72-3441

THIESSEN, Richard Eugene, 1938-
THE CHILD'S CONCEPT OF CONVEXITY.

The University of Oklahoma, Ph.D., 1971
Education, curriculum development

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

THE CHILD'S CONCEPT OF CONVEXITY

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

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Norman, Oklahoma

1971

THE CHILD'S CONCEPT OF CONVEXITY

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ACKNOWLEDGEMENT

The writer wishes to express his appreciation to Dr. Thomas J. Hill, chairman of the doctoral committee, for his guidance and encouragement during the course of this study. Appreciation is also extended to the other members of the committee: Professor Eunice M. Lewis, Dr. Donald Stafford, and Dr. Donald Reynolds.

I especially wish to thank my wife, Evelyn Jean, for her encouragement and assistance throughout the study.

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THE CHILD'S CONCEPT OF CONVEXITY

CHAPTER I

INTRODUCTION

Background of the Problem

One of the most important of the recent changes in the primary grade mathematics program is the inclusion of informal geometric concepts. During the past ten years there have been numerous articles written in The Arithmetic Teacher and in Grade Teacher encouraging the teaching of geometry at the primary level. The Cambridge Conference Report of 1963 recommended that "geometry is to be studied together with arithmetic and algebra from kindergarten on."[1] Many of the current curriculum projects are recommending that more geometry be taught in the elementary grades.[2] The SMSG series, for example, includes many geometric topics at the primary level.

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[1]Educational Services Incorporated, Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics (Boston: Houghton Mifflin Co., 1963), p. 33.

[2]Donovan A. Johnson, "Geometry for the Primary," Grade Teacher, LXXIX (April, 1962), p. 52.

Interest in the teaching of informal geometry at the primary level is not confined to recent years or even to this century. The preface of First Lessons in Geometry, written by Thomas Hill in 1854, included the following:

> I have long been seeking a Geometry for beginners, suited to my taste, and to my convictions of what is a proper foundation for scientific education. . . . Two children, one of five, the other of seven and a half, were before my mind's eye all the time of my writing; and it will be found that children of this age are quicker of comprehending first lessons in geometry than those of fifteen.[3]

The Fifth Yearbook of the National Council of Teachers of Mathematics lists, in addition, another nine books similar to the one by Hill, all written between 1895 and 1912.[4]

The results of research by Jean Piaget and others show that children have the ability to learn many geometric concepts at the primary level. This does not, as is pointed out by Johnson, mean that these concepts should be taught at this level.[5] There are, however, some good reasons for the inclusion of geometry in addition to the fact that children have the ability to learn it.

Children are interested in geometric ideas. Nearly all experiments involving the introduction of geometry at

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[3]Thomas Hill, First Lessons in Geometry (1854), Preface as quoted in The Teaching of Geometry, Fifth Yearbook of the National Council of Teachers of Mathematics (Washington, D. C.: The Council, 1930), p. 10.

[4]Ibid., p. 11.

[5]Johnson, "Geometry for the Primary," p. 52.

the primary level have shown that the children greatly enjoyed work with this aspect of mathematics.[6] Goldmark, for example, used the Hawley-Suppes Geometry Program for Primary Grades as a supplement to the prescribed arithmetic program. She found that "the children were fascinated by the very idea of doing geometry. Parents and visitors to the classroom note the obvious enthusiasm and excitement at 'geometry time.'"[7] Lamb, in a situation similar to that of Goldmark, also found "children's interest to be very high."[8]

The study of geometry provides children the opportunity to use mathematical descriptions of external reality.[9] It increases the child's awareness of size and shape as characteristics of objects in the environment that are meaningful to him.[10] Geometry provides a source of visualization for arithmetical and algebraic ideas. Robinson has suggested several ways in which geometry can be used to extend and enrich the study of arithmetic with regard to properties

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[6]Nicholas J. Vigilante, "Why Circumvent Geometry in the Primary Grades?" The Arithmetic Teacher, XII (Oxtober, 1965), p. 450.

[7]Bernice Goldmark, "Geometry in the Primary Grades," The Arithmetic Teacher, X (April, 1963), p. 191.

[8]Pose M. Lamb, "Geometry for Third and Fourth Graders," The Arithmetic Teacher, X (April, 1963), p. 193.

[9]Lewis B. Smith, "Geometry, Yes--but How?" The Arithmetic Teacher, XIV (February, 1967), p. 84.

[10]Richard K. Mastain and Bernice C. Nossoff, "Mathematics in the Kindergarten," The Arithmetic Teacher, XIII (January, 1966), p. 33.

of the natural numbers, the meaning of fractions, order properties for the natural numbers, and the concept of operation.[11]

The elementary school may provide for many children their only opportunity to study geometry. Frequently in the past the only geometry available to the student was that offered in a secondary school formal course. Brune suggests that "to defer geometry to the secondary school is sometimes to eliminate it entirely. For, being grossly ignorant of the subject, many pupils do not elect it."[12]

There is a good deal of agreement on the part of mathematics educators as to the desirability of including geometry in the primary grades.[13] There appears, however, to be considerably less agreement about which concepts should be included, the extent to which they should be developed, and the grade placement of the various concepts. The need for more research to indicate which concepts of geometry children should be encouraged to learn in the primary grades has been voiced by a number of mathematics educators. Miller suggests that, after studying the geometry curriculum in the Greek elementary school, it appears that we should

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[11]G. Edith Robinson, "The Role of Geometry in Elementary School Mathematics," The Arithmetic Teacher, XIII (January, 1966), p. 3.

[12]Irvin H. Brune, "Some K-6 Geometry," The Arithmetic Teacher XIV (October, 1967), p. 441.

[13]James E. Inskeep, Jr., "Primary-Grade Instruction in Geometry," The Arithmetic Teacher, XV (May, 1968), p. 422.

definitely include more geometry in our own elementary curriculum. According to Miller the only questions remaining to be answered are, "What topics do we teach in geometry?" and "At what grade level do we teach these topics?"[14] D'Augustine in reporting a study on teaching topics in geometry and topology calls for more research to determine what topics are teachable, suitable, and efficiently learnable at the various levels of the elementary school.[15] This concern is also apparent in the Cambridge Conference Report of 1963. This report states:

> The geometry portion of the curriculum seems to be the most difficult to design. Therefore the geometry discussed here for grades K, 1, and 2 represents a far more tentative groping than was the case for the work in real numbers.[16]

One of the recommendations made in the Cambridge Conference Report of 1963 was that children should be given experiences in the identification and naming of various geometric configurations.[17] Most contemporary geometry programs at the primary level do provide these experiences and further include the analysis of properties of these figures

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[14] G. H. Miller, "Geometry in the Elementary Grades A Comparative Study of Greek Mathematics Education," <u>The Arithmetic Teacher</u>, XI (February, 1964), p. 87.

[15] Charles H. D'Augustine, "Topics in Geometry and Point Set Topology--A Pilot Study," <u>The Arithmetic Teacher</u>, XI (October, 1964), p. 407.

[16] Educational Services Incorporated, <u>Goals</u>, p. 33.

[17] <u>Ibid</u>.

not based on measurement. Some of the figures commonly studied are circles and various polygons such as triangles, rectangles, squares, and parallelograms. For the most part the figures which are studied are convex. Should the child also have experiences with figures which are not convex? Should convexity (and non-convexity) be included in the analysis of properties of geometric configurations?

Only one primary grade textbook was found by the investigator to contain a mention of the concept of convexity. Suppes, in his book Sets and Numbers Book 2B, introduces the child to this concept and asks him to distinguish between shapes which are convex and those which are not.[18] Why is it that most primary grade textbooks do not consider convexity? Is it because the concept is too difficult for children at this level? Or is it because it is not important?

As mentioned earlier, one of the reasons for introducing geometry at the primary level is to provide children the opportunity to better describe their surroundings. Both convex and non-convex sets are represented by objects in the child's environment. This would appear to be a good reason for its inclusion in the curriculum at this level. The concept of convexity is also important in the mathematics studied at the high school and college levels. This aspect will

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[18] Patrick Suppes, Sets and Numbers Book 2B (New York: Blaisdell Publishing Company, 1963), p. 119.

be discussed in the next section.

The present study seeks to provide an answer to the question of the child's ability to deal with the concept of convex and non-convex sets by finding out what children of different ages understand about this concept, that is, what they understand without benefit of specific instruction.

Mathematical Background

A number of different approaches to the concept of convexity have been used by various authors. The approach which seems to the investigator to be most appropriate for this study is developed by Moise in his book Elementary Geometry from an Advanced Standpoint. He defines a convex set as follows: "A set A is called convex if for every two points P, Q of A, the entire segment $\overline{PQ}$ lies in A."[19]

In a sense, it is perhaps easier to see that a set is not convex (if it is not) than to see that it is convex (if it is). To see that a set C of points is not convex one need only find two points X and Y in C for which $\overline{XY}$ is not a subset of C. However, to prove that C is convex, one must show that for every pair of points X, Y of C, $\overline{XY}$ is a subset of C. The shaded regions shown in Figure 1, for example, are convex sets. One cannot find a segment with endpoints in the region unless the entire segment is in the region.

[19]Edwin Moise, Elementary Geometry from an Advanced Standpoint (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1963), p. 61.

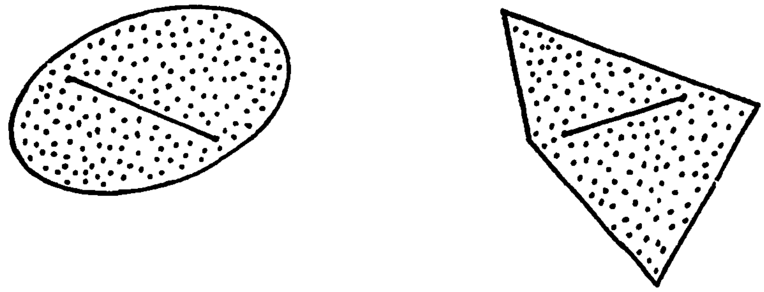

Figure 1

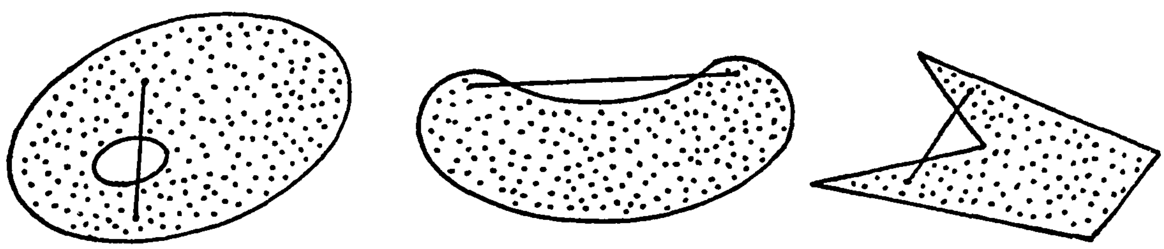

Figure 2

The shaded regions in Figure 2 are examples of sets which are not convex. For each of these regions one can find at least one segment with endpoints in the region which is not entirely contained in the region.

Sets of points other than regions may also be convex. A line, for example, is a convex set since it is certainly clear that for two points of the line the segment determined by them is contained in the line. Another example, this in three dimensions, is a sphere together with its interior. Moreover, it is clear from the definition that the empty set and a set having only one point are convex sets.

Although the convex regions in Figure 1 are bounded sets, a convex set need not be bounded. A line, a plane, and a half-plane are examples of such sets. The convex and non-convex sets which will be considered in the present study, however, will be restricted to bounded sets in two dimensions.

The concept of convexity is important in geometry as well as in other areas of mathematics such as analysis and topology. It is primarily useful in geometry in connection with the notion of separation. In the approach to plane geometry taken by Moise, this notion of separation in the plane is covered by the "Plane Separation Postulate." The use of the concept of convexity is evident in the statement of this postulate.

> Given a line and a plane containing it, the set of all points of the plane that do not lie on the line is the union of two sets such that (1) each of the sets is convex, and (2) if P belongs to one of the sets and Q belongs to the other, then the segment PQ intersects the line.

This postulate also appears in the development of geometry as presented in a number of contemporary high school geometry textbooks.

The notion of separation is fundamental to many important definitions and theorems on angles and triangles. The definition of the interior of an angle as stated by Moise, for example, is based on the Plane Separation Postulate. The theorem stating that any exterior angle of a triangle is greater than each of the remote interior angles is an example of a theorem which depends upon this postulate.

The term "convex" also occurs in plane geometry in connection with polygons. A polygon is said to be convex if and only if each side lies on a line determining a half-plane which contains the remainder of the polygon. This use of the term "convex" in geometry is inconsistent since a polygon cannot possibly form a convex set. This usage is common, however, and occurs in many textbooks at the high school and college level.

In addition to its usefulness in various areas of mathematics, the theory of convexity is now a recognized field of mathematics in its own right. Graduate and undergraduate courses dealing with convexity are being offered more frequently; in fact, one contemporary textbook for high school geometry contains an entire chapter on convexity.

A survey of the treatment of convex sets in the plane as presented in various contemporary textbooks for both high school students and undergraduates points up at least two concepts which are closely related to the concept of convexity and which, it seems to the investigator, need to be understood in order to have an understanding of convexity. These two concepts are: (1) the concept that every simple closed curve partitions the plane into three sets and (2) the concept of betweenness.

The first concept is actually a theorem called the Jordan Curve Theorem. It simply states that every simple closed curve in a plane partitions the plane into a set of points called the interior and another set called the exterior. The union of the simple closed curve, its interior and its exterior is the set of points which is the plane. The simple closed curve does not belong to either the interior or the exterior. This concept is particularly important since most of the figures considered in the study represent regions which are the union of a simple closed curve and its interior.

The second concept which was mentioned is betweenness. The meaning of the word "between" in ordinary usage is ambiguous. A building may be said to be between two others, even though it is set back from the street 50 feet farther than the others. In this instance it is not required that the three objects be in a line. The definitions used in the various approaches to plane geometry, however, do require that the objects be "on a line." Moise defines betweenness as follows:

> Let A, B, and C be three collinear points. If the distance from A to B together with the distance from B to C is equal to the distance from A to C then B is between A and C.[20]

The defining characteristic of a convex set S is that for every pair of points A and B of S, the segment AB is a subset of S. This means that for every pair of points A and B of S, the set of all points "between" A and B are also contained in S.

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[20]*Ibid.*, p. 51.

The Problem

Statement

What is the child's understanding of the concept of convexity at each of three age levels--five years, eight years, and eleven years--as evidenced by responses to various tasks presented to the children in individual interviews?

Analysis

Solution of the problem called for answers to the following questions.

1. What is the child's intuitive understanding of the defining characteristic of convex and non-convex sets?

2. What is the child's intuitive understanding of the notion that a simple closed curve partitions a plane into three sets: the simple closed curve, the interior, and the exterior?

3. What is the child's intuitive understanding of the mathematical concept of betweenness?

Limitations

1. This investigation was limited to forty-eight children selected from two elementary schools in the Norman, Oklahoma, school system. Sixteen children were selected from each of three age levels--five years, eight years, and eleven years.

2. Information for the study was obtained from individual interviews in which children were questioned

about four tasks devised by the investigator.

Related Research

A number of studies related to the study of geometry in the elementary grades are contained in the literature of mathematics education and psychology. Although there appears to be practically no research dealing directly with the child's understanding of the concept of convexity, there are a number of studies dealing with concepts closely related to it. The most impressive evidence as to the ability of primary children to understand various geometric and space concepts comes from the work of Jean Piaget. Several of Piaget's experiments, reported in The Child's Conception of Space, are of particular interest to the present study.

In the first experiment, Piaget has children manually explore various objects which are out of sight behind a screen. He then asks them to match the objects with duplicates which are visible. In a second experiment, the child is asked to draw a series of such objects. The results of these studies are summarized by Flavell as follows:

> By the time the child is 3-4 years old he can generally discriminate objects (both manually and in his drawings) on the basis of topological differentiae. For example, he can distinguish a closed from an open figure, an object with a hole in it from one without a hole, and a closed loop with something inside from one with the something outside or on the loop's boundary. But the ability to discriminate between rectilinear and curvilinear figures and, a fortiori, among figures of each type, does not develop until several years later. Thus the same child who can readily distinguish an open from a closed circle may be quite unable to discriminate between the closed circle and other, rectilinear closed

figures such as squares or diamonds.[21]

A third study by Piaget involved linear and circular order. He found that a child of four or five years of age can reproduce an order only when the situation permits of visual correspondence. For example, when the child of this age was presented with a model consisting of seven beads of different colors on a rod, he was able to reproduce the order by slipping a similar set on another rod. Piaget states that "apart from this, at this level there is absolutely no comprehension of the relationship 'between.'"[22] He further found that "towards the age of six or seven, children arrive at what may be considered a stable and rational conception of direct and reverse order."[23] With the ability to reverse order comes the ability to comprehend the relationship "between." The child can now see that if B is "between" A and C, then B is also "between" C and A.[24]

Piaget points out that in the study of order, the relation "between" is actually a particular instance of a one-dimensional surrounding. In the case of a closed curve, a segment connecting a point within the closed curve with a

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[21] John H. Flavell, The Developmental Psychology of Jean Piaget (Princeton: D. Van Nostrand Company, Inc., 1963), p. 329.

[22] Jean Piaget and Barbel Inhelder, The Child's Conception of Space (New York: W. W. Norton and Company, Inc., 1967), p. 83.

[23] Ibid., p. 101.

[24] Ibid., p. 104.

point outside the boundary must necessarily cut the boundary. Such a surrounding is thus two-dimensional. Finally, a three-dimensional surrounding is produced by the notion of something enclosed within a box in such a way that in order to grasp the object it is necessary to open the box.[25]

In the fourth of these studies, Piaget concerned himself with this notion of "surrounding" or "enclosure." In order that the tasks in the study not be overly familiar to the children, Piaget used strings with knots tied in them. He found that the main obstacle to the child's tying a knot was the transition from one dimension to another, within one and the same object. He states that:

> The problem is one of passing from the unwound string (a simple linear series with a "surrounding" in only one dimension) to the string in the form of a loop in two dimensions, and from this to finally passing one end of the same length of string through the inside of this loop (the loop and the end of string crossing its interior plane constituting a "surrounding" in three dimensions).[26]

It was found that the ability to link together the notions relating to each dimension, in so far as they depend upon the idea of "surrounding," does not begin to be developed until about six years of age.[27]

In the study of "surrounding" it was necessary that the curve which was visualized as defining the surrounding

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[25] *Ibid.*, p. 111.

[26] *Ibid.*

[27] *Ibid.*, p. 124.

be continuous and unbroken. Piaget suggests that the child's ability to draw together into an organized whole the notions considered in the four previous studies will depend upon the child's understanding of continuity.[28]

In the fifth study, Piaget examines the development of the notion of continuity from the form in which it first appears to that which it exhibits when formal thought emerges at the age of eleven or twelve. Piaget first attempts to find out how the child visualizes the subdivision of a line or figure. He then asks about the shape of the end product, to see if it is a point or not, and whether or not the point has shape. Finally, he asks about the re-creation of the line or figure out of its ultimate elements, in order to determine whether the child can conceive of the line or figure as a collection of points.[29]

For children up to seven or eight years, Piaget found the following:

> When trying to break up a line or surface he can only make a very limited number of subdivisions, . . . they end up with so-called ultimate elements of a distinctly perceptible size which, curiously enough, are of the same shape as the original, the final terms of the square being square, those of the line being lines, and so on. Finally, subdivision and reassembly are both irreversible. The line is not regarded as a collection of points and if one has been rash enough to break up the short sections into points, these are fated to remain for ever discontinuous.[30]

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[28] Ibid., p. 125.

[29] Ibid., p. 128.

[30] Ibid.

For children from seven or eight to eleven or twelve, Piaget found:

> . . . a greater flexibility in the treatment of subdivision. But though the child is now prepared to admit the possibility of a large number of subdivisions he does not regard them as being infinite. . . . they are never generalized beyond the finite, beyond visible or tangible size. While the ultimate elements are no longer thought of as isomorphic with the original whole, . . . their shape is regarded as dependent upon the particular mode of subdivision and they are never envisaged as an infinite number of points without surface area. Lastly, construction of the whole out of its constituent elements is now seen as the reverse counterpart to subdivision, but this goes no further than a purely intuitive continuity, so that the child finds himself in a dilemma, since he cannot reconcile the discontinuous nature of the points which are to be reunited, with the continuity possessed by the structure which results from this reunion.[31]

Lastly, beginning at about eleven or twelve years, he found:

> . . . subdivision is conceived of as unlimited. As for the structure of the ultimate elements, from now on this is seen to be entirely independent, of the shape of the original figure or the mode of subdivision. The points . . . no longer possess either shape or surface and, most important of all, they are all homogeneous, whether they belong to a line or any sort of figure. The synthesis of the whole is now the reverse product of unlimited subdivision, although the children still seem to find a contradiction between the discontinuous points and the continuous whole formed from them.[32]

Finally, a sixth experiment by Piaget which is of interest to the present study involves the development of the concept of the straight line. According to Piaget, the construction of a straight line may be defined as the linking of distant points by the interpolation of a series along

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[31] Ibid., p. 129.

[32] Ibid.

a direct path. For this experiment a square and a round table are used, on which stand a number of match sticks stuck into bases made of plasticine. The first and last match sticks are placed in position and the child is asked to place the remaining match sticks in a straight line between these two sticks. He found that children between the ages of four and seven can form a line more or less correctly when it runs parallel to the edge of the table, but is unable to do so when the line lies at an angle to the edge. Starting at about seven years the straight line is constructed no matter where it lies on the table and this is done by the child by sighting along the path.[33]

Replications of these studies have tended to support Piaget's findings. Dodwell (1963) replicated the second, fifth and sixth of these studies. The subjects in his study were 194 children ranging in age from five years and one month to eleven years and three months. With regard to the study involving drawings of shapes, he found that

> . . . the level of drawing competence increased with age. The youngest children frequently did not draw their shapes even topologically correctly, but were more often correct in a topological than in a projective or Euclidean sense.[34]

With regard to the remaining two studies, his findings are essentially the same as those of Piaget.

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[33] *Ibid.*, p. 158.

[34] P. C. Dodwell, "Children's Understanding of Spatial Concepts," *Canadian Journal of Psychology*, 17 (March, 1963), 141.

Lovell (1959) replicated the first four and the sixth of the studies by Piaget. The subjects in his study were 150 children between three years and six years of age. With regard to the first two studies, he found that

> There is little evidence to suggest that it is topological properties, as such, which enable a child to identify certain shapes more easily than others. It appears that gaps, holes, curves, points, corners, ins and outs, etc., in euclidean space make identification easier. Again, our data do not give much support to the view that up to four years of age children cannot distinguish between a circle, square, ellipse, etc., because these are all closed figures. But our evidence does support very strongly their view that straight sided euclidean shapes with relatively long sides and few corners (square, rhombus, quadrilateral, etc.) are hardest to identify.[35]

In the third and fourth studies, his findings differ mainly in that he found his subjects tended to perform on a higher level than Piaget's subjects of the same age. In replicating the sixth study, he found, as did Piaget, that before age six aiming does not take place to any extent. However, he disagrees as to the age at which a child is able to make straight lines. He found that 53 per cent of the children below four years of age were able to make a good straight line.[36]

Peel (1959) reports on replications of the first two of Piaget's experiments. The first study was replicated by Page who tested sixty children between the ages of 2

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[35]K. Lovell, "A Follow-Up Study of Some Aspects of the Work of Piaget and Inhelder on the Child's Conception of Space," The British Journal of Educational Psychology, XXIX (June, 1959), p. 104.

[36]*Ibid*.

years and 10 months and 7 years and 9 months. Ferns repeated the second experiment with fifty-five children between the ages 2 years and 9 months and 7 years and 9 months. Both of these studies clearly confirmed the general findings of Piaget.[37]

D'Augustine conducted an investigation with one sixth-grade class. The experimental design employed a programmed unit developed by the researcher, which attempted to teach such topics as properties of points, lines, simple closed curves, convex and non-convex regions, congruency and polygons. Each student worked independently and at his own rate. On completion of the program he was given a test on topics relating to geometry and topology.[38]

Test questions which related directly to topics taught via the program were categorized into fourteen general topics. Each of these categories was analyzed to determine the degree of success each of the top seventeen students had with each category. Eight of the fourteen categories were classified as highly teachable on the basis of this analysis. Among those meeting with only limited success was the property of convexity and non-convexity of simple closed curves. No attempt was made in this study to determine the reason

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[37]E. A. Peel, "Experimental Examination of Some of Piaget's Schemata Concerning Children's Perception and Thinking, and a Discussion of their Educational Significance," The British Journal of Educational Psychology, XXIX (June, 1959), p. 89.

[38]D'Augustine, "Topics," p. 407.

these topics met with limited success.[39]

Denmark conducted a study to assess students' concepts of a point lying between two other points. Two groups of subjects were involved in the study. One group was a class of twenty-five first graders. Each child was tested individually. The test consisted of nine items which were presented one at a time. After completing the nine items the child was asked to explain his response. From the analysis of each student's responses, test papers, and comments, the researcher found the following:

1) Seventeen of the children utilized the Euclidean concept of betweenness in responding to the test questions.
2) Five of the children consistently interpreted "betweenness" as "in the middle." By this they meant that a point was within the region bounded by parallel lines through the endpoints.
3) Three children in the class did not show evidence of possessing a well-formulated concept of betweenness.
4) The introduction of a curve through the dots and the indicated point brought out a third meaning of betweenness. The explanation given by several children was "the point is on a path between the dots."[40]

Summary

The desirability of including geometry in the primary grades is recognized by many mathematics educators. However, the need for more research to determine what topics are suitable at the various grade levels is also widely

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[39] Ibid.

[40] Tom Denmark, "An Intuitive Introduction to the Euclidean Concept of Betweenness," The Arithmetic Teacher, 15 (December, 1968), p. 683.

recognized. One possible way to obtain this information about a particular concept is to seek information about the child's intuitive understanding of the concept without benefit of specific instruction.

The concept of convex sets has been largely left out of the primary grade curriculum. There appear, however, to be some good reasons for the inclusion of this concept provided children can learn it efficiently. The present study seeks to obtain information about the child's intuitive understanding of convex and non-convex sets. The information thus obtained should be of interest and importance to mathematics educators in selecting topics for inclusion in the primary grade curriculum and in writing instructional materials involving this concept.

CHAPTER II

THE STUDY

The Sample

The sample for the study was selected from elementary schools of the Norman, Oklahoma, school system. Several considerations were involved in the selection of schools for the study. In order to include children of five years of age it was necessary to include kindergarten children in the study. This posed a special problem since the kindergarten is not a part of the public school system. Six of the eleven elementary schools in Norman do, however, have special tuition kindergartens under the sponsorship of the Norman Kindergarten Association, which are held in portable classroom buildings on the school grounds. The presence of a kindergarten then was one of the considerations in the selection of schools. A second consideration was the availability of a relatively quiet place to conduct the interviews. Finally, a third consideration was the social and economic structure of the neighborhoods from which the children in each school came. With these considerations in mind, the selection of schools for the study was discussed with Lester Reed, Norman

Superintendent of Schools. Based upon his advice, three schools--Jackson Elementary School, Cleveland Elementary School, and Monroe Elementary School--were selected for the study. Jackson Elementary School was used only for the pilot study; the other two were used for the main part of the study.

The selection of the subjects for the main part of the study was carried out in the following manner. Each principal made enrollment cards available to the investigator. Lists of students from each school who were approximately 5 years, 8 years, and 11 years of age were then obtained from these cards. More precisely, 5 years was defined as being between 5 years 3 months and 5 years 9 months. Eight years was defined as being between 7 years 9 months and 8 years 3 months, and 11 years as being between 10 years 9 months and 11 years 3 months (see Tables 1 and 2). From these lists, sixteen children were selected at random from each of the three age levels. Thus a total of forty-eight

TABLE 1

THE NUMBER OF CHILDREN FROM WHICH THE RANDOM SELECTION WAS MADE, GIVEN BY SCHOOL AND BY AGE LEVEL

School	Five Years	Eight Years	Eleven Years
Cleveland	28	38	56
Monroe	20	40	39

TABLE 2

THE AVERAGE AGE IN MONTHS OF THE SAMPLE, GIVEN BY SCHOOL AND BY AGE LEVEL

School	Five Years	Eight Years	Eleven Years
Cleveland	67	97	133
Monroe	67	97	132

children from the two schools were selected. In addition, two alternates at each age level from each school were selected at random to allow for any unexpected occurrences.

Description of Participating Schools

The following brief descriptions of the participating schools were obtained from discussions with the principals of the respective schools. All three schools are located in neighborhoods of single family dwellings. The neighborhoods served by Jackson Elementary School are made up of low to middle income families. About one-half of the families are professional and business people. Many of these are connected with the university. The remainder of the families are for the most part people working in the various trades. One or both of the parents of perhaps one-half of the children at Jackson have some college training.

The parents of children attending Monroe Elementary School are for the most part middle and upper-middle income, professional and business people. Many university professors live in this area. Most of the parents of these children have had at least some college work and many are college graduates. In addition, there are a number of very affluent families as well as several low income families living in the various neighborhoods served by Monroe.

The families of children at Cleveland Elementary School are about equally divided between middle and upper-middle income, professional and business people and low income to lower-middle income people working in the trades. One or both of the parents of perhaps one-half of the children at Cleveland have some college training.

The Tasks

Introduction

A series of four multi-step tasks were constructed to enable the child to reveal his intuitive understanding of the concept of convexity. The tasks are called, respectively, the Dog Task, the Tack Task, the Inside-Outside Task, and the Betweenness Task. The Dog Task and the Tack Task were initially developed during the spring semester of 1970 as an assignment in a course given by the investigator's major professor. These tasks were revised and refined, and the Inside-Outside Task was developed during the fall semester of 1970. The Betweenness Task was developed by Denmark[1] for use in a study which was reported in the

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[1] *Ibid.*

review of related research. During and following the formulation of the tasks, the investigator also drafted an interview schedule to be used with the tasks.

Important to the development of the tasks were the many discussions which the investigator had with members of his doctoral committee as well as with fellow doctoral students. As a result of these discussions, a number of changes and refinements were made in the various tasks.

The investigator also administered the various tasks to his own children as well as to other children in the neighborhood. This proved to be especially helpful in pointing out difficulties with regard to procedures and questioning.

Pilot Study

The pilot study was designed to provide the investigator an additional means of evaluating the tasks, of judging the appropriateness of the questioning, and of gaining experience in the techniques of interviewing children.

The pilot study sample consisted of fifteen children from Jackson Elementary School. The principal of Jackson selected five children from each of the three age levels--five years, eight years, and eleven years. The children within each age level were selected from various levels of ability in order to provide the investigator an opportunity to react to as wide a variety of responses as possible.

In order to give the provisional sequence of tasks and questions a fair run, no attempt was made during the course of the pilot run to modify the interview schedule or tasks. The interviews ranged from 22 minutes to 40 minutes in length. Each interview was taped and later evaluated in terms of a provisional rating scheme.

The results of the pilot study indicated several procedural changes which needed to be made in order to improve the interview. Perhaps the most obvious of these was the need for a quiet place to conduct the interviews. This fact was given serious consideration in the subsequent selection of schools for the remainder of the study. The pilot study also indicated that certain portions of the various tasks were not appropriate and needed to be changed or eliminated.

Reliability of the Tasks

Nine of the fifteen children--three at each age level--interviewed in the pilot study, were interviewed again after a four-week period. These children gave the same response in both instances on 89 per cent of the items, with the five-year-olds responding the same on 80 per cent, the eight-year-olds on 90 per cent, and the eleven-year-olds on 97 per cent of the items.

Validity of the Tasks

The procedure used by the investigator to establish content validity of the tasks is one suggested by Kerlinger.

> Content validation consists essentially in judgment. Alone or with others, one judges the representativeness of the items. . . .
> . . . .
> . . . each item must be judged for its presumed relevance to the property being measured, . . .[2]

The investigator's judgment of the content validity of the tasks is based upon continual evaluation during the time the tasks were being formulated and refined, as was described previously. In addition, the investigator sought the judgment of other competent individuals in mathematics and mathematics education.

Description of the Tasks

Preliminary Tasks. In order to determine that the child understood the terminology used in the questioning, two preliminary tasks were devised. The first of these involved the terminology used in the Dog Task. The child was asked to stand 7 or 8 feet in front of the investigator, and a cardboard screen was placed to one side. The child was asked, "Are you facing me? Can you see me?" He was then told to turn around and look the other way and was again asked, "Are you facing me?" After being told to again turn around and face the investigator, the child was asked if there was something he could do so that he would be facing the investigator but would not be able to see him. If the child did not suggest standing behind the screen, he was asked, "Can

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[2] Fred N. Kerlinger, Foundations of Behavioral Research (New York: Holt, Rinehart and Winston, Inc., 1964), p. 446.

you go stand behind the screen so that you are facing me?" When the child was in position behind the screen, he was asked, "Can you see me?" To complete this preliminary task, the child was asked to be seated, and two dogs were placed on the table. The child was then asked, "Can you place the dogs so they are facing each other?" He was then told to pretend that the dogs were real, and was asked, "If those were real dogs, would they be able to see each other?" If the child seemed uncertain of the terminology at any point in this task, the wording was changed and the question repeated until it was clear to the investigator that the child understood what was asked of him.

The second preliminary task involved the terminology used in both the Tack Task and the Inside-Outside Task. A hula hoop was placed on the floor and the child was asked to stand inside the hoop. The child was then asked, "Are you standing inside the hula hoop? Am I standing inside the hoop? Where am I standing?"

Dog Task. This task sought information about the child's understanding of the defining characteristics of convex and non-convex sets. A sequence of five walled enclosures of various shapes (Figure 3) was presented to the child, one enclosure at a time. The child was first asked, "Can you place the dogs in the room so that they are facing each other and can see each other?" The child was then asked, "Can you place the dogs in the room so that they are facing each other but cannot see each other?" If a child was unable to place the dogs so that they were "facing each other but could not see each other" in any of the enclosures, the investigator demonstrated how this might have been done using the first enclosure. The child was then given the fifth enclosure and asked if it would now be possible for him to place the dogs as was done in the first enclosure. If the child was successful with this enclosure, he was also questioned again about the three remaining enclosures.

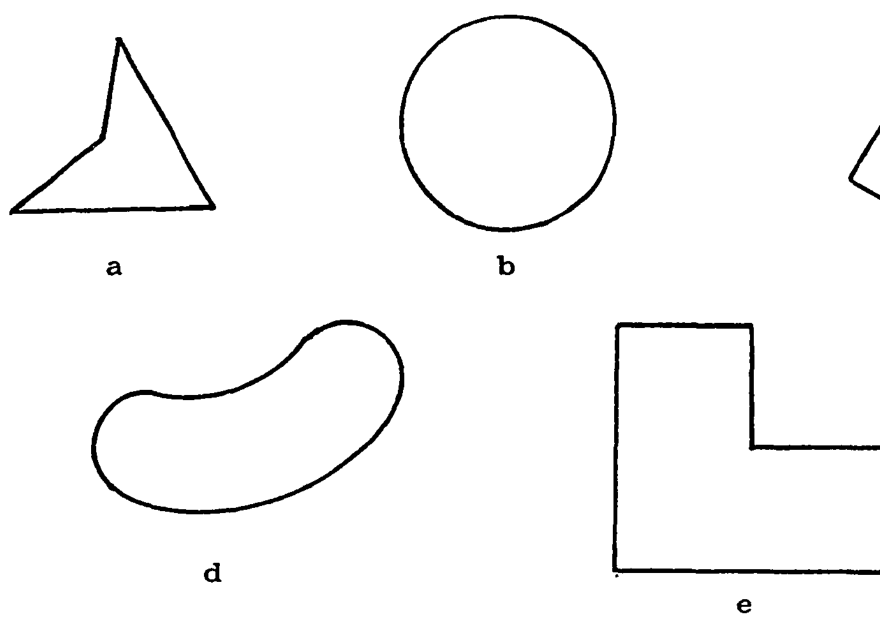

Figure 3

When the child had completed the sequence, all of the enclosures were placed within view of the child and he was asked, "Why were you able to place the dogs so that they were facing each other but could not see each other in some of the enclosures but not able to do so in others?"

<u>Tack Task</u>. The Tack Task also sought information about the child's understanding of the defining characteristics of convex and non-convex sets. This task consisted of three main parts. The first part involved a sequence of seven simple closed curves (Figure 4) each of which was drawn on a separate sheet of paper. A small tackboard was

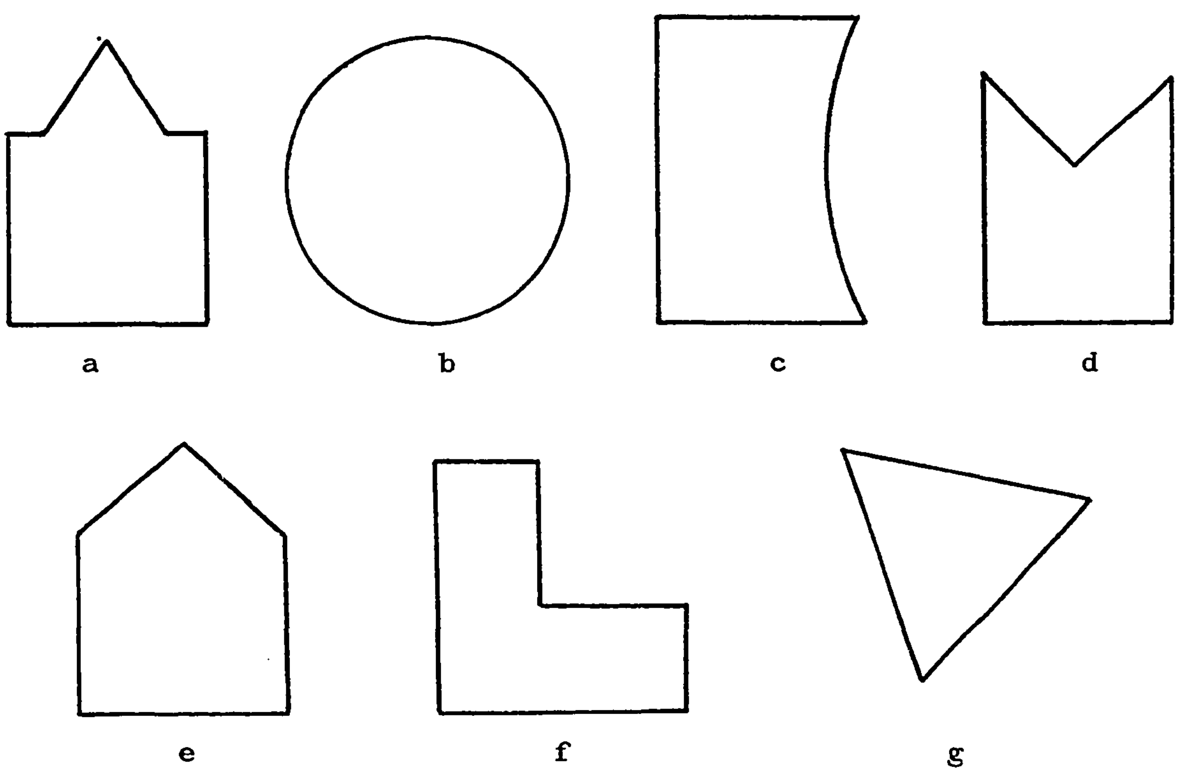

Figure 4

placed on the table in front of the child and he was given a pair of tacks which were connected by a short piece of elastic string. The elastic string made it possible to place the tacks various distances apart and still keep the string stretched between them. The seven sheets were placed in turn on the tackboard, and the child was asked, "Can you place the tacks inside the figure in such a way that the string connecting the tacks is stretched straight?" The child was then asked, "Is all of the string inside the figure too?" Finally, the investigator asked the child about placing the tacks inside the figure in such a way that the string connecting the tacks was stretched straight but a part of the string was outside the figure. After completing the seven items the child was shown the entire sequence and was asked, "Why were you able to find places to put the tacks inside some of the figures so that a part of the string got outside and on others you couldn't?"

The second part of the task involved a sequence of three figures (Figure 5) which were cut out of construction paper. These also were placed on the tackboard (in the order indicated in Figure 5) and for each figure the child was asked, "Can you place the tacks so that they are on the construction paper but so that a part of the string gets off of the construction paper?" This part of the task, although quite simple, served two purposes. It provided those children

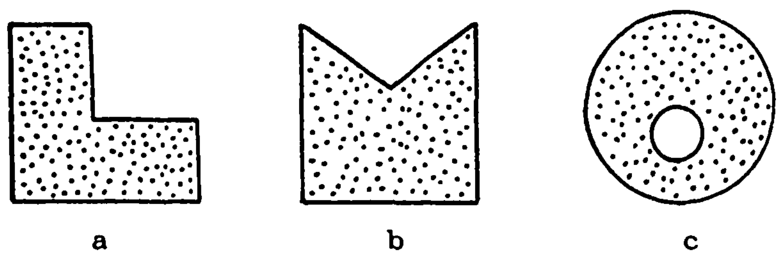

Figure 5

who had difficulty on the first part of the task an additional opportunity to respond, and it served to introduce the final part of the task.

The third part of the task also involved the three construction paper figures. The child was permitted to attempt this final part of the task only if he had been successful on both the V-shaped and L-shaped figures in the previous part. The figures were considered in the order in which they are shown in Figure 5. Beginning with the L-shaped figure the child was asked, "Can you cut this figure into two pieces and then put it back together again in such a way that for the new figure you would not be able to place the tacks on the construction paper so that a part of the string gets off the construction paper?" In order to clarify this situation, the investigator pointed out to the child that in the previous part of the task he had been able to find places to put the tacks on the construction paper so that a part of the string got off the paper. Those children who were able to do what was asked on the first figure were also asked about each of the other two figures. If a child indicated that he would be unable to do what was asked on the L-shaped figure, the investigator would do the V-shaped figure for him. The child was then asked if he could now do what was asked on the L-shaped figure. If he now succeeded with this figure, he was also asked about the circular figure.

Inside-Outside Task. This task sought information concerning the child's understanding of the notion that a simple closed curve partitions the plane into three sets: the simple closed curve, the interior, and the exterior. The task consisted of two identical sequences of simple closed curves, each of which was drawn on a separate sheet of paper (Figure 6). In addition to the curve, a red dot was placed somewhere on the sheet. The first sequence was presented to the child one sheet at a time and he was told, "As you are shown each page, look to see where the red dot is and decide if it is inside the figure or not. If you

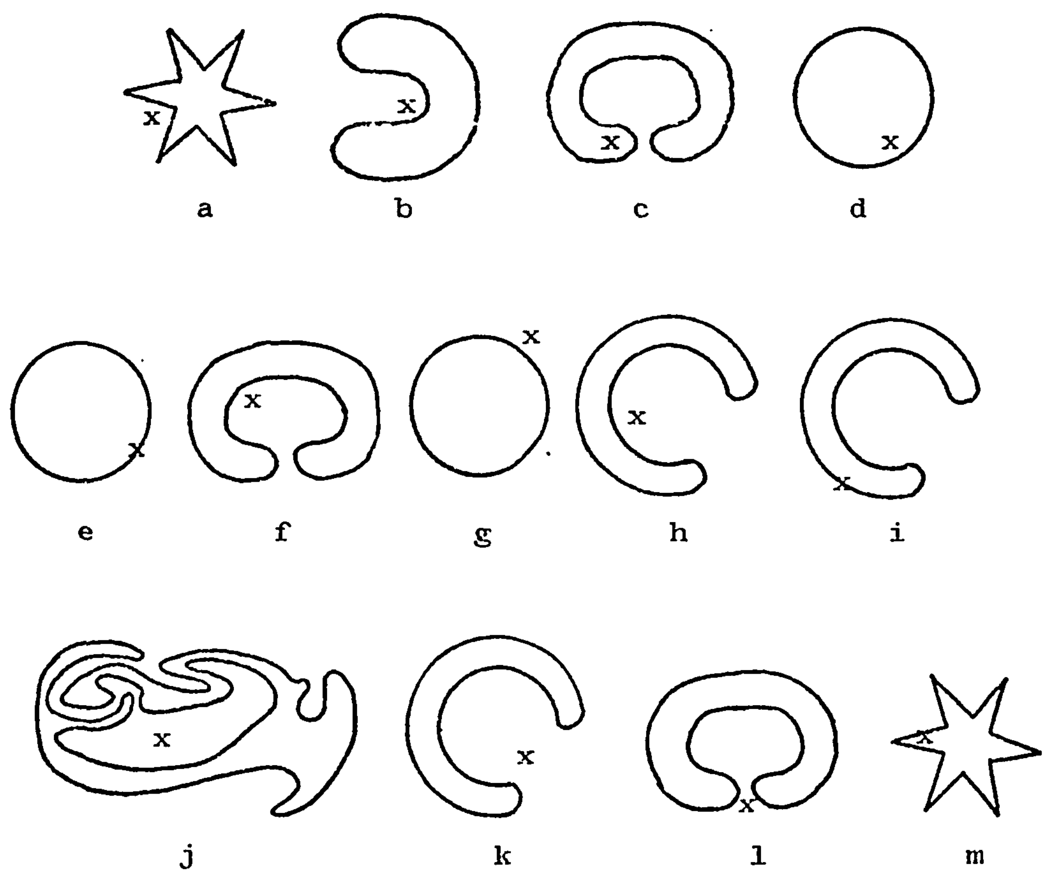

Figure 6

decide the red dot is inside the figure, put a circle around it." Upon completion of the first sequence, the child was given the second sequence which was identical to the first. This time the child was asked to circle the red dot if it was outside the figure.

After completing both sequences the child was questioned about certain of the figures. Each child was questioned about figures e, i, and m in order to find out why he did or did not circle these dots which represented points of the simple closed curve. The children were also questioned about any other responses which appeared contradictory or about which they indicated uncertainty. Whenever possible, the child was encouraged to make additional drawings on the figures in order to clarify his explanations.

Betweenness Task. This task sought information concerning the child's concept of a point lying between two other points. The task consisted of a sequence of eleven items (Figure 7) which were presented to the child. Each item consisted of two blue dots and one or more red dots (each represented by an x in Figure 7) placed on a sheet of paper. As each item was presented to the child, he was asked to circle the red dot if he thought it was between the blue dots. After the child had responded to all eleven items, he was asked to tell why certain red dots were between the blue dots and others were not. The child was again encouraged to make drawings to help clarify his explanations.

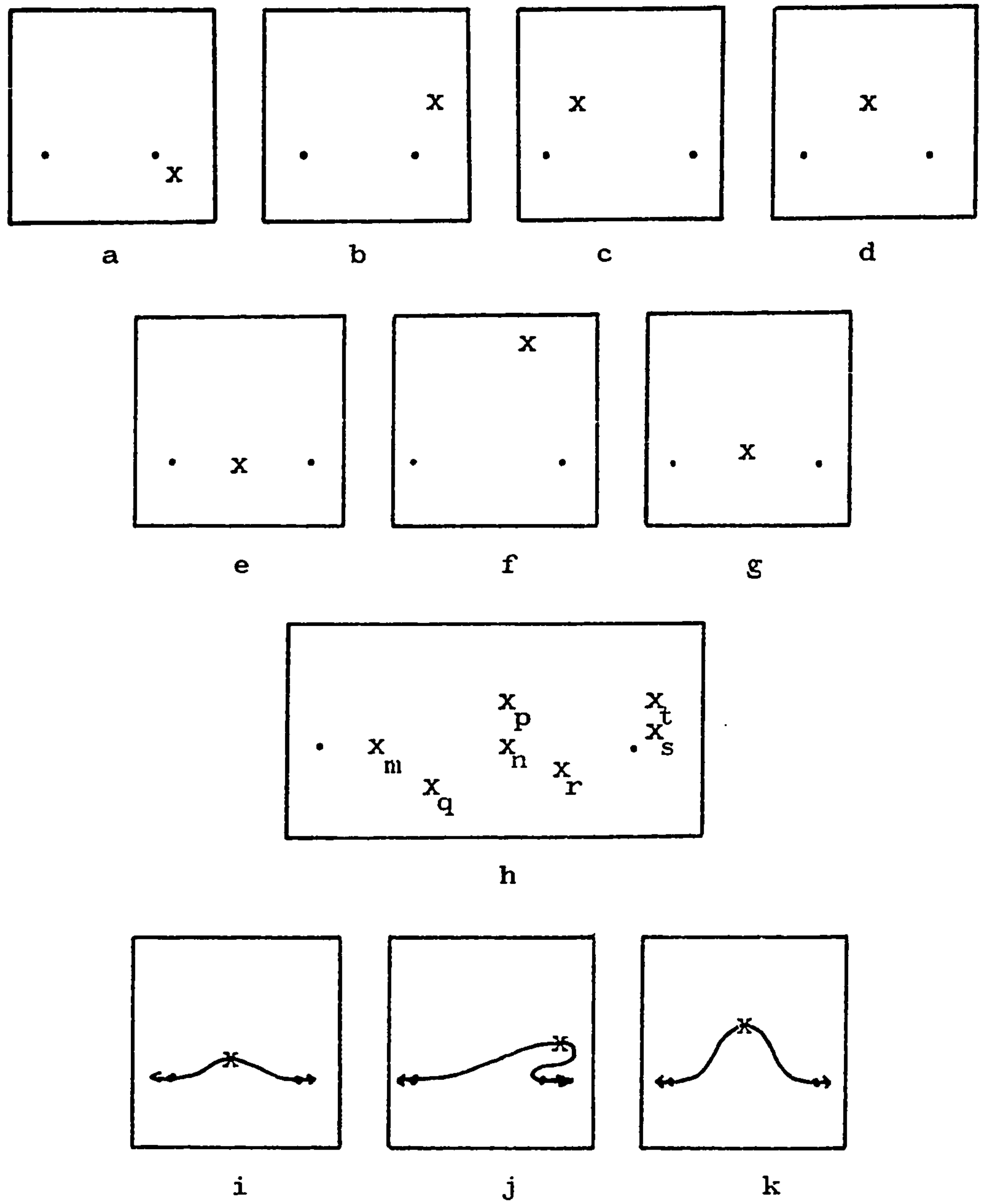

Figure 7

<u>Administration of the Tasks</u>

With the exception of the school used in the pilot study, each school provided the investigator with a quiet room in which to conduct the interviews. The eight- and eleven-year-old children in each school were called out of class individually by either the principal or her secretary. This individual also introduced the child to the investigator and explained that the investigator had devised a game and would like to have the child help him with it. Since the kindergarten was not directly connected with the other grades, it was necessary to follow a somewhat different procedure with these children. In each school the children who had been selected were introduced as a group to the investigator. The teachers told the children that they had been selected to play a game with the investigator and that it would be fun for them. In each case the children seemed eager to cooperate.

At the outset of the interview the child was seated at a table on which were placed the various materials used in the interview. The child was told, "I am going to ask you to do some things. Then I will ask you some questions about what you did. The answer is what you think it is, not what I want you to say." All of the children were presented exactly the same sequence of tasks. The order of presentation was the same as that described in the previous section.

CHAPTER III

RESULTS OF THE STUDY

The information obtained in this study was for the purpose of determining the child's understanding of the concept of convex and non-convex sets. The investigator constructed four multi-step tasks which were presented to children in individual, tape recorded interviews. Sixteen children at each of three age levels--five years, eight years, and eleven years--were used in the study.

Analysis of the Interviews

The results for each task will be presented in separate sections along with a somewhat detailed description of the manner in which that portion of the interview was analyzed. In addition, sample protocols will be presented to illustrate at least some of the categories used in the rating schemes.

The rating scheme used with three of the tasks was used previously by two different investigators. It was used by Almy[1] in a study of the understanding of the

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[1]Millie Almy, Young Children's Thinking (New York: Teachers College Press, Columbia University, 1966), p. 67.

principle of conservation among young children and also by Taback[2] in a study of children's understanding of the concept of limit. This scheme provides a total of five categories to rate the subject's responses:

Clear evidence of understanding

Some evidence of understanding

Uncertain evidence of understanding

Clear evidence of not understanding

Evidence lacking

The Betweenness Task and one portion of the Inside-Outside Task did not lend themselves to this particular rating scheme. Separate rating schemes were consequently devised for these tasks and will be described in the appropriate sections.

Reliability of the Rating Scheme

All of the interviews were rated by the investigator. In order to measure the reliability of the rating scheme, nine of the tape recordings were rated independently by a fellow doctoral student who was familiar with the study and who had followed its progress closely. These nine tape recordings were randomly selected--three at each of the three age levels.

A total of thirty-eight responses in each of the

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[2] Stanley Frederick Taback, "The Child's Concept of Limit," (unpublished Ph.D. dissertation, Columbia University, 1969), p. 48.

nine interviews were evaluated. Upon comparison, the two sets of ratings were found to agree on 84 per cent of the items.

Major Results

Results for the Preliminary Tasks

In the Dog Task the child was asked, "Can you place the dogs in the room so that they are facing each other but cannot see each other?" In order to determine the child's understanding of the terminology used in this questioning, a preliminary task involving the use of a screen placed between the subject and the investigator was utilized.

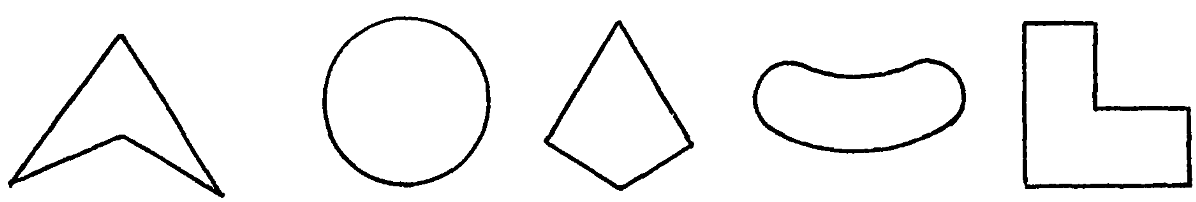
All of the eight- and eleven-year-old children understood the terminology clearly. For some of the five-year-olds, however, the words "facing each other" caused some difficulty. It was found that these children did understand the meaning of "facing in each others direction." Consequently for these children the wording for this question was changed in order to clarify the terminology. With this one adjustment in terminology, all of the children rated "Clear evidence of understanding" with regard to this preliminary task.

A second preliminary task involved the child's understanding of "inside" in a very physical situation. The child was simply asked to stand inside a hoola hoop which was placed on the floor. All of the children rated "Clear evidence of understanding" on this task.

Results for the Dog Task

The Dog Task sought information about the child's understanding of the defining characteristics of convex and non-convex sets. This task was the most physical and was most helpful in creating the impression that the tasks were more of a game than a test. This was especially true for the five-year-old children.

In the Dog Task the child was presented with five enclosures of various shapes. These were presented in the order shown in Figure 8. The child was then asked about the possibility of placing two dogs inside the enclosure in such a way that the dogs were facing each other but could not see each other. Upon completion of the task he was asked why he was able to do this in some of the enclosures and not able to do it in others.



a b c d e

Figure 8

The child's responses to each of the five enclosures was rated using the five categories indicated earlier. On the basis of these ratings and the questioning which followed completion of the task, an over-all rating was given of the child's understanding of convexity as reflected by this task, again using these five categories.

With regard to responses on individual enclosures, a rating of "Clear evidence of understanding" was given if the child responded correctly with very little hesitation. A rating of "Some evidence of understanding" was given if the correct answer was given only after considerable hesitation. "Uncertain evidence of understanding" was the rating given if the child was correct but indicated that he was quite unsure of his answer. If the child responded incorrectly he was given a rating of "Clear evidence of not understanding." Finally, "Evidence lacking" was reserved for a situation in which the response could not be determined from the tape or the investigator failed to ask for a response.

Since the over-all rating for this task took into consideration the child's explanation, in addition to responses to the individual enclosures, it was somewhat more difficult to specify in advance what a child must do to receive a particular rating. This was especially true for the second and third categories. In the discussion which follows, an attempt is made to specify, at least in general, the kind of response which was given a particular rating. In addition, examples will be given of children who fell into the various categories.

Six per cent of the five-year-olds rated "Clear evidence of understanding" on this task as compared to

56 per cent of the eight-year-olds and nearly 70 per cent of the eleven-year-olds. Most of the children receiving this rating fell into the "Clear evidence of understanding" category on enclosures a, d, and e, and rated at least "Some evidence of understanding" on enclosures b and c. DA, eight years of age, is an example of a child who received this rating. He responded correctly without hesitation on enclosures a, d, and e, but answered correctly only after some hesitation on enclosures b and c. The questioning which followed completion of the task is presented in the following protocol.

> DA (age 8). Why were you able to find a place to put the dogs so that they are facing each other but can't see each other in three of the rooms, and were unable to do so in the other two?--"Because there is no place they can do it."--In other words there is something different about these three rooms; is that what you are saying?--"Yes."--What is it that is different about the three rooms? Can you point it out?--"The ones that are kind of dented in you can do it."

More than one-third of the five-year-olds and, with one exception, the remainder of the eight- and eleven-year-olds rated "Some evidence of understanding." Many of these children received a rating of "Clear evidence of not understanding" on one of enclosures d and e. A number of these children also rated "Some evidence of understanding" or "Uncertain evidence of understanding" on enclosures b and c. Most of these children responded only after making several attempts at placing the dogs. Further, their responses were almost entirely dependent upon what they found as a result of these attempts. KC, eight years of age, is an example of a child who rated "Some evidence of understanding." She rated "Clear evidence of understanding" on enclosures a, b, c, and d, but rated "Clear evidence of not understanding" on enclosure e. The questioning which followed completion of this task is presented in the following protocol.

> KC (age 8). Can you tell me why you could do it in these two but couldn't do it in these others?--"For one thing this (referring to enclosure d) has a bump up here and you could put them behind it."--How about on number one?--"It's sort of like a triangle except it goes in."--Don't any of the other enclosures have something like that?--"No."

Five children received a rating of "Uncertain evidence of understanding." All of these children responded correctly on at least some part of the task and incorrectly on other parts. However, as was suggested earlier, it was nearly impossible to specify in advance the type of response which would receive this rating. Protocols for two of the five children will be given to illustrate responses rated in this category. LB, five years of age, rated "Clear evidence of understanding" on enclosure a, "Some evidence of understanding" on enclosures b and c, and "Clear evidence of not understanding" on enclosures d and e. The questioning following completion of the tasks follows:

> LB (age 5). Why could you do it in number one, but you couldn't do it in these other rooms?--"Because there was an easier place to put them."--There wasn't any such place in these others?--"Nope."

SN, five years of age, was another child who rated in this category. He indicated that it would not be possible to place the dogs in the first enclosure so they are "facing each other but cannot see each other." Before continuing with the task, the investigator demonstrated how this might be done; he then asked the child to attempt the fifth enclosure. SN responded correctly to this enclosure as well as to enclosures b and c. However, he responded incorrectly to enclosure d. The questioning following completion of the task follows:

> SN (age 5). You were able to do it in e, but you couldn't do it in b and c; can you tell me why? Is there something different about b and c?--"Because this one (enclosure b) is round and that one (enclosure c) is like a kite, and if you put them somewhere they couldn't see, and they would be facing each other, and they could still see."--Why were you able to do it in e? Is there something different about it?--"I don't know."

The responses of the four children who rated in the "Clear evidence of not understanding" category were nearly identical. LS, five years of age, was one of these children. She was unable to find a place to put the dogs so that they were "facing each other but could not see each other" in any of the enclosures. The investigator then demonstrated that it was possible to do so in the first enclosure, and asked the child if she could now find a place in enclosure e. After a number of attempts the child indicated that she could not do it.

The over-all results for the Dog Task are shown in Table 3. The results for each of the individual enclosures are shown in Table 4 where the number of the enclosure corresponds to the order in which the enclosures were presented as shown in Figure 8.

TABLE 3

OVER-ALL RESULTS OF THE DOG TASK

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of understanding	2	9	11
Some evidence of understanding	6	6	5
Uncertain evidence of understanding	4	1	0
Clear evidence of not understanding	4	0	0
Evidence lacking	0	0	0

Table 3 indicates that children of eight and eleven years of age perform at a considerably higher level than children of five years of age. The level of performance of eleven-year-olds, however, was not a great deal higher than that of the eight-year-olds. With the exception of one child, the eight- and eleven-year-old children all rated at least "Some evidence of understanding." Only eight of the five-year-old children rated this high, and only two rated "Clear evidence of understanding."

A number of the eight-year-olds especially had difficulty with those enclosures which represented convex sets. After succeeding without hesitation on the first enclosure, a number of them hesitated on the second and

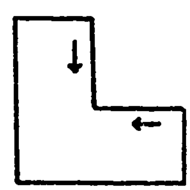
TABLE 4

RESULTS FOR THE INDIVIDUAL ENCLOSURES OF THE DOG TASK

Age	Performance	Enclosure				
		a	b	c	d	e
5 Years	Clear evidence of understanding	7	6	3	7	8
	Some evidence of understanding	1	6	7	1	2
	Uncertain evidence of understanding	0	0	2	0	1
	Clear evidence of not understanding	8	4	4	8	5
	Evidence lacking	0	0	0	0	0
8 Years	Clear evidence of understanding	15	9	12	13	11
	Some evidence of understanding	1	5	1	2	0
	Uncertain evidence of understanding	0	2	3	0	0
	Clear evidence of not understanding	0	0	0	1	5
	Evidence lacking	0	0	0	0	0
11 Years	Clear evidence of understanding	16	14	15	13	15
	Some evidence of understanding	0	2	0	0	0
	Uncertain evidence of understanding	0	0	1	0	0
	Clear evidence of not understanding	0	0	0	3	1
	Evidence lacking	0	0	0	0	0

third enclosures, making numerous attempts to find a place to put the dogs before finally saying that they would be unable to do it. DA's response to these enclosures is an illustration of this. Some of the five-year-olds simply gave up on the convex figures without concluding that it would or would not be possible to find a place to put the dogs.

Table 4 indicates that over 30 per cent of the eight-year-old children rated "Clear evidence of not understanding" on enclosure e while at the same time all of these children rated at least "Some evidence of understanding" on enclosure a. These children almost invariably placed the dogs in the positions shown in the diagram below.



They seemed unable to see that by simply turning each dog slightly they would have been able to do what was asked.

Many of the eight- and eleven-year-old children were able to respond correctly to the enclosures without even attempting to place the dogs. These children frequently looked ahead and responded to the next enclosure before it was even placed before them. This type of response was almost entirely lacking in the five-year-olds. Even those who rated "Clear evidence of understanding" on a particular enclosure generally did not respond until they had made at least one attempt at placing the dogs.

Results for the Tack Task

The Tack Task also sought information about the child's understanding of the defining characteristic of convex and non-convex sets. This task, however, was somewhat less physical than the Dog Task.

The Tack Task consisted of three main parts. The child was first presented a sequence of simple closed curves which were drawn on separate sheets of paper (Figure 9). Each figure was placed on a tackboard and the child was asked about the possibility of placing two tacks, with an elastic string connecting them, inside the figures in such a way that a portion of the string fell outside the figure. Upon completion of this portion of the task the child was

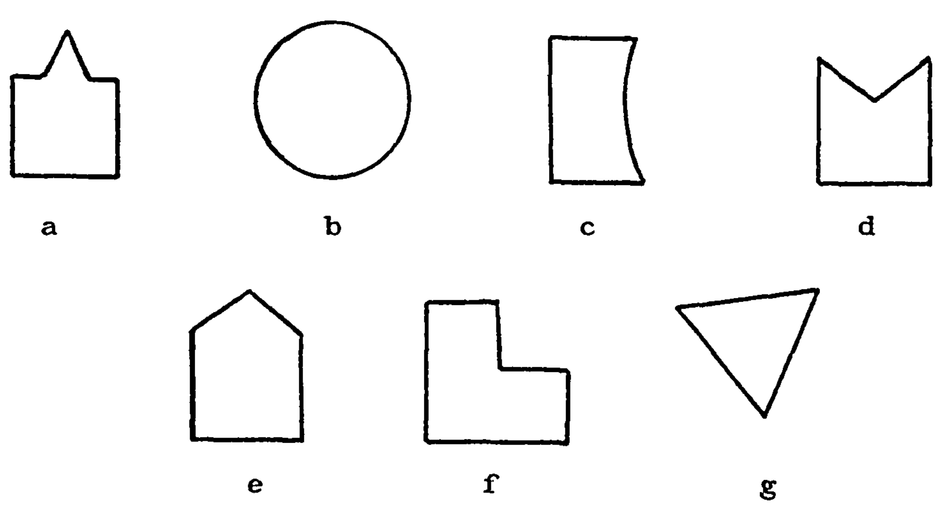


Figure 9

asked why he was able to do this on some of the figures and unable to do it on others. The second part of the task consisted of essentially repeating the first part using three construction paper figures (Figure 10) instead of the earlier drawings. Finally, in the third part of the task the child was asked about the possibility of cutting the construction paper figures into two pieces and putting them back together again in such a way that the new figure would be convex. Those children who were unable to perform correctly on at least one of the first two construction paper figures were not asked to attempt the final portion of this task.

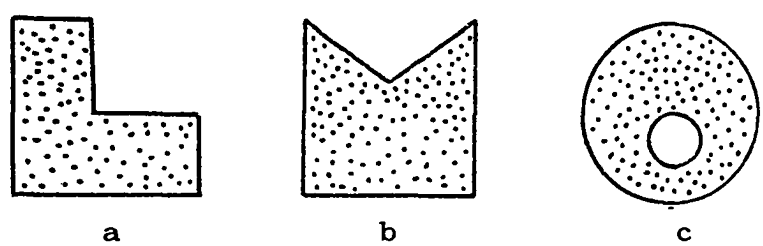


Figure 10

The criteria for rating the first two parts of this task were essentially the same as those for the Dog Task. Over-all results for each of these two parts are presented as well as the results for individual items in each part.

None of the five-year-olds and only about 30 per cent of the eight-year-olds received an over-all rating of "Clear evidence of understanding" on the first part of the

Tack Task. However, nearly all of the eleven-year-olds received this rating. All of the children in this category received a rating of "Clear evidence of understanding" on the figures which represented non-convex sets, and most of them also received this rating on the convex figures. However, a few of these children rated "Some evidence of understanding" on the convex figures. JH, eleven years of age, is an example of a child who received an over-all rating of "Clear evidence of understanding." She rated "Clear evidence of understanding" on figures a, b, c, d, and f, and rated "Some evidence of understanding on figures e and g. The questioning which followed completion of the task is presented below.

> JH (age 11). Can you explain to me why you were able to do it on some and not on others?--"Well, on these (referring to the non-convex figures) some point on the figure came inwards, so there would be a space."

Forty-three per cent of the five-year-olds and 56 per cent of the eight-year-olds rated "Some evidence of understanding" on the first part of this task. These children generally rated "Clear evidence of not understanding" on items a and c, but rated at least "Some evidence of understanding" on the other five items. Protocols of the questioning following completion of this sequence are given for several of these children.

> KC (age 8). What is different about d and f? Why could you do it there and you couldn't do it on these others?--"Some of them don't have anything like this (pointing to the indentation in figure d)."--Could you describe that?--"If it had a bottom, you could call it a triangle."

DH (age 8). Why were you able to do it on some of them, and not able to do it on others? You were able to do it on d and e, but not on the others.--"I could have done it on a!"--Will you show me how?--"I think I could have. (He is able to do this.) Yes I can!"--So you could do it on a after all. After doing it on d and f, you could see how to do it on a. Now can you tell me why you could do it on a, d, and f, but not on the others?--"These have something slanting down or in."

SN (age 5). Why could you do it on d and f, but you couldn't do it on the others?--"Well, see if you put them like this (again placing the tacks in the circular figure) the string wouldn't be out."--Can you tell me what was different about d and f, so that you could find a place?--"No."

Thirty per cent of the five-year-olds rated "Uncertain evidence of understanding." This category covered a wide variety of responses. A number of these children were unable to perform on this part of the task until the investigator had demonstrated how the tacks might be placed on one of the non-convex figures. JS, five years of age, was a child in this category for whom a demonstration was not necessary. He rated "Clear evidence of not understanding" on figure a, and "Clear evidence of understanding" on figures d and f. The convex figures caused him a good deal of difficulty. In each case he felt that it would be possible for him to place the tacks on these figures. When asked if he could place the tacks on figure b, he replied, "I think so, but it's going to be hard." After placing the tacks in various locations in this figure, he finally gave up. When figure c was placed in front of him, JC looked ahead to figure d and exclaimed, "I could do it on that one! I know I can!" After doing this, he was asked about figure e. He immediately said, "Yep. I think." After many attempts, he gave up without concluding that he would or would not be able to place the tacks on this figure. Finally, when asked why he could do it on figures d and f, but not on the others, he said, "I don't know." Three of the five-year-olds were unable to place the tacks on figure d even after being shown how this might be done on figure f. These children were rated "Clear evidence of not understanding."

The over-all results for the first part of the Tack Task are shown in Table 5. The results for each of the

TABLE 5

OVER-ALL RESULTS FOR THE SIMPLE CLOSED CURVES

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of understanding	0	5	14
Some evidence of understanding	7	9	1
Uncertain evidence of understanding	6	2	1
Clear evidence of not understanding	3	0	0
Evidence lacking	0	0	0

individual curves are shown in Table 6. The number of each curve in Table 6 corresponds to the earlier indicated order of presentation.

TABLE 6

RESULTS FOR THE INDIVIDUAL SIMPLE CLOSED CURVES OF THE TACK TASK

Age	Performance	Simple Closed Curve						
		a	b	c	d	e	f	g
5 Years	Clear evidence of understanding	0	2	2	11	3	10	5
	Some evidence of understanding	1	5	0	1	5	0	4
	Uncertain evidence of understanding	1	7	6	1	5	0	4
	Clear evidence of not understanding	14	2	8	3	2	6	2
	Evidence lacking	0	0	0	0	1	0	1
8 Years	Clear evidence of understanding	6	11	8	15	12	14	10
	Some evidence of understanding	1	4	0	0	3	0	5
	Uncertain evidence of understanding	1	1	1	0	1	1	0
	Clear evidence of not understanding	8	0	7	1	0	0	0
	Evidence lacking	0	0	0	0	0	1	1
11 Years	Clear evidence of understanding	15	13	14	16	15	16	15
	Some evidence of understanding	0	3	0	0	0	0	1
	Uncertain evidence of understanding	0	0	1	0	1	0	0
	Clear evidence of not understanding	1	0	1	0	0	0	0
	Evidence lacking	0	0	0	0	0	0	0

Table 6 indicates that children of five years of age had almost no success with the first simple closed curve, but were quite successful with the shapes in which the "indentation" was more pronounced. Figure a also caused a great deal of difficulty for the eight-year-olds. Some eight-year-olds said the reason they were unable to find a place to put the tacks inside this figure was that the sides of the indentation were not the same length, as was the case with figures d and f.

Five-year-old children experienced about the same amount of difficulty with the L-shaped figure on the Tack Task as they did on the Dog Task. This was not the case, however, with the eight-year-old children. Fourteen of them rated "Clear evidence of understanding" and no child rated "Clear evidence of not understanding." Five of the eight-year-olds rated in this lowest category on the Dog Task.

The results of the second part of the Tack Task are given in Tables 7 and 8. As was indicated previously, this portion of the task provided those children who had difficulty with the first part of the task an opportunity to respond in a somewhat simplified situation. It also provided a means for introducing the final portion of the Tack Task.

As shown in Table 8, only two of the five-year-olds rated "Clear evidence of not understanding" on the L-shaped figure which is a marked improvement over their performance

TABLE 7

OVER-ALL RESULTS FOR THE CONSTRUCTION PAPER FIGURES

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of understanding	13	16	15
Some evidence of understanding	1	0	1
Uncertain evidence of understanding	1	0	0
Clear evidence of not understanding	1	0	0
Evidence lacking	0	0	0

TABLE 8

RESULTS FOR THE INDIVIDUAL CONSTRUCTION PAPER FIGURES

Age	Performance	Figure		
		a	b	c
5 Years	Clear evidence of understanding	14	15	14
	Clear evidence of not understanding	2	1	2
8 Years	Clear evidence of understanding	16	16	16
11 Years	Clear evidence of understanding	16	16	15
	Clear evidence of not understanding	0	0	1

with the analogous figure in the first part of the task. Of the eight- and eleven-year-olds only one failed to achieve the highest rating on any of the three figures.

The results of the final portion of the Tack Task are summarized in Table 9. Those children who had difficulty with the previous portion of the task were not asked to attempt this final portion. For this reason only the results for thirteen five-year-olds are reported.

TABLE 9

RESULTS OF CUTTING THE CONSTRUCTION PAPER FIGURES

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of understanding	1	4	7
Some evidence of understanding	1	7	6
Uncertain evidence of understanding	7	4	2
Clear evidence of not understanding	4	1	1
Evidence lacking	0	0	0

Only two five-year-olds rated in the top two categories on this portion of the task, and somewhat less than 50 per cent of the eleven-year-olds rated "Clear evidence of understanding" as compared to nearly 90 per cent on the first part of the task. DL, eleven years of age, is an example of a child who rated "Clear evidence of understanding" on this portion of the task. On both the L-shaped and the V-shaped figures he indicated immediately that he would be able to do what was asked. On both of these figures he was able to immediately put the pieces together to form a convex figure. When questioned about the circular figure he responded, "You wouldn't be able to do it on that figure because you would have the circle there." The child was here referring to the circular region which had been cut out of the figure.

JM, five years of age, was the only child at this age level who rated "Some evidence of understanding." He stated immediately that he would be able to cut the L-shaped figure into two pieces and put it back together so that the resulting figure would be convex. However, after cutting the figure he made several attempts before finally finding one which was convex. Similarly, he was able after several attempts to succeed with the V-shaped figure. On the circular figure the child cut the figure into two pieces and made numerous attempts at putting the pieces back together; however, he was unable to conclude that he would or would not be able to do what was asked. The investigator finally terminated the task. At the other two age levels, the responses of the children who fell into this category were similar to those of JM.

More than 25 per cent of the children rated "Uncertain evidence of understanding" on this portion of the task. DB, eight years of age, is a typical example of a child who received this rating. When asked to cut the L-shaped figure she said, "I don't think you could." The investigator then demonstrated how this might be done with the V-shaped figure. The child was again asked about the possibility of cutting the L-shaped figure. This time, after some hesitation, she was able to cut the figure and put it back together to form the required convex figure. The child was not asked about the circular figure.

Finally, SN, five years of age, is an example of a child who rated "Clear evidence of not understanding." When asked to cut the L-shaped figure he said that he was unable to do so. After demonstrating how this might be done using the V-shaped figure, the investigator again asked the child about cutting the L-shaped figure. The child now said that he would be able to do this. However, after cutting the figure into two pieces and making numerous attempts at putting the pieces together, he indicated that he was not able to do it.

Results for the Inside-Outside Task

This task consisted of two main parts. The child was first presented with a sequence of sheets on each of which had been drawn a simple closed curve together with a dot representing a point. The child was asked to look at each sheet (Figure 11) and to circle the dot if it was "inside" the simple closed curve. The child was then given an identical sequence of sheets and this time was asked to circle the dot if it was "outside" the figure. Upon completion of these two sequences the child was questioned as to why he did or did not circle certain dots.

The child's responses to the individual items were rated according to where he considered the dot to be located. Responses fell into four categories:

Inside
Outside
Not inside and not outside
Part inside and part outside

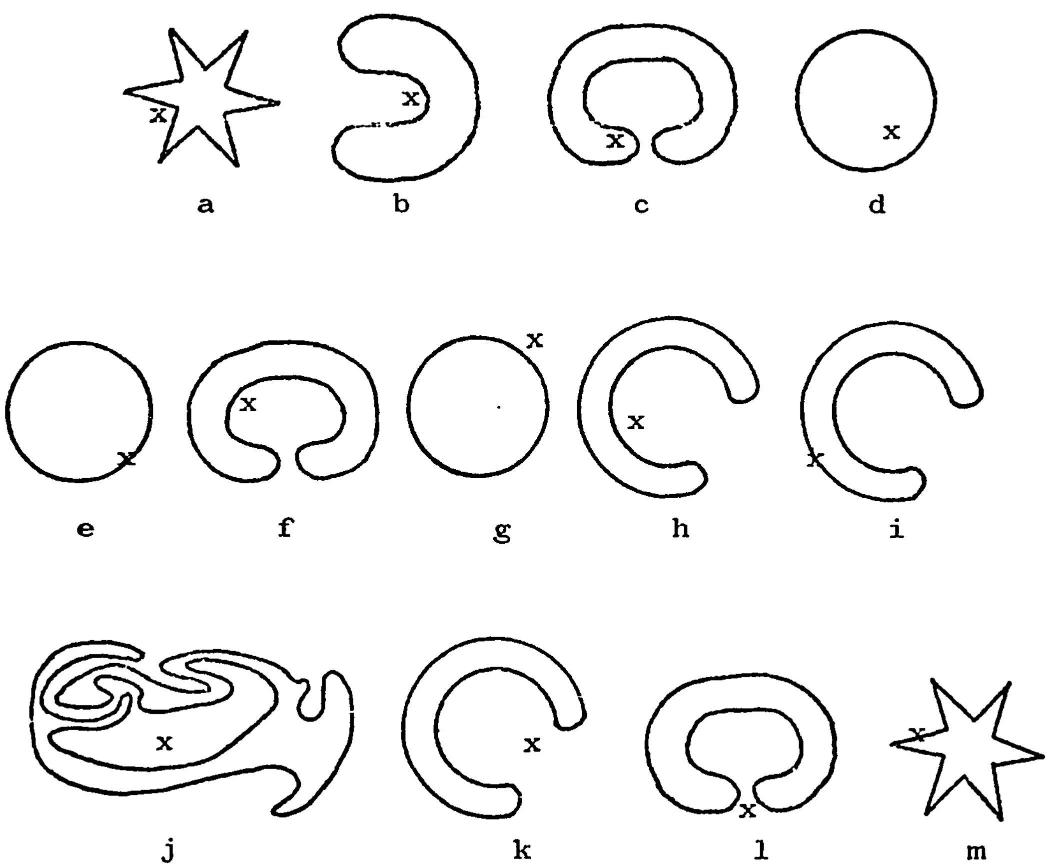


Figure 11

Each child was also given an over-all rating for his understanding of "inside" as being the interior of a simple closed curve and for his understanding of points of the curve as being disjoint from the interior points and the exterior points. The rating scheme used for these over-all ratings was the same as that used for both the Dog Task and the Tack Task.

Nearly all of the children rated "Clear evidence of Understanding" with regard to "inside" being the interior of a simple closed curve. Most children receiving this highest rating indicated only the dots in items c and d as being inside. However, because of the complexity of item j, a child who indicated that only dots on this item and on items c and d were inside, was also given this highest rating.

Only five children did not receive the highest rating with regard to "inside" being the interior of a simple closed curve. Three children rated "Some evidence of understanding; they indicated that the dots in items c, d, f, and j were all inside. Two children rated "Clear evidence of not understanding." One of these stated that the dots in items f, h, j, and k were all inside, and the other conceived of the dots in items f, h, and j as being inside. Several of the children who indicated that the dots in one or more of items f, h, and k were inside, added that these were not "inside" in the same sense as, for example, the dot in item d.

Table 10 summarizes the results of the children's understanding of "inside" as being the interior of a simple closed curve.

TABLE 10

CHILD'S UNDERSTANDING OF "INSIDE" AS THE INTERIOR OF A SIMPLE CLOSED CURVE

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of understanding	12	14	16
Some evidence of understanding	2	2	0
Uncertain evidence of understanding	0	0	0
Clear evidence of not understanding	2	0	0
Evidence lacking	0	0	0

Those children receiving a rating of "Clear evidence of understanding" of a simple closed curve as a set of points disjoint from the interior and exterior of the curve, indicated that the dots in items e, i, and m were neither inside nor outside. The following protocol is an example of a response which received this highest rating.

> DS (age 11). Will you tell me why you didn't circle the dot in item e either time?--"Because it's not inside and not outside."

A large number of the children, especially the eight-year-olds, received a rating of "Some evidence of understanding."

These children indicated that the dots in items e, i, and m were part inside and part outside. Of the seven children rating "Clear evidence of not understanding" all but one said the dots in items e, i, and m were outside. The following protocol is typical of the responses given by these children to these three items.

> JW (age 8). Why did you say the dot on item e is outside?--"Because it's on the line."--Do you mean points on the line are outside?--"Yes."

Table 11 summarizes these results.

TABLE 11

CHILD'S UNDERSTANDING OF A SIMPLE CLOSED CURVE AS A SET OF POINTS DISJOINT FROM THE INTERIOR AND THE EXTERIOR OF THE CURVE

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of understanding	11	4	11
Some evidence of understanding	2	8	5
Uncertain evidence of understanding	0	0	0
Clear evidence of not understanding	3	4	0
Evidence lacking	0	0	0

Table 12 summarizes the children's responses to each of the individual simple closed curves. Explanations regarding those dots which were not marked by the child as being

TABLE 12

RESULTS FOR THE INDIVIDUAL ITEMS OF THE INSIDE-OUTSIDE TASK

	Age											
	5 Years				8 Years				11 Years			
Figure	Inside	Outside	Not Inside and Not Outside	Part Inside and Part Outside	Inside	Outside	Not Inside and Not Outside	Part Inside and Part Outside	Inside	Outside	Not Inside and Not Outside	Part Inside and Part Outside
a	0	16	0	0	0	16	0	0	0	16	0	0
b	0	16	0	0	0	16	0	0	0	16	0	0
c	16	0	0	0	16	0	0	0	16	0	0	0
d	16	0	0	0	16	0	0	0	16	0	0	0
e	0	3	12	1	1	4	4	7	0	0	10	6
f	4	11	1	0	2	14	0	0	0	16	0	0
g	0	16	0	0	0	16	0	0	0	16	0	0
h	2	14	0	0	3	13	0	0	0	16	0	0
i	0	3	12	1	0	5	4	7	0	0	9	7
j	12	4	0	0	11	5	0	0	4	12	0	0
k	1	15	0	0	1	14	1	0	0	16	0	0
l	0	16	0	0	1	15	0	0	0	16	0	0
m	0	3	12	1	2	3	4	7	0	0	10	6

inside or outside fell into two categories. Some children said that the point was not inside and not outside; others said that the dot was part inside and part outside. These categories are shown in Table 12.

The dots in items a, b, f, h, j, k, and l were all placed in an indentation of the curve. None of the children considered the dots in items a and b to be inside. With the exclusion of item j, the simple closed curve in item f came the nearest to enclosing a region. Six children indicated that the dot in this item was inside. The curve in item h was somewhat less closed. Five children indicated that the dot in this item was inside. Item k involves the same curve but the dot is not placed so deeply in the indentation. Only two children indicated that this dot was inside.

The responses to item j were almost identical for the five- and eight-year-olds with about 25 per cent indicating that the dot was outside. About 75 per cent of the eleven-year-olds said the dot was outside. It is interesting to note the type of explanation given as to why the dot on item j was outside. The eleven-year-olds generally noted that there was a way to get out, and would indicate this with their finger or a pencil (Figure 12a). The five-year-olds and eight-year-olds more often pointed to the region which has been shaded in Figure 12b and said, "It's outside because this (pointing to the region which has been shaded) is inside."

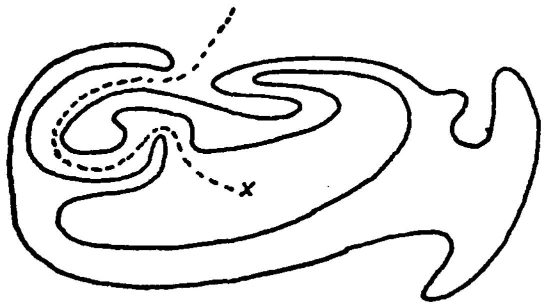


Figure 12a

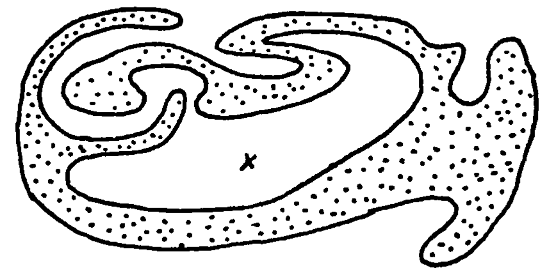


Figure 12b

<u>Results for the Betweenness Task</u>

The Betweenness Task consisted of a sequence of eleven items (Figure 13) which were presented to the child one at a time. Each item consisted of two blue dots and one or more red dots placed on a sheet of paper. As each item was presented to the child, he was asked to circle any red dot he considered to be <u>between</u> the two blue dots. After the child had responded to all eleven items, he was questioned about why he did or did not circle the various dots.

The Betweenness Task did not lend itself to the rating scheme used with the other tasks. Consequently, for this task a separate scheme was devised which permitted the investigator to take into account variations in the child's interpretation of betweenness. The scheme provides seven categories for rating the subject's responses:

Clear evidence of the Euclidean notion of betweenness

Some evidence of the Euclidean notion of betweenness

Clear evidence that the child understands betweenness in terms of the region bounded by parallel lines through the two points

Some evidence that the child understands betweenness as a region

Clear evidence that the child understands a point to be between two points provided it is on a curve drawn through the two points

Clear evidence of having no understanding of betweenness

Clear evidence of some other notion of betweenness

Sixty per cent of the children fell into one of the two categories involving the Euclidean notion of betweenness.

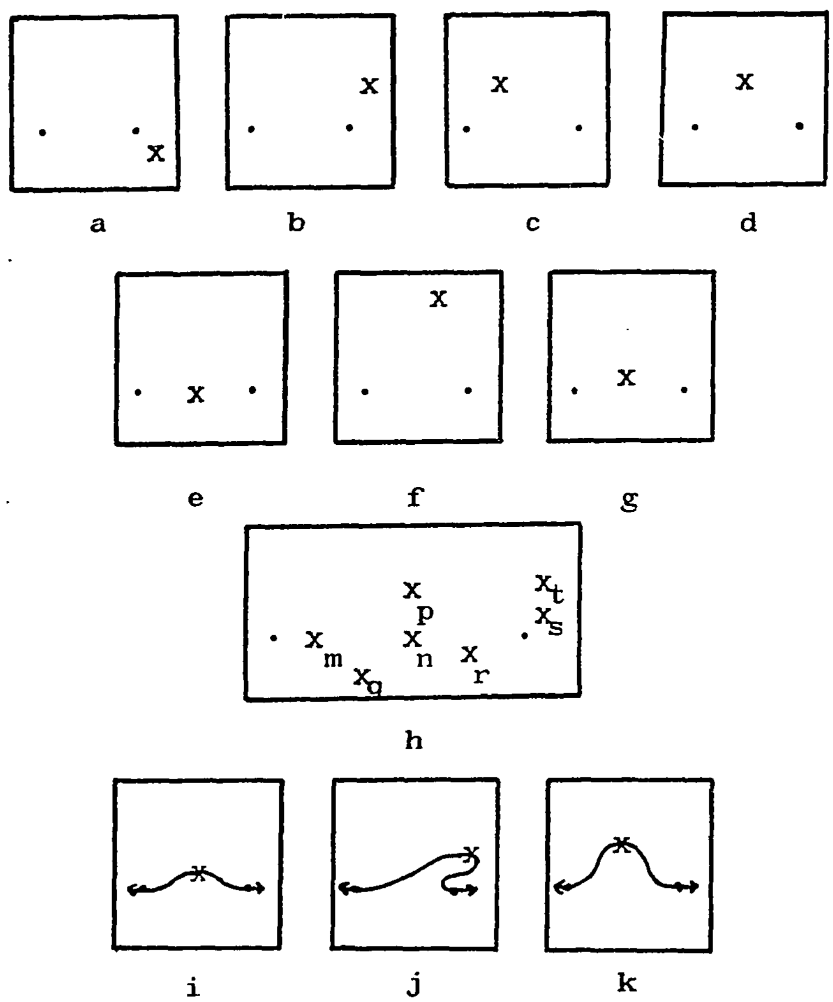


Figure 13

A child was rated "Clear evidence" if he said only the red dot in item e and dots m and n of item h were between the blue dots. A child was given a rating of "Some evidence" if in addition to the three dots mentioned above he also circled one or both of the dots in items g and i or dot r of item h.

Twenty-nine per cent of the children fell into one of the two categories which involved points in a region. JH, eleven years of age, is an example of a child rating "Clear evidence." She circled the dots on all but the first two items and dots s and t of item h. Upon being questioned, she said that for a dot to be between the blue dots it would need to be located in the region bounded by the perpendiculars to the segment joining the blue dots. Children rating "Some evidence" tended to include some but not all of the dots indicated by the children in the above category. RH is a typical example of a child who was given a rating of "Some evidence." She circled the dots on items c, d, e, g, i, j, and k. However, on item h she circled only dots m, n, and p.

Ten children, or about 20 per cent, rated "Clear evidence of understanding betweenness as a point on a curve." A child was rated in this category if he circled the red dots on all of the last three items. It was possible for a child to be rated in this category as well as in one of the previous categories. JH is an example of this situation.

As was stated above, she rated "Clear evidence of understanding betweenness as a region." However, since she circled the dots on all of the last three items, she was rated in this category also.

The results concerning the child's interpretation of betweenness are summarized in Table 13. Table 14 shows the results for the individual items.

Table 13 shows that the number of children utilizing the Euclidean notion of betweenness did not necessarily

TABLE 13

CHILD'S UNDERSTANDING OF BETWEENNESS

Performance	Age		
	5 Years	8 Years	11 Years
Clear evidence of the Euclidean notion of betweenness	2	12	5
Some evidence of the Euclidean notion of betweenness	7	0	5
Clear evidence of understanding betweenness as a region	0	4	5
Some evidence of understanding betweenness as a region	4	0	1
Clear evidence of understanding betweenness as a point on a curve	4	1	5
Clear evidence of having no understanding of betweenness	3	0	0
Clear evidence of some other notion of betweenness	0	0	0

TABLE 14

RESULTS FOR THE INDIVIDUAL ITEMS OF THE BETWEENNESS TASK

Figure	Age		
	5 Years	8 Years	11 Years
a	3	0	0
b	1	0	0
c	2	3	4
d	5	4	8
e	15	16	16
f	1	1	6
g	14	8	11
h-m	13	16	16
h-n	13	16	16
h-p	2	4	4
h-q	2	4	4
h-r	10	4	5
h-s	3	0	0
h-t	2	0	0
i	12	5	13
j	7	2	5
k	7	5	10

increase with age. In fact, the number of eight-year-olds rating "Clear evidence of having the Euclidean notion of betweenness" was much greater than for either of the other two age levels.

There was some increase with age in the number of children who interpreted betweenness as a region. However, the number of children considering a point to be between the other two points provided it was on a curve containing those points did not suggest a trend with respect to age. Only one eight-year-old rated "Clear evidence of understanding betweenness as a point on a curve" and only a little over 25 per cent of the five- and eleven-year-olds received this rating.

Finally, it should be noted that nearly all of the five-year-olds fell into the "Some evidence" category regardless of how they interpreted betweenness. Only three children rated "Clear evidence of having no understanding of betweenness."

CHAPTER IV

DISCUSSION AND CONCLUSIONS

The present study sought information about the child's intuitive understanding of the concept of convex and non-convex sets. Four multi-step tasks were devised to provide children an opportunity to reveal their insights with regard to: (1) defining characteristic of convex and non-convex sets, (2) interior and exterior of a simple closed curve, and (3) betweenness.

The sample for the study was selected from elementary schools of the Norman, Oklahoma, school system. Two schools were used in the main portion of the study. Eight children were randomly selected from each school at each of three age levels--five years, eight years, and eleven years.

The tasks were presented to the children in individual, tape recorded interviews. Each interview was subsequently evaluated by the investigator in terms of several predetermined rating schemes.

Results for each individual task were presented in the previous chapter, with results for each age level tabulated separately. The following section contains a discussion of the performance of children at each age level with regard to: (1) defining characteristic of convex and non-convex sets, (2) interior and exterior of a simple closed curve, and (3) betweenness.

Discussion of Results

Defining Characteristic of Convex and Non-convex Sets

Both the Dog Task and the Tack Task sought information about the child's intuitive understanding of the defining characteristic of convex and non-convex sets. The results for the Tack Task showed a definite improvement in performance with respect to age level. On the Dog Task, however, improvement occurred mainly between the five- and eight-year-olds. The performance of the eight- and eleven-year-olds on this task was essentially the same.

Only two five-year-olds received a rating of "Clear evidence of understanding" in the over-all results for the Dog Task and the first part of the Tack Task. In fact, more than one-half of the five-year-olds rated either "Uncertain evidence of understanding" or "Clear evidence of not understanding" on these tasks. It was characteristic of children at this age level to consider only a portion of a figure at one time. This was particularly evident for the various figures of the Tack Task. One child, for example, made a number of attempts to find a place to put the tacks in the lower part of figure 9d and then appeared surprised when he suddenly found a place to put the tacks in the upper part of the figure. Another child made perhaps twelve attempts to place the tacks inside figure 9f. In each attempt, she placed both tacks within the same rectangular portion of the figure. Consequently, she was never able to do what was asked.

The level of performance of eight-year-olds was considerably higher than that of the five-year-olds; however, the responses of children at both age levels depended to a large extent on what they found as a result of attempts to place the dogs or tacks. It is interesting to note that the children tended to make many more attempts on those figures representing convex sets than on those representing non-convex sets. In fact, a number of the five-year-olds simply gave up on the figures representing convex sets without concluding that it would or would not be possible to place the dogs or tacks.

A comparison of over-all results for eight-year-olds and eleven-year-olds on the Dog Task and the first part of the Tack Task indicates that if the top two categories are grouped together, the performance of these two age levels are nearly the same. They differ mainly with regard to the extent to which responses are dependent upon attempting to place the dogs or tacks. For many of the eleven-year-olds it was not necessary to actually attempt placing the dogs or tacks in order to respond. In fact, many of these children responded before the enclosure or figure was placed before them. This type of response occurred less often with the eight-year-olds.

The final portion of the Tack Task was the most difficult for the children. Of the thirteen five-year-olds who attempted this portion of the task, only two rated in the top two categories. The remainder of these children were able to respond only after a demonstration by the investigator. The upper two age levels showed a marked improvement in performance over the five-year-olds. It was apparent that many of the eleven-year-olds and at least some of the eight-year-olds visualized the end result before they started cutting. The greatest difficulty was experienced with the circular figure. Many of whose over-all rating was "Some evidence of understanding" seemed to feel they would be able to reconstruct this figure so that it would be convex, and did not respond until after making several attempts to do this.

Interior and Exterior of a Simple Closed Curve

In general, children had a clear understanding of "inside" as the interior of a simple closed curve. This result came as a surprise to the investigator. On the basis of questioning several pre-kindergarten children of four and five years of age, the investigator had anticipated that at least the five-year-old children would consider dots placed in an indentation of a simple closed curve to be "inside." Some children did respond in this way; however, the number doing so was small.

Children's responses to dots which represented points of the simple closed curve were much less clear. When questioned about these points, most of the children said they were either "not inside and not outside" or they were "part inside and part outside." It is interesting to note that nearly 70 per cent of both five-year-olds and eleven-year-olds said a point of the curve was "not inside and not outside," but only 25 per cent of the eight-year-olds gave this response. Regardless of which response was given, these children considered points of the curve to be different from interior points and exterior points. Of the three five-year-olds and four eight-year-olds who rated "Clear evidence of not understanding," all but one said the dot was "outside." These children apparently considered a simple closed curve to partition the plane into only two sets: the interior and the simple closed curve together with the exterior.

Betweenness

The findings of this study indicate that children interpret betweenness in different ways. In general, children consider a point to be between two other points if it is a point of the segment joining two points or if it is in the region determined by two parallel lines through the endpoints. An additional interpretation becomes apparent when a curve is drawn through the dots. These findings are in agreement with the findings of Denmark which were reported in Chapter I.

Although the Euclidean notion of betweenness is the most frequently utilized interpretation, it is interesting to note that there is no trend toward older children having this concept to a greater extent than younger children. In fact, there was a slight increase with age in the number of children who considered a point to be between if it was in the region bounded by two parallel lines. Perhaps this is an indication that the older children are more aware of the usage of betweenness in non-mathematical situations. For example, a house is said to be between the houses on either side even though it is farther back from the street than either of the other houses. It is of interest to note that Denmark found experienced teachers to utilize the Euclidean concept of betweenness to a much lesser degree than did first grade children.

The findings of the present study concerning the concept of betweenness are very similar to the findings of Denmark's study. If the results of the three age levels considered in this study are combined, the per cent of children adhering to the Euclidean concept of betweenness is 64 per cent. Denmark found 68 per cent utilizing this interpretation. Similarly, the present study found 29 per cent of the children interpreting betweenness as a region. Denmark found 20 per cent utilizing this interpretation. The results for individual items in Denmark's study are presented in Appendix B. It is apparent that these results differ somewhat on individual items from the results of this study; however, one must consider that the present study involved essentially kindergarten, second grade, and fifth grade children, whereas Denmark's study involved first graders and experienced teachers.

Conclusions

The information obtained in the present study leads to the following conclusions:

1. There was a definite improvement in performance with age on the Tack Task; however, on the Dog Task improvement occurred mainly between the five- and eight-year-olds. Five-year-olds exhibited only limited understanding on these tasks. Their responses were almost entirely dependent upon trial and error. It was characteristic of these children to focus on only a portion of a figure, rather than the entire configuration. The eight-year-olds performed at a much higher level on these tasks. They were still, however, dependent to a large extent upon the physical situation. The figures which represented convex sets tended to cause these children more difficulty than did the non-convex figures. The level of performance of the eleven-year-olds was not a great deal higher than that of eight-year-olds. Many of these

children were able to respond with little or no reliance upon the physical situation.

2. In general, children regardless of age level have a clear understanding of "inside" as being the interior of a simple closed curve. The children's conception of points of the simple closed curve, however, was much less clear. They tended to consider these points in one of two ways: (1) neither inside nor outside or (2) part inside and part outside. Although the first explanation was used most often, there was no apparent trend in the distribution of explanations with respect to age level.

3. There are differences among children's interpretations of betweenness. In general, children interpret betweenness in one of two ways: as the Euclidean notion of betweenness or in terms of the region bounded by two parallel lines. In addition, some children considered a point to be between two other points provided it was on a curve drawn through the two points. Although the Euclidean notion of betweenness was the most prevalent, there was no clear trend with regard to age level.

4. The introduction of the defining characteristic of convex sets to five-year-olds is probably

not advisable. It does appear feasible to introduce this concept to children as early as eight years of age.

Implications for Education

The investigator suggested in Chapter I that the information obtained in this study should be of interest and importance to mathematics educators concerned with the selection of topics for inclusion in the primary grade curriculum and with the writing of instructional materials involving the concepts being investigated. The findings of this study suggest several important implications for these areas.

The over-all results do not appear favorable to the introduction of the defining characteristic of convex and non-convex sets at the kindergarten level. The investigator is of the opinion, however, that children at this level could at least be introduced to figures which represent non-convex sets along with the convex figures such as circles, rectangles, triangles, and squares which are usually introduced at this level. Thus, in addition to identifying characteristics of the more usual figures, children would be discovering characteristics of figures which have indentations, holes, and parts "sticking out."

As was suggested in the discussion of results, the performance of eight-year-olds on tasks involving the defining characteristic of convex and non-convex sets was not a great deal below that of eleven-year-olds. Based upon the over-all results of the study, the investigator concludes that it is feasible to introduce the concept of convex and non-convex sets to eight-year-olds. The instructional materials, however, should take into consideration the child's intuitive understanding of this concept as well as the two related concepts.

Children in general have a clear understanding of "inside" as the interior of a simple closed curve; however, it should not be assumed that this will always be the case, especially when figures representing non-convex sets are involved. The usual instructional sequence used to teach the concepts "inside" and "outside" involve only convex figures. Perhaps the subsequent introduction of non-convex sets into this learning sequence might be used to develop an even deeper understanding of these concepts.

The child's understanding of points of a simple closed curve is perhaps of less importance to this study than his understanding of "inside." These findings suggest, however, a need to emphasize that the simple closed curve does not belong to either the interior or the exterior.

The results with regard to the concept of betweenness emphasize a need to consider the various ways in which children interpret this concept. The Euclidean notion of betweenness is clearly not the only one considered by children. Knowing the ways in which children interpret this concept, however, suggests a possible starting point for the development of a learning sequence involving the Euclidean notion of betweenness.

Finally, the findings of this study contribute to the growing body of knowledge concerning children's understanding of mathematical concepts. The importance of this knowledge is emphasized by Lovell. He suggests:

> . . . now we know--thanks to the Piaget-type research--much more about the profound aspects of the deceptively simple material in mathematics that children are called upon to learn. Again, if we take the trouble we can analyze in far greater detail the difficulties that children have in approaching such material. We also know that the development of the general ways of knowing will determine the manner in which the mathematical ideas are assimilated. Of course, we have only just made a beginning in these matters, and far more knowledge is required.[1]

Implications for Further Research

The findings of the present study suggest several areas needing further study.

1. The children who participated in the present study were from an urban, middle-class community. Studies similar to the present one are needed to provide information about children from (1) other geographic areas, (2) a wider range of social and economic backgrounds, and (3) other age levels.

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[1]Kenneth Lovell, Intellectual Growth and Understanding Mathematics, Science and Math Education Information Report, February, 1971 (Columbus, Ohio: ERIC Information Analysis Center for Science and Mathematics Education, 1971), p. 12.

2. The present study considered the child's understanding of convex and non-convex sets without benefit of specific instruction. The investigator concluded that this concept might be introduced to children eight years of age. This finding is contrary to the findings of D'Augustine reported in Chapter I. A study is needed in which an attempt is made to teach this concept to small groups of children at various age levels. Such a study should incorporate the findings of the present study as well as take into consideration the materials used by D'Augustine.

3. The possibility of introducing the concept of convexity to children is an area which is relatively unexplored. Hence, there were few guidelines to direct the investigator in devising the various tasks. Other studies are needed which would extend the present study by perhaps refining the tasks used in this study and devising other tasks which would provide further insight into the child's understanding of this concept.

BIBLIOGRAPHY

BIBLIOGRAPHY

Books

Almy, Millie. Young Children's Thinking. New York: Teachers College Press, Columbia University, 1966.

Benson, Russell V. Euclidean Geometry and Convexity. New York: McGraw-Hill Book Company, 1966.

Butler, Charles H.; Wren, F. Lynwood; and Banks, J. Houston. The Teaching of Secondary Mathematics. New York: McGraw-Hill Book Company, 1970.

Cartwright, Dorwin P. "Analysis of Qualitative Material." Research Methods in the Behavioral Sciences. Edited by L. Festinger and D. Katz. New York: The Dryden Press, 1953.

Churchill, Eileen M. Counting and Measuring. Toronto: University of Toronto Press, 1961.

Dienes, Z. P. Building Up Mathematics. London: Hutchinson Educational Ltd., 1960.

_______. Mathematics in the Primary School. Melbourne: Macmillan and Co., Ltd., 1964.

Educational Services Incorporated. Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin Co., 1963.

Flavell, J. H. The Developmental Psychology of Jean Piaget. Princeton: D. Van Nostrand Company, Inc., 1963.

Inbody, Donald. "Children's Understanding of Natural Phenomena." Research in Elementary School Curriculum. Edited by A. Montgomery Johnston and Paul C. Burns. Boston: Allyn and Bacon, Inc., 1970.

Kasner, Edward, and Newman, James. Mathematics and the Imagination. New York: Simon and Schuster, Inc., 1940.

Kay, David C. College Geometry. New York: Holt, Rinehart and Winston, Inc., 1969.

Keedy, Marvin L., and Nelson, Charles W. Geometry: A Modern Introduction. Reading, Mass.: Addison-Wesley Publishing Company, 1965.

Kelly, Paul J., and Ladd, Norman E. Geometry. Chicago: Scott, Foresman and Company, 1965.

Kerlinger, Fred N. Foundations of Behavioral Research. New York: Holt, Rinehart and Winston, Inc., 1964.

Lovell, K. The Growth of Basic Mathematical and Scientific Concepts in Children. London: University of London Press, Ltd., 1962.

Lunzer, E. A. "Children's Thinking." Educational Research in Britain. Edited by H. J. Butcher. New York: American Elsevier Publishing Company, Inc., 1968.

Moise, Edwin. Elementary Geometry from an Advanced Standpoint. Reading, Mass.: Addison-Wesley Publishing Company, 1965.

Ohmer, Merlin M. Elementary Geometry for Teachers. Reading, Mass.: Addison-Wesley Publishing Company, 1969.

Peterson, John A., and Hashisaki, Joseph. Theory of Arithmetic. 2nd ed. New York: John Wiley and Sons, Inc., 1967.

Piaget, Jean, and Inhelder, Barbel. The Child's Conception of Space. Translated by F. J. Langdon and J. L. Lunzer. The Norton Library. New York: W. W. Norton and Company, Inc., 1967.

Piaget, Jean; Inhelder, Barbel; and Szeminska, Alina. The Child's Conception of Geometry. Translated by E. A. Lunzer. New York: Basic Books, Inc., 1960.

Reeve, William David. "The Teaching of Geometry." The Teaching of Geometry. Fifth Yearbook of the National Council of Teachers of Mathematics. Edited by William David Reeve. New York: Bureau of Publications, Teachers College, Columbia University, 1930.

Rosenbloom, Paul C. "Implications of Piaget for Mathematics Curriculum." Problems in the Teaching of Elementary School Mathematics. Boston: Allyn and Bacon, Inc., 1970.

Suppes, Patrick. Sets and Numbers Book 2B. New York: Blaisdell Publishing Company, 1963.

Yaglom, I. M., and Boltyanskii, V. G. Convex Figures. Translated by Paul J. Kelly and Lewis F. Walton. New York: Holt, Rinehart and Winston, 1961.

Articles

Adler, Irving. "The Cambridge Conference Report: Blueprint or Fantasy?" The Arithmetic Teacher, XIII (March, 1966), pp. 179-86.

Brune, Irvin H. "Some K-6 Geometry." The Arithmetic Teacher, XIV (October, 1967), pp. 441-47.

D'Augustine, Charles H. "Topics in Geometry and Point Set Topology--A Pilot Study." The Arithmetic Teacher, XI (October, 1964), pp. 407-12.

________. "Factors Relating to Achievement with Selected Topics in Geometry and Topology." The Arithmetic Teacher, XIII (March, 1966), pp. 192-97.

Denmark, Tom. "An Intuitive Introduction to the Euclidean Concept of Betweenness." The Arithmetic Teacher, XV (December, 1968), pp. 683-86.

Dodwell, P. C. "Children's Understanding of Spatial Concepts." Canadian Journal of Psychology, 17 (March, 1959), pp. 141-61.

Felder, Virginia. "Geometry Concepts in Grades K-3." The Arithmetic Teacher, XII (May, 1965), pp. 356-58.

Gibney, Thomas C., and Houle, William W. "Geometry Readiness in the Primary Grades." The Arithmetic Teacher, XIV (October, 1967), pp. 470-72.

Goldmark, Bernice. "Geometry in the Primary Grades." The Arithmetic Teacher, X (April, 1963), pp. 191-92.

Huntington, Jefferson R. "Linear Measurement in the Primary Grades: A Comparison of Piaget's Description of the Child's Spontaneous Conceptual Development and the SMSG Sequence of Instruction." Journal for Research in Mathematics Education, I (November, 1970), pp. 219-32.

Inskeep, James E. "Primary-Grade Instruction in Geometry." The Arithmetic Teacher, XV (May, 1968), pp. 422-26.

Johnson, Donovan A. "Geometry for the Primary." Grade Teacher, LXXIX (April, 1962), p. 52.

Lamb, Pose M. "Geometry for Third and Fourth Graders." The Arithmetic Teacher, X (April, 1963), pp. 193-94.

Lovell, Kenneth. "A Follow-Up Study of Some Aspects of the Work of Piaget and Inhelder on the Child's Conception of Space." The British Journal of Educational Psychology, XXIX (June, 1959), pp. 104-17.

________. Intellectual Growth and Understanding Mathematics. Science and Math Education Information Report, February, 1971. Columbus, Ohio: ERIC Information Analysis Center for Science and Mathematics Education, 1971.

Mastain, Richard K., and Nossoff, Bernice C. "Mathematics in the Kindergarten." The Arithmetic Teacher, XIII (January, 1960), pp. 33-38.

Miller, G. H. "Geometry in the Elementary Grades: A Comparative Study of Greek Mathematics Education." The Arithmetic Teacher, XI (February, 1964), pp. 87-90.

Peel, E. A. "Experimental Examination of Some of Piaget's Schemata Concerning Children's Perception and Thinking, and a Discussion of their Educational Significance." British Journal of Educational Psychology, XXIX (June, 1959), pp. 89-103.

Piaget, Jean. "How Children Form Mathematical Concepts." Scientific American, 189 (November, 1953), pp. 74-79.

Robinson, G. Edith. "The Role of Geometry in Elementary School Mathematics." The Arithmetic Teacher, XIII (January, 1966), pp. 3-10.

Romberg, Thomas A., and DeVault, M. Vere. "Mathematics Curriculum: Needed Research." Journal of Research and Development in Education, I (Fall, 1967), pp. 95-110.

Skypek, Dora Helen. "Geometric Concepts in Grades 4-6." The Arithmetic Teacher, XII (October, 1965), pp. 443-49.

Smith, Lewis B. "Geometry, Yes--but How?" The Arithmetic Teacher, XIV (February, 1967), pp. 84-89.

Vigilante, Nicholas J. "Geometry for Primary Children: Considerations." The Arithmetic Teacher, XIV (October, 1967), pp. 453-59.

_______. "Why Circumvent Geometry in the Primary Grades?" The Arithmetic Teacher, XII (October, 1965), pp. 450-54.

Unpublished Materials

Taback, Stanley Frederick. "The Child's Concept of Limit." Unpublished Ph.D. dissertation, Columbia University, 1969.

APPENDIXES

APPENDIX A

The Rating Instrument

Subject______________________________ School_____________________

Age_________ Date________________ Length of interview_________

<u>Preliminary Tasks</u>

I. Preliminary task in which the child is asked about the terminology to be used in the Dog Task.

______ 1. Clear evidence of understanding
______ 2. Some evidence of understanding
______ 3. Uncertain evidence of understanding
______ 4. Clear evidence of not understanding
______ 5. Evidence lacking

II. Preliminary task in which the child is asked about standing inside a large hoop

______ 1. Clear evidence of understanding
______ 2. Some evidence of understanding
______ 3. Uncertain evidence of understanding
______ 4. Clear evidence of not understanding
______ 5. Evidence lacking

<u>Dog Task</u>

I. Rating for the individual enclosures of the Dog Task

	a	b	c	d	e
Clear evidence of understanding					
Some evidence of understanding					
Uncertain evidence of understanding					
Clear evidence of not understanding					
Evidence lacking					

II. Over-all rating for the Dog Task

______ 1. Clear evidence of understanding
______ 2. Some evidence of understanding
______ 3. Uncertain evidence of understanding
______ 4. Clear evidence of not understanding
______ 5. Evidence lacking

Tack Task

I. Rating for the individual simple closed curves of the Tack Task

	a	b	c	d	e	f	g
Clear evidence of understanding							
Some evidence of understanding							
Uncertain evidence of understanding							
Clear evidence of not understanding							
Evidence lacking							

II. Over-all rating for the portion of the Tack Task involving simple closed curves

_____ 1. Clear evidence of understanding
_____ 2. Some evidence of understanding
_____ 3. Uncertain evidence of understanding
_____ 4. Clear evidence of not understanding
_____ 5. Evidence lacking

III. Rating for the individual items of the portion of the task involving construction paper figures

	a	b	c
Clear evidence of understanding			
Some evidence of understanding			
Uncertain evidence of understanding			
Clear evidence of not understanding			
Evidence lacking			

IV. Over-all rating for the portion of the Tack Task involving construction paper figures

_____ 1. Clear evidence of understanding
_____ 2. Some evidence of understanding
_____ 3. Uncertain evidence of understanding
_____ 4. Clear evidence of not understanding
_____ 5. Evidence lacking

V. Rating for each part of the task involving cutting the construction paper figures

	a	b	c
Clear evidence of understanding			
Some evidence of understanding			
Uncertain evidence of understanding			
Clear evidence of not understanding			
Evidence lacking			

Inside-Outside Task

I. Rating for the individual items of the Inside-Outside Task

Figure	Inside	Outside	Not Inside and Not Outside	Part Inside and Part Outside
a				
b				
c				
d				
e				
f				
g				

Figure	Inside	Outside	Not Inside and Not Outside	Part Inside and Part Outside
h				
i				
j				
k				
l				
m				

II. Child's level of understanding of "inside" as being the interior of a simple closed curve

_____ 1. Clear evidence of understanding
_____ 2. Some evidence of understanding
_____ 3. Uncertain evidence of understanding
_____ 4. Clear evidence of not understanding
_____ 5. Evidence lacking

III. Child's level of understanding of the simple closed curve as being disjoint from the interior and exterior.

_____ 1. Clear evidence of understanding
_____ 2. Some evidence of understanding
_____ 3. Uncertain evidence of understanding
_____ 4. Clear evidence of not understanding
_____ 5. Evidence lacking

<u>Betweenness Task</u>

Child's understanding of betweenness

_____ 1. Clear evidence of the Euclidean notion of betweenness

_____ 2. Some evidence of the Euclidean notion of betweenness

_____ 3. Clear evidence that the child understands betweenness in terms of the region bounded by parallel lines

_____ 4. Some evidence that the child understands betweenness in terms of the region bounded by parallel lines

_____ 5. Clear evidence that the child understands a point to be between two points provided it is on a curve drawn through the two points

_____ 6. Clear evidence of having no understanding of betweenness

_____ 7. Clear evidence of some other notion of betweenness

APPENDIX B

RESULTS FOR THE INDIVIDUAL ITEMS OF DENMARK'S STUDY OF BETWEENNESS

	Per Cent	
Item	Teachers	First Graders
e	84	100
g	63	64
d	58	20
a	37	4
b	42	4
h-m	74	72
h-n	90	88
h-p	58	32
h-q	58	16
h-r	58	32
h-s	26	12
h-t	36	0
i	63	64
k	47	44
j	42	8