DEFLECTIONS OF SIMPLE BEAMS STRESSED ABOVE THE ELASTIC LIMIT WITH SELECTED INDETERMINATE BEAM ANALYSIS

By

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1963

Submitted to the Faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
August, 1964

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Report Approved:

Report Adviser

Dean of the Graduate School

ACKNOWLEDGEMENTS

The writer wishes to express his sincere appreciation to the following individuals whose assistance and encouragement have made it possible for him to complete his graduate work:

Professor Roger L. Flanders, his major adviser, for his guidance during the preparation of this report.

Dr. R. W. Little and Professor J. V. Parcher for their kind instruction during his studies.

Mr. Abdul Baki and Mr. Matthew McC. Douglas for their help and suggestions.

Mrs. Norma Monday for her excellent job of typing the manuscript.

His parents, Mr. and Mrs. Agia Ram Lamba, for their encouragement and support, making his study in the U.S.A. possible.

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CHAPTER I

INTRODUCTION

The computation of deflections is necessary in order to be sure that a structure proportioned by the plastic method will not be bent out of shape to such an extent that it becomes unserviceable. The limit up to which deflection can be tolerated is a difficult question. Even in the ordinary elastic design, more often than not, a precise answer cannot be given. Consequently, the judgment and experience of the engineer plays an important role in setting the proper deflection limit.

One of the principal advantages of plastic analysis is the simplicity with which the maximum load carrying capacity may be calculated. In case it is required to find the deflections, some of this advantage may be lost, although fairly easy procedures may be applied in certain cases. However, deflection at ultimate load is not ordinarily required any more than it is for an elastically designed structure. Therefore, since plastically designed structures are usually elastic at working loads, deflections may be computed by methods based on elastic behavior. Hence, this difficulty is not so serious as may be supposed.

The deflection requirement, however, is only a secondary one. The structure must be able to carry the assumed load, but on the other hand it must not deform too much out of shape. Therefore, our needs involving deflection computation may be satisfied with approximations.

In Chapter II of this report are solved some examples of simple beams under different types of loadings. Chapter III presents a method for computing the approximate magnitude of the deflection at ultimate load and for obtaining an upper limit to the deflection at working load. In Chapter IV is discussed the problem of superimposed loads. The fifth chapter summarizes and concludes the study.

CHAPTER II

DEFLECTIONS OF BEAMS FOR DIFFERENT LOADINGS

In the following pages are worked some examples of statically determinate beams under different types of loadings. As a preliminary step, a brief outline will be presented of the theory supporting each method of investigation.

2-1 Assumptions

The following assumptions are made:

(1) The cross-section of the beam is assumed to be rectangular.

(2) The material for the beams worked out is assumed to be perfectly plastic with the same stress-strain curve in tension and compression.

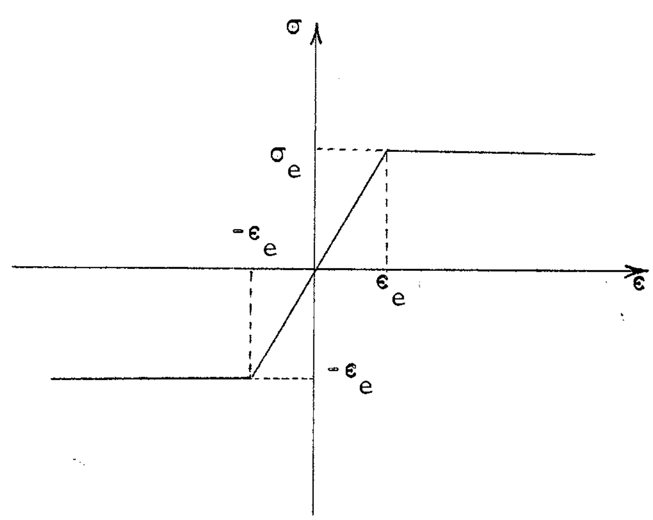


Fig. 1. Idealized Stress-Strain Curve

2-2 Area Moment Method

Theory:

Proof of the two theorems used in this method may be found in most strength of materials textbooks. The theorems may be stated as follows:

Theorem I : The change in slope between tangents drawn to the elastic curve at any two points is equal to the product of $\frac{1}{EI}$ multiplied by the area of the moment diagram between these two points.

Theorem II: When a straight beam is subjected to bending, the distance of any point on the elastic curve, measured normal to the original position of the bending axis, from a tangent drawn at any other point drawn on the elastic curve, is represented in magnitude by the moment of the area under the M/EI diagram between these two points about an ordinate through the first point.

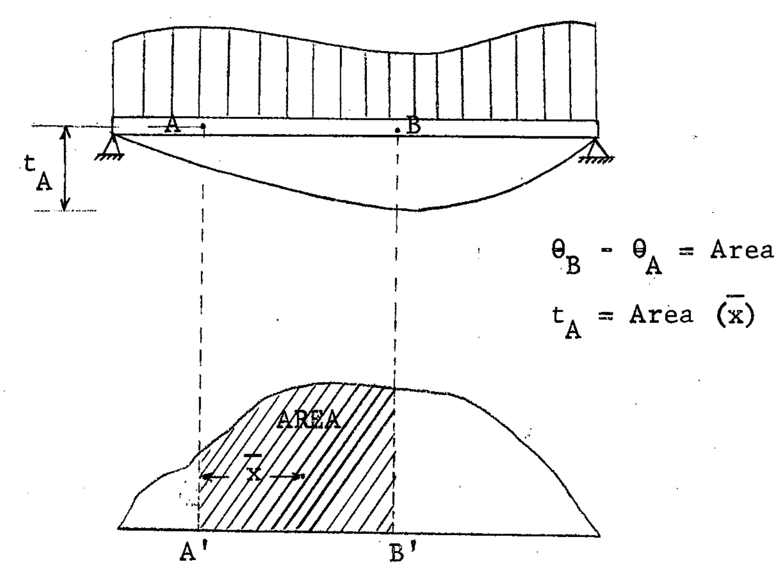


Fig. 2. Load and M/EI Diagram

Mathematically the two theorems may be expressed as:

$$\Delta\theta = \int_A^B \frac{Mdx}{EI}$$

$$t_A = \int_A^B \frac{Mxdx}{EI}$$

where M is expressed in terms of the distance, x, measured from the point A.

However, in the plastic range the rate of angle change can no longer be defined as M/EI because $f \neq \frac{My}{I}$. It may be stated, however, that the rate of angle change is M/EI_r, where I_r is the reduced moment of inertia.

EXAMPLE: SIMPLY SUPPORTED BEAM WITH A CENTRAL LOAD[4]

The figure below shows a simple beam with a central load. The treatment followed shows a procedure for determining the deflection, by making use of the Area-Moment Method.

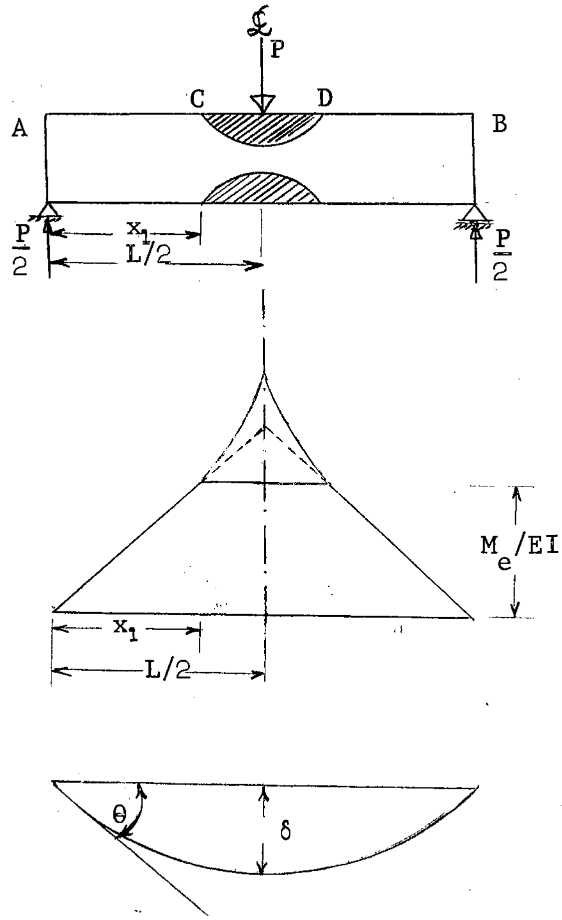


Fig. 3. M/EI Diagram and Deflection Curve

AC = Elastic CD = Elastic + Plastic DB = Elastic

$$M = \frac{Px}{2} \qquad M_{max} = \frac{PL}{4}$$

Moment at C when the plastic hinge is formed = M_e

Now $$M_e = \frac{Px_1}{2}$$

or $$x_1 = \frac{2M_e}{P}$$

Now $\theta = \theta_A - \theta_{\mathcal{L}}$

$= $ Area of M/EI diagram between A and $\mathcal{L}$

$$\theta = \frac{M_e \cdot x_1}{2EI} + \bar{A}_p \qquad \left[x_1 = \frac{2M_e}{P}\right]$$

$$= \frac{M_e^2}{PEI} + \bar{A}_p$$

Now $\lambda = \frac{K}{K_e} = \frac{\epsilon_{max}}{\epsilon_e} = \frac{h}{y_e}$

or $y_e^2 = \frac{h^2}{\lambda^2}$

Now for a rectangular section

$$\lambda = \frac{1}{\sqrt{3-2\ M/M_e}}$$

$$\therefore\ y_e^2 = 3h^2(1 - \frac{2}{3} \cdot \frac{M}{M_e})$$

Now for the plastic portion

$$y_e^2 = 3h^2(1 - \frac{2}{3}\ \frac{M}{M_e}) \qquad [\text{But } M = Px/2]$$

or $y_e^2 = 3h^2(1 - Px/3M_e)$

Now $K = K_e \cdot \frac{h}{y_e}$

$$\therefore\ \bar{A}_p = \int_{x = x_1}^{x = L/2} K dx = \frac{M_e \cdot h}{EI} \int_{x_1}^{L/2} \frac{dx}{h\sqrt{3(1 - Px/3M_e)}} = \frac{M_e}{EI} \int_{x_1}^{L/2} \frac{dx}{\sqrt{3 - Px/M_e}}$$

Substituting $3 - \frac{Px}{M_e} = s$ and integrating

$$\bar{A}_p = - \frac{M_e^2}{PEI} \int_{x_1}^{L/2} s^{-1/2} ds$$

Integrating and substituting $x_1 = \frac{2M_e}{P}$

$$\bar{A}_p = -\frac{2M_e^2}{PEI}\left[\sqrt{3 - \frac{PL}{2M_e}} - 1\right]$$

$$\therefore\ \theta = \frac{M_e^2}{PEI} - \frac{2M_e^2}{PEI}\left[\sqrt{3 - \frac{PL}{2M_e}} - 1\right] = \frac{M_e^2}{PEI}\left[3 - 2\sqrt{3 - \frac{PL}{2M_e}}\right]$$

$$\delta = \left(\frac{M_e^2}{PEI}\right)\left(\frac{2}{3}x_1\right) + \bar{A}_p \cdot x$$

Substituting $x_1 = 2M_e/P$

$$\delta = \frac{4}{3} \cdot \frac{M_e^3}{P^2 EI} + \bar{A}_p \cdot x$$

Now

$$\bar{A}_p x = \int_{x_1}^{L/2} kx dx = \frac{M_e}{EI}\int_{x_1}^{L/2} \frac{x dx}{\sqrt{3 - Px/M_e}}$$

Substituting $3 - \frac{Px}{M_e} = s$ and integrating

$$\bar{A}_p x = -\frac{M_e^3}{P^2 EI}\int_{x = x_1}^{x = L/2} S^{-1/2}(3 - S)\,dS = -\frac{M_e^3}{P^2 EI}\left[6S^{\frac{1}{2}} - \frac{2}{3}S^{3/2}\right]_{x_1}^{L/2}$$

Now $x_1 = 2M_e/P$

$$\therefore\ \bar{A}_p \cdot x = -\frac{M_e^3}{P^2 EI}\left[\sqrt{3 - PL/2M_e}\left\{4 + PL/3M_e\right\} - \frac{16}{3}\right]$$

$$\therefore\ \delta = \frac{4}{3} \cdot \frac{M_e^3}{P^2 EI} - \frac{M_e^3}{P^2 EI}\left[\sqrt{3 - PL/2M_e}\left\{4 + \frac{PL}{3M_e}\right\} - \frac{16}{3}\right]$$

$$= \frac{M_e^3}{3P^2 EI}\left[20 - \sqrt{3 - PL/2M_e}\left(12 + \frac{PL}{M_e}\right)\right]$$

2-3 Method of Double Integration

Theory:

The expression for the radius of curvature, ρ, of any curve is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Since the curvature of most (initially) straight beams is quite small when subjected to stresses below the elastic limit, the second order differential $(dy/dx)^2$ is very small and may be neglected with no appreciable error. Hence

$$\rho = \frac{1}{\frac{d^2y}{dx^2}}$$

It can be shown further that $\rho = EI/M$. By equating the two expressions for ρ, we arrive at the basic relationship $EI\frac{d^2y}{dx^2} = M$ which is the general equation for the elastic curve of a beam. M is the bending moment, expressed in terms of x, at a distance x from the origin and y is the deflection of the beam at the same point.

Example: Simply Supported Beam With a Central Load

Figure 4 shows a simple beam with a concentrated central load. The method of a double integration shall be used to determine the central deflection.

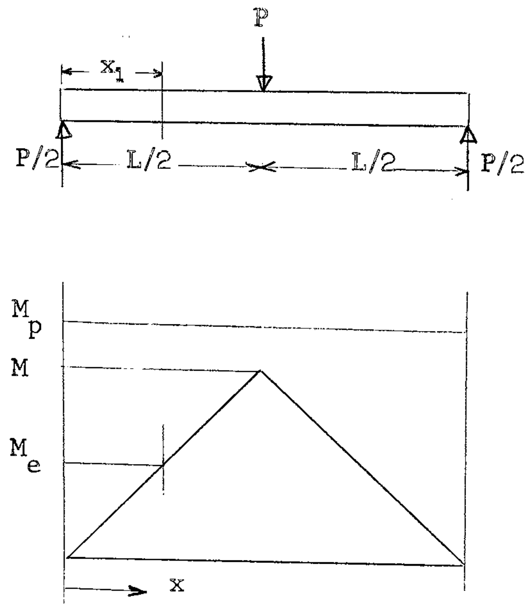


Fig. 4. Moment Diagram

$M = Px/2; \quad M_e = Px_1/2 \qquad \therefore x_1 = 2M_e/P$

Section Properties:

$0 \leq x \leq x_1$

$$k = \frac{M}{EI} = \frac{Px}{2EI}$$

$x_1 \leq x \leq L/2$

$$k = \frac{M}{EI_r} \qquad \text{but} \quad \frac{I_r}{I} = t \qquad \therefore I_r = It$$

Also for a rectangular section

$$t = \frac{3}{2\lambda}\left(1 - \frac{1}{3\lambda^2}\right)$$

and

$$\lambda = \frac{1}{\sqrt{3 - 2\frac{M}{M_e}}}$$

substituting

$$k = \frac{M}{EI} \cdot \frac{1}{\frac{3}{2\lambda}\left(1 - \frac{1}{3\lambda^2}\right)} = \frac{M_e}{EI}\left[\frac{1}{\sqrt{3 - 2\frac{M}{M_e}}}\right]$$

Integrating expressions for curvature

$0 \le x \le x_1$

$$\frac{d^2z}{dx^2} = -\frac{Px}{2EI}$$

$$\frac{dz}{dx} = -\frac{Px^2}{4EI} + \theta \qquad \text{where } \theta = \text{slope at } x = 0$$

and $$z = -\frac{Px^3}{12EI} + \theta x \qquad (\text{no constant since } z = 0,\ x = 0)$$

Evaluating at $x = x_1$ where $x_1 = 2M_e/P$

$$\frac{dz}{dx} = -\frac{M_e^2}{PEI} + \theta$$

$$z = -\frac{2M_e^3}{3P^2EI} + \theta\frac{2M_e}{P}$$

$x_1 \le x \le L/2$

$$\frac{d^2z}{dx^2} = -\frac{M_e}{EI}\left[\frac{1}{\sqrt{3 - Px/M_e}}\right]$$

$$\frac{dz}{dx} = \frac{2M_e^2}{PEI}\sqrt{3 - Px/M_e} + C_1$$

$$z = -\frac{4M_e^3}{3P^2EI}\left(3 - \frac{Px}{M_e}\right)^{3/2} + C_1x + C_2$$

Evaluating C_1 and θ

Slope at $x = L/2 = 0$

at $x = x_1 = -\frac{M_e^2}{PEI} + \theta$

then

$$\frac{2M_e^2}{PEI} + C_1 = -\frac{M_e^2}{PEI} + \theta$$

$$\therefore C_1 = -\frac{3M_e^2}{PEI} + \theta$$

and

$$\frac{2M_e^2}{PEI}\sqrt{3 - PL/2M_e} - \frac{3M_e^2}{PEI} + \theta = 0$$

$$\therefore \quad \theta = \frac{3M_e^2}{PEI} - \frac{2M_e^2}{PEI}\sqrt{3 - \frac{PL}{2M_e}}$$

$$\text{or} \quad \theta = \frac{M_e^2}{PEI}\left[3 - 2\sqrt{3 - \frac{PL}{2M_e}}\right]$$

To evaluate C_2 deflection at $x = x_1 = \frac{2M_e}{P}$ must be the same as already found above.

$$\therefore \quad -\frac{2M_e^3}{2P^2EI} + \frac{2M_e^3}{P^2EI}\left[3 - 2\sqrt{3 - \frac{PL}{2M_e}}\right]$$

$$= -\frac{4M_e^3}{3P^2EI} - \frac{6M_e^3}{P^2EI} + \frac{2M_e^3}{P^2EI}\left[3 - 2\sqrt{3 - \frac{PL}{2M_e}}\right] + C_2$$

$$\text{or} \quad C_2 = \frac{20}{3}\frac{M_e^3}{P^2EI}$$

the maximum deflection may now be evaluated at $x = L/2$, i.e.

$$z_{\mathcal{L}} = \delta = -\frac{4}{3}\frac{M_e^3}{P^2EI}\left(3 - \frac{PL}{2M_e}\right)^{3/2} - \frac{3M_e^2L}{2PEI} + \frac{M_e^2L}{2PEI}\left[3 - 2\sqrt{3 - \frac{PL}{2M_e}}\right] + \frac{20}{3}\frac{M_e^3}{P^2EI}$$

$$= \frac{M_e^3}{3P^2EI}\left[-4\sqrt{3 - \frac{PL}{2M_e}}\left(3 - \frac{PL}{2M_e}\right) - \frac{3PL}{M_e}\sqrt{3 - \frac{PL}{2M_e}} + 20\right]$$

$$= \frac{M_e^3}{3P^2EI}\left[20 - \sqrt{3 - \frac{PL}{2M_e}}\left(12 + \frac{PL}{M_e}\right)\right]$$

Summary:

Slopes:

$$M_e \le M_{max} \le M_p \qquad M_{max} = \frac{PL}{4}$$

$$\theta = \frac{M_e^2}{PEI}\left[3 - 2\sqrt{3 - \frac{PL}{2M_e}}\right]$$

max. elastic slope $P_e = 4M_e/L$, $M_e = P_eL/4$

$$\theta_e = \frac{P_eL^2}{16EI}$$

limiting plastic slope $M_p = \frac{P_pL}{4} = 1.5\,M_e = \frac{3}{8}P_eL$

$$\theta_p = \frac{P_p L^2}{12EI} = \frac{P_e L^2}{8EI}$$

Deflections:

$$\delta = \frac{M_e^3}{3P^2EI}\left[20 - \left(12 + \frac{PL}{M_e}\right)\sqrt{3 - \frac{PL}{2M_e}}\right]$$

max. elastic deflection

$$P_e = \frac{4M_e}{L} \quad , \quad M_e = \frac{P_e L}{4}$$

$$\delta_e = \frac{P_e L^3}{48EI}$$

limiting plastic deflection $\quad M_p = \frac{P_p L}{4} = 1.5\ M_e = \frac{3}{8}\ P_e L$

$$\delta_p = \frac{5}{162} \cdot \frac{P_p L^3}{EI} = \frac{5}{108} \cdot \frac{P_e L^2}{EI}$$

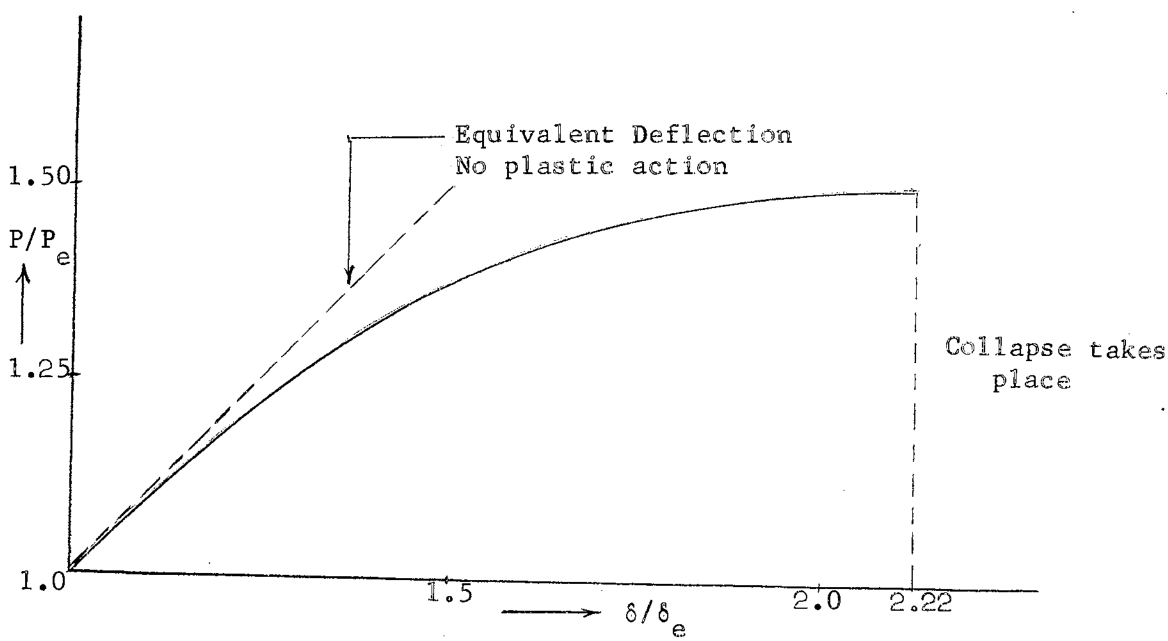


Fig. 5. Load-Deflection Curve

Collapse occurs when the deflection reaches the limiting plastic deflection of $\frac{5}{162} \frac{P_p L^3}{EI}$. The ratio δ/δ_e at that time is

$$\frac{5}{162EI} P_p L^3 / \frac{P_e L^3}{48EI} = \frac{240}{162} \cdot \frac{P_p}{P_e}$$

But for a rectangular section $P_p/P_e = 1.5$

$\therefore \quad \delta/\delta_e = 2.22$

Thus the curve indicates that the beam deflects a maximum of 1.48 times what it would assuming no plastic action.

2-4 Conjugate - Beam Method

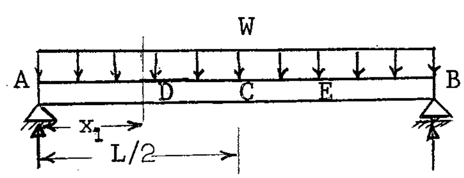
Theory:

From the similarity of the relationships $d^2M/dx^2 = W$ and, $d^2y/dx^2 = M/EI$, it may be seen that the load bears the same relationship to the moment that the M/EI bears to the deflection. Thus if the real beam is replaced by a conjugate beam (which, in some cases, differs from the real beam in type of support) and this conjugate beam then loaded with the M/EI diagram, the deflection of the real beam at a given point will be equal in magnitude to the moment in the conjugate beam at the same point.

The slope in the real beam at a given point, incidentally, will be equal in magnitude to the shear in the conjugate beam.

Example: Simply Supported Beam with Uniformly Distributed Load (4)

Figure 6 shows a simple beam loaded with a uniform-distributed load. The portions AD and EB are elastic whereas the portion DE of the beam is elastic-plastic. The conjugate beam method discussed above has been made use of for determining the central deflection of the beam.



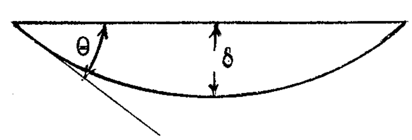


Fig. 6. Deflected Shape of Beam Uniformly Loaded

$$M = \frac{WLx}{2} - \frac{Wx^2}{2} = \frac{Wx}{2}(L - x)$$

$$M_{max} = WL^2/8$$

Moment at D when the plastic hinge is formed is M_e

$$\therefore \quad M_e = \frac{WLx_1}{2} - \frac{Wx_1^2}{2}$$

or

$$Wx_1^2 - WLx_1 + 2M_e = 0$$

or

$$x_1 = \frac{L}{2} \pm \sqrt{\frac{L^2}{4} - \frac{2M_e}{W}}$$

Now $I_r/I = t$ or $I_r = It$

Also for a rectangular section:

$$t = \frac{3}{2\lambda}\left(1 - \frac{1}{3\lambda^2}\right)$$

and

$$\lambda = \frac{1}{\sqrt{3 - 2\frac{M}{M_e}}}$$

Substituting the values of t and λ in above

$$I_r = \frac{M}{M_e} I \sqrt{3 - 2\frac{M}{M_e}} = \frac{W}{2} \cdot \frac{I}{M_e}(Lx - x^2)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}$$

Now $K = \frac{M}{EI} = \frac{d^2 z}{dx^2} = \frac{d\theta}{dx}$

$$\frac{d\theta}{dx} = \frac{M}{EI}\Big|_{x=0\to x_1} + \frac{M}{EI_r}\Big|_{x=x\to\frac{L}{2}}$$

Integrating

$$\theta_A = \frac{1}{EI}\int_0^{x_1} Mdx + \int_{x_1}^{L/2} \frac{W/2\,(Lx - x^2)\; dx}{E \cdot \frac{W}{2}\, \frac{I}{M_e}(Lx - x^2)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

$$= \frac{1}{EI} \cdot \frac{W}{2}\int_0^{x_1} (Lx - x^2)\,dx + \frac{M_e}{EI}\int_{x_1}^{L/2} \frac{dx}{\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

Integrating the above expression in two parts

$$\frac{1}{EI} \cdot \frac{W}{2}\int_0^{x_1} (Lx - x^2)\,dx = \frac{Wx_1^2}{12EI}(3L - 2x_1)$$

$$\frac{M_e}{EI}\int_{x_1}^{L/2} \frac{dx}{\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

$$= \frac{M_e}{EI} \cdot \sqrt{\frac{M_e}{W}}\left[\log\left\{2\left[\frac{W}{M_e}\left(\frac{WL^2}{4M_e} - \frac{WL^2}{2M_e} + 3\right)\right]\right\}^{\frac{1}{2}} + \frac{WL}{M_e} - \frac{WL}{M_e}\right]$$

$$- \frac{M_e}{EI} \cdot \sqrt{\frac{M_e}{W}}\left[\log\left\{2\left[\frac{W}{M_e}\left(\frac{Wx_1^2}{M_e} - \frac{WLX_1}{M_e} + 3\right)\right]\right\}^{\frac{1}{2}} + \frac{2Wx_1}{M_e} - \frac{WL}{M_e}\right]$$

$$= \frac{M_e}{EI}\sqrt{\frac{M_e}{W}}\left[\log \frac{2\left[\frac{W}{M_e}\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}\right]}{\left[\frac{2W}{M_e^2}\left(Wx_1^2 - WLx_1 + 3M_e\right)\right]^{\frac{1}{2}} - \frac{2W}{M_e}(L/2 - x_1)}\right]$$

Now $Wx_1^2 - WLx_1 + 3M_e = M_e$

$\therefore$ the above expression becomes equal to

$$= \frac{M_e}{EI} \cdot \sqrt{\frac{M_e}{W}}\left[\log \frac{\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}}{1 - \left[\frac{W}{M_e}\left(\frac{L^2}{4} - \frac{2M_e}{W}\right)\right]^{\frac{1}{2}}}\right]$$

$$\therefore \quad \theta_A = \frac{1}{EI}\left[\frac{Wx_1^2}{12}(3L - 2x_1) + M_e\sqrt{\frac{M_e}{W}}\log \frac{\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}}{1 - \left[\frac{W}{M_e}\left(\frac{L^2}{4} - \frac{2M_e}{W}\right)\right]^{\frac{1}{2}}}\right]$$

Conjugate Method For Finding the Deflection

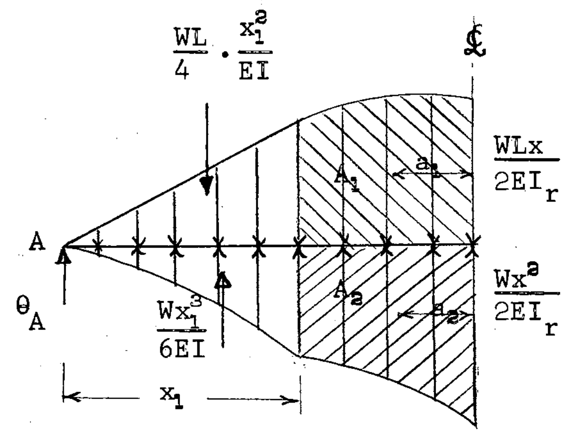


Fig. 7. Conjugate Beam

$$A_1 a_1 = \frac{WL}{2E} \int_{x_1}^{L/2} \frac{X}{I_r} (L/2 - x)\,dx = \frac{M_e L}{2EI} \int_{x_1}^{L/2} \frac{(L - 2x)}{(L - x)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}\,dx$$

Breaking it into partial fractions

$$\frac{(L - 2x)}{(L - x)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}} = \frac{A}{(L - x)} + \frac{B}{\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

where $A = -\frac{L}{\sqrt{3}}$; $B = \frac{2\sqrt{L^2/4 - 3\frac{M_e}{W}}}{\frac{L}{2} + \sqrt{L^2/4 - 3\frac{M_e}{W}}}$

$$\therefore \frac{M_e L}{2EI} \int \frac{(L - 2x)\,dx}{(L-x)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}} = -\frac{M_e L^2}{2\sqrt{3}\,EI} \int_{x_1}^{L/2} \frac{1}{(L-x)}\,dx + \frac{M_e L}{2EI} B \int_{x_1}^{L/2} \frac{dx}{\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

$$= \frac{M_e L^2}{2\sqrt{3}\,EI} \log \frac{L/2}{L - x_1} + \frac{M_e L}{2EI} \sqrt{\frac{M_e}{W}}\, B \log \frac{\sqrt{3 - \frac{WL^2}{4M_e}}}{1 - \sqrt{\frac{WL^2}{4M_e} - 2}}$$

$$A_2 a_2 = \int_{x_1}^{L/2} \frac{Wx^2 (L/2 - x)}{2EI_r} dx = \frac{M_e}{2EI} \int_{x_1}^{L/2} \frac{x(L - 2x)}{(L-x)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

Breaking it into partial fractions

$$\frac{x(L - 2x)}{(L-x)\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}} = \frac{C}{(L-x)} + \frac{D}{\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

Where $C = -\frac{L^2}{\sqrt{3}}$ and $D = \dfrac{L\sqrt{\frac{L^2}{4} - \frac{3M_e}{W}} - 2(L^2/4 - 3M_e/W)}{\frac{L}{2} + \sqrt{\frac{L^2}{4} - \frac{3M_e}{W}}}$

$$\therefore \int_{x_1}^{L/2} \frac{Wx^2 (L/2 - x)}{2EI_r} dx = \frac{M_e}{2EI} \int_{x_1}^{L/2} - \frac{L^2 dx}{\sqrt{3}\,(L-x)} + \frac{M_e}{2EI} D \int_{x_1}^{L/2} \frac{dx}{\sqrt{3 - \frac{W}{M_e}(Lx - x^2)}}$$

$$= \frac{M_e L^2}{2\sqrt{3}EI} \log\left(\frac{L/2}{L-x_1}\right) + \frac{M_e}{2EI}\sqrt{\frac{M_e}{W}}\, D \log \frac{\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}}{1 - \left(\frac{WL^2}{4M_e} - 2\right)^{\frac{1}{2}}}$$

$$\delta_c = \theta_A \cdot \frac{L}{2} + \frac{Wx_1^3}{6EI}\left(\frac{L}{2} - \frac{3}{4} x_1\right) + \frac{M_e L^2}{2\sqrt{3}\,EI} \log\left(\frac{L/2}{L-x_1}\right)$$

$$+ \frac{M_e}{2EI}\sqrt{\frac{M_e}{W}}\, D \log \frac{\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}}{1 - \left(\frac{WL^2}{4M_e} - 2\right)^{\frac{1}{2}}} - \frac{WL}{4EI} x_1^2 \left(\frac{L}{2} - \frac{2}{3} x_1\right)$$

$$- \frac{M_e L^2}{2\sqrt{3}EI} \log\left(\frac{L/2}{L-x_1}\right) - \frac{M_e L}{2EI}\sqrt{\frac{M_e}{W}}\, B \log \frac{\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}}{1 - \left(\frac{WL^2}{4M_e} - 2\right)^{\frac{1}{2}}}$$

or $$\delta_c = \frac{Wx_1^2}{24EI}\left[3L^2 - 2x_1 L + 2x_1 L - 3x_1^2 - 3L^2 + 4x_1 L\right]$$

$$+ \frac{M_e}{2EI}\sqrt{\frac{M_e}{W}} \log \frac{\left(3 - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}}}{1 - \left(\frac{WL^2}{4M_e} - 2\right)^{\frac{1}{2}}}\left[L + D - LB\right]$$

Substituting for B and D from above

$$\delta_c = \frac{1}{EI}\left[\frac{Wx_1^3}{24}(4L - 3x_1) + M_e\sqrt{\frac{M_e}{W}}\left\{\frac{L}{2}\log\frac{\sqrt{3 - \frac{WL^2}{4M_e}}}{1 - \sqrt{\frac{WL^2}{4M_e} - 2}} - \sqrt{\frac{L^2}{4} - \frac{3M_e}{W}}\right.\right.$$

$$\left.\left.\log\frac{\sqrt{3 - \frac{WL^2}{4M_e}}}{1 - \sqrt{\frac{WL^2}{4M_e} - 2}}\right\}\right]$$

Summary:

$$M_e = \frac{W_e L^2}{8} \qquad M_p = \frac{W_p L^2}{8} = 1.5\ M_e$$

$$\theta_e = \frac{1}{EI}\left[\frac{W_e L^3}{16} - \frac{W_e L^3}{48}\right] = \frac{W_e L^3}{24EI}$$

$$\theta_p = \infty$$

$$\delta_e = \frac{1}{EI}\left[\frac{W_e L^4}{48} - \frac{W_e L^4}{128}\right] = \frac{5}{384} \cdot \frac{W_e L^4}{EI}$$

$$\delta_p = \infty$$

2-5 Cantilever With An End Load[4]

Figure 8 shows a cantilever with an end load. The method of double integration has been applied to determine the deflection.

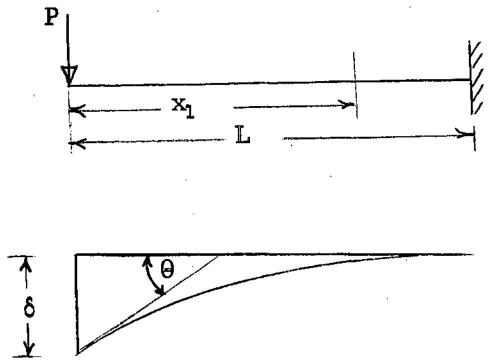


Fig. 8. Deflected Shape of Contilever With End Load

$$M = Px; \qquad M_e = Px_1 \qquad \therefore x_1 = \frac{M_e}{P}$$

Now

$$\frac{d^2z}{dx^2} = \frac{d\theta}{dx} = -\left[\frac{M}{EI} + \frac{M}{EI_r}\right]$$

$$I_r = It = I \cdot \frac{M}{M_e}\sqrt{3 - 2\frac{M}{M_e}}$$

or

$$\frac{d^2z}{dx^2} = \frac{d\theta}{dx} = -\frac{1}{EI}\left[M + \frac{M}{\frac{M}{M_e}\sqrt{3 - 2\frac{M}{M_e}}}\right]$$

$$\frac{dz}{dx} = \theta = -\frac{1}{EI}\left[\int_0^{x_1} Px\,dx + \int_{x_1}^{L} \frac{M_e}{\sqrt{3 - 2\frac{M}{M_e}}}\,dx\right] = -\frac{1}{EI}\left[\frac{Px_1^2}{2} + M_e\int_{x_1}^{L}\left(3 - 2\frac{Px}{M_e}\right)^{-\frac{1}{2}}dx\right]$$

Substituting $3 - \frac{2Px}{M_e} = s$ and integrating

$$\theta = -\frac{1}{EI}\left[\frac{P}{2}\left(\frac{M_e}{P}\right)^2 - \frac{M_e^2}{2P}\int_{x_1}^{L} s^{-\frac{1}{2}} ds\right] = -\frac{M_e^2}{2PEI}\left[3 - 2\sqrt{3 - \frac{2PL}{M_e}}\right]$$

Integrating again

$$\delta = -\frac{1}{EI}\left[\int_0^{x_1} Px^2 dx + \int_{x_1}^{L} \frac{M_e x dx}{\sqrt{3 - 2\frac{Px}{M_e}}}\right]$$

Substituting $3 - \frac{2Px}{M_e} = s$ and $x_1 = M_e/P$ and integrating

$$\delta = -\frac{M_e^3}{3P^2EI}\left[5 - \sqrt{3 - \frac{2PL}{M_e}}\left(3 + \frac{PL}{M_e}\right)\right]$$

Summary:

Slopes:

$M_e \le M_{max} \le M_p \qquad M_{max} = PL$

$$\theta = \frac{M_e^2}{2PEI}\left(3 - 2\sqrt{3 - \frac{2PL}{M_e}}\right)$$

max. elastic slope = $P_e = M_e/L \quad ; \quad M_e = P_eL$

$$\theta_e = \frac{P_eL^2}{2EI}$$

limiting plastic slope $\quad Mp = P_pL = 1.5\, M_e = 1.5\, P_eL$

$$\theta_p = \frac{P_pL^2}{1.5EI} = \frac{P_eL^2}{EI} = 2\theta_e$$

Deflections:

$$\delta = \frac{M_e^3}{3P^2EI}\left[5 - (3 + PL/M_e)\sqrt{3 - \frac{2PL}{M_e}}\right]$$

max. elastic deflection: $\quad P_e = M_e/L \quad ; \quad M_e = P_eL$

$$\delta_e = \frac{P_eL^3}{3EI}$$

limiting plastic deflection $\quad M_p = P_pL = 1.5\ M_e = 1.5\ P_eL$

$$\delta_p = \frac{40P_pL^3}{81EI} = \frac{20}{27}\frac{P_eL^3}{EI} = \frac{20}{9}\ \delta_e$$

Minimum δ_p occurs at $Mp = M_e$

2-6 Cantilever With End Moment as Shown[4]

Figure 9 shows a cantilever with an end moment. The deflection can be easily determined by making use of the method of double integration discussed earlier in the Chapter.

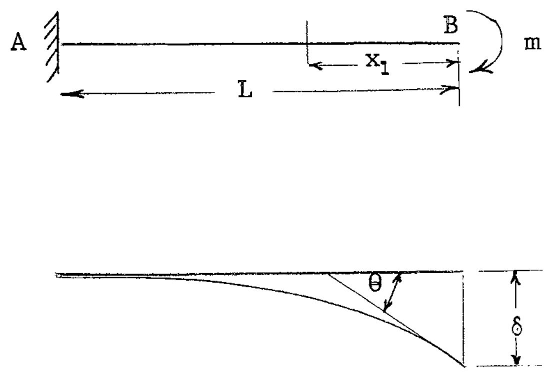


Fig. 9. Deflected Shape of Cantilever With End Moment

$$M_x = -m \qquad (M_p > m \geq M_e)$$

Now $$\frac{d^2 z}{dx^2} = \frac{d\theta}{dx} = \left[\frac{M}{EI} + \frac{M}{EI_r}\right]$$

As already shown before

$$I_r = I \cdot \frac{M}{M_e}\sqrt{3 - 2\frac{M}{M_e}}$$

$$\therefore \quad \frac{d^2 z}{dx^2} = \frac{d\theta}{dx} = \frac{1}{EI}\left[M + \frac{M}{\frac{M}{M_e}\sqrt{3-2\frac{M}{M_e}}}\right]$$

or $$\frac{dz}{dx} = \theta = \frac{1}{EI}\left[\int_0^{x_1} m\,dx + \int_{x_1}^{L} \frac{M_e}{\sqrt{3 - 2\frac{m}{M_e}}}\,dx\right]$$

or

$$\theta = \frac{1}{EI}\left[mx_1 + \frac{M_e}{\sqrt{3-2\frac{m}{M_e}}} (L - x_1) \right]$$

$\therefore$ at $x_1 = 0$

$$\theta = \frac{M_e L}{EI} \frac{1}{\sqrt{3-2\frac{m}{M_e}}}$$

Now $M_e = m$

$$\therefore \quad \theta_e = \frac{M_e L}{EI}$$

$$M_p = 1.5M_e = m$$

$$\therefore \quad \theta_p = \frac{M_e L}{EI} \cdot \frac{1}{\sqrt{3 - 2x1.5}} = \infty$$

Integrating the expression on the previous page once again

$$\delta = \frac{1}{EI}\left[\int_0^{x_1} Mx\,dx + \int_{x_1}^{L} \frac{M_e x\,dx}{\sqrt{3-2\frac{m}{M_e}}} \right] = \frac{1}{EI}\left[\frac{Mx_1^2}{2} + \frac{M_e}{2\sqrt{3-2\frac{m}{M_e}}} (L^2 - x_1^2) \right]$$

at $x_1 = 0$

$$\delta = \frac{1}{EI}\left[\frac{M_e L^2}{2\sqrt{3-2\frac{m}{M_e}}} \right] = \frac{M_e L^2}{2EI} \cdot \frac{1}{\sqrt{3-2\frac{m}{M_e}}}$$

max. elastic deflection $\quad M_e = m$

$$\delta_e = \frac{M_e L^2}{2EI}$$

Limiting plastic deflection $\quad M_p = 1.5\ M_e = m$

$$\delta_p = \frac{M_e L^2}{2EI} \frac{1}{\sqrt{3-2x1.5}} = \infty$$

2-7 Deflection of the Beam Hinged At One End and Roller On Other End and With An External Moment m as Shown By Integration Method (4)

The method of double-integration can be quite easily applied for determining the deflection of the beam shown below.

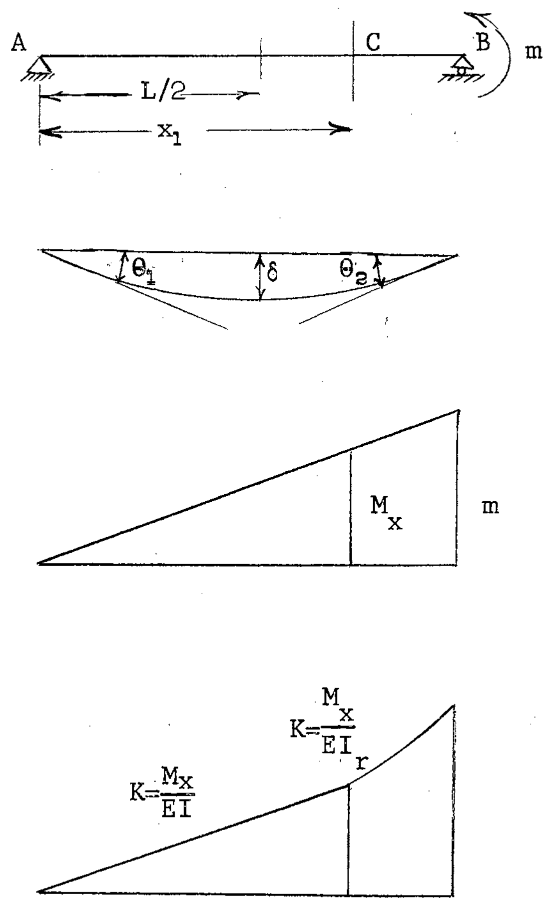


Fig. 10. Deflected Shape and Moment Diagram

Portion A - C = Elastic

C - B = Plastic

$$M_x = \frac{mx}{L} \qquad (M_p > m > M_e)$$

Now $M_e = \frac{mx_1}{L}$ or $x_1 = \frac{M_e}{m} L$

$$t = \frac{I_r}{I} = \frac{3}{2\lambda}\left(1 - \frac{1}{3\lambda^2}\right) = \frac{3}{2\lambda} - \frac{1}{2\lambda^3}$$

Now

$$\lambda = \frac{1}{\sqrt{3-2\frac{M_x}{M_e}}}$$

$$\therefore \quad I_r = It = I\left(\frac{3}{2\lambda} - \frac{1}{2\lambda^3}\right) = I\left\{\frac{3}{2}\sqrt{3-2\frac{M_x}{M_e}} - \tfrac{1}{2}\left(3-2\frac{M_x}{M_e}\right)^{3/2}\right\}$$

$$\text{let } \alpha = \left\{\frac{3}{2}\sqrt{3-2\frac{M_x}{M_e}} - \tfrac{1}{2}\left(3-2\frac{M_x}{M_e}\right)^{3/2}\right\} = \sqrt{3-2\frac{M_x}{M_e}}\left\{\frac{M_x}{M_e}\right\}$$

$$\therefore \quad I_r = I\alpha$$

Now

$$\frac{d^2z}{dx^2} = \frac{d\theta}{dx} = -\left[\frac{M_x}{EI} + \frac{M_x}{EI_r}\right]$$

$$\frac{M_x}{EI} = \frac{mx}{EIL}$$

$$\frac{M_x}{EI_r} = \frac{mx}{EIL\alpha} = \frac{mx}{EIL} \cdot \frac{1}{\frac{M_x}{M_e}\sqrt{3-2\frac{M_x}{M_e}}} = \frac{M_e}{EI}\left(3-2\frac{mx}{M_e L}\right)^{-\frac{1}{2}} \qquad \left[\because M_x = \frac{mx}{L}\right]$$

Now

$$\theta_{x=0\to x<x_1} = -\int_0^x \frac{mx}{EIL}\,dx = -\frac{mx^2}{2EIL} + C_1$$

$$\theta_{x=x_1\to x>x_1} = -\left[\int_0^{x_1}\frac{mx}{EIL}\,dx + \int_{x_1}^{x>x_1}\frac{M_e}{EI}\left(3 - \frac{2mx}{M_e L}\right)^{-\frac{1}{2}} dx\right]$$

Integrating and substituting $m = \frac{M_e}{x_1}L$

$$\theta_{x=x_1 \to x>x_1} = -\frac{3}{2}\frac{M_e^2 L}{EIm} + \frac{M_e^2 L}{EIm}\left(3 - \frac{2mx}{M_e L}\right)^{\frac{1}{2}} + C_2$$

$$\Delta_{x=0 \to x<x_1} = \int_0^x \left(-\frac{mx^2}{2EIL} + C_1\right)dx = -\frac{mx^3}{6EIL} + C_1 x + C_3$$

$$\Delta_{x=x_1 \to x>x_1} = \int_0^{x_1}\left(-\frac{mx^2}{2EIL} + C_1\right)dx + \int_{x_1}^{x>x_1}\left(-\frac{3}{2}\frac{M_e^2 L}{EIm} + \frac{M_e^2 L}{EIm}\left\{3 - \frac{2mx}{M_e L}\right\}^{\frac{1}{2}} + C_2\right) dx$$

$$= -\frac{mx_1^3}{6EIL} + C_1 x_1 - \frac{3}{2}\frac{M_e^2 L}{EIm} x + \frac{3}{2}\frac{M_e^2 L}{EIm} x_1 + \frac{M_e^2 L}{EIm} \cdot -\frac{M_e L}{2m} \cdot \frac{2}{3}\left[\left(3 - \frac{2mx}{M_e L}\right)^{3/2} - 1\right]$$

$$+ C_2 x - C_2 x_1 + C_4$$

$$= -\frac{3}{2}\frac{M_e^2 L}{EIm} x + \frac{5}{3}\frac{M_e^3 L^2}{EIm^2} - \frac{M_e^3 L^2}{3EIm^2}\left(3 - \frac{2mx}{M_e L}\right)^{3/2} + C_1\frac{M_e}{m} L + C_2 x - C_2\frac{M_e}{m} L + C_4$$

Deformation Conditions:

$\Delta_x = 0$ at $x = 0$ and $x = L$

$\Delta_{0-x_1} = \Delta_{x_1-L}$ at $x = x_1$

$\theta_{0-x_1} = \theta_{x_1-L}$ at $x = x_1$

Now at $x = 0$ $\Delta_x = 0$

$\therefore$ $C_3 = 0$; at $x = L$ $\Delta_x = 0$

$$\therefore -\frac{3}{2}\frac{M_e^2 L^2}{EIm} + \frac{5}{3}\frac{M_e^3 L^2}{EIm^2} - \frac{M_e^3 L^2}{3EIm^2}\left(3 - \frac{2mx}{M_e L}\right)^{3/2} + (C_1 - C_2)\frac{M_e}{m} L + C_2 L + C_4 = 0$$

Now $\Delta_{0-x_1} = \Delta_{x_1-L}$ at $x = x_1$

$$\therefore \frac{M_e^3 L^2}{6EIm^2} + C_1\frac{M_e}{m} L = \frac{3}{2}\frac{M_e^3 L^2}{EIm^2} - \frac{5}{3}\frac{M_e^3 L^2}{EIm^2} + \frac{M_e^3 L^2}{3EIm^2} + C_1\frac{M_e L}{m} + C_2\frac{M_e L}{m} - C_2\frac{M_e L}{m} + C_4$$

$\therefore$ $C_4 = 0$

Now $\theta_{0\to x_1} = \theta_{x_1 \to L}$ at $x = x_1$

$$\therefore \quad -\frac{M_e^2 L}{2EIm} + C_1 = -\frac{3}{2}\frac{M_e^2 L}{EIm} + \frac{M_e^2 L}{EIm}\left(3 - \frac{2mx_1}{M_e L}\right)^{\frac{1}{2}} + C_2$$

or

$$-\frac{M_e^2 L}{2EIm} + C_1 = -\frac{3}{2}\frac{M_e^2 L}{EIm} + \frac{M_e^2 L}{EIm} + C_2 \qquad \therefore C_1 = C_2$$

Now $\Delta_x = 0$ at $x = L$

$$\therefore \quad -\frac{3}{2}\frac{M_e^2 L^2}{EIm} + \frac{5}{3}\frac{M_e^3 L^2}{EIm^2} - \frac{M_e^3 L^2}{3EIm^2}\left(3 - \frac{2m}{M_e}\right)^{3/2} + C_2 L$$

$$\therefore \quad C_2 = \frac{M_e^3 L}{3EIm^2}\left[\frac{9}{2}\frac{m}{M_e} - 5 + \left(3 - \frac{2m}{M_e}\right)^{3/2}\right]$$

$$\therefore \quad \theta_1 = C_2 = \frac{M_e^3 L}{3EIm^2}\left[\frac{9}{2}\frac{m}{M_e} - 5 + \left(3 - \frac{2m}{M_e}\right)^{3/2}\right]$$

$$\theta_2 = -\frac{3}{2}\frac{M_e^2 L}{EIm} + \frac{M_e^2 L}{EIm}\left(3 - \frac{2m}{M_e}\right)^{\frac{1}{2}} + C_2$$

or

$$\theta_2 = -\frac{3}{2}\frac{M_e^2 L}{EIm} + \frac{M_e^2 L}{EIm}\left(3 - \frac{2m}{M_e}\right)^{\frac{1}{2}} + \frac{M_e^3 L}{3EIm^2}\cdot\frac{9}{2}\frac{m}{M_e} - \frac{5}{3}\frac{M_e^3 L}{EIm^2} + \frac{M_e^3 L}{3EIm^2}\cdot\left(3 - \frac{2m}{M_e}\right)^{3/2}$$

or

$$\theta_2 = -\frac{M_e^3 L}{3EIm^2}\left[5 - \left(3 - \frac{2m}{M_e}\right)^{\frac{1}{2}}\left(3 + \frac{m}{M_e}\right)\right]$$

δ at $x = L/2$

$$\delta = -\frac{mL^3}{48EIL} - C_1\frac{L}{2} + C_3 = -\frac{mL^2}{48EI} + \frac{M_e^3 L^2}{6EIM^2}\left[\frac{9}{2}\frac{m}{M_e} - 5 + \left(3 - \frac{2m}{M_e}\right)^{3/2}\right]$$

$$= \frac{M_e^3 L^2}{6EIm^2}\left[\frac{9}{2}\cdot\frac{m}{M_e} - 5 + \left(3 - \frac{2m}{M_e}\right)^{3/2}\right] - \frac{mL^2}{48EI}$$

$\theta_{1e} = \theta_1$ if $m = M_e$, $\therefore \theta_{1e} = \frac{M_e L}{3EI}\left(\frac{9}{2} - 5 + 1\right) = \frac{M_e L}{6EI}$

$\theta_{2e} = \theta_2$ if $m = M_e$; $\therefore \theta_{2e} = - \frac{M_e L}{3EI} (5-4) = - \frac{M_e L}{3EI}$

$\theta_{1p} = \theta_1$ if $m = M_p$

Now M_p/M_e = Shape Factor

for a rectangular beam $M_p/M_e = 1.5$

$\therefore \quad \theta_{1p} = \theta_1$ if $m = 1.5\ M_e$

or $\theta_{1p} = \frac{4}{27} \frac{M_e L}{EI} (\frac{27}{4} - \frac{20}{4}) = \frac{7}{27} \frac{M_e L}{EI}$

$\theta_{2p} = \theta_2$ if $m = 1.5M_e$

$\therefore \quad \theta_{2p} = - \frac{4}{27} \frac{M_e L}{EI} (5) = - \frac{20}{27} \frac{M_e L}{EI}$

also $\delta_e = \delta$ if $m = M_e$

$\therefore \quad \delta_e = \frac{M_e L^2}{6EI} (\frac{9}{2} - 5 + 1) - \frac{M_e L^2}{48EI} = \frac{M_e L^2}{16EI}$

$\delta_p = \delta$ if $m = 1.5\ M_e$

$\therefore \quad \delta_p = \frac{4M_e L^2}{54EI} (\frac{27}{4} - \frac{20}{4}) - \frac{M_e L^2}{32EI} = \frac{85}{864} \frac{M_e L^2}{EI}$

CHAPTER III

DEFLECTIONS FOR LOADS PRODUCING YIELDING BY SLOPE DEFLECTION[3]

The slope deflection equations may also be used to find out the relative deflection of segments of the structure. The moments would first be worked out from plastic analysis. The following form of slope deflection equation will be used, and the sign convention is as shown in the figure below:

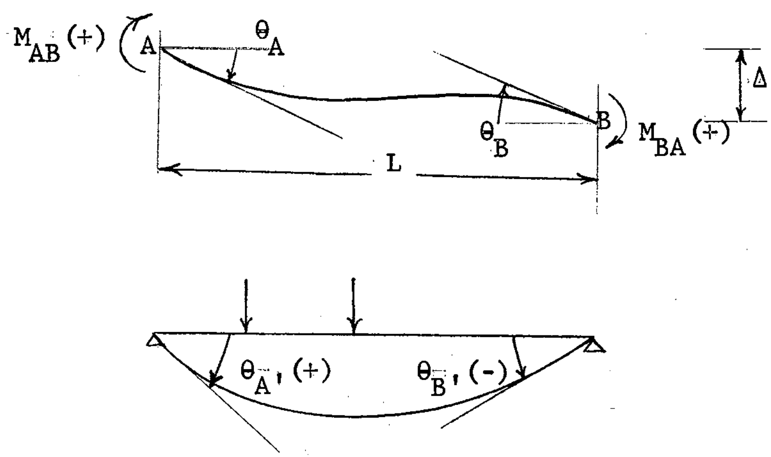


Fig. 11. Sign Convention and Nomenclature For Use In Slope-Deflection Equations

$$\theta_{AB} = \theta_{AB'} + \frac{\Delta}{L} + \frac{L}{3EI}\left(M_{AB} - \frac{M_{BA}}{2}\right)$$

where $\theta_{AB'}$ = slope at A due to a similar loading of a simply supported beam.

3-1 Procedure

(1) Obtain the ultimate load, the corresponding moment diagram, and the mechanism by plastic analysis.

(2) Compute the deflection of the various segments assuming, in turn, that each hinge is the last to form. For this:

(a) Draw free body diagram of segment.

(b) Solve slope deflection equation for the condition of continuity at the assumed last plastic hinge.

(3) Select the largest value of the deflection (this corresponding to the last plastic hinge.)

(4) Check: From the deflection calculation on the assumption as to which hinge is the last to form, compute the kinks formed due to the incorrect assumption. Remove the kinks formed due to incorrect assumption by mechanism motion and obtain the correct deflection.

3-2 Assumptions

(1) The catenary forces are neglected. These tend to decrease deflection and increase strength.

(2) The second order effects which increase deflection and decreases strength are also neglected.

(3) Any factors that influence the moment - curvature relationship are also ignored.

3-3 Case I: Deflection Analysis of a Fixed Ended Beam With a Conc. Load as Shown[3]

Let us consider the case of a beam with built-in ends with a concentrated force **P** at the middle third. The bending moment diagram has the form shown in Figure 12(b) line 1. The cross-sections of the beam have the fully plastic moments M_p. The only way this beam can collapse is by means of three yield hinges when it becomes a mechanism.

Hence the beam shall collapse when the bending moments become equal to the fully plastic moments as shown in Figure 12(b). The bending moment diagram is then shown by II.

(1) Compute the ultimate load: From the Figure 12(b) the collapse load can be calculated:

$$2M_p = \frac{2}{9} P_u L \qquad \text{or} \qquad P_u = \frac{9}{L} M_p$$

(2) The mechanism is shown in Figure 12(c).

(3) Compute the trial values of vertical deflection.

(a) Trial at Section (A): (Figure 12(d))

Slope Deflection Equation:

$$\theta_A = \theta_{A'} + \frac{\delta_{v_A}}{L} + \frac{L}{3EI} (M_{AB} - \frac{M_{BA}}{2})$$

or

$$0 = 0 + \frac{\delta_{v_A}}{L/3} + \frac{L/3}{3EI} (-M_p + \frac{M_p}{2})$$

or

$$S_{v_A} = + \frac{M_p L^2}{54EI}$$

(b) Trial at Section (B): (Figure 12(e))

Assume continuity at B so that $\theta_{BA} = \theta_{BC}$

Slope Deflection Equation:

$$\theta_{BA} = \theta_{BA'} + \frac{\delta_{v_B}}{L/3} + \frac{L/3}{3EI} (-M_p + \frac{M_p}{2})$$

or

$$\theta_{BA} = \frac{3\delta_{v_B}}{L} - \frac{M_p L}{18EI}$$

$$\theta_{BC} = \theta_{BC'} - \frac{\delta_{v_B}}{2L/3} + \frac{2L/3}{3EI} (M_p - \frac{M_p}{2})$$

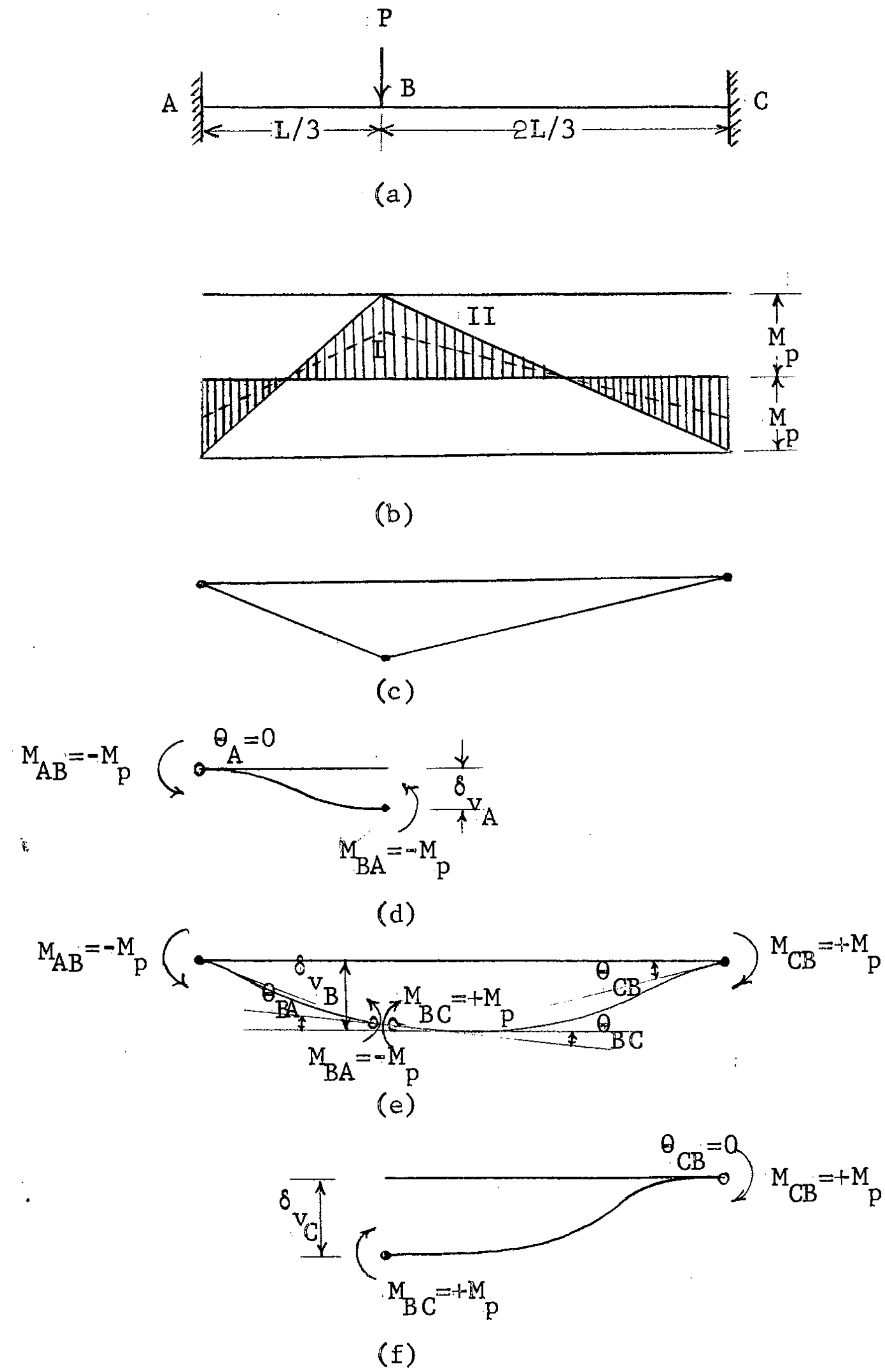


Fig. 12. Deflection Analysis of Fixed-ended Beam With Concentrated Load

or

$$\theta_{BC} = -\frac{3\delta_{v_B}}{2L} + \frac{M_pL}{9EI}$$

Now $\theta_{BA} = \theta_{BC}$

$$\therefore \quad \frac{3\delta_{v_B}}{L} - \frac{M_pL}{18EI} = -\frac{3\delta_{v_B}}{2L} + \frac{M_pL}{9EI}$$

or

$$\delta_{v_B} = +\frac{M_pL^2}{27EI}$$

(c) Trial at Section (3): (Figure 12(f))

$$\theta_{CB} = \theta_{CB'} + \frac{\delta_{v_C}}{L} + \frac{L}{3EI}\left(M_{CB} - \frac{M_{BC}}{2}\right)$$

or

$$0 = \frac{-\delta_{v_C}}{2L/3} + \frac{2L/3}{3EI}\left(M_p - \frac{M_p}{2}\right); \quad \therefore \delta_{v_C} = +\frac{2}{27}\frac{M_pL^2}{EI}$$

Of the three trials, the maximum deflection takes place when the last hinge is assumed to form at section (C). The other assumptions produce negative kinks which are not possible and produce smaller deflections.

Load Deflection Curve:

There are four distinct phases that may be uniquely determined as follows:

Phase I (O-A): (Elastic) Represents slope of deflection curve of structure (a)

Phase II (A-B): Represents slope of deflection curve of structure (b)

Phase III (B-C): Represents slope of deflection curve of cantilever structure (C)

Phase IV (C-D): Mechanism

Thus each portion of the curve represents the load deflection curve of a new structure containing one less redundant than previously.

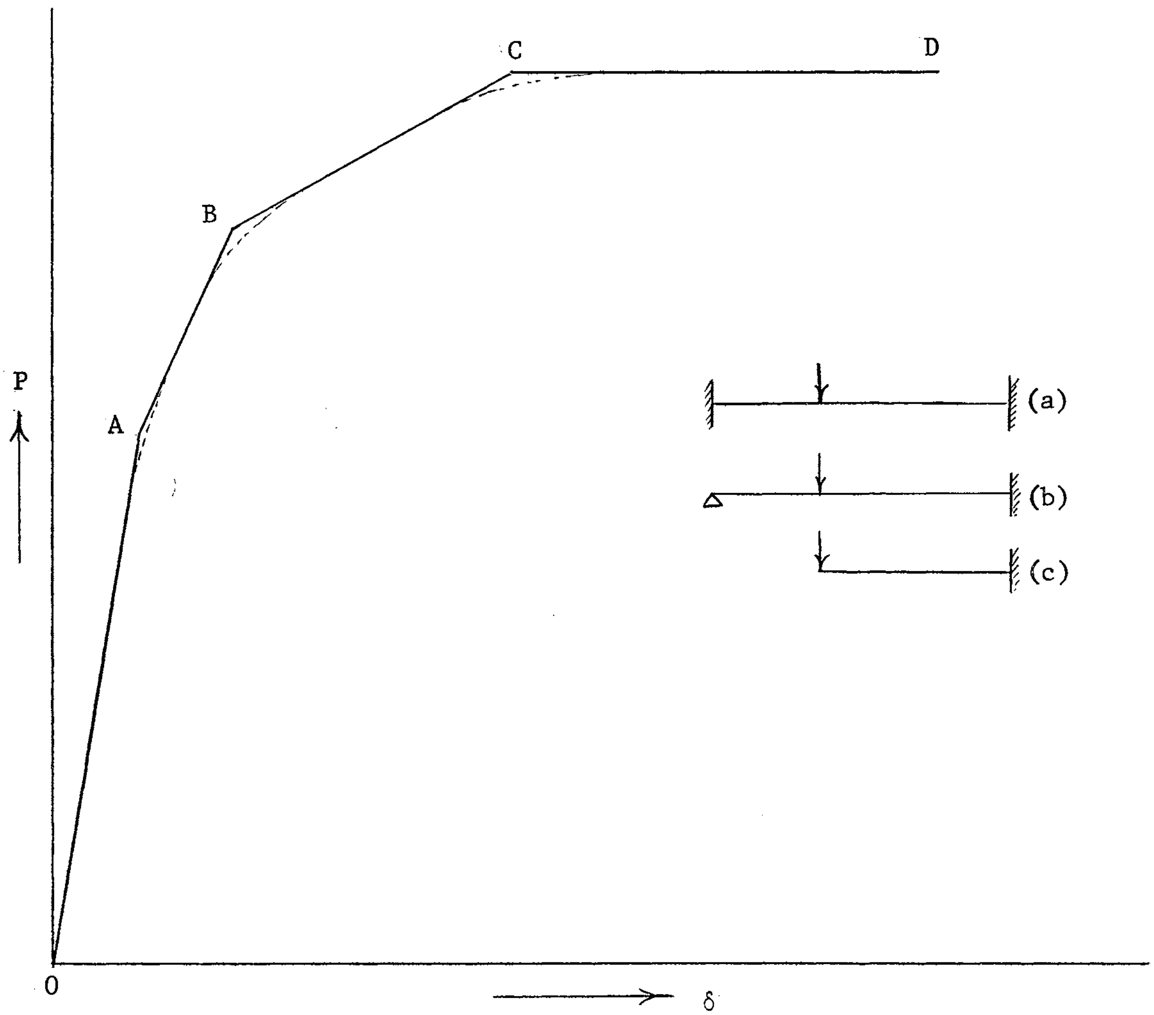


Fig. 13. Idealized Load-Deflection Curve

3-4 Deflection Analysis of Fixed Ended Uniformly Loaded Beam By Slope Deflection Method(3)

The beam with uniformly distributed load will be analyzed to determine the center deflection at ultimate load.

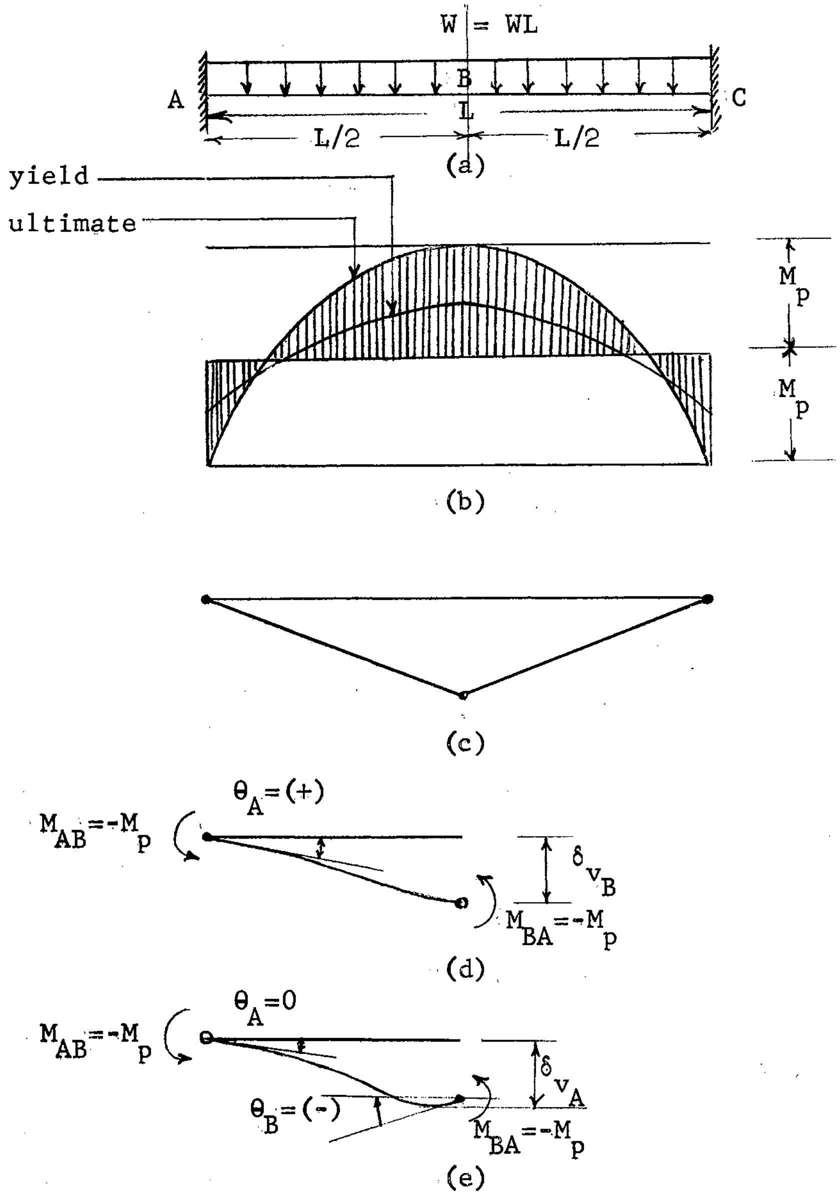


Fig. 14. Deflection Analysis of Fixed-ended Uniformly Loaded Beam

(1) Compute the ultimate load (Figure 14(b))

From the figure

$$2M_p = \frac{WL^2}{8} = \frac{W_u L}{8} \; ; \qquad \therefore W_u = \frac{16M_p}{L}$$

(2) The moment diagram and mechanism are as shown in Figures 14(b) and 14(c).

(3) Compute the trial values of vertical deflection. Looking at the problem it is quite clear that the last hinge shall form at section B. However, a trial section shall also be assumed at A and shown that this assumption is incorrect.

Trial Section at B: (Figure 14(d))

$$\theta_{BA} = \theta_{BA'} + \frac{\delta_{vB}}{L} + \frac{L}{3EI}\left(M_{BA} - \frac{M_{AB}}{2}\right)$$

$$\theta_{BA'} = \text{simple beam end rotation} = -\frac{M_p L}{12EI}$$

$$\therefore 0 = -\frac{M_p L}{12EI} + \frac{\delta_{vB}}{L/2} + \frac{L/2}{3EI}\left(-M_p + \frac{M_p}{2}\right) ; \quad \therefore \delta_{vB} = +\frac{M_p L^2}{12EI}$$

Trial Section at A: (Figure 14(e))

$$\theta_{AB} = \theta_{AB'} + \frac{\delta_{vA}}{L} + \frac{L}{3EI}\left(M_{AB} - \frac{M_{BA}}{2}\right)$$

or

$$0 = \frac{M_p L}{12EI} + \frac{\delta_{vA}}{L/2} + \frac{L/2}{3EI}\left(-M_p + \frac{M_p}{2}\right)$$

or

$$\frac{M_p L}{12EI} + 2\frac{\delta_{vA}}{L} - \frac{M_p L}{12EI} = 0 \; ; \quad \therefore \delta_{vA} = 0$$

Hence, the first trial at Section B was correct.

3-5 Comparison of the Ultimate Load W_u to the Load at First Yield W_y For a Fixed Ended Beam[3] With a Uniform Load

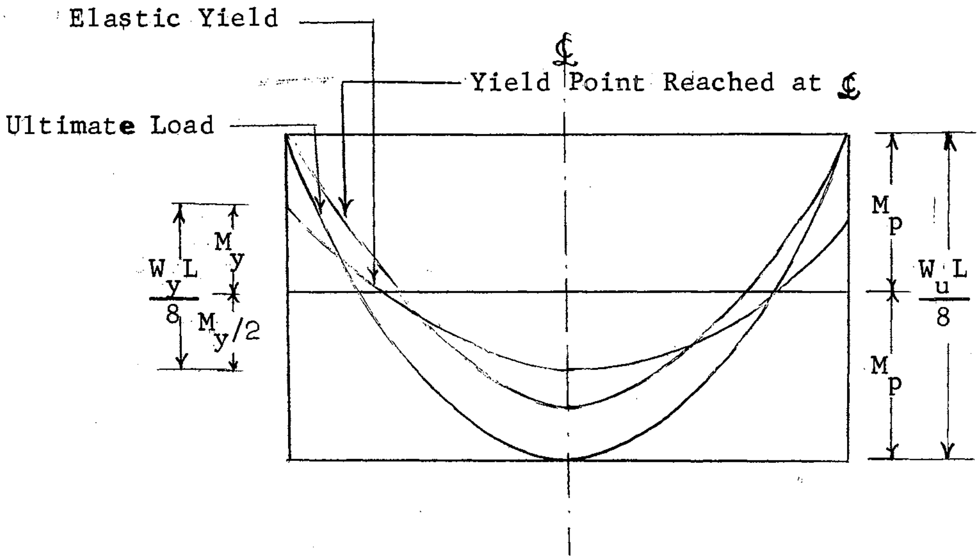


Fig. 15. Redistribution of Moments in a Fixed Ended-Beam

From the diagram above

$$\frac{W_y L}{8} = \frac{3}{2} M_y$$

or

$$W_y = \frac{12 M_y}{L}$$

$$\frac{W_u L}{8} = 2 M_p$$

or

$$W_u = \frac{16 M_p}{L}$$

$$\therefore \quad \frac{W_u}{W_y} = \frac{4}{3}\frac{M_p}{M_y} \qquad \text{where } \frac{M_p}{M_y} = \text{shape factor}$$

Thus it may be said that the reserve strength due to redistribution of moments is one-third.

Taking an average shape factor for WF sections as 1.15

$$\frac{W_u}{W_y} = \frac{4}{3} \times 1.15 = 1.53$$

Thus it may be said that the ultimate load is 53% greater than the load at first yield.

3-6 Conclusions

(1) Plastic hinges are reached first at sections subjected to greater deformations.

(2) Formation of plastic hinges allows a distribution of moment until M_p is reached at each maximum or critical section.

(3) The maximum load is attained when a mechanism forms.

3-7 Deflection at Working Load[3]

As pointed out earlier, the magnitude of the deflection is of secondary importance; a rough estimate will usually suffice. The procedure in this case is as follows:

(1) Compute the ultimate load. Divide the computed ultimate load by the load factor of safety to find the working load.

(2) Calculate the deflection at this working load.

3-8 Assumptions Made

The following assumptions were made:

(1) The effect of residual stresses was neglected.

(2) The effect of stress concentrations was neglected.

(3) The effect of gradual plastification of the cross-section was neglected.

3-9 Load v/s Deflection Curve For a Fixed End Beam With a Uniformly Distributed Load

As calculated before for this case

$$W_u = \frac{16M_p}{L}$$

$$W_y = \frac{12M_p}{L}$$

$$W_w = \frac{W_u}{F} \quad \text{where } F = \text{load factor of safety.}$$

or

$$W_w = \frac{16M_p}{FL}$$

Taking $F = 1.85$

$$W_w = \frac{16M_p}{1.85L} = 8.65\frac{M_p}{L}$$

Now $W_w < W_y$

∴ the beam is elastic.

$$\therefore \delta = \frac{WL^3}{384EI}$$

$$\delta_w = \frac{8.65M_pL^2}{384EI} = 0.0225\frac{M_pL^2}{EI}$$

Also

$$\delta_y = \frac{12M_pL^2}{384EI} = 0.03125\frac{M_pL^2}{EI}$$

Above the yield point, the slope of the curve is the same as for a simple beam.

$$\delta_{AB} = \frac{5(\Delta W)L^3}{384EI}$$

$$\Delta_w = W_u = W_y = \frac{16M_p}{L} - \frac{12M_p}{L} = \frac{4M_p}{L} = \frac{W_y}{3}$$

$$\delta_u = \delta_y + \delta_{AB}$$

$$= \frac{M_pL^2}{32EI} + \frac{5}{384}\frac{L^3}{EI}\left(\frac{4M_p}{L}\right) = 0.0833\frac{M_pL^2}{EI}$$

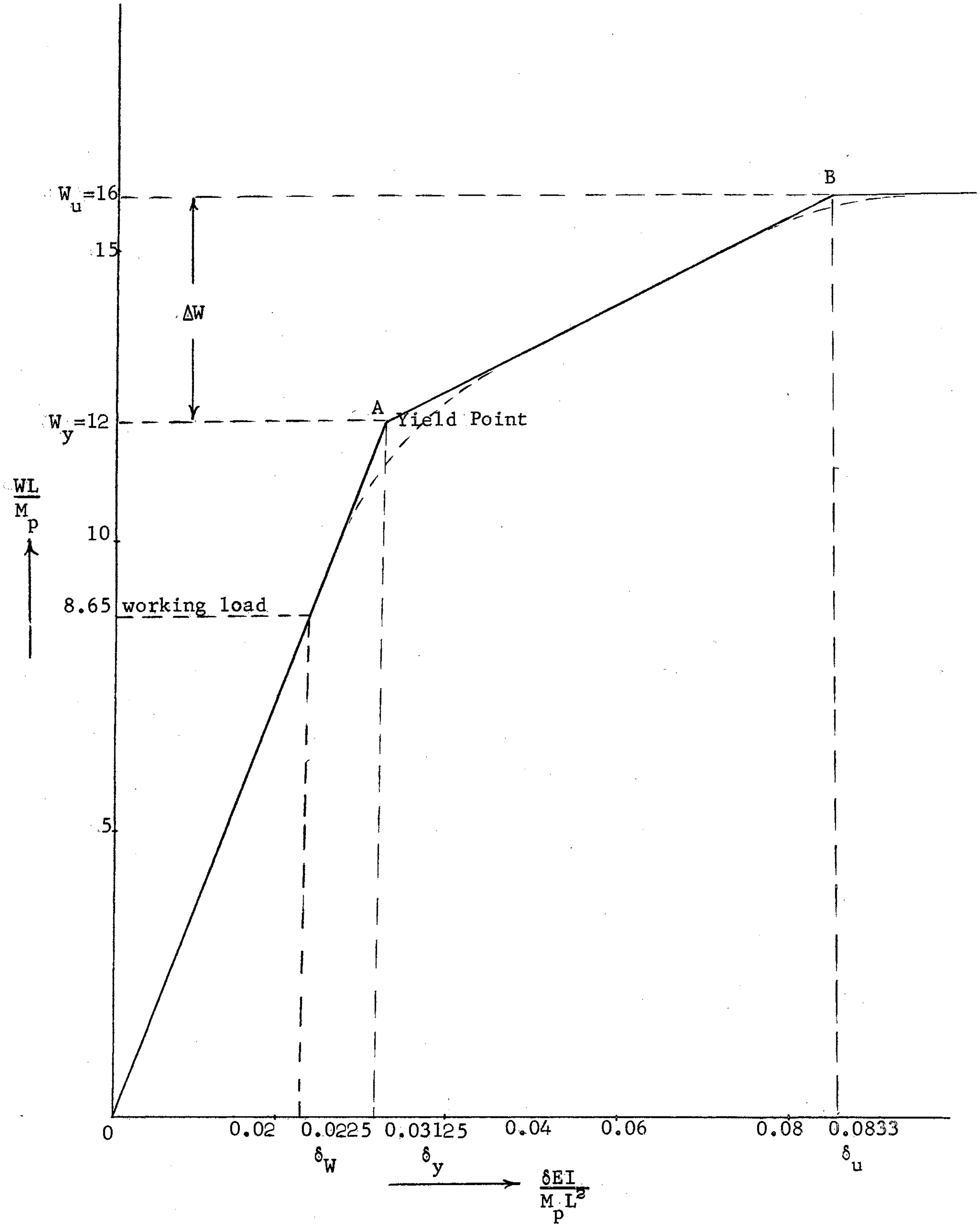


Fig. 16. Idealized Load-Deflection Curve

CHAPTER IV

DEFLECTIONS FOR SUPERIMPOSED LOADS

4-1 General

The principal of superposition does not apply to structures stressed in the plastic range. Consider the beam loaded as below.

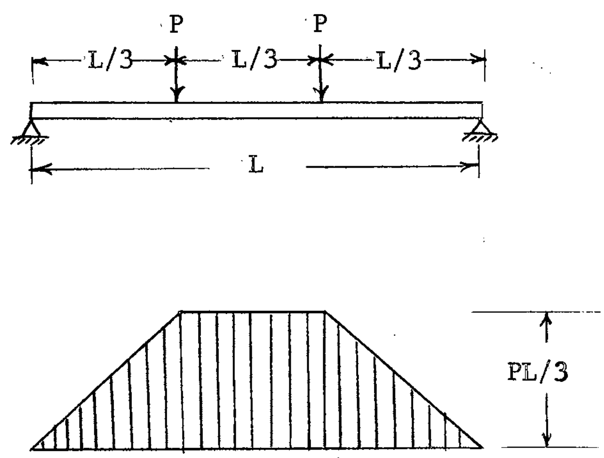


Fig. 17. Beam With Plastic Yielding

The magnitude of the loads P are such that when both the loads are applied the beam behaves plastically. However, when each of the loads P are applied separately, the beam behavior may still be in the elastic range. Hence, superposition of the effects of the two loads assumes that the beam behaves elastically under the load 2P which is not the actual case.

4-2 Beam Loaded With a Uniform Distributed Load and A Concentrated Load in the Middle

The figure below shows a simple beam loaded with a uniform-distributed load and a concentrated load in the middle. The treatment shall present a method of determining the deflection, by making use of the conjugate beam method, which has been discussed earlier.

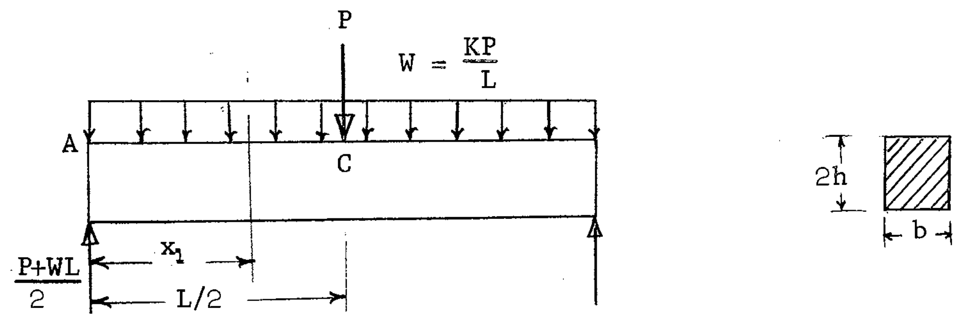


Fig. 18. Beam With Superimposed Load

$$M = \frac{WLx}{2} + \frac{Px}{2} - \frac{Wx^2}{2} - P(x - \frac{L}{2}) = (L - x)\left(\frac{Wx + P}{2}\right)$$

$$M_e = \frac{2}{3} bh^2\sigma_e$$

$$M_e = (L - x_1)\left(\frac{Wx_1 + P}{2}\right)$$

or

$$\frac{Wx_1^2}{2} - x_1\left(\frac{WL - P}{2}\right) + M_e - \frac{PL}{2} = 0$$

$$\therefore \quad x_1 = \frac{\frac{WL-P}{2} \pm \sqrt{\left(\frac{WL-P}{2}\right)^2 - 2W\left(M_e - \frac{PL}{2}\right)}}{W}$$

Now

$$K = \frac{M}{EI} = \frac{d^2z}{dx^2} = \frac{d\theta}{dx}$$

$$I_r = It = I\cdot \frac{3}{2\lambda}\left(1 - \frac{1}{3\lambda^2}\right)$$

$$= I \cdot \frac{3}{2}\sqrt{3 - 2\frac{M}{M_e}}\left[1 - \frac{1}{3}\left(3 - 2\frac{M}{M_e}\right)\right] = \frac{M}{M_e} \cdot I\sqrt{3 - 2\frac{M}{M_e}}$$

$$= \frac{(L - x)(Wx + P)}{2M_e} \; I \cdot \sqrt{3 - \frac{(Wx + P)(L - x)}{M_e}}$$

$$\frac{d\theta}{dx} = \frac{M}{EI}\Big|_{x = 0 \to x_1} + \frac{M}{EI_r}\Big|_{x = x_1 \to \frac{L}{2}}$$

$$\therefore \; \theta_A = \frac{1}{EI}\int_0^{x_1} M dx + \int_{x_1}^{L/2} \frac{(L - x(Wx + P)\; dx}{2E\frac{(L - x)(Wx + P)}{2}\frac{I}{M_e}\sqrt{3 - \frac{(Wx + P)(L - x)}{M_e}}}$$

$$= \frac{1}{2EI}\int_0^{x_1}(L - x)(Wx + P)\; dx + \frac{M_e}{EI}\int_{x_1}^{L/2}\frac{dx}{\sqrt{3 - \frac{(Wx + P)(L - x)}{M_e}}}$$

Integrating the two parts separately

$$\frac{1}{2EI}\int_0^{x_1}(L - x)(Wx + P)dx = \frac{x_1}{12EI}\left[Wx_1(3L - 2x_1) + 3P(2L - x_1)\right]$$

$$\int_{x_1}^{L/2}\frac{dx}{\sqrt{3 - \frac{(Wx + P)(L - x)}{M_e}}}$$

$$= \sqrt{\frac{M_e}{W}}\log\left[2\left\{\frac{W}{M_e}\left[\frac{W}{M_e}\cdot\frac{L^2}{4} + \left(\frac{P-WL}{M_e}\right)\frac{L}{2} + 3 - \frac{PL}{M_e}\right]\right\}^{\frac{1}{2}} + \frac{WL}{M_e} + \frac{P - WL}{M_e}\right]$$

$$- \sqrt{\frac{M_e}{W}}\log\left[2\left\{\frac{W}{M_e}\left[\frac{W}{M_e}x_1^2 + \left(\frac{P - WL}{M_e}\right)x_1 + \left(3 - \frac{PL}{M_e}\right)\right]\right\}^{\frac{1}{2}} + \frac{2W}{M_e}x_1 + \frac{P-WL}{M_e}\right]$$

$$= \sqrt{\frac{M_e}{W}}\log\left[2\left\{\frac{W}{M_e}\left(3 - \frac{PL}{2M_e} - \frac{WL^2}{4M_e}\right)\right\}^{\frac{1}{2}} + \frac{P}{M_e}\right]$$

$$- \sqrt{\frac{M_e}{W}}\log\left[2\left\{\frac{W}{M_e^2}\left(Wx_1^2 + Px_1 - WLx_1 - PL + 3M_e\right)\right\}^{\frac{1}{2}} - \frac{2W}{M_e}\left(\frac{L}{2} - x_1\right) + \frac{P}{M_e}\right]$$

But $Wx_1^2 + Px_1 - WLx_1 - PL = -2M_e$, $\therefore \displaystyle\int_{x_1}^{L/2}\frac{dx}{\sqrt{3 - \frac{(Wx+P)(L-x)}{M_e}}} =$

$$= \sqrt{\frac{M_e}{W}}\left[\log\left\{ 2\left[\frac{W}{M_e}\left(3 - \frac{PL}{2M_e} - \frac{WL^2}{4M_e}\right) + \frac{P}{M_e}\right] - \sqrt{\frac{M_e}{W}}\ \log\left[2\left(\frac{W}{M_e}\right)^{\frac{1}{2}} - \frac{2W}{M_e}\left(\frac{L}{2} - x_1\right) + \frac{P}{M_e}\right]\right.\right.$$

$$= \frac{M_e}{W}\ \log \frac{\left(3 - \frac{PL}{2M_e} - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}} + \frac{P}{\sqrt{W M_e}}}{1 - \sqrt{\frac{W}{M_e}}\left[\sqrt{\left(\frac{WL-P}{2W}\right)^2 - \frac{2}{W}\left(M_e - \frac{PL}{2}\right)} + \frac{P}{2W}\right] + \frac{P}{\sqrt{WM_e}}}$$

$$\therefore\ \theta_A = \frac{x_1}{12EI}\left[Wx_1(3L - 2x_1) + 3P(2L - x_1)\right]$$

$$+ \frac{M_e}{EI}\left[\sqrt{\frac{M_e}{W}}\ \log \frac{\left(3 - \frac{PL}{2M_e} - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}} + \frac{P}{\sqrt{WM_e}}}{1 - \sqrt{\frac{W}{M_e}}\left(\sqrt{\left(\frac{WL - P}{2W}\right)^2 - \frac{2}{W}\left(M_e - \frac{PL}{2}\right)} + \frac{P}{2}\right) + \frac{P}{\sqrt{WM_e}}}\right] \quad \ldots (1)$$

Deflection by Conjugate Beam Method

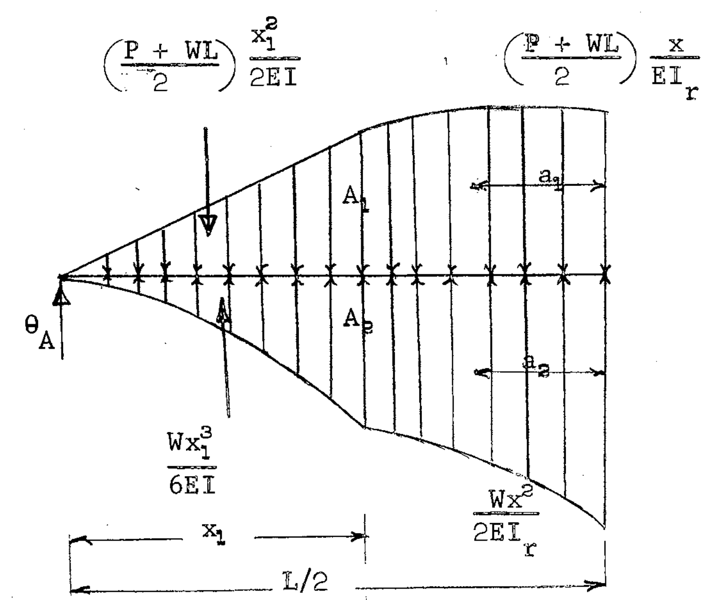


Fig. 19. Conjugate Beam

$$A_1\, a_1 = \left(\frac{P + WL}{2}\right)\frac{1}{E}\int_{x_1}^{L/2} \frac{x}{I_r}\left(\frac{L}{2} - x\right) dx$$

$$= M_e \frac{(P+WL)}{2E} \int_{x_1}^{L/2} \frac{x(L-2x)\ dx}{(L-x)(Wx+P)\sqrt{3 - \frac{(Wx+P)(L-x)}{M_e}}}$$

Breaking into Partial fractions

$$\frac{x(L-2x)}{(L-x)(Wx+P)\sqrt{3 - \frac{(Wx+P)(L-x)}{M_e}}} = \frac{A}{(L-x)} + \frac{B}{(Wx+P)} + \frac{C}{\sqrt{3 - \frac{(Wx+P)(L-x)}{M_e}}}$$

where

$$A = -\frac{L^2}{\sqrt{3}(WL+P)}$$

$$B = -\frac{P}{\sqrt{3}W} \cdot \frac{\left(L + \frac{2P}{W}\right)}{\left(L + \frac{P}{W}\right)}$$

$$C = \frac{-Lx + 2x^2}{Wx^2 + x(P-WL) + PL}$$

where

$$x = \frac{(WL - P) \pm \sqrt{(WL+P)^2 - 12WM_e}}{2W}$$

$$\therefore \quad \frac{M_e(P+WL)}{2E} \int_{x_1}^{L/2} \frac{x(L-2x)\,dx}{(L-x)(Wx+P)\sqrt{3 - \frac{(Wx+P)(L-x)}{M_e}}}$$

$$= \frac{M_e(P+WL)}{2E}\left[-A \log \frac{L/2}{L-x_1} + B\frac{1}{W} \log \frac{\frac{WL}{2} + P}{Wx_1 + P} + C \cdot \frac{M_e}{W} \log \frac{\left(3 - \frac{PL}{2M_e} - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}} + \frac{P}{\sqrt{WM_e}}}{1 - \sqrt{\frac{W}{M_e}}\left[\sqrt{\left(\frac{WL-P}{2W}\right)^2 - \frac{2}{W}\left(M_e - \frac{PL}{2}\right)} + \frac{P}{2W}\right] + \frac{P}{\sqrt{WM_e}}}\right]$$

where A, B, and C are constants defined as above.

$$A_2 a_2 = \int_{x_1}^{L/2} \frac{Wx^2}{2EI_r}(L/2 - x)\,dx = \frac{M_e W}{2EI} \int_{x_1}^{L/2} \frac{x^2(L-2x)\,dx}{(L-x)(Wx+P)\sqrt{3 - \frac{(Wx+P)(L-x)}{M_e}}}$$

Breaking into partial fractions

$$\frac{x^2(L-2x)}{(L-x)(Wx+P)\sqrt{3-\frac{(Wx+P)(L-x)}{M_e}}} = \frac{D}{(L-x)} + \frac{E}{(Wx+P)} + \frac{F}{\sqrt{3-\frac{(Wx+P)(L-x)}{M_e}}}$$

where

$$D = -\frac{L^3}{\sqrt{3}\ (WL+P)}$$

$$E = \frac{P^2}{\sqrt{3}W^2} \cdot \frac{\left(L + \frac{2P}{W}\right)}{\left(L + \frac{P}{W}\right)}$$

$$F = \frac{-Lx^2 + 2x^3}{Wx^2 + x(P-WL) + PL}$$

where

$$x = \frac{(WL - P) \pm\sqrt{(WL + P)^2 - 12WM_e}}{2W}$$

$$\therefore \quad \frac{M_e W}{2EI}\int_{x_1}^{L/2} \frac{x^2(L-2x)\,dx}{(L-x)(Wx+P)\sqrt{3-\frac{(Wx+P)(L-x)}{M_e}}} = \frac{M_e W}{2EI}\left[-D \log \frac{L/2}{L-x_1} + E\cdot\frac{1}{W} \log\right.$$

$$\frac{\frac{WL}{2}+P}{Wx_1+P} + F\cdot\frac{M_e}{W} \log \frac{\left(3-\frac{PL}{2M_e}-\frac{WL^2}{4M_e}\right)^{\frac{1}{2}} + \frac{P}{\sqrt{WM_e}}}{1-\sqrt{\frac{W}{M_e}}\left[\sqrt{\left(\frac{WL-P}{2W}\right)^2 - \frac{2}{W}\left(M_e - \frac{PL}{2}\right)} + \frac{P}{2W}\right] + \frac{P}{\sqrt{WM_e}}}$$

$$\therefore \quad \delta_c = \theta_A \cdot \frac{L}{2} + \frac{Wx_1^3}{6EI}\left(\frac{L}{2} - \frac{3}{4}x_1\right) + \frac{M_e W}{2EI}\left[-D \log\frac{L/2}{L-x_1} + E\cdot\frac{1}{W} \log \frac{\frac{WL}{2}+P}{Wx_1+P}\right.$$

$$+ F\cdot\frac{M_e}{W} \log \frac{\left(3-\frac{PL}{2M_e}-\frac{WL^2}{4M_e}\right)^{\frac{1}{2}} + \frac{P}{\sqrt{WM_e}}}{1-\sqrt{\frac{W}{M_e}}\left[\sqrt{\left(\frac{WL-P}{2W}\right)^2 - \frac{2}{W}\left(M_e - \frac{PL}{2}\right)} + \frac{P}{2W}\right] + \frac{P}{\sqrt{WM_e}}}$$

$$-\left(\frac{P+WL}{2}\right)\frac{x_1^2}{2EI}\left(\frac{L}{2} - \frac{2}{3}x_1\right) - \frac{M_e(P+WL)}{2E}\left[-A \log\frac{L/2}{L-x_1} + B\cdot\frac{1}{W} \log \frac{\frac{WL}{2}+P}{Wx_1+P}\right.$$

$$+ C \cdot \frac{M_e}{W} \log \frac{\left(3 - \frac{PL}{2M_e} - \frac{WL^2}{4M_e}\right)^{\frac{1}{2}} + \frac{P}{\sqrt{WM_e}}}{1 - \sqrt{\frac{W}{M_e}} \left[\sqrt{\left(\frac{WL+P}{2W}\right)^2 - \frac{2}{W}\left(M_e - \frac{PL}{2}\right)} + \frac{P}{2W}\right] + \frac{P}{\sqrt{WM_e}}} \Bigg]$$

where θ_A is given as in equation (1) and A, B, C, D, E and F are constants as derived on the previous pages.

CHAPTER V

SUMMARY AND CONCLUSIONS

Of the several methods available for determining the deflections of simple beams, the methods of double integration and area moments are better understood and more easily employed than any of the others. The area moment method usually gives a solution more readily than any other method.

In cases where the beam is symmetrically loaded and supported, and where it is required to find the maximum deflection, the use of double integration method yields a solution almost as easily as the area moment method. In general, it may be said that the difficulty of solving a problem by the double integration method is proportional to the difficulty of calculating the constants of integration.

There is actually very little difference between the area moment and conjugate beam methods. However, for some students the conjugate beam method may be easily remembered as it is a common everyday operation — that of finding the moment at a given section of a beam.

The procedure in Chapter III may serve as a method for computing the approximate magnitude of the deflection at ultimate load and for obtaining an upper limit to the deflection at working load.

In short, it may be said that the area moment method is the one most readily used in the greatest variety of conditions. Also it is easily recalled and, therefore, can be employed for occasional use.

APPENDIX

MOMENT-CURVATURE RELATIONSHIP[4]

In the figure below is shown a symmetric cross-section, a strain distribution and the corresponding stress distribution.

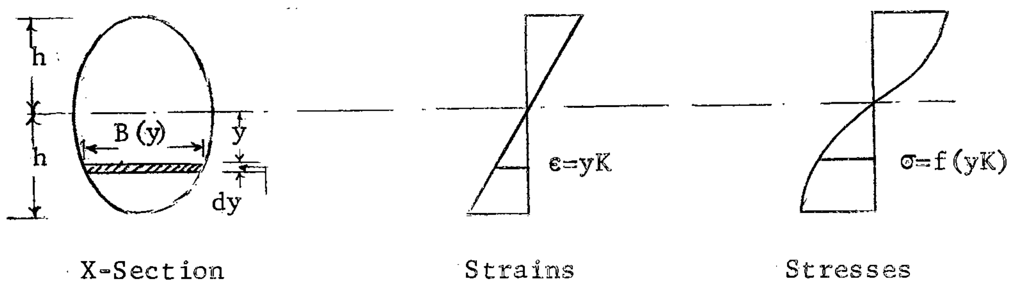


Fig. 20. General Stress and Strain Distribution

Let K = curvature

ρ = radius of curvature

y = distance from the neutral axis of the fibre with strain ϵ.

$\sigma = f(\epsilon)$ the stress-strain relation.

Then $\epsilon = yK = y/\rho$ . . . . (1)

$\sigma = f(\epsilon) = f(yK)$

The quantities ϵ, y, K and ρ are taken as positive*

Let B(y) = width of the x-section at a distance y from the neutral axis

---

* This differs from the signs adopted in strength of materials that a positive strain is tensile and a negative strain compressive.

Then the force acting on the area B(y)dy is f(yK)B(y)dy.

Statical moment of this force about the neutral axis = f(yK)B(y)ydy

∴ Statical moment of the stresses acting on the lower half of the cross-section about the N.A. is

$$\int_{y=0}^{y=h} f(yK)B(y)ydy$$

From symmetry

$$M = 2\int_{y=o}^{y=h} f(yK)B(y)ydy \quad \ldots\ldots\ldots\ldots \quad (2)$$

In case Hooke's Law governs

$$\sigma = E\epsilon$$

Then from equation (1) $\sigma = EyK$

$$\therefore \quad M = 2\int_{y=0}^{y=h} EyKBydy$$

or

$$M = 2EK\left|\frac{By^3}{3}\right|_0^h$$

or

$$M = \frac{2}{3}EKBh^3$$

or

$$M = EIK \qquad \text{Where } I = \text{moment of inertia of the cross-section} = \frac{2}{3}Bh^3$$

Let us now assume that the material of the beam is perfectly plastic with the same stress-strain curve in tension and compression.

In the following figure the entire cross-section is elastic and the moment-curvature relationship is

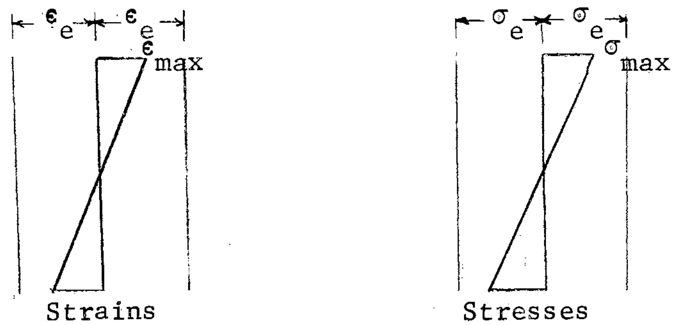


Fig. 21. Stress and Strain Distribution

$M = EIK$

When $\epsilon_{max} = \epsilon_e$

$$M_e = EIK_e$$

However, if $\epsilon_{max} > \epsilon_e$ as below, the elastic part of the cross-section lies between the two fibers while the rest of the cross-section is plastic.

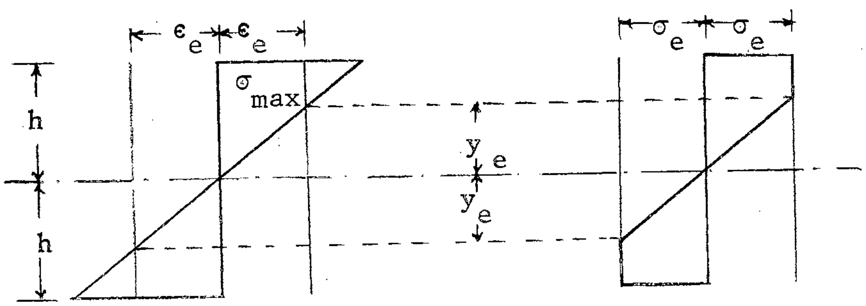


Fig. 22. Stress and Strain Distribution

$0 \le y \le y_e \qquad \sigma = E\epsilon = EyK$

$y_e \le y \le h \qquad \sigma = \sigma_e = E\epsilon_e = Ey_eK$

Substituting this in equation (2)

$$M = 2\int_{y=0}^{y=y_e} EKB(y)y^2\,dy + 2\int_{y=y_e}^{y=h} EKy_e B(y)y\,dy$$

or

$$M = 2EK\int_{y=0}^{y=y_e} y^2 B(y)\,dy + 2EKy_e\int_{y=y_e}^{y=h} yB(y)\,dy = 2EK(I_e + y_e S_p)$$

where $I_e = \int_0^{y_e} y^2 B(y)\,dy$ and $S_p = \int_{y_e}^{h} yB(y)\,dy$

or

$$M = EKI_r$$

where $I_r = 2I_e + 2y_e S_p$

= reduced moment of inertia.

From the above equations

$M_e = EIK_e$ and

$M = EI_r K$

$$\frac{M}{M_e} = \frac{I_r}{I} \cdot \frac{K}{K_e}$$

Introducing $\lambda = K/K_e$

$t = I_r/I$

$$\frac{M}{M_e} = t \cdot \lambda$$

The value of M for $\lambda \to \infty$ is called the fully plastic moment and denoted by M_p.

The ratio $\frac{M_p}{M_e} = S$ is called the shape factor.

In Figure 23 we see the M/M_e, λ lines for various cross-sections.

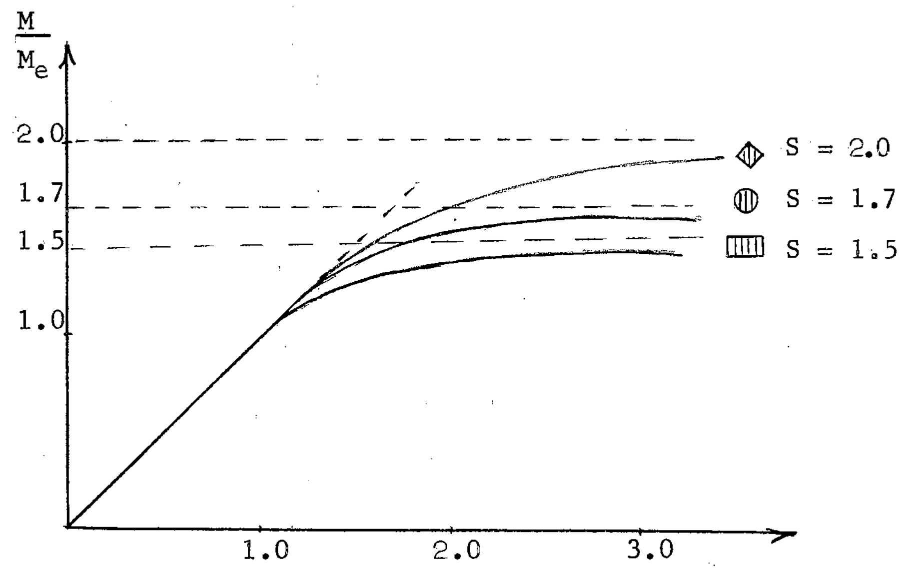
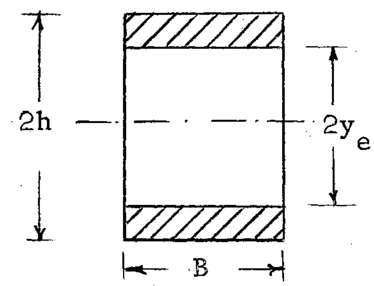


Fig. 23. Shape Factor for Various Cross-Sectional Forms

They are straight making an angle of 45° with the λ-axis for all values of $\lambda \leq 1$. For other values of λ the line is a curved one with a horizontal asymptote.

Below are given the expressions for t, M/M_e, λ, etc. for various cross-sections.[4]



$$t = \frac{3}{2\lambda}\left(1 - \frac{1}{3\lambda^2}\right)$$

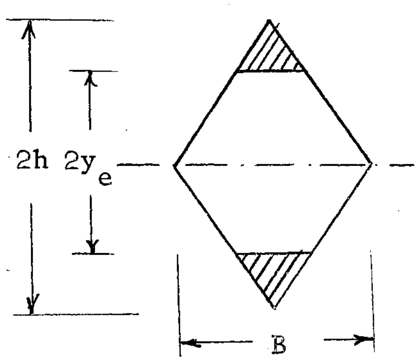
$$\frac{M}{M_e} = \frac{3}{2}\left(1 - \frac{1}{3\lambda^2}\right)$$

$$\lambda = \frac{1}{\sqrt{3 - 2\frac{M}{M_e}}}$$

$$M_e = \frac{2Bh^2\sigma_e}{3}$$

$$M_p = Bh^2\sigma_e$$

$$S = \frac{M_p}{M_e} = 1.5$$



$$t = \frac{2\lambda^3 - 2\lambda + 1}{\lambda^4}$$

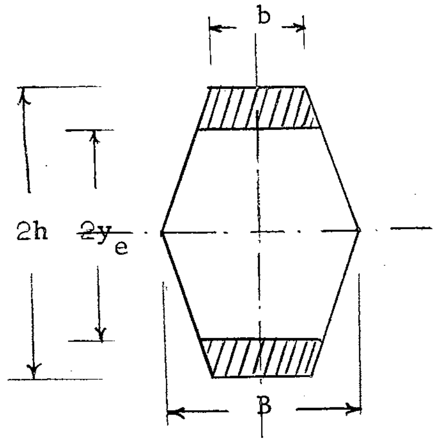
$$\frac{M}{M_e} = \frac{2\lambda^3 - 2\lambda + 1}{\lambda^3}$$

$$= 2 - \frac{2}{\lambda^2} + \frac{1}{\lambda^3}$$

$$M_e = \frac{Bh^2}{6}\sigma_e$$

$$M_p = \frac{Bh^2}{3}\sigma_e$$

$$S = \frac{M_p}{M_e} = 2$$



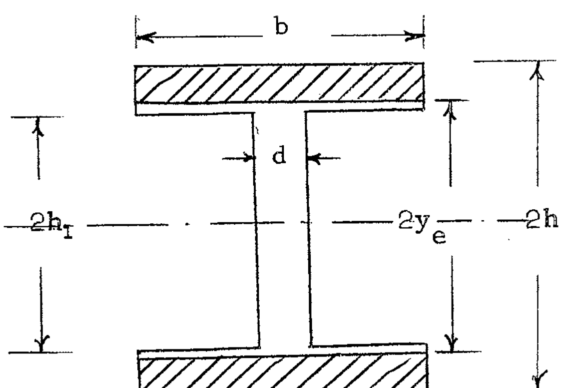
$$t = \frac{2B\lambda(3\lambda^2-1)-(B-b)(4\lambda^3-1)}{\lambda^4(B+3b)}$$

$$\frac{M}{M_e} = \frac{2B\lambda(3\lambda^2-1)-(B-b)(4\lambda^3-1)}{\lambda^3(B+3b)}$$

$$M_e = \frac{(B+3b)h^2}{6}\,\sigma_e$$

$$M_p = \frac{(B+2b)h^2}{3}\,\sigma_e$$

$$S = \frac{2(B+2b)}{(B+3b)}$$



$y_e \geq h_1$

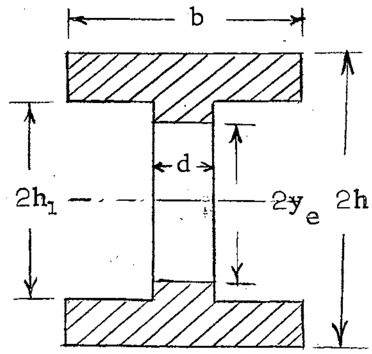
$$t = \frac{(3\lambda^2-1)-2\lambda^3\,\frac{h_1^3}{h^3}\left(1-\frac{d}{b}\right)}{2\lambda^3\left[1-\frac{h_1^3}{h^3}\left(1-\frac{d}{b}\right)\right]}$$

$$\frac{M}{M_e} = \frac{(3\lambda^2-1)-2\lambda^3\frac{h_1^3}{h^3}\left(1-\frac{d}{b}\right)}{2\lambda^2\left[1-\frac{h_1^3}{h^3}\left(1-\frac{d}{b}\right)\right]}$$

$$\lambda = \frac{1}{\sqrt{1-\frac{M}{bh^2\sigma_e}}\left(\cos\frac{\eta}{3}+3\sin\frac{\eta}{3}\right)}$$

$$\text{where } \eta = \text{arc cos}\,\frac{\frac{h_1^3}{h^3}\left(1-\frac{d}{b}\right)}{\sqrt{\left(1-\frac{M}{bh^2\sigma_e}\right)^3}}$$

$$M_e = \frac{2}{3}\left[bh^3-(b-d)h_1^3\right]\frac{\sigma_e}{h}$$



$y_e \leq h_1$

$$t = \frac{3\lambda^2\left[1 - \frac{h_1^3}{h^2}\left(1 - \frac{d}{b}\right)\right] - \frac{d}{b}}{2\lambda^3\left[1 - \frac{h_1^3}{h^3}\left(1 - \frac{d}{b}\right)\right]}$$

$$\frac{M}{M_e} = \frac{3\lambda^2\left[1 - \frac{h_1^3}{h^3}\left(1 - \frac{d}{b}\right)\right] - \frac{d}{b}}{2\lambda^2\left[1 - \frac{h_1^3}{h^3}\left(1 - \frac{d}{b}\right)\right]}$$

$$\lambda = \frac{1}{\sqrt{3}} \frac{\sqrt{\frac{d}{b}}}{\sqrt{\left[1 - \frac{h_1^2}{h^2}\left(1 - \frac{d}{b}\right)\right] - \frac{M}{bh^2\sigma_e}}}$$

$$M_p = \left[bh^2 - (b - d)h_1^2\right]\sigma_e$$

$$S = \frac{3\left[1 - \frac{h_1^2}{h^2}\left(1 - \frac{d}{b}\right)\right]}{2\left[1 - \frac{h_1^3}{h^2}\left(1 - \frac{d}{b}\right)\right]}$$

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