STATE ESTIMATION OF REENTRY VEHICLES

FROM OPTICAL TRACKING DATA

By

VINOD KUMAR BHANDARI

Bachelor of Engineering
Birla Institute of Technology and Science
Pilani, Rajasthan, India
1970

Master of Science
Rutgers University
New Brunswick, New Jersey
1971

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
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Thesis Approved:

Edward L. Shreve
Thesis Adviser

Charles M. Bacon

Craig S. Sims

Henry R. Sebesta

D. D. Durham
Dean of the Graduate College

PREFACE

This study is concerned with estimating the state of a reentry vehicle using optical tracking data with the underlying objective of developing procedures that would be applicable to any nonlinear system.

The author wishes to express his sincere appreciation to his major adviser, Dr. Edward L. Shreve for suggesting the problem, providing generous assistance and spending very many hours of valuable time. Appreciation is also expressed to other committee members, Dr. Charles M. Bacon, Dr. Craig S. Sims and Dr. Henry R. Sebesta for their assistance during this study. The author also wishes to express his appreciation to Professor Paul A. McCollum and Dr. James R. Rowland for providing valuable discussions during various phases of this study. The support as Research Assistant from the Center for System Science and as Teaching Assistant from the School of Electrical Engineering is acknowledged. A note of thanks is given to Mrs. Dixie Jennings for her excellent typing of the final copy.

Finally, special gratitude is expressed to my parents and brothers for their understanding, encouragement, support and many sacrifices.

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NOMENCLATURE

α	Drag Parameter
β	Ballistic Coefficient
ω	Earth's Sidereal Rate
μ	Geodetic Latitude
γ	Angle Between Velocity Vector and Local Horizontal
Gm	Product of Universal Gravitational Constant and Mass of Earth
θ	Geodetic Longitude
ρ	Atmospheric Density at Height h
ρ_0	Atmospheric Density at Mean Sea Level (MSL)
g	Acceleration Due to Gravity at Altitude h
g_0	Acceleration Due to Gravity at Mean Sea Level
h	Altitude
a	Distance From Earth's Center to the Earth Surface Fixed Coordinate System
r	Distance From Earth's Center to the Reentry Vehicle
v	Velocity
h_s	Height of Earth Surface Fixed (ESF) Origin Above Mean Sea Level
R_e	Radius of Earth
ν	Round Off Error Parameter
ϕ_i	i-th Natural Basis Vector
$\delta(t)$	Dirac-Delta Function
δkj	Kronecker-Delta Function

CHAPTER I

INTRODUCTION

1.1 Literature Survey

Extensive research in the area of recursive estimation has been conducted for the last fifteen years. A significant contribution to the problem of optimal estimation of the state variables of a linear dynamical system was made by Kalman and Bucy (1,2). Since then extensive research in this area has been conducted. A large number of publications and reports give the extent of research and development conducted in this area. As a next logical step the concepts of linear filtering were extended to the estimation of the states of nonlinear systems using the extended Kalman Filter. In (3,4,5) different techniques, for example least-squares, maximum-likelihood, etcetera, were used to derive the filter equations. Most of these techniques employ Taylor series expansions, neglect second- and higher-order terms, and use linearized equations to compute the pseudo conditional error covariance matrix and the filter time-varying gains.

Another approach to determine the filtering equations is based on conditional probability density and conditional expectations. Stochastic Itô calculus is used to derive the filter equations. Stochastic Itô calculus is used to derive the filter equations. This technique has been used by Kushner (6), and Denham and Pines (7).

The second-order filtering technique used in this thesis has been presented by Athans et al. (8). Mehra (9) compared several nonlinear filters for reentry vehicle tracking using radar observations. Shreve and Bhandari (10) presented a comparison of the first- and second-order filter performance for reentry vehicle tracking using optical tracker observations.

The problems of divergence, adaptive estimation, and identification of variances have received recent attention in the literature. Typical papers on this line are those of Jazwinski (11), Schlee et al. (12), Mehra (13), and Aldrich and Krabill (14).

1.2 The Problem and the Approach

The problem of estimating the state of a ballistic reentry vehicle (RV) from optical tracker observations is a highly complex problem in nonlinear filtering. Because of the nature of the optical trackers, only the endoatmospheric observations data are available; consequently all the estimates are for endoatmospheric reentry. The objective of the present research is to develop a computer software package to generate estimates of the state of a reentry vehicle using triangulated optical tracking data. Estimated quantities are: position (x,y,z), velocity $(\dot{x},\dot{y},\dot{z})$, and the aerodynamic drag parameter (α).

The geometry of tracking of a RV using optical trackers can be explained using Figure 1. An earth surface fixed (ESF) cartesian coordinate system is shown here. There are $k(k \geq 2)$ optical trackers used for tracking. Each station gathers azimuth and elevation data as a function of time. Bodwell (15) has developed an algorithm for obtaining noisy position estimates using the angle data and the optical station coordinates. Given random properties of the optical trackers, that is,

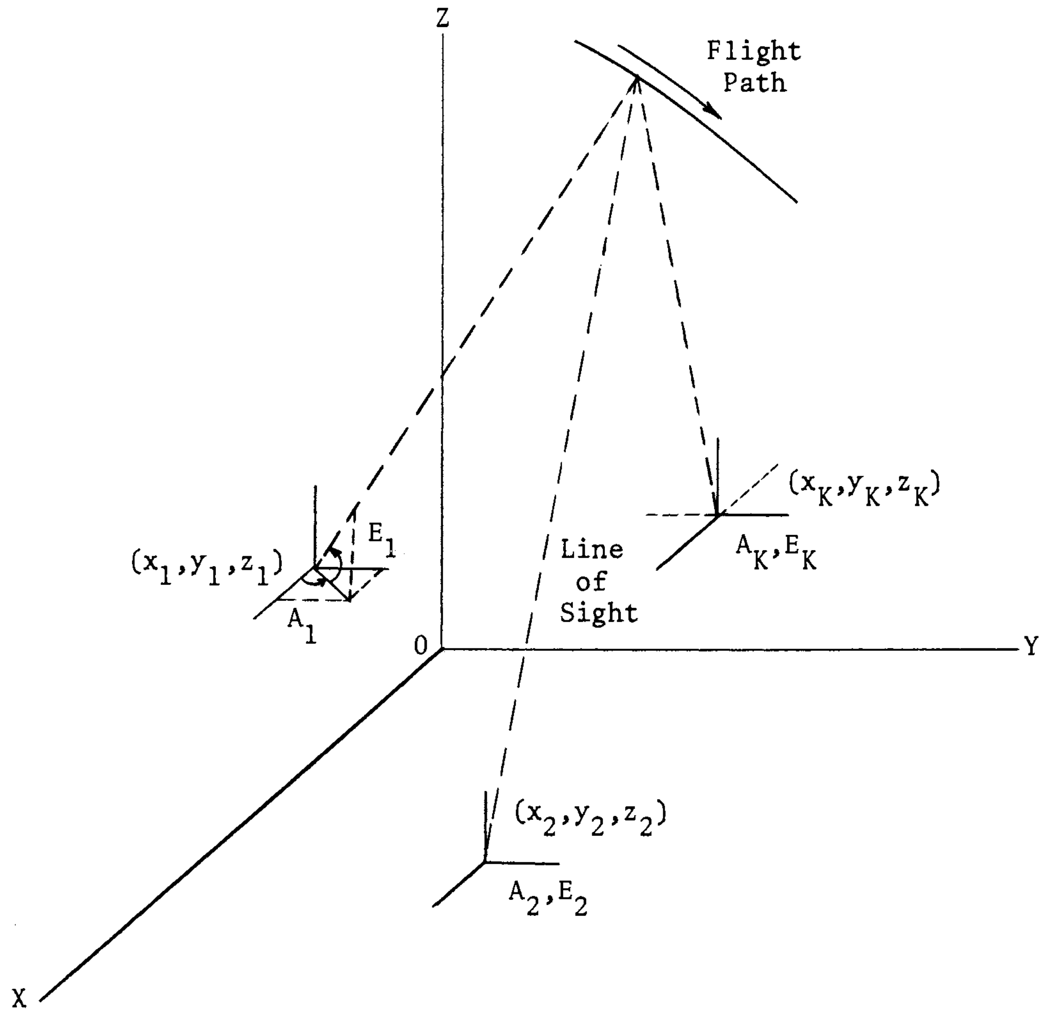

Figure 1. Tracking Network

variances of azimuth and elevation random errors, the covariance matrix associated with position estimates are obtained. For the present problem, the triangulated position observations and the corresponding error covariance matrices are available.

The objective then is to develop and implement a sequential algorithm which can be used to generate estimates of the state variables of a continuous nonlinear dynamical system from noisy observations of its output made at discrete instants of time. The motivation of this thesis was provided by the problems arising in the estimation of the state of a reentry vehicle: cartesian positions, velocities and ballistic drag parameter, using discrete optical tracker observations. The azimuth and elevation observations are triangulated to obtain position observations. Some of the most recent advances in estimation theory have been incorporated in the present development. The nonlinear dynamic model is approximated by retaining up to second-order terms in a Taylor's series expansion. In the present development what are called filtered and smoothed state estimation error covariance matrices are actually pseudo covariance matrices. The problem involves a continuous nonlinear dynamic model and a discrete linear observation model. The dynamic model for extrapolating the state of a RV is developed. This is expressed in an earth surface fixed cartesian coordinate system with x-y-z in the east-north-up directions, respectively.

Gravity and drag forces are included. The earth is assumed to be an oblate spheroid. The software package includes the capabilities of second-order filtering, and fixed-interval smoothing. The adaptive plant noise algorithm is included to solve the divergence problem.

Based on the observation error covariance matrices at observation instants, a set of error trajectories are generated by Monte Carlo techniques. When these trajectories are added to the nominal trajectory, a number of noisy observation position data sets are obtained. The performance of the software package is evaluated by processing the simulated observations with the same error covariance matrices. A double precision version of the program is used for increased accuracy of the computations over single precision. Initial values of the state estimate and estimation error covariance matrix are obtained by using a weighted-least-squares solution. The initializing technique is discussed in detail in Appendix A. The second-order filtering algorithm requires the evaluation of Jacobian and Hessian matrices of the dynamic model. The expressions for elements of these matrices are presented in Appendices B and C.

The software package inputs are triangulated position observations, observation error covariance matrices, atmospheric density model, earth and gravity parameters, and the coordinates of the ESF system origin. A description of input parameters is given in Appendix D. Appendix E briefly describes the purpose of various subroutines in the software package.

Figure 2 shows the flow chart of the software package. In this figure, k is the time index of the observation being processed, and NST is the total number of observations.

1.3 Organization

The structure of this thesis is as follows. Chapter II presents the development of the dynamic and observation models. The detailed derivation of the dynamic model (equations of motion) is included. A linear observation model is assumed.

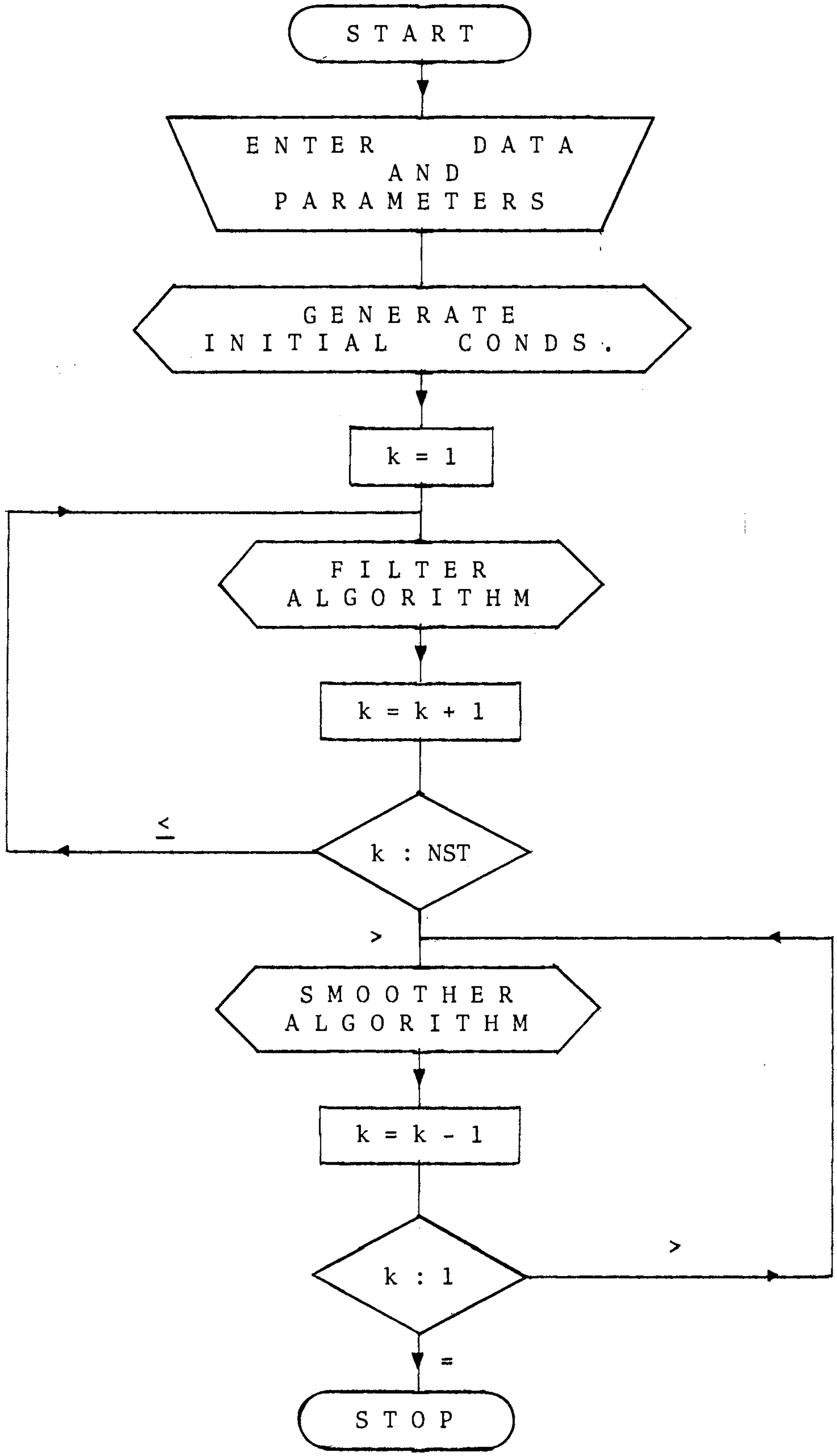

Figure 2. The Flow Chart of the Software Package

Chapter III describes the second-order filtering algorithm in the presence of plant noise. Plant noise is required to solve the problem of divergence. Different sources of plant noise and methods of dealing with the divergence problem are described in Chapter IV. In Chapter V, a fixed-interval smoothing algorithm for nonlinear systems is presented.

Chapter VI is devoted to simulation and numerical results. In this chapter filter and smoother results with and without plant noise are described. The software package consists of the filtering and smoothing algorithms, with plant noise. The performance of the package is tested for a variety of simulated noise samples. The results of statistical analysis are presented in Chapter VII. Chapter VIII contains a summary and conclusions of results obtained in the dissertation. Suggestions for further research and extensions are also included in this chapter.

CHAPTER II

DYNAMIC AND OBSERVATION MODELS

The dynamic model is used in the second-order filter to generate a priori (predicted) estimates of the state, and is also used in deriving the Jacobian and Hessian matrices described in the appendices. The noisy observation data are obtained by triangulating azimuth and elevation angles from a number of optical trackers. The two models are discussed in detail in this chapter.

2.1 Dynamic Model

The dynamic or message model is of the form

$$\dot{x}(t) = f(x(t)) + q(t) \tag{2-1}$$

where $f(\cdot)$ is a vector valued nonlinear function of the state $x(t)$, and $q(t)$ is the plant noise vector which is used to account for modeling and round off errors. The vector $q(t)$ is assumed to be a zero mean, Gaussian, white noise process. Basic assumptions in the dynamic model are: (1) observations of the reentry vehicle location are referenced to the earth surface fixed (x-y-z) system; (2) the reentry vehicle is a non-lifting point mass; (3) atmospheric density is modeled by

$$\rho = \rho_0 \exp(-kh)$$

where h is the height of the reentry vehicle above mean sea level, and

ρ_0 and k, obtained from density data, are constant over several ranges of altitude. This quantity is calculated in DENY subroutine mentioned in Appendix E; and (4) the earth is an oblate spheroid.

The dynamic model is derived here in a manner following the procedure in (16), in which range, azimuth and elevation (R,A,E) coordinates were used. The basis of the present derivation is to equate the reentry vehicle acceleration to the sum of the drag and gravitational specific forces divided by the mass of the vehicle. Identical results can be obtained using Lagrangian dynamics as in (17).

To derive the equations of motion, let $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ be the basis of an earth centered inertial frame, and $(\hat{u}'_1, \hat{u}'_2, \hat{u}'_3)$ be the basis of an x-y-z frame located on the earth's surface. The situation is shown in Figure 3.

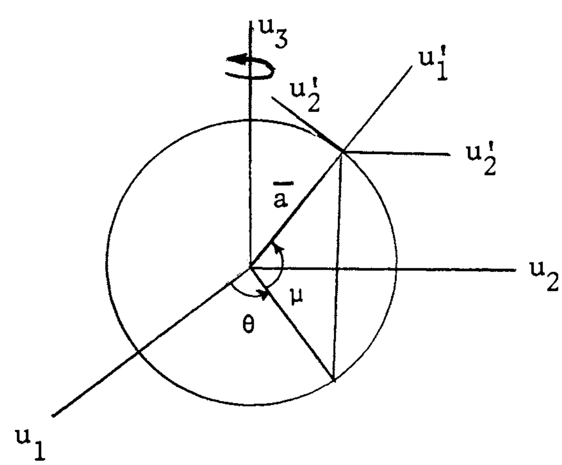

Figure 3. Relation Between Earth Surface Fixed and Earth Centered Inertial Frame

Let $\overline{R}$ be the vector to the reentry vehicle in the x-y-z system,

$$\overline{R} = x\hat{u}_1' + y\hat{u}_2' + z\hat{u}_3' \tag{2-2}$$

and if $\overline{a}$ is the vector from the earth's center to the primed origin

$$\overline{a} = a\cos\mu\cos\theta\hat{u}_1 + a\cos\mu\sin\theta\hat{u}_2 + a\sin\mu\hat{u}_3 \tag{2-3}$$

where

$$a = R_e + h_s$$

$$R_e = \text{earth's radius}$$

and

$$h_s = \text{height of ESF origin above mean sea level (MSL)} \quad .$$

In the inertial frame, the vector to the reentry vehicle is

$$\overline{r} = \overline{a} + T\overline{R} \tag{2-4}$$

where T is the transformation matrix from the primed to the unprimed system. To determine T, first rotate the primed system about the u_1' axis in a counterclockwise direction by $\pi/2 - \mu$; the resultant system is given by

$$\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\mu & -\cos\mu \\ 0 & \cos\mu & \sin\mu \end{bmatrix} \begin{bmatrix} \hat{u}_1' \\ \hat{u}_2' \\ \hat{u}_3' \end{bmatrix}$$

where

$$\mu = \text{geodetic latitude of ESF origin} \quad .$$

Now rotate the new $(\hat{i},\hat{j},\hat{k})$ system about the k axis in a clockwise direction by $\pi/2 + \theta$; the resultant coordinate system in terms of the original system is then just the unprimed, or

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\mu & -\cos\mu \\ 0 & \cos\mu & \sin\mu \end{bmatrix} \begin{bmatrix} \hat{u}_1' \\ \hat{u}_2' \\ \hat{u}_3' \end{bmatrix}$$

where

$$\theta = \text{geodetic longitude of ESF origin} \quad .$$

After multiplying the two matrices, it is seen that

$$T = \begin{bmatrix} -\sin\theta & -\sin\mu\cos\theta & \cos\theta\cos\mu \\ \cos\theta & -\sin\mu\sin\theta & \sin\theta\cos\mu \\ 0 & \cos\mu & \sin\mu \end{bmatrix} \quad . \tag{2-5}$$

The specific force equation is

$$\frac{\vec{F}}{m} = \ddot{\vec{r}}$$

or

$$\frac{\vec{F}}{m} = \ddot{\vec{a}} + T\ddot{\vec{R}} + 2\dot{T}\dot{\vec{R}} + \ddot{T}\vec{R} \tag{2-6}$$

where

$$\vec{R} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} , \quad \dot{\vec{R}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} , \quad \ddot{\vec{R}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}$$

$$\ddot{\vec{a}} = -a\omega^2\cos\mu \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$\theta = \omega t$$

F = force acting on the reentry vehicle

ω = mass of reentry vehicle and

$\ddot{\vec{r}}$ = acceleration of reentry vehicle .

T has been given above and $\dot{T}$ and $\ddot{T}$ are given below.

$$\dot{T} = \omega T \begin{bmatrix} 0 & -\sin\mu & \cos\mu \\ \sin\mu & 0 & 0 \\ -\cos\mu & 0 & 0 \end{bmatrix}$$

$$\ddot{T} = -\omega^2 T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\mu & -\sin\mu\cos\mu \\ 0 & -\sin\mu\cos\mu & \cos^2\mu \end{bmatrix} .$$

Combining the above

$$\frac{\vec{F}}{m} = -a\omega^2\cos\mu \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + T \begin{bmatrix} \ddot{x} - 2\omega\dot{y}\sin\mu + 2\omega\dot{z}\cos\mu - \omega^2 x \\ \ddot{y} + 2\omega\dot{x}\sin\mu - \omega^2 y\sin^2\mu + \omega^2 z\sin\mu\cos\mu \\ \ddot{z} - 2\omega\dot{x}\cos\mu + \omega^2 y\sin\mu\cos\mu - \omega^2 z\cos^2\mu \end{bmatrix} . \quad (2\text{-}7)$$

The forces acting on the reentry vehicle are gravity and drag forces. The lift forces are neglected. The specific forces are given by

$$\frac{\vec{F}}{m} = \frac{\vec{F}_g}{m} + \frac{\vec{F}_d}{m} \quad (2\text{-}8)$$

where $\overline{F}_g$ is the force due to gravity and $\vec{F}_d$ is due to drag. For the spherical earth

$$\frac{\vec{F}_g}{m} = -\frac{Gm}{|\vec{r}|^3}\vec{r} = -\frac{Gm}{|\vec{r}|^3}(\vec{a} + T\vec{R})$$

or

$$\frac{\vec{F}_g}{m} = -\frac{Gm}{|\vec{r}|^3}\left[a\begin{bmatrix}\cos\mu\cos\theta\\ \cos\mu\sin\theta\\ \sin\mu\end{bmatrix} + T\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right]$$

where

$$|\vec{r}|^2 = \vec{r}^T\vec{r} = (a + T\vec{R})^T(\vec{a} + T\vec{R}) = |\vec{a}|^2 + 2\vec{a}^T T\vec{R} + |\vec{R}|^2 = (z + a)^2 + x^2 + y^2$$

Gm = product of universal gravitational constant and earth's mass.

The drag force is

$$\frac{\vec{F}_d}{m} = -\frac{\frac{1}{2}g\rho|\vec{v}|\vec{v}}{\beta} = -\frac{1}{2}g\rho|\vec{v}|\vec{v}\alpha$$

where

g = acceleration due to gravity at altitude h

ρ = atmospheric density at altitude h

$$\alpha = \frac{1}{\beta}$$

β = ballistic coefficient .

In the earth surface fixed system, the velocity relative to air is

$$\vec{v} = T\dot{\vec{R}} = T\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}$$

and

$$|\vec{v}|^2 = (T\dot{\vec{R}})^T(T\dot{\vec{R}}) = \dot{\vec{R}}^T\dot{\vec{R}} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2 .$$

Thus

$$\frac{\vec{F}_d}{m} = -\frac{1}{2} g\rho\alpha(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} T \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} .$$

Combining the above results, the equations of motion become

$$\left.\begin{aligned}
\ddot{x} &= 2\omega\sin\mu\dot{y} - 2\omega\cos\mu\dot{z} + (\omega^2 - \frac{Gm}{r^3})x - \frac{1}{2} g\rho\alpha v\dot{x} \\
\ddot{y} &= -2\omega\sin\mu\dot{x} + (\omega^2\sin^2\mu - \frac{Gm}{r^3})y - \omega^2\sin\mu\cos\mu(z + a) - \frac{1}{2} g\rho\alpha v\dot{y} \\
\ddot{z} &= 2\omega\cos\mu\dot{x} - \omega^2\sin\mu\cos\mu y + (\omega^2\cos^2\mu - \frac{Gm}{r^3})(z + a) - \frac{1}{2} g\rho\alpha v\dot{z}
\end{aligned}\right] \quad (2\text{-}9)$$

where

ω = earth's sidereal rate

r = distance from earth's center to the reentry vehicle

$= (x^2 + y^2 + (z + a)^2)^{1/2}$

v = velocity = $(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$

$h = r - Re$

β = ballistic coefficient .

This model is used to derive the partial derivatives in the Jacobian and Hessian matrices. Compared to the nonspherical earth model, this approach significantly reduces program complexity and hence the program execution time. It does not however seriously affect the accuracy of its results. In state variable form, the equations of motion are

$$\dot{x}_1 = x_4 \quad (2\text{-}10)$$

$$\dot{x}_2 = x_5 \quad (2\text{-}11)$$

$$\dot{x}_3 = x_6 \tag{2-12}$$

$$\dot{x}_4 = 2\omega\sin\mu x_5 - 2\omega\cos\mu x_6 + (\omega^2 - \frac{Gm}{r^3})x_1 - \frac{1}{2}g\rho v x_4 x_7 \tag{2-13}$$

$$\dot{x}_5 = -2\omega\sin\mu x_4 - \omega^2\sin\mu\cos\mu(x_3 + a) + (\omega^2\sin^2\mu - \frac{Gm}{r^3})x_2 - \frac{1}{2}g\rho v x_5 x_7 \tag{2-14}$$

$$\dot{x}_6 = 2\omega\cos\mu x_4 - \omega^2\sin\mu\cos\mu x_2 - \frac{1}{2}g\rho v x_6 x_7 + (\omega^2\cos^2\mu - \frac{Gm}{r^3})(x_3 + a) \tag{2-15}$$

$$\dot{x}_7 = 0 \tag{2-16}$$

where

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \\ x_4 &= \dot{x} \\ x_5 &= \dot{y} \\ x_6 &= \dot{z} \\ x_7 &= \alpha = \frac{1}{\beta} \end{aligned}$$

The last differential equation assumes the ballistic coefficient is constant over one interval, but of course is updated as each data value is used to generate a new estimate of the seven element state vector. That is, in the prediction stage of the filter, the drag parameter remains unchanged. As new estimates of the state vector are generated, estimates of acceleration components are calculated using the expressions for $\dot{x}_4$, $\dot{x}_5$ and $\dot{x}_6$. A fourth-order Runge-Kutta subroutine is used to integrate the equations of motion between observations.

2.2 Observation Model

In the system being considered, the observation vector consists of noisy measurements of position, x_1, x_2 and x_3. These are obtained by triangulating azimuth and elevation angle data from a network of optical trackers. The linear observation sequence model can be described as

$$z_k = H_k x_k + v_k \tag{2-17}$$

where

$$\begin{aligned} z_k &= \text{observation vector at time (3x1)} \\ H_k &= \text{observation matrix (3x7)} \\ v_k &= \text{observation noise vector (3x1)} \end{aligned} \quad .$$

The observation matrix is

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{2-18}$$

The observation noise vector v_k is assumed to be a zero mean process whose covariance matrix is given by

$$R_k = \begin{bmatrix} \sigma^2_{x_k} & C_{xy_k} & C_{xz_k} \\ C_{xy_k} & \sigma^2_{y_k} & C_{yz_k} \\ C_{xz_k} & C_{yz_k} & \sigma^2_{z_k} \end{bmatrix} \quad . \tag{2-19}$$

The fact that the observation model is linear significantly reduces the complexity of the filter algorithm discussed in Chapter III.

2.3 Summary

The dynamic model for the propogation of the state of the reentry vehicle has been derived in this chapter. The assumptions and approximations made are stated. A linear observation model is adapted for the present study. This simplifies the filter algorithm which significantly reduces computation time.

CHAPTER III

SECOND-ORDER FILTER

3.1 Second-Order Filter With Plant Noise

The second-order filter as described in (8) is used in the program. However, modifications to allow dynamic model noise are required since the development in (8) assumes a noise free plant. The filter uses a continuous dynamic model and a discrete observation model. The following development is based on the assumption that the dynamic model (plant) noise is a zero mean, Gaussian, white noise process. The plant is described by

$$\dot{x}(t) = f(x(t)) + q(t); \; x(t_0) = x_0 \tag{3-1}$$

where $f(\cdot)$ is a nonlinear function of the state vector x, and q is the plant noise. Both x and f are seven component vectors.

The observation sequence $(k = 1,2,\cdots)$ is

$$z_k = h(x_k) + v_k$$

$$z_k = Hz_k + v_k \quad . \tag{3-2}$$

The observation noise v_k is a zero mean, Gaussian, white noise process independent of the initial state vector x_0, thus

$$E\{q(t)\} = 0 \quad \forall \, t \tag{3-3}$$

$$E\{v_k\} = 0 \qquad \forall\ k \tag{3-4}$$

$$E\{q(t)q^T(\tau)\} = Q(t)\delta(t - \tau) \tag{3-5}$$

$$E\{v_k v_j^T\} = R_k \delta_{kj} \quad . \tag{3-6}$$

Covariance matrices Q(t) and R_k are assumed known. E is the "expected value" operator.

Let $\hat{x}_k$ represent the state estimate at a given observation time t_k. Define the state error at t_k to be

$$e_k \triangleq x_k - \hat{x}_k \quad . \tag{3-7}$$

The associated error covariance matrix is then defined as

$$\Sigma_k \triangleq E\{e_k e_k^T\} \quad . \tag{3-8}$$

In the time interval $t_k \leq t < t_{k+1}$ there is no additional information until the next measurement occurs at t_{k+1}. Hence if the state equation were linear, it would be correct to estimate x(t), $t_k \leq t < t_{k+1}$, using c(t) as an estimate of the dynamical system, where

$$\dot{c}(t) = f(c(t))$$

which is the replica of the state dynamics of the plant. Because f(·) is nonlinear, this model is modified by including a vector valued function b(t) called a bias correction which will be specified so that the estimate c(t) of x(t) will be generated by

$$\dot{c}(t) = f(c(t)) + b(t);\ c(t_k) = x(t_k) \qquad t_k \leq t < t_{k+1} \quad . \tag{3-9}$$

Let e(t) denote the error during the above interval; that is

$$e(t) = x(t) - c(t);\ t_k \leq t < t_{k+1} \quad .$$

It follows that

$$e(t_k) = x_k - c(t_k) = x_k - \hat{x}_k = e_k \quad . \qquad (3\text{-}10)$$

From (3-1) and (3-9)

$$\dot{e}(t) = \dot{x}(t) - \dot{c}(t)$$

$$\dot{e}(t) = f(x(t)) - f(c(t)) - b(t) + q(t) \quad . \qquad (3\text{-}11)$$

If c(t) is "near" x(t), f(x(t)) can be expanded about c(t) using a Taylor series. Assume that by neglecting third- and higher-order terms in the Taylor series, a sufficiently accurate representation of the error dynamics is obtained. Thus

$$\dot{e}(t) = J(c(t))e(t) + \frac{1}{2}\sum_{i=1}^{7} \phi_i e^T(t)F_i(c(t))e(t) - b(t) + q(t)$$

$$\text{for } t_k \leq t < t_{k+1} \qquad (3\text{-}12)$$

where ϕ_i is the ith natural basis vector of the state space, J is the Jacobian matrix for the state model with the ijth element given by

$$[J(x_k)]_{ij} \triangleq \left.\frac{\partial f_i}{\partial x_j}\right|_{x_k} \qquad i,j = 1,2,\cdots,7$$

and F_i, the Hessian matrix for the ith row of the Jacobian matrix has its jkth element given by

$$[F_i(x_\ell)]_{jk} \triangleq \left.\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right|_{x_\ell} \qquad i,j,k = 1,2,\cdots,7 \quad .$$

Now use can be made of the mean argument to determine the vector b(t). Suppose $\hat{x}_k$ is an unbiased estimate of x_k, so that

$$E\{e(t_k)\} = 0 \quad .$$

c(t) is also required to be an unbiased estimate of x(t), so that

$$\left.\begin{aligned} E\{e(t)\} &= 0 \\ E\{\dot{e}(t)\} &= 0 \end{aligned}\right\} \quad \forall \quad t_k \leq t < t_{k+1} \quad .$$

By defining

$$S(t) = E\{e(t)e^T(t)\} \quad , \quad (3\text{-}13)$$

the expression for the bias correction is

$$b(t) = \frac{1}{2} \sum_{i=1}^{7} \phi_i \mathrm{tr}[F_i(c(t))S(t)] \quad . \quad (3\text{-}14)$$

Next a matrix differential equation which can be used to generate S(t) between observations is required. From the definition of the derivative

$$\dot{S}(t) = E\{e(t)\dot{e}^T(t) + \dot{e}(t)e^T(t)\}$$

or,

$$\dot{S}(t) = E\{J(c(t))e(t)e^T(t) + \frac{1}{2} (\sum_{i=1}^{7} \phi_i \mathrm{tr}[F_i(c(t))\{e(t)e^T(t) - S(t)\}])e^T(t)$$

$$+ q(t)e^T(t) + e(t)e^T(t)J^T(c(t)) + e(t)q^T(t)$$

$$+ e(t) \frac{1}{2} (\sum_{i=1}^{7} \phi_i \mathrm{tr}[F_i(c(t))\{e(t)e^T(t) - S(t)\}])\} \quad . \quad (3\text{-}15)$$

Now the assumption is made that e(t) is almost Gaussian with zero mean. In this case terms of the form

$$E\{\mathrm{tr}[F_i(c(t))e(t)e^T(t)]e^T(t)\} \simeq 0 \quad . \quad (3\text{-}16)$$

Hence the above differential equation reduces to

$$\dot{S}(t) = J(c(t))S(t) + S(t)J^T(c(t)) + E\{q(t)e^T(t)\} + E\{e(t)q^T(t)\}. \quad (3\text{-}17)$$

The last two terms in this equation are yet to be evaluated. In Equation (3-12)

$$b(t) = \frac{1}{2}\sum_{i=1}^{7} \phi_i \operatorname{tr}[F_i(c(t))S(t)] \quad ,$$

and the second term in (3-12) is

$$\frac{1}{2}\sum_{i=1}^{7} \phi_i e^T(t)F_i(c(t))e(t) \quad .$$

During the small time interval (t_k, t_{k+1}), $S(t)$ can be assumed to be an unbiased estimate of $e(t)e^T(t)$. Hence

$$S(t) = E\{e(t)e^T(t)\} = e(t)e^T(t) \quad .$$

This simplifies Equation (3-12) to

$$\dot{e}(t) = J(c(t))e(t) + Q(t) \quad . \quad (3\text{-}18)$$

The solution of this equation is

$$e(t) = \phi(t,t_0)e(t_0) + \int_{t_0}^{t} \phi(t,\tau)q(\tau)d\tau \quad (3\text{-}19)$$

where $\phi(t,t_0)$ is the state transition matrix of $J(c(t))$. Then

$$E\{e(t)q^T(t)\} = \phi(t,t_0)E\{e(t_0)q^T(t)\} + \int_{t_0}^{t} \phi(t,\tau)E\{q(\tau)q^T(t)\}d\tau \quad .$$

Since $e(t_0)$ and $q(t)$ are assumed independent, the preceding equation reduces to

$$E\{e(t)q^T(t)\} = \int_{t_0}^{t} \phi(t,\tau)Q(\tau)\delta(t - \tau)d\tau \quad . \quad (3\text{-}20)$$

Making use of the argument given in the appendix of (21), the above equation reduces to

$$E\{e(t)q^T(t)\} = \frac{1}{2} Q(t) = E\{q(t)e^T(t)\} \qquad . \ (3\text{-}21)$$

Since

$$\phi(t,t) = I \qquad ,$$

incorporating Equation (3-21) into (3-17) gives

$$\dot{S}(t) = J(c(t))S(t) + S(t)J^T(c(t)) + Q(t) \qquad . \ (3\text{-}22)$$

Equations (3-9) and (3-22) are integrated to propagate the state vector and error covariance matrix between observations. After propagating for t: $t_k \leq t < t_{k+1}$, the above integration gives c_{k+1} and S_{k+1}, the priori state and covariance approximation estimates. These quantities are updated based on the observation at t_{k+1}.

The Jacobian matrix for the observation model is

$$H = \frac{\partial h}{\partial x} \qquad (3\text{-}23)$$

where H is the constant matrix given by Equation (2-18), since the observation model is linear. The following expressions are based on the fact that the observation model is linear. The updated state estimate is given by

$$x_{k+1} = c_{k+1} + G_{k+1}[z_{k+1} - h(c_{k+1})] \qquad (3\text{-}24)$$

where z_{k+1} is the observation vector at t_{k+1}, and G_{k+1}, the gain matrix is given by

$$G_{k+1} = S_{k+1}H^T[HS_{k+1}H^T + R_{k+1}]^{-1} \qquad . \ (3\text{-}25)$$

R_{k+1} is the error covariance matrix associated with the observation vector z_{k+1}.

The update covariance matrix is given by

$$\Sigma_{k+1} = [I - G_{k+1}H]S_{k+1}[I - G_{k+1}H]^T + G_{k+1}R_{k+1}G_{k+1}^T \quad . \quad (3\text{-}26)$$

3.2 Summary

A derivation of the second-order filtering algorithm in the presence of dynamic model noise has been presented in this chapter. The message model is nonlinear and continuous whereas the observation mode is linear and discrete. A basic assumption made in the derivation is that the state error vector is a Gaussian, zero mean, white noise process. The effect of plant noise on the algorithm is that the differential Equation (3-22), propagating the state error covariance matrix has the additive term Q(t).

CHAPTER IV

PLANT NOISE

4.1 Introduction

For a nonlinear system subjected to Gaussian, stochastic driving functions with sampled measurements on the system corrupted by Gaussian errors, the estimate of system states can be accomplished by the recursive equations developed earlier. These equations lead to estimates of the system states and values for the estimation error covariances, provided the system is correctly and completely modeled, and the statistical parameters of the driving functions and errors known.

The filter algorithm assumes that the dynamic and observation models are completely known. The observation data consist of noisy position fixes and associated error covariance matrices.

During simulation, the second-order filter algorithm with no plant noise, exhibited divergence. By divergence is meant that the residuals (difference between observed and estimated state) keep increasing in either a positive or negative direction.

The cause of this can be traced as follows. At some stage of the filter operation, the state error covariance becomes quite small, causing the filter gain matrix to also become small. This in turn causes the difference between actual and expected observations to be weighed by a very small amount. The result is that incoming data are not reflected in the filter estimates, thereby causing divergence. A remedy to this problem is to increase the state error covariance matrix in some manner. This will cause the gain matrix to increase and alleviate the divergence problem.

One method of increasing the covariance matrix is accomplished by adding plant noise to the dynamic model. This noise is assumed to be a Gaussian, zero mean, white noise process. The modified second-order filtering algorithm has been derived in Chapter III.

4.2 Discrete Adaptive Plant Noise

Let the dynamic model be

$$x_{k+1} = \phi_{k+1,k} x_k + \Psi_{k+1,k} u_k + \omega_{k+1} \tag{4-1}$$

where

$$\begin{aligned} x_k &= \text{state vector (nxl)} \\ \phi_{k+1,k} &= \text{state transition matrix (nxn)} \\ u_k &= \text{uncertain parameter vector (pxl)} \\ \Psi_{k+1,k} &= \text{weighing matrix (nxp) and} \\ \omega_{k+1} &= \text{round off error vector (nxl)} \end{aligned} \quad .$$

Here a linear, discrete dynamic model is considered. In this section the effects of the presence of uncertain parameters u_k and round off errors ω_k on the linear filtering algorithm are studied. The results obtained are extended to a nonlinear, continuous dynamic model in the next section. The uncertain parameter vector u_k, contains those physical parameters whose values are not known exactly. The round off error vector, ω_k, accounts for the computational errors and is dependent on the machine and the complexity of the algorithm. A method of implementation of the algorithm developed here is given in Section 4.4.

Adjoining x_k and u_k vectors results in an n + pth order system

$$y_k = \begin{bmatrix} x_k \\ \hline u_k \end{bmatrix} \quad . \quad (4\text{-}2)$$

Let

$$\omega_k = 10^{-\nu} x_k$$

where ν, the round off error parameter, is selected by simulations and is dependent on the machine and complexity of the algorithm. Here it is assumed that the round off error vector is proportional to the magnitude of the state vector (12). ω_k is assumed to be a zero mean process. Round off errors in different components of the state vector are assumed independent. The covariance matrix associated with ω_k is

$$Q_{\omega k} = 10^{-2\nu} x_k x_k^T$$

with off diagonal terms set equal to zero. Round off errors and uncertain parameters are assumed to have no effect on the observation model. The dynamic model for uncertain parameter vector, u_k, is

$$u_{k+1} = u_k \quad ; \quad (4\text{-}3)$$

hence,

$$y_{k+1} = \left[\begin{array}{c|c} \phi_{k+1,k} & \Psi_{k+1,k} \\ \hline 0 & I \end{array}\right] y_k + \begin{bmatrix} \omega_{k+1} \\ \hline 0 \end{bmatrix} \quad . \quad (4\text{-}4)$$

The observation model is as before

$$z_k = H_k x_k + v_k \quad , \quad (4\text{-}5)$$

where

$$E[v_k v_j^T] = R_k \delta_{kj} \quad .$$

In terms of the augmented state vector, the observation equation becomes

$$z_k = [H_k \mid 0] y_k + v_k \quad . \quad (4\text{-}6)$$

The linear filtering algorithm is applicable to the system given by Equations (4-4) and (4-6). The filtered estimate is

$$y_{k/k} = \begin{bmatrix} x_{k/k} \\ \hline u_{k/k} \end{bmatrix} \quad ,$$

where the notation $x_{k/k}$ implies the estimate of the state x at time t_k given observation up to and including t_k. This is an n + pth order system and hence requires increased computations. An alternate method is to account for uncertain parameters without actually estimating these parameters. To accomplish this, u is assumed to be a zero mean process with covariance matrix given by

$$U = E\{[u - u_{k/k}][u - u_{k/k}]^T\} \quad .$$

The error covariance matrix for the augmented system is of the form

$$P = \left[\begin{array}{c|c} \Sigma & Cu \\ \hline Cu & U \end{array}\right]$$

where

$$Cu_{k/\ell} = E\{[x_k - x_{k/\ell}]u^T\} \quad ,$$

and

$$\Sigma_{k/\ell} = E\{[x_k - x_{k/\ell}][x_k - x_{k/\ell}]^T\} \quad .$$

From the linear filter algorithm the prediction of the state vector is given by

$$x_{k+1/k} = \phi_{k+1,k}x_k \tag{4-7}$$

and the covariance matrix is extrapolated by

$$P_{k+1/k} = E[y_{k+1/k}y^T_{k+1/k}]$$

$$= \left[\begin{array}{c|c} \phi_{k+1,k} & \Psi_{k+1,k} \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} \Sigma_{k/k} & Cu_{k/k} \\ \hline Cu_{k/k} & U_{k/k} \end{array}\right] \left[\begin{array}{c|c} \phi^T_{k+1,k} & 0 \\ \hline \Psi^T_{k+1,k} & I \end{array}\right]$$

$$+ \left[\begin{array}{c|c} Q\omega_{k+1} & 0 \\ \hline 0 & 0 \end{array}\right] \quad .$$

This yields

$$\Sigma_{k+1/k} = \phi_{k+1,k}\Sigma_{k/k}\phi^T_{k+1,k} + \Psi_{k+1,k}Cu_{k/k}\phi^T_{k+1,k}$$

$$+ \phi_{k+1,k}Cu^T_{k/k}\Psi^T_{k+1,k} + \Psi_{k+1,k}U_{k/k}\Psi^T_{k+1,k} + Q\omega_{k+1} \tag{4-8}$$

$$Cu_{k+1/k} = \phi_{k+1,k}Cu_{k/k} + \Psi_{k+1,k}U_{k/k} \tag{4-9}$$

$$U_{k+1/k} = U_{k/k} \quad . \tag{4-10}$$

The measurement matrix for the augmented system is

$$M_{k+1} = [H_{k+1} \mid 0]$$

so that the matrix to be inverted in the filter gain expression is

$$M_{k+1}P_{k+1/k}M^T_{k+1} + R_{k+1} = H_{k+1}\Sigma_{k+1/k}H^T_{k+1} + R_{k+1} \quad .$$

Now the filter estimates are given by

$$x_{k+1/k+1} = x_{k+1/k} + K_{k+1}[z_{k+1} - H_{k+1}x_{k+1/k}] \tag{4-11}$$

$$\Sigma_{k+1/k+1} = \Sigma_{k+1/k} - K_{k+1}H_{k+1}\Sigma_{k+1/k} \tag{4-12}$$

$$Cu_{k+1/k+1} = Cu_{k+1/k} - K_{k+1}H_{k+1}Cu_{k+1/k} \tag{4-13}$$

where

$$K_{k+1} = \Sigma_{k+1/k}H^T_{k+1}[H_{k+1}\Sigma_{k+1/k}H^T_{k+1} + R_{k+1}]^{-1} \quad . \tag{4-14}$$

4.3 Continuous Adaptive Plant Noise

Let the dynamic model be

$$\dot{x}(t) = f(x(t)) + d(u(t)) + \omega(t) \tag{4-15}$$

where $d(\cdot)$ is a vector valued function representing the effect of p uncertain parameters on the dynamic model, and is approximated by

$$d(u(t)) = \Psi u(t); \ \Psi = \left.\frac{\partial \dot{x}}{\partial u}\right|_{(x,u)} \quad .$$

An n component vector $\omega(t)$ is included to account for round off errors. This vector is approximated as

$$\omega(t) = 10^{-\nu}x(t) \quad .$$

The error covariance matrix associated with $\omega(t)$ is

$$Q\omega(t) = 10^{-2\nu}x(t)x^T(t) \quad .$$

The round off errors in various components of x are assumed to be independent of each other making $Q\omega$ a diagonal matrix. The value of the constant ν is determined empirically and will vary depending on the word length of the machine used.

The effect of including plant noise in the filter algorithm is a modified differential equation for the state error covariance matrix. The derivation is analogous to the discrete case. The new differential equation is

$$\dot{S} = JS + SJ^T + \phi Cu\Psi^T + \Phi Cu^T\phi^T + \Psi U\Psi^T + Q\omega \quad , \qquad (4\text{-}16)$$

where

$$U = E\{[u - \hat{u}][u - \hat{u}]^T\} \qquad (pxp)$$

$$Cu = E\{[x - x]u^T\} \qquad (nxp) \quad .$$

Equation (4-16) is a modified form of Equation (3-22) given earlier. The propagation and uptdate of the Cu matrix is given by Equation (4-9) and (4-13), respectively.

4.4 Plant Noise Implementation

When the adaptive plant noise model of Section 4.3 is to be implemented, the following quantities must be calculated or chosen.

(1) Elements of the uncertain parameter vector. This can be accomplished by inspecting the dynamic model to see what are the parameters for which exact values (within reasonable tolerance) are not available. For the present application the only element selected is density, ρ, hence p is one.

(2) The expressions for the elements of the $\Psi_{k+1,k}$ matrix.

$$[\Psi_{k+1,k}]_{ij} = \left.\frac{\partial \dot{x}_i}{\partial u_j}\right|_{(x_k,u_k)} \qquad \begin{array}{l} i = 1,\cdots,n \\ j = 1,\cdots,p \end{array} \quad . \qquad (4\text{-}17)$$

(3) The covariance matrix of the u vector. A suggested form of this matrix is

$$U = \begin{bmatrix} k_1^2u_1^2 & & & 0 \\ & k_2^2u_2^2 & & \\ & & \ddots & \\ 0 & & & k_p^2u_p^2 \end{bmatrix} \tag{4-18}$$

where the k_i's must be selected by simulation.

(4) $Cu_{0/0}$, the initial covariance matrix between the state and the uncertain parameter vector, is assumed to be zero.

(5) $Q\omega_k = 10^{-2\nu}x_kx_k^T$. This is a diagonal matrix; that is

$$Q\omega_k = 10^{-2\nu}\begin{bmatrix} x_{1k}^2 & & & 0 \\ & x_{2k}^2 & & \\ & & \ddots & \\ 0 & & & x_{7k}^2 \end{bmatrix} \tag{4-19}$$

where ν is a constant found by simulations. A typical range for ν is three to six.

For the present application p = 1. The elements of the $\Psi_{k+1,k}$ matrix are

$$\begin{aligned} [\Psi_{k+1,k}]_{i\rho} &= \frac{\partial \dot{x}_i}{\partial \rho} = 0 \text{ for } i = 1, 2, 3, 7. \\ [\Psi_{k+1,k}]_{4\rho} &= \frac{\partial \dot{x}_4}{\partial \rho} = -\frac{1}{2}g\alpha v\dot{x} \\ [\Psi_{k+1,k}]_{5\rho} &= \frac{\partial \dot{x}_5}{\partial \rho} = -\frac{1}{2}g\alpha v\dot{y} \\ [\Psi_{k+1,k}]_{6\rho} &= \frac{\partial \dot{x}_6}{\partial \rho} = -\frac{1}{2}g\alpha v\dot{z} \end{aligned} \tag{4-20}$$

The correlation between the uncertain parameters is assumed to be zero and the cross covariance term $\Psi Cu\phi^T + \phi Cu^T\Psi^T$ is set equal to zero. In order to prevent divergence in the drag parameter estimate

$$\sigma_\alpha^2 = \begin{cases} \dfrac{\alpha^2}{100} = \dfrac{x_7^2}{100} & \text{for simulated data, and} \\[2ex] \dfrac{\alpha^2}{200} = \dfrac{x_7^2}{200} & \text{for actual data} \end{cases} \quad , \qquad (4\text{-}21)$$

is added to the corresponding element of the extrapolated state error covariance matrix. This choice is based on a chi-square test as shown below.

Let $\Delta\alpha$ be the error in the drag parameter α at any time step, t_k, of the filter algorithm. The variance term for the drag parameter is Σ_α. Then compute

$$\frac{(\Delta\alpha)^2}{\Sigma_\alpha} = \chi^2 \qquad \text{(say)} \qquad .$$

From chi-square tables, for one degree of freedom and 99.5% confidence interval, the value of χ^2 is 7.88. Assume

$$\Delta\alpha = 0.2\alpha$$

$$\sigma_\alpha^2 = \frac{0.04\alpha^2}{7.88} \approx \frac{\alpha^2}{200} \qquad .$$

Addition of σ_α^2 improves the drag parameter estimate significantly.

A chi-square test is made for consistency between position residuals and the corresponding portion of the state error covariance matrix. If the test fails, plant noise is added to insure that the residual vector is consistent with the modified error covariance matrix. Let the residual vector at t_k be

$$DELS_k = [z_k - h(c_k)] \qquad (3x1)$$

and the updated estimation error covariance matrix at t_k be

$$\begin{array}{cc} \begin{array}{cc} \leftarrow 3 \rightarrow & \leftarrow 4 \rightarrow \end{array} & \\ \Sigma_k = \left[\begin{array}{c|c} T_k & - \\ \hline - & - \end{array}\right] & \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

where only T_k, the covariance matrix for the position components of the state vector, is of interest here. Define

$$(DELS_k)^T (T_k)^{-1} (DELS_k) = k^2 \quad . \tag{4-22}$$

From the chi-square tables, for three degrees of freedom and 99.5% confidence interval, the value of k^2 is 12.84. Thus if $k^2 \leq 12.84$, no plant noise is added. If $k^2 > 12.84$, plant noise is required to insure consistency between the estimation error covariance matrix and the actual error distribution. Let Q_c be the diagonal plant noise matrix added to achieve this consistency; that is

$$Q_c = \begin{bmatrix} q_c & 0 & 0 \\ 0 & q_c & 0 \\ 0 & 0 & q_c \end{bmatrix}$$

where

$$q_c = \frac{(DELS_k)^T (DELS_k)}{12.84} \quad . \tag{4-23}$$

The above matrix Q_c satisfies the equality

$$(DELS_k)^T (Q_c)^{-1} (DELS_k) = 12.84$$

and since

$$|(Q_c + T_k)^{-1}| \leq |Q_c^{-1}|$$

where both Q_c and T_k are positive definite symmetric matrices. It follows that

$$(DELS_k)^T(Q_c + T_k)^{-1}(DELS_k) \leq (DELS_k)^T(Q_c)^{-1}(DELS_k) = 12.84 \ . \quad (4\text{-}24)$$

Thus the addition of Q_c enforces consistency between the residual vector and the new matrix $(Q_c + T_k)$.

4.5 Summary

This chapter starts with tracing the origin of divergence in the filter algorithm. The main cause is that the state error covariance matrix becomes unrealistically small, leading to a small gain matrix. This results in the observations having very little effect on the estimates. A remedy to this problem is to increase the error covariance matrix so as to increase the filter gain and thus alleviate the divergence problem. Simulation results have indicated that the error covariance matrix can not be increased by arbitrary amounts.

In Section 4.2 an adaptive plant noise algorithm for the discrete case is developed. This has been adopted from reference (26). This procedure accounts for errors due to uncertain parameters and round off. In Section 4.3, the algorithm is extended to the continuous case. In Section 4.4, a method is suggested as to how the plant noise is implemented for the present problem. A method is given to select various parameters.

The estimates of drag parameter are of crucial interest. Based on a chi-square test a quantity ($\alpha^2/100$ or $\alpha^2/200$) is added to the variance term of the drag parameter to prevent divergence in drag parameter estimates. At every step of the filter algorithm, a chi-square test is made for consistency between the position residual vector and the corresponding portion of the error covariance matrix. If the test fails, plant noise is added to make the residual vector consistent with the modified error covariance matrix.

The plant noise algorithm improves the second-order filter estimates significantly as described in Chapter VI.

CHAPTER V

FIXED-INTERVAL SMOOTHING

5.1 Introduction

The smoothing problem consists of estimating the state of a process at some time t, given noisy measurements related to the process over a measurement interval which includes the time t. For tracking of reentry vehicles, the smoothing problem is the post-flight estimation of the trajectory based on noisy measurements. If the estimate of the state at any intermediate point is desired, it can be based on all the measurements including those made after the point of interest. In the present application, the smoothing process has shown to greatly improve the filter estimates. In essence, the smoothing process runs backwards in time. The fixed-interval technique of smoothing is adapted for the present problem. The estimate

$$x_{k/N}, \qquad k = 1,2,\cdots,N$$
$$N = \text{fixed point integer}$$

is termed the fixed-interval smoothed estimate. The symbol $x_{k/N}$ is the estimate of the state x at any point k based on the N observed data points.

In this chapter, a linear fixed-interval smoothing algorithm is presented and then extended for a nonlinear process. An implementation technique is given that results in a considerable saving in computation time.

5.2 Linear Fixed-Interval Smoother

Consider a discrete linear system model given by

$$x_{k+1} = \phi_{k+1,k}x_k + q_k \tag{5-1}$$

where

x_k = state vector at time instant t_k

$\phi_{k+1,k}$ = state transition matrix, and

q_k = plant noise vector .

Meditch (18) has presented an algorithm for fixed-interval smoothing for linear systems. This algorithm can be summarized by

$$x_{k/N} = x_{k/k} + A_k[x_{k+1/N} - x_{k+1/k}] \tag{5-2}$$

$$A_k = \Sigma_{k/k}\phi^T_{k+1,k}\Sigma^{-1}_{k+1/k} \tag{5-3}$$

$$\Sigma_{k/N} = \Sigma_{k/k} + A_k[\Sigma_{k+1/N} - \Sigma_{k+1/k}]A^T_k \tag{5-4}$$

where

$x_{k/N}$ = the smoothed state estimate at time t_k,

A_k = smoother gain matrix, (7x7)

$\Sigma_{k/N}$ = smoothed error covariance matrix,

$x_{k/k}$ = filtered state estimate at instant t_k,

$x_{k+1/k}$ = extrapolated value of state at instant t_{k+1} given observations up to t_k,

$\Sigma_{k/k}$ = state error covariance matrix corresponding to $x_{k/k}$, and

$\Sigma_{k+1/k}$ = state error covariance matrix corresponding to $x_{k+1/k}$.

The quantities $x_{k/k}$, $x_{k+1/k}$, $\Sigma_{k/k}$ and $\Sigma_{k+1/k}$ are calculated in the filter algorithm.

5.3 Nonlinear Fixed-Interval Smoother

Consider the nonlinear system model given by

$$\dot{x}(t) = f(x(t)) + q(t) \tag{5-5}$$

where

x(t) is an n-dimensional state vector,

f(•) is n-dimensional vector valued nonlinear function of the state, and

q(t) is the plant noise vector .

In order to develop equations for the fixed-interval nonlinear smoothing algorithm, a linear equivalent of the Equation (5-5) is developed so that results of the linear smoothing theory can be applied. The linearization is accomplished in the following manner. The right hand side of Equation (5-5) is expanded in a Taylor series about a nominal state vector x_k; that is

$$\dot{x}(t) = f(x_k) + \left.\frac{\partial f}{\partial x}\right|_{x_k} (x - x_k) + \text{HOT} + q(t) \quad .$$

Neglecting the higher order terms (HOT) and noting that

$$f(x_k) = \dot{x}_k \quad ,$$

$$\dot{x} - \dot{x}_k = \left.\frac{\partial f}{\partial x}\right|_{x_k} (x - x_k) + q(t) \quad .$$

By defining $\delta x = x - x_k$, the above equation becomes

$$\delta\dot{x} = J(x_k)\delta x + q(t) \tag{5-6}$$

where

$$J(x_k) = \frac{\partial f}{\partial x}\bigg|_{x_k}$$

is the Jacobian matrix for function f evaluated at the state vector x_k. For the system defined by the above linear differential equation, the state transition matrix satisfies the following differential equation.

$$\dot{\phi}(t,t_k) = J(x_k)\phi(t,t_k); \qquad \phi(t,t) = I$$

where $\phi(t,t_k)$ represents the state transition matrix between the time instants t_k and t. Next expand $\phi(t + \Delta t,t)$ in a Taylor series about t, so that

$$\phi(t + \Delta t,t) \approx \phi(t,t) + \dot{\phi}(t,t)\Delta t + \mathrm{HOT}(\Delta t)^2 \quad .$$

Neglecting the higher order terms,

$$\phi(t + \Delta t,t) \approx \phi(t,t) + J(\cdot)\phi(t,t)\Delta t$$

$$= (I + J(\cdot)\Delta t)\phi(t,t) \quad .$$

Since

$$\phi(t,t) = I$$

$$\phi(t + \Delta t,t) = I + J(\cdot)\Delta t \quad .$$

Letting $t + \Delta t = t_{k+1}$ and $t = t_k$, this equation becomes

$$\phi(t_{k+1},t_k) = I + J(x_k)(t_{k+1} - t_k) \quad .$$

Using the notation

$$\phi_{k+1,k} = \phi(t_{k+1}, t_k)$$

the expression for the state transition matrix becomes

$$\phi_{k+1,k} = I + J(x_k)(t_{k+1} - t_k) \quad . \tag{5-7}$$

Hence, the linearized model of Equation (5-5) can be expressed as

$$x_{k+1} = \phi_{k+1,k} x_k + q_k \tag{5-8}$$

where

x_k is n-dimensional state vector,

$\phi_{k+1,k}$ is given by Equation (5-7), and

$$q_k = q(t)\big|_{t=t_k} \quad .$$

Now the linear fixed-interval smoother algorithm given by Equations (5-2) through (5-4) is used. The boundary conditions for these equations are $x_{N/N}$ and $\Sigma_{N/N}$. $x_{k/k}$ is the filter state estimate at time instant k, given observations up to k. $x_{k+1/k}$ is the extrapolated state vector in the filter algorithm at time instant k+1 given observation data up to k. $\Sigma_{k/k}$ and $\Sigma_{k+1/k}$ are state error covariance matrices corresponding to the state $x_{k/k}$ and $x_{k+1/k}$, respectively. $x_{k+1/k}$ and $\Sigma_{k+1/k}$ are obtained in the filter routine by integrating Equations (3-9) and (3-22), with $x_{k/k}$ and $\Sigma_{k/k}$ as initial conditions. The updated values of the state $x_{k/k}$ and error covariance matrix $\Sigma_{k/k}$ are obtained by using Equations (3-24) and (3-26). In Chapter III the symbols used are x_k and Σ_k compared to $x_{k/k}$ and $\Sigma_{k/k}$ here. The index k decrements from N-1 to 0 in the implementation of the smoother algorithm.

The linear fixed-interval smoothing algorithm presented in Section 5.2 is applied to obtain smoothed state estimates and corresponding error covariance matrices. As explained before, the priori and posteriori estimates and covariance matrices $x_{k+1/k}$, $x_{k/k}$, $\Sigma_{k+1/k}$ and $\Sigma_{k/k}$ from the filtering algorithm are used for obtaining the smoothed estimates. There are two advantages of this method of implementation. First the nonlinearities of the system are taken into consideration since the linearized dynamic model of Equation (5-7) is used only to evaluate the state transition matrix. Second, this technique results in a considerable saving in computation time.

5.4 Summary

This chapter commences with the reasons for using fixed-interval smoothing for the present problem. An algorithm for fixed-interval smoothing in the lienar case is presented. For the nonlinear case, a method is presented where the dynamic model can be linearized and the linear algorithm adapted. The filtered estimates are stored and used in the smoother algorithm which results in a considerable saving in computation time. The simulation results when the smoothing algorithm is implemented, are presented in Chapter VI.

CHAPTER VI

SIMULATION AND NUMERICAL RESULTS

6.1 Introduction

The performance of the computer software package developed for estimating the state of the reentry vehicle using optical tracking data has been evaluated by extensive simulations. Three types of data (position x,y and z observations and associated error covariance matrix, R_k) were used to test the package. Two cases used simulated data where the true trajectory and noise were known. The third case was an actual data set (unknown true trajectory). The trajectories of the first two cases were similar, except in one case the reentry vehicle had a constant ballistic coefficient and in the other it was a parabolic function of altitude. The sample rate for the first two cases were 30 and 25 samples per second, respectively, while for the actual data set it was 30 samples per second.

Before presenting the performance of the filter and smoother algorithms with and without plant noise, a method of choosing program constants and parameters is presented.

6.2 Selection of Program Constants and Parameters

(1) The number of data points, N, used by XNTIAL for initializing the filter algorithm is selected by a test program; a test program is

used to select N. The criterion is that the ballistic coefficient should be positive and be in a predetermined range, and when N is raised any further, should stay essentially constant. For the three cases, N = 45, 50 and 61, respectively, are selected by this method.

(2) The atmospheric density model is required by XNTIAL, DERFUN, JACN and SHMT routines. The density model that was used to generate the simulated data is also used when processing (generating filter and smoother estimates) simulated data. The U. S. standard atmospheric density model was used (22). In the case of actual data the density versus altitude information is obtained from a rawindsonde and an exponential model fitted to the results. The standard density model used is

$$\rho = \rho_0 \exp(-kh)$$

where $_0$ and k are constants and h is altitude. For the simulated data cases,

$$\rho_0 = 0.002377; \quad k = 0.41\text{X}10^{-4} \text{ for all h} \quad .$$

For actual data

$$\rho_0 = 0.002244; \quad k = 0.3207\text{X}10^{-4} \quad h \leq 45{,}000$$

$$\rho_0 = 0.005010; \quad k = 0.4992\text{X}10^{-4} \quad 45{,}000 \leq h \leq 107{,}000$$

$$\rho_0 = 0.001930; \quad k = 0.41\text{X}10^{-4} \quad h \geq 107{,}000 \quad ,$$

where the English system of units is used.

(3) The uncertain parameter as described in Section 4.4 is density, ρ then $u_1 = \rho$. The elements of the $\Psi_{k+1,k}$ matrix are evaluated using

Equation (4-17). The constant k_1^2, chosen by simulation, describes the U matrix. The criterion of choosing k_1^2 is that it should improve the filter and smoothed state estimates for all three cases. The value of k_1^2 selected is

$$k_1^2 = 0.0025 \quad .$$

(4) The parameter describing the round off error is chosen to be 3, thus giving

$$Q\omega_k = 10^{-6} x_k x_k^T$$

with diagonal elements zero. This is described in Equation (4-19).

(5) Simulation results indicate that the variance term corresponding to x_7, the drag parameter α, varies by four orders of magnitude. This is because the gain matrix becomes very small and hence observations do not affect the α estimates. As a remedy to this problem, another term is included as plant noise. This is

$$\frac{x_7^2}{100} = \frac{\alpha^2}{100}$$

for the simulated trajectories and,

$$\frac{x_7^2}{200} = \frac{\alpha^2}{200}$$

for the actual data set. The above choice is based on a chi-square test as explained in Section 4.4. This quantity is added to the variance term of the drag parameter. The inclusion of this noise term makes considerable improvement in drag parameter estimates for all data sets.

(6) The position plant noise matrix defined in Equation (4-23) is added when the chi-square test fails. This improves position estimates

significantly.

6.3 Simulation of Observation Noise and Error Covariance Matrix

The nominal trajectory is obtained by integrating the dynamic model (equations of motion) with the initial state vector as initial conditions. The simulated observations and error covariance matrices are generated by using RCON, ADNZ, RMGN, and EIGEN subroutines. The purpose of these routines is briefly given in Appendix E. The process can be explained as follows.

(1) Consider the nominal state vector at any time

$$t_j : (xyz\dot{x}\dot{y}\dot{z}\alpha)^T \quad .$$

The error characteristics of the sensors (zero mean and σ standard deviation) are known.

(2) Using the optical station coordinates

$$((XO_i, YO_i, ZO_i), \quad i = 1, \cdots, K) \quad ,$$

the azimuth and elevation angles

$$((A_i, E_i), \quad i = 1, \cdots, K) \quad ,$$

as would be observed in the absence of any noise, are calculated. K is the number of optical stations.

(3) Two random numbers (zero mean and σ standard deviation) per optical station are generated. These numbers are added to the respective values of azimuth and elevation to obtain simulated tracker observations.

(4) Using simulated observations A_i, E_i and the optical station

coordinates, the noisy position components are obtained using the trianulation algorithm (15).

(5) The difference between the noisy positions obtained in step four and the nominal position elements gives the error vector.

Steps three, four and five are repeated MCL (selected to be 60) times. MCL is the number of Monte Carlo runs needed for generating R_j. The sample covariance matrix, R_j, associated with the MCL error vectors is the observation error covariance matrix for time instant t_j.

The next step is to generate an error vector that is consistent with the R_j matrix. The following steps are executed to achieve this.

(6) The eigenvalue vector λ (3x1) and the corresponding modal matrix Q for the R_j matrix is evaluated by using EIGEN subroutine.

(7) Three dimensional random vector (RN) with zero mean and variance λ is generated by RMGN routine.

(8) RN vector is transformed by the modal matrix to obtain the error vector NOIS.

$$\text{NOIS} = Q \times \text{RN} \quad .$$

(9) The noise vector, when added to the corresponding position components of the nominal trajectory yields the simulated observation vector, OBS.

$$\text{OBS} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \text{NOIS} \quad .$$

Steps one through nine are repeated at each time step of the trajectory. A flow chart of this process is given in Figure 4.

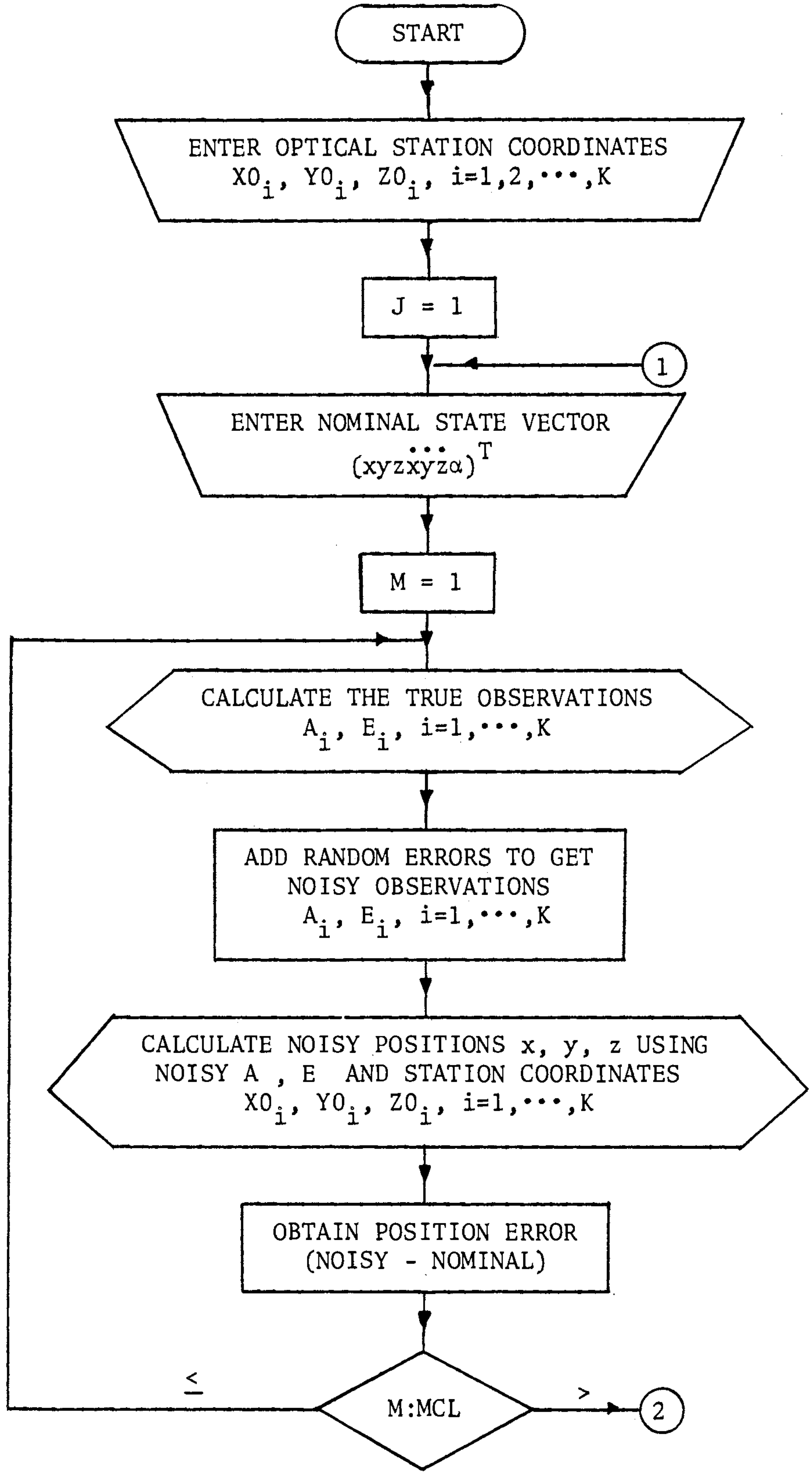

Figure 4. Generation of Simulated Observation Data

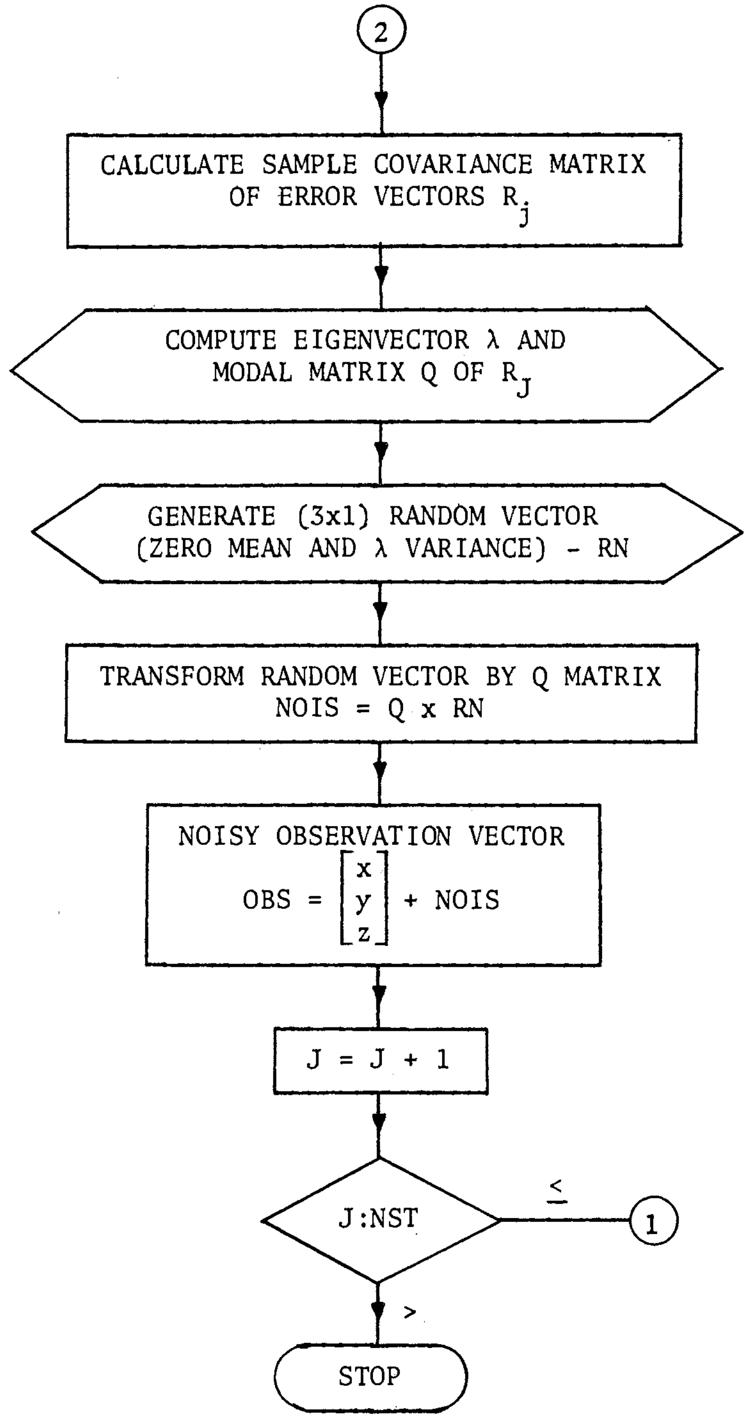

Figure 4. (Cont.)

Figures 5 through 7 show the simulated noise (observed - true) trajectories of x,y and z components for the varying ballistic coeffi cient case. The vehicle descends from an altitude of 82,000 feet, and its velocity reduces from 24,000 feet per second during the tracking time of approximately eleven seconds. In Table I, the nominal state vector components for simulated data, varying ballistic coefficient case, at the interval of 0.6 seconds are given. The position and velocity components of the nominal trajectory, for simulated data with constant ballistic coefficient (β = 2000 lbs/ft^2), are similar to the state components in Table I.

For the constant β simulated trajectory the error trajectory is very similar to the varying β case.

6.4 Filtering and Smoothing

The objective of the development of the software package is that it should produce smoothed state estimates which are "close" to the true state. In the case of actual data, where the true trajectory is not known, the software package performance is judged by the size of position residuals and error covariance matrices.

Selection of program constants and parameters constitutes a significant part of the software package development. Considerable effort is required for the calculation and programming of the Jacobian and Hessian matrix elements of the present seventh-order problem.

The second-order filter and smoother with no plant noise exhibited divergence. For the simulated observations, varying ballistic coefficient case, without plant noise, the filtered and smoother error trajectories for the β component of the state vector are shown in

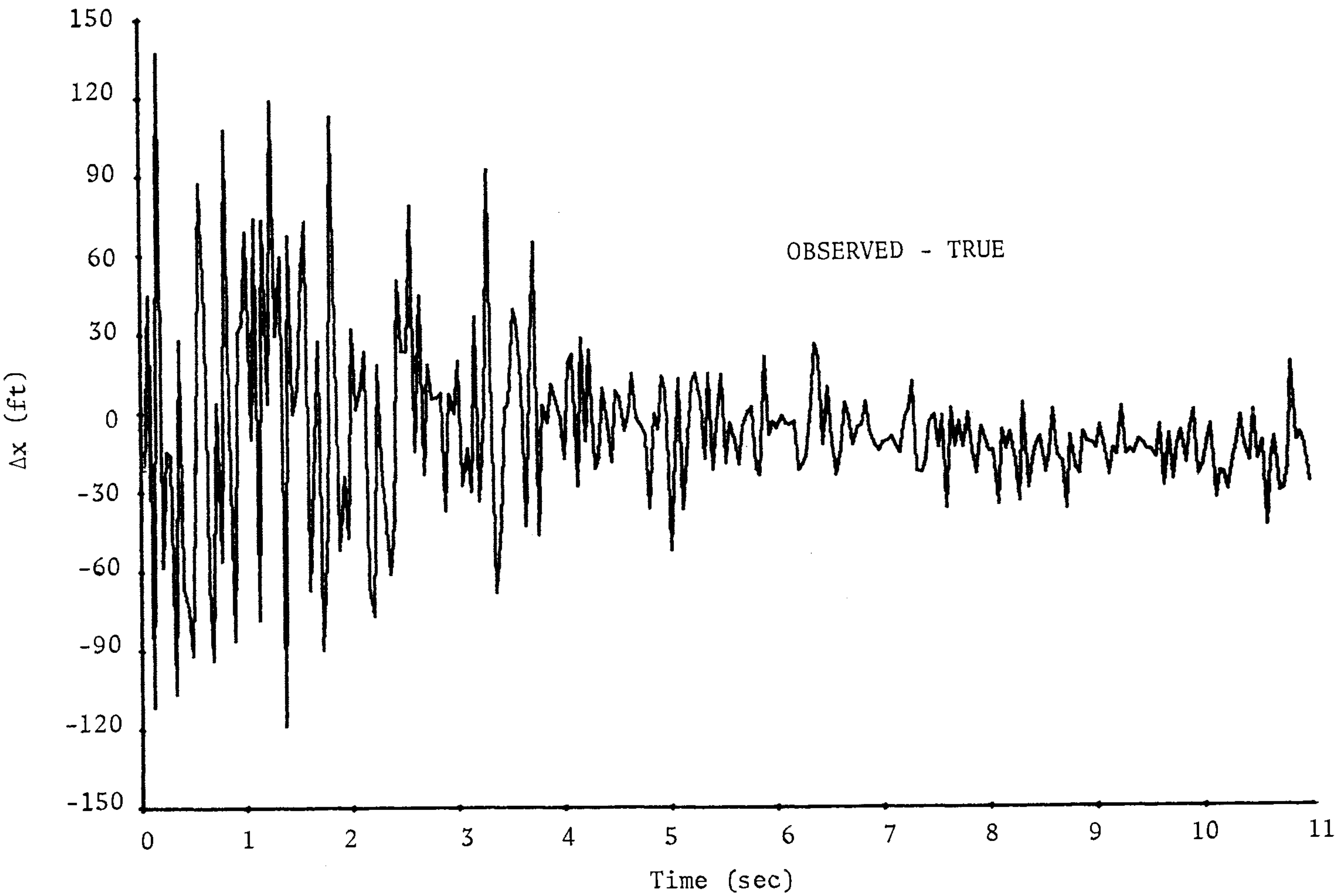

Figure 5. Noise in x Position: Varying β

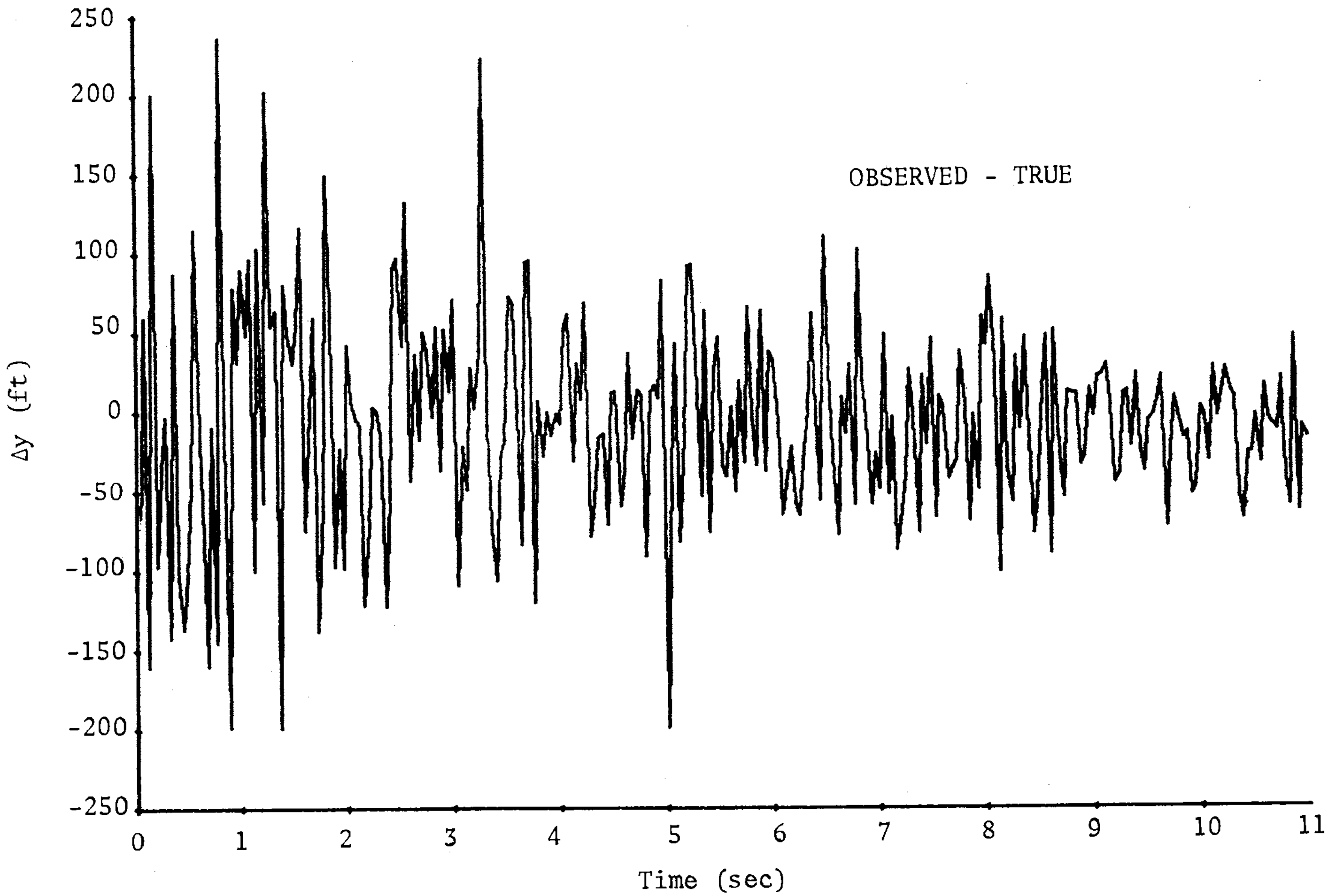

Figure 6. Noise in y Position: Varying β

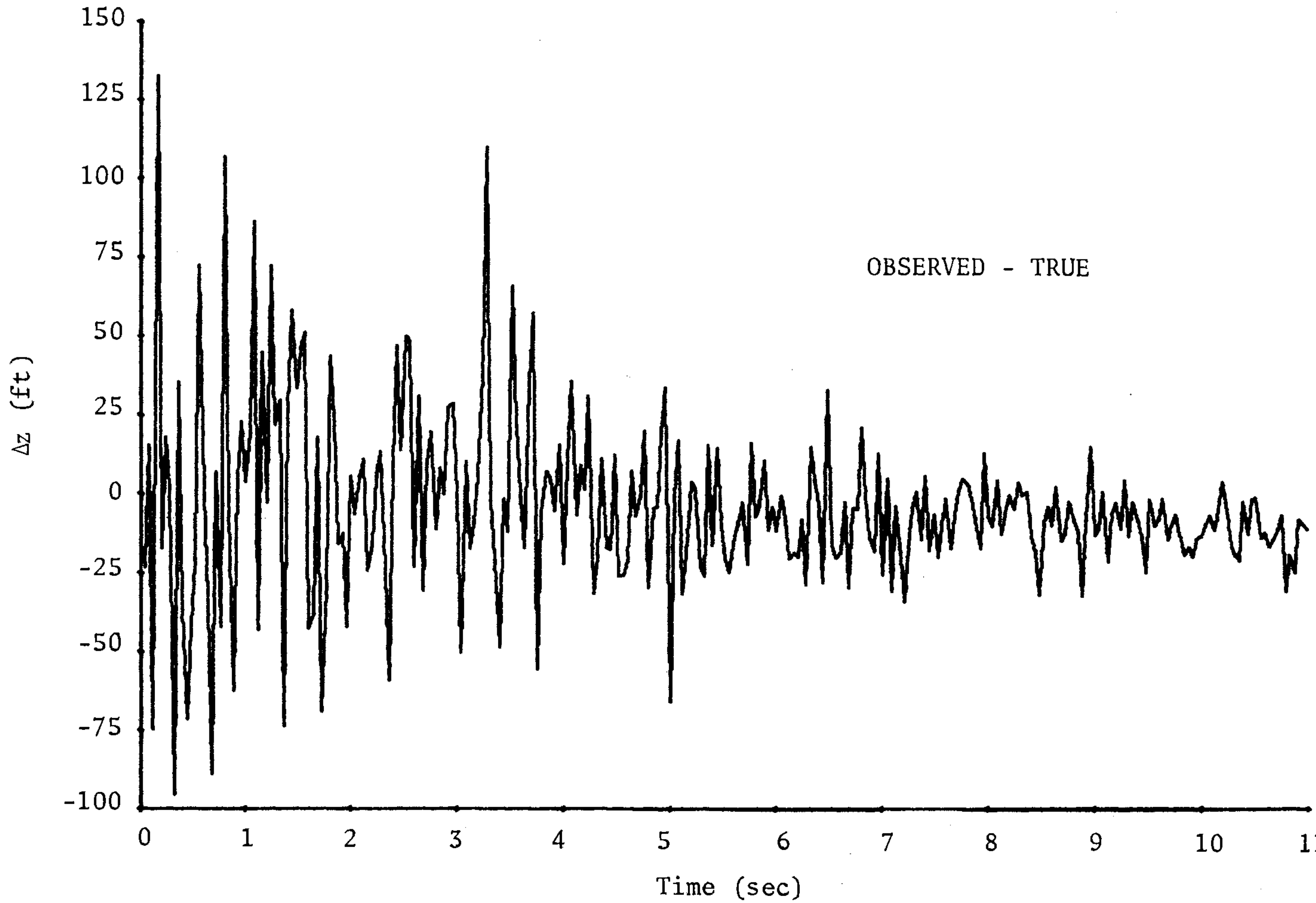

Figure 7. Noise in z Position: Varying β

TABLE I

NOMINAL TRAJECTORY FOR SIMULATED DATA, VARYING β CASE

t	x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$	β
0.040	131997.4	81728.1	80862.1	-16810.0	-10293.9	-10070.4	1988.1
0.640	121959.6	75581.1	74842.3	-16641.3	-10190.7	-9989.9	2012.1
1.240	112035.3	69503.4	68878.2	-16431.3	-10062.3	-9884.4	2024.3
1.840	102251.4	63512.0	62986.4	-16170.8	-9903.0	-9748.2	2020.2
2.440	92641.8	57627.1	57187.5	-15849.4	-9706.3	-9574.9	2008.0
3.040	83246.0	51873.1	51505.3	-15456.0	-9465.6	-9357.6	1988.1
3.640	74110.4	46278.5	45968.4	-14980.1	-9174.3	-9089.8	1953.1
4.240	65287.2	40875.0	40608.7	-14413.4	-8827.4	-8766.1	1912.0
4.840	56833.0	35697.2	35460.8	-13751.4	-8422.1	-8383.5	1865.7
5.440	48804.3	30779.9	30560.1	-12996.0	-7959.7	-7942.9	1811.6
6.040	41254.4	26155.8	25939.8	-12157.5	-7446.2	-7450.1	1754.4
6.640	34228.1	21852.3	21628.2	-11254.4	-6893.2	-6916.1	1694.9
7.240	27757.0	17888.7	17645.6	-10312.0	-6316.2	-6356.4	1634.0
7.840	21855.6	14274.0	14002.0	-9360.1	-5733.2	-5788.8	1574.8
8.440	16520.9	11006.4	10697.0	-8427.7	-5162.2	-5231.1	1517.5
9.640	7460.0	5456.3	5050.7	-6715.7	-4113.7	-4204.0	1410.4
10.240	3659.5	3128.2	2665.9	-5965.5	-3654.3	-3753.0	1362.4
10.840	285.7	1061.5	537.7	-5293.6	-3242.8	-3348.9	1317.5

Figure 8. All elements of the state vector diverged. Similar divergence is also exhibited in other data sets.

The divergence problem has been solved to a large extent with the inclusion of the plant noise algorithm of Chapter IV. Figures 9 through 15 show filter and smoother error trajectories for components of the state vector for the simulated trajectory, varying β case, with plant noise included. Figures 16 through 22 contain similar graphs for the simulated trajectory, constant β case. The position residuals and the ballistic coefficient estimate for the actual data set are shown in Figures 23 through 26.

6.5 Summary

For the simulated trajectories, the package performance for position and velocity estimation is quite good. The ballistic coefficient error is large for the filter estimates but as seen in Figures 15 and 22, the smoothing algorithm improves the ballistic coefficient estimates considerably. For the actual data set, the size of the position residuals is small and ballistic coefficient estimates are close to the expected value.

Theoretically, the filtering algorithm produces increasingly accurate estimates as additional data are processed. This is reflected by a reduction in the magnitude of the determinant of the error covariance matrix. But in actual operating conditions it is observed that the size of the residuals tends to increase with the number of observations. This is the divergence problem. A solution to this problem is to increase the covariance matrix by incorporation of the adaptive plant noise algorithm.

The drag parameter depends on the position, velocity and

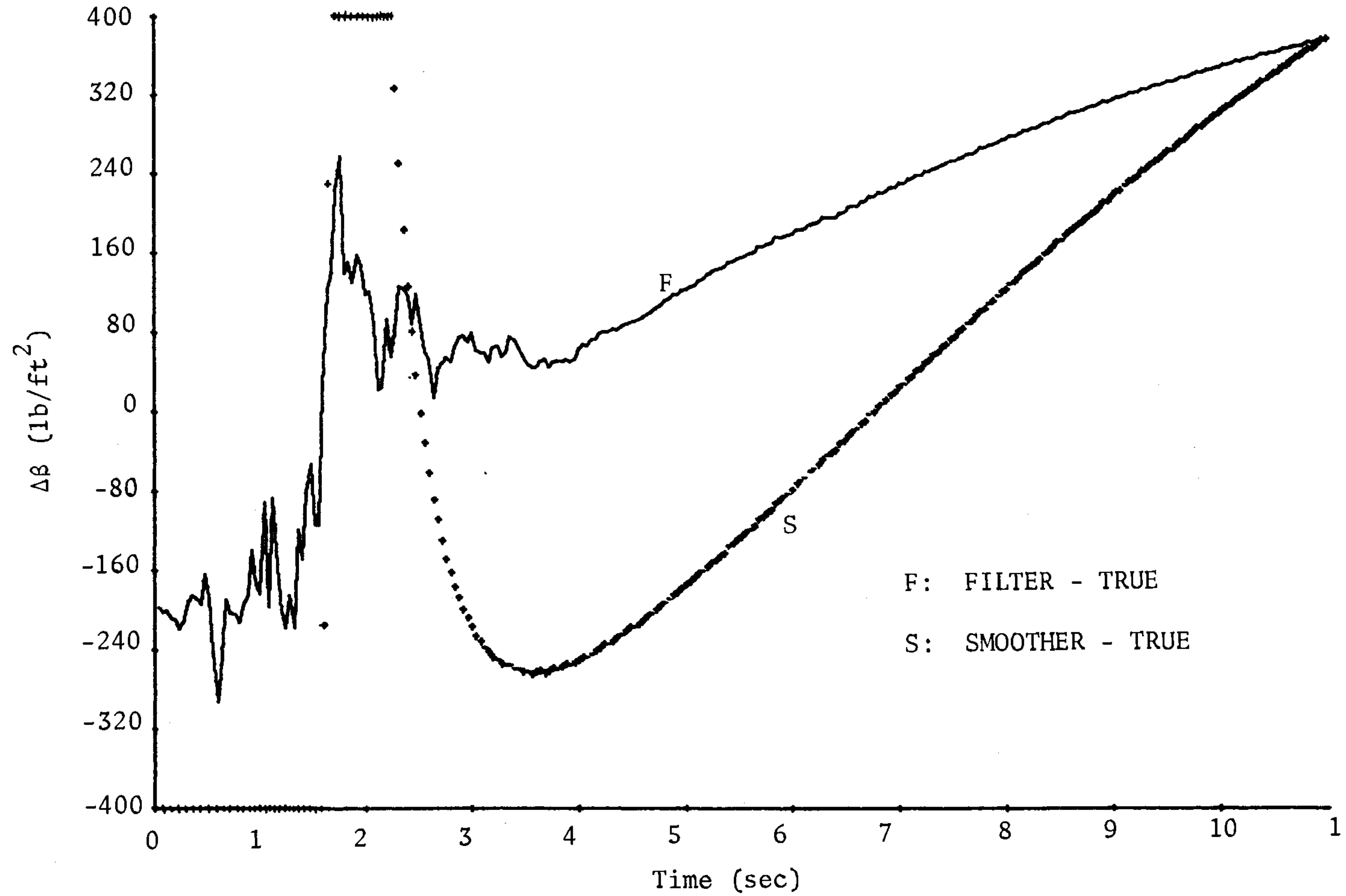

Figure 8. Error in Ballistic Coefficient: Varying β, no Plant Noise

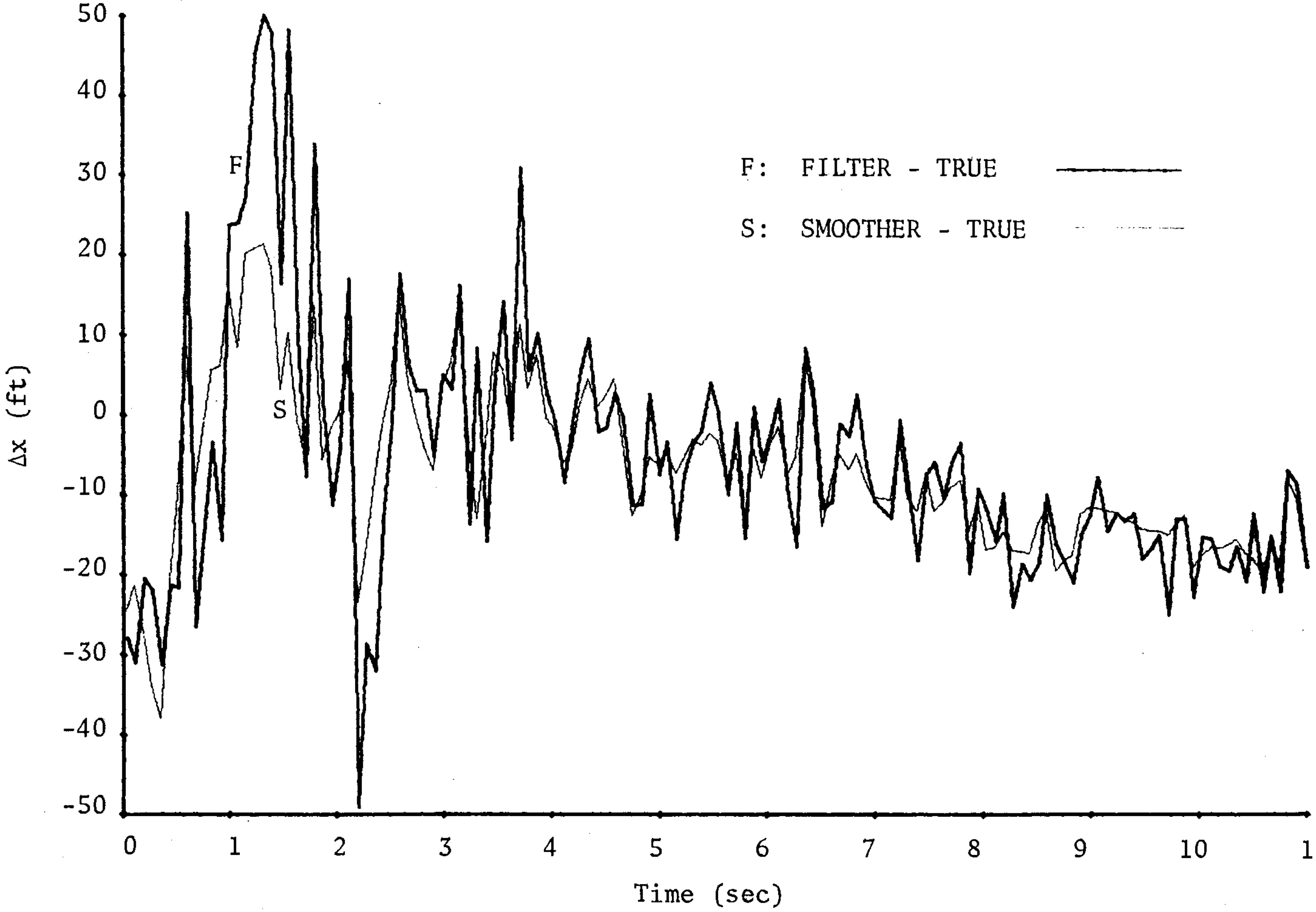

Figure 9. Error in x Position: Varying β, Plant Noise Added

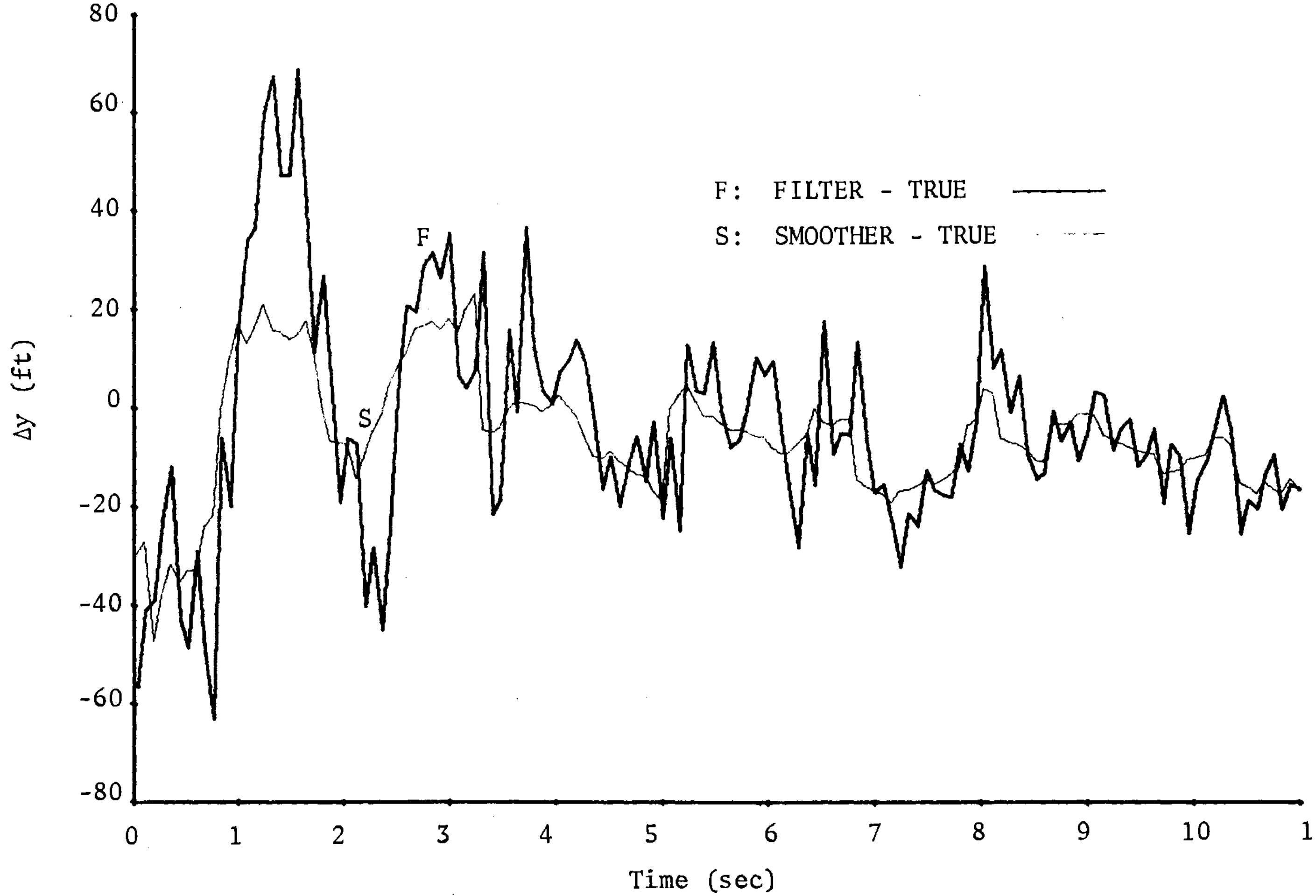

Figure 10. Error in y Position: Varying β, Plant Noise Added

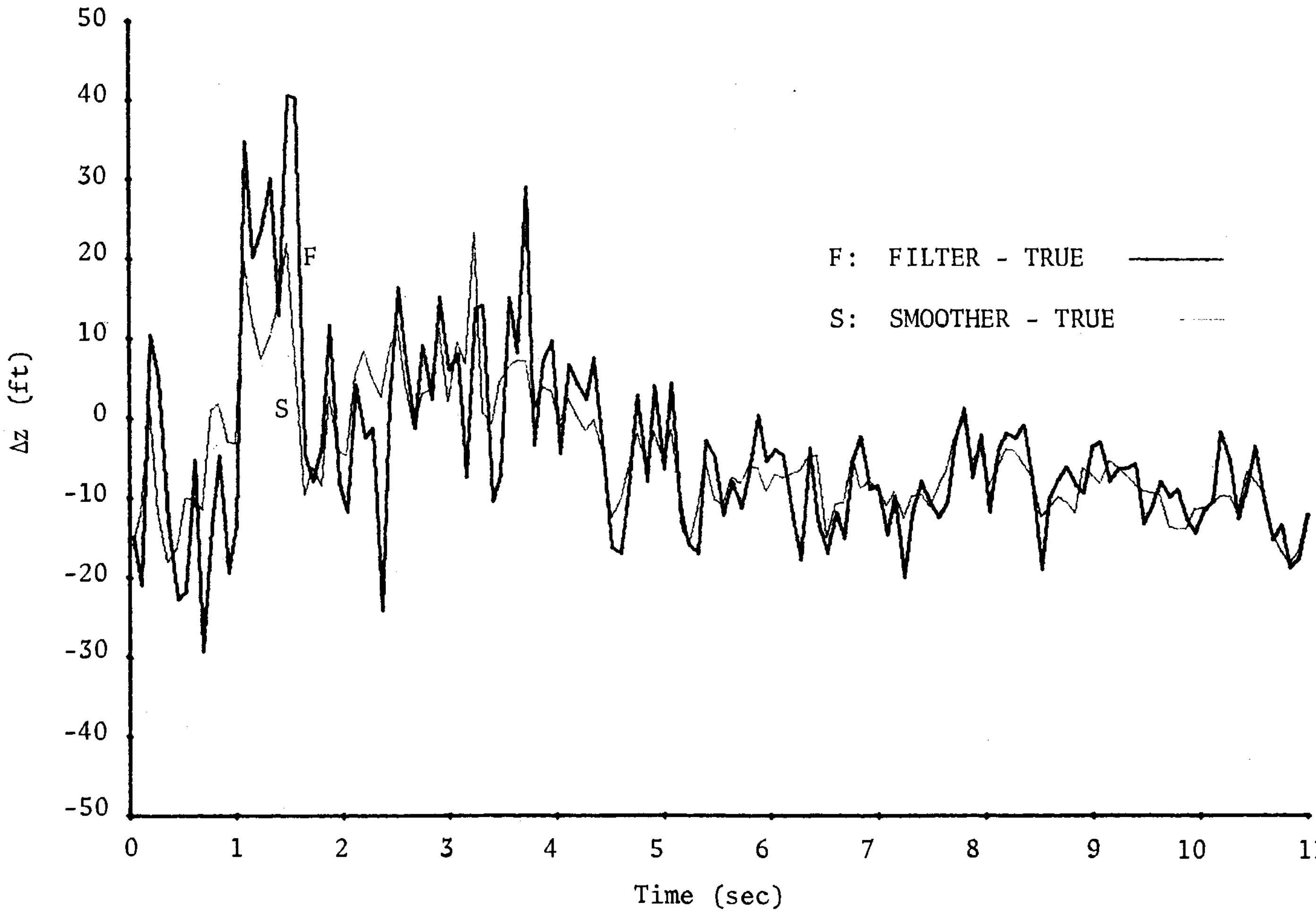

Figure 11. Error in z Position: Varying β, Plant Noise Added

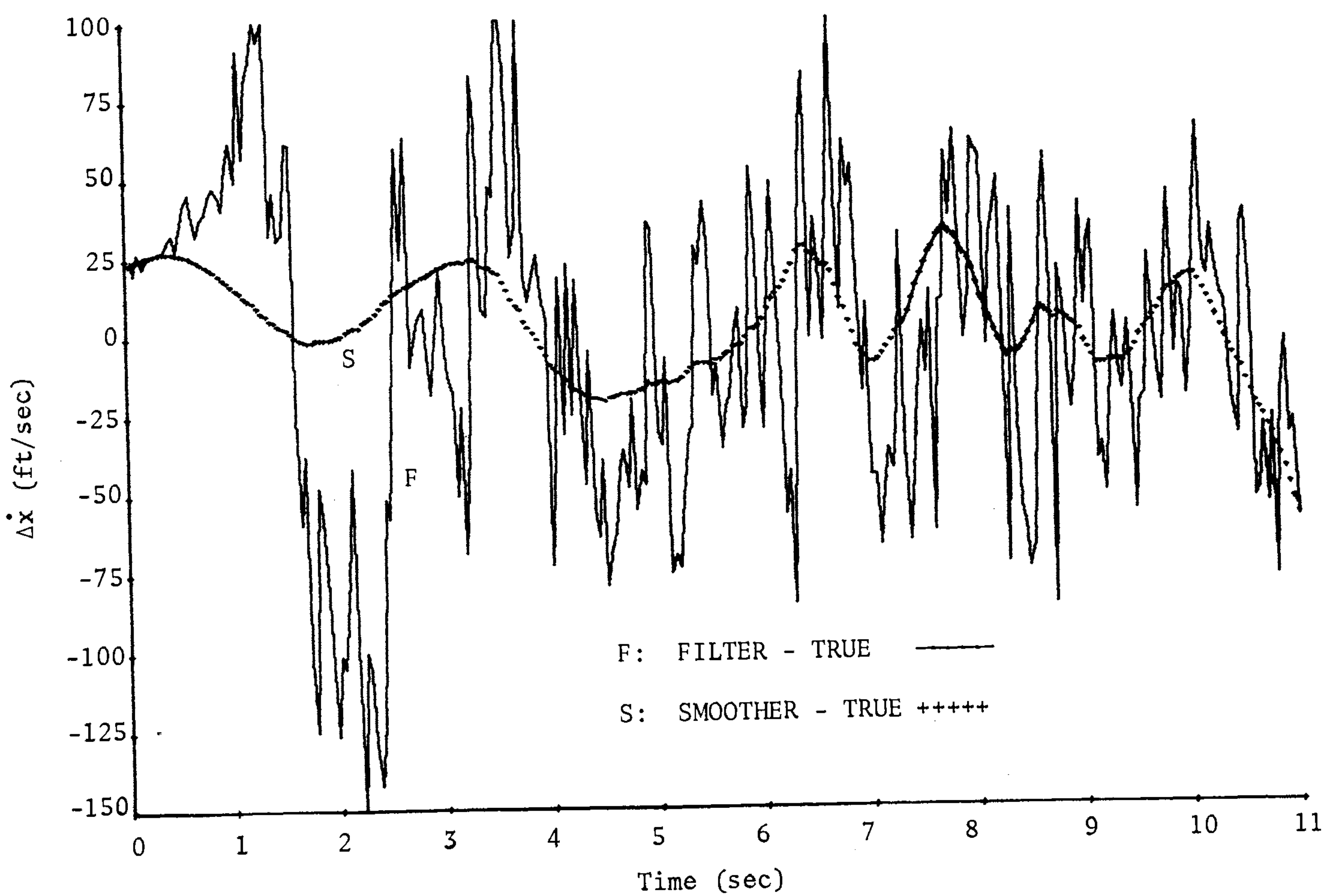

Figure 12. Error in x Velocity: Varying β, Plant Noise Added

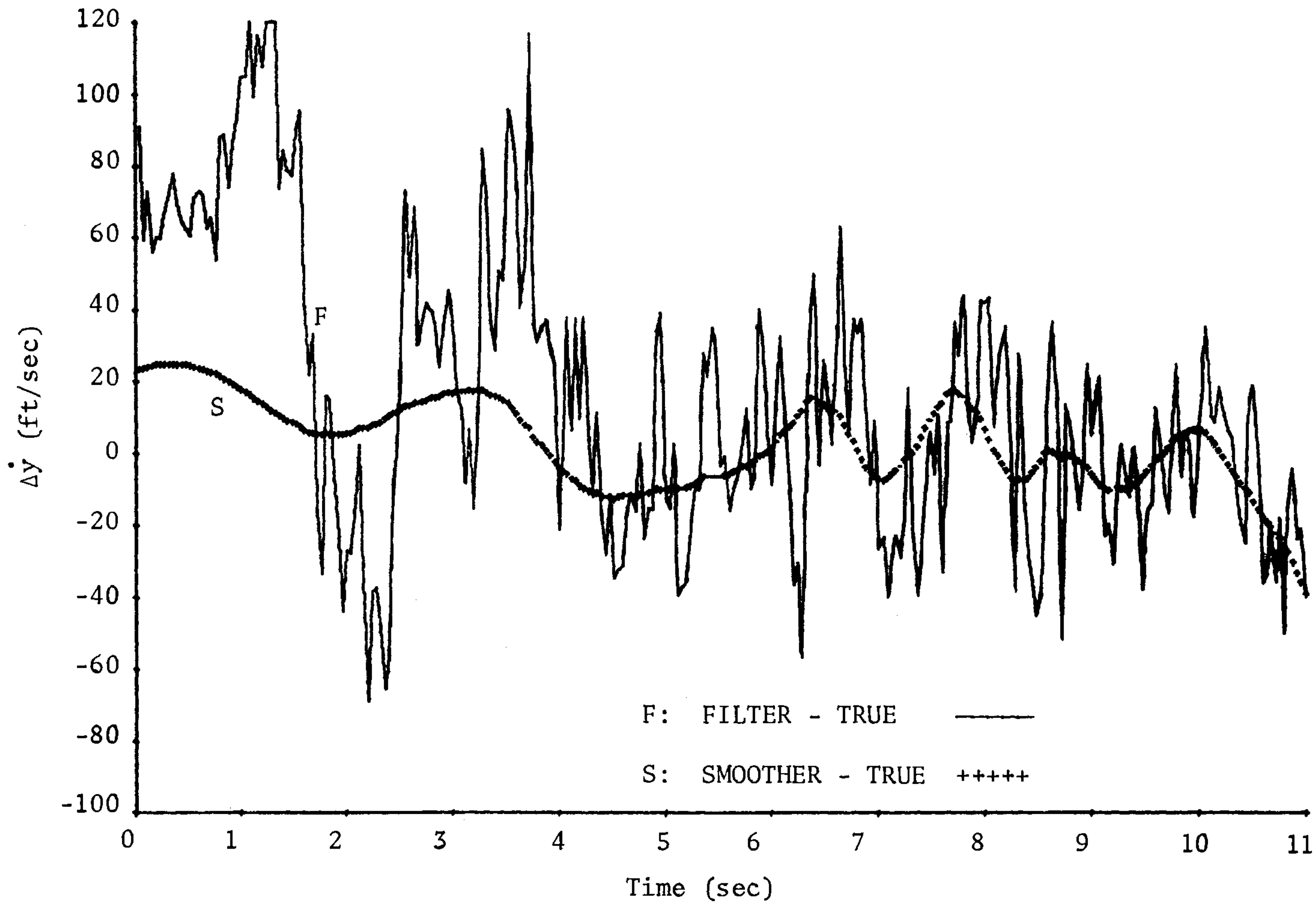

Figure 13. Error in y Velocity: Varying β, Plant Noise Added

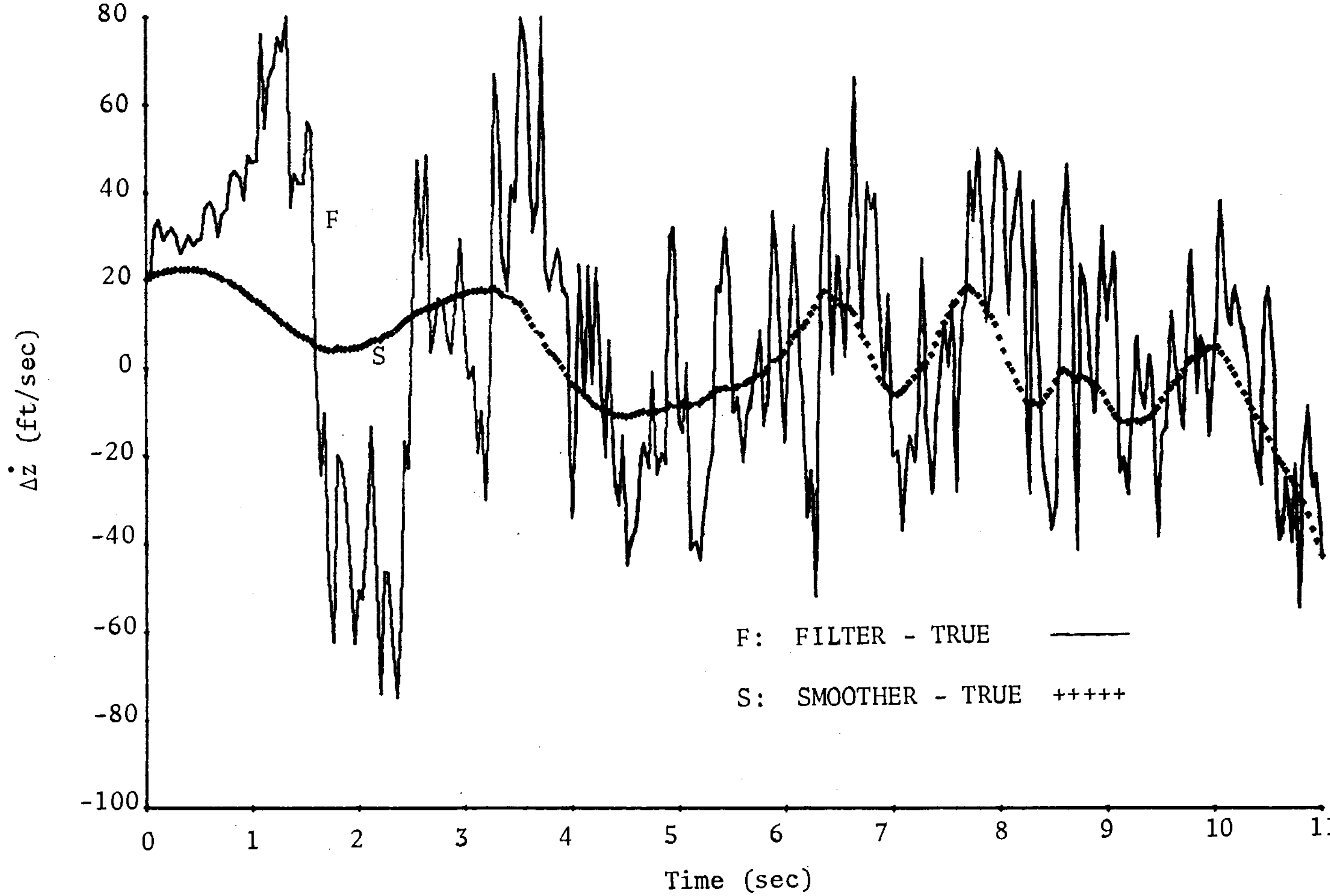

Figure 14. Error in z Velocity: Varying β, Plant Noise Added

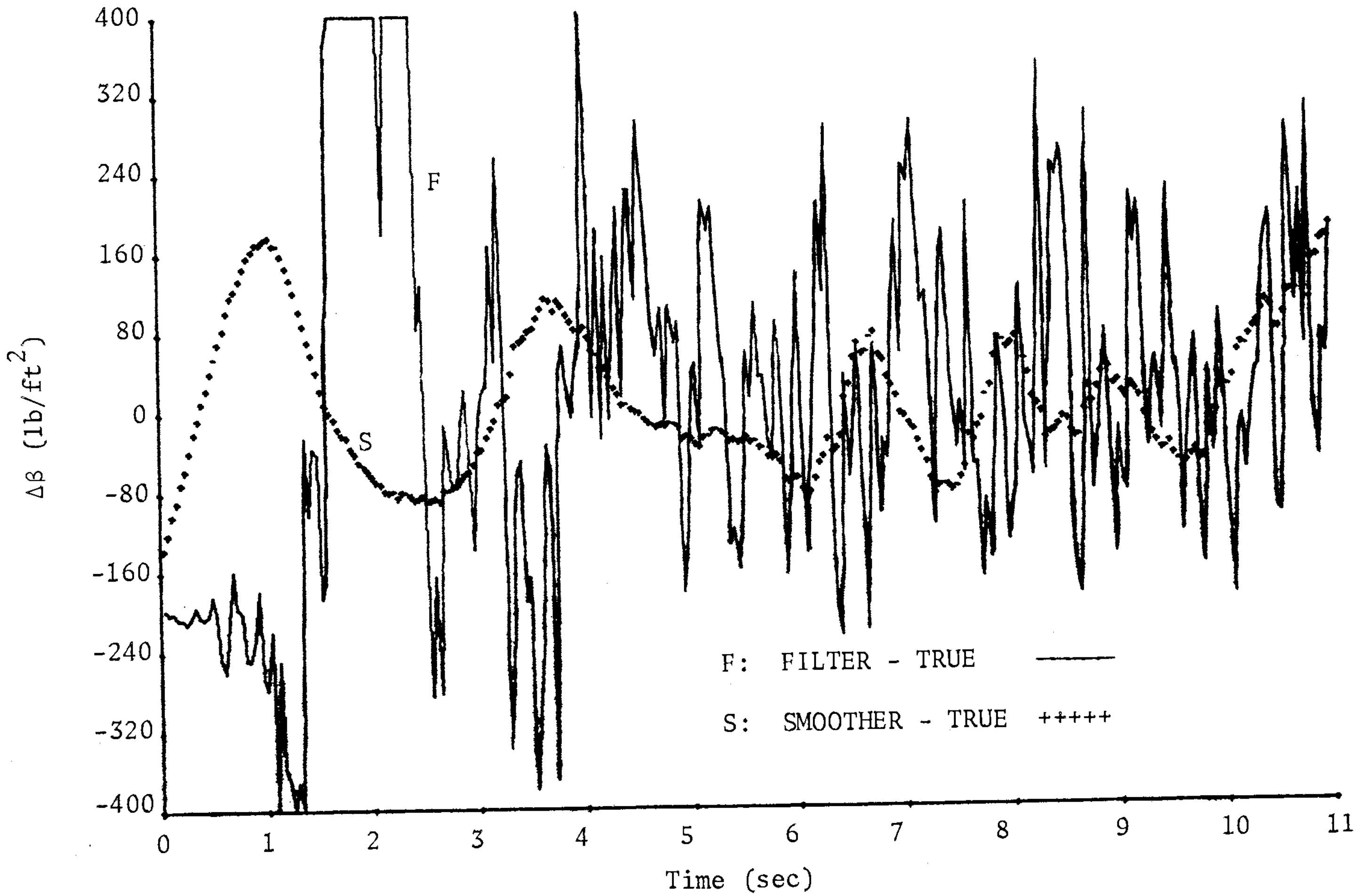

Figure 15. Error in Ballistic Coefficient: Varying β, Plant Noise Added

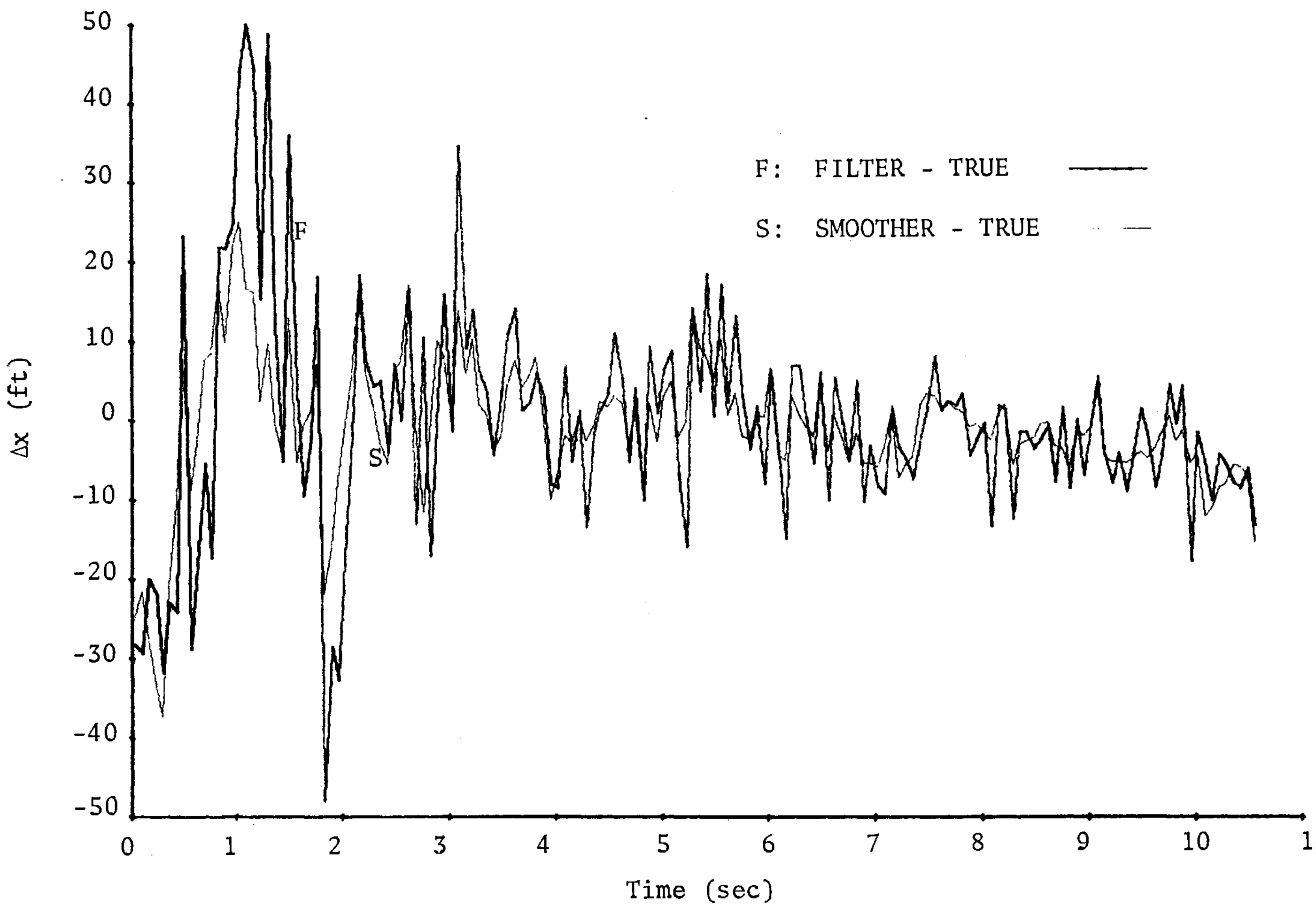

Figure 16. Error in x Position: Constant β, Plant Noise Added

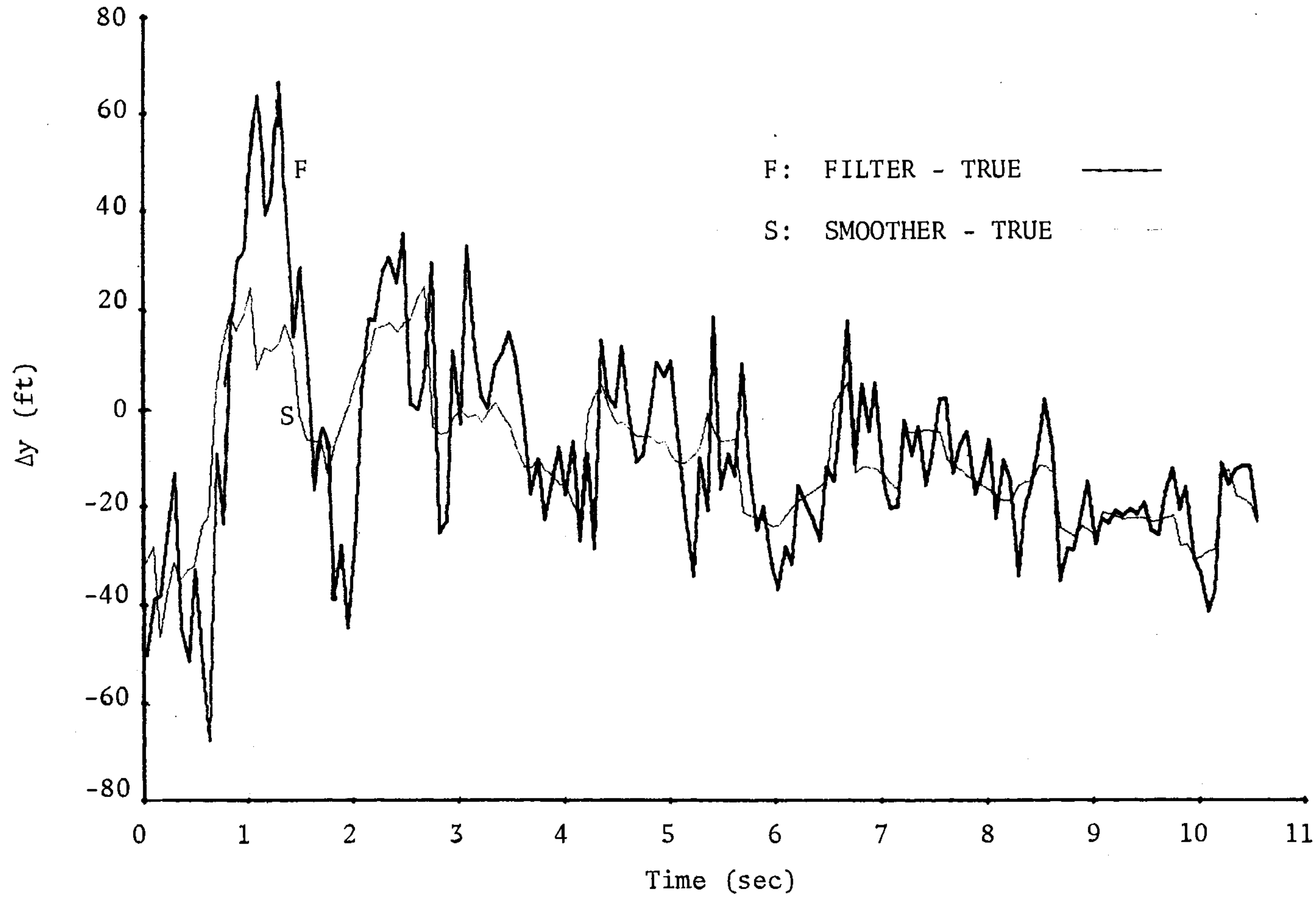

Figure 17. Error in y Position: Constant β, Plant Noise Added

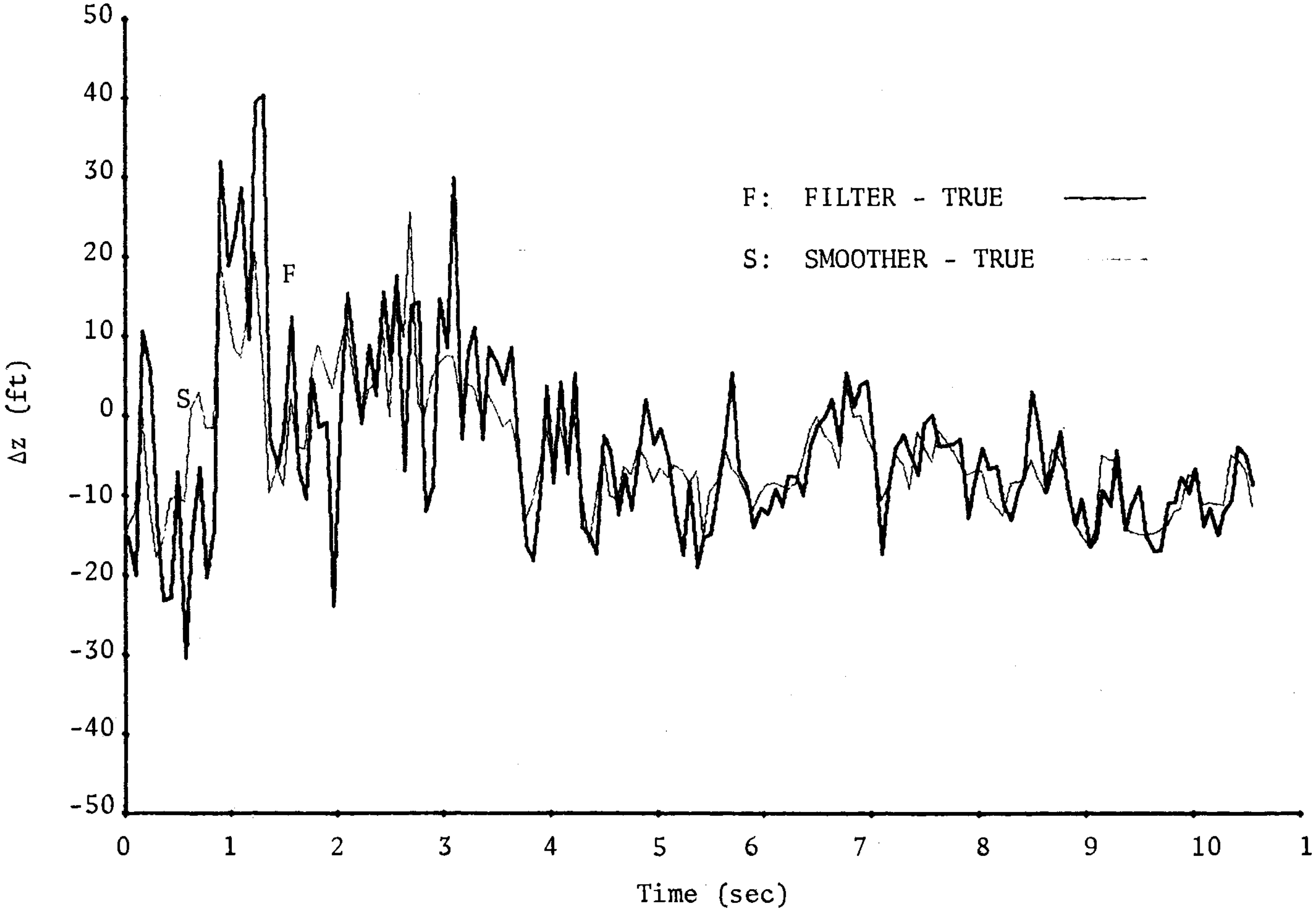

Figure 18. Error in z Position: Constant β, Plant Noise Added

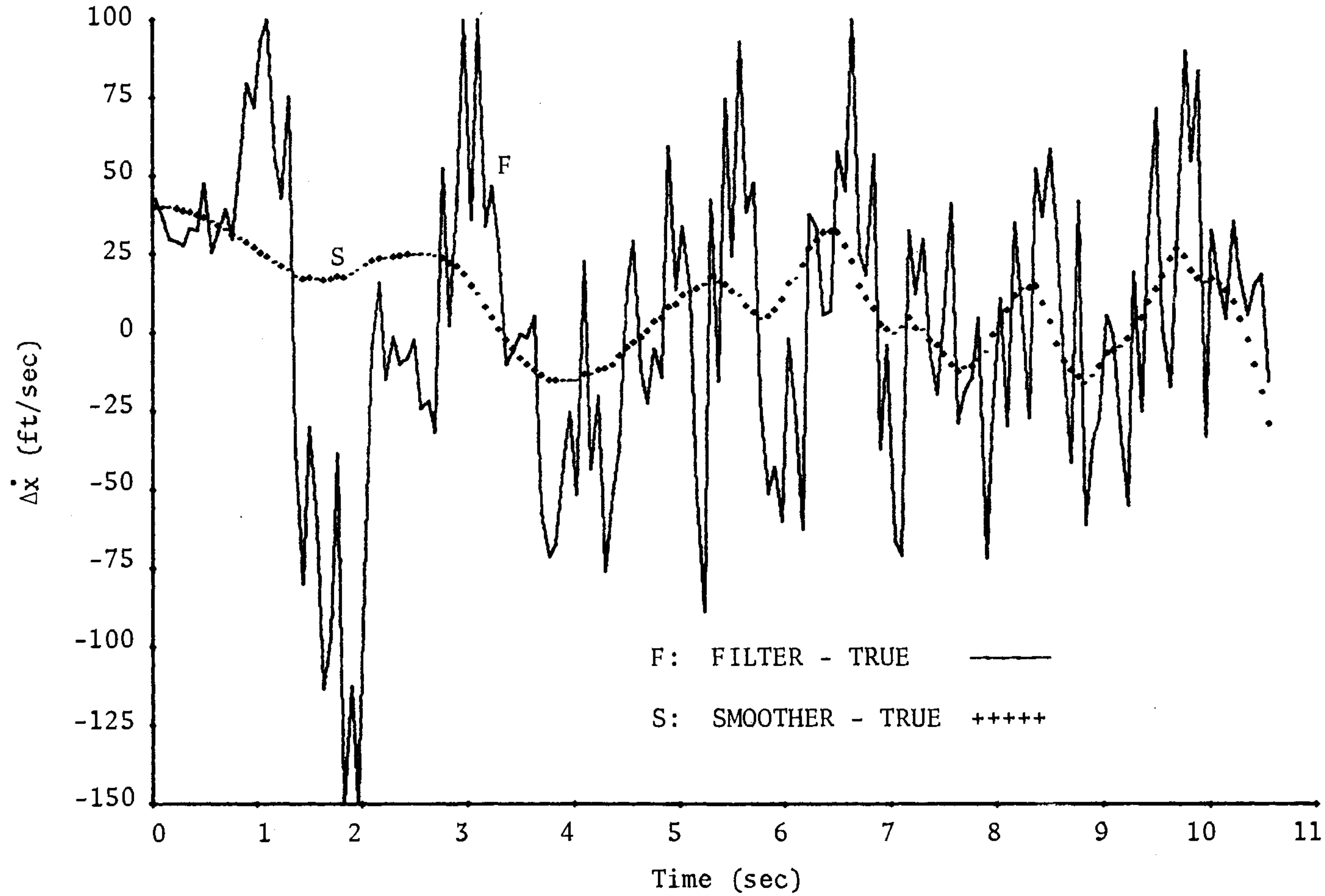

Figure 19. Error in x Velocity: Constant β, Plant Noise Added

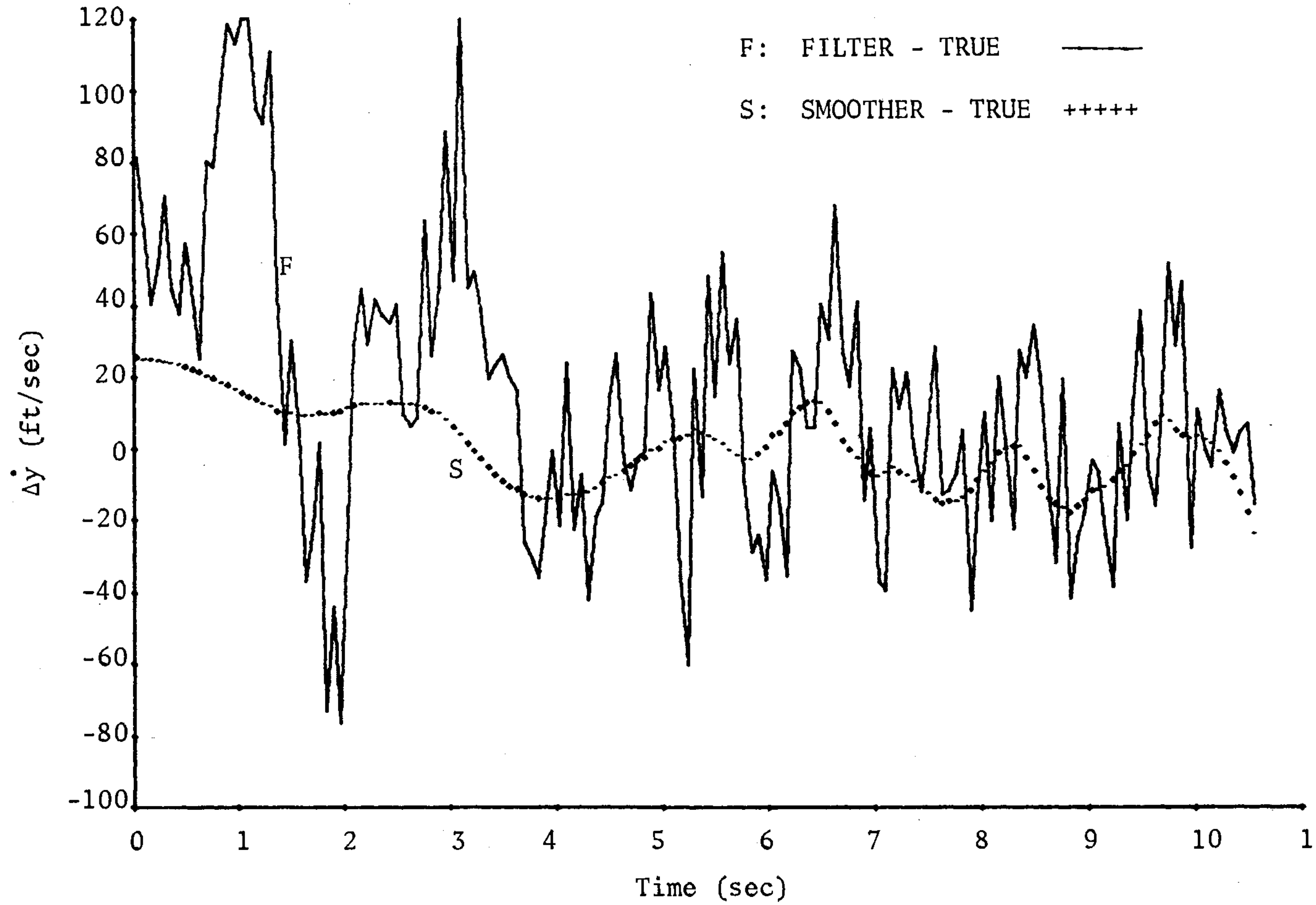

Figure 20. Error in y Velocity: Constant β, Plant Noise Added

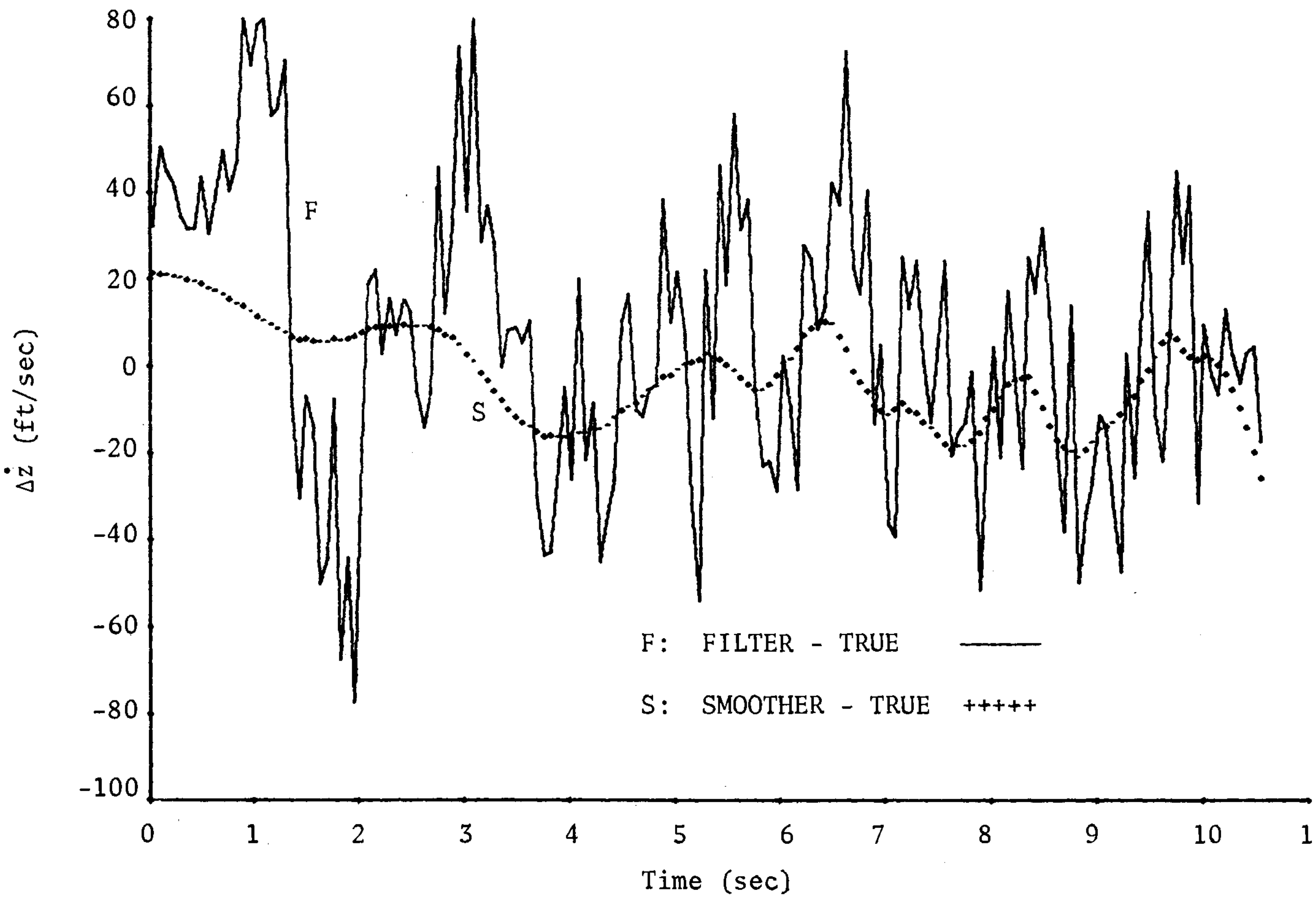

Figure 21. Error in z Velocity: Constant β, Plant Noise Added

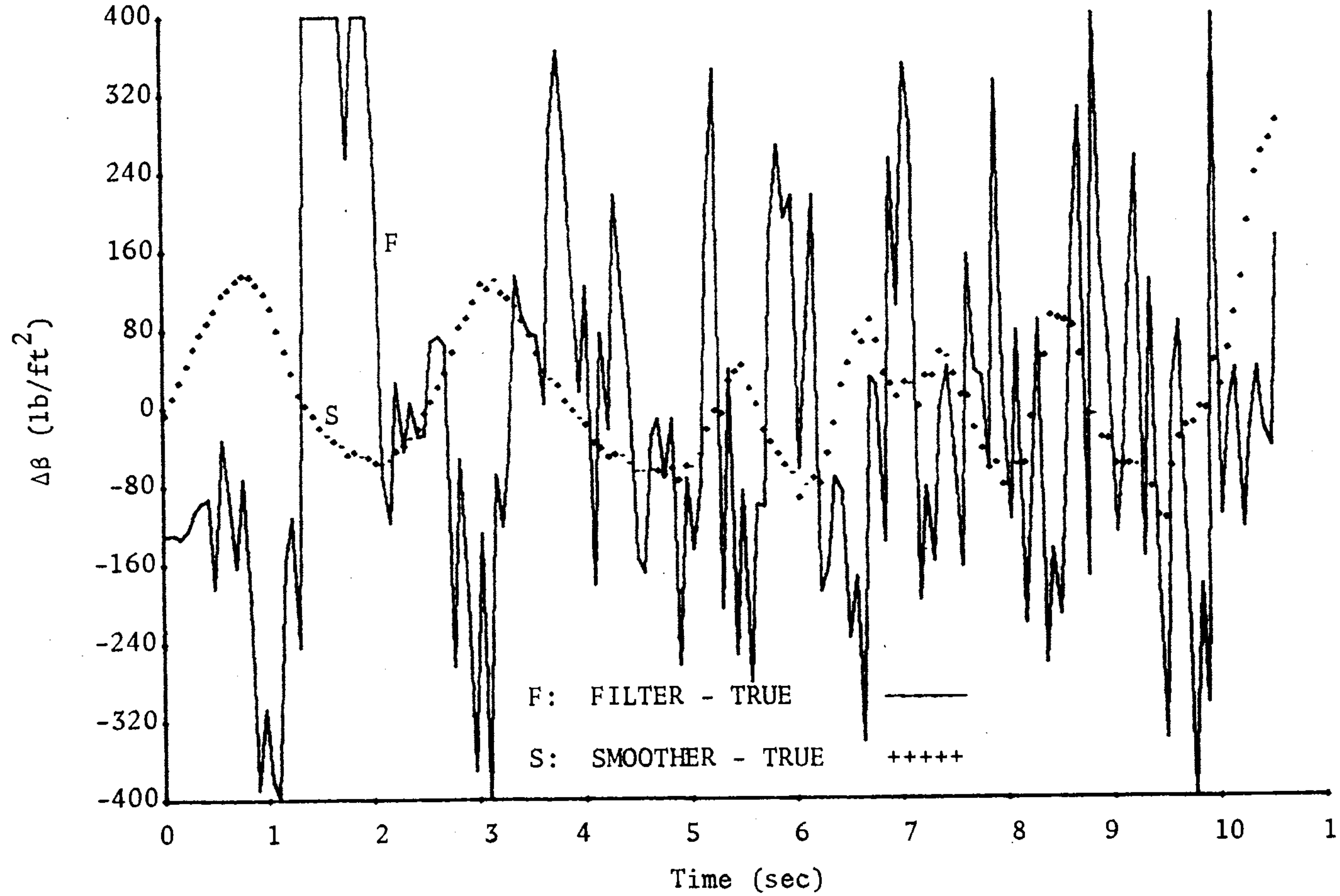

Figure 22. Error in Ballistic Coefficient: Constant β, Plant Noise Added

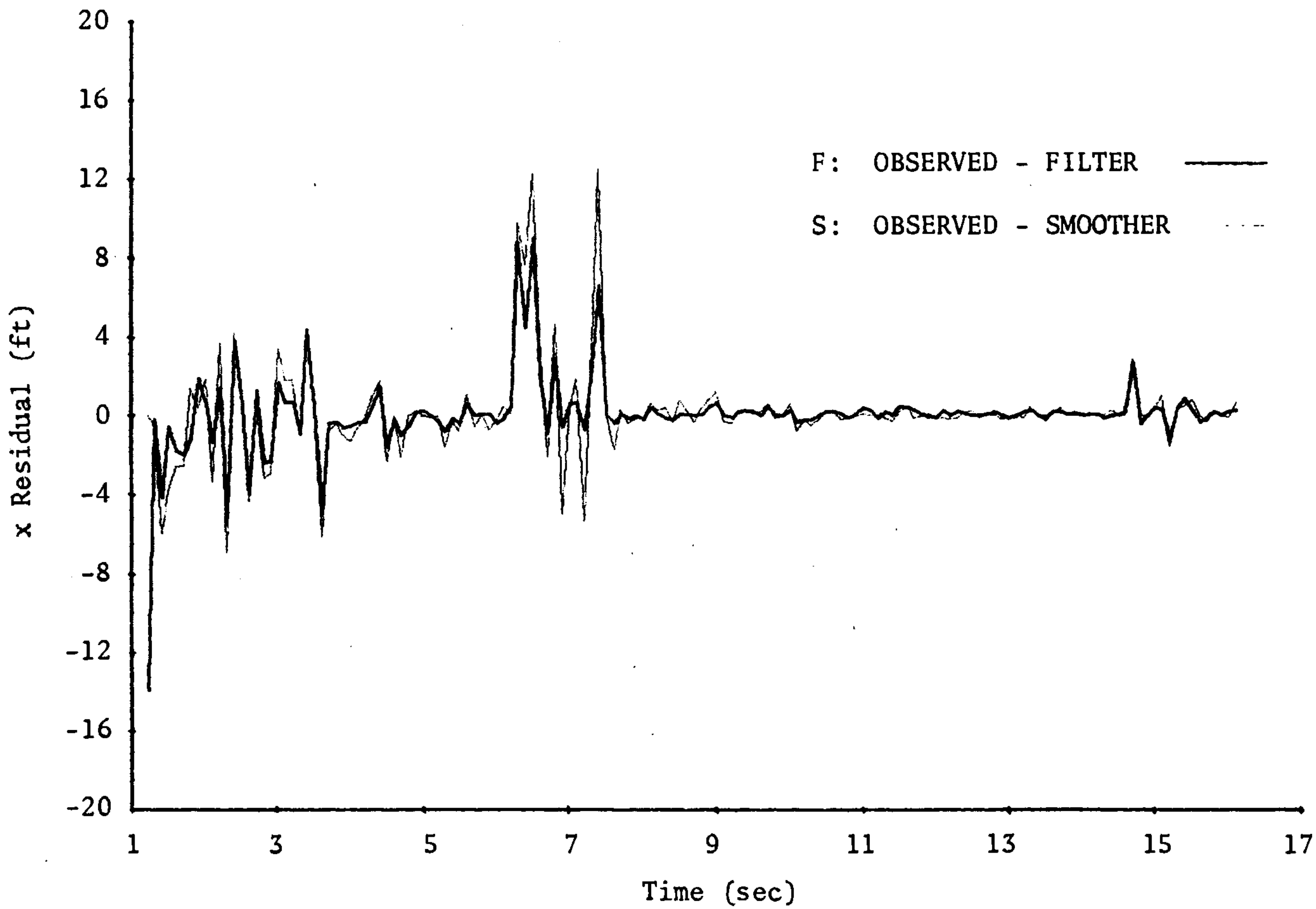

Figure 23. Position Residual x Component, Actual Data, Plant Noise Added

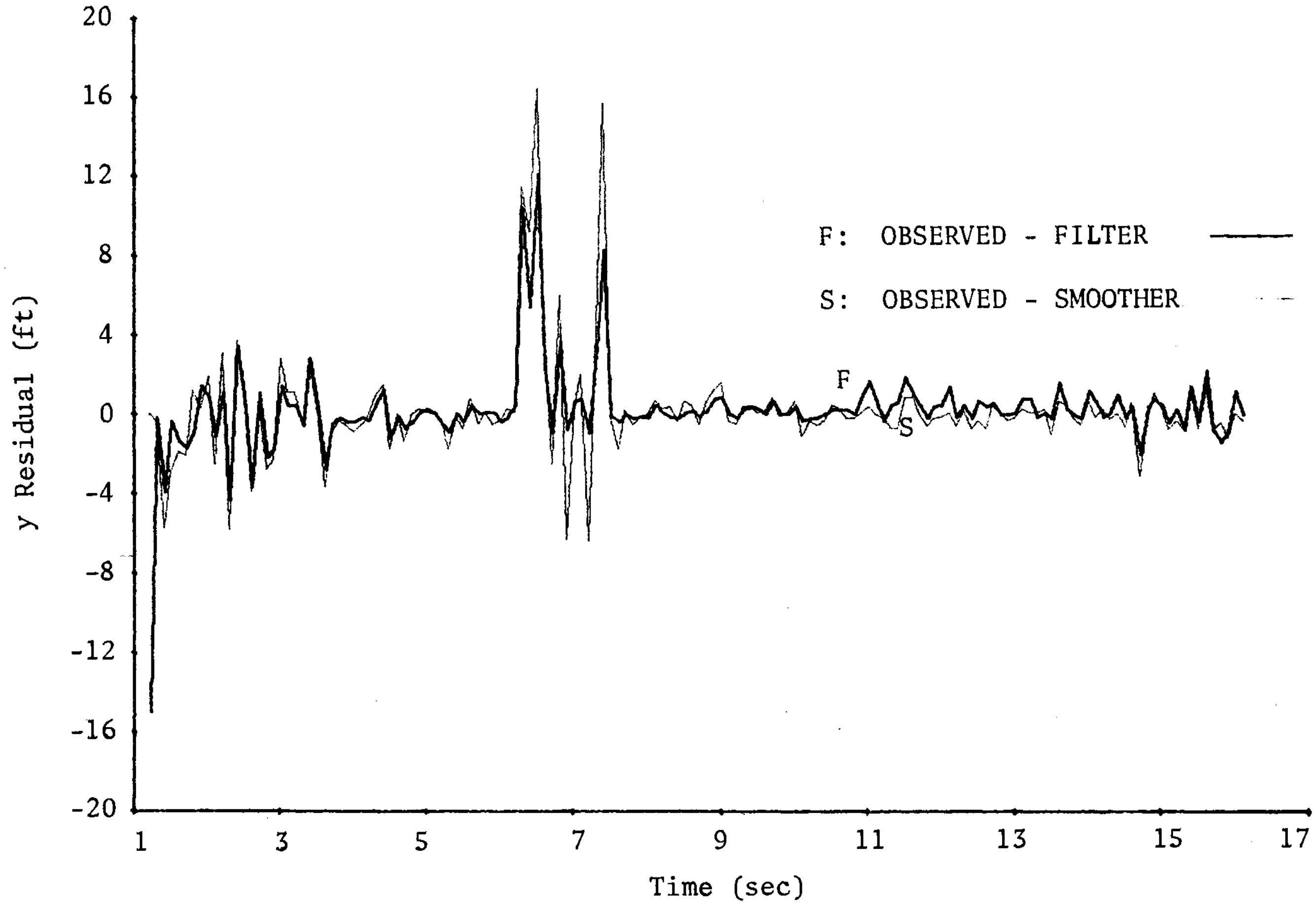

Figure 24. Position Residual y component, Actual Data, Plant Noise Added

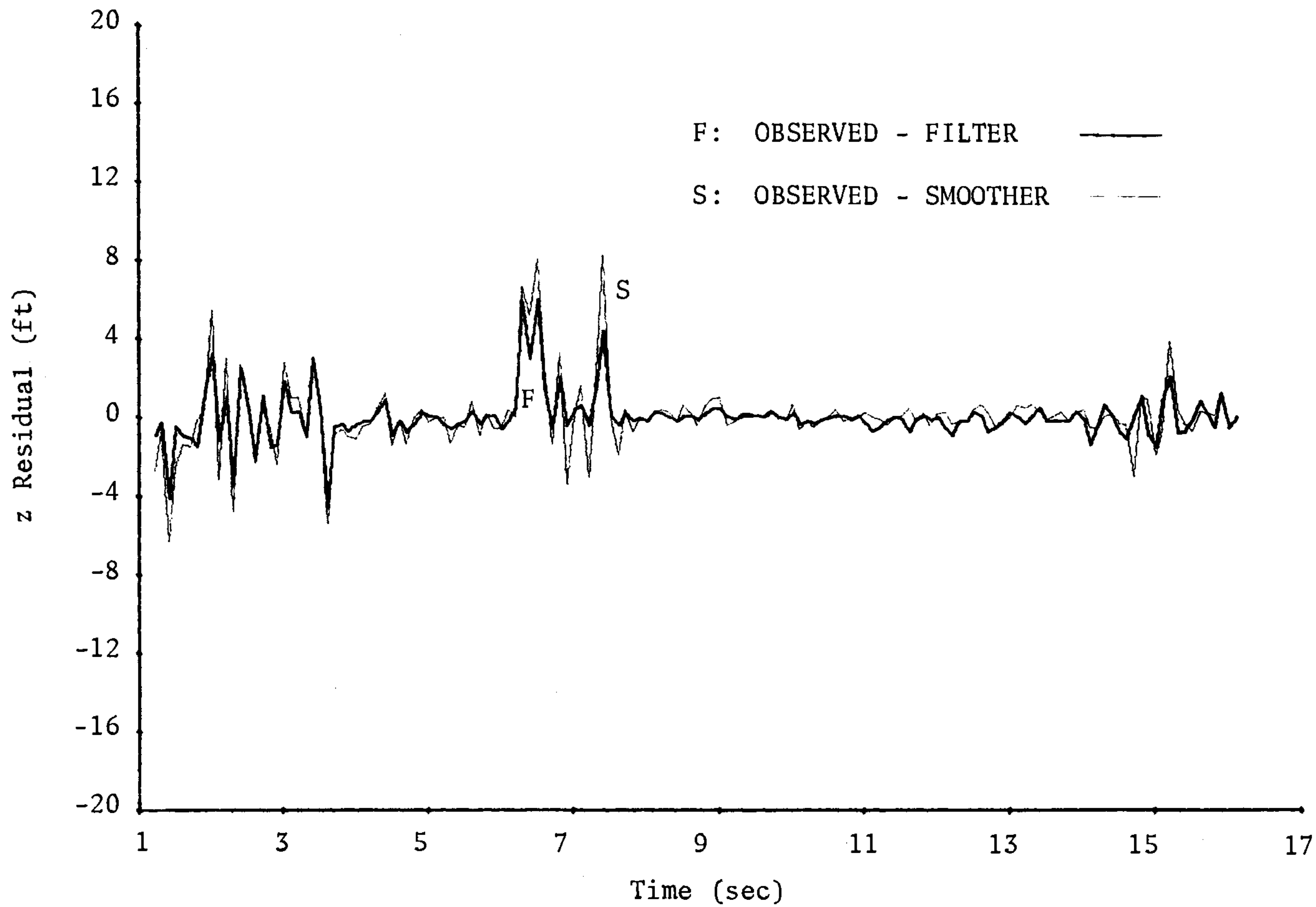

Figure 25. Position Residual z component, Actual Data, Plant Noise Added

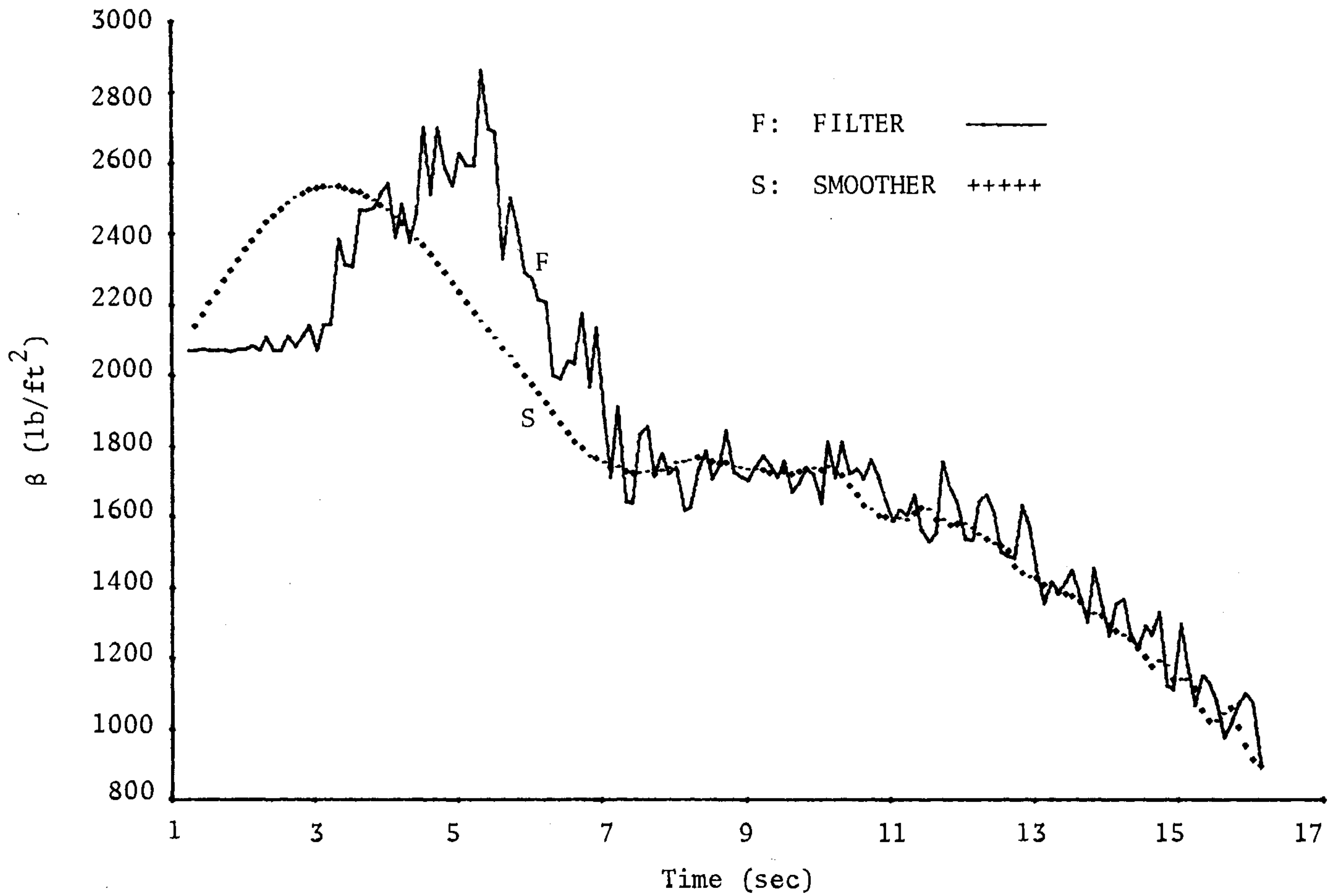

Figure 26. Ballistic Coefficient, Actual Data, Plant Noise Added

acceleration of the vehicle as described in Appendix A. The assumptions that the vehicle is a point mass and in between observations, the drag parameter stays constant, are not quite true. The density varies by four orders of magnitude from an initial altitude of 80 kilofeet to the ground level. An error in density model would be reflected in the ballistic coefficient estimates. These are some of the factors that make the ballistic coefficient very difficult to estimate.

A comparison of the performance of the software package to the results published by Athans et al. (8), Mehra (9), Jazwinski (11) and Schlee et al. (12) is made. In (8), a comparison of the performance of a first- and second-order filter as applied to a third-order problem is presented. In (9) a number of linear, extended and nonlinear filtering algorithms and their performance have been presented. Nonlinear filter performance using radar data has been presented in (11). The divergence problem is discussed in (12). The results here are comparable to those published in the literature even though the filtering problem using optical tracking data is inherently more difficult than the radar data filtering problem. The primary reason the optical tracking problem treated here is more difficult than the radar tracking problem is that when using radar, the reentry vehicle can be acquired exoatmospherically when the trajectory is still ballistic. Conseqeuntly much better initial estimates can be generated than if the reentry is acquired endoatmospherically as is the case for optical trackers. In the present problem initial estimates are generated using tracking data taken when the reentry vehicle is in a highly dynamic, nonlinear area in which the drag forces are beginning to take effect and the vehicle is oscillating about its roll, pitch and yaw axis. Also, several of the radar filtering problems treated in the literature involved doppler radars so that in addition to position fixes, range rate data was also available.

In the present study a seventh-order nonlinear problem is treated. The divergence problems encountered are solved to a large extent using an adaptive plant noise algorithm. A method based on chi-square test is presented and applied to solve the divergence problem. Also nonlinear smoothing estimates are presented along with the filter estimates.

CHAPTER VII

STATISTICAL ANALYSIS

7.1 Introduction

A second-order filtering and smoothing algorithm that exhibits satisfactory performance for three different noisy observation data sets, has been developed. An inherent question involved is how well the software package would perform when it processes some different noisy observation data set. A Monte Carlo analysis was performed to answer this question. A statistical evaluation of the program performance is the objective of this chapter. N runs of the filter and smoother each using a different noisy data set were made, and the sample means and variances of the difference between smoothed estimate and true state were computed. The results were as expected, although the small sample size required because of financial (computer time) constraint caused some noisiness in the sample means and variances. The adaptive plant noise algorithm was used for all runs.

7.2 Statistical Analysis Method

In order to evaluate the performance of the computer software package, the simulated observation data set for the varying drag vehicle was selected. The covariance matrices, R_k, for all the observation instants were held unchanged. Using the R_k matrices, a position error trajectory was generated by Monte Carlo technique. The observation position data were obtained by adding error to the nominal trajectory. This process yields a position observation data set whose covariance matrices are R_k. Different position observation data sets are obtained by using a different generator seed, or starting parameter for each Monte Carlo run. The noisy observations and the error covariance matrices are inputs to the software package. After obtaining the smoothed state vector x_s, the smoothed error vector $x_s - x_t$, as a function of time is generated and stored, where x_t is the true state vector.

The smoothed state error trajectory is obtained for N sets of position observations, where N is the number of Monte Carlo runs made for the statistical analysis. In particular, the value of N was nine due to the stated constraint. Then based on these error trajectories the statistical performance of the package is evaluated, at least to the extent possible with the small sample size. As was stated earlier, the plant noise algorithm was the same for each run, but initial estimates of the state and estimation error covariance matrix were different. The number of observations used to generate the initial values was selected separately for each Monte Carlo run.

The sample means and variances for different elements of the smoothed state vector were calculated. The sample mean of k state error trajectories is computed by

$$\bar{e}_k(t_j) = \frac{\sum_{i=1}^{k} e_i(t_j)}{k} \tag{7-1}$$

where, $e_i(t_j)$ is the error vector at time t_j on the i-th run. Eight sample mean trajectories for each component of the state vector were calculated based on two through nine runs. The sample variance of the state error trajectories is computed by

$$\sigma_k^2(t_j) = \frac{\sum_{i=1}^{k} (e_i(t_j) - \overline{e}_i(t_j))^2}{k - 1} \tag{7-2}$$

where, $\overline{e}_i(t_j)$ is calculated by Equation (7-1). As before, eight sample variance trajectories for each component of the state vector were calculated.

7.3 Analysis Results

As explained in the previous section, the sample means and variances of various components of the state error vector are calculated using k Monte Carlo runs. These quantities are plotted for two, six and nine runs. The curves for three position and velocity elements show the same trend. Thereby the results for the x component of position and velocity are presented instead of all the three x, y and z components.

Figures 27 through 29 show the sample mean error trajectories for x components of position and velocity and ballistic coefficient, β, elements of the state vector. There are three curves on each graph, identified by the number of Monte Carlo runs, as the parameter. An examination of these figures indicates that average errors become decreasingly small as more runs are made.

Figures 30 through 32 show the sample variance trajectories for three elements: x position, x velocity and the ballistic coefficient, β, of the state vector. These figures show that as more runs are made the sample variance at first increases with the number of Monte Carlo runs but later decreases. The confidence in the estimates increases with the sample size as indicated by the decrease in the sample variances.

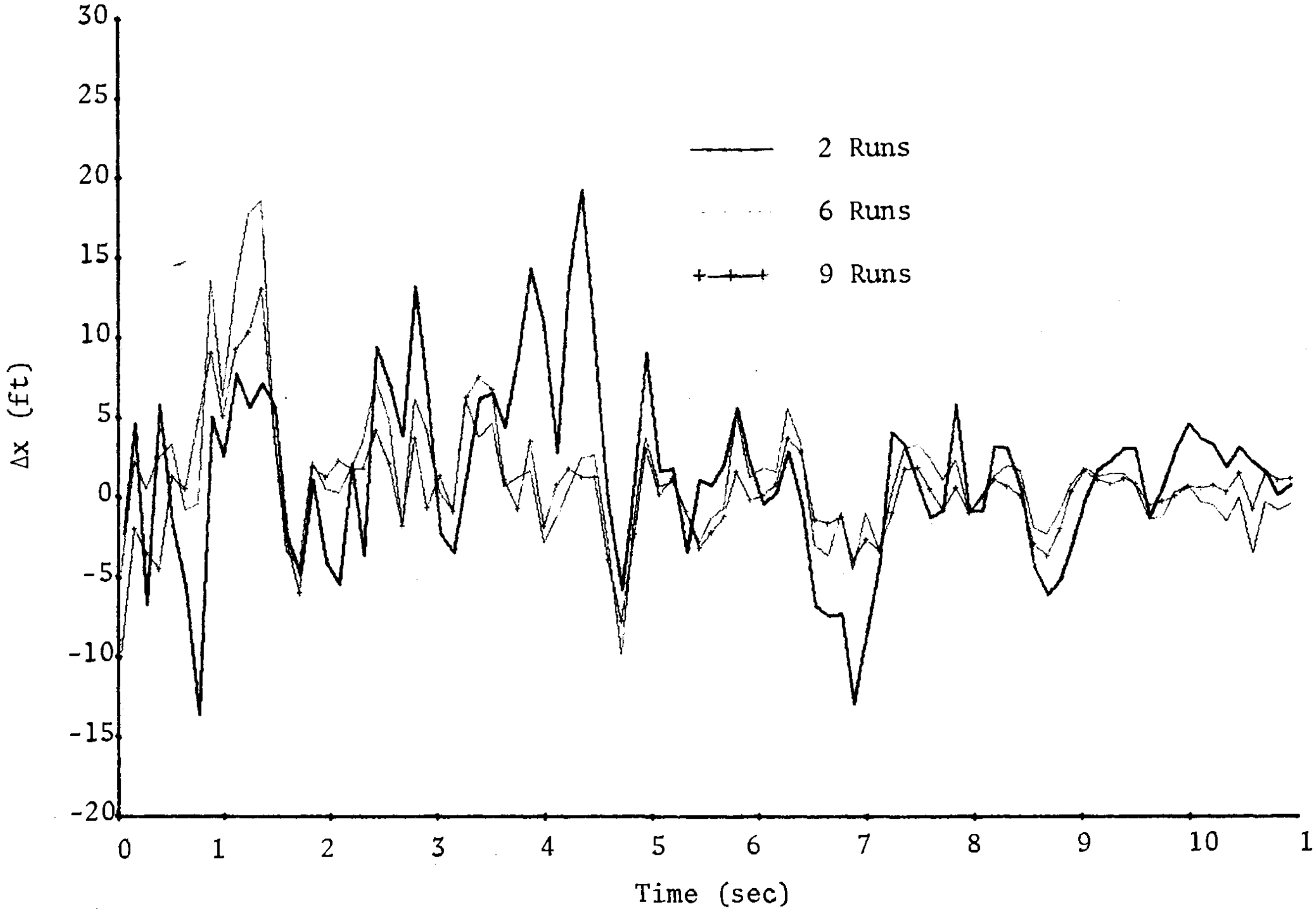

Figure 27. Sample Mean of Smoothed Error: x Component

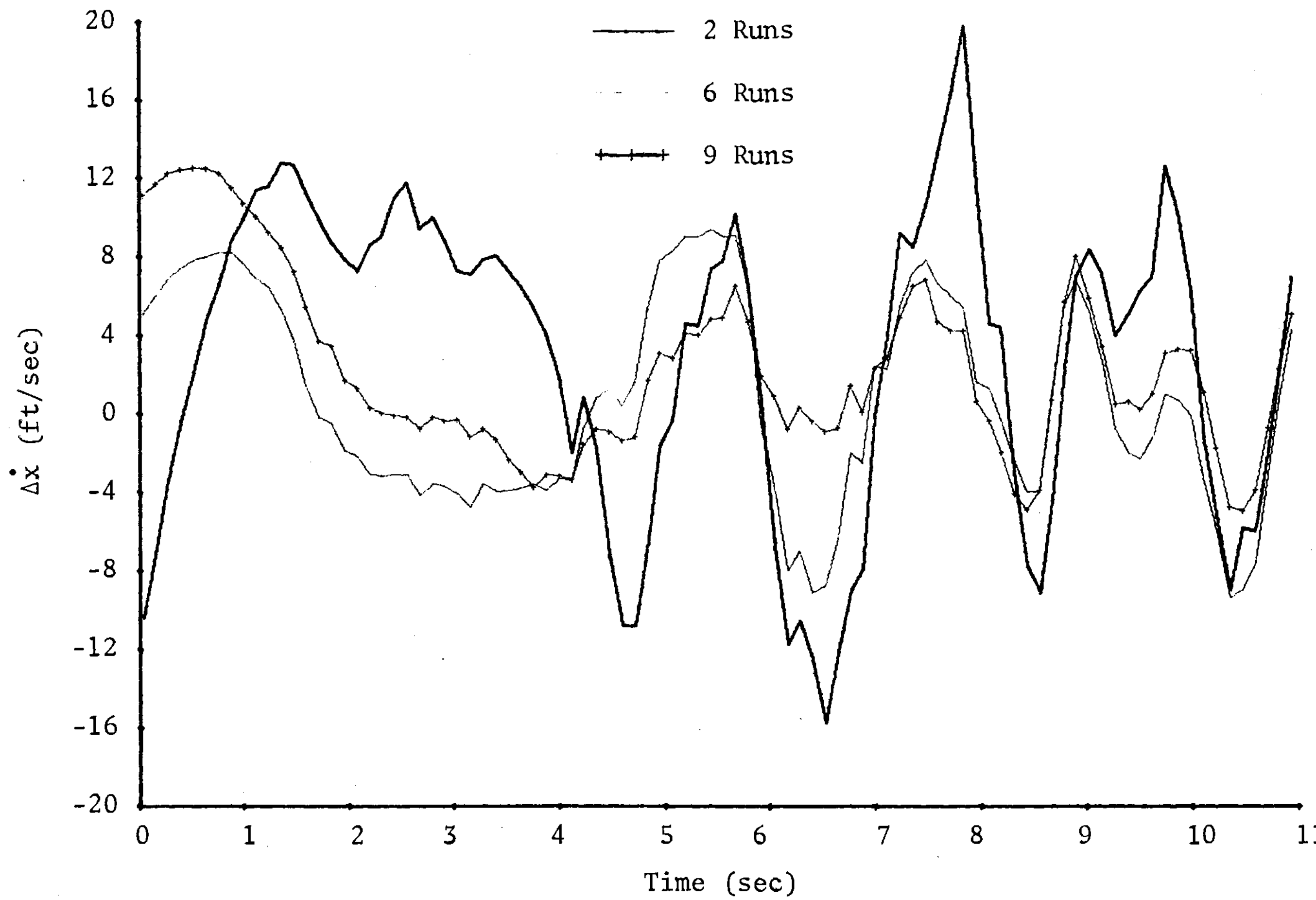

Figure 28. Sample Mean of Smoothed Error: $\dot{x}$ Component

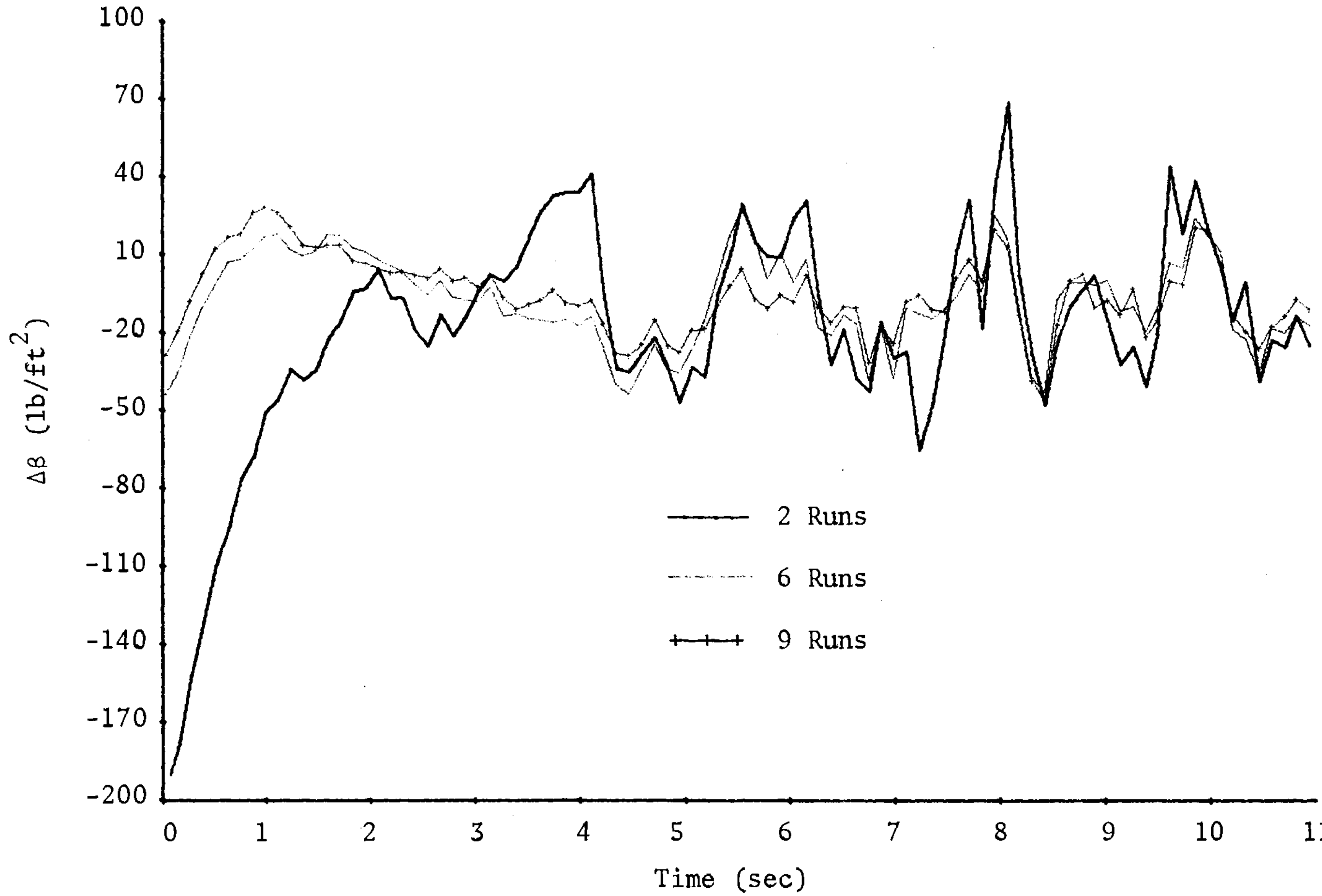

Figure 29. Sample Mean of Smoothed Error: β Component

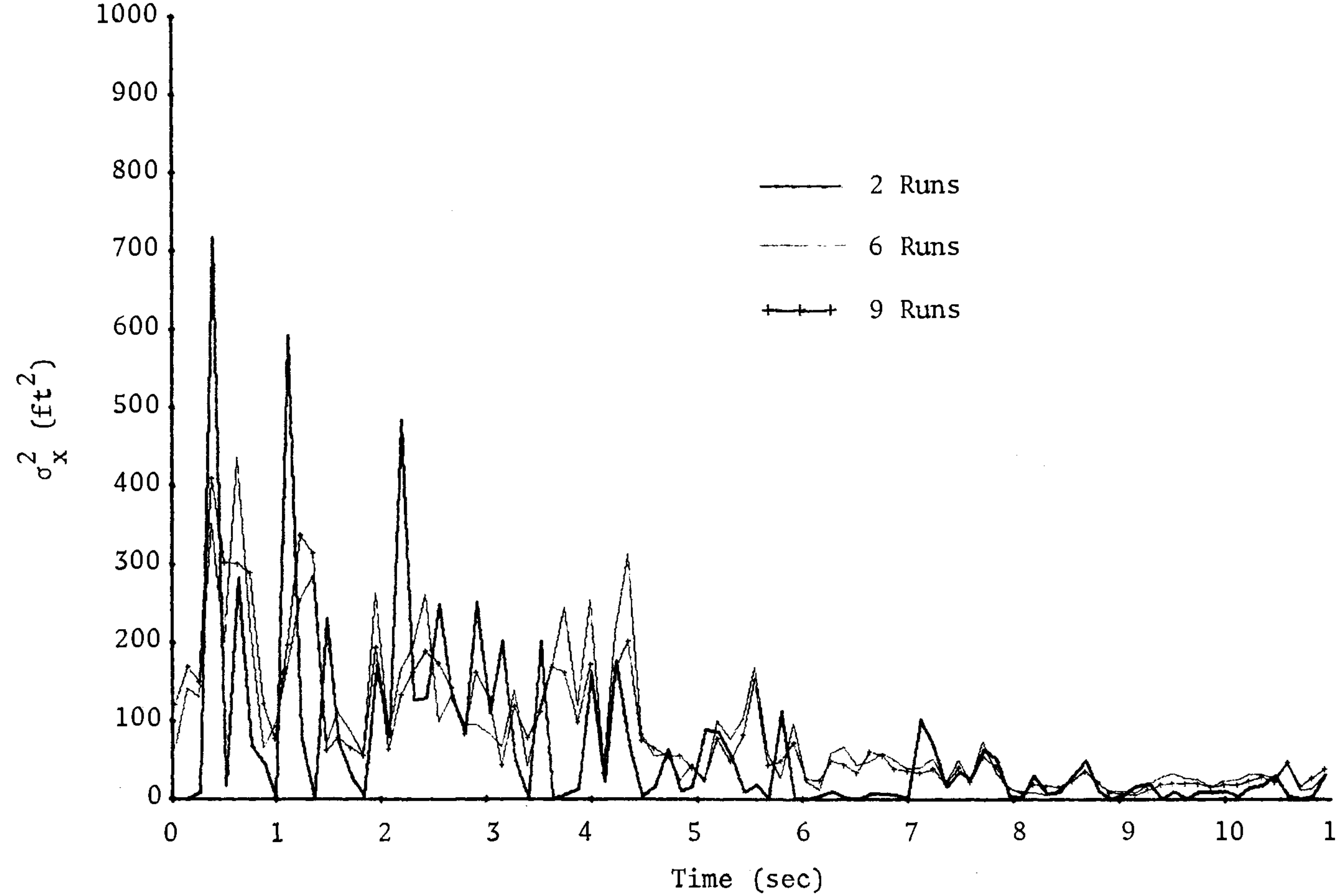

Figure 30. Sample Variance of Smoothed Error: x Component

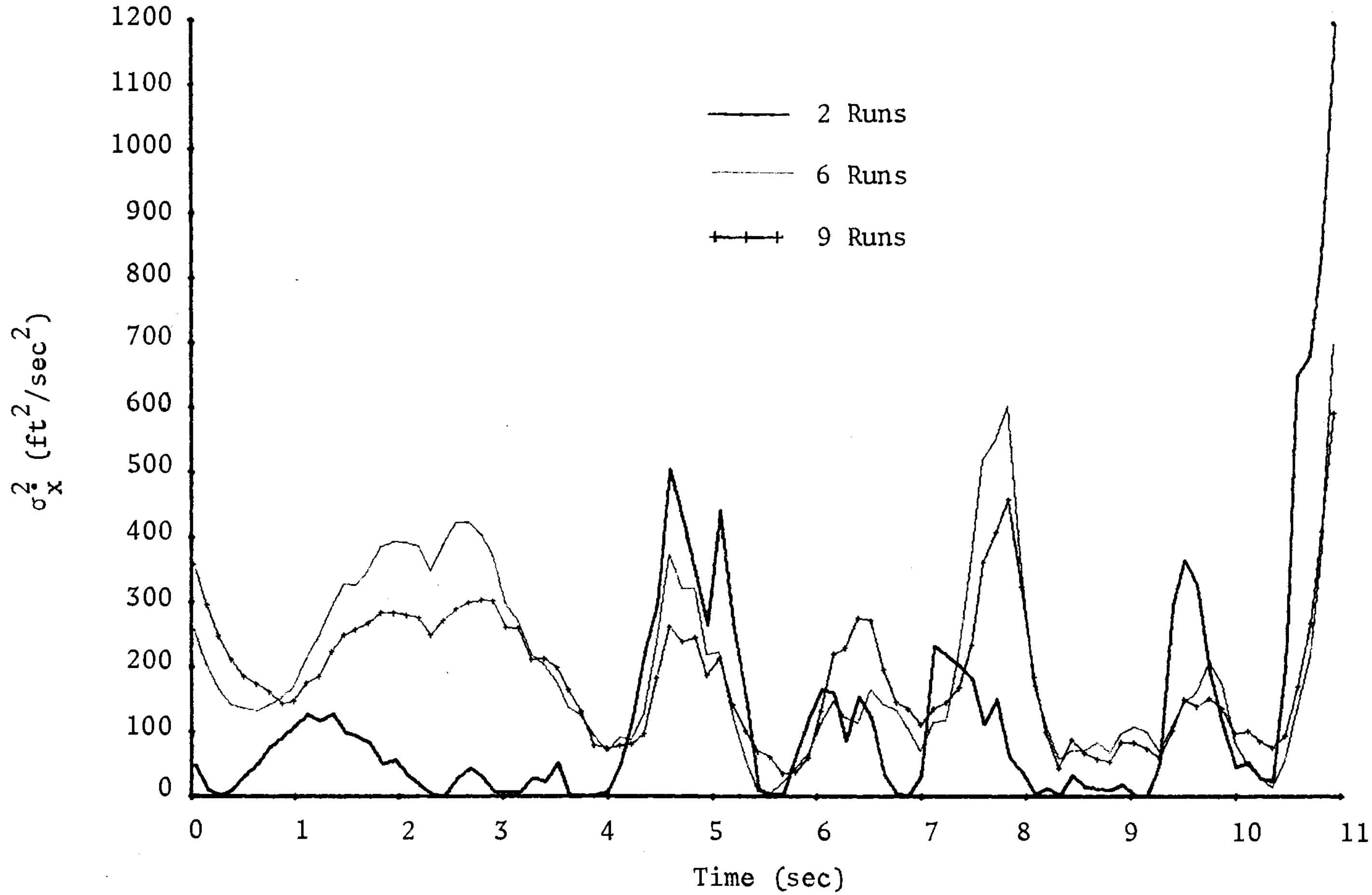

Figure 31. Sample Variance of Smoothed Error: $\dot{x}$ Component

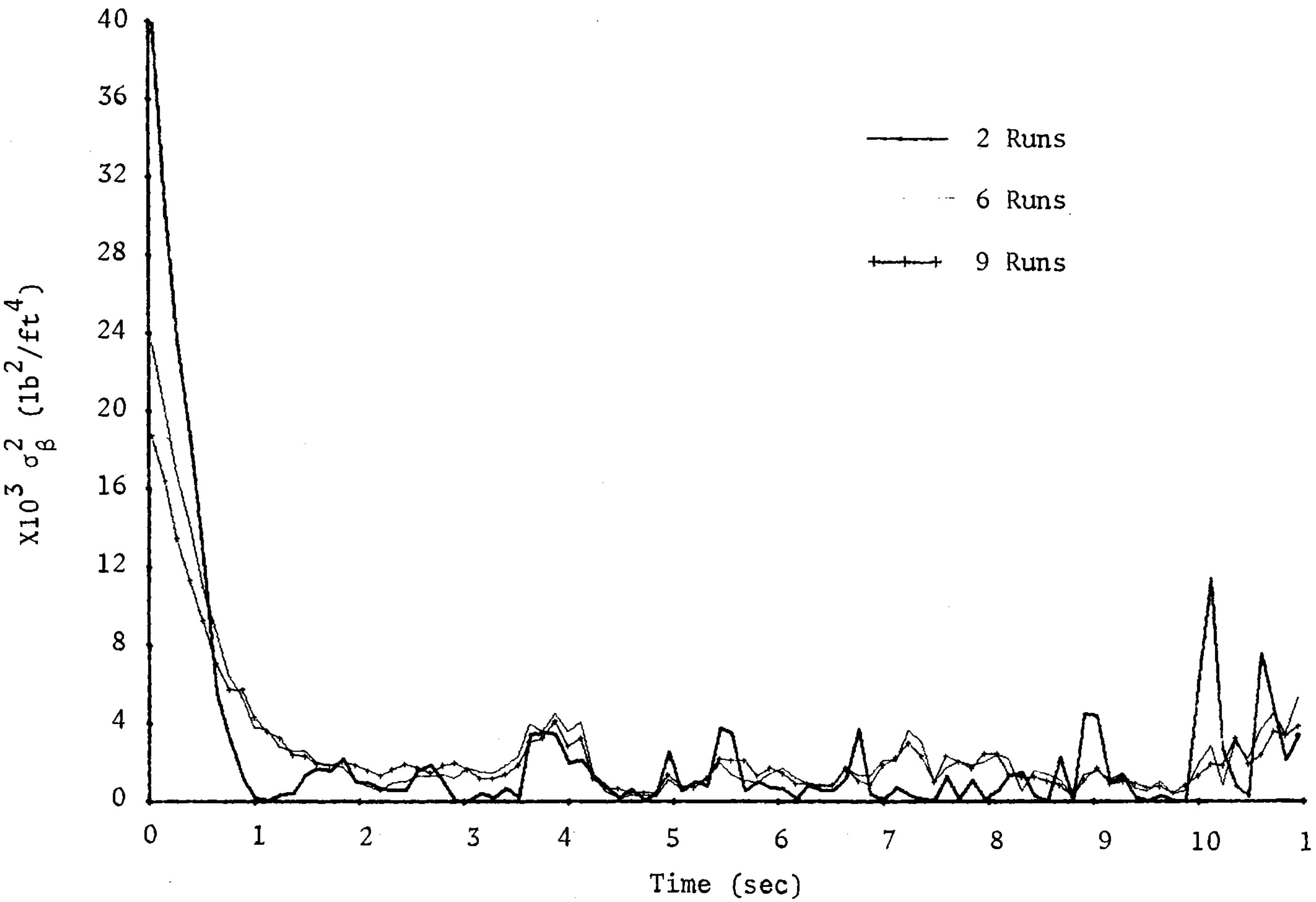

Figure 32. Sample Variance of Smoothed Error: β Component

7.4 Summary

In this chapter, the statistical analysis of the software package was performed within the stated financial constraints. Nine Monte Carlo runs were made and the sample means and variances of the smoothed error trajectories were calculated and plots made. Based on this analysis, it is concluded that if a noisy data set is processed by the software package, the smoothed state estimates would be "close" to the true state with a high degree of confidence.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

8.1 Summary

In the present study a solution to the problem of estimating the state of reentry vehicles using the optical tracking data has been presented. This involved development of a dynamic model as described in Chapter II. A linear observation model is obtained by making use of triangulated position observations.

Initially, the extended Kalman filter was found to provide unsatisfactory results because of the highly nonlinear nature of this problem. The second-order filter algorithm of Chapter III, with the plant noise algorithm improves the estimates considerably. The plant noise accounts for unmodeled errors and computation round off errors. The program was converted to double-precision to reduce effects of computational errors. A chi-square test is used to insure consistency between the position residuals and corresponding error covariance matrix. Divergence of the drag parameter estimates is eliminated by adding σ_α^2 (Equation (4-21)) to the corresponding variance element of the state error covariance matrix. The above choice is based on a chi-square test for one degree of freedom and is explained in Section 4.4. In Chapter V, a nonlinear fixed-interval smoother algorithm has been developed. An efficient method of implementation of this algorithm is also described.

The simulation and numerical results chapter describes the performance of the filter and smoother algorithms with and without the plant noise. The software package performance is described by the results obtained by its application to three data sets. A statistical method of evaluating the performance of the software package has been presented in Chapter VII.

8.2 Conclusions

The computer software package developed has shown the ability to generate much better estimates than those generated by linearized filters applied to the highly nonlinear problem of reentry. This package includes a second-order filter and a fixed-interval smoother. One main feature of the program is the incorporation of the adaptive plant noise to prevent filter divergence. The plant noise accounts for uncertainties in model parameters, unmodeled errors, incomplete dynamic model and round off errors. The parameters involved in the adaptive plant noise algorithm are selected on the basis of simulations. The fixed-interval smoother improves the filtered estimates as is shown in Chapter VI.

If a software package for some other nonlinear estimation problem is to be developed, an approach similar to the present problem is recommended. The salient features of this development would be:

(1) development of dynamic and observation models;

(2) evaluation of the Jacobian and Hessian matrix elements for dynamic and observation models;

(3) identification of uncertain and round off error parameters from simulations; and

(4) filtering and smoothing with and without plant noise.

Depending on the order of the system, the amount of computations may vary. If the order of the system is small the calculation of elements in step two may be trivial, but for any reasonable order system a large number of elements must be calculated. For example in the present seventh-order system 49 Jacobian matrix elements and 343 Hessian matrices elements must be computed. Identification of uncertain parameters and associated constants may not be a trivial matter.

The techniques developed for estimating the state of the seventh-order nonlinear system by the use of a second-order filter with adaptive plant noise and fixed-interval smoothing is a contribution to the area of nonlinear filtering applied to reentry estimation.

8.3 Suggestions for Further Work

There are many obvious extensions of this work and a few important topics are suggested here.

The development of the dynamic model with lift forces accounted for. The elliptical earth model rather than spherical, could be incorporated. Both would add to the complexity of the system and possibly call for an increase in the order of the state vector.

Further recommendations are to use as inputs the sensor angle data directly instead of the triangulated position fixes. This would have a disadvantage of increased complexity since the observation model would no longer be linear.

The parameters for the adaptive plant noise algorithm are presently chosen based on extensive simulations. If the dynamic model was more precisely known in that the above refinements were incorporated, a less complex plant noise algorithm would be required.

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APPENDIX A

INITIALIZING SUBROUTINE XNTIAL

The subroutine XNTIAL is used to generate estimates of the initial state vector and corresponding error covariance matrix. For the present problem, the observation data consists of noisy positions and associated error covariance matrices. As stated in Chapter II, the state vector has three position components (x,y,z), three velocity components $(\dot{x},\dot{y},\dot{z})$, and drag parameter (α). The inverse of the drag parameter is a quantity commonly known as ballistic coefficient, β. This is a number associated with a vehicle in motion. It is a measure of the "slipperiness" of the vehicle as it moves through air. The force on the reentry vehicle can be written as a wind pressure Q times the cross-sectional area A on which the pressure acts. Since the vehicle is not a flat plate, the coefficient of drag, C_D, to account for shape effects is used. Hence the drag force acting on the vehicle is QC_DA. The mass of the vehicle is w/g_0 where w is the vehicle weight at sea level. If the decerleation caused by the drag is a_D, then

$$QC_DA = \frac{w}{g_0} a_D$$

$$\frac{w}{C_DA} = \frac{g_0 Q}{a_D} .$$

The left hand side of the above equation is defined to be the ballistic coefficient β. That is

$$\beta = \frac{g_0 Q}{a_D}$$

when the air is sufficiently dense, the pressure Q can be written as $1/2\ \rho v^2$ where ρ is the mass density of the air and v is the vehicle velocity relative to air. Then

$$\beta = \frac{\frac{1}{2}\rho g_0 v^2}{a_D} \quad .$$

The drag acceleration is

$$a_D = g\sin\gamma - a_T$$

where a_T is the acceleration along the velocity vector and γ is the angle between local horizontal and the velocity vector. The drag parameter depends on the position, velocity and acceleration of the vehicle, gravity and atmospheric density.

At the initial time the geometry of the situation is shown in Figures 33 and 34. The quantities estimated by XNTIAL are the initial state vector

$$x_{1/1} = (x_1 y_1 z_1 \dot{x}_1 \dot{y}_1 \dot{z}_1 \alpha_1)^T$$

and the associated error covariance matrix based on the first N observations. Here subscript 1 indicates the initial time. During early reentry, the trajectory is assumed to be very close to ballistic, therefore the position components can be approximated by

$$x(t) = a_1 + a_2 t + a_3 t^2$$

$$y(t) = b_1 + b_2 t + b_3 t^2$$

$$z(t) = c_1 + c_2 t + c_3 t^2$$

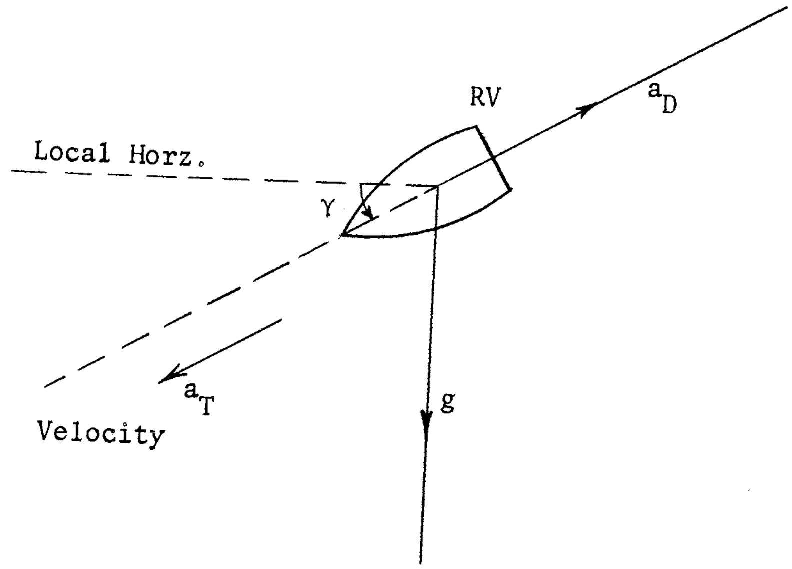

Figure 33. Geometry of Initial Reentry

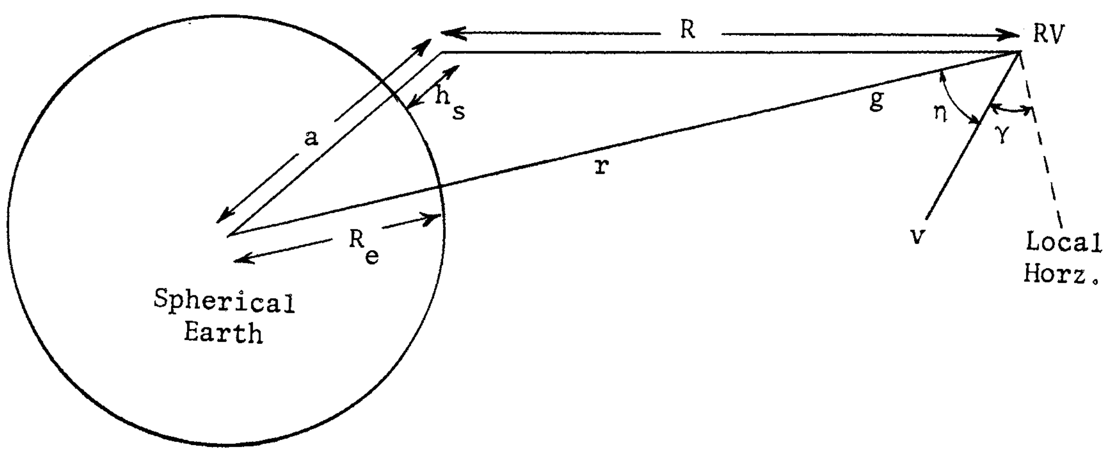

Figure 34. Reentry Vehicle Position With Respect to Earth at Initial Time

where t represents time and the coefficients a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are to be estimated.

The least-squares method of curve fitting is used to evaluate the coefficients and associated covariance matrix. The x component of N observations can be written in matrix form as

$$\begin{array}{ccccccc} X & & A & & & & n \\ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} & = & \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_N & t_N^2 \end{bmatrix} & \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} & + & & \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix} \end{array} \tag{A-1}$$

where n_i is assumed to be zero mean, Gaussian, uncorrelated noise with variance σ^2. Then the covariance matrix associated with the vector n is

$$R = \sigma^2 I_{NXN}$$

and

$$R^{-1} = \frac{1}{\sigma^2} I_{NXN} \quad .$$

Then by the least square method, the estimate of coefficients (a_1, a_2, a_3) is

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = (A^T R^{-1} A)^{-1} A^T R^{-1} X \quad .$$

This expression can be simplified to

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = (A^TA)^{-1}A^TX \quad .$$

In terms of the observed values of x position and the instants of time the preceding expression can be written as

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} N & \Sigma t_i & \Sigma t_i^2 \\ \Sigma t_i & \Sigma t_i^2 & \Sigma t_i^3 \\ \Sigma t_i^2 & \Sigma t_i^3 & \Sigma t_i^4 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma x_i \\ \Sigma x_i t_i \\ \Sigma x_i t_i^2 \end{bmatrix} \tag{A-2}$$

and the error covariance matrix associated with this estimate is

$$(A^TR^{-1}A)^{-1} = \sigma^2(A^TA)^{-1} \quad .$$

The x component of position, velocity and acceleration at initial time can be calculated by

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \end{bmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 0 & 1 & 2t_1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \quad .$$

The covariance matrix associated with this estimate is given by

$$\begin{bmatrix} \sigma_x^2 & c_{x\dot{x}} & c_{x\ddot{x}} \\ c_{x\dot{x}} & \sigma_{\dot{x}}^2 & c_{\dot{x}\ddot{x}} \\ c_{x\ddot{x}} & c_{\dot{x}\ddot{x}} & \sigma_{\ddot{x}}^2 \end{bmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 0 & 1 & 2t_1 \\ 0 & 0 & 2 \end{bmatrix} \sigma^2(A^TA)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_1^2 & 2t_1 & 2 \end{bmatrix} \tag{A-3}$$

where matrix A has been given in Equation (A-1). σ^2, the variance associated with n_i is approximated by the sample variance

$$\sigma^2 = \frac{1}{N - 1} \sum_{i=1}^{N} (x_i - \bar{x}_i)^2 \quad .$$

The mean at any time t_i is calculated by using the value of time and the coefficients given by Equation (A-2).

The procedure to evaluate initial estimates of $(x_1, \dot{x}_1, \ddot{x}_1)$ and the associated error covariance matrix has been described in detail. This procedure is repeated to estimate $(y_1, \dot{y}_1, \ddot{y}_1)$, and $(z_1, \dot{z}_1, \ddot{z}_1)$ and their associated error covariance matrices at the initial time.

At the initial point in time, the position, velocity and acceleration estimates in the x, y, z directions,

$$[x_1 y_1 z_1 \dot{x}_1 \dot{y}_1 \dot{z}_1 \ddot{x}_1 \ddot{y}_1 \ddot{z}_1]$$

have been evaluated and their covariance matrix (9x9) is given by

$$\begin{bmatrix}
\sigma_x^2 & 0 & 0 & c_{x\dot{x}} & 0 & 0 & c_{x\ddot{x}} & 0 & 0 \\
0 & \sigma_y^2 & 0 & 0 & c_{y\dot{y}} & 0 & 0 & c_{y\ddot{y}} & 0 \\
0 & 0 & \sigma_z^2 & 0 & 0 & c_{z\dot{z}} & 0 & 0 & c_{z\ddot{z}} \\
c_{x\dot{x}} & 0 & 0 & \sigma_{\dot{x}}^2 & 0 & 0 & c_{\dot{x}\ddot{x}} & 0 & 0 \\
0 & c_{y\dot{y}} & 0 & 0 & \sigma_{\dot{y}}^2 & 0 & 0 & c_{\dot{y}\ddot{y}} & 0 \\
0 & 0 & c_{z\dot{z}} & 0 & 0 & \sigma_{\dot{z}}^2 & 0 & 0 & c_{\dot{z}\ddot{z}} \\
c_{x\ddot{x}} & 0 & 0 & c_{\dot{x}\ddot{x}} & 0 & 0 & \sigma_{\ddot{x}}^2 & 0 & 0 \\
0 & c_{y\ddot{y}} & 0 & 0 & c_{\dot{y}\ddot{y}} & 0 & 0 & \sigma_{\ddot{y}}^2 & 0 \\
0 & 0 & c_{z\ddot{z}} & 0 & 0 & c_{\dot{z}\ddot{z}} & 0 & 0 & \sigma_{\ddot{z}}^2
\end{bmatrix} \quad . \qquad \text{(A-4)}$$

Here all the elements are given by Equation (A-3) and similar expressions for y and z components.

The drag parameter is given by the expression

$$\alpha = \frac{g\sin\gamma - \frac{\vec{F}}{m} \cdot \hat{v}}{\frac{1}{2} g_0 \rho v^2} \tag{A-5}$$

where g is the acceleration due to gravity at altitude h;

γ is the angle between velocity vector and local horizontal;

$\hat{v}$ is the unit velocity vector;

$\frac{\vec{F}}{m}$ is given by Equation (2-7);

g_0 is the acceleration due to gravity at mean sea level;

ρ is the atmospheric density at height h; and

v is the velocity of the vehicle.

The acceleration due to gravity at the altitude h is approximated by

$$g = g_0(1 - \frac{2h}{a})$$

where

$$a = R_e + h_s$$

$$R_e = \text{radius of spherical earth}$$

$$h_s = \text{altitude of ESF system origin} \quad .$$

Sinγ is approximated as follows. From Figure 34

$$\hat{g}\cdot\hat{v} = |\hat{g}||\hat{v}|\cos\eta = \cos\eta = \sin\gamma \tag{A-6}$$

where $\hat{g}$ is a unit vector in the direction from the vehicle to the center of the earth. $\hat{g}$ can be written as

$$\hat{g} = -\hat{r} = -\frac{\bar{r}}{|r|}$$

where $\bar{r}$ is given by Equation (2-4),

$$\bar{r} = \bar{a} + T\bar{R} \quad .$$

Using Equations (2-3) and (2-5) for $\bar{a}$ and T, respectively, the above equation can be written as

$$\bar{r} = \begin{bmatrix} a\cos\mu\cos\theta \\ a\cos\mu\sin\theta \\ a\sin\mu \end{bmatrix} + T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and

$$r = \sqrt{x^2 + y^2 + (z + a)^2} \quad .$$

The unit velocity vector $\hat{v}$ is given by

$$\hat{v} = \frac{\bar{v}}{|\bar{v}|}$$

where

$$\bar{v} = T \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

and

$$|v| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad .$$

Equation (A-6) can be written as

$$\sin\gamma = -\frac{\bar{r} \cdot \bar{v}}{|r||v|}$$

or

$$\sin\gamma = -\frac{\dot{x}x + \dot{y}y + \dot{z}(z + a)}{|r||v|} \quad . \tag{A-7}$$

$\frac{\vec{F}}{m}$ is given by Equation (2-7). Thus

$$\begin{aligned}\frac{\vec{F}}{m} \cdot \overline{v} = \frac{1}{v}[\{&\ddot{x} - 2\dot{y}\omega\sin\mu + 2\dot{z}\omega\cos\mu - \omega^2 x\}\dot{x} \\ &+ \{\ddot{y} + 2\dot{x}\omega\sin\mu - \omega^2\sin^2\mu y + (z + a)\omega^2\sin\mu\cos\mu\}\dot{y} \\ &+ \{\ddot{z} - 2\dot{x}\omega\cos\mu + \omega^2\sin\mu\cos\mu y - (z + a)\omega^2\cos^2\mu\}\dot{z}] \quad .\end{aligned} \tag{A-8}$$

The altitude at the initial time can be evaluated from the following expression using initial position estimates.

$$h = \sqrt{x_1^2 + y_1^2 + (z_1 + a)^2} - R_e \quad .$$

The atmospheric density is calculated by using the subroutine DENY. With this all the quantities in Equation (A-5) are known and the initial estimate of drag parameter can be found.

At this stage all the elements of the initial state vector have been estimated. All elements of the state error covariance matrix are known except the variance of the drag parameter. It is assumed that error in the drag parameter is uncorrelated with the error in other elements of the state vector. From Equations (A-5) through (A-8), it is clear that the drag parameter is a function of all nine positions, velocity and acceleration components; that is

$$\alpha = L(xyz\dot{x}\dot{y}\dot{z}\ddot{x}\ddot{y}\ddot{z}) \quad .$$

The error in the drag parameter is given by

$$\tilde{\alpha} = \overline{A}\tilde{x}$$

where $\overline{A}$ is a nine component gradient vector of function L, and $\tilde{x}$ is a nine component error vector with position, velocity and acceleration as elements. Then the variance of α is evaluated by

$$\begin{aligned} \sigma_\alpha^2 &= E\{\overline{A}\tilde{x}\tilde{x}^T\overline{A}^T\} \\ &= \overline{A}\ E\{\tilde{x}\tilde{x}^T\}\overline{A}^T \end{aligned} \tag{A-9}$$

where $E\{\tilde{x}\tilde{x}^T\}$ is given by Equation (A-4). The next step is to evaluate the gradient vector $\overline{A}$ using the estimates of position, velocity and acceleration at the initial time.

$$\overline{A} = [\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \cdots, \frac{\partial L}{\partial \ddot{z}}] \quad .$$

From Equation (A-5) the expression for the drag parameter can be written as

$$L = \alpha = \frac{\dfrac{-g(\dot{x}x + \dot{y}y + \dot{z}(z + a))}{rv} - \dfrac{\lambda}{v}}{\frac{1}{2}\rho g_0 v^2}$$

where

$$\lambda = v[\frac{\overline{F}}{m} \cdot \hat{v}]$$

is given by Equation (A-8). Hence

$$L = -\left[\frac{g(\dot{x}x + \dot{y}y + \dot{z}(z + a)) + \lambda r}{\frac{1}{2}\rho g_0 v^3 r}\right] \quad .$$

By defining

$$\alpha N = g(\dot{x}x + \dot{y}y + \dot{z}(z + a)) + \lambda r$$

and

$$\alpha D = \rho v^3 r$$

the expression for L reduces to

$$L = - \frac{\alpha N}{\frac{1}{2} g_0 \alpha D} \quad .$$

By using the formula for derivative of a quotient,

$$\frac{\partial L}{\partial x_i} = - \frac{2}{g_0} \frac{\frac{\partial \alpha N}{\partial x_i} \alpha D - \alpha N \frac{\partial \alpha D}{\partial x_i}}{\alpha D^2} \tag{A-10}$$

where x_i represents x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$. The Equation (A-10) requires $\partial\alpha N/\partial x_i$ and $\partial\alpha D/\partial x_i$ given below.

$$\frac{\partial \alpha N}{\partial x} = \frac{\partial g}{\partial x} [\dot{x}x + \dot{y}y + \dot{z}(z + a)] + g\dot{x} + \frac{\partial \lambda}{\partial x} r + \lambda \frac{\partial r}{\partial x}$$

$$\frac{\partial \alpha N}{\partial y} = \frac{\partial g}{\partial y} [\dot{x}x + \dot{y}y + \dot{z}(z + a)] + g\dot{y} + \frac{\partial \lambda}{\partial y} r + \lambda \frac{\partial r}{\partial y}$$

$$\frac{\partial \alpha N}{\partial z} = \frac{\partial g}{\partial z} [\dot{x}x + \dot{y}y + \dot{z}(z + a)] + g\dot{z} + \frac{\partial \lambda}{\partial z} r + \lambda \frac{\partial r}{\partial z}$$

$$\frac{\partial \alpha N}{\partial \dot{x}} = gx + \frac{\partial \lambda}{\partial \dot{x}} r$$

$$\frac{\partial \alpha N}{\partial \dot{y}} = gy + \frac{\partial \lambda}{\partial \dot{y}} r$$

$$\frac{\partial \alpha N}{\partial \dot{z}} = g(z + a) + \frac{\partial \lambda}{\partial \dot{z}} r$$

$$\frac{\partial \alpha N}{\partial \ddot{x}} = \frac{\partial \lambda}{\partial \ddot{x}} r$$

$$\frac{\partial \alpha N}{\partial \ddot{y}} = \frac{\partial \lambda}{\partial \ddot{y}} r$$

$$\frac{\partial \alpha N}{\partial \ddot{z}} = \frac{\partial \lambda}{\partial \ddot{z}} r \quad .$$

The nonzero partial derivatives of αD are

$$\frac{\partial \alpha D}{\partial x} = \frac{\partial \rho}{\partial x} v^3 r + v^3 \rho \frac{\partial r}{\partial x}$$

$$\frac{\partial \alpha D}{\partial y} = \frac{\partial \rho}{\partial y} v^3 r + v^3 \rho \frac{\partial r}{\partial y}$$

$$\frac{\partial \alpha D}{\partial z} = \frac{\partial \rho}{\partial z} v^3 r + v^3 \rho \frac{\partial r}{\partial z}$$

$$\frac{\partial \alpha D}{\partial \dot{x}} = 3v^2 \rho r \frac{\partial v}{\partial \dot{x}}$$

$$\frac{\partial \alpha D}{\partial \dot{y}} = 3v^2 \rho r \frac{\partial v}{\partial \dot{y}}$$

$$\frac{\partial \alpha D}{\partial \dot{z}} = 3v^2 \rho r \frac{\partial v}{\partial \dot{z}} .$$

The derivatives of αN and αD require the following set of equations. From Equation (A-8)

$$\frac{\partial \lambda}{\partial x} = - \omega^2 \dot{x}$$

$$\frac{\partial \lambda}{\partial y} = - \omega^2 \sin^2 \mu \dot{y} + \omega^2 \sin\mu \cos\mu \dot{z}$$

$$\frac{\partial \lambda}{\partial z} = \omega^2 \sin\mu \cos\mu \dot{y} - \omega^2 \cos^2 \mu \dot{z}$$

$$\frac{\partial \lambda}{\partial \dot{x}} = \ddot{x} - \omega^2 x$$

$$\frac{\partial \lambda}{\partial \dot{y}} = \ddot{y} - \omega^2 \sin^2 \mu y + (z + a)\omega^2 \sin\mu \cos\mu$$

$$\frac{\partial \lambda}{\partial \dot{z}} = \ddot{z} + \omega^2 \sin\mu \cos\mu y - \omega^2 \cos^2 \mu (z + a)$$

$$\frac{\partial \lambda}{\partial \ddot{x}} = \dot{x}$$

$$\frac{\partial \lambda}{\partial \ddot{y}} = \dot{y}$$

$$\frac{\partial \lambda}{\partial \ddot{z}} = \dot{z}$$

The gravity model is

$$g = g_0 (1 - \frac{2h}{a}) .$$

The nonzero derivatives of gravity are given by the following expressions.

$$\frac{\partial g}{\partial x} = - \frac{2g_0}{a} \frac{x}{r}$$

$$\frac{\partial g}{\partial y} = - \frac{2g_0}{a} \frac{y}{r}$$

$$\frac{\partial g}{\partial z} = - \frac{2g_0}{a} \frac{(z + a)}{r} .$$

The atmospheric density model is

$$\rho = \rho_0 \exp(-kh)$$

and its nonzero derivatives are given by

$$\frac{\partial \rho}{\partial x} = - k\rho \frac{x}{r}$$

$$\frac{\partial \rho}{\partial y} = - k\rho \frac{y}{r}$$

$$\frac{\partial \rho}{\partial z} = - k\rho \frac{(z + a)}{r} .$$

The velocity expression is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

and its nonzero derivatives are given by

$$\frac{\partial v}{\partial \dot{x}} = \frac{\dot{x}}{v}$$

$$\frac{\partial v}{\partial \dot{y}} = \frac{\dot{y}}{v}$$

$$\frac{\partial v}{\partial \dot{z}} = \frac{\dot{z}}{v} .$$

The distance from the vehicle to the center of the earth is

$$r = \sqrt{x^2 + y^2 + (z + a)^2}$$

and its nonzero derivatives are given by

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{(z + a)}{r} \quad .$$

With these expressions, all the elements of the gradient vector $\overline{A}$ are known and Equation (A-9) is used to evaluate the variance of the drag parameter at the initial time. Hence the covariance matrix corresponding to the initial state vector becomes

$$\begin{bmatrix} \sigma_x^2 & 0 & 0 & c_{x\dot{x}} & 0 & 0 & 0 \\ 0 & \sigma_y^2 & 0 & 0 & c_{y\dot{y}} & 0 & 0 \\ 0 & 0 & \sigma_z^2 & 0 & 0 & c_{z\dot{z}} & 0 \\ c_{x\dot{x}} & 0 & 0 & \sigma_{\dot{x}}^2 & 0 & 0 & 0 \\ 0 & c_{y\dot{y}} & 0 & 0 & \sigma_{\dot{y}}^2 & 0 & 0 \\ 0 & 0 & c_{z\dot{z}} & 0 & 0 & \sigma_{\dot{z}}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_\alpha^2 \end{bmatrix}$$

This algorithm is programmed in subroutine XNTIAL.

The method of selecting N, the number of data points used in the initializing routine for any particular trajectory should be explained. The procedure is to select N which gives initial estimate of β positive and in the approximate range of (1750, 2250) and when N is increased further β should stay essentially constant.

APPENDIX B

JACOBIAN MATRIX

As discussed in the Section 2.1, the following quantities are chosen as the state variables:

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= z \\
x_4 &= \dot{x} \\
x_5 &= \dot{y} \\
x_6 &= \dot{z} \\
x_7 &= \alpha = \frac{1}{\beta}
\end{aligned}
$$

The state equations for the spherical earth dynamic model are:

$$
\begin{aligned}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= \ddot{x} = 2\omega\sin\mu x_5 - 2\omega\cos\mu x_6 + \omega^2 x_1 - \frac{Gm}{r^3} x_1 \\
&\qquad - \frac{1}{2} g\rho v x_4 x_7 \\
\dot{x}_5 &= \ddot{y} = -2\omega\sin\mu x_4 + \omega^2\sin^2\mu x_2 - \omega^2\sin\mu\cos\mu x_3 \\
&\qquad - \frac{Gm}{r^3} x_2 - \frac{1}{2} g\rho v x_5 x_7 - a\omega^2\sin\mu\cos\mu \\
\dot{x}_6 &= \ddot{z} = 2\omega\cos\mu x_4 - \omega^2\sin\mu\cos\mu x_2 + \omega^2\cos^2\mu x_3 \\
&\qquad - \frac{Gm}{r^3}(a + x_3) - \frac{1}{2} g\rho v x_6 x_7 + a\omega^2\cos^2\mu \\
\dot{x}_7 &= 0
\end{aligned}
$$

All quantities in these equations are defined in Section 2.1. The Jacobian matrix for the above dynamic model is given by:

$$J(x) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ j_{\dot{x}x} & j_{\dot{x}y} & j_{\dot{x}z} & j_{\dot{x}\dot{x}} & j_{\dot{x}\dot{y}} & j_{\dot{x}\dot{z}} & j_{\dot{x}\alpha} \\ j_{\dot{y}x} & j_{\dot{y}y} & j_{\dot{y}z} & j_{\dot{y}\dot{x}} & j_{\dot{y}\dot{y}} & j_{\dot{y}\dot{z}} & j_{\dot{y}\alpha} \\ j_{\dot{z}x} & j_{\dot{z}y} & j_{\dot{z}z} & j_{\dot{z}\dot{x}} & j_{\dot{z}\dot{y}} & j_{\dot{z}\dot{z}} & j_{\dot{z}\alpha} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The dynamic model can be written symbolically as:

$$\dot{x} = f(x)$$

where x is a 7-component vector and f is a 7-component vector valued function of x; that is

$$f = [f_1, f_2, \cdots, f_7]^T .$$

The ijth element of the Jacobian matrix is given by:

$$[j(x)]_{ij} = \frac{\partial f_i}{\partial x_j} \qquad i,j = 1,2,\cdots,7 .$$

The elements of J denoted by j_{--}, are found to be:

$$j_{\dot{x}x} = \omega^2 - \frac{Gm}{r^3} - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\frac{x}{r}v\dot{x} + \frac{3Gm}{r^5}x^2$$

$$j_{\dot{x}y} = \frac{3Gm}{r^5}xy - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\frac{y}{r}v\dot{x}$$

$$j_{\dot{x}z} = \frac{3Gm}{r^5}x(z + a) - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\frac{(z + a)}{r}v\dot{x}$$

$$j_{\dot{x}\dot{x}} = -\frac{1}{2}\frac{g\rho}{\beta}v - \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{x}^2}{v}$$

$$j_{\dot{x}\dot{y}} = 2\omega\sin\mu - \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{x}\dot{y}}{v}$$

$$j_{\dot{x}\dot{z}} = -\ 2\omega\cos\mu - \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{x}\dot{z}}{v}$$

$$j_{\dot{x}\alpha} = -\ \frac{1}{2}\ g\rho v\dot{x}$$

$$j_{\dot{y}x} = \frac{3Gm}{r^5}\ xy - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\ v\dot{y}\ \frac{x}{r}$$

$$j_{\dot{y}y} = \omega^2\sin^2\mu - \frac{Gm}{r^3} + \frac{3Gm}{r^5}\ y^2 - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\ v\dot{y}\ (\frac{y}{r})$$

$$j_{\dot{y}z} = -\ \omega^2\sin\mu\cos\mu - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\ v\dot{y}\ \frac{(z + a)}{r} + \frac{3Gmy}{r^5}\ (z + a)$$

$$j_{\dot{y}\dot{x}} = -\ 2\omega\sin\mu - \frac{1}{2}\frac{g\rho\dot{x}\dot{y}}{\beta v}$$

$$j_{\dot{y}\dot{y}} = -\ \frac{1}{2}\frac{g\rho}{\beta}\ (v + \frac{\dot{y}^2}{v})$$

$$j_{\dot{y}\dot{z}} = -\ \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{y}\dot{z}}{v}$$

$$j_{\dot{y}\alpha} = -\ \frac{1}{2}\frac{g\rho}{\beta}\ v\dot{y}$$

$$j_{\dot{z}x} = \frac{3Gm}{r^5}\ x\ (z + a) - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\frac{x}{r}\ v\dot{z}$$

$$j_{\dot{z}y} = -\ \omega^2\sin\mu\cos\mu + \frac{3Gm}{r^5}\ y(a + z) - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\frac{y}{r}\ v\dot{z}$$

$$j_{\dot{z}z} = \omega^2\cos^2\mu - \frac{Gm}{r^3} + \frac{3Gm}{r^5}\ (a + z)^2 - \frac{1}{2}\frac{(g'\rho + g\rho')}{\beta}\frac{(z + a)}{r}\ v\dot{z}$$

$$j_{\dot{z}\dot{x}} = 2\omega\cos\mu - \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{x}\dot{z}}{v}$$

$$j_{\dot{z}\dot{y}} = -\ \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{y}\dot{z}}{v}$$

$$j_{\dot{z}\dot{z}} = -\ \frac{1}{2}\frac{g\rho}{\beta}\ (v + \frac{\dot{z}^2}{v})$$

$$j_{\dot{z}\alpha} = -\ \frac{1}{2}\ g\rho v\dot{z}$$

In the above expressions, the following quantities are used:

$$\rho = \rho_0 e^{-kh};\ \rho' = -k\rho;\ \rho'' = k^2\rho$$

$$g = g_0(1 - \frac{2h}{a});\ g' = \frac{2g_0}{a};\ g'' = 0$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\partial h}{\partial x}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\partial h}{\partial y}$$

$$\frac{\partial r}{\partial z} = \frac{(z + a)}{r} = \frac{\partial h}{\partial z}$$

$$\frac{\partial v}{\partial \dot{x}} = \frac{\dot{x}}{v}$$

$$\frac{\partial v}{\partial \dot{y}} = \frac{\dot{y}}{v}$$

$$\frac{\partial v}{\partial \dot{z}} = \frac{\dot{z}}{v}$$

These expressions are programmed in the subroutine JACN. The Jacobian is used in both the second-order filter and the fixed-interval smoother algorithm.

APPENDIX C

HESSIAN MATRICES

The Hessian matrix for each row of the Jacobian of the dynamic model is required for evaluating the second-order filter bias correction term, b(t). These are given by

$$[F_i(x)]_{\ell m} = \frac{\partial^2 f_i}{\partial x_\ell \partial x_m} \qquad \begin{array}{l} i = 1,2,\cdots,n \\ \ell,m = 1,2,\cdots,n \end{array} \quad .$$

For i = 1, 2, 3 and 7, the elements of the corresponding rows of the Jacobian matrix are constants. This leads to the 7x7 matrix

$$[F_i(x)] = 0 \qquad \text{for} \qquad i = 1,2,3,7 \quad .$$

The following notation will be used in giving elements of the Hessian matrices for other rows.

$$j_{ixx} = \frac{\partial^2 f_i}{\partial x^2}$$

$$j_{ix_\ell x_m} = \frac{\partial^2 f_i}{\partial x_\ell \partial x_m} \quad .$$

x_ℓ and x_m are elements of the state vector. The Hessian matrix is symmetric, hence it is required to calculate the n(n+1)/2 elements in the upper triangle matrix. Thus for n = 7, 28 elements of the Hessian matrices must be calculated for each of the fourth, fifth and sixth rows of the Jacobian matrix.

The Hessian matrix corresponding to the fourth row of the Jacobian matrix is:

$$j_{4xx} = \frac{\partial^2 f_4}{\partial x^2} = 9\,\frac{Gmx}{r^5} - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{v}{r}\,\dot{x} - 15\,\frac{Gm}{r^7}\,x^3 - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{x^2\dot{x}v}{r^2}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x^2\dot{x}v}{r^3}$$

$$j_{4yx} = \frac{\partial^2 f_4}{\partial y\partial x} = \frac{3Gm}{r^5}\,y - 15\,\frac{Gm}{r^7}\,x^2 y - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{xy\dot{x}v}{r^2}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{xy\dot{x}v}{r^3}$$

$$j_{4zx} = \frac{\partial^2 f_4}{\partial z\partial x} = \frac{3Gm}{r^5}\,(z + a) - \frac{15Gm}{r^7}\,x^2(z + a) - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{x(z + a)\dot{x}v}{r^2}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x(z + a)\dot{x}v}{r^3}$$

$$j_{4\dot{x}x} = \frac{\partial^2 f_4}{\partial \dot{x}\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x}{r}\left(v + \frac{\dot{x}^2}{v}\right)$$

$$j_{4\dot{y}x} = \frac{\partial^2 f_4}{\partial \dot{y}\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x\dot{x}\dot{y}}{rv}$$

$$j_{4\dot{z}x} = \frac{\partial^2 f_4}{\partial \dot{z}\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x\dot{x}\dot{z}}{rv}$$

$$j_{4\alpha x} = \frac{\partial^2 f_4}{\partial \alpha\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x\dot{x}v}{r}$$

$$j_{4yy} = \frac{\partial^2 f_4}{\partial y^2} = \frac{3Gm}{r^5}\,x - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{\dot{x}}{r}\,v - \frac{15Gmxy^2}{r^7}$$

$$- \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{y^2\dot{x}v}{r^2} + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y^2\dot{x}v}{r^3}$$

$$j_{4zy} = \frac{\partial^2 f_4}{\partial z \partial y} = -\frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{y(z+a)}{r^2}\dot{x}v + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y(z+a)}{r^3}\dot{x}v$$

$$-\frac{15Gm}{r}xy(z+a)$$

$$j_{4\dot{x}y} = \frac{\partial^2 f_4}{\partial \dot{x} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y}{r}\left(v + \frac{\dot{x}^2}{v}\right)$$

$$j_{4\dot{y}y} = \frac{\partial^2 f_4}{\partial \dot{y} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y\dot{x}\dot{y}}{rv}$$

$$j_{4\dot{z}y} = \frac{\partial^2 f_4}{\partial \dot{z} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y\dot{x}\dot{z}}{rv}$$

$$j_{4\alpha y} = \frac{\partial^2 f_4}{\partial \alpha \partial y} = -\frac{1}{2}(g'\rho + g\rho')\frac{y\dot{x}v}{r}$$

$$j_{4zz} = \frac{\partial^2 f_4}{\partial z^2} = \frac{3Gm}{r^5}x - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{\dot{x}v}{r} - \frac{15Gm}{r^7}x(z+a)^2$$

$$-\frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{(z+a)^2\dot{x}v}{r^2} + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z+a)^2\dot{x}v}{r^3}$$

$$j_{4\dot{x}z} = \frac{\partial^2 f_4}{\partial \dot{x} \partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\left(\frac{z+a}{r}\right)\left(v + \frac{\dot{x}^2}{v}\right)$$

$$j_{4\dot{y}z} = \frac{\partial^2 f_4}{\partial \dot{y} \partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z+a)\dot{x}\dot{y}}{rv}$$

$$j_{4\dot{z}z} = \frac{\partial^2 f_4}{\partial \dot{z} \partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z+a)\dot{x}\dot{z}}{rv}$$

$$j_{4\alpha z} = \frac{\partial^2 f_4}{\partial \alpha \partial z} = -\frac{1}{2}(g'\rho + g\rho')(z+a)\frac{\dot{x}v}{r}$$

$$j_{4\dot{x}\dot{x}} = \frac{\partial^2 f_4}{\partial \dot{x}^2} = -\frac{3}{2}\frac{g\rho}{\beta}\frac{\dot{x}}{v} + \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{x}^3}{v^3}$$

$$j_{4\dot{y}\dot{x}} = \frac{\partial^2 f_4}{\partial \dot{y} \partial \dot{x}} = -\frac{1}{2}\frac{g\rho}{\beta}\left(\frac{\dot{y}}{v} - \frac{\dot{x}^2\dot{y}}{v^3}\right)$$

$$j_{4\dot{z}\dot{x}} = \frac{\partial^2 f_4}{\partial \dot{z} \partial \dot{x}} = - \frac{1}{2} \frac{g\rho}{\beta} \left(\frac{\dot{z}}{v} - \frac{\dot{x}^2 \dot{z}}{v^3}\right)$$

$$j_{4\alpha\dot{x}} = \frac{\partial^2 f_4}{\partial \alpha \partial \dot{x}} = - \frac{1}{2} g\rho v - \frac{1}{2} g\rho \frac{\dot{x}^2}{v}$$

$$j_{4\dot{y}\dot{y}} = \frac{\partial^2 f_4}{\partial \dot{y}^2} = - \frac{1}{2} \frac{g\rho}{\beta} \dot{x} \left(\frac{1}{v} - \frac{\dot{y}^2}{v^3}\right)$$

$$j_{4\dot{z}\dot{y}} = \frac{\partial^2 f_4}{\partial \dot{z} \partial \dot{y}} = - \frac{1}{2} \frac{g\rho}{\beta} \frac{\dot{x}\dot{y}\dot{z}}{v^3}$$

$$j_{4\alpha\dot{y}} = \frac{\partial^2 f_4}{\partial \alpha \partial \dot{y}} = - \frac{1}{2} g\rho \frac{\dot{x}\dot{y}}{v}$$

$$j_{4\dot{z}\dot{z}} = \frac{\partial^2 f_4}{\partial \dot{z}^2} = - \frac{1}{2} \frac{g\rho}{\beta} \dot{x} \left(\frac{1}{v} - \frac{\dot{z}^2}{v^3}\right)$$

$$j_{4\alpha\dot{z}} = \frac{\partial^2 f_4}{\partial \alpha \partial z} = - \frac{1}{2} \frac{g\rho \dot{x}\dot{z}}{v}$$

$$j_{4\alpha\alpha} = \frac{\partial^2 f_4}{\partial \alpha^2} = 0$$

The Hessian matrix corresponding to the fifth row of the Jacobian matrix is:

$$j_{5xx} = \frac{\partial^2 f_5}{\partial x^2} = \frac{3Gm}{r^5} y - \frac{1}{2} \left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{v}{r} \dot{y} - \frac{15Gm}{r^7} x^2 y$$

$$- \frac{1}{2} \left(\frac{g\rho'' + 2g'\rho'}{\beta}\right) \frac{x^2 \dot{y} v}{r^2} + \frac{1}{2} \left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{x^2 \dot{y} v}{r^3}$$

$$j_{5yx} = \frac{\partial^2 f_5}{\partial y \partial x} = \frac{3Gm}{r^5} x - \frac{15Gm}{r^7} xy^2 - \frac{1}{2} \left(\frac{g\rho'' + 2g'\rho'}{\beta}\right) \frac{xy\dot{y}v}{r^2}$$

$$+ \frac{1}{2} \left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{xy\dot{y}v}{r^3}$$

$$j_{5zx} = \frac{\partial^2 f_5}{\partial z \partial x} = -\frac{15Gm}{r^7} xy(z + a) - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right) \frac{x(z + a)\dot{y}v}{r^2}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{x(z + a)\dot{y}v}{r^3}$$

$$j_{5\dot{x}x} = \frac{\partial^2 f_5}{\partial \dot{x} \partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{x\dot{x}\dot{y}}{rv}$$

$$j_{5\dot{y}x} = \frac{\partial^2 f_5}{\partial \dot{y} \partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{x}{r}\left(v + \frac{\dot{y}^2}{v}\right)$$

$$j_{5\dot{z}x} = \frac{\partial^2 f_5}{\partial z \partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{x\dot{y}\dot{z}}{rv}$$

$$j_{5\alpha x} = \frac{\partial^2 f_5}{\partial \alpha \partial x} = -\frac{1}{2}(g'\rho + g\rho') \frac{x\dot{y}v}{r}$$

$$j_{5yy} = \frac{\partial^2 f_5}{\partial y^2} = \frac{9Gm}{r^5} y - \frac{15Gm}{r^7} y^3 - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{\dot{y}v}{r}$$

$$-\frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right) \frac{y^2\dot{y}v}{r^2} + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{y^2\dot{y}v}{r^3}$$

$$j_{5zy} = \frac{\partial^2 f_5}{\partial z \partial y} = \frac{3Gm}{r^5}(z + a) - \frac{15Gm}{r^7} y^2(z + a)$$

$$-\frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right) \frac{y(z + a)}{r^2} \dot{y}v + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{y(z + a)\dot{y}v}{r^3}$$

$$j_{5\dot{x}y} = \frac{\partial^2 f_5}{\partial \dot{x} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{y}{r} \frac{\dot{x}\dot{y}}{v}$$

$$j_{5\dot{y}y} = \frac{\partial^2 f_5}{\partial \dot{y} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{y}{r}\left(v + \frac{\dot{y}^2}{v}\right)$$

$$j_{5\dot{z}y} = \frac{\partial^2 f_5}{\partial z \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right) \frac{y}{r} \frac{\dot{y}\dot{z}}{v}$$

$$j_{5\alpha y} = \frac{\partial^2 f_5}{\partial \alpha \partial y} = -\frac{1}{2}(g'\rho + g\rho') \frac{y\dot{y}v}{r}$$

$$j_{5zz} = \frac{\partial^2 f_5}{\partial z^2} = \frac{3Gm}{r^5}\, y - \frac{15Gm}{r^7}\, y(z + a)^2 - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{\dot{y}v}{r}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z + a)^2\dot{y}v}{r^3} - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{(z + a)^2\dot{y}v}{r^2}$$

$$j_{5\dot{x}z} = \frac{\partial^2 f_5}{\partial \dot{x}\partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z + a)\dot{x}\dot{y}}{rv}$$

$$j_{5\dot{y}z} = \frac{\partial^2 f_5}{\partial \dot{y}\partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\left(\frac{z + a}{r}\right)\left(v + \frac{\dot{y}^2}{v}\right)$$

$$j_{5\dot{z}z} = \frac{\partial^2 f_5}{\partial \dot{z}\partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\left(\frac{z + a}{r}\right)\frac{\dot{y}\dot{z}}{v}$$

$$j_{5\alpha z} = \frac{\partial^2 f_5}{\partial \alpha\partial z} = -\frac{1}{2}\,(g'\rho + g\rho')\,\frac{(z + a)}{r}\, v\dot{y}$$

$$j_{5\dot{x}\dot{x}} = \frac{\partial^2 f_5}{\partial \dot{x}^2} = -\frac{1}{2}\frac{g\rho}{\beta}\,\dot{y}\left(\frac{1}{v} - \frac{\dot{x}^2}{v^3}\right)$$

$$j_{5\dot{y}\dot{x}} = \frac{\partial^2 f_5}{\partial \dot{y}\partial \dot{x}} = -\frac{1}{2}\frac{g\rho}{\beta}\,\dot{x}\left(\frac{1}{v} - \frac{\dot{y}^2}{v^3}\right)$$

$$j_{5\dot{z}\dot{x}} = \frac{\partial^2 f_5}{\partial \dot{z}\partial \dot{x}} = \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{x}\dot{y}\dot{z}}{v^3}$$

$$j_{5\alpha x} = \frac{\partial^2 f_5}{\partial \alpha\partial \dot{x}} = -\frac{1}{2}\, g\rho\,\frac{\dot{x}\dot{y}}{v}$$

$$j_{5\dot{y}\dot{y}} = \frac{\partial^2 f_5}{\partial \dot{y}^2} = -\frac{3}{2}\frac{g\rho}{\beta}\frac{\dot{y}}{v} + \frac{1}{2}\frac{g\rho}{\beta}\frac{y^3}{v^3}$$

$$j_{5\dot{z}\dot{y}} = \frac{\partial^2 f_5}{\partial \dot{z}\partial \dot{y}} = -\frac{1}{2}\frac{g\rho}{\beta}\left(\frac{\dot{z}}{v} - \frac{\dot{y}^2\dot{z}}{v^3}\right)$$

$$j_{5\alpha\dot{y}} = \frac{\partial^2 f_5}{\partial \alpha\partial \dot{y}} = -\frac{1}{2}\, g\rho\left(v + \frac{\dot{y}^2}{v}\right)$$

$$j_{5\dot{z}\dot{z}} = \frac{\partial^2 f_5}{\partial \dot{z}^2} = -\frac{1}{2}\frac{g\rho}{\beta}\left(\frac{1}{v} - \frac{\dot{z}^2}{v^3}\right)\dot{y}$$

$$j_{5\alpha\dot{z}} = \frac{\partial^2 f_5}{\partial\alpha\partial\dot{z}} = -\frac{1}{2} g\rho \frac{\dot{y}\dot{z}}{v}$$

$$j_{5\alpha\alpha} = \frac{\partial^2 f_5}{\partial\alpha^2} = 0$$

The Hessian matrix corresponding to the sixth row of the Jacobian matrix is:

$$j_{6xx} = \frac{\partial^2 f_6}{\partial x^2} = \frac{3Gm}{r^5}(z + a) - \frac{15Gm}{r^7} x^2(z + a) - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{\dot{z}v}{r}$$

$$- \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{x^2\dot{z}v}{r^2} + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x^2\dot{z}v}{r^3}$$

$$j_{6yx} = \frac{\partial^2 f_6}{\partial y\partial x} = -\frac{15Gm}{r^7} xy(z + a) - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{xy\dot{z}v}{r^2} + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{xy\dot{z}v}{r^3}$$

$$j_{6zx} = \frac{\partial^2 f_6}{\partial z\partial x} = \frac{3Gm}{r^5} x - \frac{15Gm}{r^7} x(z + a)^2 - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{x(z + a)\dot{z}v}{r^2}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x(z + a)\dot{z}v}{r^3}$$

$$j_{6\dot{x}x} = \frac{\partial^2 f_6}{\partial\dot{x}\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\left(\frac{x\dot{x}\dot{z}}{rv}\right)$$

$$j_{6\dot{y}x} = \frac{\partial^2 f_6}{\partial\dot{y}\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\left(\frac{x\dot{y}\dot{z}}{rv}\right)$$

$$j_{6\dot{z}x} = \frac{\partial^2 f_6}{\partial\dot{z}\partial x} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{x}{r}\left(v + \frac{\dot{z}^2}{v}\right)$$

$$j_{6\alpha x} = \frac{\partial^2 f_6}{\partial\alpha\partial x} = -\frac{1}{2}(g'\rho + g\rho')\frac{x}{r}\dot{z}v$$

$$j_{6yy} = \frac{\partial^2 f_6}{\partial y^2} = \frac{3Gm}{r^5}(a + z) - \frac{15Gm}{r^7} y^2 (a + z) - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{\dot{z}v}{r}$$

$$- \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{y^2\dot{z}v}{r^2} + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y^2\dot{z}v}{r^3}$$

$$j_{6zy} = \frac{\partial^2 f_6}{\partial z \partial y} = \frac{3Gm}{r^5} y - \frac{15Gm}{r^7} y(a + z)^2 - \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{y(z + a)\dot{z}v}{r^2}$$

$$+ \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y(z + a)\dot{z}v}{r^3}$$

$$j_{6\dot{x}y} = \frac{\partial^2 f_6}{\partial \dot{x} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y\dot{x}\dot{z}}{rv}$$

$$j_{6\dot{y}y} = \frac{\partial^2 f_6}{\partial \dot{y} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y\dot{y}\dot{z}}{rv}$$

$$j_{6\dot{z}y} = \frac{\partial^2 f_6}{\partial \dot{z} \partial y} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{y}{r}\left(v + \frac{\dot{z}^2}{v}\right)$$

$$j_{6\alpha y} = \frac{\partial^2 f_6}{\partial \alpha \partial y} = -\frac{1}{2}(g'\rho + g\rho')\frac{y}{r} v\dot{z}$$

$$j_{6zz} = \frac{\partial^2 f_6}{\partial z^2} = \frac{9Gm}{r^5}(z + a) - \frac{15Gm}{r^7}(a + z)^3 - \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{\dot{z}v}{r}$$

$$- \frac{1}{2}\left(\frac{g\rho'' + 2g'\rho'}{\beta}\right)\frac{(z + a)^2}{r^2}\dot{z}v + \frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z + a)^2}{r^3}\dot{z}v$$

$$j_{6\dot{x}z} = \frac{\partial^2 f_6}{\partial \dot{x} \partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z + a)\dot{x}\dot{z}}{rv}$$

$$j_{6\dot{y}z} = \frac{\partial^2 f_6}{\partial \dot{y} \partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z + a)\dot{y}\dot{z}}{rv}$$

$$j_{6\dot{z}z} = \frac{\partial^2 f_6}{\partial \dot{z} \partial z} = -\frac{1}{2}\left(\frac{g'\rho + g\rho'}{\beta}\right)\frac{(z + a)}{r}\left(v + \frac{z^2}{v}\right)$$

$$j_{6\alpha z} = \frac{\partial^2 f_6}{\partial \alpha \partial z} = -\frac{1}{2}(g'\rho + g\rho')\frac{(z + a)}{r}\dot{z}v$$

$$j_{6\dot{x}\dot{x}} = \frac{\partial^2 f_6}{\partial \dot{x}^2} = -\frac{1}{2}\frac{g\rho}{\beta}\dot{z}\left(\frac{1}{v} - \frac{\dot{x}^2}{v^3}\right)$$

$$j_{6\dot{y}\dot{x}} = \frac{\partial^2 f_6}{\partial \dot{y}\partial \dot{x}} = \frac{1}{2}\frac{g\rho}{\beta}\frac{xyz}{v^3}$$

$$j_{6\dot{z}\dot{x}} = \frac{\partial^2 f_6}{\partial \dot{z}\partial \dot{x}} = -\frac{1}{2}\frac{g\rho}{\beta}\dot{x}\left(\frac{1}{v} - \frac{\dot{z}^2}{v^3}\right)$$

$$j_{6\alpha\dot{x}} = \frac{\partial^2 f_6}{\partial \alpha\partial \dot{x}} = -\frac{1}{2} g\rho\frac{\dot{x}\dot{z}}{v}$$

$$j_{6\dot{y}\dot{y}} = \frac{\partial^2 f_6}{\partial \dot{y}^2} = -\frac{1}{2}\frac{g\rho}{\beta}\dot{z}\left(\frac{1}{v} - \frac{\dot{y}^2}{v^3}\right)$$

$$j_{6\dot{z}\dot{y}} = \frac{\partial^2 f_6}{\partial \dot{z}\partial \dot{y}} = -\frac{1}{2}\frac{g\rho}{\beta}\dot{y}\left(\frac{1}{v} - \frac{\dot{z}^2}{v^3}\right)$$

$$j_{6\alpha\dot{y}} = \frac{\partial^2 f_6}{\partial \alpha\partial y} = -\frac{1}{2} g\rho\frac{\dot{y}\dot{z}}{v}$$

$$j_{6\dot{z}\dot{z}} = \frac{\partial^2 f_6}{\partial \dot{z}^2} = -\frac{3}{2}\frac{g\rho}{\beta}\frac{z}{v} + \frac{1}{2}\frac{g\rho}{\beta}\frac{\dot{z}^3}{v^3}$$

$$j_{6\alpha\dot{z}} = \frac{\partial^2 f_6}{\partial \alpha\partial z} = -\frac{1}{2} g\rho\dot{z}v$$

$$j_{6\alpha\alpha} = \frac{\partial^2 f_6}{\partial \alpha^2} = 0$$

These expressions are programmed in the subroutine SHMT.

APPENDIX D

INPUT DATA AND PARAMETERS

There are a number of physical parameters and program inputs that must be specified. These are listed below.

NX: The number of differential equations in the dynamic model which is the order of the state equations (NX=7).

E: A constant step size. The interval between successive observations.

NST: Number of observation data to be processed.

NC: NC-1 is the number of observations skipped in processing. Thus if NC=1, every data point is used; if NC=2, every other data point is used, etcetera.

N: Number of data points used by XNTIAL for initializing the filter algorithm. This is separately determined by a test program for the data to be processed.

ω: Earth's sidereal rate (radians/sec).

Gm: Gravitational constant times earth's mass (ft^3/sec^2).

h_s: Height above mean sea level of origin of x-y-z coordinate system (feet).

μ: Geodetic latitude of the origin of coordinate system (degrees).

R_e: Earth's radius (feet).

g_0: Acceleration due to gravity at mean sea level (ft/sec^2).

The parameters specifying the exponential atmospheric density model are specified by the subroutine DENY. These may vary with the data and are critical in the state estimation.

The observation sequence

$$Z_i = \{t_i;\ x_i,\ y_i,\ z_i\}, \qquad i = 1,2,\cdots,NST \qquad ,$$

and associated observation error covariance matrices R_i. This data is processed by the filter and smoother algorithms to generate the state estimates.

APPENDIX E

SUBROUTINE DESCRIPTION

The software package development involved a number of subroutines. The purpose of each of the program subroutines is briefly states in this appendix.

XNTIAL: This subroutine generates the initial state vector and the corresponding error covariance matrix. The inputs to this subroutine are physical constants and the first N observations.

DENY: The atmospheric density model is given in this subroutine. Atmospheric density and the first and second derivatives of density with respect to altitude are evaluated in this routine.

SHMT: Hessian matrices for the dynamic model are evaluated by this routine. The elements of the matrices are calculated using the state vector input to the subroutine.

AWRIT: This routine takes a matrix stored in a one dimentional array and prints it in the standard matrix form.

RK4: Fourth-order Runge-Kutta method of integration is implemented in this routine. It is used for integrating the state model.

RK2: The modified Euler's method of integration is implemented in this routine. It is used to integrate a set of first order differential equations when a large number of equations are involved.

DERFUN: A description of the dynamic model is given in this routine. Derivatives of the state vector are evaluated.

SUMS: This routine forms the differential equations for propagating the covariance matrix of the state vector. The derivatives of the covariance matrix elements are calculated by evaluating the Jacobian matrix and making use of GMTRA and MPRD routines.

SBRT: This routine generates additive bias correction terms for the nonlinear dynamic model. It uses Hessian matrices generated by SHMT.

JACN: The Jacobian matrix of the dynamic model is evaluated in this routine.

CROF: CROF adds a diagonal matrix to the extrapolated state error covariance matrix to account for round off errors.

CHTN: CHTN tests the consistency of the position residual vector with the corresponding portion of the state error covariance matrix. If the consistency requirement is not met, the error covariance matrix is increased by the amount necessary to insure that the residual vector is consistent with the new error covariance matrix.

CMPN: CMPN adds plant noise to the state error covariance matrix due to uncertain parameters. Atmospheric density is the uncertain parameter used in this routine.

SYSY: SYSY replaces a square matrix by a symmetric matrix. The corresponding off diagonal terms are replaced by their average. This is used to enforce symmetry of covariance matrices.

MPRD: The purpose of MPRD is to multiply two matrices to form a resultant matrix. This requires a subroutine LOC.

LOC: LOC computes a vector subscript for an element in a matrix of specified storage mode.

MINV: MINV inverts a matrix. The determinant of the matrix is also obtained.

GMTRA: GMTRA is used to obtain the transpose of a matrix.

EIGEN: EIGEN computes eigen values and eigen vectors of a real symmetric matrix.

MPRD, LOC, MINV, GMTRA and EIGEN subroutines have been adopted from IBM-Scientific Subroutine Package Library.

E.1 Other Programs Used for Simulation

RCON: RCON generates the estimate of position of vehicle in space based on the observed azimuth and elevation angle data collected by the optical trackers. Inputs to this program are position coordinates of the optical trackers and the azimuth and elevation angle observations.

ADNZ: This subroutine generates an error vector whose 3X3 covariance matrix is denoted by R. This is used to generate simulated noisy observation data.

RMGN: RMGN generates an array of pseudo random numbers with specified mean and standard deviation.

VITA

Vinod Kumar Bhandari

Candidate for the Degree of

Doctor of Philosophy

Thesis: STATE ESTIMATION OF REENTRY VEHICLES FROM OPTICAL TRACKING DATA

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Delhi, India, August 8, 1949, the son of Mr. and Mrs. Ram Gopal Bhandari.

Education: Graduated from B.N.S.D.Inter College, Kanpur, (U.P.) in 1965; received the Bachelor of Engineering degree in Electronics Engineering from Birla Institute of Technology and Science, Pilani, Rajasthan, India in 1970; received Master of Science in Electrical Engineering from Rutgers University in 1971; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1974.

Professional Experience: Graduate Research Assistant, School of Electrical Engineering, Oklahoma State University, from Fall, 1971 to Fall, 1973; Graduate Teaching Assistant, School of Electrical Engineering, Oklahoma State University, Spring, 1974.

Professional Societies: Student Member of the Institute of Electrical and Electronics Engineers.