UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

NORM RETRIEVAL FROM SPATIOTEMPORAL SAMPLES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

FATMA BOZKURT
Norman, Oklahoma
2018

NORM RETRIEVAL FROM SPATIOTEMPORAL SAMPLES

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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To my dear parents,

Acknowledgments

I would like to express my endless thanks to my advisor Prof. Keri Kornelson for her assistance, guidance and patience during my research. She was always present her advice and encouragement when I need. She introduced me to other research groups and gave me the opportunity to involve in the community. Working with her is a huge learning experience and I will keep her advices in my mind through all my life.

I would like to thank all faculty members in my committee, Prof. John Albert, Prof. Nikola Petrov, Prof. Christian Remling, and Prof. Thordur Runolfsson for their support and for serving as my committee members. I am also grateful to Prof. Noel Brady for his valuable suggestions to come up with a new idea for an example. Thank you also Prof. Akram Aldroubi and Longxiu Huang for their warm hospitality during the visit to Vanderbilt university.

I would like to thank all my friends in the mathematics department at the university of Oklahoma for all the great times that we have shared together. I am especially thankful to Shaoyun Yi, Mahesh Sunkula and Oreoluwa Adekoya for their friendship and support in my daily life. Thanks to them, I had wonderful days in Norman.

I feel so lucky that I had chance to meet Pat and Robert Flagler in this country. The most difficult and stressful times in my life in Norman, these two warm-hearted people were together with me and support me to overcome the obstacles.

Education is a journey from ignorance to enlightenment and Siyami Yildirim, my elementary school mathematics teacher, has been the first person who has enlightened me about the importance of education and encouraged me to study mathematics in my early ages. I never be regret that I followed his advices.

It would be impossible for me to get a PHD education in U.S. without the financial support of Ministry of national education in Turkey and Houston education attache. I would like to thank all the employees of these institutions for their help. I would also like to express my appreciation to Mustafa Oz, Erhan Aybek, Kemal Alta, Mehmet and Adile Kose for being sponsor to me. Without their support, it was impossible for me to start my PHD education.

Most importantly, I will be forever grateful to my parents Nevin and Mehmet Salih Bozkurt for their financial and most importantly emotional support through my all life. None of this work would have been accomplished without their support. Their love, sacrifice, unconditional trust and hard work gave me the opportunity to pursue my degree. I have been fortunate to have my sisters and brothers Sidika, Alev, Ipek, Sinem, Fatih and Mehmet in my life. Having brothers and sisters is the most precious gift that one can have in this world. I am grateful for their endless love and the meaning they bring to my life.

Contents

Abstract

The goal of this dissertation is to investigate norm retrievable frames having dynamical sampling structure, particularly those that fail the phase retrieval condition. We give several classifications to show how to construct norm retrievable frames dynamically, depending on the properties of the time-evolution operator. We show that norm retrievable frames generated by a single vector from a self-adjoint operator are most of the time phase retrievable frames. However, when we allow more generating vectors, there exist norm retrieval frames that do not do phase retrieval. We used two different subspace approaches to obtain these structures in real Hilbert spaces.

Chapter 1

Introduction

1.1 General Problem Formulation

A complete inner product space is called a Hilbert space. Given a signal $x \in \mathcal{H}$ in a seperable Hilbert space with a given orthonormal bases $\{e_i\}_{i \in I}$ in $\mathcal{H}$, Parseval's identity allows us to reconstruct the signal x from the measurements $\{\langle x, e_i \rangle\}_{i \in I}$. The set of coefficients $\{\langle x, e_i \rangle\}$ is unique. If a measurement is lost or scrambled, we are not able to reconstruct the signal x from remaining measurements. We can see the need for set of vectors that have a reconstruction property similar to Parseval's identity, while also allowing for some resilience to loss. If we have a redundant set of vectors $\{x_i\}_{i \in I}$ in $\mathcal{H}$, reconstruction can be solved under proper conditions. A frame $\{x_i\}_{i \in I}$ for $\mathcal{H}$ allows for redundancy while preserving a structure so that reconstruction is possible. Now, the set of measurements $\{\langle x, x_i \rangle\}_{i \in I}$ are not necessarily unique. We can think of frame vectors as generalization of orthonormal bases but the redundancy of frames makes them more adaptible than the orthonormal bases.

Frame vectors $\{x_i\}_{i\in I}$ in $\mathcal{H}$ allows us to reconstruct the signal x from the measurements $\{\langle x, x_i\rangle\}$. Suppose however, that the phase of the measurements have been lost, or cannot be measured. Setting such as tomography or crystallography can have such constraints. When we only have the phaseless measurements $\{|\langle x, x_i\rangle|\}$, we are not able to construct the exact signal x. Casazza, Balan, and Edidin ([8]) introduced the concept of **phase retrieval** for Hilbert space frames in 2006 to recover the phase of a signal given by its intensity measurements $\{|\langle x, x_i\rangle|\}$ from a redundant linear system. Note that we cannot distinguish x and cx with $|c| = 1$ from the phaseless measurements. This means in a finite dimensional real Hilbert spaces $\mathbb{R}^n$, we cannot distinguish x and $-x$ from the intensity measurements. In $\mathbb{R}^n$, they showed in [8] that we need at least $2n-1$ vectors to have phase retrieval. Phase retrieval is a stronger condition than being a frame. If a set of vectors is not a frame, than it does not satisfy phase retrieval conditions. Another condition, weaker than phase retrieval, is that of norm retrieval. Introduced in [16], a set of vectors do norm retrieval if two vectors in the Hilbert space have the same intensity measurements, then they have the same norm in the Hilbert space. The norm retrieval property relaxes the phase retrieval conditions. Every phase retrievable set is also norm retrievable set but there exits norm retrievable sets that are not phase retrievable which we are interested in. Norm retrieval requires fewer vectors than phase retrieval. Orthonormal bases for example are a norm retrievable sets but not phase retrievable.

In this thesis, we will seek to produce norm retrievable sets within a certain sampling structure. Suppose a vector $x \in \mathbb{R}^n$ is a sampled only on the orthonormal basis $\{e_i\}_{i=1}^n$. We have samples $\{\langle x, e_i\rangle\}_{i\in\Omega}$ where $\Omega \subset \{1, 2, ..., n\}$. This is not enough information to consruct x. Suupose, though, that x is evolving in some well-understood way over time. We can use repeated samples on Ω over time, and try to reconstruct the signal x.

When $\Omega \subseteq \{1, 2, ..., n\}$ is the coarse sample points in $\mathcal{H}^n$, the measurements $\{\langle x, e_i\rangle : i \in \Omega\}$ have insufficient information in general to recover the original signal x. Given an operator A on $\mathcal{H}$, suppose the signal $x \in \mathcal{H}$ varies in time increments according to the operator A. That is the signal $x \in \mathcal{H}$ evolves through the operator A over time to become $A^\ell x$ at time ℓ. Now, we can have extra information $\{A^\ell x(i) : i \in \Omega\}$ about the signal x. How many iterations do we need to reconstruct the signal x? Which sample points do we need to choose? What is the operator A? Dynamical sampling problem answers all this questions. The fundamental dynamical sampling problem ([2]) is to find conditions on Ω, A, and the number L of time increments such that measurements on the components given by course sample points Ω over times ℓ can be used to reconstruct x.

In other words, we want to construct $x \in \mathcal{H}$ from the measurements

$$\{\langle A^\ell x, e_i\rangle \, : \, \ell = 0, 1, \ldots, L; \; i \in \Omega\}. \tag{1.1.1}$$

In ([2]), Aldroubi and his collaborators recently showed that x can be recovered from the measurements in (1.1.1) if and only if the time-space samples is a set of frame vectors. In 2017, Aldroubi and his collaborators in ([4]) showed phaseless reconstruction from space-time samples.

In this paper, we will examine the intersection of these two very recent developments in frame theory. We will use samples taken in the dynamical sampling structure and attempt to show when norm retrieval is possible. Particularly, we are interested in norm retrievable sets that has the dynamical sampling structure but fails phase retrieval.

We consider the norm retrieval problem in the dynamical sampling setting in the finite dimensional real Hilbert space $\mathbb{R}^n$. The norm retrieval problem in dynamical sampling setting can be stated as follows:

The norm retrieval problem in dynamical sampling seeks to find conditions on the operator A, the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ and the time increments l_i such that the set of vectors $\{A^{\ell_i} b_i \in \mathbb{R}^n : i \in \Omega\}$ will have the norm retrieval property. That is, for two vectors in the Hilbert space which have the same intensity measurements, they have the same norm in the Hilbert space.

1.2 Organization

In Chapter 2, we give basic information about frame theory, dynamical sampling, phase retrieval and norm retrieval which are necessary to build our the norm retrieval problem in dynamical sampling setting in finite dimensional real Hilbert space $\mathbb{R}^n$.

In Chapter 3, we find results based on the structure of the time-evolution operator A in the dynamical sampling system. We begin with a diagonal operator, then give results when A is self-adjoint operator, normal operator or unitarily equivalent to Jordan form. We find the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ and the condition on the time increments $\ell_i \in \mathbb{N}$ such that the set of vectors $\{A^{\ell_i} b_i \in \mathbb{R}^n : i \in \Omega\}$ is a dynamical sampling frame and it satisfies norm retrieval without doing phase retrieval. We discover that, in some instances, norm retrieval is impossible with only one measurement vector without doing phase retrieval. We also show that if we make the iteration over more generating vectors, we can have dynamical sampling frame which satisfies norm retrieval without doing phase retrieval.

We also describe the connection between norm retrievable projections and a structure known in the frame theory literature as fusion frames. We explain how projections onto subspaces that have dynamical sampling form can give structure for finding norm retrievable vectors.

Chapter 2

Preliminary Materials

2.1 Frames

Since frame vectors are a cornerstone in our research, we give an introduction to frame theory in this chapter. In mathematics, physics and signal processing, orthonormal bases are a very important tool to represent functions. This representation is unique and we have the following perfect reconstruction and Parseval's identity for orthonormal bases. In particular, recall that the coefficients come from inner products.

Theorem 2.1.1. *(Perfect Reconstruction) If $\{e_n\}_{n\in I}$ is an orthonormal bases for a Hilbert space $\mathcal{H}$, then*

$$x = \sum_{n\in I} \langle x, e_n\rangle e_n \quad \text{for all} \quad x \in \mathcal{H}. \tag{2.1.1}$$

The sum converges in norm when $\mathcal{H}$ is infinite dimensional.

Theorem 2.1.2. *(Parseval's Identity) If $\{e_n\}_{n\in I}$ is an orthonormal bases for a*

Hilbert space $\mathcal{H}$, then

$$||x||^2 = \sum_{n\in I} |\langle x, e_n\rangle|^2 \quad \textit{for all} \quad x \in \mathcal{H}.$$

However, the conditions on orthonormal bases are very restrictive. Orthonormal bases require the vectors to be linearly independent and orthogonal to each other in an inner product space which makes it hard to satisfy any extra conditions. A frame in an inner product space is a more flexible tool which allows each element in the inner product space to be written as a linear combination of the frame elements, but the linear independence between the frame vectors is not necessary. Frames can be considered as generalizations of orthonormal bases in Hilbert spaces and the redundancy of frames makes them very useful. Frames are the vectors such that conditions are relaxed on orthonormal and have similar properties to perfect reconstruction and Parseval's identity.

Duffin and Schaeffer [23] first introduced frames for Hilbert spaces while working on a problem in non-harmonic Fourier series in 1952. Later (1986), Daubechies, Grossmann and Meyer ([22]) observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are similar to the expansions using orthonormal bases.

We refer the reader to ([28], [15],[18]) for more details about frame theory and its applications in Hilbert spaces.

Definition 2.1.3. [23] A family of vectors $\{x_i\}_{i\in I}$ in a finite or infinite dimensional Hilbert space $\mathcal{H}$ is said to be a **frame** for $\mathcal{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A||x||^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B||x||^2, \text{ for all } \quad x \in \mathcal{H}. \tag{2.1.2}$$

The positive constants A and B are called lower and upper frame bounds, respectively. They are not unique. The optimal lower frame bound is the supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds.

- A frame is called a **tight frame** if the optimal upper and lower frame bounds are equal; $A = B$.
- A frame is called a **Parseval frame** if $A = B = 1$.
- $\{x_i\}_{i \in I}$ is called an **equal norm frame** if $||x_i|| = ||x_j||$ for all $i, j \in I$ and is called a **unit norm frame** if $||x_i|| = 1$ for all $i \in I$.
- $\{x_i\}_{i \in I}$ is called a **Bessel sequence** if it satisfies the upper frame inequality in (2.1.2).

Let $\mathcal{F} = \{x_i\}_{i \in I}$ be a frame in a Hilbert space $\mathcal{H}$ and $\{e_i\}_{i \in I}$ be the standard orthonormal basis for $\ell^2(I)$. The operator $\Phi : \mathcal{H} \to \ell^2(I)$ defined by

$$\Phi(x) = \sum_{i \in I} \langle x, x_i \rangle e_i \quad \text{for all} \quad x \in \mathcal{H}.$$

is called the **analysis operator** associated with $\mathcal{F}$.

The adjoint Φ^* of the analysis operator Φ is called the **synthesis operator** of the frame $\mathcal{F}$ and is given by

$$\Phi^* : \ell^2(I) \to \mathcal{H}, \quad \Phi^*((c_i)_{i \in I}) = \sum_{i \in I} c_i x_i.$$

The operator $S = \Phi^*\Phi : \mathcal{H} \to \mathcal{H}$,

$$S(x) = \Phi^*\Phi(x) = \sum_{i \in I} \langle x, x_i \rangle x_i \tag{2.1.3}$$

is called **frame operator** of the frame $\mathcal{F}$.

Given a frame F, the frame operator S of F is a bounded, positive and invertible operator satisfying the operator inequality $AI \leq S \leq BI$, where A and B are upper and lower frame bounds and I denotes the identity operator on $\mathcal{H}$.

Remark 2.1.4. The lower frame condition ensures that a frame is complete. On the other hand, the upper frame condition ensures that the analysis operator is well-defined.

For any $x \in \mathcal{H}$, Parseval frames $\{x_i\}_{i \in I}$ in $\mathcal{H}$ give us a specific set of coefficients which allows us to recontruct x from the set of vectors $\{x_i\}_{i \in I}$. Similar to Equation (2.1.1), the coefficients come from inner products.

Proposition 2.1.5. *[18] A collection of vectors $\{x_i\}_{i \in I}$ is a Parseval frame for a Hilbert space $\mathcal{H}$ if and only if the following formula holds for every $x \in \mathcal{H}$:*

$$x = \sum_{i \in I} \langle x, x_i \rangle x_i \tag{2.1.4}$$

Equation (2.1.4) is called the **recontruction** formula for a Parseval frame $\{x_i\}_{i \in I}$ in $\mathcal{H}$ similar to perfect reconstruction for an orthonormal basis. Even though every orthonormal bases is a Parseval frame, there exist Parseval frames which are not orthonormal bases.

Example 2.1.6. *Consider the collection of vectors $\{x_1, x_2, x_2\}$ in $\mathbb{R}^2$*

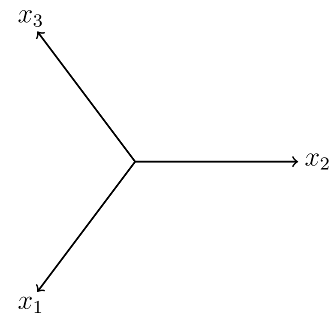

Figure 2.1: Mercedes-Benz Frame

$$\{x_1, x_2, x_2\} = \left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right\}.$$

The set of vectors $\{x_1, x_2, x_2\}$ satisfies (2.1.4) and a Parseval frame but not an orthonormal bases. For any $x \in \mathbb{R}^2$, $x = \langle x, x_1 \rangle x_1 + \langle x, x_2 \rangle x_2 + \langle x, x_3 \rangle x_3$ but the set of vectors $\{x_1, x_2, x_2\}$ is linearly dependent. Therefore, this set of vectors does not form a basis in $\mathbb{R}^2$ and the coefficients $\{\langle x, x_i \rangle\}$ are not unique.

The reconstruction formula for a general frame $\{x_i\}_{i \in I}$ in $\mathcal{H}$ is a little bit different. Let S be the frame operator of $\{x_i\}_{i \in I}$ defined in (2.1.3), then for any vector $x \in \mathcal{H}$,

$$Sx = \sum_{i \in I} \langle x, x_i \rangle x_i$$

Theorem 2.1.7. *[18] Let $\{x_i\}_{i \in I}$ be a frame for a Hilbert space $\mathcal{H}$ with frame operator S and the lower and upper frame bounds A and B. Then $\{S^{-1}x_i\}_{i \in I}$ is also a frame for $\mathcal{H}$ that has the lower and upper frame bounds $\frac{1}{B}$ and $\frac{1}{A}$.*

Given a vector $x \in \mathcal{H}$, the representation problem is to find coefficients c_n such that

$$x = \sum_{i \in I} c_n x_i$$

.

Since S is a self-adjoint, bounded and invertible operator on $\mathcal{H}$, by replacing x with $S^{-1}x$ in (2.1.3), the representation problem can be solved by setting

$$x = \sum_{i \in I} \langle x, S^{-1}x_i \rangle x_i \quad \forall x \in \mathcal{H}.$$

Given the coefficient $\{\langle x, x_i \rangle\}_{i \in I}$, the reconstruction problem attempts to find x.

If we apply S^{-1} to both sides of (2.1.3), the reconstruction problem can be solved by setting

$$x = \sum_{i \in I} \langle x, x_i \rangle S^{-1}x_i \quad \forall x \in \mathcal{H}.$$

Combining these two results, we have a representation such that

$$x = \sum_{i \in I} \langle x, S^{-1}x_i \rangle x_i = \sum_{i \in I} \langle x, x_i \rangle S^{-1}x_i \quad \forall x \in \mathcal{H}. \tag{2.1.5}$$

Definition 2.1.8. Let $\{x_i\}_{i \in I}$ be a frame for a Hilbert space $\mathcal{H}$. A sequence $\{y_i\}_{i \in I}$ in $\mathcal{H}$ is called a **dual frame** for $\{x_i\}_{i \in I}$ if $\{y_i\}_{i \in I}$ satisfies the reconstruction formula:

$$x = \sum_{i \in I} \langle x, y_i \rangle x_i = \sum_{i \in I} \langle x, x_i \rangle y_i \quad \forall x \in \mathcal{H}. \tag{2.1.6}$$

If $y_i = S^{-1}x_i \quad \forall i \in I$, (2.1.5) shows this is a dual frame. We call the frame $\{S^{-1}x_i\}_{i \in I}$ **the canonical dual** of the frame $\{x_i\}_{i \in I}$. If $\{y_i\}_{i \in I}$ is not the canonical dual frame, then it is called an **alternate dual** frame.

We can state a relationship between frames and orthogonal projections as follows:

Proposition 2.1.9. *[18] Let $\{x_i\}_{i\in I}$ be a sequence in a Hilbert space $\mathcal{H}$, and let P denote the orthogonal projection of $\mathcal{H}$ onto a closed subspace V. Then the following hold:*

1. *if $\{x_i\}_{i\in I}$ is a frame in $\mathcal{H}$ with frame bounds A, B, then $\{Px_i\}_{i\in I}$ is a frame for V with frame bounds A, B.*

2. *if $\{x_i\}_{i\in I}$ is a frame in V with frame operator S, then the orthogonal projection of $\mathcal{H}$ onto V is given by*

$$Px = \sum_{i\in I}\langle x, S^{-1}x_i\rangle x_i, \quad x \in \mathcal{H}.$$

Theorem 2.1.10. *[28] Suppose that $\mathcal{H}^n$ is n-dimensional Hilbert space and $\{x_i\}_{i=1}^m$ is a finite collection of vectors from $\mathcal{H}^n$. Then*

$\{x_i\}_{i=1}^m$ is a frame for $\mathcal{H}^n$ if and only if span $\{x_i\}_{i=1}^m = \mathcal{H}^n$.

Proof. (1) $\Rightarrow$ (2) : To prove by contrapositive, suppose $\{x_i\}_{i=1}^m$ does not span $\mathcal{H}^n$. So, there exists a nonezero vector $y \in \mathcal{H}^n$ which is orthogonal to each vector in span $\{x_i\}_{i=1}^m$. This says that $\sum\limits_{i=1}^{m} |\langle y, x_i\rangle|^2 = 0$ and the set of vectors $\{x_i\}_{i=1}^m$ would not have a positive lower frame bound. Thus $\{x_i\}_{i=1}^m$ would not be a frame in $\mathcal{H}^n$.

(2) $\Rightarrow$ (1) : Again to prove by contrapositive, suppose $\{x_i\}_{i=1}^m$ is not a frame in $\mathcal{H}^n$. Since the upper frame bound condition always holds for finite sequences, $\{x_i\}_{i=1}^m$ is not a frame in $\mathcal{H}^n$ if the lower frame bound condition is violated. In this case, for each positive integer k, there exists an element $y_k \in \mathcal{H}^n$ such that $||y_k|| = 1$ and

$$\sum_{i=1}^{m} |\langle y_k, x_i\rangle|^2 < \frac{1}{k}$$

.

Since $\{y_k\}_{k=1}^{\infty}$ is a bounded sequence, $\{y_k\}_{k=1}^{\infty}$ must have a convergent subsequence, $\{y_{k_j}\}_{j=1}^{\infty}$, from the Bolzano- Weierstrass Theorem.

Let y be the limit of $\{y_{k_j}\}$, so $||y_{k_j} - y|| \to 0$ as $j \to \infty$. Hence, we have

$$0 = \lim_{j\to\infty} \sum_{i=1}^{m} |\langle y_{k_j}, x_i\rangle|^2 = \sum_{i=1}^{m} |\langle y, x_i\rangle|^2.$$

This shows that y is orthogonal to every vector in the set $\{x_i\}_{i=1}^{m}$. In this case, either $y = 0$ or $\text{span}\{x_i\}_{i=1}^{m} \neq \mathcal{H}^n$. Since each $||y_{k_j}|| = 1$ and $||y_{k_j} - y|| \to 0$, we know that $||y|| = 1$. This proves that $\{x_i\}_{i=1}^{m}$ does not span $\mathcal{H}^n$.

□

We see in Theorem (2.1.10) that in finite dimensions, the frames in $\mathcal{H}^n$ are exactly the spanning sets.

We will use two particular frame constructions of fusion frames and scalable frames in later sections. We give their definitions here for reference. A fusion frame consists of subspaces rather than vectors that satisfy a frame-like condition.

Definition 2.1.11. [14] Let I be an index set and $\{v_i\}_{i\in I}$ be a family of weights. That is $v_i > 0$ for all $i \in I$. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$ and P_{Wi} is the orthogonal projection onto the subspace Wi for each $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a **fusion frame** for $\mathcal{H}$, if there exists constants $0 < A \leq B < \infty$ such that

$$A||x||^2 \leq \sum_{i\in I} {v_i}^2 ||P_{W_i}(x)||^2 \leq B||x||^2, \text{ for all } \quad x \in \mathcal{H}. \tag{2.1.7}$$

A and B are called the fusion frame bounds. The family (W_i, v_i) is called a **Parseval fusion frame** if $A = B = 1$ and a **tight fusion frame** if $A = B$.

Definition 2.1.12. [33]

A frame $\{x_i\}_{i\in I}$ for a Hilbert space $\mathcal{H}$ is called **scalable frame** if there exists scalars $\{c_i\}_{i\in I}$ such that $\{c_i x_i\}_{i\in I}$ is a Parseval frame. If there exists $\delta > 0$, such that $c_i > \delta$ for all $i \in I$, then $\{x_i\}_{i\in I}$ is called a **strictly scalable frame**.

Remark 2.1.13. It is easy to see that every tight frame is a strictly scalable frame. If $\{x_i\}_{i\in I}$ is a tight frame with bound A, then for any $x \in \mathcal{H}$, we have

$$A||x||^2 = \sum_{i\in I} |\langle x, x_i\rangle|^2 \quad \text{and} \quad x = \sum_{i\in I} \langle x, \frac{x_i}{\sqrt{A}}\rangle \frac{x_i}{\sqrt{A}}.$$

This shows that $\left\{\frac{x_i}{\sqrt{A}}\right\}_{i\in I}$ is a Parseval frame in $\mathcal{H}$ and $\{x_i\}_{i\in I}$ is a strictly scalable frame with coefficients $c_i = \dfrac{1}{\sqrt{A}}$ for all i.

2.2 Dynamical sampling

Over the last 6 years, a new type of sampling, involving both space and time samples, has been evolving. One motivation in the development of the dynamical sampling framework is Wireless Sensor Networks (WSN). In WSN, a group of spatially dispersed sensors are distributed to the field for monitoring and getting information about the physical conditions of the environments like temperature, wind, humidity, sound, pollution or many other conditions.

The place of sensors are very important in WSN to get an accurate information. Sometimes, placing sensors at desired locations might not be possible or expensive. By reducing number of sensor devices and activating them more frequently in time, we might still get the same information. This idea of the spatiotemporal trade off was studied in heat diffusion processes by Lu and Vetterli in ([34]).

The mathematical system was created by Aldroubi and his collaborators in 2012 with results appearing in ([2],[3]) and others.

In the dynamical sampling problem, a signal $x \in \mathcal{H}$ is reconstructed from a set of fixed spatial that are represented at samples Ω at different time intervals ℓ. The idea is to place the sensors at location Ω and get the information over multiple times ℓ to reconstruct an unknown status. The combination of space and time samples makes the dynamical sampling problem different from standard sampling.

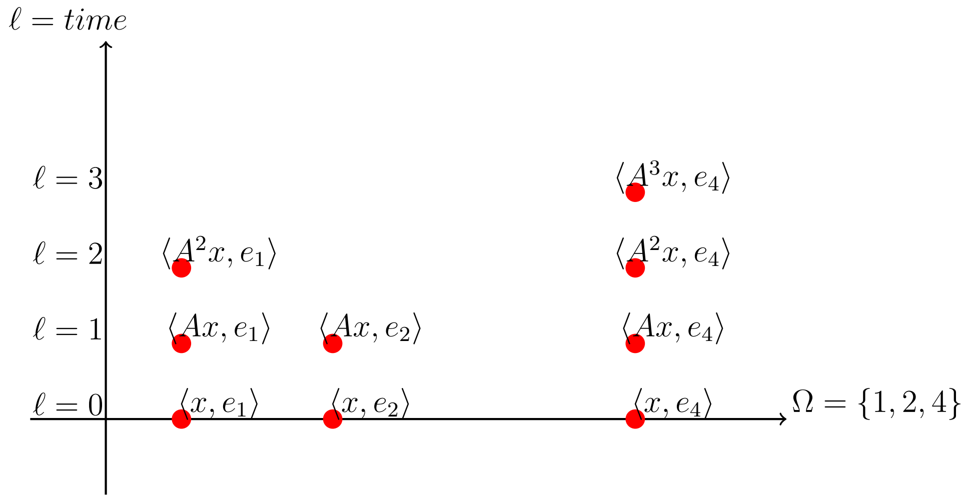

Figure 2.2: Time-space dynamical sampling pattern

Let $\mathcal{H}$ be a real or complex Hilbert space. Suppose that a signal $x \in \mathcal{H}$ varies in time increments according to the operator A on $\mathcal{H}$. Knowing how the system evolves over time is the crucial component in dynamical sampling.

$$\begin{aligned} x_0 &= x \\ x_1 &= Ax \\ x_2 &= A(Ax) = A^2x \\ &\vdots \\ x_L &= A^Lx \end{aligned}$$

The fundamental dynamical sampling problem ([2]) is to find conditions on Ω, A, and the number L of time increments such that measurements on the components given by course sample points Ω over times ℓ can be used to reconstruct x.

In other words, we want to construct $x \in \mathcal{H}$ from the measurements

$$\{\langle A^\ell x, e_i\rangle \,:\, \ell = 0, 1, \dots, L;\ i \in \Omega\}. \tag{2.2.1}$$

shown in Figure 2.2

The dynamical sampling problem has a connection to the frame theory. Since we want to construct $x \in \mathcal{H}$ from the measurements in (2.2.1), the set of vectors in (2.2.2) must be a frame in $\mathcal{H}$.

Lemma 2.2.1 ([2]). *We can reconstruct x from the sampling set indexed by Ω over times $\ell = 0, 1, \dots, L$ if and only if the set*

$$\{A^{*\ell} e_i \,:\, i \in \Omega, \ell = 0, 1, \dots L\} \tag{2.2.2}$$

is a frame for $\mathcal{H}^n$.

Proof. Let the set $\{A^{*\ell} e_i \,:\, i \in \Omega, \ell = 0, 1, \dots L\}$ be a frame for $\mathcal{H}$ and S be its frame operator, then

$$S(x) = \sum_{i \in \Omega, \ell = 0, 1, \dots L} \langle x, A^{*\ell} e_i\rangle A^{*\ell} e_i. \tag{2.2.3}$$

Since the frame operator S is invertible, we have

$$x = S^{-1}S(x) = \sum_{i,\ell} \langle x, A^{*\ell} e_i\rangle S^{-1}(A^{*\ell} e_i) = \sum_{i,\ell} \langle A^\ell x, e_i\rangle S^{-1}(A^{*\ell} e_i). \tag{2.2.4}$$

The result follows from the Equality in (2.2.4). $\square$

If A is a diagonazible operator, then it can be decomposed as $A = B^{-1}DB$, where D is diagonal and B is invertible. In this case, we can state an equivalent version of dynamical sampling:

We consider whether $\{D^\ell b_i\}$ is a frame for $\mathbb{C}^n$, where $b_i = Be_i$. We see this by observing that:

$$A^\ell e_i = B^{-1} D^\ell B e_i = B^{-1} D^\ell b_i.$$

We have that frames are preserved under bounded invertible operators, for that reason $\{A^\ell e_i\}_{i\in\Omega,\ell=0,1,\dots L}$ is a frame if and only if $\{D^\ell b_i\}_{i\in\Omega,\ell=0,1,\dots L}$ is a frame.

Let A be a matrix that can be writen as $A^* = B^{-1}DB$ where D is diagonal and B is invertible. Let $\{\lambda_j\}$ be distinct eigenvectors of D and P_j denote the orthogonal projection in $\mathcal{H}^n$ onto the eigenspace E_j of D associated to the eigenvalue λ_j. Then we have the following result.

Theorem 2.2.2. *[2, Thm: 2.2] Let $\Omega \subseteq \{1, 2, \dots, n\}$ and $\{b_i : i \in \Omega\}$ be vectors in $\mathbb{C}^n$. Let D be a diagonal matrix and r_i be the degree of the D-annihilator of b_i. Then $\{D^j b_i : i \in \Omega; j = 0, 1, ..., l_i; l_i = r_i - 1\}$ is a frame of $\mathbb{C}^n$ if and only if $\{P_j(b_i) : i \in \Omega\}$ is a frame of E_j for all j.*

Theorem (2.2.2) states that when D is a diagonal operator with distinct nonzero eigenvalues λ_j and $b \in \mathbb{C}^n$ with no zero components, then we can have dynamical sampling frame with a single vector. Higher dimensional eigenspaces require more vectors to have dynamical sampling frame. If the number of sampling vectors $|\Omega|$ is less than maximum of the dimension of eigenspaces, we cannot have dynamical sampling frame even if we increase time measurements.

The authors of ([2]) have also extended Theorem (2.2.2) to non-diagonalizable operators. We use the same notation in ([2]).

A matrix $J \in \mathbb{C}^{nxn}$ is in Jordan form if

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix} \tag{2.2.5}$$

For $s = 1, 2, ..n$, $J_s = \lambda_s I_s + N_s$ where I_s is an $r_s \times r_s$ identity matrix and N_s is a $r_s \times r_s$ nilpotent block-matrix of the form:

$$N_s = \begin{pmatrix} N_{s_1} & 0 & \cdots & 0 \\ 0 & N_{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{s_{\gamma_s}} \end{pmatrix} \tag{2.2.6}$$

Each N_{si} is a $r_i^s \times r_i^s$ cyclic nilpotent matrix of the form:

$$N_{si} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.2.7}$$

with $r_1^s \geq r_2^s \geq ...$, and $r_1^s + r_2^s + ... + r_s^s = r_s$. The matrix J has distinct eigenvalues λ_i, $i = 1, 2, ..n$ and $r_1 + r_2 + ... + r_n = N$.

Let k_{sj} denote the index corresponding to the first row of the cyclic nilpotent matrix N_{sj} (3.5.3), and let $e_{k_{sj}}$ be the corresponding standard orthonormal basis of $\mathbb{C}^n$. We define $W_s = \text{span}\{e_{k_{sj}} : j = 1, 2, ..., \gamma_s\}$ and P_s denote the orthogonal projection onto W_s.

Theorem 2.2.3. *[2, Thm 2.6] Let J be a matrix in Jordan form as in 3.5.1. Let $\Omega \subseteq \{1, 2, \dots, n\}$ and $\{b_i : i \in \Omega\}$ be vectors in $\mathbb{C}^n$, r_i be the degree of the J-annihilator of the vector b_i and $l_i = r_i - 1$. Then the following propositons are equivalent.*

1. *The set of vectors $\{J^j b_i : i \in \Omega,\ j = 0, 1, ..., l_i, \}$ is a frame for $\mathbb{C}^n$.*

2. *For every $s = 1, .., n$, $\{P_s(b_i) : i \in \Omega\}$ is a frame for W_s.*

Theorem (2.2.3) gives a necessary and sufficient condition for dynamical sampling reconstruction for any operator A on $\mathbb{C}^n$.

2.3 Phase Retrieval and Norm Retrieval

Signal reconstruction has a wide variety of application in many engineering areas but recovering a signal when there is a partial loss of information is a big challenge. The signal reconstruction in the case of partial loss of information is only possible under special conditions.

If the frame vectors are redundant, they have the advantage of possibly reconstructing the signal in the case of partial loss of information, which is not possible using orthonormal bases. The signal reconstruction problem has been studied widely in physics, signal processing and mathematics. Recovering the phase of a signal given by its intensity measurements from a redundant linear system is different then signal reconstruction. Casazza, Balan, and Edidin ([8]) introduced the concept of phase retrieval for Hilbert space frames in 2006 to recover the phase of a signal given by its intensity measurements from a redundant linear system.

The problem occurs in speech recognition ([27]), optics applications such as X-ray crystallography ([17],[37]), quantum state tomography ([35]), and electron microscopy ([40], [31]). The notion of norm retrieval is more recent construction. It was introduced in ([5]) as a relaxation of phase retrieval. The idea is to be able to reproduce the norm of a vector x given its phaseless measurements. Norm retrieval is a very new concept and we are just beginning to understand the possible applications.

In this chapter, we will give basic informations about phase retrieval and norm retrieval. We refer the reader ([26],[6],[10],[9],[11],[16]) for more information about phase retrieval and norm retrieval for Hilbert spaces.

Definition 2.3.1. [8] A set of vectors $\{x_i\}_{i-1}^{M}$ in $\mathbb{R}^n$ yields **phase retrieval** if for all $x, y \in \mathbb{R}^n$ satisfying $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i = 1, .., M$, then $x = cy$ where $c = \pm 1$ in $\mathbb{R}^n$.

Definition 2.3.2. [5] A set of vectors $\{x_i\}_{i-1}^M$ in $\mathbb{R}^n$ does **norm retrieval**, if for $x, y \in \mathbb{R}^n$ satisfying $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i = 1, ..., M$, then $||x|| = ||y||$.

Here, we only ask to recover the magnitude of the vector from phaseless measurements.

There is also a notion of phase retrieval and norm retrieval by projections which align with our previous definitions when the projections are one-dimensional. is similar to one dimensional case.

Definition 2.3.3. [5] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^M$ (or $\{P_i\}_{i=1}^M$) yields **phase retrieva**l if for $x, y \in \mathbb{R}^n$ satisfying $||P_i x|| = ||P_i y||$ for all $i = 1, ..., M$, then $x = cy$ for some scalar c with $c = \pm 1$.

Definition 2.3.4. [5] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^M$ (or $\{P_i\}_{i=1}^M$) yields **norm retrieva**l if for $x, y \in \mathbb{R}^n$ satisfying $||P_i x|| = ||P_i y||$ for all $i = 1, ..., M$, then $||x|| = ||y||$.

Definition 2.3.5. [8] A frame $\{x_i\}_{i-1}^M$ in $\mathbb{R}^n$ satisfies the **complement property** if for any index set $I \subset \{1, ...M\}$, either $\text{span}\{x_i\}_{i\in I} = \mathbb{R}^n$ or $\text{span}\{x_i\}_{i\in I^c} = \mathbb{R}^n$.

In the real Hilbert space, a fundamental paper ([8]) classifies phase retrieval by the complement property as follows.

Theorem 2.3.6. *[8] A frame $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$ yields phase retrieval if and only if it has the complement property. In particular, a full spark frame with $2n-1$ vectors yields phase retrieval. If $\{x_i\}_{i-1}^M$ yields phase retrieval in $\mathbb{R}^n$, then $M \geq 2n-1$. In other words, there is no set of $2n-2$ vectors that yields phase retrieval.*

Norm retrieval differs from phase retrieval. A set of vectors in $\mathbb{R}^n$ needs at least $2n-1$ vectors to satisfy phase retrieval but we can have norm retrieval with less number of vectors. For example, orthonormal bases are norm retrievable sets, but they are not sets that accomplish phase retrieval.

Lemma 2.3.7. *[13] If the set of vectors $\{x_i\}_{i=1}^n$ does norm retrieval in $\mathbb{R}^n$, then the vectors $\{x_i\}_{i=1}^n$ are orthogonal.*

A classification of norm retrievable vectors in $\mathbb{R}^n$ is given by author of ([29]) in Theorem (2.3.8). Since this classification plays an important role in our problem, we also include the proof to make it clear for readers.

Theorem 2.3.8. *[29] A frame set $\{x_i\}_{i=1}^M \in \mathbb{R}^n$ is a norm retrievable frame if and only if any partition of $I \subset [1..M]$ index set, we have*

$$span\{x_i\}_{i\in I}^{\perp} \perp span\{x_i\}_{i\in I^c}^{\perp}. \tag{2.3.1}$$

Proof. ($\Longrightarrow$) Suppose $\{x_i\}_{i=1}^M \in \mathbb{R}^n$ be a norm retrievable frame and $I \subset [1..M]$ be a partition of index set. For any $x \in span\{x_i\}_{i\in I}^{\perp}$ and $y \in span\{x_i\}_{i\in I^c}^{\perp}$, we have

$$\langle x, x_i \rangle = 0 \quad \text{for all } i \in I \quad \text{and} \quad \langle y, x_i \rangle = 0 \quad \text{for all } i \in I^c$$

which gives us

$$\langle x+y, x_i\rangle = -\langle x-y, x_i\rangle \quad \text{for all } i \in I \quad \text{and} \quad \langle x+y, x_i\rangle = \langle x-y, x_i\rangle \quad \text{for all } i \in I^c.$$

Now, we can write

$$|\langle x+y, x_i\rangle| = |\langle x-y, x_i\rangle| \quad \text{for all } i \in [1..M].$$

Since $\{x_i\}_{i=1}^M \in \mathbb{R}^n$ is a norm retrievable frame, by definition (2.3.2), we have $||x+y|| = ||x-y||$ and

$$||x||^2 + 2\langle x, y\rangle + ||y||^2 = ||x+y||^2 = ||x-y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2$$

which implies that $\langle x, y\rangle = 0$ and span$\{x_i\}_{i\in I}^\perp \perp$ span $\{x_i\}_{i\in I^c}^\perp$.

($\Longleftarrow$) Suppose span$\{x_i\}_{i\in I}^\perp \perp$ span $\{x_i\}_{i\in I^c}^\perp$ for any partition $I \subset [1..M]$ of index set and

$$|\langle x, x_i\rangle| = |\langle y, x_i\rangle| \quad \text{for all } i \in [1..M].$$

Then we can make a partition $I = \{i \in [1,2,..M] : \langle x, x_i\rangle = -\langle y, x_i\rangle\}$ and $I^c = [1,2,..M] \setminus I$ so that

$$x+y \in span\{x_i\}_{i\in I}^\perp \quad \text{and} \quad x-y \in span\{x_i\}_{i\in I^c}^\perp$$

By assumption, we have span$\{x_i\}_{i\in I}^\perp \perp$ span $\{x_i\}_{i\in I^c}^\perp$. Hence, we can write

$$0 = \langle x+y, x-y\rangle = ||x||^2 - ||y||^2 \quad \text{and} \quad ||x||^2 = ||y||^2$$

.

$\square$

Remark 2.3.9. Let $I \subset [1..M]$ be a partition of index set. Theorem (2.3.8) implies that $\{x_i\}_{i=1}^M \in \mathbb{R}^n$ does norm retrieval if and only if (span $\{x_i\}_{i\in I})^{\perp} \subset \text{span}\{x_i\}_{i\in I^c}$ as shown in ([13]). The condition of phase retrieval has a defining property in $\mathbb{R}^n$ parallel to (2.3.1). If the complementary property (2.3.5) is satisfied, we can see that (2.3.1) is also satisfied, so phase retrieval is a stronger condition than norm retrieval.

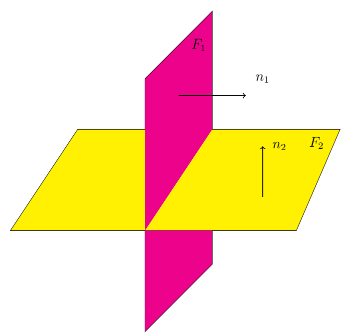

Figure 2.3

Example 2.3.10. *We want to understand the condition in (2.3.1) better.*

Let $F = \{x_i \in \mathbb{R}^3 : i \in I; \quad |I| = 4\}$ *be a set of full spark vectors that spans* $\mathbb{R}^3$*. Theorem (2.3.8) states that* F *does norm retrieval in* $\mathbb{R}^3$ *if and only if for any partition* F_1, F_2 *of* F *into two subsets,* $(spanF_1)^{\perp} \perp (spanF_2)^{\perp}$.

For any partition F_1, F_2 *of* F*, we have two possibilities for dimension of* $spanF_i$ *for* $i = 1, 2$.

Either $dim(spanF_1) = dim(spanF_2) = 2$ *or* $dim(spanF_i) = 3$ *for one of* $i = 1$ *or* $i = 2$*. Without lose of generality, assume* $dim(spanF_1) = 3$*. Then* $spanF_1 = \mathbb{R}^3$ *and the complementary condition in (2.3.5) is satisfied. Hence,* F *may possibly do norm retrieval in* $\mathbb{R}^3$.

If dim(spanF_1) = dim(spanF_2) = 2, then the complementary condition in (2.3.5) fails. So we must check the condition (2.3.1).

The norm retrieval property in (2.3.1) states that if normal vectors n_1, n_2 of the planes spanF_1 and spanF_2 respectively are orthognal as shown in Figure 2, then $F = \{x_i \in \mathbb{R}^3 : i \in I; \quad |I| = 4\}$ does norm retrieval in $\mathbb{R}^3$. If, on the other hand, normal vectors n_1, n_2 of spanF_1 and spanF_2 are not orthogonal as shown in Figure 3, then $F = \{x_i \in \mathbb{R}^3 : i \in I; \quad |I| = 4\}$ does not do norm retrieval in $\mathbb{R}^3$.

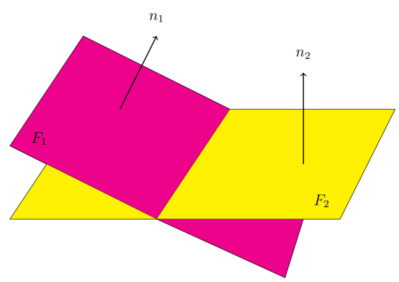

Figure 2.4

Given a set of vectors $\{x_i\}_{i=1}^M$ in $\mathbb{R}^n$. The complementary property in (2.3.5) gives a classification of phase retrievable vectors in $\mathbb{R}^n$. Theorem (2.3.8) also gives a classification of norm retrievable vectors in $\mathbb{R}^n$.

We now move on to describe the conditions for phase and norm retrieval of subspaces.

Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. Phase retrieval and norm retrieval definitions for projections $\{P_i\}_{i=1}^M$ are defined in (2.3.3) and (2.3.4) respectively.

The classification of phase retrieval by projections in $\mathbb{R}^n$ were found by Edidin in ([24]) in terms of the spans of $\{P_i x\}_{i=1}^M$, for $x \in \mathbb{R}^n$.

Theorem 2.3.11. *[24] Let $\{W_i\}_{i=1}^M$ be a collection of subspaces in $\mathbb{R}^n$ and define $\{P_i\}_{i=1}^M$ to be the orthogonal projections onto each of these subspaces. The collection of projections $\{P_i\}_{i=1}^M$ does phase retrieval if and only if for any nonzero vector $x \in \mathbb{R}^n$, span$\{P_i x\}_{i=1}^M = \mathbb{R}^n$.*

Authors in [12] gave a classification of norm retrieval by projections in $\mathbb{R}^n$ similar to the Edidin Theorem 2.3.11. This classification generalizes Theorem 2.3.8 from norm retrieval of vector to do norm retrieval of projections.

Theorem 2.3.12. *[12] Let $\{P_i x\}_{i=1}^M$ be projections on $\mathbb{R}^n$, then the following are equivalent:*

1. *The projections $\{P_i\}_{i=1}^M$ do norm retrieval.*
2. *For every nonzero vector $x \in \mathbb{R}^n$, (span$\{P_i x\})^\perp \subset \{x\}^\perp$.*
3. *For every nonzero vector $x \in \mathbb{R}^n$, $x \in$ span$\{P_i x\}_{i=1}^M$.*

Proof. (1)$\Longrightarrow$(2): To prove by contrapositive, suppose there exists a nonzero vector $u \in \mathbb{R}^n$ such that $(\text{span}\{P_i u\})^\perp \not\subset u^\perp$. Then there exists a nonzero vector $w \in (\text{span}\{P_i u\})^\perp$ such that u and w are not orthogonal and $w \perp P_i u$ for all i.

Let $x = \dfrac{1}{2}(u + w)$ and $y = \dfrac{1}{2}(u - w)$. Since u and w are not orthogonal, we have $||x|| \neq ||y||$. Since $w \perp P_i u$ for all i, we have

$$
\begin{aligned}
||P_i(u+w)||^2 &= \langle P_i u + P_i w, P_i u + P_i w\rangle \\
&= ||P_i u||^2 + ||P_i w||^2 \\
&= \langle P_i u - P_i w, P_i u - P_i w\rangle \\
&= ||P_i u||^2 - ||P_i w||^2
\end{aligned}
$$

Hence, $||P_i u|| = ||P_i w||$ for all i but $||x|| \neq ||y||$. Which says that the projections $\{P_i\}_{i=1}^M$ does not do norm retrieval.

(2)$\Longrightarrow$(1): Again by contrapositive, suppose the projections $\{P_i\}_{i=1}^M$ does not do norm retrieval. Then there are vectors $x, y \in \mathbb{R}^n$ such that $||P_i u|| = ||P_i w||$ for all i but $||x|| \neq ||y||$. Let $u = x + y$ and $w = x - y$, then u and w are not orthogonal. Which implies that $w \notin u^\perp$ but $w \in (\text{span}\{P_i u\})^\perp$. So, property (2) fails.

(2)$\Longrightarrow$(3): To prove by contrapositive, suppose $x \notin \text{span}\{P_i x\}_{i=1}^M$. Then $x = x_1 + x_2$ where $x_1 \in \text{span}\{P_i x\}_{i=1}^M$ and $x_2 \notin \text{span}\{P_i x\}_{i=1}^M$. This shows that $\langle x, x_2 \rangle \neq 0$ and the condition $(\text{span}\{P_i x\})^\perp \subset \{x\}^\perp$ fails. This proves (2)$\Longrightarrow$(3).

(3)$\Longrightarrow$(2): Since $x \in \text{span}\{P_i x\}_{i=1}^M$ implies $(\text{span}\{P_i x\})^\perp \subset \{x\}^\perp$. This part is obvious.

$\square$

The set of projections $\{P_i\}_{i=1}^M$ onto W_i which does norm retrieval gives us opportunity to construct norm retrievable vectors in $\mathbb{R}^n$ using orthonormal bases in W_i.

By using projections, the authors of [13] have an extended version of the classification of norm retrieval as shown in the following theorem.

Theorem 2.3.13. *[13] Let $\{P_i\}_{i=1}^M$ be the projections onto subspaces $\{W_i\}_{i=1}^M$ of $\mathbb{R}^n$. The set of projections $\{P_i\}_{i=1}^M$ does norm retrieval if and only if for any orthonormal bases $\{u_{ij}\}_{j=1}^{r_i}$ of W_i, the set of vectors $\{u_{ij}\}_{(i,j)}$ does norm retrieval in $\mathbb{R}^n$.*

Norm retrieval is preserved under rescaling. That is, if we rescale each vector in any norm retrievable set, we will have a new norm retrievable set.

Lemma 2.3.14. *[13] All scalable frames do norm retrieval.*

Proof. Let $\{x_i\}_{i\in I}$ be a scalable frame in a real or complex Hilbert space H. To show that $\{x_i\}_{i\in I}$ does norm retrieval in $\mathcal{H}$, suppose given $x, y \in \mathcal{H}$, we have

$$|\langle x, x_i\rangle| = |\langle y, x_i\rangle| \quad \text{for all} \quad i \in I.$$

Since $\{x_i\}_{i\in I}$ is a scalable frame in $\mathcal{H}$, by Definition (2.1.12), there exists scalars $\{c_i\}_{i\in I}$ such that $\{c_i x_i\}_{i\in I}$ is a Parseval frame. Parseval identity shows that for any $x \in \mathcal{H}$, we have

$$||x||^2 = \sum_{i\in I} |\langle x, c_i x_i\rangle|^2 \quad \text{for all} \quad i \in I.$$

For any scalar c_i, we have

$$|\langle x, c_i x_i\rangle| = |\langle y, c_i x_i\rangle| \quad \text{for all} \quad i \in I.$$

This implies that $||x|| = ||y||$.

Authors in [1] showed that when A is a unitarily diagonalizable normal operator, we can get a scalable frames with the dynamical sampling structure under proper conditions. In Chapter 3, we show a similar structure to build norm retrievable sets in the dynamical sampling setting that is not scalable frame.

□

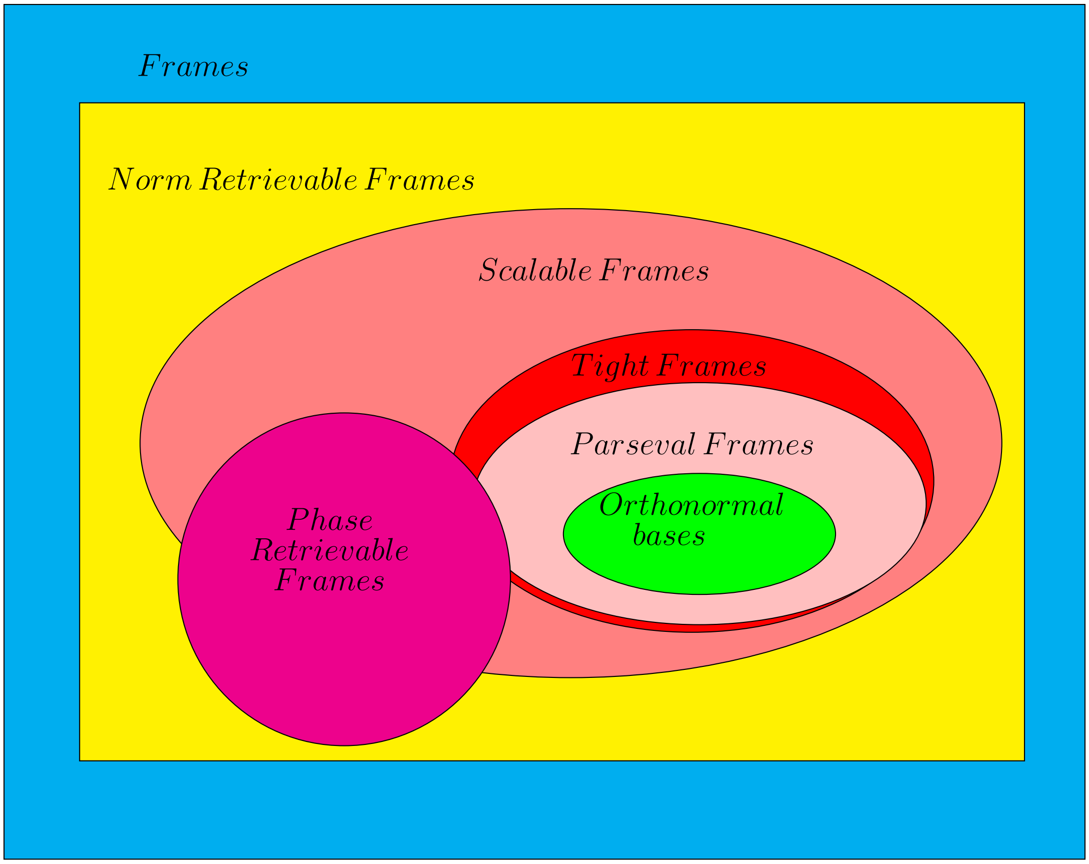

Figure 2.5

In the next example, we show a set of vectors in $\mathbb{R}^n$ for any $n \geq 2$ that does norm retrieval. In each set, there are $2n-2$ vectors, so these sets cannot do phase retrieval in $\mathbb{R}^n$.

Example 2.3.15. *Let $\{e_i\}_{i=1}^n$ be the standard orthonormal basis in $\mathbb{R}^n$. Then the set of vectors $\{\alpha e_n \pm e_i\}_{i=1}^{n-1}$ does norm retrieval when $\alpha = \pm\dfrac{1}{\sqrt{n-1}}$.*

Proof. Given $x = [x_1.x_2, ...x_n]^T, y = [y_1, y_2, ..., y_n]^T \in \mathbb{R}^n$, suppose we have $|\langle x, \alpha e_n \pm e_i\rangle| = |\langle y, \alpha e_n \pm e_i\rangle|$ for all $i = 1, 2, ..n-1$. Then,

$$|\langle x, \alpha e_n \pm e_i\rangle|^2 = |\langle y, \alpha e_n \pm e_i\rangle|^2$$

$$\alpha^2 {x_n}^2 + {x_i}^2 = \alpha^2 {y_n}^2 + {y_i}^2$$

for $i = 1, 2, ..n - 1$. This shows that

$$(n-1)\alpha^2 {x_n}^2 + \sum_{i=1}^{n-1} {x_i}^2 = (n-1)\alpha^2 {y_n}^2 + \sum_{i=1}^{n-1} {y_i}^2$$

and

$$((n-1)\alpha^2 - 1){x_n}^2 + ||x||^2 = ((n-1)\alpha^2 - 1){y_n}^2 + ||y||^2.$$

If $(n-1)\alpha^2 - 1 = 0$, then $||x||^2 = ||y||^2$. Hence, $\{\alpha e_n \pm e_i\}_{i=1}^{n-1}$ does norm retrieval when $\alpha = \pm \frac{1}{\sqrt{n-1}}$.

$\square$

In $\mathbb{R}^3$, the set of vectors $\{\alpha e_3 \pm e_i\}_{i=1}^2$ is full spark. We know in reference that the set of vectors $\{\alpha e_3 \pm e_i\}_{i=1}^2$ does norm retrieval if and only if for any partition F_1, F_2 of the set of vectors $\{\alpha e_3 \pm e_i\}_{i=1}^2$, $(\text{span} F_1)^\perp \perp (\text{span } F_2)^\perp$. Since the set of vectors $\{\alpha e_3 \pm e_i\}_{i=1}^2$ is full spark, when cardinality of $|F_1| = 3$ or $|F_2| = 3$, then $\text{span} F_1^\perp \perp \text{span } F_2^\perp$ and we are done.

When cardinality of $|F_1| = 2$ and $|F_2| = 2$, as shown in Figure 5, normal vectors n_1, n_2 of the planes $\text{span} F_1$ and $\text{span} F_2$ respectively will be orthogonal with $\alpha = \frac{1}{\sqrt{2}}$. This holds for all pairs of planes.

Remark 2.3.16. In Example 2.3.15, we show that the set of vectors $\{\alpha e_n \pm e_i\}_{i=1}^{n-1}$ in $\mathbb{R}^n$ does norm retrieval when $\alpha = \pm \frac{1}{\sqrt{n-1}}$. Actually, the set of vectors $\{\alpha e_n \pm e_i\}_{i=1}^{n-1}$ also has a dynamical sampling structure. To see this, define an operator A on $\mathbb{R}^n$ such that

$$\begin{aligned} Ae_i &= e_{i+1} \quad \text{for} \quad i = 1, 2, ..., n-2 \\ Ae_{n-1} &= -e_1 \\ Ae_n &= e_n. \end{aligned}$$

Now, we can write the set of vectors $\{\alpha e_n \pm e_i\}_{i=1}^{n-1}$ in $\mathbb{R}^n$ in the dynamical sampling structure by a single generator. For $b = \alpha e_n - e_1$, we have

$$\begin{aligned} A^\ell b &= \alpha e_n - e_{\ell+1} \quad \text{for} \quad \ell = 0, 1, 2, ..., n-3 \\ A^\ell b &= \alpha e_n + e_{\ell+1} \quad \text{for} \quad \ell = n-2, ..., 2n-3. \end{aligned}$$

Hence, $\{\alpha e_n \pm e_i\}_{i=1}^{n-1} = \{A^\ell b\}_{\ell=0}^{2n-3}$ when $b = \alpha e_n - e_1$. Since $|\ell| = 2n-2$, we do not have enough vectors to do phase retrieval. Recall that we need at least $2n-1$ vectors in $\mathbb{R}^n$ to have a phase retrievable set.

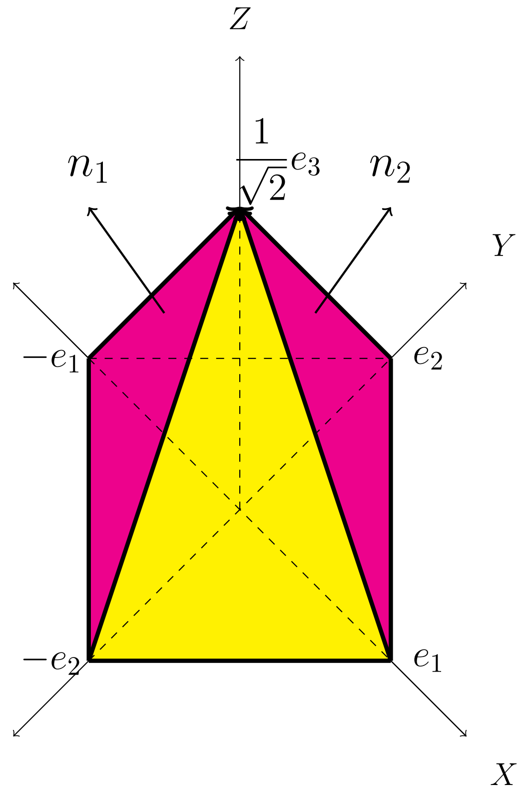

Figure 2.6: lllustration of Example 2.3.15 in $\mathbb{R}^3$.

Chapter 3

Norm Retrieval of Vectors in Dynamical Sampling Form

3.1 Description of Problem

We begin by setting up a classical dynamical sampling system in $\mathbb{R}^n$ with an operator A. Suppose that A is a linear operator on $\mathbb{R}^n$ and the signal $x \in \mathbb{R}^n$ varies in time increments according to the operator A. At time $\ell \in \mathbb{N}$, the signal $x \in \mathbb{R}^n$ evolves through the operator A to become $A^\ell x = x_\ell$. Let $\Omega \subseteq \{1, 2, ..., n\}$ be the sample points and $\{e_i\}_{i=1}^n$ be the standard orthonormal bases in $\mathbb{R}^n$.

We represent $A^\ell x(i)$ as the time-space sample at time $\ell \in \mathbb{N}$ and location $i \in \Omega$. That is

$$A^\ell x(i) = \langle A^\ell x, e_i \rangle.$$

Then $\Omega \subseteq \{1, 2, ..., n\}$ gives the sample points. As we showed in Chapter 2, the measurements $\{x(i) : i \in \Omega\}$ have insufficient information in general to recover the original signal x. Representing samples over time from fixed positions in space, we will have extra information $\{A^\ell x(i) : i \in \Omega\}$ about the signal x. We give basic informations about the dynamical sampling problem in chapter 2. Figure (2.2) gives an illustration of time-space samples in dynamical sampling.

The fundamental dynamical sampling problem ([2]) is to find conditions on Ω, A, and the number L of time increments such that measurements on the components given by coarse sample points Ω over times ℓ can be used to reconstruct x. In other words, we want to construct x from the measurements

$$\{\langle A^\ell x, e_i\rangle \,:\, \ell = 0, 1, \ldots, L;\; i \in \Omega\}. \tag{3.1.1}$$

In ([2]), Aldroubi and his collaborators recently showed that x can be recovered from the measurements $\{\langle A^\ell x, e_i\rangle \,:\, \ell = 0, 1, \ldots, L;\; i \in \Omega\}$ if and only if $\{A^{*\ell} e_i \,:\, \ell = 0, 1, \ldots, L;\; i \in \Omega\}$ is a frame in $\mathcal{H}$ (real or complex Hilbert space). They gave the conditions on $A, \Omega, \text{and}\, \ell$ in Theorem (2.2.2) and Theorem (2.2.3), which are stated in Chapter 2, such that $\{A^{*\ell} e_i \,:\, \ell = 0, 1, \ldots, L;\; i \in \Omega\}$ is a frame in $\mathcal{H}$.

In this paper, we show constructions of norm retrievable sets that arise from dynamical sampling system in the finite dimensional real Hilbert space $\mathbb{R}^n$. By the Definition (2.3.2), a set of vectors $\{x_i\}_{i\in I} \in \mathbb{R}^n$ does norm retrieval if any given $x, y \in \mathbb{R}^n$, $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i \in I$ implies that $||x|| = ||y||$.

The norm retrieval problem in a dynamical sampling setting can be stated as follows:

Problem: The norm retrieval problem in dynamical sampling seeks to find conditions on the operator A, the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ and the time increments ℓ_i such that the set of vectors $\{A^{\ell_i} b_i \in \mathbb{R}^n : i \in \Omega\}$ will have the norm retrieval property as stated in Definition (2.3.2). Recall that the collection must necessarily be a frame. We are particularly interested in the set of vectors $\{A^{\ell_i}b_i \in \mathbb{R}^n : i \in \Omega\}$ which does norm retrieval but not phase retrieval.

We show in the Theorem (3.1.1) that a set of vectors F does norm retrieval in $\mathbb{R}^n$ if the identity operator in $\mathbb{R}^n$ is in the spanning set of the rank one projections of the vectors in F.

Theorem 3.1.1. *Let A be an operator on $\mathbb{R}^n$, $\{e_j\}_{j=1}^n$ be the standard orthonormal bases and $\{b_i \in \mathbb{R}^n : i \in \Omega,\ |\Omega| < n\}$. The set of vectors $\{A^\ell b_i\}_{\{\ell\in\{1,2,\ldots,\ell_i\}, i\in\Omega\}}$ does norm retrieval in $\mathbb{R}^n$ if there is a solution $\{C_{\ell,i}\}$ to the system of linear equations:*

$$\sum_{\ell,i} C_{\ell,i}|\langle e_j, A^\ell b_i\rangle|^2 = 1 \tag{3.1.2}$$

$$\sum_{\ell,i} C_{\ell,i}\langle e_j, A^\ell b_i\rangle\langle e_k, A^\ell b_i\rangle = 0 \tag{3.1.3}$$

for all $j, k = 1, 2, ..n$ with $j \neq k$.

Proof. Suppose given the operator A on $\mathbb{R}^n$ and the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$, we know the measurements $|\langle x, A^\ell b_i\rangle| = |\langle y, A^\ell b_i\rangle| \quad \forall\, \ell \in \{0, 1, .., \ell_i\}, i \in \Omega$ for fixed $x, y \in \mathbb{R}^n$.

Then,

$$\langle x - y, A^\ell b_i\rangle = 0 \quad \text{or} \quad \langle x + y, A^\ell b_i\rangle = 0 \quad \forall \ell, i$$

So,

$$\langle x - y, \langle x + y, A^\ell b_i\rangle A^\ell b_i\rangle = \langle x - y, A^\ell b_i(A^\ell b_i)^*(x + y)\rangle = 0 \quad \forall \ell, i$$

Given any scalar value $C_{\ell,i}$, we have $C_{\ell,i}\langle x - y, A^\ell b_i(A^\ell b_i)^*(x + y)\rangle = 0 \quad \forall \ell, i$.

If $I \in \text{span}\{A^\ell b_i(A^\ell b_i)^*\}_{\{\ell,i\}}$, then $\langle x-y, x+y\rangle = 0$ and $||x|| = ||y||$.

Now, we show that $I \in \text{span}\{A^\ell b_i(A^\ell b_i)^*\}_{\{\ell,i\}}$ if and only if equations (3.1.2) and (3.1.3) have a solution.

Let $\{e_j\}_{j=1}^n$ be the standard orthonormal bases in $\mathbb{R}^n$. Any vector $A^\ell b_i \in \mathbb{R}^n$ can be writen as,

$$A^\ell b_i = \begin{bmatrix} \langle e_1, A^\ell b_i\rangle \\ \langle e_2, A^\ell b_i\rangle \\ \vdots \\ \langle e_n, A^\ell b_i\rangle \end{bmatrix}, \text{ then we have}$$

$$A^\ell b_i(A^\ell b_i)^* = \begin{bmatrix} |\langle e_1, A^\ell b_i\rangle|^2 & \langle e_1, A^\ell b_i\rangle\langle e_2, A^\ell b_i\rangle & \cdots \langle e_1, A^\ell b_i\rangle\langle e_n, A^\ell b_i\rangle \\ \langle e_2, A^\ell b_i\rangle\langle e_1, A^\ell b_i\rangle & |\langle e_2, A^\ell b_i\rangle|^2 & \cdots \langle e_2, A^\ell b_i\rangle\langle e_n, A^k b_i\rangle \\ \vdots & \vdots & \vdots \\ \langle e_n, A^\ell b_i\rangle\langle e_1, A^\ell b_i\rangle & \langle e_n, A^\ell b_i\rangle\langle e_2, A^{\ell_i} b_i\rangle & \cdots |\langle e_n, A^\ell b_i\rangle|^2 \end{bmatrix}$$

The system of linear equations in (3.1.2) and (3.1.3) has a solution if and only if $I \in \text{span}\{A^\ell b_i(A^\ell b_i)^*\}_{\{\ell,i\}}$. In that case, we also have $\{A^\ell b_i\}_{\ell,i}$ does norm retrieval in $\mathbb{R}^n$ as shown in Example (3.5.4).

□

When A is an $n \times n$ diagonal operator, the authors in ([1, Thm.3]) showed that the set of vectors $\{A^\ell b_i\}_{\{\ell,i\}}$ is a scalable frame if and only if there exists a positive solution $\{C_{\ell,i}\}$ to the system of equations in (3.1.2) and (3.1.3). We know that all scalable frames do norm retrieval by the Theorem (2.3.14). Theorem (3.1.1) shows that there exists norm retrievable frames $\{A^k b_i\}_{\{k,i\}}$, that are not scalable frames, whenever the solution $\{C_{k,i}\}$ to the system of equations in (3.1.2) and (3.1.3) is not a positive solution. Theorem (3.1.1) does not give the conditions on the operator A, the set of sample points $\{b_i \in \mathbb{R}^n : i \in \Omega,\ |\Omega| < n\}$ and time increments ℓ but we show later how it works to obtain dynamical sampling frame which does norm retrieval .

Recall our definitions of norm retrieval of vectors and projections given in (2.3.2) and (2.3.4) respectively. In 2017, Peter G. Casazza and his research group in ([13]) give a classification of norm retrievable sets in $\mathbb{R}^n$ in terms of projections as follows: Let $\{P_i\}_{i=1}^M$ be the projections onto subspaces $\{W_i\}_{i=1}^M$ of $\mathbb{R}^n$. The set of projections $\{P_i\}_{i=1}^M$ does norm retrieval if and only if for any orthonormal bases $\{u_{ij}\}_{j=1}^{r_i}$ of W_i, the set of vectors $\{u_{ij}\}_{(i,j)}$ does norm retrieval in $\mathbb{R}^n$.

We are able to write a more general version of the norm retrieval classification in ([13]), we will use this extensively to obtain dynamical sampling frames which do norm retrieval in $\mathbb{R}^n$.

Proposition 3.1.2. *Let $\{P_i\}_{i=1}^M$ be the projections onto the subspaces $\{W_i\}_{i=1}^M$ of $\mathbb{R}^n$. If the set of vectors $\{b_{ij}\}_{j=1}^{n_i}$ does norm retrieval in W_i for all $i = 1, 2, .., M$ and the projections $\{P_i\}_{i=1}^M$ do norm retrieval in $\mathbb{R}^n$, then the set of vectors $\{b_{ij} : i = 1, 2, .., M, \quad j = 1, 2, ..n_i\}$ does norm retrieval in $\mathbb{R}^n$.*

Proof. Given $x, y \in \mathbb{R}^n$, suppose $|\langle x, b_{ij}\rangle| = \langle y, b_{ij}\rangle|$ for all i, j. Then $|\langle x, b_{ij}\rangle| = \langle y, b_{ij}\rangle|$ for all $j = 1, 2, .., n_i$. By assumption, $\{b_{ij}\}_{j=1}^{n_i}$ does norm retrieval in W_i for all $i = 1, 2, .., M$. This implies that $||P_i x|| = ||P_i y||$ for all $i = 1, 2, .., M$. Since the projections $\{P_i\}_{i=1}^M$ do norm retrieval in $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

This Proposition allows for many of our constructions of norm retrievable frames in dynamical sampling setting.

3.2 Self Adjoint Operators

In this section, we are interested in obtaining dynamical sampling frames which are norm retrievable sets but not phase retrievable. We assume that A is a self-adjoint operator and try to find conditions on the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ and the time increments ℓ such that $\{A^\ell b_i\}_{\{\ell \in \{0,1,..,\ell_i\}, i \in \Omega\}}$ does norm retrieval in $\mathbb{R}^n$ but fails to do phase retrieval.

First, we show when it is possible to obtain norm retrievable sets by a single generator. Given a vector $b \in \mathbb{R}^n$, the subspace spanned by the vectors $\{b, Ab, A^2b, ...A^{r-1}b\}$ is called the Krylov subspace $K(A, b)$ generated by an operator A on $\mathbb{R}^n$, where r is the degree of the A-annihilator of b.

$$K(A, b) = \text{span}\{b, Ab, ...A^{r-1}b\}$$

Since self-adjoint operators are unitarily equivalent to diagonal operators, we can restrict our efforts to finding diagonal operators that give norm retrieval. We begin with D on $\mathbb{R}^2$ and a single generating vector b.

Lemma 3.2.1. *Let*

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

with non-zero $\lambda_1, \lambda_2, b \in \mathbb{R}^2$. Then $\{b, Db\}$ does norm retrieval but not phase retrieval in $\mathbb{R}^2$ if and only if $\lambda_1 b_1^2 + \lambda_2 b_2^2 = 0$.

Proof. ($\Longrightarrow$) Since we only have two vectors $\{b, Db\}$ in $\mathbb{R}^n$, they do norm retrieval if they are orthogonal to each other by the Lemma (2.3.7). This implies that

$$0 = \langle b, Db \rangle = \lambda_1 b_1^2 + \lambda_2 b_2^2.$$

($\Longleftarrow$) If $\lambda_1 b_1^2 + \lambda_2 b_2^2 = 0$, then $\{b, Db\}$ are orthogonal to each other. This says that $\left\{ \frac{b}{||b||}, \frac{Db}{||Db||} \right\}$ are orthonormal bases in $\mathbb{R}^n$. Hence, they do norm retrieval in $\mathbb{R}^n$.

The set $\{b, Db\}$ fails the complementary property (2.3.5) since it does not have enough vectors, hence fails to do phase retrieval in $\mathbb{R}^n$.

$\square$

Lemma (3.2.1) is unique in that the 2×2 case is the only diagonal operator that generates norm retrievable sets which are not phase retrievable from one generating vector. When $n \geq 3$ and D is a diagonal operator on $\mathbb{R}^n$, for any non-zero vector $b \in \mathbb{R}^n$, we do not have norm retrievable sets which are not phase retrievable in $\mathbb{R}^n$ by a single generator b.

Lemma 3.2.2. *Let D be a diagonal operator*

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

with $\lambda_1, \lambda_2, \lambda_3$ in $\mathbb{R}^3$. For any non-zero vector b in $\mathbb{R}^3$, the set of vectors

$$F = \{b, Db, D^2b, .., D^\ell b\}$$

cannot do norm retrieval when $\ell \le 3$.

Proof. Let $b = [b_1\, b_2\, b_3]^T$ be a nonzero vector in $\mathbb{R}^3$.

For the set $F = \{b, Db, D^2, .., D^\ell b\}$ to be able to do norm retrieval in $\mathbb{R}^3$, they first should span $\mathbb{R}^3$ by the Theorem (2.1.10), hence $\ell \ge 2$.

Suppose $F = \{b, Db, D^2b\}$ spans $\mathbb{R}^3$. By Lemma (2.3.7), $F = \{b, Db, D^2b\}$ does norm retrieval if the vectors are pairwise orthogonal to each other. However,

$$\langle b, D^2b\rangle = \langle Db, Db\rangle = ||Db||^2 > 0$$

for any D and $b \neq 0$. Thus, $F = \{b, Db, D^2b\}$ does not do norm retrieval in $\mathbb{R}^3$.

Next, consider the set of vectors $F = \{b, Db, D^2b, D^3b\}$. By the complement property (2.3.5), the set of vectors does norm retrieval if and only if for any partition $\{F_1, F_2\}$ of F, we have $(\text{span}F_1)^\perp \perp (\text{span}F_2)^\perp$. In particular, consider $F_1 = \{b, Db\}$ and $F_2 = \{D^2b, D^3b\}$. Taking the cross products, we see

$$(\text{span}F_1)^\perp = span \begin{bmatrix} (\lambda_3 - \lambda_2)b_2b_3 \\ -(\lambda_3 - \lambda_1)b_1b_3\} \\ (\lambda_2 - \lambda_1)b_1b_2 \end{bmatrix}, \; (\text{span}F_2)^\perp = span \begin{bmatrix} (\lambda_2^2\lambda_3^3 - \lambda_2^3\lambda_3^2)b_2b_3 \\ -(\lambda_1^2\lambda_3^3 - \lambda_1^3\lambda_3^2)b_1b_3 \\ (\lambda_1^2\lambda_2^3 - \lambda_1^3\lambda_2^2)b_1b_2 \end{bmatrix}.$$

and $(\text{span}F_1)^\perp \perp (\text{span}F_2)^\perp$ if and only if we have

$$(\lambda_2\lambda_3)^2(\lambda_3 - \lambda_2)^2(b_2b_3)^2 + (\lambda_1\lambda_3)^2(\lambda_3 - \lambda_1)^2(b_1b_3)^2 + (\lambda_1\lambda_2)^2(\lambda_2 - \lambda_1)^2(b_1b_2)^2 = 0.$$

This implies that $\lambda_1 = \lambda_2 = \lambda_3$. But in this case, $F = \{b, Db, D^2b, D^3b\}$ does not span $\mathbb{R}^3$ and thus fails to do norm retrieval. Hence, we do not have any vector $b \in \mathbb{R}^3$ such that $F = \{b, Db, D^2b, .., D^\ell b\}$ does norm retrieval when $\ell \le 3$.

When $\ell \geq 3$, F has 5 or more vectors. In this situation, it is possible to have phase retrieval, hence norm retrieval.

$\square$

We can generalize the Lemma (3.2.2) to self-adjoint operators as follows.

Theorem 3.2.3. *Let A be a self-adjoint operator on $\mathbb{R}^n$. For any given non-zero vector $b \in \mathbb{R}^n$ with $n \geq 3$, the following conditions hold;*

1. *If n is odd and $k \leq 2n-3$, then the set $F = \{b, Ab, A^2b, ..., A^kb\}$ does not do norm retrieval in $\mathbb{R}^n$.*

2. *If n is even and $k \leq 2n-4$, then the set $F = \{b, Ab, A^2b, ..., A^kb\}$ does not do norm retrieval in $\mathbb{R}^n$.*

Proof. The set $F = \{b, Ab, A^2b, ..., A^\ell b\}$ does norm retrieval in $\mathbb{R}^n$ if and only if the norm retrieval condition (2.3.1) holds. That is for any partition F_1, F_2 of F, $(\text{span}F_1)^\perp \perp (\text{span}F_2)^\perp$. An equivalent statement to (2.3.1) in the Remark (2.3.9) is that the set $F = \{b, Ab, A^2b, ..., A^kb\}$ does norm retrieval if for any partition F_1, F_2 of F, we have $(\text{span } F_1)^\perp \subseteq \text{span } F_2$.

If a set does norm retrieval, by adding more vectors to this set we still have norm retrieval. Therefore, we cannot obtain a norm retrievable set by removing vectors from a set which does not do norm retrievel. For that reason, it is enough to look at the cases $\ell = 2n-3$ when n is odd and $\ell = 2n-4$ when n is even.

Case 1: When n is odd and $\ell = 2n-3$, we can have the following partition of the set F.

$$F_1 = \{b, Ab, ..., A^{n-2}b\}$$

$$F_2 = \{A^{n-1}b, A^nb, ..., A^{2n-3}b\}$$

For any nonzero $x \in$ span F_2, let $x = A^{n-1}(c_0 b + c_1 Ab + ... + c_{n-2}A^{n-2}b)$ for some scalars $\{c_j\}_{j=0}^{n-2}$. Take $y = c_0 b + c_1 Ab + ... + c_{n-2}A^{n-2}b$. Then $y \in$ span F_1 and,

$$\langle x, y\rangle = \langle A^{n-1}y, y\rangle = \langle A^{(n-1)/2}y, A^{(n-1)/2}y\rangle = ||A^{(n-1)/2}y||^2 > 0.$$

This implies that we cannot have any non-zero vector $x \in F_2$ that can be in (span $F_1)^{\perp}$. There is a maximum of $n-1$ linearly independent vectors in span F_1. That is (span $F_1)^{\perp} \neq \{\emptyset\}$ and span $F_1 \neq \mathbb{R}^n$. This contradicts (span $F_1)^{\perp} \subseteq$ span F_2.

Case 2: When n is even and $k = 2n-4$, similar to the first case, we have the following partition of the set F.

$$F_1 = \{b, Ab, ..., A^{n-2}\}$$

$$F_2 = \{A^{n-1}b, A^n b, ..., A^{2n-4}\}$$

For any $x \in$ span F_2, $x = A^{n-2}(d_1 Ab + ... + d_{n-2}A^{n-2}b)$ for some scalars $\{d_j\}_{j=1}^{n-2}$ and $z = d_1 Ab + ... + d_{n-2}A^{n-2}b \in$ span F_1 but $\langle x, z\rangle = ||A^{(n-2)/2}z||^2 > 0$. Again this contradicts (span $F_1)^{\perp} \subseteq$ span F_2 since (span $F_1)^{\perp} \neq \{\emptyset\}$ and every non-zero vector $x \in F_2$ has some $y \in F_1$ with $\langle x, y\rangle > 0$. □

This theorem eliminates a number of possibilities , but only applies to dynamical sampling systems with a single generating vector.

Next, we describe properties from the recent paper ([4]) that found conditions for phase retrieval in dynamical sampling structure.

Definition 3.2.4. [4] Suppose that a bounded operator $A \in B(\mathcal{H})$ has a minimal polynomial p^A. A nonzero polynomial p is a **k-partial annihilator** of $A, k \in \mathbb{N}$, if p and p^A have a common divisor of degree k.

Definition 3.2.5. [4] Let A be an $n \times n$ matrix. If for all $k \in \mathbb{N}$, any k-partial annihilator of A which has degree at most $r = \max\{1, 2k-2\}$ has at least $k+1$ nonzero coefficients,then the matrix A is called **iteration regular**.

In ([4]), the authors show that A being iteration regular ensures that the vectors $\{x, Ax, A^2x, ...\}$ are full spark, as shown here.

Proposition 3.2.6. *[4] Let $K = span\{x, Ax, A^2x...\}$ with $dim = k$ in $\mathbb{R}^n$. If A is iteration regular, then any k vectors from the system of $\{x, Ax, ...A^rx\}$ with $r = max\{1, 2k-2\}$, form a basis in $K(A, x)$.*

Proof. Assume that A is iteration regular and $x \in \mathbb{R}^n$ is a nonzero vector. Let p_x^A be the A-annihilator of x. That is p_x^A is the monic polynomial of the smallest degree such that $p_x^A(A)x = 0$. The dimension k of the maximal Krylov subspace $K_m(A, x) = \{x, Ax, A^2x, ...\}$ is equal to the degree of the polynomial p_x^A.

When $k = 1$, $r = 1$ and the claim is obvious.

When $k \geq 2$, suppose we have the k vectors $\{A^{\ell_i}x : i = 1, 2, ...k\}$ from the set $\{x, Ax, A^2x, ...A^{2k-2}x\}$. We want to show that the set of vectors $\{A^{\ell_i}x : i = 1, ...k\}$ is linearly independent. Suppose there exists some coefficients $\{c_i\}$ such that

$$\sum_{i=1}^{k} c_i A^{\ell_i} x = 0.$$

Then $\sum_{i=1}^{k} c_i A^{\ell_i} x = q(A)x$ is a polynomial of degree $\leq 2k-2$. Since $q(A)x = 0$ and p_x^A be the A-annihilator of x, p_x^A divides q. Therefore, q has k roots in common with p_x^A. The polynomial q has at most k non-zero coefficients. Since A is iteration regular, this implies that all its coefficients $\{c_i\}$ must be zero. Hence, any k vectors from the system $\{x, Ax, ...A^rx\}$, $r = \max\{1, 2k-2\}$, form a basis in $K(A,x)$.

$\square$

Remark 3.2.7. As shown in Proposition (3.2.6), any partition of $\{x, Ax, ...A^{2k-2}x\}$ will have a spanning set for $K(A,x)$ when A is iteration regular. This shows that we can still get norm retrievable frame generated by a single vector b in $\mathbb{R}^n$ with a self-adjoint operator A if we get $F = \{b, Ab, A^2b, ..., A^kb\}$ to be a phase retrievable frame. In this case, the number of iterations k is at least $2n-2$.

However, there exist invertible operators A that do generate norm retrievable frames which are not phase retrievable by iteration on a single vector:

Example 3.2.8. *Consider the operator A and vector b in $\mathbb{R}^2$,*

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \quad and \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then

$$F = \{b, Ab\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$$

and

$$I = \frac{1}{2}bb^* + \frac{1}{8}Ab(Ab)^*$$

This implies that $F = \{b, Ab\}$ does norm retrieval because the vectors are orthogonal, but we do not have enough vectors to do phase retrieval in $\mathbb{R}^2$.

We showed that a self-adjoint operator A on $\mathbb{R}^n$ cannot produce a norm retrievable frame in $\mathbb{R}^n$ with fewer than $2n-3$ iterations on a single generating vector $b \in \mathbb{R}^n$.

If $\operatorname{span}\{b, Ab, A^2b, ...\} = \mathbb{R}^n$ and A is iteration regular as defined in the Definition (3.2.5), then the set $F = \{b, Ab, A^2b, ..., A^kb\}$ does norm retrieval in $\mathbb{R}^n$ for $k = 2n-2$ as shown in Proposition (3.2.6).

A self-adjoint operator A on $\mathbb{R}^n$ can generate norm retrievable frames with fewer than $2n-2$ iterations if we use more generating vectors. (This corresponds to using more than one sensor to sample).

Suppose we have 4 vectors $\{z_i\}_{i=1}^4$ that are full spark in $\mathbb{R}^3$. If we want them to do norm retrieval, any partition must satisfy condition (2.3.1). This means any subset of 3 vectors spans the space. In addition, we must also have partitions that split into 2 dimensional spaces satisfy (2.3.1). Since the set is full spark, we know any 2 vectors are linearly independent, hence span a plane. The spans of the vectors in one of these partitions yield 2 planes. Recall from our earlier Example (2.3.10) that property (2.3.1) means that the normal vectors to these 2 planes must be orthogonal as shown in Figure 7.

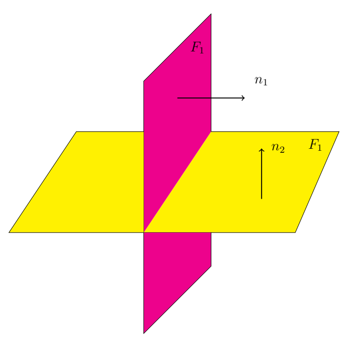

Figure 3.1

Example 3.2.9. *We now give an explicit example of a set of 4 vectors that do norm retrieval in $\mathbb{R}^3$. We can use two of the coordinate planes as our spans for one set of partitions. We accomplish this by choosing the 4 vectors to be of the* *form:*

$$\{z_i\}_{i=1}^4 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \beta \end{bmatrix} \right\}$$

By construction $\langle z_1 \times z_2, z_3 \times z_4 \rangle = 0$*. We now need to find conditions on* α, β *to make the two remaining pairs of planes have orthogonal normal vectors.*

Computing the necessary cross products gives

$$\begin{aligned} z_1 \times z_3 &= \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T \\ z_2 \times z_4 &= \begin{bmatrix} -\alpha\beta & \alpha & \beta \end{bmatrix}^T \\ z_1 \times z_4 &= \begin{bmatrix} -\beta & 1 & \beta \end{bmatrix}^T \\ z_2 \times z_3 &= \begin{bmatrix} -\alpha & \alpha & 1 \end{bmatrix}^T \end{aligned}$$

Taking appropriate inner products shows that we have orthogonal inner products of the planes when we satisfy:

$$\alpha\beta + \alpha + \beta = 0.$$

Solutions to this equation form a hyperbola in α *and* β*, but there are nonzero integer solutions* $\alpha = \beta = -2$*.*

The vectors $\{z_i\}_{i=1}^4$ *do not contain an orthonormal basis, and are not a tight frame. It is clear from observation that the set does not contain an orthonormal basis. To see that it is not a tight frame, we compute the frame operator by* *recalling that the analysis operator Φ is represented by the matrix with the vectors as rows. The analysis operator is $S = \Phi^*\Phi$.*

$$S = \Phi^*\Phi = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 5 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

Since the frame operator is not a multiple of the identity, the frame $\{z_i\}_{i=1}^4$ is not tight.

Remark 3.2.10. The vectors $\{z_i\}_{i=1}^4$ in our example (3.2.9) can be expressed as a set coming from dynamical samples with a diagonal operator.

Let b_1, b_2, and diagonal matrix D be the following:

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Then the vectors $\{b_1, Db_1, b_2, Db_2\}$ make up the elements of our example for norm retrieval frame in $\mathbb{R}^3$ with $\alpha = -2$ and $\beta = -2$.

Example (3.2.9) shows that when A is a self-adjoint operator on $\mathbb{R}^n$, there exists some vectors $\{b_i \in \mathbb{R}^n : i \in \Omega; |\Omega| < n\}$ such that $\{A^{\ell_i} b_i : i \in \Omega,\ \ell_i = 0, 1, ...l\}$ does norm retrieval in $\mathbb{R}^n$.

Now, we will show for which vectors $\{b_i \in \mathbb{R}^n : i \in \Omega; |\Omega| < n\}$, the set of vectors $\{A^\ell b_i : i \in \Omega,\ \ell = 0, 1, ...\ell_i\}$ does norm retrieval in $\mathbb{R}^n$.

We start with a diagonal operator D on $\mathbb{R}^n$ with $n \geq 3$.

Lemma 3.2.11. *Let D be a diagonal operator on $\mathbb{R}^n$*

$$D = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ & & & \lambda_s I_s \end{bmatrix} \tag{3.2.1}$$

with distinct eigenvalues λ_j for all $j = 1, 2, ..., s$. I_i is a $r_j \times r_j$ identity matrix for $j = 1, 2, ..s$. If D is iteration regular, then there exists orthogonal vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ such that $\{D^\ell b_i \in \mathbb{R}^n : i \in \Omega \quad \ell = 2t_i - 2\}$ does norm retrieval but not phase retrieval in $\mathbb{R}^n$ where t_i is the degree of D-annihilator of b_i.

Proof. Suppose D be a diagonal operator on $\mathbb{R}^n$ given by (3.2.1).

By rearranging the order if it is necessary, we can write $r_1 \geq r_2 \geq r_3 \geq ... \geq r_s$. Let E_j be the eigenspace corresponding to the real eigenvalue λ_j for all $j = 1, 2, ..., s$.

We have $\dim E_j = r_j$ for all $j = 1, 2, ..., s$. Let $\{e_{jk}\}_{k=1}^{r_j}$ be the standard orthonormal basis vectors such that $E_j = \text{span}\{e_{jk}\}_{k=1}^{r_j}$. Assume $e_{jk} = 0$ when $k > r_j$. For $1 \leq i \leq r_1$, we define

$$b_i = \sum_{j=1,2,..s} e_{ji} \tag{3.2.2}$$

Hence, we have a set of orthogonal vectors $\{b_i\}_{i=1}^{r_1}$ such that
$D^\ell b_i \in \text{span}\{e_{1k}, ...e_{sk}\}$ for all $\ell \in \mathbb{N}$. The Krylov subspace of b_i satisfies that $K(D, b_i) = span\{e_{1i}, ...e_{si}\}$. Let t_i be the degree of the D-annihilator of b_i. By proposition in (3.2.6), when D is iteration regular, then $\{D^\ell b_i\}$ does phase retrieval (hence norm retrieval) for $\ell = 2t_i - 2$ in $K(D, b_i)$ for all i.
$\mathbb{R}^n = K(D, b_1) \oplus ... \oplus K(D, b_{r_1})$ by choice of vectors b_i. Let P_i be the orthogonal

projection onto $K(D, b_i)$ for all i. Then

$$\sum_{i=1}^{r_1} P_i = I. \tag{3.2.3}$$

To show that $\{D^\ell b_i; i \in \Omega, \quad \ell = 2t_i - 2\}$ does norm retrieval in $\mathbb{R}^n$, suppose

$$|\langle x, D^\ell b_i\rangle| = |\langle y, D^\ell b_i\rangle| \quad \forall i, \ell$$

for given $x, y \in \mathbb{R}^n$, Then $|\langle x, D^\ell b_i\rangle| = |\langle y, D^\ell b_i\rangle|$ for all i and $\{D^\ell b_i\}_\ell$ does phase retrieval (hence norm retrieval) in $K(D, b_i)$ for all i since D is iteration regular. Hence, we have $||P_i x|| = ||P_i y||$ for all i. Since $||x||^2 = \sum_{i=1}^{r_1} ||P_i x||$ for all $x \in \mathbb{R}^n$ by the equality in (3.2.3). The set of vectors $\{D^\ell b_i; i \in \Omega, \quad \ell = 2t_i - 2\}$ does norm retrieval in $\mathbb{R}^n$.

□

Remark 3.2.12. The set of vectors $\{D^\ell b_i; i \in \Omega, \quad \ell = 2t_i - 2\}$ defined in Lemma (3.2.11) does norm retrieval but it fails the complementary property to do phase retrieval in $\mathbb{R}^n$.

Next, we give an explicit example in $\mathbb{R}^4$ to demonstrate this construction.

Example 3.2.13. *Let D be a diagonal operator on $\mathbb{R}^4$ with nonzero distinct eigenvalues λ_1, λ_2.*

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_2 \end{bmatrix} \in \mathbb{M}_n(\mathbb{R}) \tag{3.2.4}$$

Choose $b_1 = e_1 + e_3, \quad b_2 = e_2 + e_4$ as described in Lemma (3.2.11). The *set of vectors* $\{b_i, Db_i, D^2b_i\}$ *is full spark and does phase retrieval in* $K(D, b_i)$ *for* $i = 1, 2$. *The Krylov subspaces* $K(D, b_i)$ *are 2-dimensional and orthogonal to each other. For that reason, the orthogonal projections* P_i *onto* $K(D, b_i)$ *do norm retrieval. By Lemma (3.1.2), the set of vectors* $F = \{b_1, Db_1, D^2b_1, b_2, Db_2, D^2b_2\}$ *does norm retrieval in* $\mathbb{R}^4$. *Since the number of vectors in* F *is less then* $2n-1 = 7$ *for* $n = 4$, F *does not do phase retrieval in* $\mathbb{R}^4$.

This example shows that phase retrieval does not have an analog to our Proposition 3.1.2.

Let A be a self-adjoint operator defined on $\mathbb{R}^n$. Then there exists vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ such that $\mathbb{R}^n$ can be written as orthogonal direct sum of Krylov subspaces $\{K(A, b_i) : i \in \Omega\}$ that are generated as follows.

Choose an arbitrary vector $b_1 \in \mathbb{R}^n$. The Krylov subspace generated with A and b_1 can be written as

$$K(A, b_1) = \text{span}\{b_1, Ab_1, ...A^{r_1-1}b_1\}$$

where r_1 is the degree of A-annihilator of b_1. Since $K(A, b_1)$ is a closed subspace of $\mathbb{R}^n$, we can write $\mathbb{R}^n = K(A, b_1) \oplus K(A, b_1)^{\perp}$ as orthogonal direct sum of $K(A, b_1)$ and $K(A, b_1)^{\perp}$.

If $K(A, b_1)^{\perp} \neq \{\emptyset\}$, then choose a nonzero vector $b_2 \in K(A, b_1)^{\perp}$.

Since A is a self-adjoint operator and $\langle A^{k_1}b_1, A^{k_2}b_2\rangle = \langle A^{k_1+k_2}b_1, b_2\rangle = 0$ for any $k_1, k_2 \in \mathbb{N}$, we have $K(A, b_2) \subset K(A, b_1)^{\perp}$. Now, we have the orthogonal direct sum $K(A, b_1) \oplus K(A, b_2)$.

If $\mathbb{R}^n = K(A, b_1) \oplus K(A, b_2)$, then we are done. Otherwise, choose a nonzero vector $b_3 \in \mathbb{R}^n$ such that b_3 is orthogonal to both $K(A, b_1)$ and $K(A, b_2)$. Since A

is a self-adjoint operator, we have $K(A, b_1) \oplus K(A, b_3)$ and $K(A, b_2) \oplus K(A, b_3)$. Thus, $K(A, b_1) \oplus K(A, b_2) \oplus K(A, b_3)$.

Since $\mathbb{R}^n$ is finite dimensional, we can continue to write orthogonal direct sum of Krylov subspaces until $\mathbb{R}^n = K(A, b_1) \oplus K(A, b_2) \oplus ... \oplus K(A, b_r)$ for some $r \in \mathbb{N}$.

Theorem 3.2.14. *Let A be a self-adjoint operator defined on $\mathbb{R}^n$ that is iteration regular. Given the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$, if $\mathbb{R}^n = K(A, b_1) \oplus K(A, b_2) \oplus ... \oplus K(A, b_r)$ for some $r \in \mathbb{N}$, then $\{A^\ell b_i : i \in \Omega = \{1, 2, ..., r\}; 0 \leq \ell \leq 2r_i - 2\}$ does norm retrieval in $\mathbb{R}^n$ where r_i is degree of the A-annihilator of b_i.*

Proof. Suppose $\mathbb{R}^n = K(A, b_1) \oplus K(A, b_2) \oplus ... \oplus K(A, b_r)$ for the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$. Since A is iteration regular, for each nonzero vector $b_i \in \mathbb{R}^n$, any r_i vectors from the system $\{b_i, Ab_i, ...A^\ell b_i\}$, $\ell = \max\{1, 2r_i - 2\}$, form a basis in $K(A, b_i)$ by Proposition (3.2.6). This says that the set $\{A^\ell b_i\}_{\ell=0}^{2r_i-2}$ is full spark in $K(A, b_i)$ with $2r_i - 1$ vectors and satisfies complement property. Hence the set of vectors $\{A^\ell b_i\}_{\ell=0}^{2r_i-2}$ does phase retrieval (hence norm retrieval) in $K(A, b_i)$ for all i. Let P_i be the orthogonal projections onto the subspaces $K(A, b_i)$, then $\sum_{i=1}^{r_1} P_i = I$ and $\sum_{i=1}^{r_1} ||P_i x||^2 = I||x||^2$ for any $x \in \mathbb{R}^n$. Which implies that $\{A^\ell b_i : i \in \Omega = \{1, 2, ..., r\}; 0 \leq \ell \leq 2r_i - 2\}$ does norm retrieval in $\mathbb{R}^n$ where r_i is degree of the A-annihilator of b_i.

$\square$

3.3 Normal Operators

Let A be a normal operator on $\mathbb{R}^n$. That is $AA^* = A^*A$, and $A^* = A^\top$ in $\mathbb{R}^n$.

The eigenvalues of A are not necessarily all real values. For that reason, in the Jordan decomposition of $A = BJB^{-1}$, B may not be a real matrix when we have a real normal matrix A. A strictly real version of Schur decomposition will have that desired preservation of real entries.

Theorem 3.3.1. *[30] (Real Schur decomposition) If A is a real $n \times n$ matrix, there is a real orthogonal matrix B such that $A = BTB^\top$*

$B^\top$ is transpose of B and T is an upper triangular matrix given by

$$T = \begin{bmatrix} T_1 & * & * & \cdots & * \\ & T_2 & * & \cdots & * \\ & & \ddots & & \vdots \\ & & & & T_k \end{bmatrix} \in \mathbb{M}_n(\mathbb{R}), \quad 1 \leq k \leq n \tag{3.3.1}$$

where each T_j is either a real 1×1 matrix or a real 2×2 matrix $T_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$ corresponding to the complex eigenvalues $\lambda_j = \alpha_j + i\beta_j$ and $\bar{\lambda}_j = \alpha_j - i\beta_j$ of A for which $\alpha_j, \beta_j \in \mathbb{R}$.

Example 3.3.2. *For the given normal operator N on $\mathbb{R}^3$, there does not exist any $b \in \mathbb{R}^3$ such that $F = \{b, Nb, N^2b, N^3b\}$ does norm retrieval in $\mathbb{R}^3$.*

$$N = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

F does norm retrieval if and only if for any partion F_1, F_2 of F, $spanF_1^\perp \perp spanF_2^\perp$. For $F_1 = \{b, Nb\}$ and $F_2 = \{N^2b, N^3b\}$, we have $spanF_1^\perp \perp spanF_2^\perp$ if and only if $5(b_1^2 + b_2^2)b_3^2 + 8(b_1^2 + b_2^2)^2 = 0$.
There are no nonzero solutions, hence no $b \in \mathbb{R}^3$ such that $F = \{b, Nb, N^2b, N^3b\}$

does norm retrieval in $\mathbb{R}^3$*. However, for* $b_1 = e_1, b_2 = e_3$ $F = \{b_1, Nb_1, N^2b_1, b_3\}$ *does norm retrieval in* $\mathbb{R}^3$ *but does not do phase retrieval since it fails complementary property.*

We are trying to find norm retrievable sets which are not phase retrievable. For that reason, we have the following theorem for real normal operators as a result of the real Schur decomposition. Since U is orthogonal, we reduce the normal case to operators of the form J in (3.3.2).

Theorem 3.3.3. *[30] Let* A *be an* $n \times n$ *matrix with real entries. Then* A *is normal if and only if there is a real orthogonal matrix* U *and a block diagonal matrix* J *such that* $U^\top AU = J$*.* $U^\top$ *is the transpose of the operator* U*.*

J is given by

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \in \mathbb{M}_n(\mathbb{R}), \quad 1 \leq k \leq n \tag{3.3.2}$$

where each J_j *is either a real* 1×1 *matrix or a real* 2×2 *matrix of the form*

$$J_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}, \quad \alpha_j, \beta_j \in \mathbb{R}.$$

We may restrict our work on operators in the block diagonal form J, since U is real orthogonal (unitary).

Since the main diagonal blocks J_j in (3.3.2) can be arranged in any order, we can write

$$J = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

where D_1 is a diagonal matrix with real eigenvalues $\lambda_1, \lambda_2, ...\lambda_s$ of A

$$D_1 = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ & & & \lambda_s I_s \end{bmatrix} \tag{3.3.3}$$

For $j = 1, 2, ..s$, I_i is a $r_j \times r_j$ identity matrix.

D_2 is a block diagonal matrix with each block has the from $J_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$ with respect to pair of complex eigenvalues $\lambda_j = \alpha_j + \beta_j$, $\bar{\lambda}_j = \alpha_j - \beta_j$ of A where $\alpha_j, \beta_j \in \mathbb{R}^n$.

$$D_2 = \begin{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} & & & \\ & \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} \alpha_s & \beta_s \\ -\beta_s & \alpha_s \end{bmatrix} \end{bmatrix} \tag{3.3.4}$$

Note: In the notation we used in (3.3.4), if we have repeated complex eigenvalues, $\lambda_j = \lambda_s$, then the respective block diagonal matrixes J_j and J_s in (3.3.4) are same .

Lemma 3.3.4. *Let D_2 be a block diagonal matrix on $\mathbb{R}^{2n}$ which has the form in (3.3.4) and $\{e_i\}_{i=1}^{2n}$ be the orthonormal bases in $\mathbb{R}^{2n}$. Then $\{b_k, Db_k, D^2b_k\}_{k=1}^n$*

does norm retrieval in $\mathbb{R}^{2n}$ if $b_k = e_{2k-1}$ or $b_k = e_{2k}$.

Proof. Let $N_j = \text{span}\{D_2 e_{2j-1}, D_2 e_{2j}\}$ in $\mathbb{R}^{2n}$ for $1 \leq j \leq n$.
Then $\mathbb{R}^{2n} = N_1 \oplus N_2 \oplus ... \oplus N_n$. If $\alpha_j = 0$ for all j, then $\{b_i, Db_i, D^2 b_i\}_{i=1}^n$ is an orthogonal set in $\mathbb{R}^{2n}$ for $b_i = e_{2i-1}$ or $b_i = e_{2i}$ and hence does norm retrieval. If $\alpha_j \neq 0$, then $\{e_{2j-1}, De_{2j-1}, D^2 e_{2j-1}\}$ is a full spark set in N_j. Then $\{e_{2j-1}, De_{2j-1}, D^2 e_{2j-1}\}$ does phase retrieval (and hence norm retrieval) in N_j. By the Lemma (3.1.2), the set of vectors $\{e_{2j-1}, De_{2j-1}, D^2 e_{2j-1}\}_{j=1}^n$ does norm retrieval in $\mathbb{R}^{2n}$. □

Theorem 3.3.5. *Let A be a normal operator on $\mathbb{R}^n$ with the decomposition in the Theorem (3.3.3). Then $\{A^{\ell_i} b_i, A^{\ell_j} c_j\}$ does norm retrieval in $\mathbb{R}^n$ if the set of vectors $\{A^{\ell_i} b_i\}$ does norm retrieval in diagonal D_1 part of A and $\{A^{\ell_j} c_j\}$ does norm retrieval in the non-diagonal D_2 part of A.*

Proof. The proof follows from Lemma (3.2.11) and Lemma (3.3.4). □

Next, we show a different method to show that there exist set of vectors $W = \{b_i \in \mathbb{R}^n\}$ such that $\{A^\ell b_i \in \mathbb{R}^n\}$ does norm retrieval in $\mathbb{R}^n$. In this case, the sum of orthogonal projections onto $A^\ell W$ does not need to be the identity.

Theorem 3.3.6. *([5]) Let $\{x_i\}_{i=1}^M$ be a set of vectors in a Hilbert space $\mathcal{H}^n$. The following are equivalent:*

(1) $\{x_i\}_{i=1}^M$ yields phase retrieval in $\mathcal{H}^n$.

(2) $\{Ax_i\}_{i=1}^M$ yields phase retrieval for all invertible operators A on $\mathcal{H}^n$.

(3) $\{Ax_i\}_{i=1}^M$ yields norm retrieval for all invertible operators A on $\mathcal{H}^n$.

Remark 3.3.7. We have that phase retrieval is preserved under invertible operators as shown in Theorem (3.3.6). This is another instance where norm retrieval is harder to manage. We can see readily that norm retrieval is preserved under unitary operators but not all invertible operators.

For example, orthonormal bases do norm retrieval. An invertible operator A might send an orthonormal basis to a non-orthogonal set. This illustrates that A does not preserve norm retrieval since it fails Lemma (2.3.7).

Given a finite set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ in a Hilbert space.

Let $W = span\{b_i \in \mathbb{R}^n; i \in \Omega\}$ be a subspace of $\mathbb{R}^n$. For each $\ell \in \mathbb{N}$, we can define; $A^\ell W = span\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega\} \subset \mathbb{R}^n$. Let P_ℓ be orthogonal projection from $\mathbb{R}^n$ onto $A^\ell W$ for each $\ell \in \mathbb{N}$. The previous theorem tells us that if the set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ does phase retrieval in W, then $\{A^\ell b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ does phase retrieval in $A^\ell W$ for each $\ell \in \mathbb{N}$ when A is an invertible operator on $\mathbb{R}^n$.

Suppose there exist $m \in \mathbb{N}$ such that $\mathbb{R}^n = span\{A^\ell b_i\}_{i\in\Omega,\,\ell=0,1,\dots m}$. The set of vectors $\{A^\ell b_i\}_{i\in\Omega}$ is phase retrievable in $A^\ell W$ for each $\ell = 0, 1, \dots m$ but it does not imply that $\{A^\ell b_i\}_{i\in\Omega\,\ell=0,1,\dots m}$ does phase retrieval in $\mathbb{R}^n$.

Example 3.3.8. *Let $\{e_i\}_{i=1}^3$ be the standard orthonormal basis in $\mathbb{R}^3$.*

Define $W = span\{e_1, e_2, e_1 + e_2\}$.

Let A be an invertible operator on $\mathbb{R}^3$ such that $Ae_1 = e_2$ and $Ae_2 = e_3$. Then we have $AW = span\{e_2, e_3, e_2 + e_3\}$. Both $\{e_1, e_2, e_1 + e_2\}$ and $\{e_2, e_3, e_2 + e_3\}$ do phase retrieval in W and AW respectively but $\{e_1, e_2, e_3, e_1 + e_2, e_2 + e_3\}$ fails the complementary property (2.3.5), and thus not do phase retrieval in $\mathbb{R}^3$.

The following theorem gives us sufficient conditions on the set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ and the orthogonal projections P_ℓ onto $A^\ell W$ such that $\{A^\ell b_i\}_{i\in\Omega\,\ell=0,1,\dots m}$ does norm retrieval in $\mathbb{R}^n$.

Theorem 3.3.9. *Let the set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ do phase retrieval in $W \subset \mathbb{R}^n$ and let A be an invertible operator on $\mathbb{R}^n$. Then the set of vectors $\{A^\ell b_i\}_{i\in\Omega\, \ell=0,1,...m}$ does norm retrieval in $\mathbb{R}^n$ if the set of orthogonal projections $\{P_\ell\}_{\ell=0}^m$ onto the subspaces $A^\ell W = span\{\{A^\ell b_i\}_{i\in\Omega}\}$ does norm retrieval in $\mathbb{R}^n$.*

Proof. Given $x, y \in \mathbb{R}^n$, suppose $|\langle x, A^\ell b_i\rangle| = |\langle y, A^\ell b_i\rangle|$ for all $i \in \Omega,\ \ell = 0, 1, ...m$. For fixed ℓ, define P_ℓ to be the orthogonal projection onto $A^\ell W$.

We have $P_\ell A^\ell b_i = A^\ell b_i$ and $|\langle P_\ell x, P_\ell A^\ell b_i\rangle| = |\langle P_\ell y, P_\ell A^\ell b_i\rangle|$ for all $i \in \Omega$. By Theorem (3.3.6), since A is an invertible operator and the set of vectors $\{b_i \in \mathbb{R}^n; i \in \Omega,\ |\Omega| < n\}$ does phase retrieval in W, $\{A^\ell b_i\}_{i\in\Omega}$ does phase retrieval (hence norm retrieval) in $A^\ell W$ for each ℓ. This implies that $||P_\ell x|| = ||P_\ell y||$ for all $\ell = 0, 1, ...M$. Since we assumed the set of orthogonal projections $\{P_\ell\}_{\ell=0}^m$ does norm retrieval in $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

3.4 Unitary operator iteration

If our dynamical sampling operator is unitary, this gives us a smoother way to do norm retrieval. Let $\Omega \subset \{1, 2, ..., n\}$ be an index set and $\{e_i\}_{i=1}^n$ be an orthonormal bases of R^n. Assume U is a unitary operator on R^n.

Let $W = span\{e_i; i \in \Omega\}$ and $U^j W = span\{U^j e_i; i \in \Omega\}$ for any $j \in \mathbb{N}$. For any given $j \in \mathbb{N}$, since U is an unitary operator and unitary operators preserve the inner product, we have $\langle U^j e_i, U^j e_k\rangle = \langle e_i, e_k\rangle = 0$ for any $i \neq k$. That is $\{U^j e_i\}_{i\in\Omega}$ is an orthonormal basis for $U^j W$ for each j.

Lemma 3.4.1. *Let $\mathbb{R}^n$ be the real Hilbert space and $W = span\{e_i; i \in \Omega\}$,*

$U^jW = span\{U^je_i; i \in \Omega\}$ for any integer $j \geq 0$ and P_j be the orthogonal projection onto U^jW for any $j \geq 0$. Suppose U is a unitary operator on $\mathbb{R}^n$. If the set of projections $\{P_j\}_{j=0}^M$ does norm retrieval on $\mathbb{R}^n$, then the set of vectors $\{U^je_i\}_{i\in\Omega,j=0,1,..M}$ does norm retrieval in $\mathbb{R}^n$.

Proof. For any given vectors $x, y \in \mathbb{R}^n$
Suppose $|\langle x, U^je_i\rangle| = |\langle y, U^je_i\rangle|$ for any $i \in \Omega$ and $j = 0, 1, ..M$. Since $U^je_i \in U^jW$ for any $j = 0, 1, ..M$, we have $P_jU^je_i = U^je_i$, and hence

$$\begin{aligned}|\langle x, U^je_i\rangle| = |\langle y, U^je_i\rangle| &\Longrightarrow |\langle x, P_jU^je_i\rangle| = |\langle y, P_jU^je_i\rangle| \\ &\Longrightarrow |\langle P_jx, U^je_i\rangle| = |\langle P_jy, U^je_i\rangle|.\end{aligned}$$

Since P_j is a projection on U^jW. For each fixed j, Since $\{U^je_i\}_{i\in\Omega}$ is an orthonormal basis in U^jW, we have

$$||P_jx|| = \sum_{i\in\Omega} | < P_jx, U^je_i > |^2 = \sum_{i\in\Omega} | < P_jy, U^je_i > |^2 = ||P_jy|| \qquad (3.4.1)$$

By assumption, Since the projections $\{P_j\}_{j=0}^M$ do norm retrieval on $\mathbb{R}^n$, we have $||x|| = ||y||$.

□

Note: In the above lemma, we used orthonormality of $\{U^je_i\}_{i\in\Omega}$ in U^jW to show norm retrievability using that U is a unitary operator. Hence, this lemma also holds for any operator which is an isometry.

U being a unitary is a strong condition, however we can relax this assumption as shown in the following lemma.

Corollary 3.4.2. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ be a set of vectors in $\mathbb{R}^n$. Define $W = span\{b_i \in \mathbb{R}^n; i \in \Omega\}$. If the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ does norm retrieval in W, then for any given $k \in \mathbb{N}$, $\{U^k b_i\}_{i\in\Omega}$ does norm retrieval in $U^k W$.*

Proof. Fix $k \in \mathbb{N}$, supose we have

$$|\langle x, U^j b_i\rangle| = |\langle y, U^j b_i\rangle| \quad \forall i \in \Omega \quad \text{for given} \quad x, y \in U^k W.$$

Then,

$$|\langle x, U^j b_i\rangle| = |\langle U^{*j} x, b_i\rangle| = |\langle U^{*j} y, b_i\rangle| = |\langle x, U^j b_i\rangle|.$$

Since the set of vectors $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ does norm retrieval in W, we have $||U^{*j}x|| = ||U^{*j}y||$ and therefore $||x|| = ||y||$ (Since U is a unitary operator, U^* is also a unitary operator). □

Let $\{P_j\}$ be an orthogonal projection onto the subspace $U^j W$ for each j. We can now give a condition that will ensure that the set of projections $\{P_j\}_j$ does norm retrieval in $\mathbb{R}^n$. It connects to the fusion frames we defined in (2.1.11). Recall that fusion frames are the set of projections $\{P_j\}_j$ with positive weights $\{v_j\}$ such that there exist constants $0 < A \leq B < \infty$ and

$$A||x||^2 \leq \sum_{i\in I} {v_i}^2 ||P_{W_i}(x)||^2 \leq B||x||^2, \text{ for all } \quad x \in \mathbb{R}^n.$$

Theorem 3.4.3. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i \in \mathbb{R}^n : i \in \Omega \quad |\Omega| < n\}$ be a set of orthonormal vectors in $\mathbb{R}^n$. The set of vectors $\{U^j b_i : i \in \Omega, \quad j = 0, 1, ...\ell\}$ does tight frame in $\mathbb{R}^n$ if and only if the set of projections $\{P_j\}_j$ onto the* *subspaces $U^jW = \{U^j\ b_i : i \in \Omega\}$ is a tight fusion frame with weights $v_j = 1$ for all j.*

Proof. ($\Longrightarrow$) Suppose the set of vectors $\{U^jb_i : i \in \Omega,\ j = 0, 1, ...\ell\}$ does tight frame in $\mathbb{R}^n$ with frame bound $C > 0$. Then given any $x \in \mathbb{R}^n$, we can write

$$||x||^2 = \frac{1}{C}\sum_{i,j} |\langle x, U^jb_i\rangle|^2.$$

Since $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ is a set of orthogonarmal vectors in $\mathbb{R}^n$ and U is a unitary operator, $\{U^jb_i : i \in \Omega\}$ is also orthonormal set of vectors in U^jW for each j. Hence, the orthogonal projection P_j onto the subspaces $U^jW = \{U^j\ b_i : i \in \Omega\}$ can be written as

$$P_j(x) = \sum_{i\in\Omega}\langle x, U^jb_i\rangle U^jb_i.$$

Thus,

$$||x||^2 = \frac{1}{C}\sum_{i,j} |\langle x, U^jb_i\rangle|^2 = \frac{1}{C}\sum_{j} ||P_j(x)||^2$$

and the set orthogonal projections $\{P_j\}_j$ is a C -tight fusion frame with weights $v_j = 1$. ($\Longleftarrow$) This follows from definition of tight fusion frame with weights $v_k = 1$ for all k.

□

If $\{b_i \in \mathbb{R}^n : i \in \Omega\}$ is a set of vectors that are orthogonal but not orthonormal in $\mathbb{R}^n$, then $\{U^jb_i : i \in \Omega,\ j = 0, 1, ...\ell\}$ is not necessarily a tight frame in $\mathbb{R}^n$ anymore. In this case, we have the following corollary that follows from Theorem (2.3.12), Lemma (3.1.2) and Lemma (2.3.7).

Corollary 3.4.4. *Let U be a unitary operator on $\mathbb{R}^n$ and $\{b_i : i \in \Omega\}$ be a set* *of orthogonal vectors in $\mathbb{R}^n$. The set of vectors $\{U^j b_i : i \in \Omega,\ j = 0, 1, ...\ell\}$ does norm retrieval in $\mathbb{R}^n$ if $x \in span\{P_j(x)\}_{j=0}^{\ell}$, for any $x \in \mathbb{R}^n$.*

3.5 Jordan Form

In this section, we are interested in the linear operator A on $\mathbb{R}^n$ which has all real eigenvalues and unitarily similar to Jordan form. We want to construct subspaces in $\mathbb{R}^n$ which are not necessarily orthogonal to each other but projections onto these subspaces will do norm retrieval in the dynamical sampling structure. We use the notation from ([2]) to set up our next construction.

Let $J \in \mathbb{R}^{nxn}$ be Jordan matrix which has all real eigenvalues, then we have

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix} \tag{3.5.1}$$

For $j = 1, 2, ..s,\ J_j = \lambda_j I_j + N_j$ where I_j is an $r_j \times r_j$ identity matrix and N_j is a $r_j \times r_j$ nilpotent block-matrix of the form:

$$N_j = \begin{pmatrix} N_{j_1} & 0 & \cdots & 0 \\ 0 & N_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{j_i} \end{pmatrix} \tag{3.5.2}$$

Each N_{ji} is a $r_j^i \times r_j^i$ cyclic nilpotent matrix of the form:

$$N_{ji} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.5.3}$$

with $r_j^1 \geq r_j^2 \geq ... \geq r_j^i$ and $r_j^1 + r_j^2 + ... + r_j^i = r_j$. The matrix J has distinct eigenvalues λ_j, $j = 1, 2, ..s$ and $r_1 + r_2 + ... + r_s = n$.

Let k_{ji} denote the index corresponding to the first row of the cyclic nilpotent matrix N_{ji} in (3.5.3), and let $e_{k_{ji}}$ be the corresponding standard orthonormal bases vector of $\mathbb{R}^n$ corresponding to index k_{ji}.

We define $W_j = span\{e_{k_{ji}} : j = 1, 2, ..., s\}$.

Example 3.5.1. *Let* $J = \lambda I + N \in \mathbb{R}^4$*,*

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

where

$$N_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for $i = 1, 2$ *and* $W = span\{e_1, e_3\}$*. Then*

$$JW = span\{\lambda e_1 + e_2, \lambda e_3 + e_4\}$$
$$J^2W = span\{\lambda^2 e_1 + 2\lambda e_2, \lambda^2 e_3 + \lambda e_4\}$$
$$J^3W = span\{\lambda^3 e_1 + 3\lambda^2 e_2, \lambda^3 e_3 + 3\lambda^2 e_4\}$$
$$J^4W = span\{\lambda^4 e_1 + 4\lambda^3 e_2, \lambda^4 e_3 + 4\lambda^3 e_4\}$$

Let P_ℓ be the orthogonal projection onto the subspace $J^\ell W$ for each $\ell \in \mathbb{N}$. For fixed $\ell \in \mathbb{N}$,

$$||J^\ell W e_1||^2 = \lambda^{2\ell} + \ell^2 \lambda^{2(\ell-1)} = ||J^\ell W e_3||^2.$$

Let $c_\ell = \lambda^{2\ell} + \ell^2\lambda^{2(\ell-1)}$ for each $\ell \in \mathbb{N}$, then the orthogonal projection P_ℓ onto the subspace $J^\ell W$ for each $\ell \in \mathbb{N}$ can be written as:

$$P_\ell(x) = \frac{1}{c_\ell}\sum_{i=1,3}\langle x, J^\ell W e_i\rangle J^\ell W e_i \quad and \quad ||P_\ell(x)||^2 = \frac{1}{c_\ell}\sum_{i=1,3}|\langle x, J^\ell W e_i\rangle|^2.$$

This implies that the set of vectors $\{J^\ell W e_i\}_{i,\ell}$ does norm retrieval in $\mathbb{R}^n$ if and only if $I = \sum_\ell c_\ell P_\ell$ since the coefficients $\{c_\ell\}$ are independent from choice of x.

Theorem 3.5.2. *Let $W_j = span\{e_{k_{ji}} : j = 1, 2, ..., s\}$, $l = 0, 1, ..., r^i_j$ and $\{P^\ell_i\}$ be the orthogonal projection onto $J^l W_i$. Suppose order r^i_j of N_{ji} are same for all i, j. Then the set of vectors $\{J^l e_{k_{ij}}\}$ does norm retrieval in $\mathbb{R}^n$ if $I = \sum_\ell c_{\ell i} P^\ell_i$.*

Proof. By choice of $e_{k_{ji}}$ as standard basis corresponding to the first row of N_{ji} , $J^l e_{k_{ji}}$ forms an orthogonal basis for $J^l W_i$. As shown on Example (3.5.1), for fixed

l, $||J^l e_{k_{ij}}|| = c^l$ for all i, j.

The orthogonal projection $\{P_i^\ell\}$ onto $J^l W_i$ can be define as

$$P_\ell^i(x) = \frac{1}{c_{\ell i}} \sum_{\ell,i} \langle x, J^\ell W e_{k_{ji}} \rangle J^\ell W e_{k_{ji}}.$$

This implies $\{J^l e_{k_{ij}}\}$ does norm retrieval in $\mathbb{R}^n$ if and only if $I = \sum_\ell c_{\ell,i} P_\ell^i$.

$\square$

Let A be a linear operator on $\mathbb{R}^n$ and p be the annihilator polynomial of A. That is $p(A)x = 0$ for all $x \in \mathbb{R}^n$.

Lemma 3.5.3. *Let $F = \{x_i\}_{i=1}^m$ be a frame in $\mathbb{R}^n$ and p be the annihilator polynomial of A. Let F_1, F_2 be a partition of F and p_1, p_2 be the annihilator polynomial of F_1, F_2 respectively. If p/p_1p_2 , then the set of vectors $F = \{x_i\}_{i=1}^m$ does norm retrieval in $\mathbb{R}^n$.*

Proof. $F = \{x_i\}_{i=1}^m$ does norm retrieval in $\mathbb{R}^n$ if and only if for any partition $I \subseteq \{1, ..., m\}$, $(spanF_1)^\perp \subset (spanF_2)$. For that reason, its enough to show that if p/p_1p_2, then $(spanF_1)^\perp \subset (spanF_2)$.
To prove by the contrapositive, suppose there exists $x \in (spanF_1)^\perp$ such that $x \notin spanF_2$. Since $x \in spanF_1^\perp$, we also have $x \notin spanF_1$. The set of frames $F = \{x_i\}_{i=1}^m$ spans the space $\mathbb{R}^n$ and we have $\mathbb{R}^n = (spanF_1) + (spanF_2)$. Hence, such an x will exists if we can write $x = x_1 + x_2$, where $x_1 \in (spanF_1)$ and $x_2 \in (spanF_2)$ but $x_2 \notin (spanF_1)$, where both x_1, x_2 are non-zero vectors.

On the other hand, $p(A)x = 0$ but $p_1(A)x \neq 0$, and $p_2(A)x \neq 0$ since $x \notin spanF_1$, $x \notin spanF_2$ and p_1, p_2 are annihilator polynomials of the sets F_1, F_2 respectively. So, p does not divide p_1p_2.

$\square$

We give the theorems that diagonal and self-adjoint operators on $\mathbb{R}^n$ do not have any norm retrievable frame generated from a single vector for fewer vectors than phase retrieval. On the other hand, the following example shows the existence of operators which do norm retrieval with a single generator.

Example 3.5.4. *Let*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

then the set

$$F = \{b, Ab, A^2b, A^3b\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

contains an orthogonal basis, and hence it does norm retrieval. Since the number of vectors is less then 5, it does not do phase retrieval in $\mathbb{R}^3$. We know from Lemma (2.3.14) that scalable frames all do norm retrieval. F is a scalable frame but it does not a stricly scalable frame. To see this , note that span of the rank one operators generated by the vectors $\{b, Ab, A^2b, A^3b\}$ contains the identity operator.

$$bb^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Ab(Ab)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2b(A^2b)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A^3b(A^3b)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$I = bb^* + \frac{1}{2}A^2b(A^2b)^* + \frac{1}{2}A^3b(A^3b)^*.$$

Chapter 4

Future Work

4.1 Future Work

Norm retrieval and dynamical sampling are two newly-emerging research areas in the frame theory. In this paper, we give a method in real Hilbert spaces to construct norm retrievable sets with dynamical sampling structure.

We now describe some areas for future work. We see that the dynamical sampling structure also exists in infinite dimensional Hilbert spaces in ([19],[20],([21]). Authors in ([20]) proved that every frame can be represented in the dynamical sampling form with finitely many vectors and bounded operators if the frame is norm-bounded below. In other words, there exist finitely many vectors b_i and bounded operators A_i for any given frame that is norm bounded below such that the frame can be represented as a finite union of sequences $\{(A_i)^n b_i\}_{n=0}^{\infty}$ for some $i = 1, 2, ..., m$. Recently, Aldroubi and his collaborators in ([4]) also showed that phase retrieval is possible in the dynamical sampling structure in the infinite dimensional Hilbert spaces. Our next research project will be looking for norm retrievable sets in the infinite dimensional real Hilbert spaces that is generated by dynamical sampling method.

Norm retrieval in complex Hilbert spaces requires a different set of criteria to verify that a set of vectors do norm retrieval, as described in paper [25]. Finite or infinite dimensional complex Hilbert spaces are another places where we can construct norm retrievable sets in the dynamical sampling form. In real Hilbert spaces, the complementary property completely classify phase retrievable conditions but in complex Hilbert spaces, complementary property is necessary but not sufficient to classify phase retrievable sets. Similar problems occurs when we try to figure out which sets do norm retrieval in finite complex Hilbert spaces. The authors in [25] have defined a classification of norm retrievable frames in finite dimensional complex Hilbert spaces as follows: Let $\{x_i\}_{i=1}^m$ be a frame in $\mathbb{C}^n$. Given a bounded linear operator $\mathcal{K} : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{C}^m$ defined by $\mathcal{K}(\mathcal{H}) := [\langle Tx_i, x_i\rangle]_{1\leq i\leq m}$, the set of vectors $\{x_i\}_{i=1}^m$ does norm retrieval in $\mathbb{C}^n$ if and only if any operator $T \in \operatorname{Ker}(\mathcal{K}) \cap S^{1,1}$ has trace zero. Where $S^{1,1} = \{T \in \mathbb{B}(\mathcal{H}) : T = T^*$, $\operatorname{rank}(T) \leq 2$, and $\sigma(T)$ is the set of eigenvalues of T and $\lambda_{max}, \lambda_{min}$ are largest and smallest eigenvalues of T. In [7], Balan showed that the set of vectors $\{x_i\}_{i=1}^m$ in $\mathbb{C}^n$ do phase retrieval if and only if $\operatorname{Ker}(\mathcal{K}) \cap S^{1,1} = 0$. These two classification are quite challenging to generate dynamical sampling frames that are phase retrievable and norm retrieval.

The authors in ([32],[38],[39]) have defined frames in Quaternionic Hilbert spaces. Many frame properties carry over to the quaternionic setting. This means that phase retrievable and norm retrievable sets also can be obtain in the Quaternionic Hilbert spaces. The author in [36] showed that phase retrievable is possible in Quaternionic Hilbert spaces but norm retrieval is still an open question in these spaces. We will examine conditions for vectors to do norm retrieval and phase retrieval on these spaces and perhaps also try to set up dynamical sampling. One might hope to get dynamical sampling frames in Quaternionic Hilbert spaces.

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