6z^2}{t^2} + \frac{1}{v^2}}}{\frac{q^6z^2}{t^2} + \frac{1}{v^2}} \qquad x_1 = \frac{\frac{-yq^3z}{t^2}}{\frac{q^6z^2}{t^2} + \frac{1}{v^2}}$$

$$y_2 = \frac{\frac{n}{tv}\sqrt{\frac{q^6z^2}{t^2} + \frac{1}{v^2}}}{\frac{y^2}{t^2v^2} + \frac{q^6z^2}{t^2} + \frac{1}{v^2}} \qquad x_2 = \frac{\frac{ny}{t^2v^2}}{\frac{y^2}{t^2v^2} + \frac{q^6z^2}{t^2} + \frac{1}{v^2}}$$

$$x_3 = \frac{\frac{-nq^3z}{t^2}}{\frac{q^6z^2}{t^2} + \frac{1}{v^2}}$$

or, since $t, v > 0$, we have

$$y_1 = \frac{tv\sqrt{y^2 + q^6z^2v^2 + t^2}}{q^6z^2v^2 + t^2} \qquad x_1 = \frac{-yq^3zv^2}{q^6z^2v^2 + t^2}$$

$$y_2 = \frac{n\sqrt{q^6z^2v^2 + t^2}}{y^2 + q^6z^2v^2 + t^2} \qquad x_2 = \frac{ny}{y^2 + q^6z^2v^2 + t^2}$$

$$x_3 = \frac{-nq^3zv^2}{q^6z^2v^2 + t^2}.$$

Thus, by using the fact that

$$W\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) = \mathrm{e}(x_1 + x_2)W(\tau)$$

we have

$$\left|W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1 \end{pmatrix}\right)\right|$$

$$= \left|W\left(\begin{pmatrix} \frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2} & \frac{n\sqrt{q^6z^2v^2+t^2}}{y^2+q^6z^2v^2+t^2} & \\ & \frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2} & \\ & & 1 \end{pmatrix}\right)\right|$$

$$\times \left|\mathrm{e}\left(\frac{-yq^3zv^2}{q^6z^2v^2+t^2} + \frac{ny}{y^2+q^6z^2v^2+t^2}\right)\right|$$

$$= \left|W\left(\begin{pmatrix} \frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2} & \frac{n\sqrt{q^6z^2v^2+t^2}}{y^2+q^6z^2v^2+t^2} & \\ & \frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2} & \\ & & 1 \end{pmatrix}\right)\right|.$$

Now, by Theorem 2.11,

$$\left|W\left(\begin{pmatrix} \frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2} & \frac{n\sqrt{q^6z^2v^2+t^2}}{y^2+q^6z^2v^2+t^2} & \\ & \frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2} & \\ & & 1 \end{pmatrix}\right)\right.$$

$$\left.\times \left(\frac{tv\sqrt{y^2+q^6z^2v^2+t^2}}{q^6z^2v^2+t^2}\right)^{n_1} \left(\frac{n\sqrt{q^6z^2v^2+t^2}}{y^2+q^6z^2v^2+t^2}\right)^{n_2}\right|$$

is bounded for $n_1 > N_1(\nu_1,\nu_2)$ and $n_2 > N_2(\nu_1,\nu_2)$. Thus, taking n_1 large, we see that

$$W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1 \end{pmatrix}\right)$$

is rapidly decreasing as $y \to \pm\infty$. Taking n_2 large, we see that this function is bounded as $y \to 0$ and that

$$\sum_{n\neq 0} \frac{a_{1,n}\chi(n)}{|n|} W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1 \end{pmatrix}\right)$$

is absolutely convergent. ∎

Combining Lemmas 3.3 and 3.4 we obtain:

Corollary 3.5. *Let χ be a primitive character mod q and F a cusp form, then for $z \in \mathbf{R}$, $t, v > 0$*

$$\sum_{u=1}^{q} \overline{\chi}(u) \sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right)$$

$$= \frac{q}{\tau(\chi)} \int_{-\infty}^{\infty} \sum_{n\neq 0} \frac{a_{1,n}\chi(n)}{|n|} W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1 \end{pmatrix}\right) dy$$

where $(a,q) = d_a$, $u\bar{u} \equiv 1 \bmod q$, *and* $aa' \equiv -d_a \bmod q$. *The integral and sum are both absolutely convergent.*

We can extend, at least in terms of w, the region for Lemma 3.2. We observe

Lemma 3.6.

$$\int_0^\infty \int_0^\infty \sum_{a=1}^{q^2} \widetilde{F}^0_{d_a} \left(\left(\begin{matrix} 1 & z & \\ & 1 & \\ & & 1 \end{matrix} \right) \left(\begin{matrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{matrix} \right) \left(\begin{matrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{matrix} \right) \right) \Bigg|_{z=0} t^{w-1} v^{s-1} \frac{dt}{t} \frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ *and* $\mathrm{Re}(s-1) > N_1(\nu_1,\nu_2)$; *and*

$$\int_0^\infty \int_0^\infty \frac{\partial}{\partial z} \left(\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a} \left(\left(\begin{matrix} 1 & z & \\ & 1 & \\ & & 1 \end{matrix} \right) \left(\begin{matrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{matrix} \right) \left(\begin{matrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{matrix} \right) \right) \right) \Bigg|_{z=0} t^{w-1} v^{s-1} \frac{dt}{t} \frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ *and* $\mathrm{Re}(s+1) > N_1(\nu_1,\nu_2)$.

Proof: From Lemma 3.2 we have that

$$\int_0^\infty \int_0^\infty \sum_{a=1}^{q^2} \widetilde{F}^0_{d_a} \left(\left(\begin{matrix} 1 & z & \\ & 1 & \\ & & 1 \end{matrix} \right) \left(\begin{matrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{matrix} \right) \left(\begin{matrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{matrix} \right) \right) \Bigg|_{z=0} t^{w-1} v^{s-1} \frac{dt}{t} \frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) > \max\left(\widetilde{N}_2(\nu_1,\nu_2)+1, 2\right)$, and thus

$$\int_0^\infty \int_1^\infty \sum_{a=1}^{q^2} \widetilde{F}^0_{d_a} \left(\left(\begin{matrix} 1 & z & \\ & 1 & \\ & & 1 \end{matrix} \right) \left(\begin{matrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{matrix} \right) \left(\begin{matrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{matrix} \right) \right) \Bigg|_{z=0} t^{w-1} v^{s-1} \frac{dt}{t} \frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$. Similarly

$$\int_0^\infty \int_0^\infty \frac{\partial}{\partial z} \left(\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a} \left(\left(\begin{matrix} 1 & z & \\ & 1 & \\ & & 1 \end{matrix} \right) \left(\begin{matrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{matrix} \right) \left(\begin{matrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{matrix} \right) \right) \right) \Bigg|_{z=0} t^{w-1} v^{s-1} \frac{dt}{t} \frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) > \max\left(\widetilde{N}_2(\nu_1,\nu_2)+1, 3\right)$, and thus

$$\int_0^\infty \int_1^\infty \frac{\partial}{\partial z}\left(\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$. So we are left to consider

$$\int_0^\infty \int_0^1 \sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

and

$$\int_0^\infty \int_0^1 \frac{\partial}{\partial z}\left(\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}.$$

We have, by Corollary 3.5, that

$$\sum_{u=1}^{q}\overline{\chi}(u)\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)$$
$$= \frac{q}{\tau(\chi)}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)dy.$$

Thus

$$\sum_{u=1}^{q}\overline{\chi}(u)\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\Bigg|_{z=0}$$
$$= \frac{q}{\tau(\chi)}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1\end{pmatrix}\right)dy$$

and by Lemma 2.4 we have

$$\frac{\partial}{\partial z}\left(\sum_{u=1}^{q}\overline{\chi}(u)\sum_{a=1}^{q^2} \widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0}$$
$$= \frac{q}{\tau(\chi)}\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1\end{pmatrix}\right)\frac{-2\pi i y v^2 q^3}{t^2}dy.$$

So in either case, we are interested in

$$W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1\end{pmatrix}\right),$$

and the coordinates of this matrix are, by Lemma 2.3,

$$y_1 = \frac{\sqrt{\frac{y^2}{t^2v^2}+\frac{1}{v^2}}}{\frac{1}{v^2}} = \frac{v\sqrt{y^2+t^2}}{t} \qquad x_1 = 0$$

$$y_2 = \frac{\frac{n}{tv}\sqrt{\frac{1}{v^2}}}{\frac{y^2}{t^2v^2}+\frac{1}{v^2}} = \frac{nt}{y^2+t^2} \qquad x_2 = \frac{\frac{ny}{t^2v^2}}{\frac{y^2}{t^2v^2}+\frac{1}{v^2}} = \frac{ny}{y^2+t^2}$$

$$x_3 = 0.$$

Thus using the fact that

$$W\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) = \mathrm{e}(x_1+x_2)W(\tau)$$

we have

$$\left|W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1 \end{pmatrix}\right)\right| = \left|W\left(\begin{pmatrix} \frac{nv}{\sqrt{y^2+t^2}} & & \\ & \frac{nt}{y^2+t^2} & \\ & & 1 \end{pmatrix}\right)\mathrm{e}\left(\frac{ny}{y^2+t^2}\right)\right|$$

$$= \left|W\left(\begin{pmatrix} \frac{nv}{\sqrt{y^2+t^2}} & & \\ & \frac{nt}{y^2+t^2} & \\ & & 1 \end{pmatrix}\right)\right|$$

and by Theorem 2.11, for $n_1 > N_1(\nu_1,\nu_2)$ and $n_2 > N_2(\nu_1,\nu_2)$ we have that

$$\left|W\left(\begin{pmatrix} \frac{nv}{\sqrt{y^2+t^2}} & & \\ & \frac{nt}{y^2+t^2} & \\ & & 1 \end{pmatrix}\right)\left(\frac{v\sqrt{y^2+t^2}}{t}\right)^{n_1}\left(\frac{nt}{y^2+t^2}\right)^{n_2}\right|$$

is bounded. So, taking n_1 large, we see

$$W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1 \end{pmatrix}\right)$$

is rapidly decreasing as $y \to \pm\infty$ and $v \to \infty$. Whereas taking n_2 large we see this function is bounded as $y \to 0$, $t \to 0$, and that

$$\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1 \end{pmatrix}\right)$$

is absolutely convergent.

Thus

$$\int_0^\infty\int_0^1\sum_{u=1}^{q}\overline{\chi}(u)\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{q^3} & & \\ & v & \\ & & 1 \end{pmatrix}\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

$$= \frac{q}{\tau(\chi)}\int_0^\infty\int_0^1\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1 \end{pmatrix}\right)t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s-1) > N_1(\nu_1,\nu_2)$ and

$$\int_0^\infty \int_0^1 \frac{\partial}{\partial z}\left(\sum_{u=1}^{q}\overline{\chi}(u)\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} \times t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

$$= \frac{-2\pi i q^4}{\tau(\chi)}\int_0^\infty \int_0^1 \int_{-\infty}^\infty \sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{\frac{nv}{t}}{\frac{yv}{t}} & & \\ & v & \\ & & 1\end{pmatrix}\right) yt^{w-3}v^{s+1}dy\frac{dt}{t}\frac{dv}{v}$$

is absolutely convergent for $\mathrm{Re}(s+1) > N_1(\nu_1,\nu_2)$. ∎

We can now prove:

Theorem 3.7. *For χ a primitive character mod q and F a cusp form, $L_\chi(w,F)$ extends to an entire function of w. If χ is even, we have*

$$\frac{q^{3w-1}\tau(\chi)}{(\tau(\overline{\chi}))^2}L_{\overline{\chi}}(w,\widetilde{F})\widetilde{\Phi}(s-1,w-1) = L_\chi(1-w,F)\Phi(s-1,-w),$$

and if χ is odd, we have

$$\frac{q^{3w-1}\tau(\chi)}{(\tau(\overline{\chi}))^2}L_{\overline{\chi}}(w,\widetilde{F})\widetilde{\Phi}(s-1,w) = L_\chi(1-w,F)\widehat{\Phi}(s+1,-w).$$

Proof: We can use Lemma 3.6 to extend the L-function. Recall that

$$\sum_{n\neq 0}\frac{\widetilde{a}_{1,n}\overline{\chi}(n)\delta_n^\chi}{|n|^w} = \begin{cases}2L_{\overline{\chi}}(w,\widetilde{F}) & \text{if } \chi \text{ is even}\\ 2L_{\overline{\chi}}(w-1,\widetilde{F}) & \text{if } \chi \text{ is odd.}\end{cases}$$

We first assume that χ is even. By Lemma 3.2,

$$\frac{2q^{3w}\chi(-1)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)L_{\overline{\chi}}(w,\widetilde{F})$$

$$= \sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty \int_0^\infty \sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\Bigg|_{z=0} \times t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v},$$

and by Lemma 3.6 the integrals are absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(s-1) > N_1(\nu_1,\nu_2)$. Thus for any choice of w we can choose s large enough so that we can extend $L_{\overline{\chi}}(w,\widetilde{F})$, and by Lemma 2.14, $\widetilde{\Phi}(s-1,w-1)$ only has isolated zeros. So by choosing s to miss the zeros of $\widetilde{\Phi}(s-1,w-1)$, we see that $L_{\overline{\chi}}(w,\widetilde{F})$ extends to an entire function of w. Now applying Corollary 3.5 we obtain

$$\frac{q}{\tau(\chi)}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)\Bigg|_{z=0} t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}.$$

In this case, we are interested in

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1\end{pmatrix}\right) t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$

which by a change of variables equals

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ yv & v & \\ & & 1\end{pmatrix}\right) t^{w}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}.$$

Applying Lemma 2.2 we obtain

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{|n|v}{t} & & \\ yv & v & \\ & & 1\end{pmatrix}\right) t^{w}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}.$$

Now by interchanging the order of summation and integration, which may change the region of convergence, and a change of variables, the above equals

$$\sum_{n\neq 0}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}tv & & \\ yv & v & \\ & & 1\end{pmatrix}\right)\left(\frac{|n|}{t}\right)^{w} v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$

or

$$2L_\chi(1-w,F)\Phi(s-1,-w).$$

By Lemma 2.15 and the fact that $a_{1,n}=O(|n|)$, this function is absolutely convergent for $\mathrm{Re}(s-1)>N_1(\nu_1,\nu_2)$, $\mathrm{Re}(-w)>N_2(\nu_1,\nu_2)$, and $\mathrm{Re}(1-w)>2$. Thus we have

$$\begin{aligned}&\frac{2q^{3w}\chi(-1)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)L_{\overline{\chi}}(w,\widetilde{F})\\ &\quad=\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\Bigg|_{z=0}\\ &\qquad\times t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}\end{aligned}$$

for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(s-1) > N_1(\nu_1,\nu_2)$, and

$$\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$
$$= \frac{2q}{\tau(\chi)}L_\chi(1-w,F)\Phi(s-1,-w)$$

for $\mathrm{Re}(s-1) > N_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) < \min\Big(-1,-N_2(\nu_1,\nu_2)\Big)$. Thus, since $\chi(-1)=1$, we have

$$\frac{q^{3w-1}\tau(\chi)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)L_{\overline{\chi}}(w,\widetilde{F}) = L_\chi(1-w,F)\Phi(s-1,-w)$$

for $\mathrm{Re}(w) < \min\Big(-1,-N_2(\nu_1,\nu_2)\Big)$, $\mathrm{Re}(s-1) > N_1(\nu_1,\nu_2)$, and $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$; and by meromorphic continuation this completes the case where χ is even.

If χ is odd, we have

$$\frac{2q^{3w}\chi(-1)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)L_{\overline{\chi}}(w-1,\widetilde{F})$$
$$= \frac{1}{2\pi i}\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty$$
$$\times\frac{\partial}{\partial z}\left(\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$
$$= \frac{q}{2\pi i\tau(\chi)}\int_0^\infty\int_0^\infty$$
$$\times\frac{\partial}{\partial z}\left(\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)dy\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

and the sum and integrals are absolutely convergent for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(s+1) > N_1(\nu_1,\nu_2)$. Again, just as in the even case, $L_{\overline{\chi}}(w-1,\widetilde{F})$ extends to an entire function of w. Since the sums and integrals are absolutely convergent, the above equals

$$\frac{q}{2\pi i\tau(\chi)}\int_0^\infty\int_0^\infty\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}\frac{\partial}{\partial z}\left(W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}.$$

In this case, we observe, by Lemma 2.4 that

$$\frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_{-\infty}^{\infty}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}\frac{\partial}{\partial z}\left(W\left(\begin{pmatrix}\frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ -\frac{q^3zv}{t} & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$

$$= \frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|}\frac{2\pi i y v^2}{t} W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1\end{pmatrix}\right)\frac{-q^3}{t}t^{w-1}v^{s-1}dy\frac{dt}{t}\frac{dv}{v}$$

which in turn equals

$$-q^3\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|} W\left(\begin{pmatrix} \frac{nv}{t} & & \\ \frac{yv}{t} & v & \\ & & 1\end{pmatrix}\right) y t^{w-3}v^{s+1}dy\frac{dt}{t}\frac{dv}{v}.$$

Now, by a change of variables this equals

$$-q^3\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{|n|} W\left(\begin{pmatrix} \frac{nv}{t} & & \\ yv & v & \\ & & 1\end{pmatrix}\right) y t^{w-1}v^{s+1}dy\frac{dt}{t}\frac{dv}{v}.$$

Again, we interchange the order of integration and summation, which may change the region of convergence, and applying Lemma 2.2 we have

$$-q^3\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{n}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty W\left(\begin{pmatrix} \frac{|n|v}{t} & & \\ yv & v & \\ & & 1\end{pmatrix}\right) y t^{w-1}v^{s+1}dy\frac{dt}{t}\frac{dv}{v},$$

which by a change of variables equals

$$-q^3\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{n|n|^{1-w}}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty W\left(\begin{pmatrix} tv & & \\ yv & v & \\ & & 1\end{pmatrix}\right) y t^{1-w}v^{s+1}dy\frac{dt}{t}\frac{dv}{v}.$$

We observe, since χ is odd that

$$\begin{aligned}\sum_{n\neq 0}\frac{a_{1,n}\chi(n)}{n|n|^{1-w}} &= \sum_{n=1}^\infty \frac{a_{1,n}\left(\chi(n)-\chi(-n)\right)}{n^{2-w}}\\ &= 2\sum_{n=1}^\infty\frac{a_{1,n}\chi(n)}{n^{2-w}}\\ &= 2L_\chi(2-w),F).\end{aligned}$$

So the above equals

$$-2q^3L_\chi(2-w),F)\widehat{\Phi}(s+1,1-w).$$

By Lemma 2.16 and the fact that $a_{1,n}=O(|n|)$, this function is absolutely convergent for $\mathrm{Re}(s+1)>N_1(\nu_1,\nu_2)$, $\mathrm{Re}(1-w)>N_2(\nu_1,\nu_2)$, and $\mathrm{Re}(2-w)>2$. Thus we have

$$\begin{aligned}&\frac{2q^{3w}\chi(-1)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)L_{\overline{\chi}}(w-1,\widetilde{F})\\ &= \frac{1}{2\pi i}\sum_{u=1}^q\overline{\chi}(u)\int_0^\infty\int_0^\infty\\ &\times\frac{\partial}{\partial z}\left(\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\bar{u}}{qd_a}\\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0} t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}\end{aligned}$$

for $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$ and $\mathrm{Re}(s+1) > N_1(\nu_1,\nu_2)$, and

$$\frac{1}{2\pi i}\sum_{u=1}^{q}\overline{\chi}(u)\int_0^\infty\int_0^\infty \frac{\partial}{\partial z}\left(\sum_{a=1}^{q^2}\widetilde{F}^0_{d_a}\left(\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}d_a & \frac{a'}{q} & \\ & \frac{1}{d_a} & \frac{a\overline{u}}{qd_a} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{t}{q^3} & & \\ & v & \\ & & 1\end{pmatrix}\right)\right)\Bigg|_{z=0}$$

$$\times t^{w-1}v^{s-1}\frac{dt}{t}\frac{dv}{v}$$

$$= \frac{-2q^4}{\tau(\chi)}L_\chi(2-w),F)\widehat{\Phi}(s+1,1-w)$$

for $\mathrm{Re}(s+1) > N_1(\nu_1,\nu_2)$ and $\mathrm{Re}(w) < \min\Big(1-N_2(\nu_1,\nu_2),0\Big)$. Thus, since $\chi(-1)=-1$, we have

$$\frac{q^{3w-4}\tau(\chi)}{(\tau(\overline{\chi}))^2}\widetilde{\Phi}(s-1,w-1)L_{\overline{\chi}}(w-1,\widetilde{F}) = L_\chi(2-w),F)\widehat{\Phi}(s+1,1-w)$$

for $\mathrm{Re}(w) < \min\Big(1-N_2(\nu_1,\nu_2),0\Big)$, $\mathrm{Re}(s+1) > N_1(\nu_1,\nu_2)$, and $\mathrm{Re}(s+w-2) > \widetilde{N}_1(\nu_1,\nu_2)$.

So by meromorphic continuation and replacing w by $w+1$ we are done. ∎

Now applying Lemmas 2.14, 2.15, and 2.16, Theorem 3.7 becomes, for χ even

$$\frac{q^{3w-1}\tau(\chi)}{\pi^{3w-\frac{3}{2}}(\tau(\overline{\chi}))^2}L_{\overline{\chi}}(w,\widetilde{F})\Gamma\left(\frac{w+\alpha}{2}\right)\Gamma\left(\frac{w+\beta}{2}\right)\Gamma\left(\frac{w+\gamma}{2}\right)$$
$$= L_\chi(1-w,F)\Gamma\left(\frac{1-w-\alpha}{2}\right)\Gamma\left(\frac{1-w-\beta}{2}\right)\Gamma\left(\frac{1-w-\gamma}{2}\right)$$

and for χ odd

$$\frac{q^{3w-1}\tau(\chi)}{i\pi^{3w-\frac{3}{2}}(\tau(\overline{\chi}))^2}L_{\overline{\chi}}(w,\widetilde{F})\Gamma\left(\frac{1+w+\alpha}{2}\right)\Gamma\left(\frac{1+w+\beta}{2}\right)\Gamma\left(\frac{1+w+\gamma}{2}\right)$$
$$= L_\chi(1-w,F)\Gamma\left(\frac{2-w-\alpha}{2}\right)\Gamma\left(\frac{2-w-\beta}{2}\right)\Gamma\left(\frac{2-w-\gamma}{2}\right).$$

Thus letting

$$\epsilon_\chi = \begin{cases}\dfrac{(\tau(\overline{\chi}))^2}{\tau(\chi)\sqrt{q}} & \text{if } \chi \text{ is even}\\[2ex] \dfrac{i\,(\tau(\overline{\chi}))^2}{\tau(\chi)\sqrt{q}} & \text{if } \chi \text{ is odd}\end{cases}$$

and

$$\Gamma^q_{\nu_1,\nu_2}(w) = \left(\frac{q}{\pi}\right)^{\frac{3w}{2}}\Gamma\left(\frac{w-\alpha}{2}\right)\Gamma\left(\frac{w-\beta}{2}\right)\Gamma\left(\frac{w-\gamma}{2}\right),$$

so

$$\widetilde{\Gamma}^q_{\nu_1,\nu_2}(w) = \left(\frac{q}{\pi}\right)^{\frac{3w}{2}}\Gamma\left(\frac{w+\alpha}{2}\right)\Gamma\left(\frac{w+\beta}{2}\right)\Gamma\left(\frac{w+\gamma}{2}\right)$$

we have:

Corollary 3.8. *For χ a primitive character mod q and F a cusp form, $L_\chi(w,F)$ extends to an entire function of w. If χ is even, we have*

$$L_{\overline{\chi}}(w,\widetilde{F})\widetilde{\Gamma}^q_{\nu_1,\nu_2}(w) = \epsilon_\chi L_\chi(1-w,F)\Gamma^q_{\nu_1,\nu_2}(1-w),$$

and if χ is odd, we have

$$L_{\overline{\chi}}(w,\widetilde{F})\widetilde{\Gamma}^q_{\nu_1,\nu_2}(1+w) = \epsilon_\chi L_\chi(1-w,F)\Gamma^q_{\nu_1,\nu_2}(2-w).$$

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APPENDIX

REMARKS ON WEIL'S THEOREM

In this appendix we will present some remarks on Weil's converse theorem for $GL(2, \mathbf{R})$ modular forms. We let $G = GL(2, \mathbf{R})^+$ be the subgroup of $GL(2, \mathbf{R})$ with positive determinant. In the introduction we considered H, the upper half plane, as a quotient space; however, here we will consider it in the classic sense. We define the *action of G on H* as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \text{ and } z \in H.$$

With this action we define the *stroke* of a function on H. Let f be a function on H and define, for $k \in \mathbf{Z}$,

$$f(z)|[\gamma]_k = (\det \gamma)^{\frac{k}{2}} (cz+d)^{-k} f(\gamma z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Before we can define a modular form we must define a subgroup of $\Gamma = SL(2, \mathbf{Z})$. For a positive integer N, the *principal congruence subgroup of level N* is given by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \, : \, a \equiv d \equiv 1 \bmod N \text{ and } b \equiv c \equiv 0 \bmod N \right\}.$$

With this we can define a modular form. Let k be any integer and let Γ' be any subgroup of Γ containing $\Gamma(N)$. A function ϕ on H is a *modular form of weight k on Γ'* if

(a) ϕ is analytic on H,

(b) $\phi|[\gamma]_k = \phi$ for all $\gamma \in \Gamma'$, and

(c) for each $\gamma \in \Gamma$, $\phi(z)|[\gamma]_k$ can be written in the form $\sum_{n=0}^{\infty} a_n \mathrm{e}\left(\frac{nz}{N}\right)$.

We say ϕ is a *cusp form of weight k on* Γ', if in addition $a_0 = 0$ for all $\gamma \in \Gamma$. We let $M_k(\Gamma')$ denote the set of all modular forms of weight k on Γ' and $S_k(\Gamma')$ denote the set of all cusp forms of weight k on Γ'.

In order to state Weil's theorem we must define a restricted set of modular forms. We let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,:\, c \equiv 0 \bmod N \right\} \text{ and}$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,:\, a \equiv 1 \bmod N \right\}$$

and note that trivally we have $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma$. Since $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \Gamma_1(N)$, condition (b) in the definition of a modular form implies that for any $\phi \in M_k(\Gamma_1(N))$

$$\phi(z) = \phi(z)|[T]_k = \phi\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} z\right) = \phi(z+1).$$

Thus ϕ has an expansion of the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n \mathrm{e}(nz).$$

In fact, $|a_n| = O(n^c)$ for some $c \in \mathbf{R}$ ([O] p. IV-43). For any function f which can be written in the form $f(z) = \sum_{n=0}^{\infty} a_n \mathrm{e}(nz)$, the *L-function associated with f* is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Also, for any character χ we have the *twist of f by χ*

$$f_\chi(z) = \sum_{n=0}^{\infty} a_n \chi(n) \mathrm{e}(nz)$$

and the *twisted L-function associated with f*

$$L_\chi(s,f) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}.$$

Finally, for χ a character mod N we let

$$M_k(N,\chi) = \left\{ \phi \in M_k(\Gamma_1(N)) \,:\, \phi|[\gamma]_k = \chi(d)\phi \text{ for all } \gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(N) \right\}$$

and

$$S_k(N,\chi) = M_k(N,\chi) \cap S_k(\Gamma_1(N)).$$

With this we can now state Weil's converse theorem.

Theorem A.1. [W] *Fix N and k positive integers and ε a character mod N. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $|a_n| = O(n^c)$ for some $c \in \mathbf{R}$ and let $\phi(z) = \sum_{n=0}^{\infty} a_n e(nz)$. Further suppose $\phi|[H_N]_k = \frac{1}{\omega i^k}\phi$ for $\omega = 1$ or -1 and $H_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}$. If $\phi \in M_k(N,\chi)$ we have the following:*

(a) The function

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s,\phi) + a_0\left(\frac{1}{s} + \frac{\omega}{k-s}\right)$$

extends to an entire function which is bounded in every vertical strip and we have the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s,\phi) = \omega\left(\frac{\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)L(k-s,\phi).$$

(b) For every character χ mod q where $(q,N) = 1$ we have

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s,\phi)$$

extends to an entire function which is bounded in every vertical strip and we have the functional equation

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s,\phi) = C_\chi\left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)L_{\overline{\chi}}(k-s,\phi)$$

where

$$C_\chi = \frac{\omega\varepsilon(q)\chi(-N)\tau(\chi)}{\tau(\overline{\chi})}.$$

Conversely, let $\mathcal{Z}$ be a subset of the integers meeting every arithmetic progression of the form $\{u+nv\}_{n\in\mathbf{Z}}$ with $(u,v) = 1$. If condition (a) is satisfied and if for all $q \in \mathcal{Z}$ condition (b) is satisfied, then $\phi \in M_k(N,\chi)$. If in addition, $L(s,\phi)$ converges absolutely for $\mathrm{Re}(s) > k-\alpha$ *for some $\alpha > 0$, then $\phi \in S_k(N,\chi)$.*

The goal of the following work is to establish a functional equation for the L-function twisted by a character mod q, where q and N are not necessarily coprime. In Theorem A.6 we will prove a portion of this theorem which demonstrates the necessity of the requirement that $(q, N) = 1$ in this setting. The remainder of the proof of Theorem A.1 is not related to this matter and will not be reproduced. A proof can be found in [O].

Before we can prove Theorem A.6 we need to establish several lemmas. We begin with an easy result.

Lemma A.2. *If χ is a primitive character mod q and f is any function which can be written in the form $f(z) = \sum_{n=1}^{\infty} a_n \mathrm{e}(nz)$ where $|a_n| = O(n^c)$ for some $c \in \mathbf{R}$, then*

$$f_\chi(z) = \frac{\tau(\chi)\chi(-1)}{q} \sum_{b=1}^{q} \overline{\chi}(b) f\left(z + \frac{b}{q}\right).$$

Proof: By the definition of f_χ and Lemma 3.1 we have

$$\begin{aligned} f_\chi(z) &= \sum_{n=1}^{\infty} a_n \chi(n) \mathrm{e}(nz) \\ &= \sum_{n=1}^{\infty} a_n \mathrm{e}(nz) \frac{\tau(\chi)\chi(-1)}{q} \sum_{b=1}^{q} \overline{\chi}(b) \mathrm{e}\left(\frac{bn}{q}\right). \end{aligned}$$

Since $|a_n| = O(n^c)$ the sum is absolutely convergent so we can rearrange the sum and obtain

$$\frac{\tau(\chi)\chi(-1)}{q} \sum_{b=1}^{q} \overline{\chi}(b) \sum_{n=1}^{\infty} a_n \mathrm{e}\left(n\left(z + \frac{b}{q}\right)\right)$$

which clearly equals

$$\frac{\tau(\chi)\chi(-1)}{q} \sum_{b=1}^{q} \overline{\chi}(b) f\left(z + \frac{b}{q}\right). \qquad \blacksquare$$

With this result we can express $\Gamma(s) L_\chi(s, f)$ as a Mellin transform of f_χ.

Lemma A.3. *If χ is a primitive character mod q and and f is any function which can be written in the form $f(z) = \sum_{n=1}^{\infty} a_n \mathrm{e}(nz)$ where $|a_n| = O(n^c)$ for some $c \in \mathbf{R}$, then*

$$(2\pi)^{-s} \Gamma(s) L_\chi(s, f) = \int_0^\infty f_\chi(ir) r^s \frac{dr}{r} \quad \text{for } \mathrm{Re}(s) > \max(c + 1, 0).$$

Proof: In the region $\mathrm{Re}(s) > \max(c+1, 0)$ the L-function is absolutely convergent as is $\Gamma(z) = \int_0^\infty e^{-r} r^z \frac{dr}{r}$. Thus

$$\begin{aligned}\Gamma(s)L_\chi(s,f) &= \int_0^\infty e^{-r} r^s \frac{dr}{r} \sum_{n=1}^\infty \frac{a_n \chi(n)}{n^s} \\ &= \sum_{n=1}^\infty a_n \chi(n) \int_0^\infty e^{-r} \left(\frac{r}{n}\right)^s \frac{dr}{r}\end{aligned}$$

which by a change of variables equals

$$\sum_{n=1}^\infty a_n \chi(n) \int_0^\infty e^{-2\pi n r} (2\pi r)^s \frac{dr}{r}.$$

Since we are in the region in which the integral and the sum are absolutely convergent we can interchange the order of summation and integration and obtain

$$(2\pi)^s \int_0^\infty \sum_{n=1}^\infty a_n \chi(n) \mathrm{e}(irn) r^s \frac{dr}{r}$$

which equals

$$(2\pi)^s \int_0^\infty f_\chi(ir) r^s \frac{dr}{r}.$$

∎

If, as before, we let $H_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}$ we have

Lemma A.4. *Let ϕ be a function on H such that for some ε, a character mod N, we have $\phi|[\gamma]_k = \varepsilon(d)\phi$ for all $\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(N)$ and $\phi|[H_N]_k = \frac{1}{\omega i^k}\phi$ where $\omega = 1$ or -1. If $(q,N) = 1$, $(b,q) = 1$ and u and v are integers such that $qv - bNu = 1$, then*

$$\phi\left(z + \frac{b}{q}\right) = \frac{\omega i^k \varepsilon(q)}{(\sqrt{N} qz)^k} \phi\left(\frac{u}{q} - \frac{1}{Nq^2 z}\right).$$

Proof: We first note that such u and v exist since we have

$$\begin{matrix}(q,N)=1 \\ (b,q)=1\end{matrix} \Rightarrow (bN, q) = 1.$$

We let $\gamma = \begin{pmatrix} q & -b \\ -Nu & v \end{pmatrix}$ and evaluate $\phi\left(z+\frac{b}{q}\right)|[H_N\gamma]_k$ in two ways. First noting $\gamma \in \Gamma_0(N)$ we have

$$\begin{aligned}\phi\left(z+\frac{b}{q}\right)|[H_N\gamma]_k &= \left(\phi\left(z+\frac{b}{q}\right)|[H_N]_k\right)|[\gamma]_k \\ &= \frac{1}{\omega i^k}\phi\left(z+\frac{b}{q}\right)|[\gamma]_k \\ &= \frac{\varepsilon(v)}{\omega i^k}\phi\left(z+\frac{b}{q}\right).\end{aligned}$$

We also observe $H_N\gamma = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} q & -b \\ -Nu & v \end{pmatrix} = \begin{pmatrix} Nu & -v \\ Nq & -Nb \end{pmatrix}$. Thus

$$\begin{aligned}\phi\left(z+\frac{b}{q}\right)|[H_N\gamma]_k &= \frac{(Nqv-N^2ub)^{\frac{k}{2}}}{(Nqz)^k}\phi\left(\frac{Nu(z+\frac{b}{q})-v}{Nq(z+\frac{b}{q})-Nb}\right) \\ &= \frac{(N(qv-Nub))^{\frac{k}{2}}}{(Nqz)^k}\phi\left(\frac{Nuz+\frac{1}{q}(Nub-qv)}{Nqz+Nb-Nb}\right) \\ &= \frac{N^{\frac{k}{2}}}{(Nqz)^k}\phi\left(\frac{Nuz-\frac{1}{q}}{Nqz}\right) \\ &= \frac{1}{(\sqrt{N}qz)^k}\phi\left(\frac{u}{q}-\frac{1}{Nq^2z}\right).\end{aligned}$$

Combining these two equations we obtain

$$\phi\left(z+\frac{b}{q}\right) = \frac{\omega i^k}{\varepsilon(v)(\sqrt{N}qz)^k}\phi\left(\frac{u}{q}-\frac{1}{Nq^2z}\right).$$

Since $qv - bNu = 1$ we have $qv \equiv 1 \bmod N$, so $\varepsilon(q)\varepsilon(v) = 1$. Thus

$$\phi\left(z+\frac{b}{q}\right) = \frac{\omega i^k\varepsilon(q)}{(\sqrt{N}qz)^k}\phi\left(\frac{u}{q}-\frac{1}{Nq^2z}\right).$$

∎

Lemma A.5. *Let N and k be positive integers, ε be a character mod N, and $\phi(z) \in S_k(N,\varepsilon)$. Further suppose $\phi|[H_N]_k = \frac{1}{\omega i^k}\phi$ for $\omega = 1$ or -1. Then for every character χ mod q where $(q,N) = 1$*

$$\phi_\chi(z) = \frac{\omega\chi(-N)\varepsilon(q)\tau(\chi)}{\tau(\overline{\chi})}\left(\frac{i}{\sqrt{N}qz}\right)^k\phi_{\overline{\chi}}\left(\frac{-1}{Nq^2z}\right).$$

Proof: As noted earlier, since $\phi \in S_k(N,\chi)$ we have

$$\phi(z) = \sum_{n=1}^{\infty} a_n \mathrm{e}(nz)$$

where $|a_n| = O(n^c)$ for some $c \in \mathbf{R}$. Thus, by Lemma A.2

$$\phi_\chi(z) = \frac{\tau(\chi)\chi(-1)}{q} \sum_{b=1}^{q} \overline{\chi}(b)\phi\left(z + \frac{b}{q}\right).$$

Since $\chi(b) = 1$ unless $(b,q) = 1$ we have by Lemma A.4

$$\overline{\chi}(b)\phi\left(z + \frac{b}{q}\right) = \frac{\omega i^k \varepsilon(q)}{(\sqrt{N}qz)^k}\overline{\chi}(b)\phi\left(\frac{u}{q} - \frac{1}{Nq^2z}\right)$$

where $-bNu \equiv 1 \bmod q$. Thus $\overline{\chi}(b) = \chi(-Nu)$ and as b runs through a complete set of residues modulo q so does u. Thus

$$\phi_\chi(z) = \frac{\tau(\chi)\chi(-1)}{q}\frac{\omega i^k \varepsilon(q)\chi(-N)}{(\sqrt{N}qz)^k} \sum_{u=1}^{q} \chi(u)\phi\left(\frac{u}{q} - \frac{1}{Nq^2z}\right).$$

Finally, applying Lemma A.2 again this equals

$$\frac{\tau(\chi)\chi(-1)}{\tau(\overline{\chi})\overline{\chi}(-1)}\frac{\omega i^k \varepsilon(q)\chi(-N)}{(\sqrt{N}qz)^k}\phi_{\overline{\chi}}\left(\frac{-1}{Nq^2z}\right).$$

∎

With this we can now prove the part of Theorem A.1 which will be investigated. We restrict our attention to cusp forms in order to simplify the computations.

Theorem A.6. *Let N and k be positive integers, ε be a character mod N, and $\phi(z) \in S_k(N,\varepsilon)$. Further suppose $\phi|[H_N]_k = \frac{1}{\omega i^k}\phi$ for $\omega = 1$ or -1. Then for every character χ mod q where $(q,N) = 1$*

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s,\phi)$$

extends to an entire function and satisfies the functional equation

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s,\phi) = C_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)L_{\overline{\chi}}(k-s,\phi)$$

where

$$C_\chi = \frac{\omega\varepsilon(q)\chi(-N)\tau(\chi)}{\tau(\overline{\chi})}.$$

Proof: Since $\phi \in S_k(N,\varepsilon)$ we can apply Lemma A.3 and obtain

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s) = \left(q\sqrt{N}\right)^s \int_0^\infty \phi_\chi(ir)r^s\frac{dr}{r}.$$

For the integral we observe

$$\int_0^\infty \phi_\chi(ir)r^s\frac{dr}{r} = \int_0^{\frac{1}{q\sqrt{N}}} \phi_\chi(ir)r^s\frac{dr}{r} + \int_{\frac{1}{q\sqrt{N}}}^\infty \phi_\chi(ir)r^s\frac{dr}{r}.$$

Applying Lemma A.5 to the first summand we have

$$\begin{aligned}\int_0^{\frac{1}{q\sqrt{N}}} \phi_\chi(ir)r^s\frac{dr}{r} &= \frac{\omega\chi(-N)\varepsilon(q)\tau(\chi)}{\tau(\overline{\chi})}\int_0^{\frac{1}{q\sqrt{N}}} \left(\frac{i}{\sqrt{N}qir}\right)^k \phi_{\overline{\chi}}\left(\frac{-1}{Nq^2ir}\right) r^s\frac{dr}{r}\\ &= \frac{C_\chi}{\left(q\sqrt{N}\right)^k}\int_0^{\frac{1}{q\sqrt{N}}} \phi_{\overline{\chi}}\left(\frac{i}{Nq^2r}\right) r^{s-k}\frac{dr}{r}.\end{aligned}$$

Making the change of variables $r \to \dfrac{1}{rNq^2}$ this becomes

$$\frac{C_\chi}{\left(q\sqrt{N}\right)^k}\int_{\frac{1}{q\sqrt{N}}}^\infty \phi_{\overline{\chi}}(ir)\left(\frac{1}{Nq^2r}\right)^{s-k}\frac{dr}{r}$$

which equals

$$\frac{C_\chi}{\left(q\sqrt{N}\right)^{2s-k}}\int_{\frac{1}{q\sqrt{N}}}^\infty \phi_{\overline{\chi}}(ir)r^{k-s}\frac{dr}{r}.$$

Thus

$$\int_0^\infty \phi_\chi(ir)r^s\frac{dr}{r} = C_\chi\left(q\sqrt{N}\right)^{k-2s}\int_{\frac{1}{q\sqrt{N}}}^\infty \phi_{\overline{\chi}}(ir)r^{k-s}\frac{dr}{r} + \int_{\frac{1}{q\sqrt{N}}}^\infty \phi_\chi(ir)r^s\frac{dr}{r}$$

and so we have

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s) = C_\chi\left(q\sqrt{N}\right)^{k-s}\int_{\frac{1}{q\sqrt{N}}}^\infty \phi_{\overline{\chi}}(ir)r^{k-s}\frac{dr}{r} + \left(q\sqrt{N}\right)^s\int_{\frac{1}{q\sqrt{N}}}^\infty \phi_\chi(ir)r^s\frac{dr}{r}.$$

These integrals converge for all values of s since $\phi(ir) = \sum_{n=1}^{\infty} a_n \mathrm{e}(nir)$ decays exponentially as $r \to \infty$. Thus $\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_\chi(s)$ extends to an entire funcion of s. Also, replacing s by $k-s$, χ by $\overline{\chi}$, and multiplying by C_χ we obtain

$$\begin{aligned} C_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} & \Gamma(k-s)L_{\overline{\chi}}(k-s) \\ &= C_\chi C_{\overline{\chi}} \left(q\sqrt{N}\right)^s \int_{\frac{1}{q\sqrt{N}}}^{\infty} \phi_\chi(ir)r^s \frac{dr}{r} + C_\chi \left(q\sqrt{N}\right)^{k-s} \int_{\frac{1}{q\sqrt{N}}}^{\infty} \phi_{\overline{\chi}}(ir)r^{k-s}\frac{dr}{r}. \end{aligned}$$

Thus the proof will be complete if we show $C_\chi C_{\overline{\chi}} = 1$. We observe

$$C_\chi C_{\overline{\chi}} = \omega^2 \left(\varepsilon(q)\right)^2 \chi(-N)\overline{\chi}(-N) = \left(\varepsilon(q)\right)^2.$$

So it will suffice to show $\left(\varepsilon(q)\right)^2 = 1$.

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have $ad - bc = 1$ with $c|N$ which implies $ad \equiv 1 \bmod N$. Also $H_N\gamma H_N^{-1} = \begin{pmatrix} d & -\frac{c}{N} \\ -Nb & a \end{pmatrix} \in \Gamma_0(N)$. With this we evaluate $\phi|[H_N\gamma]_k$ in two ways. First

$$\phi|[H_N\gamma]_k = \left(\phi|[H_N]_k\right)|[\gamma]_k = \frac{1}{\omega i^k}\phi|[\gamma]_k = \frac{\varepsilon(d)}{\omega i^k}\phi.$$

On the other hand

$$\phi|[H_N\gamma]_k = \left(\phi|[H_N\gamma H_N^{-1}]_k\right)|[H_N]_k = \varepsilon(a)\phi|[H_N]_k = \frac{\varepsilon(a)}{\omega i^k}\phi.$$

Thus $\varepsilon(a) = \varepsilon(d)$, which implies, since $ad \equiv 1 \bmod N$, that $\overline{\varepsilon}(a) = \varepsilon(a)$. In particular, $\left(\varepsilon(q)\right)^2 = 1$. ∎

Relaxing the restriction that $(q, N) = 1$ in the hypothesis of this theorem is our goal. Most of the following results have been proven using representation theory ([L]). However, all of the following proofs will be done classically. Note that in the previous discourse the significance of $(q, N) = 1$ appeared in the proof of Lemma A.4. More precisely in the choice of γ. We will first prove a simple version of Lemma A.4 where $(q, N) > 1$ and the transformation law for the action of H_N on ϕ is not required.

Theorem A.7. *Let ϕ be a function on H such that for some ε, a character mod N, we have $\phi|[\gamma]_k = \varepsilon(d)\phi$ for all $\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(N)$. If for $q, b, \bar{b} \in \mathbf{Z}$ we have $N|q$, $(b,q) = 1$, and $\bar{b}b \equiv 1 \bmod q$ then*

$$\phi\left(z + \frac{b}{q}\right) = \frac{\varepsilon(-\bar{b})}{(qz)^k}\phi\left(\frac{-1}{q^2 z} - \frac{\bar{b}}{q}\right).$$

Proof: Since $\bar{b}b \equiv 1 \bmod N$ there exists a $v \in \mathbf{Z}$ such that $b\bar{b} - qv = 1$. We let $\gamma = \begin{pmatrix} -\bar{b} & v \\ q & -b \end{pmatrix}$ and as in the proof of Lemma A.4 we evaluate $\phi\left(z + \frac{b}{q}\right)|[\gamma]_k$ in two ways. First, we observe that $\gamma \in \Gamma_0(N)$ since $N|q$. Thus

$$\phi\left(z + \frac{b}{q}\right)|[\gamma]_k = \varepsilon(-b)\phi\left(z + \frac{b}{q}\right).$$

On the other hand, we observe

$$\begin{aligned} \phi\left(z + \frac{b}{q}\right)|[\gamma]_k &= (qz)^{-k}\phi\left(\frac{-\bar{b}(z + \frac{b}{q}) + v}{q(z + \frac{b}{q}) - b}\right) \\ &= (qz)^{-k}\phi\left(\frac{-\bar{b}z - \frac{1}{q}(b\bar{b} - qv)}{qz}\right) \\ &= (qz)^{-k}\phi\left(\frac{-\bar{b}z - \frac{1}{q}}{qz}\right) \\ &= \frac{1}{(qz)^k}\phi\left(\frac{-1}{q^2 z} - \frac{\bar{b}}{q}\right). \end{aligned}$$

Thus

$$\frac{1}{(qz)^k}\phi\left(\frac{-1}{q^2 z} - \frac{\bar{b}}{q}\right) = \varepsilon(-b)\phi\left(z + \frac{b}{q}\right).$$

∎

Note that this is not quite the form which is needed since $\varepsilon(-\bar{b})$ depends on b. This problem is eliminated by taking ε to be the trivial character. Thus our hypothesis becomes $\phi|[\gamma]_k = \phi$ for all $\gamma \in \Gamma_0(N)$. Noting that $-I \in \Gamma_0(N)$ and $\phi|[-I]_k = (-1)^k\phi$ we observe that k must be even. So we have

Corollary A.8. *Let ϕ be a function on H such that $\phi|[\gamma]_{2k} = \phi$ for all $\gamma \in \Gamma_0(N)$. If for $q, b, \bar{b} \in \mathbf{Z}$ we have $N|q$, $(b,q) = 1$, and $\bar{b}b \equiv 1 \bmod q$, then*

$$\phi\left(z + \frac{b}{q}\right) = \frac{1}{(qz)^{2k}}\phi\left(\frac{-1}{q^2 z} - \frac{\bar{b}}{q}\right).$$

With this result we have the analogs of Lemma A.5 and Theorem A.6.

Lemma A.9. *Let N and k be positive integers and $\phi(z) \in S_{2k}(N,1) = S_{2k}(\Gamma_0(N))$. If χ is a character mod q, where $N|q$, then*

$$\phi_\chi(z) = \frac{\chi(-1)\tau(\chi)}{(qz)^{2k}\tau(\overline{\chi})}\phi_{\overline{\chi}}\left(\frac{-1}{q^2 z}\right).$$

Proof: Since $\phi \in S_{2k}(N,1)$ we have, by Lemma A.2

$$\phi_\chi(z) = \frac{\chi(-1)\tau(\chi)}{q}\sum_{b=1}^{q}\overline{\chi}(b)\phi\left(z + \frac{b}{q}\right).$$

Observing $\chi(b) = 0$ unless $(b,q) = 1$ we can apply Corollary A.8 and obtain

$$\frac{\chi(-1)\tau(\chi)}{q}\sum_{b=1}^{q}\frac{\overline{\chi}(b)}{(qz)^{2k}}\phi\left(\frac{-1}{q^2 z} - \frac{\overline{b}}{q}\right)$$

which in turn equals

$$\frac{\tau(\chi)}{q(qz)^{2k}}\sum_{b=1}^{q}\chi(-\overline{b})\phi\left(\frac{-1}{q^2 z} - \frac{\overline{b}}{q}\right).$$

Again, applying Lemma A.2 we obtain

$$\frac{\tau(\chi)}{(qz)^{2k}\tau(\overline{\chi})\overline{\chi}(-1)}\phi_{\overline{\chi}}\left(-\frac{1}{q^2 z}\right).$$

∎

By using this lemma and the methods used in the proof of Theorem A.6, *mutatis mutandis*, one can show

Corollary A.10. *Let N and k be positive integers and $\phi(z) \in S_{2k}(N,1) = S_{2k}(\Gamma_0(N))$. If χ is a character mod q, where $N|q$, then*

$$\left(\frac{q}{2\pi}\right)^s \Gamma(s)L_\chi(s,\phi)$$

extends to an entire function and satisfies the functional equation

$$\left(\frac{q}{2\pi}\right)^s \Gamma(s)L_\chi(s,\phi) = \widetilde{C}_\chi\left(\frac{q}{2\pi}\right)^{2k-s}\Gamma(2k-s)L_{\overline{\chi}}(2k-s,\phi)$$

where

$$\widetilde{C}_\chi = \frac{\chi(-1)\tau(\chi)}{i^{2k}\tau(\overline{\chi})}.$$

Up to this point we have established a functional equation for the L-function twisted by characters mod q, where $(q,N)=1$ and $(q,N)=N$. In order to reduce the restrictions on q and N further we recall a result of Atkin and Lehner ([AL]). For $\phi(z)=\sum_{n=1}^{\infty} a_n e(nz)$, we extend a_n to $\mathbf{Q}$ by setting $a_\alpha = 0$ for $\alpha \notin \mathbf{Z}$. We define Atkin and Lehner's Hecke operators T_p and U_q as follows:

$$(T_p\phi)(z) = \sum_{n=1}^{\infty}(a_{np} + p^{2k-1}a_{\frac{n}{p}})e(nz)$$
$$(U_q\phi)(z) = \sum_{n=1}^{\infty} a_{nq}e(nz).$$

For $\varphi \in S_{2k}(\Gamma_0(N))$ we say φ is a *newform of weight* $2k$ *and level* N if for primes $(p,N)=1$ and $q|N$, there exist $\alpha,\beta,\lambda,\mu \in \mathbf{C}$ such that $T_p\varphi = \alpha\varphi$, $U_q\varphi = \beta\varphi$, $\varphi|[H_N]_{2k} = \lambda\varphi$, and $\overline{\varphi(-\overline{z})} = \mu\varphi(z)$. This definition is equivalent to that of Atkin and Lehner ([L] Theorem 9). We say $\varphi(z) = \sum_{n=1}^{\infty} a_n e(nz)$ is *normalized* if $a_1 = 1$. Let $\mathcal{N}_{2k}(\Gamma_0(N))$ denote the set of all normalized newforms of weight $2k$ and level N. Finally, if q is a prime dividing N such that $q^\alpha \| N$, we let W_q denote any matrix of the form

$$W_q = \begin{pmatrix} q^\alpha x & y \\ Nu & q^\alpha v \end{pmatrix} \quad \text{where } x,y,u,v \in \mathbf{Z} \text{ such that } \det W_q = q^\alpha.$$

We now state, without proof, a theorem of Atkin and Lehner.

Theorem A.11. [AL] *If* $\varphi(z) = \sum_{n=1}^{\infty} a_n e(nz) \in \mathcal{N}_{2k}(\Gamma_0(N))$, *$p$ is a prime with* $(p,N)=1$ *and q is a prime dividing N such that* $q^\alpha \| N$, *then*

(a) $T_p\varphi = a_p\varphi$

(b) $U_q\varphi = a_q\varphi$

(c) $\varphi|[W_q]_{2k} = \lambda(q)\varphi$ *where* $\lambda(q) = \pm 1$.

With this result we can relax the condition on q and N by assuming φ is a newform of even weight. We begin with

Lemma A.12. *Let $\varphi \in \mathcal{N}_{2k}(\Gamma_0(N))$, let q be a prime dividing N, and let α be an integer such that $q^\alpha \| N$. Further suppose that φ satisfies $\varphi|[H_N]_{2k} = \frac{1}{\omega i^{2k}}\varphi$ where $\omega = 1$ or -1. Then if $(b,q) = 1$ and β is a nonnegative integer we have*

$$\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right) = \frac{\omega i^{2k}}{\lambda(q)\left(\sqrt{\widetilde{N}}q^{\alpha+\beta}z\right)^{2k}}\varphi\left(\frac{u}{q^{\alpha+\beta}} - \frac{1}{\widetilde{N}q^{2(\alpha+\beta)}z}\right)$$

where $N = q^\alpha\widetilde{N}$, $q^{\alpha+\beta}v - ub\widetilde{N} = 1$ and λ is from Theorem A.11(c).

Proof: We are under the hypothesis that $(b,q) = 1$ and $(q,\widetilde{N}) = 1$, thus we can find $u, v \in \mathbf{Z}$ such that $q^{\alpha+\beta}v - ub\widetilde{N} = 1$. Letting $W_q = \begin{pmatrix} q^{\alpha+\beta} & -b \\ -q^\alpha\widetilde{N}u & q^\alpha v \end{pmatrix}$, we will evaluate $\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right)|[H_N W_q]_{2k}$ in two ways. By Theorem A.11 we have

$$\begin{aligned}
\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right)|[H_N W_q]_{2k} &= \left(\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right)|[H_N]_{2k}\right)|[W_q]_{2k} \\
&= \frac{1}{\omega i^{2k}}\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right)|[W_q]_{2k} \\
&= \frac{\lambda(q)}{\omega i^{2k}}\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right).
\end{aligned}$$

On the other hand, we observe

$$H_N W_q = \begin{pmatrix} q^\alpha\widetilde{N}u & -q^\alpha v \\ q^{2\alpha+\beta}\widetilde{N} & -q^\alpha b\widetilde{N} \end{pmatrix}.$$

Thus

$$\begin{aligned}
\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right)|[H_N W_q]_{2k} &= \frac{(q^{2\alpha}\widetilde{N})^k}{(q^{2\alpha+\beta}\widetilde{N}z)^{2k}}\varphi\left(\frac{q^\alpha\widetilde{N}u\left(z + \frac{b}{q^{\alpha+\beta}}\right) - q^\alpha v}{q^{2\alpha+\beta}\widetilde{N}\left(z + \frac{b}{q^{\alpha+\beta}}\right) - q^\alpha b\widetilde{N}}\right) \\
&= \frac{1}{\left(q^{\alpha+\beta}\sqrt{\widetilde{N}}z\right)^{2k}}\varphi\left(\frac{q^\alpha\widetilde{N}uz + \frac{1}{q^\beta}(b\widetilde{N}u - q^{\alpha+\beta}v)}{q^{2\alpha+\beta}\widetilde{N}z}\right) \\
&= \frac{1}{\left(q^{\alpha+\beta}\sqrt{\widetilde{N}}z\right)^{2k}}\varphi\left(\frac{u}{q^{\alpha+\beta}} - \frac{1}{q^{2\alpha+2\beta}\widetilde{N}z}\right).
\end{aligned}$$

Whence

$$\varphi\left(z + \frac{b}{q^{\alpha+\beta}}\right) = \frac{\omega i^{2k}}{\lambda(q)\left(\sqrt{\widetilde{N}}q^{\alpha+\beta}z\right)^{2k}}\varphi\left(\frac{u}{q^\alpha} - \frac{1}{q^{2(\alpha+\beta)}\widetilde{N}z}\right). \qquad \blacksquare$$

We could now proceed to prove the analogs of Lemma A.5 and Theorem A.6, however we will first generalize Lemma A.12 further. First we need to establish the following result.

Lemma A.13. *If* $\varphi(z) = \sum_{n=1}^{\infty} a_n \mathrm{e}(nz) \in \mathcal{N}_{2k}(\Gamma_0(N))$ *and* q *is a prime dividing* N, *then*

$$a_q a_n = a_{nq} \quad \text{for all } n \geq 1$$

and

$$\varphi(q^m z) = \frac{1}{(qa_q)^m} \sum_{r=0}^{q^m-1} \varphi\left(z + \frac{r}{q^m}\right) \quad \text{for all } m \geq 0.$$

Proof: By Theorem A.11(b) we observe

$$a_q \varphi(z) = (U_q \varphi)(z) = \sum_{n=1}^{\infty} a_{nq} \mathrm{e}(nz).$$

But trivially

$$a_q \varphi(z) = \sum_{n=1}^{\infty} a_q a_n \mathrm{e}(nz).$$

So by the uniqueness of the Fourier expansion

$$a_q a_n = a_{nq} \quad \text{for all } n \geq 1.$$

We prove the remaining statement by induction. We first note that the result is trivial for $m = 0$. However, we must prove the result for $m = 1$ which is used in the general case.

If $m = 1$ we have

$$\begin{aligned} \sum_{r=0}^{q-1} \varphi\left(z + \frac{r}{q}\right) &= \sum_{r=0}^{q-1} \sum_{n=1}^{\infty} a_n \mathrm{e}\left(n\left(z + \frac{r}{q}\right)\right) \\ &= \sum_{n=1}^{\infty} a_n \mathrm{e}(nz) \sum_{r=0}^{q-1} \mathrm{e}\left(\frac{nr}{q}\right). \end{aligned}$$

Now since

$$\sum_{r=0}^{q-1} \mathrm{e}\left(\frac{nr}{q}\right) = \begin{cases} q & \text{if } q|n \\ 0 & \text{otherwise,} \end{cases}$$

we have the above equals

$$q \sum_{n=0}^{\infty} a_{nq} \mathrm{e}(nqz) = q a_q \varphi(qz).$$

Now assuming the result is true for m we have

$$\begin{aligned}\varphi(q^{m+1}x) &= \varphi\left(q^m(qz)\right)\\ &= \frac{1}{(qa_q)^m}\sum_{s=0}^{q^m-1}\varphi\left(qz+\frac{s}{q^m}\right)\\ &= \frac{1}{(qa_q)^m}\sum_{s=0}^{q^m-1}\varphi\left(q\left(z+\frac{s}{q^{m+1}}\right)\right).\end{aligned}$$

Applying the result for $m=1$ we have that this equals

$$\begin{aligned}\frac{1}{(qa_q)^m}\sum_{s=0}^{q^m-1}\frac{1}{qa_q}\sum_{r=0}^{q-1}\varphi\left(z+\frac{s}{q^{m+1}}+\frac{r}{q}\right) &= \frac{1}{(qa_q)^{m+1}}\sum_{s=0}^{q^m-1}\sum_{r=0}^{q-1}\varphi\left(z+\frac{s+q^m r}{q^{m+1}}\right)\\ &= \frac{1}{(qa_q)^{m+1}}\sum_{r'=0}^{q^{m+1}-1}\varphi\left(z+\frac{r'}{q^{m+1}}\right).\end{aligned}$$

So by induction we have the desired result for all $m \geq 0$. ∎

With this we prove the following lemma.

Lemma A.14. *Let $\varphi \in \mathcal{N}_{2k}(\Gamma_0(N))$ and let q be a prime such that $q^\alpha|N$. Further suppose that φ satisfies $\varphi|[H_N]_{2k} = \frac{1}{\omega i^{2k}}\varphi$ where $\omega = 1$ or -1. Then if $(b,q)=1$ we have*

$$\varphi\left(z+\frac{b}{q^\alpha}\right) = \frac{\omega i^{2k}}{\lambda(q)\left(q^\alpha\sqrt{\widetilde{N}}z\right)^{2k}}\varphi\left(\frac{u}{q^\alpha}-\frac{1}{q^{2\alpha}\widetilde{N}z}\right)$$

where $ub\widetilde{N} \equiv -1 \bmod q^\alpha$, $q^{\alpha+\beta}\|N$ and $N = q^{\alpha+\beta}\widetilde{N}$.

Proof: By Lemma A.13

$$\begin{aligned}\varphi\left(z+\frac{b}{q^\alpha}\right) &= \varphi\left(q^\beta\left(\frac{z}{q^\beta}+\frac{b}{q^{\alpha+\beta}}\right)\right)\\ &= \frac{1}{(qa_q)^\beta}\sum_{r=0}^{q^\beta-1}\varphi\left(\frac{z}{q^\beta}+\frac{b}{q^{\alpha+\beta}}+\frac{r}{q^\beta}\right)\\ &= \frac{1}{(qa_q)^\beta}\sum_{r=0}^{q^\beta-1}\varphi\left(\frac{z}{q^\beta}+\frac{rq^\alpha+b}{q^{\alpha+\beta}}\right).\end{aligned}$$

Now since $(b,q)=1$ and q is a prime we have $(rq^\alpha+b,q)=1$, so we apply Lemma A.12 and obtain that the above equals

$$\frac{\omega i^{2k}}{(qa_q)^\beta\lambda(q)\left(\sqrt{\widetilde{N}}q^\alpha z\right)^{2k}}\sum_{r=0}^{q^\beta-1}\varphi\left(\frac{u}{q^{\alpha+\beta}}-\frac{1}{\widetilde{N}q^{2\alpha+\beta}z}\right)$$

where $u(rq^\alpha + b)\widetilde{N} \equiv -1 \bmod q^{\alpha+\beta}$ and λ is from Theorem A.11(c). Thus, we can replace u by $-\overline{rq^\alpha + b}\,\overline{\widetilde{N}}$ where $(rq^\alpha + b)\overline{rq^\alpha + b} \equiv 1 \bmod q^{\alpha+\beta}$ and $\widetilde{N}\overline{\widetilde{N}} \equiv \bmod q^{\alpha+\beta}$. Since we are summing over $0 \leq r \leq q^\beta - 1$, an elementary number theory argument implies the above equals

$$\frac{\omega i^{2k}}{(qa_q)^\beta \lambda(q)\left(\sqrt{\widetilde{N}}q^\alpha z\right)^{2k}} \sum_{r=0}^{q^\beta-1} \varphi\left(\frac{-(rq^\alpha + \overline{b})\overline{\widetilde{N}}}{q^{\alpha+\beta}} - \frac{1}{\widetilde{N}q^{2\alpha+\beta}z}\right)$$
$$= \frac{\omega i^{2k}}{(qa_q)^\beta \lambda(q)\left(\sqrt{\widetilde{N}}q^\alpha z\right)^{2k}} \sum_{r=0}^{q^\beta-1} \varphi\left(\frac{-r\widetilde{N}}{q^\beta} - \frac{\overline{b}\overline{\widetilde{N}}}{q^{\alpha+\beta}} - \frac{1}{\widetilde{N}q^{2\alpha+\beta}z}\right).$$

Now as r runs through $\{0, \ldots, q^\beta - 1\}$ so does $-r\overline{\widetilde{N}}$ modulo q^β. Thus, the above equals

$$\frac{\omega i^{2k}}{(qa_q)^\beta \lambda(q)\left(\sqrt{\widetilde{N}}q^\alpha z\right)^{2k}} \sum_{r'=0}^{q^\beta-1} \varphi\left(\frac{r'}{q^\beta} - \frac{\overline{b}\overline{\widetilde{N}}}{q^{\alpha+\beta}} - \frac{1}{\widetilde{N}q^{2\alpha+\beta}z}\right)$$

which by Lemma A.13 equals

$$\frac{\omega i^{2k}}{\lambda(q)\left(\sqrt{\widetilde{N}}q^\alpha z\right)^{2k}} \varphi\left(q^\beta\left(\frac{-\overline{b}\overline{\widetilde{N}}}{q^{\alpha+\beta}} - \frac{1}{\widetilde{N}q^{2\alpha+\beta}z}\right)\right) = \frac{\omega i^{2k}}{\lambda(q)\left(\sqrt{\widetilde{N}}q^\alpha z\right)^{2k}} \varphi\left(\frac{-\overline{b}\overline{\widetilde{N}}}{q^\alpha} - \frac{1}{\widetilde{N}q^{2\alpha}z}\right).$$

Since $-\overline{b}\overline{\widetilde{N}}b\widetilde{N} \equiv -1 \bmod q^{\alpha+\beta}$ implies $-\overline{b}\overline{\widetilde{N}}b\widetilde{N} \equiv -1 \bmod q^\alpha$ we have the desired results. ∎

Before we proceed, we combine Lemma A.12 and Lemma A.14.

Corollary A.15. *Let $\varphi \in \mathcal{N}_{2k}(\Gamma_0(N))$ and let q be a prime dividing N such that $q^\alpha \| N$, and write $N = q^\alpha\widetilde{N}$. Further suppose that φ satisfies $\varphi|[H_N]_{2k} = \frac{1}{\omega i^{2k}}\varphi$ where $\omega = 1$ or -1. Then if $(b, q) = 1$ and Q is any power of q we have*

$$\varphi\left(z + \frac{b}{Q}\right) = \frac{\omega i^{2k}}{\lambda(q)\left(Q\sqrt{\widetilde{N}}z\right)^{2k}} \varphi\left(\frac{u}{Q} - \frac{1}{Q^2\widetilde{N}z}\right)$$

where u is such that $ub\widetilde{N} \equiv -1 \bmod Q$ and λ comes from Theorem A.11(c).

We can now prove the analog of Lemma A.5 in this generality.

Lemma A.16. *Let $\varphi \in \mathcal{N}_{2k}(\Gamma_0(N))$ and let q be a prime dividing N such that $q^\alpha \| N$. Further suppose that φ satisfies $\varphi|[H_N]_{2k} = \frac{1}{\omega i^{2k}}\varphi$ where $\omega = 1$ or -1. If Q is any power of q and χ is a primitive character mod Q we have*

$$\varphi_\chi(z) = \frac{\omega\chi(-\widetilde{N})\tau(\chi)}{\lambda(q)\tau(\overline{\chi})}\left(\frac{i}{\sqrt{\widetilde{N}}Qz}\right)^{2k}\varphi_{\overline{\chi}}\left(\frac{-1}{\widetilde{N}Q^2z}\right)$$

where $N = q^\alpha\widetilde{N}$ and λ is from Theorem A.11(c).

Proof: By Lemma A.2 and Corollary A.15

$$\begin{aligned}\varphi_\chi(z) &= \frac{\tau(\chi)\chi(-1)}{Q}\sum_{b=1}^{Q}\overline{\chi}(b)\varphi\left(z+\frac{b}{Q}\right)\\ &= \frac{\tau(\chi)\chi(-1)}{Q}\sum_{b=1}^{Q}\overline{\chi}(b)\frac{\omega i^{2k}}{\lambda(q)\left(\sqrt{\widetilde{N}}Qz\right)^{2k}}\varphi\left(\frac{u}{Q}-\frac{1}{\widetilde{N}Q^2z}\right)\end{aligned}$$

where $ub\widetilde{N} \equiv -1 \bmod Q$ and λ is from Theorem A.11(c). Thus $\chi(b)\chi(-u\widetilde{N}) = 1$ which implies $\overline{\chi}(b) = \chi(-u\widetilde{N})$. Also, as b runs through a complete set of representatives modulo Q, so does u. Therefore, the above equals

$$\frac{\omega\tau(\chi)\chi(\widetilde{N})}{\lambda(q)Q}\left(\frac{i}{\sqrt{\widetilde{N}}Qz}\right)^{2k}\sum_{u=1}^{Q}\chi(u)\varphi\left(\frac{u}{Q}-\frac{1}{\widetilde{N}Q^2z}\right)$$

which by Lemma A.2 equals

$$\frac{\omega\tau(\chi)\chi(\widetilde{N})}{\lambda(q)Q}\left(\frac{i}{\sqrt{\widetilde{N}}Qz}\right)^{2k}\frac{Q}{\overline{\chi}(-1)\tau(\overline{\chi})}\varphi_{\overline{\chi}}\left(\frac{-1}{\widetilde{N}Q^2z}\right)$$

$$= \frac{\omega\tau(\chi)\chi(-\widetilde{N})}{\lambda(q)\tau(\overline{\chi})}\left(\frac{i}{\sqrt{\widetilde{N}}Qz}\right)^{2k}\varphi_{\overline{\chi}}\left(\frac{-1}{\widetilde{N}Q^2z}\right). \blacksquare$$

By using this lemma and the methods used in the proof of Theorem A.6, *mutatis mutandis*, one can show our desired goal.

Theorem A.17. *Let $\varphi \subset \mathcal{N}_{2k}(\Gamma_0(N))$ and let q be a prime dividing N such that $q^\alpha \| N$, and write $N = q^\alpha\widetilde{N}$. Further suppose that φ satisfies $\varphi|[H_N]_{2k} = \frac{1}{\omega i^{2k}}\varphi$ where $\omega = 1$ or -1. If Q is any power of q and χ is a primitive character mod Q we have*

$$\left(\frac{Q\sqrt{\widetilde{N}}}{2\pi}\right)^s\Gamma(s)L_\chi(s,\varphi)$$

extends to an entire function of s and satisfies

$$\left(\frac{Q\sqrt{\widetilde{N}}}{2\pi}\right)^{s}\Gamma(s)L_{\chi}(s,\varphi)=\widehat{C}_{\chi}\left(\frac{Q\sqrt{\widetilde{N}}}{2\pi}\right)^{2k-s}\Gamma(2k-s)L_{\overline{\chi}}(2k-s,\varphi)$$

where

$$\widehat{C}_{\chi}=\frac{\omega\chi(-\widetilde{N})\tau(\chi)}{\lambda(q)\tau(\overline{\chi})}.$$

VITA

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