AN INVESTIGATION OF CORRELATION BETWEEN OBSERVATIONS IN THE LATIN SQUARE AND THE BALANCED INCOMPLETE BLOCK DESIGNS

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Preface

In the latin square and balanced incomplete block designs one is frequently interested in the effects of within block (row or column) correlation. The conventional analysis is set up under the assumption that the observations are mutually independent.

This thesis shows that the conventional analysis of variance technique holds for equal within block variances as well as an overall 'mean' correlation. The usual exact test of significance holds. A slightly different technique is needed if the within block variances are unequal between blocks, although the test of significance is still exact. An investigation is conducted on possible limitations on these within block variances, and the variance of a contrast is found.

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CHAPTER I

USUAL MODEL AND PREVIOUS EXTENSIONS

The analysis of variance procedures, developed by Fisher, are widely used in analysis of experiments. Therefore the basic requirements are considered when the experiment is designed. These include additivity of effects and uncorrelated treatment and plot errors which are distributed normally and have a constant variance. This may also be stated as

$$E\ (e_{ijk})^2 = \sigma^2$$

$$E\ (e_{ijk}\ \ e_{i'j'k'}) = 0 \qquad (i,\ j,\ k \text{ not all equal to } i',j',k')$$

Dr. F. A. Graybill (1957, 1958) has investigated conditions for which the analysis of variance procedure will be valid. One example has constant variance

$$E\ (e_i)^2 = \sigma^2$$

and constant covariance

$$E\ (e_i e_j) = p\sigma^2 \qquad (i \neq j).$$

He has also shown that the randomized blocks analysis is valid if the observations, y_{ij}, have constant within block variance

$$E\ (e_{ij})^2 = k_i\sigma^2,$$

if the covariance of any two y_{ij} within a block is constant (not necessarily zero)

$$E\ (e_{ij}e_{ij'}) = pk_i\sigma^2 \qquad (j \neq j'),$$

and the y_{ij} are independent if they are in different blocks

$$E(e_{ij}e_{i'j'}) = 0.$$

These variances may differ from block to block. The purpose of this thesis is to apply Dr. Graybill's theorems to the Latin square and balanced incomplete block designs.

Notation

In general, the notation used in Kempthorne's book and Graybill's papers will be used. A model $y_{ijk} = \mu + t_i + b_j + c_k + e_{ijk}$ where $i = 1, \ldots, t$ or n; $k = 1, \ldots, c$; $j = 1, \ldots, b$; will have the e_{ijk} distributed normally with mean μ and variance-covariance matrix V. The null matrix, φ, will be a matrix or vector with elements all zero. The matrices denoted by V, A_1, A_2, $\ldots$, will be symmetric and their dimension will be denoted by $n \times n$, $1 \times n$, $t \times t$, etc. The observation vector, Y, $(n \times 1)$, will have elements y_{ijk}, and the order of the elements y_{ijk} in Y will usually be obvious. $Y_{\cdot\cdot\cdot}$ will indicate that the y_{ijk} have been summed over i, j, and k and $y_{\cdot\cdot\cdot}$ will indicate that the observations have been summed and averaged over the indices.

The statement, w is distributed $N(\mu,V)$, will mean w is distributed as the multivariate normal with vector mean μ and variance-covariance matrix V. The statement "W is distributed as $x'^2(\lambda,n)$" will mean that the random variable w is distributed as the non-central chi-square with non-centrality equal to λ and degrees of freedom n. The statement "w is distributed $x^2(n)$" will mean w is distributed as the central chi-square with n degrees of freedom.

Necessary Theorems

Theorem 1. If Y is distributed as $N_n(\mu,V)$ where V is an $n \times n$ positive definite symmetric matrix, and if $Y'AY = \sum_{i=1}^{k} Y'A_iY$ where the rank of A_i is p_i and the rank of A is p, then any one of the six conditions, C_1, C_2, C_3, C_4, C_5, C_6, is necessary and sufficient that the $Y'A_iY$ be independently distributed as $x'^2(p_i, \lambda)$ where $\lambda_i = \frac{1}{2}\mu'A_i\mu$.

C_1: AV be idempotent and $\sum_{i=1}^{k} p_i = p$.

C_2: AV and each A_iV be idempotent.

C_3: AV be idempotent and $A_iVA_j = \varphi$ for all $i \neq j$.

C_4: $Y'AY$ be distributed as $x'^2(p, \lambda)$ and $p = \sum_{i=1}^{k} p_i$ $(\lambda = \frac{1}{2}\mu'A\mu)$.

C_5: $Y'AY$ be distributed as $x'^2(p,\lambda)$ and A_iV be idempotent $(\lambda = \frac{1}{2}\mu'A\mu)$.

C_6: $Y'AY$ be distributed as $x'^2(p,\lambda)$ and $A_iVA_j = \varphi$ for $i \neq j$, $(\lambda = \frac{1}{2}\mu'A\mu)$.

The above theorem is due to Graybill (1957).

Theorem 2. If y is distributed $N_n(\mu,V)$ where $V = k(A_4 + A_5) + k_1A_1 + k_2A_2 + k_3A_3$ and if $\Sigma A_i = I$ and $\Sigma p_i = p$, where $k_i > 0$ and p_i is the rank of A_i, and if $A_iA_j = \varphi$ $(i \neq j)$ and $A_iA_i' = A_i$ for all i and A_i symmetric, then the conditions of theorem 1 are satisfied except that some of the factors are multiplied by a non-zero constant. (Some of the above conditions are implied by others, but for convenience, they are all listed.)

The above theorem and the following proof is similar to a theorem by Graybill (1958).

Proof. By Theorem 1 above, we know that

(a) $Y'A_4Y \sim k \cdot x'^2(\lambda^*, p_4)$ iff $\frac{A_4V}{k}$ is idempotent. From above, $\frac{A_4V}{k} = \frac{kA_4}{k} = A_4$ and A_4 is idempotent. $(\lambda^* = \frac{\sigma^2\mu' A_4 \mu}{2})$

(b) $Y'A_5Y$ is distributed as $k \cdot x^2(p_3)$ iff $\frac{A_5V}{k}$ is idempotent.

$$\frac{A_5V}{k} = \frac{kA_5}{k} = A_5 \qquad \text{and} \quad A_5' \, A_5 = A_5,$$

$\lambda = 0$ since $\frac{\mu' A_5 \mu}{2k} = 0$.

The conditions k, $k_i > 0$ insure that V is a positive definite matrix and thus qualifies for a covariance matrix.

Since the blocks, columns and treatments are mutually orthogonal, the expectations of the mean squares may be obtained by using Theorem 1. However, due to the constant factors multiplying the A_i, the results are altered slightly. If

$$Y'A_iY \sim x'^2(p_i, \lambda_i)$$

where p_i is the degrees of freedom associated with each A_i and $\lambda_i = \frac{\mu' A_i \mu}{2}$; then

$$k_i Y'A_iY \sim k_i x'^2(p_i, \lambda_i).$$

For A_5, $\frac{\mu' A_5 \mu}{2} = \varphi$, so $k \; Y'A_5Y \sim k \; x^2(p_5)$.

Analysis of Variance Table

Source	D. F.	Sum of Squares	Mean Square	Expectation
total	p	$Y'Y$		
mean	p_1	$Y'A_1Y$	$(S.\ S.)/p_1$	
blocks	p_2	$Y'A_2Y$	$(S.\ S.)/p_2$	$k_2x'^2(p_2, \lambda_2)$
columns	p_3	$y'A_3Y$	$(S.\ S.)/p_3$	$k_3x'^2(p_3, \lambda_3)$
treatments	p_4	$Y'A_4Y$	$(S.\ S.)/p_4$	$k\ x'^2(p_4, \lambda_4)$
error	p_5	$Y'A_5Y$	$(S.\ S.)/p_5$	$k\ x'^2(p_5, \lambda_5)$

Then (M. S. treatments)/(M. S. error) $\sim F'(\lambda_4, n_4, p_5)$.

CHAPTER II

LATIN SQUARE MODEL

Given a Latin Square with additive model

$$y_{ijk} = \mu + r_i + c_j + t_k + e_{ijk} \qquad (i, j, k = 1, 2, \ldots, t)$$

each j, k appears once in each row and each i, k appears once in each column. From the identity

$$y_{ijk} = y_{\cdot\cdot\cdot} + (y_{i\cdot\cdot} - y_{\cdot\cdot\cdot}) + (y_{\cdot j\cdot} - y_{\cdot\cdot\cdot}) + (y_{\cdot\cdot k} - y_{\cdot\cdot\cdot}) +$$

$$(y_{ijk} - y_{i\cdot\cdot} - y_{\cdot j\cdot} - y_{\cdot\cdot k} + 2y_{\cdot\cdot\cdot})$$

the sums of squares estimates may be obtained.

Total S.S. = Mean S.S.+ Row S.S. + Column S. S. +

$$\sum_{i,j=1}^{t} y_{ijk}^2 = \frac{(Y_{\cdot\cdot\cdot})^2}{t^2} + \sum_{i=1}^{t} \frac{(Y_{i\cdot\cdot})^2}{t} - \frac{(Y_{\cdot\cdot\cdot})^2}{t^2} + \sum_{j=1}^{t} \frac{(Y_{\cdot j\cdot})^2}{t} - \frac{(Y_{\cdot\cdot\cdot})^2}{t^2} +$$

Treatment S. S. + Error S. S.

$$\sum_{k=1}^{t} \frac{(Y_{\cdot\cdot k})^2}{t} - \frac{(Y_{\cdot\cdot\cdot})^2}{t^2} + \sum_{i,j,k=1}^{t} y_{ijk}^2 - \sum_{i=1}^{t} \frac{(Y_{i\cdot\cdot})^2}{t} - \sum_{j=1}^{t} \frac{(Y_{\cdot j\cdot})^2}{t} - \sum_{k=1} \frac{(Y_{\cdot\cdot k})^2}{t} +$$

$$2\,\frac{(Y_{\cdot\cdot\cdot})^2}{t^2}.$$

Sum of Squares Matrices

A general sum of squares matrix for the mean, row, and column effects is easily obtained. Because of the many latin squares available, there is no absolutely general treatment sum of squares matrix, but one can be easily constructed for each value of t and each specific square chosen.

The general notation for the sums of squares due to the mean will be $Y'A_1Y$ in matrix notation, where A_1 is $t^2 \times t^2$ and according to the Graybill (1957) notation, may be represented by $1/t^2 \; J(t^2)$, where $J(t^2)$ means a $t^2 \times t^2$ matrix of ones.

$$A_1 = \frac{1}{t^2} \begin{bmatrix} 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 \end{bmatrix}$$

The uncorrected row sum of squares is $Y'A_2Y$ where A_2 is $t^2 \times t^2$.

$$A_2 = \frac{1}{t} \begin{bmatrix} \alpha & \varphi & \varphi & \cdot & \cdot & \cdot & \varphi \\ \varphi & \alpha & \varphi & \cdot & \cdot & \cdot & \varphi \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi & \cdot & \cdot & \cdot & \cdot & \varphi & \alpha \end{bmatrix}$$

The diagonal elements α are $J(t)$ matrices and the φ matrices are $t \times t$ matrices with all elements zero. The row sum of squares matrix is $A_2^* = A_2 - A_1$.

A_3 is $t^2 \times t^2$ (as are all the A_i) and is a matrix of the form

$$\frac{1}{t} \begin{bmatrix} \delta & \delta & \cdot & \cdot & \delta \\ \delta & \delta & \cdot & \cdot & \delta \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta & \delta & \cdot & \cdot & \delta \end{bmatrix}$$

where each δ is a $t \times t$ diagonal matrix with ones on the diagonal and zeros for off-diagonal elements.

$$\delta = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

The column sum of squares matrix is $A_3^* = A_3 - A_1$.
The A_4 matrix depends on the particular arrangement of treatments in rows and columns, and $A_5^* = I - A_1^* - A_2^* - A_3^* - A_4^*$.

It is easy to show that the matrices are idempotent and that their cross products are null. The prime or transpose notation will not be used since all the matrices are symmetric.

For A_1A_1, each element reduces to $t^2(1/t^2)^2 = 1/t^2$ which is the same as A_1.

For A_2A_2, each element of the product in the $t \times t$ sub-matrices on the diagonal reduces to $t(1/t)(1/t) + (t^2 - t)\ 0 \cdot 0 = 1/t$ and the elements in the off-diagonal submatrices reduce to $t \cdot 0(1/t) + t(1/t) \cdot 0 + (t^2 - 2t) \cdot 0 \cdot 0 = 0$.

For A_3A_3, the diagonal elements in the sub-matrices reduce to $t(1/t)(1/t) + (t^2 - t) \cdot 0 \cdot 0 = 1/t$ and the off-diagonal elements of the sub-matrices δ reduce to $t(1/t) \cdot 0 + t \cdot 0(1/t) + (t^2 - 2t) \cdot 0 \cdot 0 = 0$.

Thus A_1, A_2 and A_3 are indempotent.

Before the conditions for Theorem 2 may be verified, a few more matrix products are needed.

Multiplying A_1A_2, a matrix is obtained with all elements equal to $t(1/t^2)(1/t) + (t^2 - t)(1/t^2) \cdot 0 = 1/t^2$. Due to symmetry, A_2A_1 is the same, A_1.

The products A_1A_3 and A_3A_1 also yield A_1.

Multiplying A_2A_3, elements of the form $(1/t)(1/t) + (t - 1)(1/t)\cdot 0 + (t - 1)(1/t)\cdot 0 + (t^2 - 2t + 1)\cdot 0\cdot 0 = 1/t^2$ are obtained. A_1 is also obtained from the product A_3A_2.

Rather than working with A_4^* and A_5^*, the matrix $I - A_1^* - A_2^* - A_3^* = A_4^* + A_5^*$ will be used. However, it is not necessary to discover whether or not this form of the matrix is idempotent since the form that is usually being considered is $A_4^* + A_5^* = I - A_1 - A_2^* - A_3^*$. Later A_4 and A_5 will be considered separately.

$$A_2^*A_2^* = (A_2 - A_1)(A_2 - A_1) = A_2 - A_1 - A_1 + A_1 = A_2 - A_1 = A_2^*.$$

$$A_3^*A_3^* = (A_3 - A_1)(A_3 - A_1) = A_3 - A_1 - A_1 + A_1 = A_3 - A_1 = A_3^*.$$

$$(I - A_1 - A_2^* - A_3^*)(I - A_1 - A_2^* - A_3^*) =$$

$$(I - A_2 - A_3 + A_1)(I - A_2 - A_3 + A_1) = I - A_2 - A_3 + A_1 - A_2 + A_2 + A_1 - A_1 - A_3 + A_1 + A_3 - A_1 + A_1 - A_1 - A_1 + A_1 =$$

$$I - A_2 - A_3 + A_1 = I - A_1 - A_2^* - A_3^*.$$

$$A_2^*A_1 = (A_2 - A_1)\, A_1 = A_1 - A_1 = \varphi.$$

$$A_1A_2^* = A_1(A_2 - A_1) = A_1 - A_1 = \varphi.$$

$$A_3^*A_1 = (A_3 - A_1)\, A_1 = A_1 - A_1 = \varphi.$$

$$A_1A_3^* = A_1(A_3 - A_1) = A_1 - A_1 = \varphi.$$

$$A_2^*A_3^* = (A_2 - A_1)(A_3 - A_1) = A_1 - A_1 - A_1 + A_1 = \varphi.$$

$$A_3^*A_2^* = (A_3 - A_1)(A_2 - A_1) = A_1 - A_1 - A_1 + A_1 = \varphi.$$

$$A_1(I - A_1 - A_2^* - A_3^*) = A_1(I - A_2 - A_3 + A_1) = A_1 - A_1 - A_1 + A_1 = \varphi.$$

$$(I - A_2 - A_3 + A_1)\, A_1 = A_1 - A_1 - A_1 + A_1 = \varphi.$$

$$(A_2 - A_1)(I - A_2 - A_3 + A_1) = A_2 - A_2 - A_1 + A_1 - A_1 + A_1 + A_1 - A_1 = \varphi.$$

$$(I - A_2 - A_3 + A_1)(A_3 - A_1) = A_3 - A_3 - A_1 + A_1 - A_1 + A_1 + A_1 - A_1 = \varphi.$$

Therefore all the matrices are idempotent and their cross products are null.

The Three by Three Latin Square

Since the three by three (3 x 3) latin square has only one reduced form, it will be considered for an example. No matter what the field plan was, the data can be arranged in the following form,

$$\begin{bmatrix} \text{I } (y_{11}) & \text{II } (y_{12}) & \text{III } (y_{13}) \\ \text{II } (y_{21}) & \text{III } (y_{22}) & \text{I } (y_{23}) \\ \text{III } (y_{31}) & \text{I } (y_{32}) & \text{II } (y_{33}) \end{bmatrix}$$

where I, II, III refer to the treatments and the y_{ij} denote the positions of the symbols that will be used to set up the sum of squares matrices. The nine y_{ij} may be arranged in a 9 x 1 observation vector:

$$Y' = (y_{11}\ y_{12}\ y_{13}\ y_{21}\ y_{22}\ y_{23}\ y_{31}\ y_{32}\ y_{33})$$

Although the above symbols have only two subscripts denoting row and column, a treatment effect is also present. In the three by three case, adding the third subscript would be simple, but in higher dimension latin squares this would be impossible to do for a general case since there is more than one reduced form. Therefore, in the latin square, y_{ij} will refer to an observation in the i^{th} row and the j^{th} column with the k subscript being added to indicate the treatment in the standard form.

The total sum of squares is $\sum_{ij} y_{ij}^2 = Y'Y = Y'IY$, where I is the 9 x 9 identity matrix.

$$I = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

The sum of squares for mean is $(Y_{\circ\circ})^2/9 = (Y'J(9)Y)/9$ where $J(9)$ is a 9×9 matrix consisting of ones.

$$Y' \frac{J(9)}{9} Y = (1/9)\, Y' \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \end{bmatrix} Y = Y'A_1^{*}Y = Y'A_1Y.$$

The row sum of squares is $\sum_i (Y_{i\circ})^2/3 - (Y_{\circ\circ})^2/9$, but as in the general case, $\sum_i (Y_{i\circ})^2$ will be considered first separately:

$$Y'A_2\, Y = (1/3)Y' \begin{bmatrix} \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} & \varphi & \varphi \\ \varphi & \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} & \varphi \\ \varphi & \varphi & \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} \end{bmatrix} Y$$

where $\varphi = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$.

The column sum of squares matrix is $A_3 - A_1$. $\sum_j (Y_{\circ j\circ})^2/3 = Y'A_3\, Y.$

$$A_3 = \frac{1}{3}\begin{bmatrix} 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}$$

All of the above are described in the section on the general sum of squares matrices.

The treatment sum of squares matrix may be found by considering the arrangement of the reduced latin square. Two rows are exchanged or two columns are exchanged in order to have one treatment appear on the diagonal. This is not necessary, but will be done for convenience. The following array might be the result.

$$\begin{bmatrix} \text{I} & \text{II} & \text{III} \\ \text{III} & \text{I} & \text{II} \\ \text{II} & \text{III} & \text{I} \end{bmatrix}$$

Then a three by three matrix denoting the position of the treatment in the array may be written for each treatment.

$$\text{I}, \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \quad \text{II}, \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{bmatrix} \quad \text{III}, \begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix}$$

The treatment sum of squares matrix will be composed of the above sub-matrices. Also one might observe that the first, sixth, and eighth observations are in treatment I, and by definition of $Y'A_4 Y$, observe that the first, sixth, and eighth rows and columns should have zeros in all positions except for ones in the first, sixth, and eighth positions, and so forth for the rest of the treatments.

$$\sum_k \frac{Y_{\cdot\cdot k}^2}{3} = \frac{1}{3} Y' \begin{bmatrix} 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \end{bmatrix} Y$$

The row sum of squares, written out in detail, is

$Y'A_2^*Y = Y'A_2\ Y - Y'A_1Y =$

$$(1/9)\ Y' \begin{bmatrix} 2&2&2&-1&-1&-1&-1&-1&-1 \\ 2&2&2&-1&-1&-1&-1&-1&-1 \\ 2&2&2&-1&-1&-1&-1&-1&-1 \\ -1&-1&-1&2&2&2&-1&-1&-1 \\ -1&-1&-1&2&2&2&-1&-1&-1 \\ -1&-1&-1&2&2&2&-1&-1&-1 \\ -1&-1&-1&-1&-1&-1&2&2&2 \\ -1&-1&-1&-1&-1&-1&2&2&2 \\ -1&-1&-1&-1&-1&-1&2&2&2 \end{bmatrix} Y.$$

The column sum of squares may be represented similarly.

$$Y'A_3^*Y = (1/9)\ Y' \begin{bmatrix} 2&-1&-1&2&-1&-1&2&-1&-1 \\ -1&2&-1&-1&2&-1&-1&2&-1 \\ -1&-1&2&-1&-1&2&-1&-1&2 \\ 2&-1&-1&2&-1&-1&2&-1&-1 \\ -1&2&-1&-1&2&-1&-1&2&-1 \\ -1&-1&2&-1&-1&2&-1&-1&2 \\ 2&-1&-1&2&-1&-1&2&-1&-1 \\ -1&2&-1&-1&2&-1&-1&2&-1 \\ -1&-1&2&-1&-1&2&-1&-1&2 \end{bmatrix} Y.$$

The treatment sum of squares, $Y'A_4^*Y$, is

$$(1/9)Y' \begin{bmatrix} 2&-1&-1&-1&-1&2&-1&2&-1 \\ -1&2&-1&2&-1&-1&-1&-1&2 \\ -1&-1&2&-1&2&-1&2&-1&-1 \\ -1&2&-1&2&-1&-1&-1&-1&2 \\ -1&-1&2&-1&2&-1&2&-1&-1 \\ 2&-1&-1&-1&-1&2&-1&2&-1 \\ -1&-1&2&-1&2&-1&2&-1&-1 \\ 2&-1&-1&-1&-1&2&-1&2&-1 \\ -1&2&-1&2&-1&-1&-1&-1&2 \end{bmatrix} Y.$$

The error sum of squares quadratic form $Y'A_5^*Y =$

$$(1/9)Y' \begin{bmatrix} 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 \end{bmatrix} Y.$$

or the above might be written

$$Y'A_5Y = Y'Y - Y'A_2Y - Y'A_3Y - Y'A_4Y + 2Y'\dot{A}_1Y.$$

The matrices in the example may be tested for independence and idempotence by multiplying the matrices together. Some representative cases are the following:

$$A_4^*A_5^* = \varphi$$

$$\begin{bmatrix} 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 \end{bmatrix} (1/9)^2 = \varphi$$

$$A_1A_2^* = \varphi$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \end{bmatrix} (1/9)^2 = \varphi$$

Similarly

$A_1^*A_3^* = \varphi$, $A_1^*A_4^* = \varphi$, $A_1^*A_5^* = \varphi$, $A_2^*A_3^* = \varphi$, $A_2^*A_4^* = \varphi$, $A_2^*A_5^* = \varphi$, $A_3^*A_4^* = \varphi$, and $A_3^*A_5^* = \varphi$, or stated more simply, $A_i^*A_j^* = \varphi \ (i \neq j)$. Similarly $A_j^*A_i^* = \varphi$, $i \neq j$. This implies that the sum of squares matrices are independent.

$$A_1A_1 = A_1$$

$$(1/9)\begin{bmatrix} 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1 \end{bmatrix} (1/9)\begin{bmatrix} 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1 \end{bmatrix} =$$

$$(1/9)^2\begin{bmatrix} 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9\\ 9&9&9&9&9&9&9&9&9 \end{bmatrix}$$

$A_3^*A_3^* = A_3^*$, $A_4^*A_4^* = A_4^*$, $A_5^*A_5^* = A_5^*$. Therefore the sum of squares matrices are symmetric, idempotent, and their cross products are null. Also $A_4^* + A_5^* = A_4 + A_5$ is idempotent.

$$(A_4^* + A_5^*)(A_4^* + A_5^*) = A_4^*A_4^* + A_5^*A_4 + A_4^*A_5^* + A_5^*A_5^* = A_4^* + A_5^*.$$

A Closer Look at the Variance-Covariance Matrix

The analysis of variance procedure will go through if the variance-covariance matrix is broken up into

$$V = k_1A_1 + k_2A_2^* + k_3A_3^* + k(A_4 + A_5).$$

If $k = k_1 = k_2 = k_3 = \sigma^2$, the variance matrix is the usual case, $\sigma^2 I$.

$k_1A_1 + k_2A_2^* + k_3A_3^* + k(A_4 + A_5) =$

$k_1A_1 + k_2(A_2 - A_1) + k_3\ (A_3 - A_1) + k(I + A_1 - A_2 - A_3) =$

$kI + (k + k_1 - k_2 - k_3)\ A_1 + (k_2 - k)A_2 + (k_3 - k)\ A_3.$

For the three by three case, the V matrix may be written as shown on the following page. For example, the first term in the matrix (first row and first column) is

$$(1/9)\ [9k + (k + k_1 - k_2 - k_3) + 3(k_2 - k) + 3(k_3 - k)] =$$
$$k + (1/9)(k + k_1 - k_2 - k_3) + (1/3)(k_2 - k) + (1/3)(k_3 - k) =$$
$$(4/9)k + (1/9)k_1 + (2/9)k_2 + (2/9)k_3.$$

The matrix of general interest is the following which has equal within blocks and within column covariances.

$$V = \begin{bmatrix} a & b & b & c & d & d & c & d & d \\ b & a & b & d & c & d & d & c & d \\ b & b & a & d & d & c & d & d & c \\ c & d & d & a & b & b & c & d & d \\ d & c & d & b & a & b & d & c & d \\ d & d & c & b & b & a & d & d & c \\ c & d & d & c & d & d & a & b & b \\ d & c & d & d & c & d & b & a & b \\ d & d & c & d & d & c & b & b & a \end{bmatrix}$$

After comparing the two matrices, the four equations may be set up, subject to the conditions that $a,\ k,\ k_1,\ k_2 > 0$.

$$(4/9)\ k + (1/9)\ k_1 + (2/9)\ k_2 + (2/9)\ k_3 = a$$
$$-\ (2/9)\ k + (1/9)\ k_1 + (2/9)\ k_2 - (1/9)\ k_3 = b$$
$$-\ (2/9)\ k + (1/9)\ k_1 - (1/9)\ k_2 + (2/9)\ k_3 = c$$
$$(1/9)\ k + (1/9)\ k_1 - (1/9)\ k_2 - (1/9)\ k_3 = d.$$

k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$	k	$k + k_1 - k_2 - k_3$	$k_2 - k$	$k_3 - k$
9	1	3	3		1	3			1	3			1		3		1				1				1		3		1				1		
	1	3		9	1	3	3		1	3			1				1		3		1				1				1		3		1		
	1	3			1	3		9	1	3	3		1				1				1		3		1				1				1		3
	1		3		1				1			9	1	3	3		1	3			1	3			1		3		1				1		
	1				1		3		1				1	3		9	1	3	3		1	3			1				1		3		1		
	1				1				1		3		1	3			1	3		9	1	3	3		1				1				1		3
	1		3		1				1				1		3		1				1			9	1	3	3		1	3			1	3	
	1				1		3		1				1				1		3		1				1	3		9	1	3	3		1	3	
	1				1				1		3		1				1				1		3		1	3			1	3		9	1	3	3

(1/9)

If a solution to the equations exists, some conditions might possibly be imposed on the a's, b's, c's and d's of the various latin square variance-covariance matrices.

The equations may also be put in matrix form.

$$\begin{bmatrix} 4 & 1 & 2 & 2 \\ -2 & 1 & 2 & -1 \\ -2 & 1 & -1 & 2 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} k \\ k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 9a \\ 9b \\ 9c \\ 9d \end{bmatrix}$$

The solutions are

$$k = a - b - c + d$$

$$k_1 = a + 2b + 2c + 4d$$

$$k_2 = a + 2b - c - 2d$$

$$k_3 = a - b + 2c - 2d.$$

The variance of the y_{ij} is represented by the a element and is a constant. The b element refers to the constant covariance of elements in the same block, which is constant from block to block. The c element represents the covariance of elements within columns, and this is constant for all columns. The d element represents the mean effect. Another way of stating this, where i represents the row coefficient and j represents the column coefficient is

$$E(e_{ij})^2 = a$$

$$E(e_{ij}e_{ij'}) = b \qquad (j \neq j')$$

$$E(e_{ij}e_{i'j}) = c \qquad (i \neq i')$$

$$E(e_{ij}e_{i'j'}) = d \qquad (i \neq i, \quad j \neq j')$$

<u>EXAMPLE 1</u>: To obtain the identity matrix, $V = I$, the values $a = 1$, $b = c = d = 0$ are substituted in the system of solutions. The k_i values are consistent with the condition that they be non-zero.

$$k = k_1 = k_2 = k_3 = 1$$

EXAMPLE 2: To obtain the matrix, $(a - b)I + bJ(9)$, the values a, $b = c = d$ are substituted in the system of solutions. Then

$$k = a - b \qquad k_1 = 8b + a \qquad k_2 = a - b \qquad k_3 = a - b.$$

To satisfy the restrictions, a must be greater than b and greater than $-8b$, as well as greater than zero.

Other examples that might be considered are $b = c$, $b = d$, $c = d$, however there will be a restriction on b, c, or d relative to a, but this will be discussed later. Since a must always be greater than b, c, or d, many possible combinations may be eliminated.

Some Special Variance-Covariance Matrices

When Latin squares of side n are considered, the $n^2 \times n^2 A_i$ matrices are combined in the usual manner to form V.

$$V = k_1 A_1 + k_2 A_2^* + k_3 A_3^* + k(I - A_1 - A_2^* - A_3^*)$$

The variance elements consist of

$$A = k + (1/n^2)(k + k_1 - k_2 - k_3) + (1/n)(k_2 - k) + (1/n)(k_3 - k)$$
$$= k(1 + 1/n^2 - 2/n) + k_1(1/n^2) + k_2(1/n - 1/n^2) + k_3(1/n - 1/n^2).$$

The covariance between elements in the same row is represented by

$$B = (1/n^2)(k + k_1 - k_2 - k_3) + (1/n)(k_2 - k)$$
$$= k(1/n^2 - 1/n) + k_1(1/n^2) + k_2(1/n - 1/n^2) - k_3(1/n^2).$$

The covariance between elements in the same column is

$$C = (1/n^2)(k + k_1 - k_2 - k_3) + (1/n)(k_3 - k)$$
$$= k(1/n^2 - 1/n) + k_1(1/n^2) - k_2(1/n^2) + k_3(1/n - 1/n^2).$$

The covariance between all other elements is

$$D = (1/n^2)(k + k_1 - k_2 - k_3).$$

In matrix form the set of equations is

$$\begin{bmatrix} 1 + 1/n^2 - 2/n & 1/n^2 & 1/n - 1/n^2 & 1/n - 1/n^2 \\ 1/n^2 - 1/n & 1/n^2 & 1/n - 1/n^2 & -1/n^2 \\ 1/n^2 - 1/n & 1/n^2 & -1/n^2 & 1/n - 1/n^2 \\ 1/n^2 & 1/n^2 & -1/n^2 & -1/n^2 \end{bmatrix} \begin{bmatrix} k \\ k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

The set of solutions is

$$k = A - B - C + D$$

$$k_1 = A + (n - 1)B + (n - 1)D + (n^2 - 2n + 1)D$$

$$k_2 = A + (n - 1)B - C - (n - 1)D$$

$$k_3 = A - B + (n - 1)(C - D)$$

where A and each k_i must be positive.

<u>EXAMPLE 1</u>: To obtain $V = I$, $A = 1$, $B = C = D = 0$. Substituting into the equations, $k = k_1 = k_2 = k_3 = 1$.

<u>EXAMPLE 2</u>: To obtain $V = I + (1/n)J(n^2)$, $A = 1$, $B = C = D = 1/n$. Then $k = k_2 = k_3 = 1 - 1/n = (n - 1)/n$, $k_1 = 1 + (n^2 - 1)/n$.

<u>EXAMPLE 3</u>: $A = 1$, $B = C = 1/n$, $D = 1/n^2$. The k_i solutions are $k = (n - 1)^2/n^2$, $k_1 = (2 - 1/n)^2$, $k_2 = k_3 = (2 - 1/n)(1 - 1/n)$.

<u>EXAMPLE 4</u>: $A = 1$, $B = C = 1/n$, $D = 0$. $k = 1 - 2/n$, $k_1 = 3 - 2/n$, $k_2 = k_3 = 2(1 - 1/n)$.

All of the examples are admissible cases since they satisfy the restrictions that A, k, k_1, k_2, and k_3 are greater than zero. No additional restrictions need be imposed on the k's for the A elements to be greater than the B, C, or D elements.

<u>EXAMPLE 5</u>: $B = C = D$ if $(-A)/(n^2 - 1) < B < A$. The conditions are obtained by using the k and k_1 equations, and the restriction that k and k_1 are greater than zero.

EXAMPLE 6: $B = C$ if either $[-A + (n-1)D]/(n-2) < B$ or $-A/2(n-1) - D(n+1)/2 < B$, whichever is larger, and $B < (A + D)/2$.

EXAMPLE 7: $B = D$ if $A > C$ and $-[A + (n-1)C]/n(n-1) < B < [A + (n-1)C]/n$.

EXAMPLE 8: Is symmetric with example 7. $C = D$ if $A > B$ and $-[A + (n-1)B]/n(n-1) < C < [A + (n-1)B]/n$.

It has been stated that A must always be greater than B, C, or D. This is obvious by inspection of the A, B, C and D equations.

$A = B + k + k_3/n$ where the k_i are positive. Similar equations may be obtained in terms of C and D.

$A = C + k + k_2/n$ and

$A = D + k(1 - 2/n) + k_2/n + k_3/n$.

Unequal Block Variances in the Latin Square

Suppose, rather than the usual analysis of variance, the identity

$$y_{ijk} = y_{i\cdot\cdot} + (y_{\cdot j\cdot} - y_{\cdot\cdot\cdot}) + (y_{\cdot\cdot k} - y_{\cdot\cdot\cdot}) + (y_{ijk} - y_{i\cdot\cdot} - y_{\cdot\cdot k} - y_{\cdot j\cdot} + 2y_{\cdot\cdot\cdot})$$

is used for the analysis of variance. The same terms are the treatment and error sum of squares, so their distribution is unchanged. The sum of squares may again be broken into

$$\begin{aligned} I &= A_2 + A_3^* + A_4^* + (I - A_2 - A_3^* - A_4^*) \\ &= A_2 + A_3^* + (I - A_2 - A_3^*). \end{aligned}$$

A_2 has rank n and can be represented as the sum of n idempotent matrices of rank one. These matrices will be called E_i and each represents the sum of squares of the i^{th} row. Each E_i is $n^2 \times n^2$ with n $n \times n$ blocks along the diagonal. All elements of E_i are 0 except the elements of the i^{th} $n \times n$ block which are all $1/n$. The variance-covariance matrix may be represented as

$$V = \sum_{i=1}^{n} k_i E_i + k_{n+1} A_3^* + k(I - A_2 - A_3^*) \quad \text{where} \quad \sum_{i=1}^{n} E_i = A_2.$$

$(k, k_i) > 0$. By considering different values of k_i $(i = 1, \ldots n)$, different V's may be considered. The correlation between elements in the same blocks may be nearly negligible for k_i nearly 0, or it may be very large (near + 1) for k_i larger.

The $2n + 2$ equations in n unknowns are

$$k_i/n + k_{n+1}(n - 1)/n^2 + k(1 - 1/n)^2 = A_i \qquad i = 1, \ldots, n$$

$$k_i/n + k_{n+1}/n^2 - k[(1/n) + 1/n^2] = B_i \qquad i = 1, \ldots, n$$

$$k_{n+1}(n - 1)/n^2 - k(n - 1)/n^2 = C$$

$$-k_{n+1}/n^2 + k/n^2 = D.$$

The solutions are

$$k = A_i - B_i + nD = m + nD$$

$$k_{n+1} = A_i - B_i + nC = m + nC = m - n(n - 1)D$$

$$k_i = A_6 + (n - 1)B_i + 2nD = m + nB_i + 2nD$$

where $D = - C/(n - 1)$ and $A_i - B_i$ is a positive constant, say m.

Other restrictions are

$|A_i| > |B_i|$, $|A_i| > |C|$, $|A_i| > |B_i + D|$, $A_i > 0$, $- m/n < D < m/n(n - 1)$ and $- (m/n + 2D) < B_i < A_i + nD$.

The columns may be considered in the same manner; however, both the columns and the rows cannot be treated in the above manner. There are n independent rows and there would be n independent columns, giving a total of 2n independent parameters although there are only 2n - 1 independent parameters or degrees of freedom.

The Variance of a Contrast

The next logical step might be to consider the variance of the estimates or contrasts under the new assumptions on the V matrix. The variance of the difference of two treatments occurring in the same row is $A + A - 2B = 2(A - B) = 2[k(1 - 1/n) + k_3/n]$. This will be smaller than the usual variance if the within blocks variance is positive, otherwise it will be larger. Similarly, the variance of the difference of any two elements in the same column is $2(A - C) = 2[k(1 - 1/n) + k_2/n]$. The variance of the difference of two observations not in the same row or the same column is $2(A - D) = 2[k(1 - 2/n) + k_2/n + k_3/n]$.

Each of the n treatments occurs n times, and a quantity that would be of more interest than the variance of the difference of two single observations is the variance of the difference of the sum of two treatments. This variance, T is equal to

$$E[\sum_{j=1}^{n}(y_{1j} - \mu_{1j}) - \sum_{j=1}^{n}(y_{2j} - \mu_{2j})]^2$$

where $i = 1, \ldots, n$ and refers to treatments, $j = 1, \ldots, n$ and refers to blocks.

$$y_{1j} - \mu_{1j} = (\mu + r_j + c_i + t_1 + e_{1j}) - (\mu + r_j + c_i + t_1) = e_{1j}.$$

Also $y_{2j} - \mu_{2j} = e_{2j}$. Then

$$T = E[\sum_{j=1}^{n} e_{1j}^2 - 2\sum_{j=1}^{n} e_{1j} \sum_{j=1}^{n} e_{2j} + \sum_{j=1}^{n} e_{2j}^2]$$

$$= E[\sum_{j=1}^{n} e_{1j}^2 + \sum_{\substack{j=1 \\ j\neq j'}}^{n} e_{1j}\, e_{1j'} - 2\sum_{j=1}^{n} e_{1j}\, e_{2j}$$

$$- 2\sum_{\substack{j=1 \\ j\neq j'}}^{n} e_{1j}\, e_{2j'} + \sum_{j=1}^{n} e_{2j}^2 + \sum_{\substack{j=1 \\ j\neq j'}}^{n} e_{2j}e_{2j'}.]$$

When the expectations of the terms are taken $T =$

$$nA + n(n - 1)D - 2nB - 2nC - 2n(n - 2)D + nA + n(n - 1)D =$$

$$2n(A - B - C + D) = 2nk.$$

If the contrast of the means is wanted, the variance would be the result above divided by n. Since the variance is equal to $2nk$, a constant for all values of B, C, and D, their particular values then do not matter for a contrast over all rows and columns.

A contrast of more general interest is $\sum_{i=1}^{n} \lambda_i Y_{i\cdot}$, where $Y_{i\cdot} = \sum_{j=1}^{n} y_{ij}$, $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 1$. The expected value of the contrast,

$$E(\sum_{i=1}^{n} \lambda_i Y_{i\cdot}) = \sum_{i=1}^{n} n\lambda_i \mu_i = n \sum_{i=1}^{n} \lambda_i \mu_i$$

in found by taking the expected value of each observation. If the μ_i are all equal, $E(\sum_{i=1}^{n} \mu_i Y_{i\cdot})$ is zero. The variance of the contrast is found by examining

$$E[\sum_{i=1}^{n} \lambda_i (Y_{i\cdot} - n\mu_{i\cdot})]^2 = E[\sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{n} y_{ij} - n\mu_{i\cdot})]^2 = E[\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} (y_{ij} - \mu_{ij})]^2 =$$

$$E[\sum_{i=1}^{n} \lambda_i^2 (\sum_{j=1}^{n} y_{ij} - n\mu_{i\cdot})^2] + E[\sum_{\substack{i=1 \\ k\neq i}}^{n} \lambda_i \lambda_k \sum_{j=1}^{n} (y_{ij} - \mu_{ij}) \sum_{j=1}^{n} (y_{ij} - \mu_{ij})] =$$

$$E[\sum_{i=1}^{n} \lambda_i^2 \sum_{j=1}^{n} (y_{ij} - \mu_{ij})^2] + E[\sum_{i=1}^{n} \lambda_i^2 \sum_{\substack{j=1 \\ j\neq j'}}^{n} (y_{ij} - \mu_{ij})(y_{ij'} - \mu_{ij})] +$$

$$E[\sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_i \lambda_k \sum_{j=1}^{n} (y_{ij} - \mu_{ij})(y_{kj} - \mu_{kj})] + E[\sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_i \lambda_k \sum_{\substack{j=1 \\ j\neq j'}}^{n} (y_{ij} - \mu_{ij})(y_{kj'} - \mu_{kj})].$$

If it is assumed that the variance in different blocks and columns is constant, the expectations will be (whether the model definition is substituted or not)

$$n \sum_{i=1}^{n} \lambda_i^2 A + n(n-1) \sum_{i=1}^{n} \lambda_i^2 D + n \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_i\,{}_k B + n \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_k\,{}_i C + n(n-2) \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_k \lambda_i D =$$

$$nA \sum_{i=1}^{n} \lambda_i^2 + nD[(n-1) \sum_{i=1}^{n} \lambda_i^2 + (n-2) \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_i \lambda_k] + nB \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_k \lambda_i + nC \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_k \lambda_i =$$

$$nA + nD[(n-1) - (n-2)] - nB - nC = n(A + D - B - C) = nk.$$

The variance of a contrast of the means may be obtained by dividing the above by n.

Other assumptions that might be considered are

$$E(y_{ij} - \mu_{ij})^2 = A_j, \quad E(y_{ij} - \mu_{ij})(y_{ij'} - \mu_{ij'}) = D$$

$$E(y_{ij} - \mu_{ij})(y_{kj} - \mu_{kj}) = B_j \ (k \neq i) \qquad \text{and}$$

$$E(y_{ij} - \mu_{ij})(y_{kj'} - \mu_{kj'}) = C \ \text{ or } \ D \ (i \neq k, \ \ j \neq j'),$$

although the B's might have been assumed constant and the C's variable. Then the expected value of the variance of the contrast would be

$$\sum_{i=1}^{n} \lambda_i^2 \sum_{j=1}^{n} A_j + n(n-1) \sum_{i=1}^{n} \lambda_i^2 D + \sum_{\substack{i=1 \\ i \neq k}}^{n} \lambda_i \lambda_k \sum_{j=1}^{n} B_j + n \sum_{\substack{i=1 \\ i \neq k}}^{n} \lambda_i \lambda_k C +$$

$$n(n-2) \sum_{\substack{i=1 \\ k \neq i}}^{n} \lambda_i \lambda_k D = \sum_{j=1}^{n} A_j + n(n-1)\ D - \sum_{j=1}^{n} B_j - nC - n(n-2)D =$$

$$n(D - C) + \sum_{j=1}^{n} (A_j - B_j).$$

It has previously been noted that $A_j - B_j = m$, a constant.

Since $C = (-n + 1)D$, the variance of the contrast can be expressed as $n(D + nD - D) + \sum_{j=1}^{n} m = n^2 D + nm = n(nD + m) = nk$. Therefore, the variance of a contrast of treatments which are in all the blocks is a constant.

The same thing can be done for the columns. Since the blocks and columns are symmetric, B and C may be substituted in the definitions. Also the meaning attached to the notation would change. However, the variance of the contrast would still be nk, where k is the constant multiplying A_4 and A_5.

CHAPTER III

BALANCED INCOMPLETE BLOCK DESIGN

The balanced incomplete block design consists of t treatments arranged in b blocks with k elements per block in r replicates. Each treatment occurs in the same block with every other treatment $\lambda = r(m - 1)/(t - 1)$ times. In order to avoid notational confusion, m, rather than k, will be used for the number of elements per block.

The analysis of variance that will be considered is the simple one using the adjusted treatment sum of squares.

Analysis of Variance

Source of Variation	D.F.	S.S.	Expected Mean Square
Total	mb	$Y'Y$	
Mean	1	$Y'A_1Y$	
Block (ignoring treatment)	$b - 1$	$Y'A_2^*Y$	
Treatment (eliminating block)	$t - 1$	$Y'A_3^*Y$	$\sigma^2_e + r\sigma^2_t$
Error (intrablock)	$(m - 1)(b - 1)$	$Y'A_4Y$	σ^2_e

The model may be written, ignoring replications, as

$$y_{ijk} = \mu + T_i + B_j + e_{ijk}$$

$$i = 1, 2, \ldots, t; \quad j = 1, 2, \ldots, b; \quad k = n_{ij};$$

$$n_{ij} = 1 \text{ if block } j \text{ contains treatment } i,$$

$$= 0 \text{ if block } j \text{ does not contain treatment } i.$$

The sum of squares for blocks is found by $\sum_{j=1}^{b} Y_{\cdot j \cdot}^2/m - Y_{\cdot\cdot\cdot}^2/mb$.
The adjusted treatment sum of squares is $\sum_{i=1}^{t} (mY_{i\cdot\cdot} - B_i')^2/mt$ where B_i' is the total of all blocks in which the treatment appears. The adjusted treatment total is $mY_{i\cdot\cdot} - B_i' = m Y_{i\cdot\cdot} - \sum_j n_{ij} Y_{\cdot j}$.

The Structure of the Variance-Covariance Matrix

The assumptions on the distribution of the variances will be

$$E(e_{ij})^2 = a$$

$$E(e_{ij}e_{i'j}) = b, \qquad (i \neq i')$$

$$E(e_{ij}e_{i'j'}) = c \qquad (j \neq j')$$

where a, b and c are constants.

The total sum of squares, $Y'Y$, may be divided into

$$Y'A_1Y + Y'A_2^*Y + Y'A_4^*Y + Y'A_5Y$$

from the analysis of variance table above. Each of the A_i are $mb \times mb$. The Y vector is $mb \times 1$. Because $rt = mb$ in the balanced incomplete block design, $A_1 = [1/(rt)^2]J(rt) = [1/(mb)^2]J(mb)$.

$$A_2^* = A_2 - A_1 \quad \text{where} \quad A_2 = \begin{bmatrix} p & \varphi & \varphi & \cdot & \cdot & \cdot & \varphi \\ \varphi & p & \varphi & \cdot & \cdot & \cdot & \varphi \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \varphi \\ \varphi & \cdot & \cdot & \cdot & & & p \end{bmatrix} \quad \text{and} \quad p = \frac{1}{m} J(m).$$

By the previous arguments

$$A_1A_1 = A_1, \qquad A_2A_2 = A_2, \qquad A_1A_2 = A_2A_1 = A_1,$$

and it may be shown that the matrices are idempotent so that Theorem 2 may be applied. $A_1A_2^* = A_1(A_2 - A_1) = A_1 - A_1 = A_2^*A_1 = \varphi$. The quadratic forms are independent if their products are null. The treatment and error matrix is $I - A_1 - A_2^* = I - A_2$. It is idempotent since

$$(I - A_2)(I - A_2) = I - A_2 - A_2 + A_2 = I - A_2.$$

The mean, blocks, and treatment plus error sums of squares can be shown to be independent by showing that their products are null.

$$A_1(I - A_2) = A_1 - A_1 = \varphi = (I - A_2)A_1$$

$$(A_2 - A_1)(I - A_2) = A_2 - A_1 - A_2 + A_1 = (I - A_2)(A_2 - A_1) = \varphi.$$

Then, because of Theorem 2, the observation vector Y may be said to be distributed $N(\mu, V)$, where

$$\begin{aligned} V &= k_1 A_1 + k_2 A_2^* + k(A_4^* + A_5) \\ &= k_1 A_1 + k_2(A_2 - A_1) + k(I - A_2) \\ &= kI + (k_1 - k_2)A_1 + (k_2 - k)A_2. \end{aligned}$$

From the definitions of a, b, c, V and the A_i, three equations in three unknowns may be solved.

$$a = k + \frac{1}{mb}(k_1 - k_2) + \frac{1}{m}(k_2 - k) = k(1 - \frac{1}{m}) + \frac{k_1}{mb} + \frac{k_2}{m}(1 - \frac{1}{b})$$

$$b = \frac{1}{mb}(k_1 - k_2) + \frac{1}{m}(k_2 - k) = -\frac{k}{m} + \frac{k_1}{mb} + \frac{k_2}{m}(1 - \frac{1}{b})$$

$$c = \frac{1}{mb}(k_1 - k_2) = \frac{k_1}{mb} - \frac{k_2}{mb}$$

The solutions are

$$\begin{aligned} k_1 &= a + b(m - 1) + c(mb - m) \\ k_2 &= a + b(m - 1) - mc \\ k &= a - b. \end{aligned}$$

The k's are defined to be positive. If $V = \sigma^2 I$, $a = \sigma^2$ and $b = c = 0$. The variances a must be greater than b or c, but b may equal c; b and c may be either positive or negative, but a must be positive.

The Variance of a Contrast

The estimate of a contrast of treatment effects will be expressed by $\sum_{i=1}^{t} \lambda_i(T_i - T.)$ where $\sum \lambda_i = 0$ and $\sum \lambda_i^2 = 1$. Then

$\Sigma\, \lambda_i(T_i - T.) = \Sigma\, \lambda_i T_i$. In order to find the variance of the contrast, the variance-covariance matrix of the $(T_i - T.)$ is needed. Then the following theorem of Graybill (1955) will be used. If the $(t + 1) \times 1$ vecter $\overline{T}$ with elements $T_i - T.$ $(i = 1, \ldots, t)$ and δ has the multivariate normal distribution with mean μ and variance $\overline{B}$, then any linear combination of the $T_i - T.$, say $\sum_{i=1}^{t} \lambda_i(T_i - T.)$ where the λ_i are the above constants, has the univariate normal distribution with mean $\Sigma\, \lambda_i \mu_i$ (the μ_i are elements of the vector μ) and variance $\sum_{i=1}^{t} \sum_{j=1}^{t} \lambda_i \lambda_j b_{ij}$, where b_{ij} designates the ij^{th} elements of $\overline{B}$.

The model is rewritten

$$Y = X_1 B + X_2 T + e$$

where the mean is incorporated into the B's. The X_1 matrix refers to the arrangement of the observations into blocks. Its dimensions are $mb \times b$. The B vector is $b \times 1$. The dimensions of X_2 are $mb \times t$ and indicate the allotment of the treatments to the y_{ijk}. The T vector is $t \times 1$, and both Y and e are $mb \times 1$.

The normal equations are

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{B} \\ \hat{T} \end{bmatrix} = \begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

which gives

$$[X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2]\hat{T} = [X_2' - X_2'X_1(X_1'X_1)^{-1}X_1']Y.$$

This is denoted by

$$A\hat{T} = Q = CY \qquad \text{where} \qquad A = CX_2.$$

A is $t \times t$, $\hat{T}$ is $t \times 1$, C is $t \times mb$ and Y is $mb \times 1$. Since $A\,J(t) = \varphi$, A does not have an inverse, but if it is augmented by j, a $t \times 1$ vector of ones, the resulting matrix does have an inverse.

$$\begin{bmatrix} A & j \\ j' & 0 \end{bmatrix} \begin{bmatrix} \hat{T}* \\ \delta \end{bmatrix} = \begin{bmatrix} A_1 Y \\ 0 \end{bmatrix}$$

The inverse may be shown to be

$$\begin{bmatrix} B & \frac{1}{t}j \\ \frac{1}{t}j' & 0 \end{bmatrix}$$

Consequently $\hat{T}* = BCY$. Adding the row and column of ones changes the quantity estimated from $\hat{T}$ to $\hat{T}*$.

$$\hat{T}* = \hat{T} - \frac{1}{t} J(t)\ \hat{T} = [I - \frac{1}{t}J(t)]\ \hat{T} = J_1\hat{T} = BC\hat{T}.$$

The covariance of $\hat{T}*$ is

$$\begin{aligned} E(\hat{T}* - E(\hat{T}*))(\hat{T}* - E(\hat{T}*)) &= E(BCY - J_1T)(BCY - J_1T)' \\ &= E(BCT + BCe - BCT)(BCT + BCe - BCT)' \\ &= E(BCe)(BCe)' = E(BCee'C'B') \\ &= BC\ E(ee')\ C'B' \end{aligned}$$

Since $E(ee') = sX_1X_1' + rJ(mb) + kI$, the above expression becomes $BC\ (sX_1X_1' + rJ(mb) + kI)\ C'B'$ where $r = \frac{1}{mb}(k_1 - k_2)$ and $s = \frac{1}{m}(k_2 - k)$. The expression may be multiplied out, and each of the three terms may be considered separately.

$$\begin{aligned} sBCX_1X_1'C'B' &= sB\ \ [X_2' - X_2'X_1(X_1'X_1)^{-1}X_1']\ \ X_1X_1'C'B' \\ &= sB\ \ [X_2'X_1X_1' - X_2'X_1(X_1'X_1)^{-1}(X_1'X_1)X_1']\ \ C'B' \\ &= sB\ C'B' = \varphi. \end{aligned}$$

$$rBCJ(mb)C'B' = \varphi$$

Since $A = CX_2$, $AJ = CX_2J$, $X_2J = J$, $AJ = \varphi = CJ$.

$$kBCIC'B' = kBCC'B' = kBAB' = k(I - \frac{1}{t}J)B' = kB'$$

since $CC' = A$ and $JB' = \varphi$.

Therefore the covariance matrix is kB'. Rather than using k, σ^2 might be used. B is symmetric, so the quantity can be called σ^2B.

$$A = \begin{bmatrix} r - \frac{r}{m} & -\frac{\lambda}{m} & -\frac{\lambda}{m} & \cdots & \cdots \\ -\frac{\lambda}{m} & r - \frac{r}{m} & -\frac{\lambda}{m} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{\lambda}{m} & \cdots & \cdots & -\frac{\lambda}{m} & r - \frac{r}{m} \end{bmatrix}$$

or

$$A = \frac{r(m-1)}{m} \begin{bmatrix} 1 & \frac{-1}{t-1} & \frac{-1}{t-1} & \cdots & \cdots \\ \frac{-1}{t-1} & 1 & \frac{-1}{t-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{-1}{t-1} & \cdots & \cdots & \frac{-1}{t-1} & 1 \end{bmatrix}$$

It can be shown by multiplication of matrices that

$$B = \frac{m}{t(t-2)\lambda} \begin{bmatrix} t-1 & 1 & 1 & \cdots & \cdots \\ 1 & t-1 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & t-1 \end{bmatrix}$$

Unequal Block Variances

The logical extension of the previous model is to one where the blocks may have unequal variances. The model would then be

$$Y'Y = Y'A_2Y + Y'A_4^*Y + Y'A_5Y$$

Where $Y'A_2Y$ is the uncorrected sum of squares for blocks and $Y'A_4^*Y$ and $Y'A_5Y$ refer to the sum of squares for treatment and error, respectively. The results of Theorem 2 may be used if Y is distributed $N(\mu,V)$ where

$$V = k(A_4^* + A_5) + \sum_{i=1}^{b} k_i E_i,$$

$A_4^* + A_5 + \Sigma E_i = I$, $\sum_{i=1}^{b} E_i = A_2$ and the E_i each have rank one and A_2 has rank b.

$$E_i = \begin{bmatrix} c_{11} & c_{12} & c_{13} \cdots\cdots & c_{1b} \\ c_{21} & c_{22} & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ c_{i1} & c_{i2} & \cdots c_{ii} \cdots & c_{ib} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ c_{b1} & \cdots & \cdots\cdots & c_{bb} \end{bmatrix}$$

where each c_{jk} is m x m and c_{ii} is an m x m submatrix of elements 1/m and all other $c_{jk} = \varphi$. The E_i are idempotent and $E_i E_j = \varphi$ $(i \neq j)$. Other properties of matrix products follow.

$A_4^* + A_5 = I - A_2$. However, A_2 has already been shown to be idempotent.

$(I - A_2)(I - A_2) = I - A_2 - A_2 + A_2 = I - A_2$.

$E_i A_2 = E_i (\sum_i E_i) = E_i$.

$E_i(I - A_2) = E_i - E_i = \varphi$.

The variance of a treatment observation in the i^{th} block is

$$E(e_{ij})^2 = \sigma_{a_i}^2 = a_i.$$

The covariance of two observations in the same block is

$$E(e_{ij}e_{ij'}) = \sigma_{b_i}^2 = b_i \quad (j \neq j').$$

The covariance of two observations not in the same block is

$$E(e_{ij}e_{i'k}) = 0 \quad (i \neq i'),$$

i.e., the observations in different blocks are uncorrelated.

The elements of the variance-covariance matrix may be expressed as linear combinations of the k_i and known constants.

$$a_i = k + k_i/m - k/m = k(m - 1)/m + k_i/m$$

$$b_i = k_i/m - k/m = (k_i - k)/m.$$

$$a_i - b_i = k > 0$$

$$(m - 1)b_i + a_i = k_i > 0$$

The sign of b_i is determined by the quantity $k_i - k$.

The variance of a contrast under the above assumptions is the same as in the previous case. The same argument may be constructed for finding the variance except that some new symbols are necessary. In this case

$$E(ee') = \sum_{i=1}^{b} r_i E_i + kI$$

where $r_i = (k_i - k)/m$ and the E_i are the same as previously mentioned except that these are multiplied by m.

$$E_i = X_{1i}X_{1i}' \quad \text{and} \quad E_i = X_1X_1'.$$

The X_{1i} may best be described by considering the $mb \times b$ X_1 matrix. X_1 consists of b $m \times b$ submatrices X_{1i}.

$$X_1' = [X_{11}' \ X_{12}' \ X_{13}' \ \ldots\ldots\ldots \ X_{1b}']$$

All elements of the X_{1i} matrices are zero except for the i^{th} column which consists of ones. Using the above definitions, more new terms may be considered. The terms are not important in themselves, but the matrix products they represent will appear in the variance of the contrast.

$$\underset{(b \times mb)}{X_1'}\underset{(mb \times b)}{X_{1i}} = \underset{(b \times b)}{P_i}$$

which is a matrix consisting of zeros except for an m in the i^{th} place.

$$(X_1'X_1)^{-1}X_1'X_{1i} = Q_i$$

which is all zeros except for a one in the i^{th} place.

$$X_1Q_i = X_{1i}$$

since only the elements in the i^{th} column will be non-zero, and these will be only the elements in the i^{th} column of X_1.

Therefore the variance of the contrast can be found by considering

$$BA_1(\sum_{i=1}^{b} r_iE_i + kI)A_1'B'.$$

The second half of the term reduces to kB, as it did previously.

The rest of the expression is

$$\begin{aligned} BA_1(\Sigma\, r_iE_i)A_1'B' &= \Sigma\, r_i\, BA_1E_iA_1'B' \\ &= \Sigma\, r_iB[X_2' - X_2'X_1(X_1'X_1)^{-1}X_1']\, X_{1i}X_{1i}'A_1'B \\ &= \Sigma\, r_iB[X_2'X_{1i}X_{1i}' - X_2'X_1Q_iX_{1i}']\, A_1'B' \\ &= \Sigma\, r_iB[X_2'X_{1i}X_{1i}' - X_2'X_{1i}X_{1i}']\, A_1'B' \\ &= \varphi. \end{aligned}$$

Therefore the variance of the treatment contrasts is a constant.

CHAPTER IV

SUMMARY AND CONCLUSIONS

In this thesis the usual analysis of variance technique and exact test of significance has been shown to hold if

$$E(e_{ij}e_{i'j'}) \neq 0 \quad \text{for some} \quad (i \neq i') \quad \text{or} \quad (j \neq j').$$

Y may be distributed $N(\mu, V)$, rather than $N(\mu, \sigma^2 I)$ if V has a specific form. Some of the characteristics of this V matrix have been considered.

Latin Square

For the latin square, if i refers to rows and j refers to columns, some assumptions on the V matrix might be

$$E(e_{ij})^2 = a$$

$$E(e_{ij}e_{i'j}) = b \qquad (i \neq i')$$

$$E(e_{ij}e_{ij'}) = c \qquad (j \neq j')$$

$$E(e_{ij}e_{i'j'}) = d \qquad (i \neq i') \quad \text{and} \quad (j \neq j'),$$

and an exact test of the treatment effects is available. If the assumptions are

$$E(e_{ij})^2 = a_i$$

$$E(e_{ij}e_{ij'}) = b_i \qquad (j \neq j')$$

$$E(e_{ij}e_{i'j}) = c \qquad (i \neq i')$$

$$E(e_{ij}e_{i'j'}) = \frac{-c}{n-1} \; (i \neq i') \quad \text{and} \quad (j \neq j'),$$

then a modified analysis of variance technique using the uncorrected sum of squares for rows will lead to an exact test for treatments. Similarly, if

$$E(e_{ij})^2 = a_j$$
$$E(e_{ij}e_{ij'}) = b \qquad (j \neq j')$$
$$E(e_{ij}e_{i'j}) = c_j \qquad (i \neq i')$$
$$E(e_{ij}e_{i'j'}) = \frac{-b}{n-1} \ (i \neq i') \text{ and } (j \neq j'),$$

then the uncorrected sum of squares for column is used in the analysis.

Balanced Incomplete Block

The assumptions that are needed if the usual analysis is to be used are, if i refers to blocks and j refers to treatments,

$$E(e_{ij})^2 = a$$
$$E(e_{ij}e_{ij'}) = b \qquad (j \neq j')$$
$$E(e_{ij}e_{i'j'}) = c \qquad (i \neq i').$$

If the block variances are unequal, i.e.,

$$E(e_{ij})^2 = a_i$$
$$E(e_{ij}e_{ij'}) = b_i \qquad (j \neq j')$$
$$E(e_{ij}e_{i'j'}) = 0 \qquad (i \neq i'),$$

then the uncorrected blocks ignoring treatment sum of squares is used. However, this does not change the treatment or error sum of squares since the uncorrected sum of squares is used for total.

Significance

Kempthorne (1952) states that the correlation between plot errors within a block is $-1/(k - 1)$, where k is the size of block, in the randomized block design. It therefore seems reasonable that there is a correlation present in latin squares and incomplete blocks. Therefore the application of the above results would seem to be quite extensive.

Orthogonal contrasts are especially important in factorial treatment arrangements, and the variance of these contrasts is used in construction of confidence intervals. Even though the assumptions on V are changed, the variance of a contrast is unchanged. It is a constant, σ^2, which is estimated by the error mean square term in the analysis of variance table. Therefore a different estimate of the variance is not needed for an alternate assumption.

Suggestions for Future Study

The variance-covariance matrices suggested in this thesis are not the only ones to which Theorem 2 will apply. Those considered herein are the ones that the author felt would have practical application. Possibly other combinations would have application.

Another future extension might be to other designs, such as the general two-way classification, the partially balanced incomplete block, latin cube, hierarchal design, split plot, multi-dimensional lattices, or even Box designs. The effect of the lack of independence among observations on the size of the error might also be investigated.

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