ALGEBRAIC AND TOPOLOGICAL METHODS IN THE COHOMOLOGY THEORY OF ELEMENTARY GROUPS

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PREFACE

This dissertation makes no pretense to originality. It is, rather, an initial investigation of the essential complexities involved in the cohomology theory of groups. For instance, celebrated pioneering works of Eilenberg-MacLane ([12],[12],[14]) on the groups $H(\pi,n)$ and the groundwork of Hochschild-Serre [19] on the cohomology of group extensions should, in principle, enable us to compute the cohomology algebra of groups. Practically, however, it seems to be impossible to do so in general. Hence, we focus attention on the most effective computational techniques, supplementing and enriching by cohomology operations the algebraic structure of the cohomology module.

This dissertation has developed out of a year course on homological algebra taught by Professor Hiroshi Uehara. It aism to give an overview of the theory and practice of the cohomological method. It is a collection of many results which until now were scattered through the literature, and some have been proved in detail while others have only been stated "with reference to sources".

In Chapter I we present the definitions and results of algebraic topology which play important roles in calculating the cohomology algebra of a pair of topological spaces. We include here the direct limit of homotopy groups that has not been available in existing texts.

In Chapter II we consider the theory of cohomology of groups in the general setting of modules over R-algebras. We make use of results in relative homological algebra [15] to define the extension functor Ext and to introduce a generalized form of the classical comparison theorem. This setting allows us to define the cup product relative to an A-pairing [30].

Chapter III provides the results that link Chapters I and II together. We show the result $H^*(K(\pi,1),N) \cong H^*(\pi,N)$ when π operates trivially on the abelian group N.

In Chapter IV we use the results obtained in Chapters I, II, and III to compute the cohomology ring $H^*(\pi,N)$ for various groups π. In particular, we compute $H^*(RP^\infty,Z_2) \cong H^*(Z_2,Z_2)$, and $H^*(S^1,R) \cong H^*(Z,R)$ using the algebraic topology tools. We define cup product in $\hom_{Z(Z_3)}(Z_3,Z)$ and compute the cohomology ring $H^*(Z_3,Z)$.

Chapter V formulates the mechanism of spectral sequences via exact couples (Massey [23]) and then proceeds to several applications in the cohomology of group extensions using the Hochschild-Serre spectral sequence [19].

The final chapter culminates in making detailed computations of the cohomology algebras $H^*(Z_2,Z_2)$, $H^*(Z_3,Z)$, $H^*(Z_3 \times Z_2,Z)$ and $H^*(S_3,Z)$.

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CHAPTER I

ELEMENTS OF ALGEBRAIC TOPOLOGY

In Chapter III the fact is discussed that the cohomology of a group π can be interpreted as the cohomology of an arcwise-connected aspherical space (all higher homotopy groups vanish) with fundamental group π. Consequently, the methods of algebraic topology are utilized in Chapter IV for the computation of the cohomology algebra of a group. This chapter is a collection of those definitions and results which play important roles in calculating the cohomology algebra of a pair of topological spaces. The reader is assumed to know the elementary parts of relative singular cohomology (homology) theory and some basic properties of homotopy groups. The notation is that of [16]. For proofs, the reader is referred to [16] or [25]. Throughout this thesis, R will denote a commutative ring with unit. Unless noted otherwise, the tensor product notation $\otimes$ stands for $\otimes_R$.

The Cohomology Functor $H^*:\mathcal{T}\to\mathcal{A}$

A key feature of cohomology which distinguishes it from homology is the existence of a natural multiplication called cup product ($\smile$- product). For a pair of spaces (X,A), this multiplication makes the graded R-module $H^*(X,A)$ into a graded R-algebra and establishes a finer invariant than the cohomology module itself.

First define a cup product at the relative singular cochain level. For subspaces A_1, A_2 of X, it is required that

$$a \cup b \; \varepsilon \; \frac{S^{p+q}(X)}{(S^{p+q}(A_1) + S^{p+q}(A_2))}$$

if $a \; \varepsilon \; S^p(X,A_1)$, $b \; \varepsilon \; S^q(X,A_2)$.

Define a $(p + q)$-singular cochain $a \cup b$ by setting

$$(a \cup b)(\sigma) = a(\sigma^{(p+q)\cdots(p+1)}) \cdot b(\sigma^{(p-1)\cdots(0)}) \qquad ,$$

for any singular $(p + q)$-simplex σ of X, where $\sigma^{(i)}$ denotes the i-th face of σ and

$$\sigma^{(i_1)(i_2)\cdots(i_k)} = [\cdots[[\sigma^{(i_1)}]^{(i_2)}]\cdots]^{(i_k)} \qquad .$$

Since a and b vanish on $S_p(A_1)$ and $S_q(A_2)$ respectively, it follows that $a \cup b$ vanishes on $S_{p+q}(A_1) + S_{p+q}(A_2)$. If, in addition, $\{A_1,A_2\}$ is an excised pair in X, then the inclusion map $i:S(A_1) + S(A_2) \to S(A_1 \cup A_2)$ is a chain equivalence inducing a graded R-isomorphism

$$H^*((S(X)/(S(A_1) + S(A_2)))^{\#}) \approx H^*((S(X)/S(A_1 \cup A_2))^{\#}) = H^*(X,A_1 \cup A_2) \quad .$$

With this identification, the cohomology class $[a \cup b]$ is considered as an element in $H^{p+q}(X,A_1 \cup A_2)$.

<u>Definition 1.1</u>: If $\{A_1,A_2\}$ is an excised pair in X, then for $\alpha = [a] \; \varepsilon \; H^p(X,A_1)$, $\beta = [b] \; \varepsilon \; H^q(X,A_2)$, the <u>cup product</u> $\alpha \cup \beta \; \varepsilon \; H^{p+q}(X,A_1 \cup A_2)$ is defined by $\alpha \cup \beta = [a \cup b]$.

<u>Proposition 1.1</u>: The cup product has the following properties:

(1) <u>Bilinearity</u>. If $\{A_1,A_2\}$ is an excised pair in X, then the map $\cup:H^*(X,A_1) \otimes H^*(X,A_2) \to H^*(X,A_1 \cup A_2)$ defined by $\cup(\alpha \otimes \beta) = \alpha \cup \beta$

is a morphism of graded R-modules.

(2) Naturality. If $\{A_1,A_2\}$ and $\{B_1,B_2\}$ are excised pairs in X and Y respectively, and if $h:X \to Y$ is a continuous map such that $h(A_i) \subset B_i$ (i=1,a), then $h^*(\gamma_1 \smile \gamma_2) = h^*(\gamma_1) \smile h^*(\gamma_2)$, for $\gamma_i \in H^*(Y,B_i)$ (i=1,2).

(3) Associativity. Let A_i (i=1,2,3) be subspaces of X such that $\{A_1,A_2\}$, $\{A_2,A_3\}$, $\{A_1 \cup A_2,A_3\}$, and $\{A_1,A_2 \cup A_3\}$ are excised pairs in X. Then for $\alpha_i \in H^*(X,A_i)$ (i=1,2,3),

$$(\alpha_1 \smile \alpha_2) \smile \alpha_3 = \alpha_1 \smile (\alpha_2 \smile \alpha_3) \in H^*(X,A_1 \cup A_2 \cup A_3) \quad .$$

(4) Commutativity. If $\{A_1,A_2\}$ is an excised pair in X, then $\alpha_1 \smile \alpha_2 = (-1)^{pq}\alpha_2 \smile \alpha_1$, for $\alpha_1 \in H^p(X,A_1)$ and $\alpha_2 \in H^q(X,A_2)$.

(5) Unit. For $\alpha \in H^*(X,A)$, $1 \smile \alpha = \alpha \smile 1 = \alpha$, where $1 \in H^0(X)$ is the cohomology class of the augmentation $\varepsilon:S_0(X) \to R$.

For the category of pairs of topological spaces $\mathfrak{T}$, and for the category of graded R-algebras $\mathfrak{A}$, we define the cohomology functor

$$H^*:\mathfrak{T} \to \mathfrak{A}$$

by $H^*(X,A)$. By (1) and (3) of Proposition 1.1, $H^*(X,A)$ is a graded R-algebra, and (2) implies H^* is a contravariant functor. Note that the algebras in $\mathfrak{A}$ are not assumed to have a unit. However, in the absolute case $A = \phi$, $\mathfrak{A}$ is the usual category of graded R-algebras with units.

Relations Between Homology and Cohomology

Given a pair of spaces (X,A), there exists a bilinear pairing (called the Kronecker product)

$$H^q(X,A) \otimes H_q(X,A) \to R$$

given by the formula $\langle\alpha,\omega\rangle = \langle u,a\rangle = u(a)$, for $\alpha = [u] \in H^q(X,A)$, $\omega = [a] \in H_q(X,A)$, and where $\langle u,a\rangle$ denotes the value $u(a)$ of the cocycle u on the cycle a. It is obvious that the definition does not depend upon the choice of representatives.

The Kronecker product enters in the following <u>universal-coefficient theorem for cohomology</u>.

<u>Theorem 1.1</u>: Given a pair of spaces (X,A) with $H_{q-1}(X,A)$ free, then an R-isomorphism

$$\zeta: H^q(X,A) \to \hom_R(H_q(X,A),R)$$

is defined by $\zeta(\alpha)(\omega) = \langle\alpha,\omega\rangle$, for $\alpha \in H^q(X,A)$, $\omega \in H_q(X,A)$.

There is another product, called cap product ($\frown$- product), closely related to cup product, that associates homology and cohomology classes together. For subspaces A_1, A_2 of X, $u \in S^q(X,A_2)$, and $a = \sum_i r_i\sigma_i \in S_p(X)/(S_p(A_1) + S_p(A_2))$, we define

$$u \frown a = \sum_i (r_i u(\sigma_i^{(p-q-1)\cdots(0)}))\sigma_i^{(p)\cdots(p-q+1)} \in S_{p-q}(X,A_1) \quad .$$

This definition induces a R-homomorphism (mapping $[u] \otimes [a]$ to $[u \frown a]$)

$$H^q(X,A_2) \otimes H_p\left(\frac{S(X)}{S(A_1) + S(A_2)}\right) \to H_{p-q}(X,A_1) \quad .$$

If $\{A_1,A_2\}$ is an excised pair in X, this yields a R-homomorphism

$$\frown : H^q(X,A_2) \otimes H_p(X,A_1 \cup A_2) \to H_{p-q}(X,A_1)$$

called the <u>cap product</u>.

Definition 1.2: If $\{A_1,A_2\}$ is an excised pair in X, then for $\alpha = [u] \in H^q(X,A_2)$, $\omega = [a] \in H_p(X,A_1 \cup A_2)$, the cap product $\alpha \cap \omega \in H_{p-q}(X,A_2)$ is defined by $\alpha \cap \omega = [u \cap a]$.

Proposition 1.2: The cap product has the following properties:

(1) Naturality. If $\{A_1,A_2\}$ and $\{B_1,B_2\}$ are excised pairs in X and Y respectively, and if $h:X \to Y$ is a continuous map such that $h(A_i) \subset B_i$ $(i=1,2)$, then $h_*(h^*(\alpha) \cap \omega) = \alpha \cap h_*(\omega)$, for $\alpha \in H^*(Y,B_2)$, $\omega \in H_*(X,A_1 \cup A_2)$.

(2) Duality. If $\{A_1,A_2\}$ is an excised pair in X, then $\langle\alpha,\beta \cap \omega\rangle = \langle\alpha \cup \beta,\omega\rangle$, for $\alpha \in H^q(X,A_1)$, $\beta \in H^p(X,A_2)$, and $\omega \in H_{p+q}(X,A_1 \cup A_2)$.

Direct Limits of Homology and Homotopy

Let $\{X_\lambda,f_\lambda^u\}$ be a direct system of topological spaces X_λ and let $\sum_\lambda X_\lambda$ be the topological direct sum of the spaces X_λ. Define the direct limit $\varinjlim X_\lambda$ of the system by the quotient space $\sum_\lambda X_\lambda/\sim$, where $\sim$ is the usual equivalence relation: for $x_\lambda \in X_\lambda$ and $x_\mu \in X_\mu$, $x_\lambda \sim x_\mu$ iff there exists γ such that $\lambda < \gamma$, $\mu < \gamma$, and $f_\lambda^\gamma(x_\lambda) = f_\mu^\gamma(x_\mu)$. Denoting the quotient map by $f:\sum_\lambda X_\lambda \to \varinjlim X_\lambda$, we have $f_\lambda = f|_{X_\lambda}:X_\lambda \to \varinjlim X_\lambda$.

Given a sequence of topological spaces $X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$, there is a direct system of spaces $\{X_n,i_n^m\}$ where $i_n^m:X_n \to X_m$ is the inclusion map. Evidently, $\varinjlim X_n$ is the union $\bigcup_n X_n$ whose topology is given by the property that A contained in $\bigcup_n X_n$ is open iff $A \cap X_n$ is open in X_n for any n. Hence, X_n is a subspace of $\varinjlim X_n$. The following theorem is well known.

Theorem 1.2: If $X_1 \subset X_2 \subset \cdots \quad X_n \subset X_{n+1} \subset \cdots$ is a sequence of

Hausdorff spaces, then there exists an isomorphism

$$\varinjlim i_{n_*}:\varinjlim H_q(X_n) \approx H_q(\varinjlim X_n) \quad ,$$

where $i_{n_*}:H_q(X_n) \to H_q(\varinjlim X_n)$ is the homomorphism induced by the inclusion $i_n:X_n \to \varinjlim X_n$.

Note that the proof of this theorem and the following homotopy theorem is based on the fact that any compact subset in $\varinjlim X_n$ is contained in some X_n.

<u>Theorem 1.3</u>: If $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$ is a sequence of Hausdorff spaces, then there exists an isomorphism

$$\varinjlim i_{n\#}:\varinjlim \pi_p(X_n,*) \approx \pi_p(\varinjlim X_n,*) \quad ,$$

where $*$ is a base point in X_1 and $i_{n\#}:\pi_p(X_n,*) \to \pi_p(\varinjlim X_n,*)$ is the homomorphism induced by the inclusion $i_n:X_n \to \varinjlim X_n$.

<u>Proof</u>: Although a proof of this theorem is elementary, it is sketched here because it is not contained in [16] or [25]. Since $i_{m\#}\, i^m_{n\#} = i_{n\#}$ for arbitrary integers m,n with $m > n$, it follows that $\varinjlim i_{n\#}$ is well defined. Assume that $i_{n\#}(\alpha_n) = i_{m\#}(\alpha_m)$ for $\alpha_n = [f] \in \pi_p(X_n,*)$ and $\alpha_m = [g] \in \pi_p(X_m,*)$. Then the two maps $i_n \circ f, i_m \circ g:(I^p,\partial I^p) \to (\varinjlim X_n,*)$ are homotopic. Hence there exists a homotopy $h:(I^p \times I,\partial I^p \times I) \to (\varinjlim X_n,*)$ between $i_n \circ f$ and $i_m \circ g$. Since $h(I^p \times I)$ is compact in $\varinjlim X_n$, there exists $h':(I^p \times I, \partial I^p \times I) \to (X_\ell,*)$ such that $h = i_\ell \circ h'$ and $\ell > m \geq n$. For any $\lambda \in I^p$, we have $i_\ell h'(\lambda,0) = h(\lambda,0) = i_n f(\lambda) = i_\ell i^\ell_n f(\lambda)$, so that $h'(\lambda,0) = i^\ell_n f(\lambda)$. Similarly, $h'(\lambda,1) = i^\ell_m g(\lambda)$. Therefore h' is a homotopy between representatives $i^\ell_n f$, $i^\ell_m g$ of $i^\ell_{n\#}(\alpha_n)$, $i^\ell_{m\#}(\alpha_m)$, respectively. It follows that

$i^{\ell}_{n\#}(\alpha_n) = i^{\ell}_{m\#}(\alpha_m)$ and thus $\varinjlim i_{n\#}$ is a monomorphism. Let $\alpha = [\zeta]$ be an arbitrary element of $\pi_p(\varinjlim X_n,*)$. Then $\zeta(I^p)$ is a compact subset in $\varinjlim X_n$ and so there exists $\eta : I^p \to X_n$ with the property $i_n \eta = \zeta$. Hence $\varinjlim i_{n\#}$ is an epimorphism. This completes the proof. ▲

Covering Spaces

Let $p:E \to B$ be a continuous map. An open subset $U \subset B$ is said to be <u>evenly covered</u> by p iff $p^{-1}(U)$ is the disjoint union of open subsets of E each of which is mapped homeomorphically onto U by p. If U is evenly covered by p, it is clear that any open subset of U is also evenly covered by p. A continuous map $p:E \to B$ is called a <u>covering projection</u> iff each point $b \in B$ has an open neighborhood evenly covered by p. E is called a <u>covering space</u> of B and B the <u>base space</u> of the covering.

In the sequel (see IV.1 and IV.2) two elementary covering spaces are considered; one is the 2-fold covering of the n-dimensional real projective space RP^n by the n-sphere S^n, and the other is the covering space R^1 of the 1-sphere S^1.

Let B be a connected locally arcwise-connected space. The category C of connected covering spaces of B has objects which are covering projections $p:E \to B$ and morphisms f which satisfy commutative triangles

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "P_1"'] & & E_2 \arrow[dl, "P_2"] \\ & B & \end{tikzcd}$$

It is easy to see that every morphism in C is itself a covering projection. The following result is well known.

Theorem 1.4: Let $p_1:E_1 \to B$ and $p_2:E_2 \to B$ be objects in the category C of connected covering spaces of a connected locally arcwise-connected space B. Then there is a morphism $f:E_1 \to E_2$ iff there exist $e_1 \in E_1$ and $e_2 \in E_2$ with $p_1(e_1) = p_2(e_2)$ such that $p_{1\#}\pi_1(E_1,e_1)$ is conjugate to a subgroup of $p_{2\#}\pi_1(E_2,e_2)$ in $\pi_1(B,p_1(e_1))$.

It follows that two objects in C are equivalent iff their fundamental groups are mapped to conjugate subgroups of the fundamental group of the base space.

By a covering transformation or deck transformation of an object $p:E \to B$ in the category C we mean an isomorphism $h:E \to E$ in C. Hence, a deck transformation h is a fibre preserving homeomorphism of E. The set of deck transformations of $p:E \to B$ forms a group under composition. It is called the group of deck transformations and is denoted by $\mathfrak{T}(E,p)$.

An object $p:E \to B$ in C is called a regular (normal) covering iff $p_\#\pi_1(E,e)$ is normal in $\pi_1(B,p(e))$ for some $e \in E$. Since a normal subgroup is equal to each of its conjugate subgroups, the condition of regularity for a covering is independent of the base point e. The following theorem describes the fundamental group of the base space in terms of a universal covering space.

Theorem 1.5: Let $p:E \to B$ be a regular covering in C. Then the group (E,p) of deck transformations is isomorphic to the quotient group $\pi_1(B,p(e))/p_\#\pi_1(E,e)$. If E is a universal covering space (namely, $\pi_1(E,e) = 0$), then $\pi_1(B,p(e)) \simeq \mathfrak{T}(E,p)$.

This section is concluded by stating a long exact homotopy sequence associated with a covering. It is well known that the covering homotopy theorem holds true for any object $p:E \to B$ in C; that is, if a continuous map $f:Y \to B$ has a lifting map $f':Y \to E$ (namely, $pf' = f$), then any homotopy $h:Y \times I \to B$ with $h(y,0) = f(y)$ for all $y \in Y$ can be lifted to a homotopy $H:Y \times I \to E$ such that $H(y,0) = f'(y)$ for all $y \in Y$. This is illustrated by the commutative diagram

$$\begin{tikzcd}
& & E \arrow[drr, "p"] & & \\
Y \arrow[urr, "f'"] \arrow[rr, "i"] & & Y \times I \arrow[u, dashed, "H"] \arrow[rr, "h"] & & B
\end{tikzcd}$$

where i is injection and $hi = f$. Hence, there is an exact sequence of a covering $p:E \to B$ with a fibre F:

$$\cdots \to \pi_n(F) \xrightarrow{i_\#} \pi_n(E) \xrightarrow{p_\#} \pi_n(B) \xrightarrow{\Delta} \pi_{n-1}(F) \to \cdots \to \pi_2(B)$$
$$\xrightarrow{\Delta} \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{p_\#} \pi_1(B) \quad .$$

Since F is a discrete space, the following theorem holds true.

Theorem 1.6: If $p:E \to B$ is a covering, then $\pi_n(E,e) \cong \pi_n(B,p(e))$ for $n \geq 2$, and $p_\#$ maps $\pi_1(E,e)$ isomorphically into $\pi_1(B,p(e))$.

Duality of Manifolds

In this section the Poincaré Duality Theorem is stated. It is presented here as preparation for the computation of the cohomology algebra of the infinite dimensional real projective space (see IV.1).

Theorem 1.7: Let X be a compact orientable n-manifold and let $\omega \in H_n(X,\partial X)$ be the fundamental class of X. Then the maps

$$\cap\omega : H^q(X) \to H_{n-q}(X,\partial X)$$

and

$$\frown\omega : H^q(X,\partial X) \to H_{n-q}(X)$$

are isomorphisms.

This theorem is called the Lefschetz Duality Theorem. In the case when $\partial X = \phi$, the theorem is called the Poincaré Duality Theorem.

CHAPTER II

ELEMENTS OF HOMOLOGICAL ALGEBRA

The prerequisites for the study of the cohomology algebra of a group have been developed in several treatises (see [19], [22], and [20]) on certain topics in homological algebra. It is the purpose of this chapter to provide the reader with a direct access to this somewhat specialized material.

Types of Algebras and Modules Over Algebras

Definition 2.1: A graded Hopf Algebra A (over R) is a graded R-module which is both a graded R-algebra with product $A \otimes A \xrightarrow{m} A$ and unit $R \xrightarrow{e} A$ and a graded R-coalgebra for a coproduct $A \xrightarrow{\Delta} A \otimes A$ and counit $A \xrightarrow{\varepsilon} R$ such that (1) the unit e is a morphism of graded coalgebras; (2) the counit ε is a morphism of graded algebras; and (3) the product m is a morphism of graded coalgebras.

Observe that if A and B are graded Hopf algebras over R, then $A \otimes B$ is a graded Hopf algebra over R with coproduct the composition

$$A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes t \otimes 1} A \otimes B \otimes A \otimes B$$

(where t is the twisting homomorphism), and with counit the composition

$$A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} R \otimes R \xrightarrow{\approx} R \quad .$$

Some elementary examples of Hopf algebras are now given.

(1) The group ring $Z(\pi)$ of a group π is a trivially graded Hopf algebra over Z with coproduct $\Delta: Z(\pi) \to Z(\pi) \otimes Z(\pi)$ defined by $\Delta(g) = g \otimes g$ $(g \in \pi)$, and with counit the usual augmentation $\varepsilon: Z(\pi) \to Z$.

(2) Let $P_R[x]$ be the graded polynomial algebra on one generator x of even degree. $P_R[x]$ is a Hopf algebra with

$$\Delta(x^n) = \sum_{p+q=n} \binom{p+q}{p} x^p \otimes x^q \quad , \varepsilon(1) = 1 \qquad .$$

(3) Let $E_R[x]$ be the exterior algebra on one generator x of degree 1. Then $E_R[x]$ is a Hopf algebra with

$$\Delta(x) = 1 \otimes x + \otimes x\ 1 \quad , \varepsilon(1) = 1 \qquad .$$

By taking tensor products of Hopf algebras, it follows from a previous observation that the polynomial algebra $P_R[x_1, \cdots, x_n]$ on n generators each of even degree or the exterior algebra $E_R[x_1, \cdots, x_m]$ on m generators each of degree 1 is a Hopf algebra.

Definition 2.2: Let A be a graded R-algebra. Then M is said to be a left A-module (or, just A-module when no confusion can occur) iff (1) M is a graded R-module, and (2) there is a morphism of graded R-modules ${}_M\Phi: A \times M \to M$ of degree zero such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m \otimes 1"] \arrow[d, "1 \otimes {}_M\Phi"'] & A \otimes M \arrow[d, "{}_M\Phi"] \\ A \otimes M \arrow[r, "{}_M\Phi"] & M \end{tikzcd} \qquad \begin{tikzcd} R \otimes M \arrow[r, "\simeq"] \arrow[d, "e \otimes 1"'] & M \\ A \otimes M \arrow[ur, "{}_M\Phi"'] & \end{tikzcd}$$

commute.

One defines comodules over coalgebras by dualizing the above definition.

Definition 2.3: If M and N are left A-modules, then $f:M \to N$ is called an A-module homomorphism of degree d provided that f is a graded R-homomorphism of degree d and the diagram

$$\begin{array}{ccc} A \otimes M & \xrightarrow{1 \otimes f} & A \otimes N \\ \Big\downarrow {\scriptstyle M^{\Phi}} & & \Big\downarrow {\scriptstyle N^{\Phi}} \\ M & \xrightarrow{f} & N \end{array}$$

commutes; in other words, such that always
$f(ax) = (-1)^{(\deg f)(\deg a)} af(x)$.

The set of all A-module homomorphisms $f:M \to N$ of degree d forms a R-module, which is denoted by $\mathrm{Hom}_A^d(M,N)$. Then the class of all left A-modules forms a category ${}_A\mathfrak{M}$ with morphisms $\mathrm{hom}_A(\ ,) = \mathrm{Hom}_A^0(\ ,\ )$.

Some important left modules by pull-back are now considered which will be used in later sections. If $\alpha:A \to B$ is a morphism of graded R-algebras and N is a B-module, then N can be considered as an A-module by pull-back along α. Hereafter, this A-module is denoted by ${}_\alpha N$.

(1) Given left modules M and N over a Hopf algebra A, then ${}_\Delta(M \otimes N)$ is an A-module by pull-back along the coproduct $\Delta:A \xrightarrow{\Delta} A \otimes A$.

(2) If a graded R-algebra A is augmented by $\varepsilon:A \to R$, then ${}_\varepsilon R$ is an A-module by pull-back along the augmentation ε.

Definition 2.4: Let A and B be graded R-algebras and let $\alpha:A \to B$ be a morphism of algebras. For $M \in {}_A\mathfrak{M}$ and $N \in {}_B\mathfrak{M}$ a R-homomorphism $f:M \to N$ is called an α-morphism of modules iff f is a morphism of

A-modules considering ${}_{\alpha}N \in {}_{A}\mathcal{M}$.

It follows immediately that f is an α-morphism of modules iff the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\alpha \otimes f"] \arrow[d, "M^{\Phi}"'] & B \otimes N \arrow[d, "N^{\Phi}"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

is commutative; or, equivalently, $f(ax) = \alpha(a)f(x)$ for any $a \in A$ and for any $x \in M$. Note that if $g:N \rightarrow N'$ is a morphism of B-modules and $\alpha:A \rightarrow B$ is a morphism of algebras, then g is also a morphism of A-modules by considering ${}_{\alpha}N$, ${}_{\alpha}N' \in {}_{A}\mathcal{M}$.

<u>Definition 2.5</u>: Let M, N, and L be left modules over a Hopf algebra A. M and N are said to be <u>paired</u> with respect to L iff there exists a morphism of left A-modules (called an <u>A-pairing</u>)

$$\theta : {}_{\Delta}(M \otimes N) \rightarrow L \qquad ,$$

where $\Delta:A \rightarrow A \otimes A$ is the coproduct of A.

Consideration of the diagram

$$\begin{tikzcd}
 & M \otimes N \arrow[r, "\theta"] & L \\
A \otimes M \otimes A \otimes N \arrow[ur, "M^{\Phi} \otimes N^{\Phi}"] & & \\
A \otimes A \otimes M \otimes N \arrow[u, "1 \otimes t \otimes 1"] & & \\
 & A \otimes M \otimes N \arrow[ul, "\Delta \otimes 1 \otimes 1"] \arrow[uuu, "M \otimes N^{\Phi}"'] \arrow[r, "1 \otimes \theta"] & A \otimes L \arrow[uuu, "L^{\Phi}"']
\end{tikzcd}$$

shows that θ is a R-homomorphism satisfying

$$\sum (-1)^{|x||a''|}\theta(a'x \otimes a''y) = a\theta(x \otimes y) \qquad ,$$

where $\Delta(a) = \sum a' \otimes a''$.

Definition 2.6: Let A be a Hopf algebra over R. Then M is said to be an A-module algebra iff (1) M is an A-module; (2) M is a R-algebra; and (3) the diagram

$$\begin{tikzcd}
M \otimes M \arrow[r, "m"] & M \\
A \otimes M \otimes M \arrow[u, "M \otimes M^{\Phi}"] \arrow[r, "1 \otimes m"] & A \otimes M \arrow[u, "M^{\Phi}"']
\end{tikzcd}$$

is commutative.

Condition (3) states that the multiplication $m: {}_{\Delta}(M \otimes M) \to M$ is a morphism of A-modules. In view of Definition 2.5, it follows that if M is an A-module algebra, M and M are paired with respect to M by the A-pairing m.

In the sequel, the following special pairing is used. If M, N, and L are left modules over the Hopf algebra $Z(\pi)$ (group ring), then M and N are paired with respect to L iff there exists a $Z(\pi)$-pairing $\theta: M \otimes N \to L$ such that $\theta(ax \otimes ay) = a\theta(x \otimes y)$, for $a \in \pi$, $x \in M$, $y \in N$.

Definition 2.7: Let A be a Hopf algebra over R. Then N is said to be an A-module coalgebra iff (1) N is an A-module;
(2) N is a R-coalgebra; and (3) the diagram

$$\begin{tikzcd}
N \otimes N & N \arrow[l, "d"'] \\
A \otimes N \otimes N \arrow[u, "N \otimes N"] & A \times N \arrow[l, "1 \otimes d"'] \arrow[u, "N"']
\end{tikzcd}$$

is commutative.

Note that the dual of an A-module coalgebra is an A-comodule algebra.

$$\mathrm{Ext}_A^{**}(M,N)$$

The texts that give direct access to the theory of cohomology of groups, for example [5], [24], and [33], all use the "classical" projectivity. However, the work of S. Eilenberg and J. C. Moore [15] gives us the notion of a "new" projectivity. In this section both concepts are defined and compared. Then this "new" projectivity is used to define the extension functor Ext.

The definition of Eilenberg and Moore's "new" projectivity is given first.

Definition 2.8: An object $P \varepsilon {}_A\mathcal{M}$ is said to be $\mathcal{E}_0$-projective iff for any R-split exact sequence $E: M' \xrightarrow{f} M \xrightarrow{g} M''$ in ${}_A\mathcal{M}$ and for any morphism of A-modules $\alpha: P \to M$ with $g\alpha = 0$, there exists a morphism of A-modules $h: P \to M'$ such that the diagram

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "h"'] \arrow[d, "\alpha"] & \\
M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \\
 & K \arrow[u, "k"'] \arrow[ul, Leftrightarrow, "c", "\ell"'] &
\end{tikzcd}$$

is commutative; where k is the kernel of g and c is a R-homomorphism with the property $\ell c = 1_K$. It should be noted that in the diagram all maps except c are morphisms of A-modules.

Now the definition of "classical" projectivity is given.

Definition 2.9: An object $P \in {}_A\mathcal{M}$ is said to be $\mathcal{E}_1$-projective iff for any exact sequence $E: M' \xrightarrow{f} M \xrightarrow{g} M''$ in ${}_A\mathcal{M}$ and for any morphism of A-modules $\alpha: P \to M$ with $g\alpha = 0$, there exists a morphism of A-modules $h: P \to M'$ such that $fh = \alpha$.

$\mathcal{P}_0$ is used to denote the class of $\mathcal{E}_0$-projective modules in ${}_A\mathcal{M}$, and the notation $\mathcal{P}_1$ is used to represent the class of $\mathcal{E}_1$-projective modules in ${}_A\mathcal{M}$.

The following simple example shows the fact that, in general, $\mathcal{P}_1 \subsetneqq \mathcal{P}_0$. Let A = R = Z, then it is easy to see that $\mathcal{P}_0$ is the class of all abelian groups, while $\mathcal{P}_1$ is the class of all free abelian groups.

Proposition 2.1: The A-module P is in the class $\mathcal{P}_0$ iff P is a retract of an extended A-module $A \otimes M$.

Hereafter, projective means $\mathcal{E}_0$-projective when no confusion can occur.

Definition 2.10: The left complex $\mathfrak{X}$ over M (in notation $\varepsilon: \mathfrak{X} \to M$) is said to be projective resolution of $M \in {}_A\mathcal{M}$ iff (1) for $i \geq 0$, $X_i \in \mathcal{P}_0$; (2) $X_{i-1} \xleftarrow{\partial_{i-1}} X_i \xleftarrow{\partial_i} X_{i+1}$ for $i \geq 0$, and $0 \leftarrow M \xleftarrow{\varepsilon} X_0$ are R-split exact.

Let $M \in {}_A\mathcal{M}$. The unit $e: R \to A$ gives coker $e = A/\mathrm{Im}e = A/R1_A$, which is denoted by A/R. For each $n \geq 0$ construct the extended A-module

$$B_n(A;M) = A \otimes \underbrace{A/R \otimes \cdots \otimes A/R}_{n\text{-factors}} \otimes M = A \otimes (A/R)^n \otimes M \quad .$$

As a R-module, $B_n(A;M)$ is spanned by elements $x = a \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes y$ which are, following the notation of Eilenberg and MacLane, customarily written as $x = a[a_1|\cdots|a_n]y$, or as $x = a(a_1,\cdots,a_n)y$. In particular, elements of $B_0(A;M)$ are written as $a[\ ]y$ or $a(\ )y$. Construct the left complex $\varepsilon:B(A:M) \to M$

$$M \xleftarrow{\varepsilon} B_0(A;M) \xleftarrow{\partial_0} \cdots \leftarrow B_{n-1}(A;M) \xleftarrow{\partial_{n-1}} B_n(A;M) \leftarrow \cdots \ :B(A;M)$$

where ε and ∂_{n-1} ($n \geq 1$) are defined by

$$\varepsilon(a[\ ]y) = ay$$

$$\partial_{n-1}(a[a_1|\cdots|a_n]y) = aa_1[a_2|\cdots|a_n]y + \sum_{i=1}^{n-1} (-1)^i a[a_1|\cdots|a_ia_{i+1}|\cdots|a_n]y$$
$$+ (-1)^n a[a_1|\cdots|a_{n-1}]y \quad .$$

This complex is called the <u>normalized bar resolution</u> of the A-module M.

<u>Theorem 2.1</u>: For each A-module M, $\varepsilon:B(A;M) \to M$ is a projective resolution of M.

Now apply the normalized bar resolution to ${}_{\varepsilon}R \in {}_{A}\mathfrak{M}$ when A is augmented by $\varepsilon:A \to R$. Observing that $B_n(A;{}_{\varepsilon}R) \cong A \otimes (A/R)^n$, it is noted that $B_n(A;{}_{\varepsilon}R)$ is spanned by elements $a[a_1|\cdots|a_n]$, while ∂ is given by the previous formula with the "outside" factor y omitted. In particular, when $A = Z(\pi)$ (group ring), $B_n(Z(\pi);{}_{\varepsilon}Z)$ is the free $Z(\pi)$-module with generators $[x_1|\cdots|x_n]$ all n-tuples of elements $x_1 \neq 1, \cdots, x_n \neq 1$ of π, setting $[x_1|\cdots|x_n] = 0$ if any one $x_i = 1$. Also, the notation $B_n(Z(\pi);{}_{\varepsilon}Z) = B_n(\pi)$ and $B(Z(\pi),{}_{\varepsilon}Z) = B(\pi)$ is adopted.

If $M \in {}_A\mathfrak{M}$, a variant of the normalized bar resolution $B(A;M)$, the non-normalized bar resolution $\beta(A;M)$ has

$$\beta_n(A;M) = A \otimes \underbrace{A \otimes \cdots \otimes A}_{\text{n-factors}} \otimes M = A \otimes A^n \otimes M \quad .$$

The boundary and ε are given by the formulas for $B(A;M)$ with each $a[a_1|\cdots|a_n]y$ replaced by $a \otimes a_1 \otimes \cdots \otimes a_n \otimes y$. The projection $\eta:\beta(A;M) \to B(A;M)$ is a chain equivalence of complexes of A-modules. In particular, when $A = Z(\pi)$, $\beta_n(Z(\pi);{}_\varepsilon Z) = \beta_n(\pi)$ is the free $Z(\pi)$-module generated by all n-tuples $x_1 \otimes \cdots \otimes x_n$ of elements of π (no normalized condition).

In Chapter III a chain complex of Z-modules called the reduced non-normalized bar construction $\overline{\beta}(\pi)$ is used for a group π. Although $\overline{\beta}(\pi)$ is not a resolution, it can be used to compute cohomology of groups for some important special cases. For $n > 0$ let $\overline{\beta}(\pi) = \underbrace{Z(\pi) \otimes \cdots \otimes Z(\pi)}_{\text{n-factors}}$ be the free abelian group generated by all n-tuples $x_1 \otimes \cdots \otimes x_n$ of elements of π. Set $\overline{\beta}_0(\pi) = Z$ and define $\partial_{n-1}:\overline{\beta}_n(\pi) \to \overline{\beta}_{n-1}(\pi)$ by

$$\partial_{n-1}(x_1 \otimes \cdots \otimes x_n) = x_2 \otimes \cdots \otimes x_n$$
$$\sum_{i=1}^{n-1} (-1)^i x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n + (-1)^n x_1 \otimes \cdots \otimes x_{n-1} \quad .$$

In order to introduce the cup-product in derived functors [1], [15], [28] the presentation of a comparison theorem in relative homological algebra in the following form is now given.

Theorem 2.2: Let $\varepsilon:\mathfrak{X} \to M$ be a projective resolution of M in the category ${}_A\mathfrak{M}$ and let $\eta:\mathfrak{Y} \to N$ be a projective resolution of N in the category ${}_B\mathfrak{M}$. Then, for any α-morphism of modules $f: M \to N$ there exists an α-chain map extension $F: \mathfrak{X} \to \mathfrak{Y}$ of f in the sense that (1) for each $n \geq 0$, $F_n: X_n \longrightarrow Y_n$ is an α-morphism of modules, and (2) $d_{n-1}F_n = F_{n-1}\partial_{n-1}$ for $n \geq 1$ and $f\varepsilon = \eta F_0$, where

$$\begin{tikzcd}
M \arrow[d, "f"'] & X_0 \arrow[l, "\varepsilon"'] \arrow[d, "F_0"'] & X_1 \arrow[l, "\partial_0"'] \arrow[d, "F_1"'] & \cdots \arrow[l] & X_n \arrow[l, "\partial_{n-1}"'] \arrow[d, "F_n"'] & \cdots \arrow[l] & :\mathfrak{X} \arrow[d, "F"] \\
N & Y_0 \arrow[l, "\eta"] & Y_1 \arrow[l, "d_0"] & \cdots \arrow[l] & Y_n \arrow[l, "d_{n-1}"] & \cdots \arrow[l] & :\mathfrak{Y}
\end{tikzcd}$$

and (3) if F, F' are α-chain map extensions of f, then there exists an α-chain homotopy $h: \mathfrak{X} \to \mathfrak{Y}$ connecting F with F'.

Proof: First observe that the theorem is the usual comparison theorem in the case when $A = B$ and α is the identity map. The following remarks enable us to reduce this theorem to the classical case; 1) the B-modules N and Y_i $(i \geq 0)$ can be considered as A-modules by pull-back along α, 2) the B-module morphisms η and d_i $(i \geq 0)$ can be regarded as morphisms in ${}_A\mathfrak{M}$ by considering ${}_\alpha N$, ${}_\alpha Y_i \in {}_A\mathfrak{M}$, 3) by definition 2.4, $f: M \to {}_\alpha N$ is a morphism in ${}_A\mathfrak{M}$. From 1) and 2), $\eta: \mathfrak{Y} \to N$ can be considered as an acyclic complex in ${}_A\mathfrak{M}$. It follows from the usual comparison theorem that there exists a chain map extension F of f in ${}_A\mathfrak{M}$. By definition 2.4, F is an α-chain map. It is immediate to see the rest of the proof. This proves the theorem. ▲

Let M and N denote arbitrarily given left A-modules. Select any projective resolution $\varepsilon: \mathfrak{X} \to M$ of M

$$M \xleftarrow{\varepsilon} X_0 \xleftarrow{\partial_0} X_1 \leftarrow \cdots \xleftarrow{\partial_{n-1}} X_n \xleftarrow{\partial_n} \cdots \quad :\mathfrak{X}$$

where ε and ∂_n $(n \geq 0)$ are morphisms of graded A-modules. Consider

$\mathrm{Hom}_A(\mathfrak{X},N)$ which is the graded R-cochain complex

$$\mathrm{Hom}_A(\mathfrak{X},N):\ \mathrm{Hom}_A(X_0,N) \xrightarrow{\delta^0} \mathrm{Hom}_A(X_1,N) \xrightarrow{\delta^1} \cdots \to \mathrm{Hom}_A(X_n,N) \xrightarrow{\delta^n} \cdots$$

with a graded R-module $\mathrm{Hom}_A(X_n,N) = \{\mathrm{Hom}_A^p(X_n,N) \mid p = 0, \pm 1, \pm 2, \cdots\}$ and a morphism of graded R-modules $\delta^n = \mathrm{Hom}_A(\partial_n,N)$. For every integer $n \geq 0$, the n-dimensional cohomology module $H^n(\mathrm{Hom}_A^p(\mathfrak{X},N))$ of a R-cochain complex $\mathrm{Hom}_A^p(\mathfrak{X},N)$ for each grading p will be referred to as the <u>n-dimensional extension functor</u> over A of the given A-module M with coefficients in the A-module N and will be denoted by the symbol $\mathrm{Ext}_A^{n,p}(M,N)$, where n refers to the homological dimension and p denotes the grading. Thus for the category of left A-modules ${}_A\mathfrak{M}$ and for the category of graded R-modules ${}_R\mathfrak{M}$ we define the contravariant functor

$$\mathrm{Ext}_A^{**}(\ ,N):{}_A\mathfrak{M} \to {}_R\mathfrak{M}$$

by $\mathrm{Ext}_A^{**}(M,N) = \{\mathrm{Ext}_A^{n,p}(M,N)\}$.

The cohomology $H^*(\pi,N)$ of a group π with coefficients in a $Z(\pi)$-module N provides an important example of the functor $\mathrm{Ext}_A^*(\ ,N)$ with A the group ring $Z(\pi)$. These cohomology groups may be defined directly in terms of the extension functor by

$$H^*(\pi,N) = \mathrm{Ext}^*_{Z(\pi)}({}_\varepsilon Z,N) \quad ,$$

considering ${}_\varepsilon Z$ as a $Z(\pi)$-module by the augmentation $\varepsilon:Z(\pi) \to Z$. Thus, the <u>n-th cohomology of a group π</u> with coefficients in the $Z(\pi)$-module N is defined by $H^n(\pi,N) = \mathrm{Ext}^n_{Z(\pi)}({}_\varepsilon Z,N)$. It should be noted that since all modules involved in this case are trivially graded, the second asterisk on the shoulder of Ext is dropped.

When the definition of the reduced non-normalized bar construction

$\overline{\beta}(\pi)$ was given, it was mentioned that although $\overline{\beta}(\pi)$ is not a resolution it can be used to compute cohomology of groups for some important special cases. This is now made precise.

Proposition 2.2: If N is a trivial $Z(\pi)$-module, that is, $gx = x$ for all $g \in \pi$, $x \in N$, then

$$H^n(\pi,N) = H^n(\mathrm{Hom}_Z(\overline{\beta}(\pi),N)) \quad ,$$

where $\overline{\beta}(\pi)$ is the reduced non-normalized bar construction.

Cup Products for A-Pairings

The main purpose of this section is to establish an algebraic analogy to the first section in Chapter I. For a detailed account of this section, refer to [30].

Let A be a Hopf algebra over R with coproduct $\Delta: A \to A \otimes A$, and let M be a left A-module coalgebra with coproduct $d: M \to M \otimes M$. If $\varepsilon: \mathfrak{X} \to M$ is a projective resolution of M, then $\varepsilon \otimes \varepsilon: \mathfrak{X} \otimes \mathfrak{X} \to M \otimes M$ is a projective resolution of $M \otimes M$. Since d is a Δ-morphism of modules, by the comparison theorem 2.2 there exists a Δ-chain map extension $h: \mathfrak{X} \to \mathfrak{X} \otimes \mathfrak{X}$ of d which preserves both the grading and the homological dimension. If P, Q, and S are A-modules such that P and Q are paired with respect to S by the A-pairing θ, then there exist morphisms of R-modules

$$\mathrm{Hom}_A(\mathfrak{X},P) \otimes \mathrm{Hom}_A(\mathfrak{X},Q) \xrightarrow{\Lambda} \mathrm{Hom}_{A\otimes A}(\mathfrak{X} \otimes \mathfrak{X}, P \otimes Q) \xrightarrow{\chi(h,\theta)} \mathrm{Hom}_A(\mathfrak{X},S)$$

where Λ is defined by

$$\Lambda(f \otimes g)(x \otimes y) = f(x) \otimes g(y) \quad ,$$

for

$$f \in \mathrm{Hom}_A(\mathfrak{X},P),\ g \in \mathrm{Hom}_A(\mathfrak{X},Q),\ x \otimes y \in \mathfrak{X} \otimes \mathfrak{X} \qquad ;$$

and $\chi(h,\theta)$ is defined by

$$\chi(h,\theta)(\rho) = \theta\rho h, \text{ for } \rho \in \mathrm{Hom}_{A\times A}(\mathfrak{X} \otimes \mathfrak{X}, P \otimes Q) \qquad .$$

For $a \in A$ and $x \in \mathfrak{X}$,

$$\begin{aligned}
(\chi(h,\theta)(\rho))(ax) &= (\theta\rho h)(ax) \\
&= \theta(\Delta(a)\rho h(x)) \\
&= \theta((({}_P\Phi \otimes {}_Q\Phi)\circ(1 \otimes t \otimes 1)\circ(\Delta \otimes 1 \otimes 1))(a \otimes \rho h(x))) \\
&= {}_S\Phi((1 \otimes \theta)(a \otimes \rho h(x))) \\
&= {}_S\Phi(a \otimes \theta\rho h(x)) \\
&= a(\theta\rho h(x)) \qquad .
\end{aligned}$$

Thus, $\chi(h,\theta) \in \mathrm{Hom}_A(\mathfrak{X},S)$. The reader can see here that the definition 2.5 of an A-pairing is effectively used.

It is immediate to see the composition $\smile_\theta = \chi(h,\theta)\Lambda$ satisfies the coboundary formula

$$\delta(f \smile_\theta g) = \delta f \smile_\theta g + (-1)^{|f|} f \smile_\theta \delta g \qquad .$$

The composition $\smile_\theta$ induces a morphism of graded R-modules (also denoted by $\smile_\theta$, or just $\smile$ when no confusion can occur)

$$\smile_\theta : \mathrm{Ext}^{n,s}(M,P) \otimes \mathrm{Ext}_A^{m,t}(M,\theta) \longrightarrow \mathrm{Ext}^{n+m,s+t}(M,S)$$

called the <u>cup product</u> with respect to the A-pairing θ. Note that cup product depends on the pairing θ but not on the particular projective resolution of M.

If N is an A-module algebra, then N and N are paired with respect to N by the A-pairing $m :_\Delta (N \otimes N) \to N$. Thus, if M is an A-module coalgebra, the cup product

$$\smile : \mathrm{Ext}_A^{n,*}(M,N) \otimes \mathrm{Ext}_A^{m,*}(M,N) \to \mathrm{Ext}_A^{n+m,*}(M,N)$$

gives $\mathrm{Ext}_A^{**}(M,N)$ the structure of a graded R-algebra.

In the special case when $M = {}_{\varepsilon}Z$, $A = Z(\pi)$, and N is a $Z(\pi)$-module algebra, then the graded Z-module $\mathrm{Ext}^*_{Z(\pi)}(Z,N) = H^*(\pi,N)$ becomes a graded ring when multiplication is defined in terms of the cup product; it is called the <u>cohomology ring</u> of π with coefficients in the $Z(\pi)$-module algebra N.

Characteristic Class and Lower Dimensional Cohomology

Cohomology groups of a group were formally defined in the 1940's. However, these groups in low dimensions had been studied earlier as part of the general body of group theory. For example, the 1-dimensional cohomology groups (crossed homomorphisms modulo principal homomorphisms) had been long known; the 2-dimensional cohomology groups, in the form of factor sets, had appeared as early as 1926. This section is devoted to the task of calculating the cohomology groups $H^n(\pi,N)$, for $n = 0,1$; and, to a discussion of the cohomology group $H^2(\pi,N)$ in relation to the characteristic class of a group extension.

For a group π and a $Z(\pi)$-module N (<u>π-module for short</u>), $H^*(\pi,N) = \mathrm{Ext}^*_{Z(\pi)}(Z,N)$. By using the non-normalized bar resolution $\beta(\pi)$ (see II.2), there is for each $n \geq 0$ a Z-isomorphism

$$\hom_{Z(\pi)}(\beta_n(\pi),N) \simeq \hom_Z(\tilde{\beta}_n(\pi),N),\ \tilde{\beta}_n(\pi)) = \underbrace{Z(\pi) \otimes \cdots \otimes Z(\pi)}_{n\text{-factors}}$$

defined by the maps

$$\hom_{Z(\pi)}(\beta_n(\pi),N) \underset{\lambda_n}{\overset{\mu_n}{\rightleftarrows}} \hom_Z(\tilde{\beta}_n(\pi),N)$$

with

$$\mu_n(g)(x) = g(1 \otimes x) \ , \text{ for } g \in \hom_{Z(\pi)}(\beta_n(\pi),N) \ , \ x \in \tilde{\beta}_n(\pi) \quad ,$$

and

$$\lambda_n(f)(a \otimes x) = af(x) \ , \text{ for } f \in \hom_Z(\tilde{\beta}_n(\pi),N) \ , \ a \otimes x \in \beta_n(\pi) \quad .$$

Hence, the cochain complex $\mathrm{Hom}_{Z(\pi)}(\beta(\pi),N)$ is isomorphic to the cochain complex

$$\mathrm{Hom}_Z(\tilde{\beta}(\pi),N)\colon \hom_Z(Z,N) \xrightarrow{d^0} \hom_Z(\tilde{\beta}_1(\pi),N) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \hom_Z(\tilde{\beta}_n(\pi),N) \xrightarrow{d^n} \cdots$$

with coboundary $d^{n-1} = \mu_n \delta^{n-1} \lambda_{n-1}$ having the explicit formulation

$$\begin{aligned} d^{n-1}(f)(g_1 \otimes \cdots \otimes g_n) &= (\mu_n \delta^{n-1} \lambda_{n-1})(g_1 \otimes \cdots \otimes g_n) \\ &= \mu_n(\lambda_{n-1} f \partial_{n-1})(g_1 \otimes \cdots \otimes g_n) \\ &= (\lambda_{n-1} f \partial_{n-1})(1 \otimes g_1 \otimes \cdots \otimes g_n) \\ &= g_1 f(g_2 \otimes \cdots \otimes g_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i f(g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n) \\ &\quad + (-1)^n f(g_1 \otimes \cdots \otimes g_{n-1}) \quad , \end{aligned}$$

for $f \in \hom_Z(\tilde{\beta}_{n-1}(\pi),N)$, $g_1 \otimes \cdots \otimes g_n$ a Z-base element in $\tilde{\beta}_n(\pi)$. The cohomology group $H^n(\pi,N)$ may thus be calculated by considering $H^n(\mathrm{Hom}_Z(\tilde{\beta}(\pi), N)) = \mathrm{Ker}\ d^n/\mathrm{Im}\ d^{n-1}$.

Since $N \simeq \hom_Z(Z,N)$, an element $x \in N$ is identified with the o-cochain f_x, where $f_x\colon Z \to N$ is defined by $f_x(1) = x$. Then for $g \in \pi$

$$(d^0 f_x)(g) = x - xg \quad .$$

Therefore, the kernel of d^0 is the subgroup

$$N^\pi = \{x \in N \mid g\,x = x \text{ for all } g \in \pi\}$$

of elements of N invariant under the action of π. If f is a 1-cochain, then

$$(d^1 f)(g_1 \otimes g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1) \quad .$$

The cochain f is therefore a cocycle iff

$$f(g_1 g_2) = g_1 f(g_2) + f(g_1) \quad .$$

The mappings of π into N subject to this condition are called crossed homomorphisms of π to N, and the group of crossed homomorphisms of π to N is denoted by $Z^1_c(\pi,N)$. The 1-cochain f belongs to the image of d^0 iff there exists an element $a \in N$ such that

$$f(g) = ga - a$$

for all $g \in \pi$. Such crossed homomorphisms are called principal, and the group of principal crossed homomorphisms is denoted by $B^1_c(\pi,N)$. These facts are summarized in the following proposition.

Proposition 2.3: For a group π and a π-module N, $H^0(\pi,N) = N^\pi$ and $H^1(\pi,N) = Z^1_c(\pi,N)/B^1_c(\pi,N)$. In particular, when π acts trivially on N, $H^0(\pi,N) = N$ and $H^1(\pi,N) = \hom_Z(\pi,N)$.

The relation between $H^2(\pi,N)$ and extensions of the abelian group N by the group π is now investigated.

Definition 2.11: A group extension is a short exact sequence $E: 0 \to N \xrightarrow{i} G \xrightarrow{p} \pi \to 1$ where N is an abelian group; it is convenient to write the group composition in 0 and N as addition, that in G, π, and 1 multiplication. G is called an extension of N by π. The extension E splits iff there is a homomorphism $\gamma:\pi \to G$ with $p\gamma = 1_\pi$.

Let Aut N denote the group of automorphisms of N, with group multiplication the composition of automorphisms. Conjugation in G yields a homomorphism $\theta: G \to \text{Aut } N$ under which the action of each $\theta(g)$ on any $x \in N$ is given by

$$\theta(g)(x) = i^{-1}(gi(x)g^{-1}) \ , \ g \in G, \ x \in N \qquad ,$$

or, one simply considers i as the inclusion map and writes

$$\theta(g)(x) = g\,x\,g^{-1}$$

when no confusion occurs. Since N is abelian, observe that $(\theta \circ i)(x) = 1_N : N \to N$ for all $x \in N$. Hence θ induces a homomorphism $\Phi:\pi \to \text{Aut } N$ defined by

$$\Phi(\sigma)(x) = \theta(g)(x) \ , \ x \in N, \ \sigma \in \pi \qquad ,$$

where $g \in G$ is such that $p(g) = \sigma$. It is easy to see that Φ is independent of the choice of a representative g for σ. Then Φ gives N the structure of a π-module; for $\sigma \in \pi$ and $x \in N$, define an action of π on N by $\sigma \cdot x = \Phi(\sigma)(x)$. This proves

Proposition 2.4: An extension of N by π furnishes N with the structure of a π-module.

Proposition 2.5: If N is an abelian group and π is a group, then there exists a one-to-one correspondence between the set $\{\Phi:\pi \to \text{Aut } N \mid \Phi$ is a group homomorphism$\}$ and the set of all possible π-module structures of N.

For each group homomorphism $\Phi:\pi \to \text{Aut } N$ construct a multiplicative, but not necessarily abelian, group $N \times_{\Phi} \pi$ called the <u>semi-direct product</u> of N and π relative to Φ. As a set, $N \times_{\Phi} \pi = \{(x,\sigma) \mid x \in N,\ \sigma \in \pi\}$. Multiplication is defined by

$$(x,\sigma)(x_1,\sigma_1) = (x + \Phi(\sigma)(x_1),\sigma\sigma_1)\ ,\ x,\ x_1 \in N,\ \sigma,\sigma_1 \in \pi\ .$$

One proves that this is a group with the "identity" element $1 = (0,1)$ and inverse $(x,\sigma)^{-1} = (-\Phi(\sigma^{-1})(x),\sigma^{-1})$.

<u>Proposition 2.6</u>: Any split group extension

$$0 \to N \xrightarrow{i} G \underset{\gamma}{\overset{p}{\rightleftarrows}} \pi \to 1$$

yields an isomorphism $G \simeq N \times_{\Phi} \pi$ for a $\Phi:\pi \to \text{Aut } N$.

<u>Proof</u>: First observe that Im i is normal in G, $1 = \text{Im } i \cap \text{Im } \gamma$, and every element in G is a product xy for $x \in N$ and $y \in \pi$. Next, for $y \in \text{Im } \gamma$ the inner automorphism τ_y of G ($\tau_y:G \to G$ is defined by $\tau_y(g) = ygy^{-1}$) restricted to Im i is an inner automorphism of Im i. Thus $\Phi:\text{Im } \gamma \to \text{Aut Im } i$ defined by $\Phi(y) = \tau_y|_{\text{Im } i}$ is a group homomorphism, and $\eta:\text{Im } i \times_{\Phi} \text{Im } \gamma \to G$ defined by $\eta(x,y) = xy$ is an isomorphism. Then identifying Im i with N and Im γ with π, the proof is complete. ▲

<u>Definition 2.12</u>: If N is a π-module, an extension $E:0 \to N \xrightarrow{i} G \xrightarrow{p} \pi \to 1$ is said to be <u>compatible</u> with the π-module structure of N iff the π-module structure obtained from the extension E coincides

with the given π-module structure.

Given π and a π-module N, there is at least one extension, the semi-direct product, compatible with the π-module structure of N. If $\Phi:\pi \to \mathrm{Aut}\, N$ is the homomorphism corresponding to the π-module structure of N, then there is a short exact sequence

$$E: 0 \to N \xrightarrow{i} N \times_{\Phi} \pi \xrightarrow{p} \pi \to 1$$

where i is the monomorphism given by $i(x) = (x,1)$ and p is $p(x,\sigma) = \sigma$. Recalling that the π-module structure of N furnished by E is defined by

$$\sigma\cdot x = i^{-1}(gi(x)g^{-1}) \quad ,$$

where $p(g) = \sigma$, setting $g = (0,\sigma)$, then

$$\begin{aligned} \sigma\cdot x &= i^{-1}((0,\sigma)(x,1)(0,\sigma)^{-1}) \\ &= i^{-1}((\sigma x,\sigma)(0,\sigma^{-1})) \\ &= i^{-1}(\sigma x,1) \\ &= \sigma x \quad . \end{aligned}$$

Whence

Proposition 2.7: If N is a π-module with structure corresponding to $\Phi:\pi \to \mathrm{Aut}\, N$, then the extension

$$0 \to N \xrightarrow{i} N \times_{\Phi} \pi \xrightarrow{p} \pi \to 1$$

is compatible with the given π-module structure of N.

If N is a π-module, $E: 0 \to N \to G \to \pi \to 1$ and $E': 0 \to N \to G' \to \pi \to 1$ are two compatible extensions of N by π, then E is said to be related to E', in notation $E \sim E'$, iff there exists an isomorphism $\gamma: G \to G'$

such that the diagram

$$\begin{tikzcd}
 & & G \arrow[dd, dashed, "\gamma"] \arrow[dr] & & \\
0 \arrow[r] & N \arrow[ur] \arrow[dr] & & \pi \arrow[r] & 1 \\
 & & G \arrow[ur] & &
\end{tikzcd}$$

is commutative. This relation is an equivalence relation. Let $\mathcal{E}(\pi,N)$ denote the equivalence classes of compatible extensions of N by π modulo this relation.

Suppose N is a π-module and $E: 0 \to N \xrightarrow{i} G \xrightarrow{p} \pi \to 1$ is a compatible extension. Construct a function $s:\pi \to G$ such that $s(1) = 0$ and $ps = 1_\pi$ (such a function is called a <u>section</u> of π in G). Then $(s(\sigma_1)\cdot s(\sigma_2)\cdot s(\sigma_1\sigma_2)^{-1}) \varepsilon$ Im i for $\sigma_1, \sigma_2 \in \pi$. Let $h_{\sigma_1,\sigma_2} \in N$ be that unique element such that $i(h_{\sigma_1,\sigma_2}) = s(\sigma_1)\cdot s(\sigma_2)\cdot s(\sigma_1\sigma_2)^{-1}$. Corresponding to the fixed section s, there is a map

$$f_s:\pi \otimes \pi \to N$$

defined by

$$f_s(\sigma_1,\sigma_2) = h_{\sigma_1,\sigma_2} \quad .$$

Then f_s determines a 2-cochain (also denoted by f_s)

$$f_s:B_2(\pi) \to N$$

where $B(\pi)$ is the normalized bar resolution (see II.2), defined by

$$f_s([\sigma_1|\sigma_2]) = f_s(\sigma_1,\sigma_2) \quad .$$

Moreover,

$$\begin{aligned}(s(\sigma_1)\cdot s(\sigma_2))\cdot s(\sigma_3) &= (i(h_{\sigma_1,\sigma_2})\cdot s(\sigma_1\sigma_2))\cdot s(\sigma_3)\\ &= i(h_{\sigma_1,\sigma_2})\cdot(s(\sigma_1\sigma_2)\cdot s(\sigma_3))\\ &= i(h_{\sigma_1,\sigma_2})\cdot(i(h_{\sigma_1\sigma_2,\sigma_3})\cdot s(\sigma_1\sigma_2\sigma_3)) \quad .\end{aligned}$$

Similarly,

$$\begin{aligned}s(\sigma_1)\cdot(s(\sigma_2)\cdot s(\sigma_3)) &= s(\sigma_1)\cdot(i(h_{\sigma_2,\sigma_3})\cdot s(\sigma_2\sigma_3))\\ &= (s(\sigma_1)\cdot i(h_{\sigma_2,\sigma_3})\cdot s(\sigma_1)^{-1})\cdot(s(\sigma_1)\cdot s(\sigma_2\sigma_3))\\ &= i(\sigma_1\cdot h_{\sigma_2,\sigma_3})\cdot(i(h_{\sigma_1,\sigma_2\sigma_3})\cdot s(\sigma_1\sigma_2\sigma_3)) \quad .\end{aligned}$$

Whence by associativity

$$i(h_{\sigma_1,\sigma_2})\cdot i(h_{\sigma_1\sigma_2,\sigma_3}) = i(\sigma_1\cdot h_{\sigma_2,\sigma_3})\cdot i(h_{\sigma_1,\sigma_2\sigma_3})$$

and thus

$$f_s(\sigma_1,\sigma_2) + f_s(\sigma_1\sigma_2,\sigma_3) = \sigma_1\, f_s(\sigma_2,\sigma_3) + f_s(\sigma_1,\sigma_2\sigma_3) \quad .$$

Consequently, f_s is a 2-cocycle and $[f_s] \in H^2(\pi,N)$.

If s' is another section of π in G, then corresponding to each $\sigma \in \pi$ there exists a unique element (denote it) $x_\sigma \in N$ such that $s(\sigma) = i(x_\sigma)\cdot s'(\sigma)$. Define $g:B_1(\pi) \to N$ by $g([\sigma]) = x_\sigma$. Then for $\sigma_1,\ \sigma_2 \in \pi$,

$$\begin{aligned}s(\sigma_1)\cdot s(\sigma_2) &= i(x_{\sigma_1})\cdot s'(\sigma_1)\cdot i(x_{\sigma_2})\cdot s'(\sigma_2)\\ &= i(x_{\sigma_1})\cdot(s'(\sigma_1)\cdot i(x_{\sigma_2})\cdot s'(\sigma_1)^{-1})\cdot s'(\sigma_1)\cdot s'(\sigma_2)\\ &= i(x_{\sigma_1})\cdot i(\sigma_1\cdot x_{\sigma_2})\cdot s'(\sigma_1)\cdot s'(\sigma_2)\\ &= i(x_{\sigma_1})\cdot i(\sigma_1\cdot x_{\sigma_2})\cdot i(h'_{\sigma_1,\sigma_2})\cdot s'(\sigma_1\sigma_2) \quad ,\end{aligned}$$

while

$$i(h_{\sigma_1,\sigma_2})\cdot s(\sigma_1\sigma_2) = i(h_{\sigma_1,\sigma_2})\cdot i(x_{\sigma_1\sigma_2})\cdot s'(\sigma_1\sigma_2) \quad .$$

Since $s(\sigma_1)\cdot s(\sigma_2) = i(h_{\sigma_1,\sigma_2})\cdot s(\sigma_1\sigma_2)$,

$$\sigma_1 g([\sigma_2]) - g([\sigma_1\sigma_2]) + g([\sigma_1]) = f_s([\sigma_1,\sigma_2]) - f_{s'}([\sigma_1,\sigma_2])$$

so that $\delta' g = f_s - f_{s'}$ and hence $[f_s] = [f_{s'}]$. Setting $[f_s] = [f_E]$, there is a well-defined map

$$\Phi\colon (\pi,N) \to H^2(\pi,N)$$

with $\Phi([E]) = [f_E]$. The cohomology class $[f_E]$ is called the <u>characteristic class</u> of the group extension E.

The map Φ is injective, for suppose $E: 0 \to N \xrightarrow{i} G \xrightarrow{p} \pi \to 1$ and $E': 0 \to N \xrightarrow{i'} G' \xrightarrow{p'} \pi \to 1$ are two extensions of N by π such that $\Phi([E']) = \Phi([E])$. Let s be a section of π in G and let s' be a section of π in G'. Then there exists a 1-cochain $g: B_1(\pi) \to N$ such that

$$f_{s'} - f_s = \delta' g \quad .$$

Define $s''\colon \pi \to G$ by

$$s''(\sigma) = i(g([\sigma]))\cdot s(\sigma) \ , \qquad \sigma \in \pi \quad .$$

Then s" is a section of π in G satisfying the property that

$$f_{s''}(\sigma_1,\sigma_2) = f_{s'}(\sigma_1,\sigma_2) \ , \qquad \sigma_1,\sigma_2 \in \pi \quad .$$

If $g \in G$, then g can be uniquely written as the product $i(x)\cdot s''p(g)$ for $x \in N$. Define $\alpha: G \to G'$ by

$$\alpha(g) = i'(x)\cdot s'p(g) \quad .$$

Then α is a homomorphism and the diagram

$$\begin{tikzcd} & & G \arrow[dd, "\alpha"] \arrow[dr, shift left, "p"] & & \\ 0 \arrow[r] & N \arrow[ur, "i"] \arrow[dr, "i'"'] & & \pi \arrow[r] \arrow[ul, shift left, "s"] \arrow[ul, bend right, "s''"'] \arrow[dl, shift left, "s'"] & 1 \\ & & G' \arrow[ur, shift left, "p'"] & & \end{tikzcd}$$

is commutative; and hence by the 5-Lemma, α is an isomorphism and $E \sim E'$.

Now Φ is shown to be surjective. Suppose $[f] \in H^2(\pi,N)$. Construct the group $N {}_f\!\times \pi$. Its elements are all pairs (x,σ) with the product

$$(x_1,\sigma_1)\cdot(x_2,\sigma_2) = (x_1 + \sigma_1 x_2 + f(\sigma_1,\sigma_2),\ \sigma_1\sigma_2) \quad .$$

One proves that this is a group with the "identity" element $1 = (0,1)$ and inverse $(x_1,\sigma_1)^{-1} = (-\sigma_1^{-1}x_1 - \sigma_1^{-1}f(\sigma_1,\sigma_2),\ \sigma_1^{-1})$ and there is a short exact sequence

$$E''0 \to N \xrightarrow{i} N {}_f\!\times \pi \xrightarrow{p} \pi \to 1$$

where i is the homomorphism given by $i(x) = (x,1)$, p is given by $p(x,\sigma) = \sigma$. The section s of π in $N {}_f\!\times \pi$ defined by

$$s(\sigma) = (0,\sigma)$$

is such that $\Phi([f_s]) = [f]$. Summarizing, the following theorem has been proven.

<u>Theorem 2.3</u>: Given a π-module N then $\Phi: \mathcal{E}(\pi,N) \to H^2(\pi,N)$ is a one-to-one correspondence between the equivalence classes of compatible extensions of N by π and the elements of $H^2(\pi,N)$.

To illustrate Theorem 2.3, $\mathcal{E}(\pi,N)$ is computed for some elementary groups π and N.

(1) Take $\pi = Z_2$ with generator t and $N = Z$. There is only one

Z_2-module structure for Z, Proposition 2.5, and it is obtained by letting Z_2 act trivially on Z. If f is the 2-cochain that maps $[t|t]$ to 1 and $[t^i|t^j]$ to 0, for i, j $\neq$ 1, then the odd multiples of f are the 2-cocycles that are not coboundaries. Hence $H^2(Z_2,Z)$ is a cyclic group of order two and $\mathcal{E}(Z_2,Z) = \{[E],[E']\}$ where

$$E:0 \to Z \xrightarrow{i} Z \oplus Z_2 \to Z_2 \to 1$$

$$E':0 \to Z \xrightarrow{2} Z \to Z_2 \to 1$$

with $i(n) = (n,0)$ and 2 maps Z onto 2Z in Z.

(2) Take $\pi = Z_2$ with generator t and $N = Z_3$ with generator β. There are two possible Z_2-module structures for Z_3 and they are obtained by $t\beta = \beta^q$ where q = 1 or 2. In either case, the set of 2-cochains is a cyclic group of order three generated by the cochain f_β that maps $[t|t]$ to β.

(2a) If Z_2 acts on Z_3 by $t\beta = \beta$, then every 2-cochain is a cocycle and a coboundary. Hence $H^2(Z_2,Z_3) = 0$ and $\mathcal{E}(Z_2,Z_3) = \{[E]\}$ where

$$E:0 \to Z_3 \to Z_3 \oplus Z_2 \simeq Z_6 \to Z_2 \to 1 \qquad .$$

(2b) If Z_2 acts on Z_3 by $t\beta = \beta^2$, then f_β and $2f_\beta = f_{\beta^2}$ are coboundaries. Thus $H^2(Z_2,Z_3) = 0$ and $\mathcal{E}(Z_2,Z_3)$ consists of the single equivalence class represented by the extension

$$E:0 \to Z_3 \xrightarrow{i} S_3 \to Z_2 \to 1 \qquad ,$$

where S_3 denotes the symmetric group of degree 3 and $i(\beta) = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ for the generator β of Z_3.

(3) Take $\pi = Z_3$ with generator t and N = Z. Again, by Proposition 2.5 the only Z_3-module structure for Z is obtained by letting Z_3 act

trivially on Z. If f is the 2-cochain defined by

$$f([t^i|t^j]) = \begin{cases} 1 \ ; \ i + j \geq 3 \\ \\ 0 \ ; \ \text{otherwise} \end{cases}$$

then f, and 2f are the 2-cocycles that are not coboundaries, while 3nf for any integer n is a coboundary. Thus $H^2(Z_3,Z)$ is a cyclic group of order three and $\mathcal{E}(Z_3,Z) = \{[E_1],[E_2],[E_3]\}$ where

$$E_1 : 0 \to Z \to Z \oplus Z_3 \to Z_3 \to 1 \qquad ;$$

$$E_2 : 0 \to Z \xrightarrow{3} Z \to Z_3 \to 1 \qquad ,$$

$$E_3 : 0 \to Z \xrightarrow{-3} Z \to Z_3 \to 1 \qquad .$$

CHAPTER III

$$\mathrm{Ext}^*_{Z(\pi)}(Z,N) \simeq H^*(K(\pi,1),N)$$

It was proved by Hurewicz [20] that if X is an arcwise-connected aspherical space then the fundamental group $\pi_1(X)$ determines all the homology and cohomology groups of X. Eilenberg and MacLane [10] showed this determination in a purely algebraic fashion and further proved $H^*(K(\pi,1),N) \simeq H^*(\pi,N)$ for an abelian group N as coefficient group. The purpose of this chapter is to make a quick review of this important link between algebraic topology and the cohomology theory of groups.

Let X be an arcwise-connected aspherical space whose fundamental group $\pi_1(X)$ is isomorphic to a given abstract group π. We call such a space <u>$K(\pi,1)$</u>. Let $x_0 \in X$ be a fixed point of X which is chosen as the base point for the fundamental group $\pi_1(X)$. Denote by $\tilde{S}(X)$ the subcomplex of the total singular complex $S(X;Z)$ obtained by considering only those singular simplices whose vertices are mapped into x_0. It was shown in [8] that there exist chain maps ρ, i

$$S(X) \underset{i}{\overset{\rho}{\rightleftarrows}} \tilde{S}(X)$$

with i the injection map, such that $\rho i = 1_{\tilde{S}(X)}$ and $i\rho$ is chain homotopic to the identity map $1_{S(X)}$; $i\rho \sim 1_{S(X)}$. Thus $S(X)$ and $\tilde{S}(X)$ are chain equivalent and the cohomology groups of the complexes $S(X)$ and $\tilde{S}(X)$ are isomorphic.

It is clear the complex $\tilde{S}(X)$ is more closely connected with the fundamental group $\pi_1(X)$ than is the larger complex $S(X)$. In fact, any 1-face of every singular simplex in $\tilde{S}(X)$ determines a unique element of $\pi_1(X)$.

Now define chain maps κ, η

$$S(X) \underset{\eta}{\overset{\kappa}{\rightleftarrows}} \overline{\beta}(\pi)$$

where $\overline{\beta}(\pi)$ is the reduced non-normalized bar construction for π (see II.2).

Let $\Delta^n = [e_0,\cdots,e_n]$ be a standard n-simplex whose vertices are $e_0,\cdots,e_n$, and let $T:\Delta^n \to X$ be a singular n-simplex in $\tilde{S}(X)$. Since every vertex e_i of Δ^n is mapped into the base point $x_0 \in X$, each edge e_ie_j $(i \leq j)$ of Δ^n maps into a closed path in X and therefore determines uniquely an element $\alpha_{i,j}$ of π. If $i = j$, $\alpha_{i,j} = 1$ can be defined. Then

$$\kappa_n:S_n(X) \to \overline{\beta}_n(\pi)$$

is defined by $\kappa_n(T) = \alpha_{0,1} \otimes \alpha_{1,2} \otimes \cdots \otimes \alpha_{n-1,n}$. Now

$$\begin{aligned}(\partial_{n-1}\kappa_n)(T) &= \partial_{n-1}(\alpha_{0,1} \otimes \alpha_{1,2} \otimes \cdots \otimes \alpha_{n-1,n}) \\ &= \alpha_{1,2} \otimes \alpha_{2,3} \otimes \cdots \otimes \alpha_{n-1,n} \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \alpha_{0,1} \otimes \cdots \otimes \alpha_{i-1,i}\alpha_{i,i+1} \otimes \cdots \otimes \alpha_{n-1,n} \\ &\quad + (-1)^n \alpha_{0,1} \otimes \alpha_{1,2} \otimes \cdots \otimes \alpha_{n-2,n-1} \quad ,\end{aligned}$$

while

$$
\begin{aligned}
(\kappa_{n-1}\partial_{n-1})(T) &= \sum_{0<i<n} (-1)^i \kappa_{n-1}(T^{(i)}) \\
&= \kappa_{n-1}(T^{(0)}) + \sum_{i=1}^{n-1} (-1)^i \kappa_{n-1}(T^{(i)}) + (-1)^n \kappa_{n-1}(T^{(n)}) \\
&= (\partial_{n-1}\kappa_n)(T) \qquad ,
\end{aligned}
$$

where $T^{(i)}$ denotes the i-th face of T. Hence, $\partial\kappa = \kappa\partial$ and κ is therefore a chain map.

Define $\eta_0:\overline{\beta}_0(\pi) \to \tilde{S}_0(X)$ by $\eta_0(1) = T$, where $T:\Delta^0 \to X$ is given by $T(e_0) = x_0$. Let α be a base element of $\overline{\beta}_1(\pi)$. Let $T:\Delta^1 \to X$ be a continuous function mapping Δ^1 into a closed path about x_0 belonging to the element α of the fundamental group. Define $\eta_1(\alpha) = T$. Next, let $(\Delta^2)^{(i)}$ denote the 1-face of Δ^2 opposite the i-th vertex e_i $(i=0,1,2)$. In notation, set $\alpha^{(0)} = \alpha_2$, $\alpha^{(1)} = \alpha_1\alpha_2$, and $\alpha^{(2)} = \alpha_1$. Since η has already been defined for elements of $\overline{\beta}_1(\pi)$, there are three mappings

$$
\eta_1(\alpha^{(i)}):(\Delta^2)^{(i)} \to X, \quad i = 0,1,2
$$

which give closed paths about x_0 belonging to the elements α_2, $\alpha_1\alpha_2$, α_1 of π, respectively. Jointly these three mappings give a mapping $T:\partial\Delta^2 \to X$ of the boundary $\partial\Delta^2$ of Δ^2. This map is null homotopic because $\alpha_1\alpha_2(\alpha_1\alpha_2)^{-1} = 1$. Consequently, T can be extended to a mapping $T:\Delta^2 \to X$. Define $\eta_2(\alpha) = T$.

From now on, the procedure is by induction. Suppose that η_k has been defined for $k < p$ $(k > 2)$. Let $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_p$ be a base element of $\overline{\beta}_p(\pi)$ and notationally let

$$
\begin{aligned}
\alpha^{(0)} &= \alpha_2 \otimes \cdots \otimes \alpha_p \qquad , \\
\alpha^{(i)} &= \alpha_1 \otimes \cdots \otimes \alpha_i\alpha_{i+1} \otimes \cdots \otimes \alpha_p, \qquad i \neq 0,p \\
\alpha^{(p)} &= \alpha_1 \otimes \cdots \otimes \alpha_{p-1} \qquad .
\end{aligned}
$$

If $(\Delta^p)^{(i)}$ is the i-th face of Δ^p, there are p+1 mappings

$$\eta_{p-1}(\alpha^{(i)}):(\Delta^p)^{(i)} \to X \qquad ,$$

for i = 0,1, ,p. By virtue of the inductive construction of η, these mappings agree on the common faces of any two (p-1)-faces of Δ^p. Consequently, they combine and give a map $T:\partial\Delta^p \to X$. Since $\partial\Delta^p$ is homeomorphic to a (p-1)-sphere and $\pi_{p-1}(X) = 0$ because $p-1 > 1$, the map T can be extended to a map$T:\Delta^p \to X$. Define $\eta_p(\alpha) = T$, and the definition of η is complete. If one observes that $\eta(\alpha) = T$ implies $\eta(\alpha^{(i)}) = T^{(i)}$, it follows that η and ∂ commute so that η is a chain map.

The chain maps κ and η are such that $\kappa\eta = 1_{\overline{\beta}(\pi)}$ and $\eta\kappa \sim 1_{\tilde{S}(X)}$. Hence

<u>Proposition 3.1</u>: Let X be a $K(\pi,1)$ space. Then the complexes S(X), $\tilde{S}(X)$, and $\overline{\beta}(\pi)$ are all chain equivalent.

As has previously been observed (see II.2), passing to cohomology, this chapter is concluded by

<u>Theorem 3.1</u>: $H^*(\pi,N) \simeq H^*(K(\pi,1),N)$ if π acts trivially on N.

CHAPTER IV

COMPUTATION OF GROUPS

In this chapter the tools developed in Chapters I, II, and III are applied to the computation of the cohomology ring $H^*(\pi, N)$ for various groups π.

$RP^\infty = K(Z_2, 1)$

In this section the computation of $H^*(Z_2, Z_2)$ will be carried out by constructing a space RP^∞ whose fundamental group $\pi_1(RP^\infty)$ is isomorphic with Z_2 and whose higher homotopy groups vanish, and then applying to this space Theorem 3.1.

The <u>n-dimensional real projective space</u> RP^n $(n \geq 0)$ is defined to be $(R^{n+1} - \{0\})/\sim$ where $\sim$ is the usual equivalence relation: $x \sim y$ iff $x = ry$ for some $r \neq 0 \in R$. Using the quotient map $\pi : R^{n+1} - \{0\} \to RP^n$, RP^n is topologized. Write $[x_1, \cdots, x_{n+1}]$ for the equivalence class of $(x_1, \cdots, x_{n+1}) \in R^{n+1} - \{0\}$.

First observe that RP^n is a n-dimensional connected compact closed manifold. Consider a diagram

$$\begin{tikzcd} R^{n+1} - \{0\} \arrow[r, "\pi"] & RP^n \\ S^n \arrow[u, "i"] \arrow[ur, "p"'] & \end{tikzcd}$$

where S^n is the unit n-sphere in $R^{n+1} - \{0\}$ and i is the inclusion map.

Define p by the composition πi. It is thus immediate to see that p is the identification map such that the inverse image by p of any point $[x] \in RP^n$ consists of exactly two antipodal points $\pm x/|x|$. It follows that RP^n is a connected compact Hausdorff space. For $i = 1,\cdots,n+1$, let $V_i = \{[x_1,\cdots,x_{n+1}]\} \in RP^n \mid x_i \neq 0\}$ and define $\Psi_i : V_i \to R^n$ by $\Psi_i([x_1,\cdots,x_{n+1}]) = (x_1/x_i,\cdots,x_{i-1}/x_i,x_{i+1}/x_i,\cdots,x_{n+1}/x_i)$. Since Ψ_i is a homeomorphism and since RP^n is the union of the V_i's, RP^n is a n-dimensional connected compact closed manifold.

Next, decompose RP^n into a CW-complex so that one can calculate the cohomology (homology) of RP^n by the standard method for CW-complexes. Define $\lambda : R^n - \{0\} \to R^{n+1} - \{0\}$ by $\lambda(x_1,\cdots,x_n) = (x_1,\cdots,x_n,0)$, and consider RP^{n-1} as a subspace of RP^n by the identification $[x] = [\lambda(x)]$. Define $f_n : (D^n,\partial D^n) \to (RP^n,RP^{n-1})$, where D^n is the n-dimensional unit disk and ∂D^n its boundary, by

$$f_n(x_1,\cdots,x_n) = \left[x_1,\cdots,x_n, \sqrt{1 - \sum_{i=1}^{n} x_i^2} \right] .$$

Then $f_n|_{D^n-\partial D^n}$ is a homeomorphism onto $e^n = RP^n - RP^{n-1}$. Thus inductively there is a cell decomposition of RP^n,

$$RP^n = e^0 \cup e^1 \cup \cdots \cup e^n ,$$

where e^i $(i = 0,1,\cdots,n)$ denotes an i-cell.

Define RP^∞ to be the direct limit $\varinjlim RP^n$ (see II.3), where $RP^0 \subset RP^1 \subset RP^2 \subset \cdots \subset RP^n \subset \cdots$, and call RP^∞ the <u>infinite dimensional real projective space</u>.

Proposition 4.1: For $n > 2$,

$$\pi_\rho(RP^n) = \begin{cases} Z & ; \text{for } \rho = 0 \\ Z_2 & ; \text{for } \rho = 1 \\ 0 & ; \text{for } n > \rho > 1 \end{cases}.$$

Proof: Since $p: S^n \to RP^n$ is a 2-fold covering, by Theorem 1.6 $\pi_\rho(S^n) \approx \pi_\rho(RP^n)$ for every $\rho \geq 2$. If $\rho < n$, then $\pi_\rho(S^n) = 0$. Hence $\pi_\rho(RP^n) = 0$ for any ρ with $n > \rho \geq 2$. If $n \geq 2$, then S^n is simply connected. By Theorem 1.5, the group $\mathcal{J}(S^n,p)$ of deck transformations is isomorphic to $\pi_1(RP^n)$. Since a deck transformation $f: S^n \to S^n$ preserves each fibre consisting of antipodal points, f is the antipodal map or the identity. This proves that $\pi_1(RP^n) \approx Z_2$ for $n \geq 2$. ▲

For each $\rho \geq 2$, $\pi_\rho(RP^n) = 0$ when $n > \rho$. By Theorem 1.3, $\pi_\rho(RP^\infty) \approx \varinjlim \pi_\rho(RP^n) = 0$, and $\pi_1(RP\ \ ) \approx \varinjlim \pi_1(RP^n) = Z_2$. It follows that $RP^\infty = K(Z_2,1)$.

It is well known that for Z_2 coefficients, RP^n is an orientable connected compact n-manifold without boundary and

$$H_q(RP^n;Z_2) \approx H^q(RP^n;Z_2) = \begin{cases} Z_2 & ; \text{for } 0 \leq q \leq n \\ 0 & ; \text{for } q > n \end{cases}.$$

Moreover, the cell decomposition of RP^n shows that, if $i: RP^{n-1} \to RP^n$ is the inclusion map, $i_*: H_q(RP^{n-1};Z_2) \to H_q(RP^n;Z_2)$ is an isomorphism for $0 \leq q \leq n-1$. These facts together with the Poincaré Duality Theorem and properties of cap product (see I.2) are used to prove the following theorem.

Theorem 4.1: The cohomology algebra $H^*(RP^n;Z_2)$ is the truncated polynomial algebra $P_{Z_2}[\alpha]/(\alpha^{n+1})$ on one generator α of degree 1 and height n+1.

Proof: Let ω_n denote the fundamental class of RP^n and define the continuous mappings

$$i_{j,k}:RP^j \to RP^k, \quad 1 \leq j \leq k \leq n$$

by letting $i_{j,k}$ be the inclusion map for $j \neq k$, and $i_{j,k}$ be the identity for $j = k$. The proof proceeds by induction on n, the dimension of RP^n. If $n = 1$, then the result is obviously true. For $n = 2$, $i_{1,2*}(\omega_1)$ is a generator of $H_1(RP^2;Z_2)$. Define the generator $\alpha_2 \in H^1(RP^2;Z_2)$ by $\langle\alpha_2,i_{1,2*}(\omega_1)\rangle = 1$. Then $\alpha_2 \smile \alpha_2 = \alpha_2^2$ is a generator of $H^2(RP^2;Z_2)$ because $\langle\alpha_2 \smile \alpha_2,\omega_2\rangle = \langle\alpha_2,\alpha_2 \frown \omega_2\rangle = \langle\alpha_2,i_{1,2*}(\omega_1)\rangle = 1$. Thus the theorem holds true in this case.

Assume the result to be true for any real projective space of dimension less than n; that is, assume for $k < n$, $\langle\alpha_k,i_{1,k*}(\omega_1)\rangle = 1$ and $\langle\alpha_k^\ell,i_{\ell,k*}(\omega_\ell)\rangle = 1$ for all $\ell \leq k$. Define the generator $\alpha_n \in H^1(RP^n;Z_2)$ by $\langle\alpha_n,i_{1,n*}(\omega_1)\rangle = 1$. Then $\alpha_n \smile \alpha_n = \alpha_n^2$ is a generator of $H^2(RP^n;Z_2)$ because

$$\begin{aligned}\langle\alpha_n^2,i_{2,n*}(\omega_2)\rangle &= \langle\alpha_n^2,(i_{n-1,n}\circ i_{2,n-1})_*(\omega_2)\rangle \\ &= \langle\alpha_n^2,i_{n-1,n*}(i_{2,n-1*}(\omega_2)\rangle \\ &= \langle i^*_{n-1,n}(\alpha_n^2),i_{2,n-1*}(\omega_2)\rangle \\ &= \langle(i^*_{n-1,n}(\alpha_n))^2,i_{2,n-1*}(\omega_2)\rangle \\ &= \langle\alpha_{n-1}^2,i_{2,n-1*}(\omega_2)\rangle \\ &= 1 \quad .\end{aligned}$$

By a similar procedure, one can show α_n^p $(p < n)$ is a generator of $H^p(RP^n;Z_2)$. The last step for $p = n$ follows easily from the fact that $\langle\alpha_n^n,\omega_n\rangle = \langle\alpha_n^n,\alpha_n^{n-1} \frown \omega_n\rangle = \langle\alpha_n,i_{1,n*}(\omega_1)\rangle = 1$. This proves the theorem. ▲

Theorem 4.2: $H^*(RP^\infty;Z_2) \simeq H^*(Z_2,Z_2)$ is the polynomial algebra $P_{Z_2}[\S]$ over Z_2 on one generator $\S$ of degree 1.

Proof: Since $RP^\infty = K(Z_2,1)$, it follows that $H^*(RP^\infty;Z_2) \simeq H^*(Z_2,Z_2)$. From Theorem 1.2, $H^*(RP^\infty;Z_2) = H^*(\varinjlim RP^n;Z_2) \simeq \varinjlim H^*(RP^n;Z_2) = P_{Z_2}[\S]$. This completes the proof. ▲

$$S^1 = K(Z,1)$$

The universal covering space of a circle S^1 is the real line R^1 with the projection $p(x) = e^{2\pi ix}$. Then a deck transformation $h:R^1 \to R^1$ preserves fibres so that $e^{2\pi ih(x)} = e^{2\pi ix}$. Thus $h(x) - x$ is an integer for all $x \in R^1$. Since the map $k:R^1 \to Z$ defined by $k(x) = h(x) - x$ is continuous, and since R^1 is connected, $h(x) - x = c$ for some fixed integer c; that is $h(x) = x + c$ for some integer c. Hence h is a translation of the integer c. It follows that the group $\mathcal{J}(S^1,p)$ is Z. Since the covering space R^1 of S^1 is contractible to a point, by Theorem 1.5, $\pi_1(S^1) \simeq Z$. From Theorem 1.6, $\pi_\rho(R^1) \simeq \pi_\rho(S^1)$ for $\rho \geq 2$. Thus $\pi_\rho(S^1) = 0$ for any $\rho \geq 2$. This shows that S^1 is a $K(Z,1)$-space.

Theorem 4.3: $H^*(S^1,R) \simeq H^*(Z,R)$ for a ring R. Hence $H^*(Z,R)$ is isomorphic to the exterior algebra $E_R[\alpha]$ over R on one generator α of degree 1.

Proof: Evident. ▲

Obviously the proof of Theorem 4.3 is a topological one. A purely algebraic computation of the cohomology algebra $H^*(Z,R)$ is now given.

Consider the exact sequence of $Z(Z)$-modules

$$E: 0 \to \ker \varepsilon \xrightarrow{i} Z(Z) \xrightarrow{\varepsilon} Z \to 0$$

with i the inclusion and ε the usual augmentation. It is well known that $\ker \varepsilon$ is a free $Z(Z)$-module. By Proposition 2.1, $\ker \varepsilon$, $Z(Z)$, and 0 are projective objects. Hence the left complex $\mathfrak{X}$ over ${}_{\varepsilon}Z$

$$Z \leftarrow Z(Z) \xleftarrow{i} \ker \varepsilon \leftarrow 0 \leftarrow 0 \leftarrow \cdots \quad : \mathfrak{X}$$

is a $Z(Z)$-module resolution of ${}_{\varepsilon}Z$ (see II.2). It follows immediately that $H^n(Z,R) = 0$ for $n \geq 2$. By Proposition 2.3, $H^0(Z,R) = R$ and $H^1(Z,R) = \hom_Z(Z,R) = R$. Summarizing, the module structure of $H^*(Z,R)$ is given by

$$H^n(Z,R) = \begin{cases} R \ ; \ n = 0,1 \\ 0 \ ; \ n > 1 \end{cases} .$$

Evidently, $H^*(Z,R)$ is isomorphic to the exterior algebra $E_R[\alpha]$ over R on one generator α of degree 1.

Cohomology of Cyclic Groups

The definition $H^*(\pi,N) = \mathrm{Ext}^*_{Z(\pi)}({}_{\varepsilon}Z,N)$ was given in Chapter II. Thus one may calculate the cohomology modules of a particular group π by using a $Z(\pi)$-module resolution of ${}_{\varepsilon}Z$ suitably adapted to the structure of the group π.

Let $\pi = Z_p$ be the multiplicative cyclic group of order p with generator t. The group ring $Z(Z_p)$ is the ring of all polynomials $P_Z(t)$

taken modulo the relation $t^p = 1$; thus, $Z(Z_p) = P_Z(t)/(t^p)$ $= \{a_0 + a_1 t + \cdots + a_{p-1}t^{p-1} \mid a_i \in Z\}$. For $n \geq 0$, let $X_n = Z(Z_p)$ and define $\partial_n : X_{n+1} \to X_n$ by $\partial_n(\sum_{i=0}^{p-1} a_i t^i) = (t - 1)(\sum_{i=0}^{p-1} a_i t^i)$, for n even, and

$\partial_n(\sum_{i=0}^{p-1} a_i t^i) = (\sum_{i=0}^{p-1} t^i)(\sum_{i=0}^{p-1} a_i t^i)$, for n odd, where $x = \sum_{i=0}^{p-1} a_i t^i$ is an arbitrary element in $Z(Z_p)$. Clearly $\partial_n \partial_{n+1} = 0$. If $\partial_{2n}(x) = 0$, then $a_0 = a_1 = \cdots = a_{p-1}$ and $\partial_{2n+1}(a_0) = x$. If $\partial_{2n-1}(x) = 0$, then $a_0 + a_1 + \cdots + a_{p-1} = 0$, and $x = -\partial_{2n}(a_0 + (a_1 + a_0)t + \cdots + (a_{p-1} + \cdots + a_0)t^{p-1})$. The usual augmentation $\varepsilon : Z(Z_p) \to Z$ has $\varepsilon\partial_0 = 0$ and $\ker \varepsilon = \operatorname{im} \partial_0$. All told, the left complex Γ over ${}_\varepsilon Z$

$$Z \xleftarrow{\varepsilon} X_0 \xleftarrow{\partial_0} \cdots \leftarrow X_{n-1} \xleftarrow{\partial_{n-1}} X_n \xleftarrow{\partial_n} X_{n+1} \leftarrow \cdots \quad : \Gamma$$

provides a $Z(Z_p)$-module projective resolution of ${}_\varepsilon Z$. This resolution was originally due to Steenrod [26].

For any $Z(Z_p)$-module N, the isomorphism $\hom_{Z(Z_p)}(Z(Z_p), N) \simeq N$ maps $f \in \hom_{Z(Z_p)}(Z(Z_p), N)$ into $f(1)$. Hence the cochain complex

$$N : N \xrightarrow{d} N \xrightarrow{h} N \xrightarrow{d} N \xrightarrow{h} \cdots ,$$

where $d(x) = \partial_2(x)$ and $h(x) = \partial_1(x)$, is isomorphic to the cochain complex $\mathrm{Hom}_{Z(Z_p)}(\Gamma, N)$. The cohomology modules of Z_p with coefficients in N are those of the cochain complex N.

<u>Proposition 4.2</u>: For a cyclic group of order p with a generator t and a Z_p-module N, the cohomology modules of Z_p with coefficients in N are: $H^0(Z_p, N) = \{a \in N \mid ta = a\}$, $H^{2n}(Z_p, N) = \{a \in N \mid ta = a\}/\operatorname{im} h$, with $n > 0$, $H^{2n+1}(Z_p, N) = \{a \in N \mid h(a) = 0\}/\operatorname{im} d$, with $n > 0$.

Cohomology Ring $H^*(Z_3,Z)$

Let $\pi = Z_p$ be the multiplicative cyclic group of order p with a generator t. Let $\Delta:Z(Z_p) \to Z(Z_p) \otimes Z(Z_p)$ be the coproduct of the Hopf algebra $Z(Z_p)$. The Δ-morphism of modules $f:Z \to Z \otimes Z$ defined by $f(1) = 1 \otimes 1$ can be extended to a Δ-chain map $h:\Gamma \to \Gamma \otimes \Gamma$ by the direct application of Theorem 2.2 (the comparison theorem), where Γ is the Steenrod resolution of the $Z(Z_p)$-module ${}_\varepsilon Z$ discussed in the previous section.

Steenrod [27] computed h explicitly in this case. Define

$$h_n:X_n \to \sum_{j=0}^{n} X_j \otimes X_{n-j} = (\Gamma \otimes \Gamma)_n \text{ by } h_n(e_{2i}) = \sum_{j=0}^{i} e_{2j} \otimes e_{2i-2j}$$

$$+ \sum_{j=0}^{i-1} \sum_{0 \le k < \ell \le n-1} t^k e_{2j+1} \otimes t^\ell e_{2i-2j-1} \text{ for } n = 2i, \text{ and}$$

$$h_n(e_{2i+1}) = \sum_{j=0}^{i} (e_{2j} \otimes e_{2i-2j+1} + e_{2j+1} \otimes te_{2i-2j}), \text{ for } n = 2i + 1,$$

where, notationally,

$$X_n = \{a_0 e_n + a_1 te_n + \cdots + a_{p-1} t^{p-1} e_n \mid a_i \in Z\} \quad .$$

By Proposition 4.2, the module structure of $H^*(Z_3,Z)$ is given by

$$H^n(Z_3,Z) = \begin{cases} Z & ; n = 0 \\ Z/3Z & ; n \text{ even} \\ 0 & ; \text{otherwise} \end{cases} \quad .$$

Since Z is a $Z(Z_3)$-module algebra (see II.1), Z and Z are paired with respect to Z by the $Z(Z_3)$-pairing $\theta:Z \otimes Z \to Z$ defined by $\theta(n \otimes m) = nm$. Then the cup product is a morphism of graded Z-modules

$$\smile : H^n(Z_3,Z) \otimes H^m(Z_3,Z) \to H^{n+m}(Z_3,Z) \quad .$$

<u>Theorem 4.4</u>: The cohomology ring $H^*(Z_3,Z)$ is

$$Z(\alpha^0) \otimes P_{Z_3}[\alpha]/(\alpha^0)$$

where $P_{Z_3}[\alpha]$ is the polynomial algebra over Z_3 on one generator α of degree 2.

<u>Proof</u>: If α_{2n} is a generator of $H^{2n}(Z_3,Z)$, then at the cochain level α_{2n} is represented by $f_{2n} \in \hom_{Z(Z_3)}(X_{2n},Z)$ where $f_{2n}(e_{2n}) = 1$. The proof of the theorem proceeds by induction on n. If $n = 1$, at the cochain level one has $\hom_{Z(Z_3)}(X\ ,Z) \otimes \hom_{Z(Z\ )}(X\ ,Z) \xrightarrow{\Lambda}$

$\hom_{Z(Z_3)\otimes Z(Z_3)}(X_2 \otimes X_2, Z \otimes Z) \xrightarrow{\chi(h,\theta)} \hom_{Z(Z_3)}(X_4,Z)$, where $\smile = \chi(h,\theta)$. Thus

$$\begin{aligned} f_2^2(e_4) = (f_2 \smile f_2)(e_4) &= (\chi(h,\theta)\Lambda)(f_2 \otimes f_2)(e_4) \\ &= (\theta\Lambda(f_2 \otimes f_2))h(e_4) \\ &= (\theta\Lambda(f_2 \otimes f_2))(\sum_{j=1}^{2} e_{2j} \otimes e_{2i-2j} \\ &\quad + \sum_{0 \le k < \ell \le 3} t^k e_1 \otimes t^\ell e_3) \\ &= (f_2(e_2) \otimes f_2(e_2)) \\ &= 1 \end{aligned} \quad ,$$

and $\alpha_2 \smile \alpha_2$ generates $H^4(Z_3,Z)$. Assume that for $k < n$ $\alpha_2^k = \alpha_2 \smile \cdots \smile \alpha_2$ (k-factors) generates $H^{2k}(Z_3,Z)$. Consequently,

$$
\begin{aligned}
f_2^n(e_{2n}) = (f_2 \smile f_2^{n-1})(e_{2n}) &= (\chi(h,\theta)\Lambda)(f_2 \otimes f_2^{n-1})(e_{2n}) \\
&= (\theta\Lambda(f_2 \otimes f_2^{n-1}))h(e_{2n}) \\
&= (\theta\Lambda(f_2 \otimes f_2^{n-1}))(\sum_{j=0}^{n} e_{2j} \otimes e_{2n-2j} \\
&\quad + \sum_{j=0}^{n-1} \sum_{0 \le k < \ell \le 2n-1} t^k e_{2j+1} \otimes t^\ell e_{2n-2j-1}) \\
&= (f_2(e_2) \otimes f_2^{n-1}(e_{2n-2})) \\
&= 1 .
\end{aligned}
$$

This proves the theorem. ▲

CHAPTER V

SPECTRAL SEQUENCES

If N is a normal subgroup of the group π, the cohomology module $H^*(\pi,N)$ can be calculated by successive approximations from the cohomology of N and that of π/N. These successive approximations are codified in the notion of a spectral sequence. In this chapter the mechanism of these sequences is formulated via "exact couples" (Massey [23]). A filtered cochain complex is associated with a spectral sequence and the Hochschild-Serre spectral sequence [19] with E_2 term $H(\pi/N,H(N))$ is derived. The chapter is completed by giving some results which will be applied to the calculation of the ring $H^*(\pi,N)$ in Chapter VI.

Exact Couples

An <u>exact couple</u> $C^1 = \{D_1,E_1,i_1,j_1,k_1\}$ is a pair of modules D_1, E_1 together with three homomorphisms i_1, j_1, k_1

$$C^1 \qquad \begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

which form an exact triangle in the sense that kernel = image at each vertex. The modules D_1 and E_1 in an exact couple may be graded R-modules or Z-bigraded R-modules; in the latter case, each of i_1, j_1, k_1 has some bidegree.

The exactness of C^1 shows that the composition $d_1 = j_1k_1:E_1 \to E_1$ is such that $d_1d_1 = 0$; hence d_1 is a differential operator on E_1. Let

(i) $D_2 = \text{im } i_1$,

(ii) $E_2 = \ker d_1/\text{im } d_1$,

(iii) $i_2:D_2 \to D_2$ be defined by $i_2 = i_1|_{D_2}$,

(iv) $j_2:D_2 \to E_2$ be defined by $j_2(x) = \overline{j_1(y)}$, where $y \varepsilon D_1$ is such that $i_1(y) = x$, and $\overline{j_1(y)}$ denotes the coset represented by $j_1(y)$,

(v) $k_2:E_2 \to D_2$ be defined by $k_2(\overline{x}) = k_1(x)$. Then the triangle

$$C^2 \qquad \begin{tikzcd} D_2 \arrow[rr, "i_2"] & & D_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd}$$

is exact. Call $C^2 = \{D_2,E_2,i_2,j_2,k_2\}$ the <u>derived exact couple</u> of C^1 (for notational reasons, C^2 is called the 2-nd derived exact couple).

It is clear that this process of derivation can be applied to the derived exact couple C^2 to obtain the 3-rd derived exact couple $C^3 = \{D_3,E_3,i_3,j_3,k_3\}$, and so on. In general, denote the n-th derived exact couple by $C^n = \{D_n,E_n,i_n,j_n,k_n\}$. The sequence $\{E_n,d_n \mid n \geq 1\}$ is called the Koszul-Leray <u>spectral sequence</u> of the exact couple C^1.

<u>Proposition 5.1</u>: $E_n \cong k_1^{-1}(\text{im } i_1^{n-1})/j_1(\ker i_1^{n-1})$, $n \geq 1$, where i_1^{n-1} is the (n - 1)-fold iteration of i_1.

A proof of this result is sketched here because it is not found in texts (for example, [5] and [22]) covering this subject. For $n \geq 1$, let

(i) $E_{n,n} = \{x \varepsilon E_n \mid d_n(x) = 0\}$,

(ii) $\tau:E_{n,n} \to E_n$ be the inclusion map,

(iii) $\kappa^n_{n+1}:E_{n,n} \to E_{n+1}$ be defined by $\kappa^n_{n+1}(x) = \overline{x}$.

These definitions are illustrated in the diagram

$$\begin{tikzcd}
E_{1,1} \arrow[d, "\tau"] \arrow[dr, "\kappa_2^1"] & E_{2,2} \arrow[d, "\tau"] \arrow[dr, "\kappa_3^2"] & E_{3,3} \arrow[d, "\tau"] & \cdots & E_{n,n} \arrow[d, "\tau"] \arrow[dr, "\kappa_{n+1}^n"] & E_{n+1,n+1} \arrow[d, "\tau"] & \cdots \\
E_1 \arrow[d, "d_1"] & E_2 \arrow[d, "d_2"] & E_3 \arrow[d, "d_3"] & \cdots & E_n \arrow[d, "d_n"] & E_{n+1} \arrow[d, "d_{n+1}"] & \cdots \\
E_1 & E_2 & E_3 & \cdots & E_n & E_{n+1} & \cdots
\end{tikzcd}$$

Let

(i)^1 $E_{n,n+1} = \{x \in E_n \mid d_n(x) = 0,\ d_{n+1}\kappa_{n+1}^n(x) = 0\}$,

(ii)^1 $\tau: E_{n,n+1} \to E_{n,n}$ be the inclusion map,

(iii)^1 $\kappa_{n+1}^n: E_{n,n+1} \to E_{n,n}$ be defined by $\kappa_{n+1}^n(x) = \overline{x}$; note the use of the same notation as in (iii),

(iv)^1 $\kappa_{n+2}^n = \kappa_{n+2}^{n+1}\kappa_{n+1}^n: E_{n,n+1} \to E_{n+2}$.

Continuing in this fashion, let

(i)^k $E_{n,n+k} = \{x \in E_n \mid d_n(x) = 0,\ d_{n+1}\kappa_{n+1}^n(x) = 0,\ \cdots$

$d_{n+1}\kappa_{n+k}^n(x) = 0\}$,

(ii)^k $\tau: E_{n,n+k} \to E_{n+1,n+k}$ be the inclusion map,

(iii)^k $\kappa_{n+1}^n: E_{n,n+k} \to E_{n+1,n+k}$ be defined by $\kappa_{n+1}^n(x) = \overline{x}$,

(iv)^k $\kappa_{n+2}^n = \kappa_{n+2}^{n+1}\kappa_{n+1}^n: E_{n,n+2} \to E_{n+2,n+2}$

$\kappa_{n+3}^n = \kappa_{n+3}^{n+2}\kappa_{n+2}^{n+1}\kappa_{n+1}^n: E_{n,n+3} \to E_{n+3}$

$\vdots$

$\kappa_{n+k}^n = \kappa_{n+k}^{n+k-1} \cdots \kappa_{n+1}^n: E_{n,n+k} \to E_{n+k}$.

In diagram form, one has

$$
\begin{tikzcd}
E_{1,n-1} \arrow[d, "\tau"] \arrow[dddddrrrr, dashed, "\kappa^1_{n-1}"] \arrow[ddddddrrrrr, dashed, "\kappa^1_n"] & & & & & & \\
E_{1,n-2} \arrow[d, "\tau"] & & & & & & \\
\vdots \arrow[d] & & & & & & \\
E_{1,3} \arrow[d, "\tau"] \arrow[dr, "\kappa^1_2"] & & & & & & \\
E_{1,2} \arrow[d, "\tau"] \arrow[dr, "\kappa^1_2"] & E_{2,3} \arrow[d, "\tau"] \arrow[dr, "\kappa^2_3"] & & & & & \\
E_{1,1} \arrow[d, "\tau"] \arrow[dr, "\kappa^1_2"] & E_{2,2} \arrow[d, "\tau"] \arrow[dr, "\kappa^2_3"] & E_{3,3} \arrow[d, "\tau"] & \cdots & E_{n-1,n-1} \arrow[d, "\tau"] \arrow[dr, "\kappa^{n-1}_n"] & & \\
E_1 \arrow[d, "d_1"] & E_2 \arrow[d, "d_2"] & E_3 \arrow[d, "d_3"] & \cdots & E_{n-1} \arrow[d, "d_{n-1}"] & E_n \arrow[d, "d_n"] & \cdots \\
E_1 & E_2 & E_3 & \cdots & E_{n-1} & E_n & \cdots
\end{tikzcd}
$$

Since each of the maps κ are surjective, $E_n \simeq E_{1,n-1}/\ker \kappa^1_n$. First, one shows $E_{1,n-1} = k_1^{-1}(\text{im } i_1^{n-1})$. Now $E_{n-1,n-1} = \ker d_{n-1} = \ker (j_{n-1}k_{n-1})$ $= k^{-1}_{n-1}(\ker j_{n-1}) = k^{-1}_{n-1}(\text{im } i_{n-1}) = k^{-1}_{n-1}(\text{im } i_1^{n-1})$. Also, for all $x \in E_{1,n-1}$, $\kappa^1_{n-1}(x) \in E_{n-1,n-1}$; so that $k_{n-1}\kappa^1_{n-1}(x) = i_1^{n-1}(\zeta)$ for $\zeta \in D_1$. Hence, $i_1^{n-1}(\zeta) = k_{n-1}\kappa^1_{n-1}(x) = k_{n-1}\kappa^{n-2}_{n-1}\kappa^1_{n-2}(x) = k_{n-2}\kappa^1_{n-2}(x)$ $= k_{n-2}\kappa^{n-3}_{n-2}\kappa^1_{n-3}(x) = \cdots = k_1(x)$, and it follows that $E_{1,n-1} \subset k_1^{-1}(\text{im } i_1^{n-1})$. Similarly, $k_1^{-1}(\text{im } i_1^{n-1}) \subset E_{1,n-1}$. Next, one shows $\ker \kappa_n^1 = j_1(\ker i_1^{n-1})$. If $x \in \ker \kappa_n^1$ then $\kappa_{n-1}^1(x) = j_{n-1}(\zeta_1)$, where $\zeta_1 \in D_{n-1}$ and $\zeta_1 = i_1^{n-2}(a_1)$ for $a_1 \in \ker i_1^{n-1}$. Thus $\kappa_{n-2}^1(x)$ $= j_{n-2}(\zeta_2)$, where $\zeta_2 = i_1^{n-3}(a_2)$ for $a_2 \in \ker i_1^{n-1}$ and it follows that $\kappa_{n-2}^1(x) = j_{n-2}(i_1^{n-3}(a_1 + a_2))$. Continuing in this fashion, one obtains $\kappa_2^1(x) = j_2(i_1(a_1 + \cdots + a_{n-2}))$ for $a_{n-2} \in \ker i_1^2$, so that $x = j_1(a_1 + \cdots + a_{n-2} + a_{n-1})$ for $a_{n-1} \in \ker i_1$. Hence $x \in j_1(\ker i_1^{n-1})$, and $\ker \kappa_n^1 \subset j_1(\ker i_1^{n-1})$. Similarly, $j_1(\ker i_1^{n-1}) \subset \ker \kappa_n^1$ and the proof is complete. ▲

In view of Proposition 5.1, the terms E_n of the spectral sequence can be considered as successive approximations to E_∞, which is defined as

$$E_\infty = k_1^{-1}\left(\bigcap_{n=1}^{\infty} \operatorname{im} i_1^{n-1}\right)/j_1\left(\bigcup_{n=1}^{\infty} \ker i_1^{n-1}\right) \quad .$$

Spectral Sequences Associated With a Filtered Cochain Complex

Definition 5.1: A graded cochain complex $G = \{C,\delta,F\}$ with a decreasing filtration F is

(1) a graded cochain complex over R:

$$C: C^0 \to C^1 \to \cdots \to C^n \xrightarrow{\delta^n} C^{n+1} \to \cdots \quad ,$$

where $\delta^n: C^n \to C^{n+1}$ is a morphism of graded R-modules, and

(2) for each integer p, F^pC is a subcomplex of C and $F^{p+1}C$ is a subcomplex of F^pC (in notation, $F^pC \supset F^{p+1}C$). If, in addition, $F^pC = C$ for $p \leq 0$, and $F^pC^n = 0$ for $p > n$, then the filtration F is said

to be <u>strictly convergent</u>.

It is now shown that an exact couple can be associated with a graded cochain complex $G = \{C,\delta,F\}$ with decreasing filtration F. The short exact sequence of cochain complexes

$$0 \to F^{p+1}C \xrightarrow{\ell} F^pC \xrightarrow{\rho} F^pC/F^{p+1}C \to 0$$

yields the usual long exact cohomology sequence

$$\cdots \to H^n(F^{p+1}C) \xrightarrow{i} H^n(F^pC) \xrightarrow{j} H^n(F^pC/F^{p+1}C) \xrightarrow{k} H^{n+1}(F^{p+1}C) \to \cdots$$

where i is induced by the injection ℓ, j by the projection ρ, and k is the cohomology connecting homomorphism. These sequences for all p combine to give the <u>zig-zag exact pile-up</u> in Figure 1. In this display, each sequence consisting of a vertical step i, followed by two horizontal steps j and k, followed by a vertical step i, $\cdots$ is exact.

Let

$$D^{p,q} = H^{p+q}(F^pC), \qquad E_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C)$$

$$D_1 = \{D_1^{p,q}\} \quad , \qquad E_1 = \{E_1^{p,q}\} \quad .$$

Then the zig-zag exact pile-up implies the exactness of the triangle

$$C^1 \qquad \begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

where i_1, j_1, k_1 are induced by the families of maps $\{i\}$, $\{j\}$, and $\{k\}$, respectively. C^1 is called the <u>exact couple associated with G</u>.

Through iteration, one obtains the n-th derived exact couple

$$
\begin{tikzcd}
 & & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & & \\
\cdots \arrow[r] & H^{p+q-1}\left(\frac{F^{p}C}{F^{p+1}C}\right) \arrow[r, "k"] & H^{p+q}(F^{p+1}C) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}\left(\frac{F^{p+1}C}{F^{p+2}C}\right) \arrow[r, "k"] & H^{p+q+1}(F^{p+2}C) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q+1}\left(\frac{F^{p+2}C}{F^{p+3}C}\right) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{p+q-1}\left(\frac{F^{p-1}C}{F^{p}C}\right) \arrow[r, "k"] & H^{p+q}(F^{p}C) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}\left(\frac{F^{p}C}{F^{p+1}C}\right) \arrow[r, "k"] & H^{p+q+1}(F^{p+1}C) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q+1}\left(\frac{F^{p+1}C}{F^{p+2}C}\right) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{p+q-1}\left(\frac{F^{p-2}C}{F^{p-1}C}\right) \arrow[r, "k"] & H^{p+q}(F^{p-1}C) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}\left(\frac{F^{p-1}C}{F^{p}C}\right) \arrow[r, "k"] & H^{p+q+1}(F^{p}C) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q+1}\left(\frac{F^{p}C}{F^{p+1}C}\right) \arrow[r] & \cdots \\
 & & \vdots & & \vdots & &
\end{tikzcd}
$$

Figure 1. Zig-Zag Exact Pile-Up

$C^n = \{D_n, E_n, i_n, j_n, k_n\}$ associated with G. D_n and E_n are bigraded R-modules with maps

$$i_n \text{ of bidegree } (-1,1) \quad ,$$
$$j_n \text{ of bidegree } (n-1,-n+1) \quad ,$$
$$k_n \text{ of bidegree } (1,0) \quad .$$

The differential bigraded R-module E_n has the differential operator $d_n = j_n k_n$ of bidegree $(n,-n+1)$. The spectral sequence $\{E_n, d_n \mid n \geq 1\}$ is said to be <u>derived from G</u>.

<u>Proposition 5.2</u>: If $\{E_n, d_n \mid n \geq 1\}$ is derived from $G = \{C, \delta, F\}$ with strictly convergent filtration F, then

(1) $E_n^{p,q} = 0$ if $p < 0$ or $q < 0$,

(2) for $n > \max(p,q+1)$

$$E_\infty^{p,q} \simeq E_n^{p,q} \simeq \frac{\operatorname{im}\{i: H^{p+q}(F^pC) \to H^{p+q}(C)\}}{\operatorname{im}\{i: H^{p+q}(F^{p+1}C) \to H^{p+q}(C)\}} \quad .$$

<u>Proof</u>: Part (2) is proved first. The short exact cochain complexes

$$0 \to F^{p+n}C \to F^{p+1}C \to F^{p+1}C/F^{p+n}C \to 0 \quad ,$$

$$0 \to F^{p+1}C/F^{p+n}C \to F^pC/F^{p+n}C \to F^pC/F^{p+1}C \to 0 \quad ,$$

$$0 \to F^pC \to F^{p-n+1}C/F^pC \to F^{p-n+1}C/F^pC \to 0 \quad ,$$

yield, for $n \geq 1$, the long exact cohomology sequences

$$\cdots \xrightarrow{i'} H^{p+q+1}(F^{p+1}C) \xrightarrow{j'} H^{p+q+1}(F^{p+1}C/F^{p+n}C) \xrightarrow{k'} \cdots \quad ,$$

$$\cdots \xrightarrow{i''} H^{p+q}(F^pC/F^{p+n}C) \xrightarrow{j''} H^{p+q}(F^pC/F^{p+1}C) \xrightarrow{k''} \cdots \quad ,$$

$$\cdots \xrightarrow{i'''} H^{p+q-1}(F^{p-n+1}C) \xrightarrow{j'''} H^{p+q-1}(F^{p-n+1}C/F^pC) \xrightarrow{k'''} \cdots \quad ,$$

respectively. It is easily verified that im i_1^{n-1} = ker j',
ker i_1^{n-1} = ker i'', and the diagram

$$\begin{tikzcd}
H^{p+q}(F^pC/F^{p+1}C) \arrow[rr, "k''"] \arrow[dr, "k"'] & & H^{p+q+1}(F^{p+1}C/F^{p+n}C) \\
& H^{p+q+1}(F^{p+1}C) \arrow[ur, "j'"'] &
\end{tikzcd}$$

is commutative. Therefore

$$E_n^{p,q} \simeq \frac{\text{im }\{H^{p+q}(F^pC/F^{p+n}C) \xrightarrow{j''} H^{p+q}(F^pC/F^{p+1}C)\}}{\text{im }\{H^{p+q-1}(F^{p-n+1}C/F^pC) \xrightarrow{k'''} H^{p+q}(F^pC) \xrightarrow{j} H^{p+q}(F^pC/F^{p+1}C)\}} .$$

If n > max (p,q+1), then $D_n^{p+1,q} = 0$ and thus

$$E_\infty^{p,q} = k^{-1}\left(\bigcap_{n=1}^{\infty} \text{im } i^{n-1}\right)/j\left(\bigcup_{n=1}^{\infty} \text{ker } i^{n-1}\right)$$

$$= \frac{\text{ker } k}{j(\text{ker }\{i^{n-1}:H^{p+q}(F^pC) \to H^{p+q}(C)\})}$$

$$= \frac{\text{im }\{H^{p+q}(F^pC) \xrightarrow{j} H^{p+q}(F^pC/F^{p+1}C)\}}{\text{im }\{H^{p+q-1}(C/F^pC) \xrightarrow{k'''} H^{p+q}(F^pC) \xrightarrow{j} H^{p+q}(F^pC/F^{p+1}C)\}}$$

$$\simeq E_n^{p,q} .$$

Furthermore, the zig-zag exact pile-up gives

$$\begin{tikzcd}
\cdots \arrow[r, "k_n"] & D_n^{p-n+2,q+n-2} \arrow[d, "i_n"] & & \\
& D_n^{p-n+1,q+n-1} \arrow[r, "j_n"] & E_n^{p,q} \arrow[r, "k_n"] & D_n^{p+1,q} \arrow[d, "i_n"] \\
& & & \vdots
\end{tikzcd}$$

and since $D_n^{p-n+2,q+n-2} = H^{p+q}(C)$, $D_n^{p-n+1,q+n-1} = H^{p+q}(C)$,

$D_n^{p+1,q} = H^{p+q+1}(F^{p+1}C) = 0$, it follows that

$$E_n^{p,q} \simeq \frac{\mathrm{im}\ \{i : H^{p+q}(F^pC) \to H^{p+q}(C)\}}{\mathrm{im}\ \{i : H^{p+q}(F^{p+1}C) \to H^{p+q}(C)\}} .$$

This proves (2) of the proposition.

By Proposition 5.1, $E_n^{p,q} \simeq k_1^{-1}(\mathrm{im}\ i_1^{n-1})/j_1(\ker\ i_1^{n-1})$. This fact is represented by the following diagram:

$$\begin{tikzcd}
 & & H^{p+q+1}(F^{p+n}C) \arrow[d, "i"] \\
 & & \vdots \arrow[d, "i"] \\
 & & H^{p+q+1}(F^{p+2}C) \arrow[d, "i"] \\
H^{p+q}(F^pC) \arrow[r, "j"] \arrow[d, "i"] & H^{p+q}(F^pC/F^{p+1}C) \arrow[r, "k"] & H^{p+q+1}(F^{p+1}C) \\
H^{p+q}(F^{p-1}C) \arrow[d] & & \\
\vdots \arrow[d, "i"] & & \\
H^{p+q}(F^{p-n+1}C) & &
\end{tikzcd}$$

If $p < 0$, then $F^pC/F^{p+1}C$ is the trivial cochain complex so that $H^{p+q}(F^pC/F^{p+1}C) = 0$. Thus, $E_n^{p,q} = 0$. If $q < 0$, then consideration of the cochain complex

$$F^pC/F^{p+1}C : (F^pC/F^{p+1}C)^0 \to \cdots \to (F^pC/F^{p+1}C)^{p+q} = 0 \to \cdots$$

shows immediately that $H^{p+q}(F^pC/F^{p+1}C) = 0$, and again $E_n^{p,q} = 0$. This completes the proof of the proposition. ▲

Proposition 5.3: If $\{E_n, d_n \mid n \geq 1\}$ is derived from $G = \{C, \delta, F\}$ with strictly convergent filtration F, then $H^*(C)$ is filtered by E_∞ by defining $F^rH^n(C) = \text{im}\,\{i: H^n(F^rC) \to H^n(C)\}$, so that

$$H^n(C) = F^0H^n(C) \supset F^1H^n(C) \supset \cdots \supset F^nH^n(C) \supset F^{n+1}H^n(C) = 0$$

is a finite sequence of submodules with

$$F^rH^n(C)/F^{r+1}H^n(C) \approx E_\infty^{r,n-r}$$

for $n \geq r \geq 0$.

It should be noted that the above filtration for $H^*(C)$ enables one to compute the cohomology of C, up to module extension, in terms of E_∞.

Pairing of Spectral Sequences

Let N be a subgroup of a group π, and let P be a π-module. Define a decreasing filtration $\tilde{F}$ of the cochain complex $C = \text{Hom}_{Z(\pi)}(B(\pi), P)$ as follows: $\tilde{F}^pC = C$ for $p \leq 0$. For $p > 0$, one sets $0 = \tilde{F}^pC^n$ if $p > n$, and for $p \leq n$ define $\tilde{F}^pC^n$ to be the group of all elements $f \in C^n$ for which $f(\gamma_1, \cdots, \gamma_n) = 0$ whenever $n-p+1$ of the arguments belong to the subgroup N. Evidently, $\tilde{F}$ is a strictly convergent decreasing filtration for C.

Let P, Q, S be π-modules such that P and Q are paired with respect to S by the $Z(\pi)$-pairing θ (see II.1). Let $C(P) = \text{Hom}_{Z(\pi)}(B(\pi), P)$, $C(Q) = \text{Hom}_{Z(\pi)}(B(\pi), Q)$, $C(S) = \text{Hom}_{Z(\pi)}(B(\pi), S)$. Then the $\smile_\theta$-product at the cochain level

$$\smile_\theta : C^p(P) \otimes C^q(Q) \to C^{p+q}(S)$$

is defined explicitly by the formula

$$(f \smile_\theta g)(\gamma_1,\cdots,\gamma_{p+q}) = \theta(f(\gamma_1,\cdots,\gamma_p) \otimes \gamma_1\cdots\gamma_p g(\gamma_{p+1},\cdots,\gamma_{p+q})) \quad ,$$

for $f \in C^p(P)$ and $g \in C^q(Q)$. Furthermore, the filtration $\tilde{F}$ is <u>compatible with cup products</u> in the sense that if the complexes $C(P)$, $C(Q)$, $C(S)$ are filtered by $\tilde{F}$ described above, then

$$\smile_\theta : \tilde{F}^p C(P) \otimes \tilde{F}^q C(Q) \to \tilde{F}^{p+q} C(S) \quad .$$

In particular, suppose $f \in \tilde{F}^p C^{p+r}(P)$, $g \in \tilde{F}^q C^{q+s}(Q)$, and $r+s+1$ of the elements $\gamma_1,\cdots,\gamma_{p+r+q+s}$ are in N. Then $(f \smile_\theta g)(\gamma_1,\cdots,\gamma_{p+r+q+s}) = 0$, because if at least $r+1$ of the elements $\gamma_1,\cdots,\gamma_{p+r}$ are in N then $f(\gamma_1,\cdots,\gamma_{p+r}) = 0$; if the listing $\gamma_1,\cdots,\gamma_{p+r}$ contains less than $r+1$ elements of N, then at least $s+1$ of the elements $\gamma_{p+r+1},\cdots,\gamma_{p+r+q+s}$ are in N and it follows that $g(\gamma_{p+r+1},\cdots,\gamma_{p+r+q+s}) = 0$. Therefore, $f \smile_\theta g \in F^{p+q} C^{p+r+q+s}(S)$.

Let $\{E_r, d_r \mid r \geq 1\}$, $\{E_r', d_r' \mid r \geq 1\}$, $\{E_r'', d_r'' \mid r \geq 1\}$ denote the spectral sequences derived from $G = \{C(P), \delta_P, F_P\}$, $G' = \{C(Q), \delta_Q, F_Q\}$, $G'' = \{C(S), \delta_S, F_S\}$, respectively. Since $\smile_\theta$ satisfies the usual coboundary formula, it induces a pairing

$$\smile : E_r^{p,q}(P) \otimes E_r^{p',q'}(Q) \to E_r^{p+p',q+q'}(S)$$

defined by $(\alpha \otimes \beta) = \overline{f \smile_\theta g}$, where $\alpha = \overline{f} \in E_r^{p,q}$ and $\beta = \overline{g} \in E_r^{p',q'}$. In notation, $\smile(\alpha \otimes \beta) = \alpha \smile \beta$.

Hochschild-Serre Spectral Sequence

In the case where N is normal in π, the Hochschild-Serre spectral sequence is introduced by defining a second filtration F of the cochain complex $C = \mathrm{Hom}_{Z(\pi)}(B(\pi), P)$. The filtration F has the defect of not being compatible with cup product but it is most useful in computations.

Again define $F^pC = C$ for $p \leq 0$. For $p > 0$, set $F^pC^n = 0$ if $p > n$ and for $p \leq n$ define F^pC^n to be the group of all elements $f \in C^n$ for which $f(\gamma_1,\cdots,\gamma_n)$ depends only on $\gamma_1,\cdots,\gamma_{n-p}$ and the cosets $\overline{\gamma}_{n-p+1},\cdots,\overline{\gamma}_n$. It is easy to see that F^pC is a subcomplex of $\tilde{F}^pC$ for all p.

Proposition 5.4: If E_r, $\tilde{E}_r$ denote the terms of the spectral sequences derived from the filtrations F, $\tilde{F}$, respectively, then the injections $i:F^pC \to \tilde{F}^pC$ induce isomorphisms $E_r \simeq \tilde{E}_r$, for each $r \geq 1$.

Proof: See Proposition 1 in [19].

If P is a π-module, the cochain isomorphism $\mathrm{Hom}_{Z(\pi)}(B(\pi),P) \simeq \mathrm{Hom}_{Z(\pi)}(\overline{B}(\pi),P)$ has previously been calculated (see II.4). For fixed $p \geq 0$, let $\mathfrak{X}^p = \mathrm{Hom}_Z(\overline{B}_p(\pi/N),\mathrm{Hom}_Z(\overline{B}(N),P))$ be the cochain complex

$$\mathfrak{X}^p : X^{p,0} \overset{\delta^0}{\to} X^{p,1} \to \cdots \to X^{p,q} \overset{\delta^q}{\to} X^{p,q+1} \to \cdots$$

with $X^{p,q} = \mathrm{hom}_Z(\overline{B}_p(\pi/N),\mathrm{hom}_Z(\overline{B}_q(N),P))$, and $\delta f = \delta_N f$, for $f \in \mathfrak{X}^p$, where δ_N is the coboundary operator for $\mathrm{Hom}_Z(\overline{B}(N),P)$. Define the cochain map $r_p:F^pC \to \mathfrak{X}^p$ by

$$(r_pf)(\overline{g}_1,\cdots,\overline{g}_p)(h_1,\cdots,h_q) = f(h_1,\cdots,h_q,g_1,\cdots,g_p)$$

for $f \in F^pC^{p+q}$ where $g_i \in \pi$ is a representative of the coset $\overline{g}_i$ and $h_j \in N$. If $\Psi \in F^{p+1}C^{p+q}$ then

$$(r_p\Psi)(\overline{g}_1,\cdots,\overline{g}_p)(h_1,\cdots,h_q) = \Psi(h_1,\cdots,h_q,g_1,\cdots,g_p) = 0$$

because $\Psi(h_1,\cdots,h_q,g_1,\cdots,g_p)$ depends only on $h_1,\cdots,h_{q-1}$ and the cosets $\overline{h}_q = \overline{0}$, $\overline{g}_1,\cdots,\overline{g}_p$. Thus the restriction of r_p to the subcomplex $F^{p+1}C \subset F^pC$ is the trivial map. Hence r_p induces a cochain map $R_p:F^pC/F^{p+1}C \to \mathfrak{X}^p$ defined by $R_p(\overline{f}) = \overline{r_p(f)}$, $f \in F^pC/F^{p+1}$, and consequently a homomorphism $\Phi : H^{p+q}(F^pC/F^{p+1}C) \to H^q(\mathfrak{X}^p)$. In fact, it is shown in [19] that Φ is an isomorphism.

Now the claim is that $H^q(\mathfrak{X}^p) \simeq \hom_{Z(\pi/N)}(B_p(\pi/N), H^q(N,P))$. Since $Z(\pi) \simeq Z(N) \otimes Z(\pi/N)$, $B(\pi)$ is a projective resolution of the trivial N-module Z. By pull-back along the injection $i : N \to \pi$, ${}_iP$ is a N-module. It follows that

$$H^*(N,P) \simeq H^*(\mathrm{Hom}_{Z(N)}(B(\pi),P)) \quad .$$

However, $\mathrm{Hom}_{Z(N)}(N(\pi),P)$ is a π/N-cochain complex by defining $(\bar{g}\Psi)(x) = g\Psi(g^{-1}x)$, for $\Psi \in \mathrm{Hom}_{Z(N)}(B(\pi),P)$, $\bar{g} \in \pi/N$, $x \in B(\pi)$. Hence $H^*(N,P)$ is a π/N-module. Thus

$$\begin{aligned} H^q(\mathfrak{X}^p) &= \hom_Z(\bar{B}_p(\pi/N), H^q(\hom_Z(\bar{B}(N),P))) \\ &= \hom_Z(\bar{B}_p(\pi/N), H^q(N,P)) \\ &\simeq \hom_{Z(\pi/N)}(B_p(\pi/N), H^q(N,P)) \quad . \end{aligned}$$

<u>Theorem 5.1:</u> There exists an isomorphism

$$\Phi : E_1^{p,q} \simeq \hom_{Z(\pi/N)}(B_p(\pi/N), H^q(N,P)) \quad ,$$

where Φ is induced by the cochain map $r_p : F^pC \to \mathfrak{X}^p$.

An investigation of $E_2^{p,q}$ is now made. Consider the diagram

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{\Phi} & \hom_{Z(\pi/N)}(B_p(\pi/N), H^q(N,P) \\ {\scriptstyle d_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta_{\pi/N}} \\ E_1^{p+1,q} & \xrightarrow{\Phi} & \hom_{Z(\pi/N)}(B_{p+1}(\pi/N), H^q(N,P)) \end{array}$$

where d_1 is the 1-st differential of the Hochschild-Serre spectral sequence and $\delta_{\pi/N}$ is the coboundary operator of the cochain complex $\mathrm{Hom}_{Z(\pi/N)}(B(\pi/N), H^q(N,P))$ induced by the boundary of $B(\pi/N)$. It is shown in [5] that

$$\Phi d_1 = (-1)^q \delta_{\pi/N}\Phi \quad .$$

The following is then readily verified.

Theorem 5.2: The isomorphism Φ of E_1 onto $\mathrm{Hom}_{Z(\pi/N)}(B(\pi/N), H^*(N,P))$ induces an isomorphism

$$E_2^{p,q} \simeq H^p(\pi/N, H^q(N,P)) \quad .$$

Some Theorems Involving E

First recall a few facts about the maps restriction and inflation. If N is a subgroup of π and P is a π-module, the injection $\kappa: N \to \pi$ induces a homomorphism

$$\mathrm{res}: H^n(\pi,P) \to H^n(N,P)$$

called restriction. If N is normal in π, P^N (see II.4) is a π/N-module. The projection $\sigma: \pi \to \pi/N$ and the injection $j: P^N \to P$ together induce a homomorphism

$$\mathrm{inf}: H^n(\pi/N, P^N) \to H^n(\pi,P)$$

called inflation. Furthermore, for N normal in π, the image of restriction lies in the π/N-module $H^n(N,P)^{\pi/N}$ (see V.4).

In the Hochschild-Serre spectral sequence, the edge terms are

$$E_2^{p,0} \simeq H^p(\pi/N, H^0(N,P)) \simeq H^p(\pi/N, P^N) \quad ,$$

$$E_2^{0,q} \simeq H^0(\pi/N, H^q(N,P)) \simeq H^q(N,P)^{\pi/N}$$

and

$$E_1^{0,q} \simeq \hom_{Z(\pi/N)}(B_0(\pi/N), H^q(N,P)) \simeq H^q(N,P) \quad .$$

There exist maps

$$H^n(\pi,P) \to E_\infty^{0,n} \to E_2^{0,n}$$

with the first map epic, the second map monic, and the composition is the restriction map. There are also maps

$$E_2^{n,0} \to E_\infty^{n,0} \to H^n(\pi,P)$$

with the first map epic, the second map monic, and the composition is the inflation map.

Let $\{E_n, d_n \mid n \geq 1\}$ be the spectral sequence derived from $G = \{C,\delta,F\}$ with strictly convergent filtration F. Associated with the short exact sequence

$$0 \to F^1C \to C \to C/F^1C \to 0$$

one has the following commutative diagram with exact rows and exact columns ($q > 1$):

$$\begin{tikzcd}
\cdots \arrow[r] & D_1^{0,q-1} \arrow[r, "j_1"] \arrow[dd, "\sigma"] & E_1^{0,q-1} \arrow[rr, "k_1"] & & D_1^{1,q-1} \arrow[rr, "i_1"] & & D_1^{0,q} \arrow[r] & \cdots \\
& & & & & E_2^{q,0} \arrow[ul, "\gamma = \mu e_B"'] \arrow[dl, "e_B"] & & \\
0 \arrow[r] & E_{q+1}^{0,q-1} \arrow[r, "h"] \arrow[dd] & E_q^{0,q-1} \arrow[rr, "d_q"] \arrow[uu, "e_F"'] & & E_q^{q,0} \arrow[rr, "\xi"] \arrow[uu, "\mu"'] \arrow[dl] & & E_{q+1}^{q,0} \arrow[r] \arrow[uu, "\tau"'] & 0 \\
& & & 0 & & & & \\
& 0 & 0 \arrow[uu] & & 0 \arrow[uu] & & 0 \arrow[uu] &
\end{tikzcd}$$

where the maps σ, e_B, ξ are projections, the maps h, e_F, μ, τ are injections, and d_q is the q-th differential of the spectral sequence.

Let

$$T = \{x \in E_1^{0,q-1} \mid k_1(x) \in \text{im } \gamma\} \quad .$$

The elements of T are called transgressive elements of $E_1^{0,q-1}$. Note that T is a submodule of $E_1^{0,q-1}$ and that $T = \text{im } e_F$.

<u>Definition 5.2</u>: A transgression t is a homomorphism $t: T \to E_2^{q,0}/\ker \gamma$ such that $t(x) = \overline{\gamma^{-1}k_1(x)}$ for all transgressive elements x.

Since $\gamma = \mu e_B$, $T = \text{im } e_F$, and $f: E_2^{q,0}/\ker \gamma \to E_q^{q,0}$ defined by $f(\overline{y}) = e_B(y)$ is an isomorphism, a transgression t is a homomorphism $t: T \to E_q^{q,0}$ satisfying $t(x) = d_q(z)$, where $z \in E_q^{0,q-1}$ is such that $e_F(z) = x$. Thus, a transgression t is essentially the q-th differential $d_q: E_q^{0,q-1} \to E_q^{q,0}$.

The material introduced in this section can be combined to get the following theorems for an analysis of $H^*(\pi,P)$. Proofs of the theorems are straightforward and can be found in [5], [19], or [22].

The following result is the well known <u>decomposition theorem</u>.

<u>Theorem 5.3</u>: If N is a normal subgroup of the finite group π with index $p = [\pi:N]$ relatively prime to its order $q = [N:1]$, then for each π-module P and each $n > 0$, there is a split exact sequence

$$0 \to H^n(\pi/N,P^N) \xrightarrow{\text{inf}} H^n(\pi,P) \xrightarrow{\text{res}} H^n(N,P)^{\pi/N} \to 0$$

which gives an isomorphism $H^n(\pi,P) \simeq H^n(\pi/N,P^N) \oplus H^n(N,P)^{\pi/N}$. Moreover, this decomposition is multiplicative with respect to cup products.

Multiplicative with respect to cup products has the following meaning. Choose integers a and b such that $ap + bq = 1$. If $x \in H^n(\pi,P)$, set $\alpha(x) = apx$, $\beta(x) = bqx$, so that $x = \alpha(x) + \beta(x)$. If P is a $Z(\pi)$-module algebra (see II.1), $x \in H^n(\pi,P)$, $y \in H^m(\pi,P)$, then

$$x \cup y = \alpha(x) \cup \alpha(y) + \beta(x) \cup \beta(y) \quad ,$$

or

$$\alpha(x \cup y) = \alpha(x) \cup \alpha(y)$$

and

$$\beta(x \cup y) = \beta(x) \cup \beta(y) \quad .$$

If $m \geq 1$ and $H^n(N,P) = 0$ for $0 < n < m$, then $\inf: H^n(\pi/N,P^N) \to H^n(\pi,P)$ is an isomorphism for $n < m$. Moreover, the transgression t in dimension m corresponds canonically to the $(m+1)$-differential d_{m+1} of the Hochschild-Serre spectral sequence. These observations yield <u>Serre's 5-term exact sequence</u>.

<u>Theorem 5.4</u>: Let N be normal in π and let P be a π-module. Let $m \geq 1$, and assume $H^n(N,P) = 0$ for $0 < n < m$. Then the 5-term sequence

$$0 \to H^m(\pi/N,P^N) \xrightarrow{\inf} H^m(\pi,P) \xrightarrow{\text{res}} H^m(N,P)^{\pi/N} \xrightarrow{t_m} H^{m+1}(\pi/N,P^N) \xrightarrow{\inf} H^{m+1}(\pi,P)$$

is exact.

Another 5-term exact sequence is given by

<u>Theorem 5.5</u>: Let $m \geq 1$ and assume that $H^n(N,P) = 0$ for $1 < n < m$. For $0 < n < m$ there is an exact sequence

$$H^n(\pi/N,P^N) \xrightarrow{\inf} H^n(\pi,P) \to H^{n-1}(\pi/N,H^1(\pi,P)) \to H^{n+1}(\pi/N,P^N) \xrightarrow{\inf} H^{n+1}(\pi,P) \quad .$$

The following form of the <u>cup product reduction theorem</u> of

Eilenberg-MacLane [11] is due to Hochschild-Serre [19].

Theorem 5.6: Let π be a group, N a normal subgroup of π which operates trivially on the π-module P. Let d_2' denote the homomorphism of $H^{n-1}(\pi/N,\hom_Z(N,P))$ into $H^{n+1}(\pi/N,P)$ which corresponds to $d_2:E_2^{n-1,1} \to E_2^{n+1,0}$. Let ζ be the characteristic class of the group extension

$$0 \to N/[N,N] \to \pi/[N,N] \to \pi/N \to 1 \qquad .$$

Then, for every $\alpha \in H^{n-1}(\pi/N,\hom_Z(N,P))$, $d_2(\alpha) = -\zeta \cup \alpha$.

Given the hypothesis of Theorem 5.6, in principle it should be possible to compute the cohomology ring $H^*(\pi,P)$. Practically, however, this seems to be impossible in general.

The computation of E_2 by Hochschild-Serre shows that E_2 depends only on (A) the groups N and π/N, and (B) the structure of $H^*(N,P)$ as a π/N-module. Thus E_2 is a rather crude approximation to $H^*(\pi,P)$. Charlap and Vasquez [6] determined the 2-nd differential d_2 (and hence E_3) of the Hochschild-Serre spectral sequence and showed that it depends not only on (A) and (B), but also on a characteristic class $\alpha \in H^2(\pi/N,N)$. The following discussion is a generalization of their work.

Let N be a subgroup of π and let P be a π-module such that $P^N = P$. Then P, $N/[N,N]$, and $H^1(N/[N,N],P)$ can all be considered as π/N-modules. For $f \in H^1(N/[N,N],P)$, $\overline{x} \in N/[N,N]$, the map

$$\theta:H^1(N/[N,N],P) \otimes N/[N,N] \to P$$

defined by $\theta(f \otimes \overline{x}) = f(\overline{x})$ is a $Z(\pi/N)$-pairing. Thus cup product is a pairing (see V.3)

$$\smile : E_r^{p,q}(H^1(N/[N,N],P)) \otimes E_r^{p,q}(N/[N,N]) \to E_r^{p+p,q+q}(P) \quad .$$

By Theorem 5.2,

$$E_2^{m-1,1}(P) \approx H^{m-1}(\pi/N, H^1(N/[N,N],P)) \quad ,$$

and

$$E_2^{m-1,0}(H^1(N/[N,N],P)) \approx H^{m-1}(\pi/N, H^0(N/[N,N], H^1(N/[N,N],P))).$$

Consequently, if τ is the isomorphism

$$\tau : E_2^{m-1,1}(P) \to E_2^{m-1,0}(H^1(N/[N,N],P))$$

then $d_2 : E_2^{m-1,1}(P) \to E_2^{m+1,0}(P)$ satisfies the property

$$d_2(\alpha) = (-1)^m \tau(\alpha) \smile \zeta \quad ,$$

for

$$\alpha \in E_2^{m-1,1}(P) \quad ,$$

where ζ is the characteristic class of the group extension

$$0 \to N/[N,N] \to \pi/[N,N] \to \pi/N \to 0 \quad .$$

CHAPTER VI

FURTHER COMPUTATIONS OF COHOMOLOGY ALGEBRAS FOR FINITE GROUPS

In this chapter detailed computations are made of the cohomology algebra of groups by employing the techniques developed in previous chapters.

Cohomology Algebra $H^*(Z_2,Z_2)$

In Chapter IV topological methods were used to compute the cohomology algebra $H^*(Z_2,Z_2)$. This section is devoted to a purely algebraic computation involving the Hochschild-Serre spectral sequence (see V.4).

Consider the group extension (see Definition 2.11)

$$E:0 \to Z \xrightarrow{2} Z \to Z_2 \to 1$$

where 2 maps Z onto 2Z in Z. By Proposition 2.5, there is only one Z_2-module structure for the coefficient module Z_2 and it is obtained by letting Z_2 act trivially on Z_2. By Proposition 2.3

$$H^n(Z,Z_2) = \begin{cases} Z_2 \ ; \ n = 0,1 \\ 0 \ ; \ \text{otherwise} \end{cases} .$$

Let ξ and η denote the generators of $H^1(Z,Z_2)$ and of $H^0(Z,Z_2)$, respectively. If $\{E_r, d_r \mid r \geq 1\}$ is the Hochschild-Serre spectral sequence associated with the group extension E, then $E_1^{p,q}$ is isomorphic to $\hom_Z(B_p(Z_2), H^q(Z,Z_2))$ by Theorem 5.1. Hence

$$E_1^{p,q} \simeq \begin{cases} \hom_Z(B_p(Z_2), Z_2(\eta)) & ; q = 0,\ p \geq 0 \\ \hom_Z(B_p(Z_2), Z_2(\xi)) & ; q = 1,\ p \geq 0 \\ 0 & ; \text{otherwise} \end{cases} .$$

Since $B_p(Z_2)$ is normalized, the generators of $E_1^{p,q}$ for $q = 0,1$ are given by maps

$$\iota_p(\underbrace{[t|t|\cdots|t]}_{\text{p-factors}}) = \eta$$

and

$$\tau_p(\underbrace{[t|t|\cdots|t]}_{\text{p-factors}}) = \xi$$

where t is the generator of Z_2. Hence

$$E_1^{p,q} \simeq \begin{cases} Z_2(\iota_p) & ; q = 0,\ p \geq 0 \\ Z_2(\tau_p) & ; q = 1,\ p \geq 0 \\ 0 & ; \text{otherwise} \end{cases} .$$

Consider the first differentials $d_1 : E_1^{p,0} \to E_1^{p+1,0}$ and $d_1 : E_1^{p,1} \to E_1^{p+1,1}$. Since

$$d_1(\iota_p)(\underbrace{[t|\cdots|t]}_{(p+1)\text{-factors}}) = t\iota_p([t|\cdots|t]) + (-1)^{p+1}\iota_p([t|\cdots|t]) = 0 \quad ,$$

and similarly $d_1(\tau_p) = 0$, all 1-st differentials d_1 are trivial. It follows that

$$E_2^{p,q} \simeq \begin{cases} Z_2([\iota_p]) & ; q = 0,\ p \geq 0 \\ Z_2([\tau_p]) & ; q = 1,\ p \geq 0 \\ 0 & ; \text{otherwise} \end{cases} .$$

By Theorem 5.2, $E_2^{p,q} \simeq H^p(Z_2, H^q(Z,Z_2))$, so that $H^p(Z_2,Z_2) \simeq Z_2$ for every $p \geq 0$. This is exactly what was calculated in Theorem 4.2 and Proposition 4.2.

Now investigate higher differentials of the spectral sequence. The claim is that d_r is trivial for $r \geq 3$. The sequences

$$E_r^{p-r,r} \xrightarrow{d_r} E_r^{p,1} \xrightarrow{d_r} E_r^{p+r,2-r}$$

and

$$E_r^{p-r,r-1} \xrightarrow{d_r} E_r^{p,0} \xrightarrow{d_r} E_r^{p+r,1-r} ,$$

give $E_r^{p-r,2-r} = E_r^{p-r,r-1} = 0$ for $r \geq 3$, and $E_r^{p-r,r} = E_r^{p+r,1-r} = 0$ for $r \geq 2$. Therefore $d_r = 0$ for $r \geq 3$, so that $E_3^{p,q} = E_\infty^{p,q}$. However, $H^{p+q}(Z,Z_2) = 0$ for $p+q \geq 2$ and, by Proposition 5.3,

$$F^pH^{p+q}(Z,Z_2)/F^{p+1}H^{p+q}(Z,Z_2) \simeq E_\infty^{p,q}$$

for $p+q \geq 2$. It follows that for $p \geq 0$, $d_2: E_2^{p,1} \to E_2^{p+2,0}$ maps the generator $[\tau_p]$ to the generator $[\iota_{p+2}]$.

Defining the pairing $\theta: Z_2(\xi) \otimes Z_2(\eta) \to Z_2(\xi)$ in the obvious way, one can define the cup product

$$\cup_\theta : E_2^{p,0} \otimes E_2^{0,1} \to E_2^{p,1}$$

by $[\iota_p] \cup [\tau_0]$. The equalities

$$
\begin{aligned}
(\iota_p \smile \tau_0)([t|\cdots|t[) &= \theta(\iota_p([t|\cdots|t]) \otimes \tau_0([\ ])) \\
&= \theta(\xi \otimes \eta) \\
&= \xi
\end{aligned}
$$
,

establish $[\iota_p] \smile [\tau_0] = [\tau_p]$.

Now let $Z_2(\eta)$ and $Z_2(\eta)$ be paired with respect to $Z_2(\eta)$ by the pairing $\theta':Z_2(\eta) \otimes Z_2(\eta) \to Z_2(\eta)$ with θ' defined by $\theta'(\eta \otimes \eta) = \eta$. Then

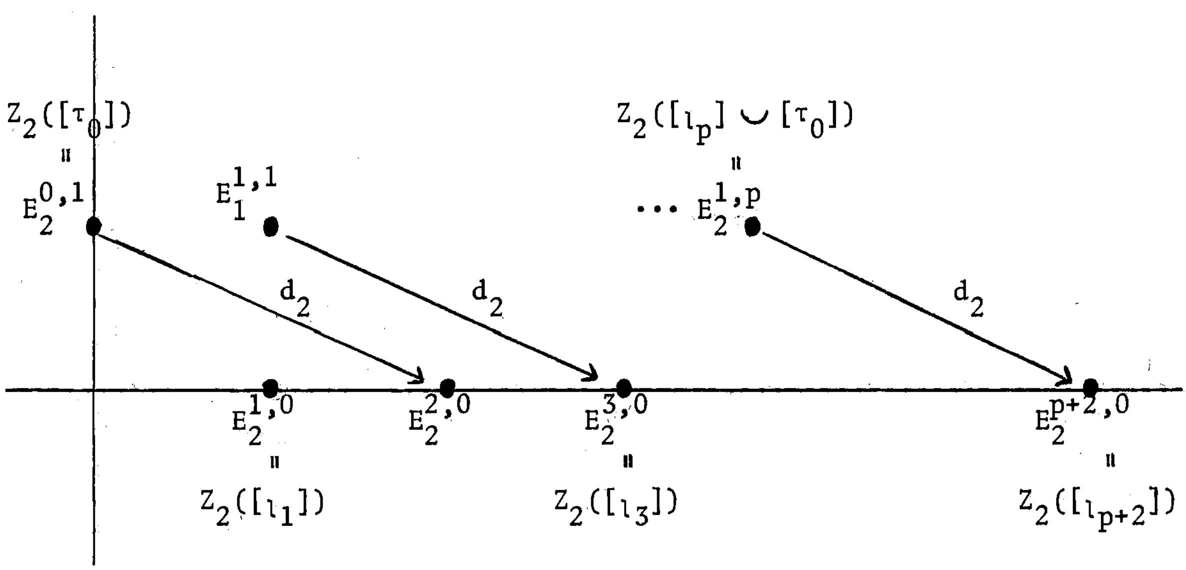
$$(\iota_1 \smile \iota_1)([t|t]) = \theta'(\iota_1([t]) \otimes \iota_1([t])) = \eta$$
,

and thus $[\iota_1]^2 = [\iota_2]$. Assume that $[\iota_1]^p = [\iota_p]$. Since cup product satisfies the coboundary formula (see II.3),

$$d_2([\iota_p] \smile [\tau_0]) = [\iota_p] \smile d_2([\tau_0]) = [\iota_p] \smile [\iota_2]$$
.

However, $[\iota_p] \smile [\tau_0] = [\tau_p]$ and $d_2([\tau_p]) = [\iota_{p+2}]$. Hence $[\iota_p] \smile [\iota_2] = [\iota_1]^p \smile [\iota_1]^2 = [\iota_1]^{p+2}$.

These facts are summarized in the following E_2-table:



Consequently, $H^*(Z_2,Z_2)$ is the polynomial algebra $P_{Z_2}[\eta]$ over Z_2 on one generator η of degree 1.

Cohomology Ring $H^*(Z_3,Z)$

In Chapter IV the cohomology ring $H^*(Z_3,Z)$ was computed using the Steenrod resolution Γ and the explicit formula for the Δ-chain map $h:\Gamma \to \Gamma \otimes \Gamma$. This ring is now calculated using the tools developed in Chapter V. It is noted here that the procedure used in the previous section cannot practically be applied to this seemingly simple case; the reason being that there are more maps than one can effectively work with.

Consider the group extension

$$E:0 \to 3Z \to Z \to Z_3 \to 0 \qquad ,$$

and let Z be a Z_3-module by letting Z_3 act trivially on Z. By Proposition 2.3,

$$H^n(Z,M) = \begin{cases} M \ ; \ n = 0,1 \\ \\ 0 \ ; \ \text{otherwise} \end{cases} \qquad ,$$

for any Z(Z)-module M.

First the module structure of $H^*(Z_3,Z)$ is computed by employing methods different from those used in Chapter IV. By Theorem 5.5, for all $m \geq 2$, one has the 5-term exact sequence

(1)

$$H^m(Z_3,Z) \to H^m(Z,Z) \to H^{m-1}(Z_3,H^1(3Z,Z)) \xrightarrow{d_2'} H^{m+1}(Z_3,Z) \to H^{m+1}(Z,Z) \quad .$$

Setting m = 2, the last four terms yield

$$0 \to H^1 \ (Z_3,Z) \to H^3(Z_3,Z) \to 0 \qquad .$$

Thus $H^1(Z_3,Z) \approx H^3(Z_3,Z)$. Setting $m = 4,6,\cdots,2n$ $(n \geq 2)$, one obtains

$$H^1(Z_3,Z) \simeq H^3(Z_3,Z) \simeq H^5(Z_3,Z) \simeq \cdots \simeq H^{2n+1}(Z_3,Z) \quad .$$

Similarly, by setting $m = 3,5,7,\cdots 2n+1$ $(n \geq 1)$, one obtains

$$H^2(Z_3,Z) \simeq H^4(Z_3,Z) \simeq H^6(Z_3,Z) \simeq \cdots \simeq H^{2n}(Z_3,Z) \quad .$$

By Proposition 2.3, $H^1(Z_3,Z) = \hom_Z(Z_3,Z) = 0$, so that $H^{2n+1}(Z_3,Z) = 0$ for all $n \geq 0$. Theorem 5.4 gives Serre's 5-term exact sequence

(2)

$$0 \to H^1(Z_3,Z) \xrightarrow{\text{inf}} H^1(Z,Z) \xrightarrow{\text{res}} H^1(3Z,Z)^{Z_3} \xrightarrow{d_2'} H^2(Z_3,Z) \xrightarrow{\text{inf}} H^2(Z,Z) \quad .$$

Since $H^1(Z_3,Z) = H^2(Z,Z) = 0$, $H^1(Z,Z) = \hom_Z(Z,Z)$, and $H^1(3Z,Z)^{Z_3} = H^1(3Z,Z) = \hom_Z(3Z,Z)$, then $H^2(Z_3,Z) \simeq Z_3$. Then the module structure of $H^*(Z_3,Z)$ is given by

$$H^n(Z_3,Z) \simeq \begin{cases} Z & ; n = 0 \\ Z_3 & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases} \quad .$$

Next observe that d_2 in the exact sequences (1), (2) is $d_2 : E_2^{m-1,1}(Z) \to E_2^{m+1,0}(Z)$ up to isomorphism for $m \geq 1$. Hence $d_2 : E_2^{0,1}(Z) \to E_2^{2,0}(Z)$ is epic, while $d_2 : E_2^{m-1,1}(Z) \to E_2^{m+1,0}(Z)$ is an isomorphism for $m > 1$. By the cup product reduction theorem (Theorem 5.6), $d_2(\beta) = \tau(\beta) \smile \alpha$ for any $\beta \in H^{m-1}(Z_3, H^1(3Z,Z)) = E_2^{m-1,1}(Z)$, where $\alpha \in H^2(Z_3,3Z) = E_2^{2,0}(3Z)$ is the characteristic class of the group extension E and

$$\tau : E_2^{m-1,1}(Z) \to E_2^{m-1,0}(H^1(3Z,Z)) = H^{m-1}(Z_3, H^0(3Z, H^1(3Z,Z)))$$

is the canonical isomorphism. Note that a pairing

$$\theta : H^1(3Z,Z) \otimes 3Z \to Z$$

defined by $\theta(f \otimes 3k) = f(3k)$ for $f \in H^1(3Z,Z) = \hom_Z(3Z,Z)$, is used in the cup product

$$\smile_\theta : E_2^{m-1,0}(H^1(3Z,Z)) \otimes E_2^{2,0}(3Z) \to E_2^{m+1,0}(Z) \quad .$$

Let γ be a generator of $E_2^{2,0}(Z) = Z_3$ such that $-3f = f'$ where f and f' are representative cocycles for γ and α, respectively. It is now shown that $d_2(\beta) = \gamma^2$ for a generator β of $E_2^{2,1}(Z)$. If g is a representative cocycle for β, then $\tau(\beta)$ is represented by $h \in \hom_Z(\overline{B}_2(Z_3), H^1(3Z,Z))$ such that $g(x) = h(x)([\ ])$ for any $x \in \overline{B}_2(Z_3)$. Since $\tau(\beta)$ is a generator of $E_2^{2,0}(H^1(3Z,Z))$, h can be chosen as a cocycle satisfying the property $h(x)([\ ])(3) = f(x)$ for any $x \in \overline{B}_2(Z_3)$. Then for x, $y \in \overline{B}_2(Z_3)$,

$$\begin{aligned}
-(h \smile f')(x \otimes y) &= -h(x)([\ ])(f'(y)) \\
&= -h(x)([\ ])(-3f(y)) \\
&= f(y) \cdot h(x)([\ ])(3) \\
&= f(y) \cdot f(x) \\
&= (f \smile f)(x \otimes y) \quad .
\end{aligned}$$

Hence $-\tau(\beta) \smile \alpha = \gamma^2$ so that γ^2 is a generator of $E_2^{4,0}(Z)$.

Let $\beta \in E_2^{4,1}(Z)$ be a generator and let h be a representative of $\tau(\beta) \in E_2^{4,0}(H^1(3Z,Z))$. Since γ^2 is a generator of $E_2^{4,0}(Z)$, h can be chosen as $h(x)([\ ])(3) = (f \smile f)(x)$ for $x \in \overline{B}_4(Z_3)$. Then for $x \in \overline{B}_4(Z_3)$ and $y \in \overline{B}_2(Z_3)$,

$$\begin{aligned}-(h \smile f')(x \otimes y) &= -h(x)([\ ])(f'(y)) \\ &= h(x)([\ ])(3f(y)) \\ &= f(y)\cdot h(x)([\ ])(3) \\ &= (f \smile f)(x)\cdot f(y) \\ &= (f \smile f \smile f)(x \otimes y) \quad .\end{aligned}$$

Hence $d_2(\beta) = -\tau(\beta) \smile \alpha = \gamma^3$, which is a generator of $E_2^{6,0}(Z)$.

Continuing this process, it is concluded that

$$H^*(Z_3,Z) \approx Z(\gamma^0) \oplus P_{Z_3}[\gamma]/(\gamma^0) \quad ,$$

where $P_{Z_3}[\gamma]$ is the polynomial algebra over Z_3 with one generator γ of degree 2.

Cohomology Ring $H^*(Z_3 \times Z_2,Z)$

It has previously been shown (see II.4) that

$$E: 0 \to Z_3 \xrightarrow{i} Z_3 \times Z_2 \xrightarrow{p} Z_2 \to 1$$

is exact, where $i(t) = (t,1)$ and $p(t,\sigma) = \sigma$ for generators t, σ of Z_3, Z_2, respectively. If $Z_3 \times Z_2$ acts trivially on Z, then because the index $2 = [Z_3 \times Z_2 : Z_3]$ is relatively prime to the order of Z_3, the decomposition theorem (Theorem 5.3), gives an isomorphism

$$H^n(Z_3 \times Z_2,Z) \approx H^n(Z_2,Z) \oplus H^n(Z_3,Z)^{Z_2} \quad .$$

In Chapter V the usual method of giving the cohomology modules $H^n(Z_3,Z)$ the structure of a Z_2-module was discussed. However, in general, this method is a very cumbersome one. The introduction of a procedure developed by Charlap-Vasquez [7] will allow the use of the Steenrod resolution (see IV.3) to give $H^*(Z_3,Z)$ a Z_2-module structure.

This method is discussed in more generality than is necessary for this particular case because the same technique will be used in the next section.

Let Z_r act on Z_s by $\sigma t = t^q$ where $q^r \equiv 1 \pmod{s}$. Here σ and t are the generators of multiplicative cyclic groups Z_r and Z_s of order r and s, respectively. Let

$$\alpha = 1 + t + \cdots + t^{q-1} \in Z(Z_s)$$

and let Γ

$$Z \xleftarrow{\varepsilon} X_0 \xleftarrow{\partial} X_1 \leftarrow \cdots \xleftarrow{\partial} X_n \leftarrow \cdots \quad :\Gamma$$

be the Steenrod resolution for the Z_s-module Z. Define

$$A_n : Z_r \to \hom_Z(X_n, X_n)$$

by

$$A_{2k}(\sigma)(1) = \alpha^k \ , \ A_{2k+1}(\sigma)(1) = \alpha^{k+1} \quad .$$

Define $A_n(\sigma^i)$ = i-fold iteration of $A_n(\sigma)$ for $0 \leq i < r$. For $f \in \mathrm{Hom}_{Z(Z_s)}(X_n, Z)$ define σf by

$$(\sigma f)(x_n) = \sigma(f(A_n(\sigma^{-1})x_n)) \quad .$$

Then this action induces an action of Z_r on $H^*(Z_s, Z)$; furthermore, this action coincides with the action defined in Chapter V. For a proof, see Proposition 2 in [7].

For the particular problem of this section, $Z_2 = \{1, \sigma\}$ acts on $Z_3 = \{1, t, t^2\}$ by $\sigma t = t$. Then $\alpha = 1$ and it follows easily that $H^n(Z_3, Z)^{Z_2} = H^n(Z_3, Z)$.

The same techniques previously used to calculate the cohomology ring

$H^*(Z_3,Z)$ (see IV.4 or VI.2) can be used to show $H^*(Z_2,Z)$ is $Z(\beta^0) \oplus P_{Z_2}[\beta]/(\beta^0)$, where $P_{Z_2}[\beta]$ is the polynomial algebra over Z_2 on one generator β of degree 2. Thus the module structure of $H^*(Z_3 \times Z_2,Z)$ is given by

$$H^n(Z_3 \times Z_2,Z) \simeq \begin{cases} Z & ; n = 0 \\ Z_2 \oplus Z_3 \simeq Z_6 & ; n \text{ even} \\ 0 & ; \text{otherwise} \end{cases} .$$

Since the decomposition given by Theorem 5.3 is multiplicative with respect to cup product, it follows that

$$H^*(Z_3 \times Z_2,Z) \simeq Z(\beta^0) \oplus (P_{Z_2}[\beta]/(\beta^0) \oplus P_{Z_3}[\alpha]/(\alpha^0))$$

where $P_{Z_2}[\beta]$ is the polynomial algebra over Z_2 on one generator β of degree 2, and $P_{Z_3}[\alpha]$ is the polynomial algebra on one generator α of degree 2 over Z_3.

Cohomology Ring $H^*(S_3,Z)$

Consider the short exact sequence

$$0 \to Z_3 \xrightarrow{i} S_3 \to Z_2 \to 1 ,$$

where S_3 denotes the symmetric group of degree 3 and

$$i(t) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

for the generator t of Z_3.

If S_3 acts on Z trivially, then because the index $2 = [S_3;Z_3]$ is relatively prime to the order of Z_3, the decomposition theorem (Theorem 5.3) gives an isomorphism

$$H^n(S_3,Z) \simeq H^n(Z_2,Z) \oplus H^n(Z_3,Z)^{Z_2} \quad .$$

Using the method of Charlap-Vasquez [7] discussed in the previous section, one has in this case $Z_2 = \{1,\sigma\}$ acting on $Z_3 = \{1,t,t^2\}$ by $\sigma t = t^2$. Then $A_n : Z_2 \to \hom_Z(X_n, X_n)$ is defined by

$$A_{2k}(\sigma)(1) = (1+t)^k \ , \ A_{2k+1}(\sigma)(1) = (1+t)^{k+1} \quad .$$

Since $H^n(Z_3,Z) = 0$ for odd n (see V.2), one need not consider the odd cases. First consider $H^{4k+2}(Z_3,Z)$, $k = 0,1,\cdots$. The generator τ of $H^{4k+2}(Z_3,Z)$ has coset representation $\overline{f}$ where $f \in \hom_{Z(Z_3)}(X_{4k+2},Z)$ is defined by $f(1) = 1$. Then $\sigma f : X_{4k+2} \to Z$ is defined by

$$(\sigma f)(1) = \sigma f(A_{4k+2}(\sigma)(1)) = f((1+t)^{2k+1}) = 2^{2k+1} \quad .$$

Now $(\sigma f - f)(1) = (\sigma f)(1) - f(1) = 2^{2k+1} - 1$, and 2^{2k+1} is not a multiple of 3. Since $\delta^{4k}(g)(1)$ is a multiple of 3 for all $g \in \hom_{Z(Z_3)}(X_{4k},Z)$, it follows that $\sigma f - f \notin \operatorname{im} \delta^{4k}$. Hence

$$H^{4k+2}(Z_3,Z)^{Z_2} = 0 \ , \ k = 0,1,2,\cdots \quad .$$

Next consider $H^{4k+4}(Z_3,Z)^{Z_2}$, for $k = 0,1,2,\cdots$. Again, the generator τ of $H^{4k+4}(Z_3,Z)$ has coset representation $\overline{f}$ where $f \in \hom_{Z(Z_3)}(X_{4k+4},Z)$ is defined by $f(1) = 1$. Then $\sigma f : X_{4k+4} \to Z$ is defined by

$$(\sigma f)(1) = \sigma f(A_{4k+4}(\sigma)(1)) = f((1+t)^{2k+2}) = 2^{2k+2} \quad ,$$

and $(\sigma f - f)(1) = 3m$ for some positive integer m. If one defines $g : X_{4k+3} \to Z$ by $g(1) = m$, then $\delta^{4k+3} g = \sigma f - f$. Whence $H^{4k+4}(Z_3,Z)^{Z_2} = H^{4k+4}(Z_3,Z)$, and the module structure for $H^*(S_3,Z)$ is given by

$$H^n(S_3,Z) \simeq \begin{cases} Z & ; n = 0 \\ Z_2 & ; n = 4k+2,\ k = 0,1,2, \\ Z_2 + Z_3 \simeq Z_6 & ; n = 4k+4,\ k = 0,1,2, \\ 0 & ; \text{otherwise} \end{cases}$$

Since the decomposition given by Theorem 5.3 is multiplicative with respect to cup product, it follows that

$$H^*(S_3,Z) \simeq Z(\beta^0) \oplus (P_{Z_2}[\beta]/(\beta^0) \oplus P_{Z_3}[\alpha]/(\alpha^0))$$

where $P_{Z_2}[\beta]$ is the polynomial algebra over Z_2 on one generator β of degree 2, and $P_{Z_3}[\alpha]$ is the polynomial algebra over Z_3 on one generator α of degree 4.

CHAPTER VII

CONCLUSION

This dissertation has centered around two main objectives, an investigation of the essential complexities involved in the cohomology theory of groups and the derivation of most of the known tools that facilitate the calculation of cohomology algebras. This chapter is concerned with proposed research topics and/or problems which are closely allied to the material presented in this dissertation.

As was shown in Chapter III, the cohomology of a group π can be interpreted as the cohomology of an arcwise-connected aspherical space with fundamental group π. It is therefore natural to utilize the previous work of Adams [1] and Uehara [28] in an investigation pertaining to the special features of the cohomology of a group which are induced by the behavior of the Steenrod p-th powers.

Chapters V and VI document the important role played by spectral sequences in the cohomology theory of groups. In order to increase the applicative utility of these sequences, it is proposed to determine whether or not the terms of the Lyndon spectral sequence [21] are isomorphic to those of the Hochschild-Serre spectral sequence [19].

In relation to the recent work in homology theory by Hilton-stammbach [17], it is proposed to dualize and extend to the cohomology case (with cup product) their study of higher differentials in the Lyndon and Hochschild-Serre spectral sequences associated with a group extension.

Calculation of the cohomology algebra of finite groups has occupied the central position in Chapters IV and VI. Wall [31], although successful in the homology case, could not determine the cohomology algebra of meta-cyclic groups. In his celebrated paper "Characters and Cohomology of Finite Groups", Atiyah [2], [3] considered relations between the integral cohomology ring $H^*(\pi,Z)$ and the ring $R(\pi)$ of unitary characters. In conjunction with this work, Wall [32] failed to compute the complete algebraic structure of the cohomology groups of $\pi = Z_4 \oplus Z_2$. It is therefore proposed to find more effective computational tools to facilitate the calculation of cohomology algebras of finite groups.

As an immediate step towards this goal, a specific problem to investigate the $Z(\pi)$-module structure of the cohomology of a $Z(\pi)$-module N for a group ring $Z(\pi)$ is proposed. In the author's opinion, in spite of efforts made by Charlap and Vasquez [7], this problem has remained unsettled to the extent that a general theory in relative homological algebra will be developed in order to fully find a proper solution for the problem.

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