CURRENT RESULTS IN OSCILLATION THEORY OF THE THIRD ORDER LINEAR DIFFERENTIAL EQUATION

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CHAPTER I

ORDINARY THIRD ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

Introduction

This thesis is concerned with results concerning the oscillatory properties of the third order ordinary linear homogeneous differential equation which have appeared since 1967. In particular those results concerning existence of oscillatory (and nonoscillatory solutions), solution space properties, and the relation between the third order linear differential equation and its adjoint are studied. Also, techniques which were used to obtain these results will be presented.

The purpose of this thesis is to gather these results into a single unified work which would be of use to those interested in differential equations. Also, many of the techniques used to obtain these results will be displayed for the benefit of the novice in differential equations. This thesis is a partial extension of Swanson's book [22] which summarizes many of the results in this area of differential equations which were obtained prior to 1968. Consequently, this thesis brings the work of Swanson up to date.

The study of the third order linear differential equation began with Birkhoff's paper [6] in 1911 which dealt with separation and comparison theorems. Most of the oscillatory theory of third order linear differential equations has been developed during the past two decades. Some of those responsible for the early results concerning the oscillation theory of third order equations are Gregus, Hanan, Lazer, Rab, Svec and Villari. Much of the early work in this area was motivated by linear third order equations with constant coefficients. Recently there has been some interest in the relationship between the oscillatory behavior of the third order linear equation and the oscillatory behavior of its adjoint.

The body of this thesis is divided into three chapters. Chapter II is devoted to the study of the oscillatory properties of fundamental sets of the third order linear differential equation. For example, if one considers the differential equation which has as a fundamental set $\{e^t, \sin t, \cos t\}$ then any oscillatory solution is a linear combination of $\sin t$ and $\cos t$. A generalization of this example is given in Chapter II.

In Chapter III the existence of oscillatory solutions and the asymptotic behavior of solutions and their derivatives are studied. The techniques used in this chapter are of special interest. Some of the results and techniques were motivated by Lazer's paper [15]. The criterion which are considered to guarantee the existence of oscillatory solutions or nonoscillatory solutions are the following:

(i) The signs of the coefficients of y, y' and y'',

(ii) The integrability of the coefficients of y, y' and y''',

(iii) The characteristic equation.

However, some oscillatory properties of the constant coefficient case do not generalize as will be shown by example.

Chapter IV contains results which display the connection between the oscillatory nature of the third order linear differential equation and its formal adjoint. Much of this material is very recent and some has yet to appear in the literature. The relationships between the differential equation and its adjoint with respect to the properties R, RN, RO, oscillatory, nonoscillatory and weakly oscillatory are considered.

Some of the proofs given in the literature were corrected (for example Lemma 4.5) or shortened (see Lemma 4.19) by using the results of Birkhoff and Polya. In fact much of the work in this chapter follows from the work of Birkhoff, Polya and Lazer. Also, several examples using the results of the theory developed in Chapters II, III and IV are given.

Third Order Linear Differential Equations

Consider the differential equation

$$y''' + p(t)\, y'' + q(t)\, y' + r(t)\, y = 0 , \qquad (1)$$

where $p(t)$, $q(t)$, $r(t) \in C[a, \infty)$, and a is a real number. The formal adjoint of (1) is the differential equation given by

$$y''' - (p(t)y)'' + (q(t)y)' - r(t)y = 0 , \qquad (1)^*$$

With the assumption that $p(t)$, $q(t)$, $r(t) \in C[a, \infty)$, the solutions of equation (1) as well as $(1)^*$ form a vector space of dimension three over the reals. Also, with the above assumptions there exists a unique solution satisfying the initial conditions $y(t_0) = a_1$, $y'(t_0) = a_2$, $y''(t_0) = a_3$, where $t_0 \in [a, \infty)$. The solution space of (1) will be denoted by Ω and the solution space of $(1)^*$ by Ω^*. The substitution

$$y(t) = z(t) \exp\left[-\frac{1}{3}\int_a^t p(s)ds\right]$$

transforms (1) into the differential equation

$$z''' + P(t)z' + Q(t)z = 0 . \qquad (2)$$

The oscillatory properties remain invariant under this substitution. Thus, the oscillatory nature of equation (1) can be considered without loss of generality by studying oscillation theory of equations of the form (2).

In the remainder of this paper a solution of (1) shall mean a nontrivial solution. A solution of (1) is said to be oscillatory (or is said to oscillate) if its set of zeros is unbounded above. Solutions which are not oscillatory are called nonoscillatory.

Preliminary Concepts

The following definitions and theorems are necessary to understand the remainder of this thesis. The proofs of the theorems are omitted.

Definition 1.1 A subspace of the space Ω of the solutions of the differential equation (1) is said to be oscillatory if it contains at least one oscillatory solution. A subspace is said to be weakly oscillatory if it contains both an oscillatory and a nonoscillatory solution.

Definition 1.2 A subspace of Ω is said to be nonoscillatory [strongly oscillatory] if none [all] of the solutions in the subspace oscillate.

<u>Definition 1.3</u> The differential equation (1) is said to have <u>property</u> R on $[a, \infty)$ if it is weakly oscillatory and it has two solutions y_1 and y_2 such that $W(y_1, y_2)(t) \neq 0$ for $t \in [a, \infty)$, where $W(y_1, y_2)$ represents the Wronskian of y_1 and y_2.

<u>Theorem 1.4</u> (Polya [17]). If (1) has solutions y_1 and y_2 such that $y_1(t) \neq 0$ on $[a, \infty)$ and $W(y_1, y_2)(t) \neq 0$ on $[a, \infty)$, then no solution of (1) can have more than two zeros on $[a, \infty)$ (counting multiplicities). This says that the solution space is nonoscillatory.

<u>Remark 1.5</u> Let equation (1) have property R. It follows from Theorem 1.4 that the solutions y_1 and y_2 in Definition 1.4 oscillate. For, suppose $y_1(t) \neq 0$ on $[b, \infty)$ where $b \geq a$. Since $W(y_1, y_2)(t) \neq 0$ on $[b, \infty)$, no solution of (1) can oscillate by Theorem 1.4. This is a contradiction since equation (1) has property R; hence an oscillatory solution. Thus, $y_1(t)$ oscillates. Similarly, $y_2(t)$ oscillates.

<u>Definition 1.6</u> The differential equation (1) is said to have <u>property</u> RO if it has property R and a solution of (1) is oscillatory if and only if it is a nontrivial linear combination of y_1 and y_2, where y_1 and y_2 are as in Definition 1.3. Equation (1) is said to have <u>property</u> RN if it has property R and every nonoscillatory solution of (1) is a constant multiple of a fixed nonoscillatory solution. It follows directly that properties RN and RO are mutually exclusive.

The following examples are intended to illustrate some of the above concepts.

Example 1.7 Consider the differential equation

$$y''' + y' = 0 \tag{3}$$

which has $\{1, \sin t, \cos t\}$ as a fundamental set. Now $y(t) \equiv 1$, $y_1(t) = \sin t$ are nonoscillatory and oscillatory solutions respectively. Since $W(\cos t, \sin t) \equiv 1$, (3) has property R. But $z_1(t) = \frac{1}{2} + \sin t$ is oscillatory and $z_2(t) = 2 + \sin t$ is nonoscillatory; hence, (3) has neither property RO nor RN.

The next example characterizes the properties RO and RN in the case where (1) has constant coefficients.

Example 1.8 Consider the equation

$$y''' + py'' + qy' + ry = 0, \tag{4}$$

where p, q, and r are constants. The adjoint of (4) is

$$y''' - py'' + qy' - ry = 0. \tag{5}$$

If the characteristic equation of (4) does not have imaginary roots, then (4) cannot have an oscillatory solution. Hence, it cannot have property R. So, suppose the characteristic equation of (4) has roots α, $\mu + i\beta$, and $\mu - i\beta$, $\beta \neq 0$. Consider the solutions $e^{\alpha t}$, $e^{\mu t}\sin\beta t$, and $e^{\mu t}\cos\beta t$.

Case I. Suppose $\mu < \alpha$. The solution

$$\begin{aligned} y(t) &= c_1 e^{\alpha t} + c_2 e^{\mu t}\sin\beta t + c_3 e^{\mu t}\cos\beta t \\ &= e^{\alpha t}(c_1 + c_2 e^{(\mu - \alpha)t}\sin\beta t + c_3 e^{(\mu - \alpha)t}\cos\beta t) \end{aligned}$$

oscillates if and only if $c_1 = 0$ since $\mu - \alpha < 0$ and

$$c_2{}^{(\mu - \alpha)t}(\sin\beta t + \cos\beta t) = K\, e^{(\mu - \alpha)t} \sin(\beta t + \gamma) .$$

Therefore, (4) has property RO.

Case II. Let $\mu > \alpha$, then

$$\begin{aligned} y(t) &= c_1 e^{\alpha t} + c_2 e^{\mu t} \sin\beta t + c_3 e^{\mu t} \cos\beta t \\ &= e^{\mu t}(c_1 e^{(\alpha - \mu)t} + \sqrt{c_2^2 + c_3^2}\ \sin(\beta t + \delta) \end{aligned}$$

is nonoscillatory if and only if $c_2^2 + c_3^2 = 0$, as $\alpha - \mu < 0$. Thus (4) has property RN.

Case III. Suppose $\alpha = \mu$. Then the solution

$$y(t) = e^{\alpha t}(\frac{1}{2} + \sin\beta t)$$

is oscillatory, and the solution

$$y(t) = e^{\alpha t}(2 + \sin\beta t)$$

is nonoscillatory. Hence, (t) has neither property RO nor RN.

The properties RN and RO may be charcterized by the following conditions on μ and α:

(i) equation (4) has property RO if and only if $\mu < \alpha$,

(ii) equation (4) has property RN if and only if $\mu > \alpha$.

It is easily shown that $-\alpha$, $-\mu - i\beta$, $-\mu + i\beta$ are roots of the characteristic equation of (5). If equation (4) has property RO, then

$\mu < \alpha$. Hence $-\alpha > -\mu$, and (5) has property RN. Similarly, if (4) has property RN then (5) has property RO. This relationship between the differential equation and its adjoint will be considered again in Chapter IV.

Definition 1.9 The differential equation (1) is said to be (2, 1) disconjugate if for any nontrivial solution $y(t)$ of (1) and any number $b \in [a, \infty)$, $y(b) = y'(b) = 0$ implies $y(t) \neq 0$ for $t > b$.

Remark 1.10 The differential equation in Example 1.7 has property R. The solution

$$y(t) = -1 + \sin t$$

is an oscillatory solution with double zeros. So property R does not imply (2, 1) disconjugacy.

Lemma 1.11 (Hanan [12]). If $u(t)$ and $v(t)$ are linearly independent solutions of equation (2) such that $u(b) = v(b) = 0$ for some $b > a$ and equation (2) is (2, 1) disconjugate, then the zeros of $u(t)$ and $v(t)$ separate in $[a, b)$.

Theorem 1.12 (Mammana [16]). Equation (2) is (2, 1) disconjugate if $2Q(t) - P'(t) < 0$.

Lemma 1.13 (Lazer [15]). If $P(t) \leq 0$, $Q(t) < 0$, $2Q(t) - P'(t) \leq 0$ in equation (2) then the derivative of any oscillatory solution of (2) is bounded.

Theorem 1.14 (Lazer [15]). If $P(t) \leq 0$, and $Q(t) > 0$, then equation (2) has a solution $z(t)$ such that

$$z'''(t)\, z''(t)\, z'(t)\, z(t) \neq 0, \quad t \in [a, \infty),$$

$$\operatorname{sgn} z(t) = \operatorname{sgn} z''(t) \neq \operatorname{sgn} z'(t) = \operatorname{sgn} z'''(t),$$

$$\lim_{t\to\infty} z''(t) = \lim_{t\to\infty} z'(t) = 0,$$

and $z(t)$ is asymptotic to a fixed constant.

<u>Theorem 1.15</u> (Lazer [15]). If $P(t) \le 0$, $Q(t) > 0$,

$$2\,\frac{P(t)}{Q(t)} + \frac{d^2}{dt^2}\left(\frac{1}{Q(t)}\right) \le 0,$$

and equation (2) has one oscillatory solution, then all solutions of (2) oscillate except constant multiples of the nonvanishing solution whose existence is asserted in Theorem 1.14.

<u>Theorem 1.16</u> If in equation (2) $P(t) \le 0$, $Q(t) \le 0$ and $y(t)$ is a solution of (2) such that

$$y(t_0) \ge 0, \quad y'(t_0) \ge 0, \quad \text{and} \quad y''(t_0) > 0$$

for some $t_0 \in [a, \infty)$, then $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ for all $t > t_0$ and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \infty,$$

<u>Theorem 1.17</u> (Birkhoff [6]). If $y_1(t)$, $y_2(t)$, $y_3(t)$ form a fundamental set for (1) then

$$z_1(t) = \frac{W(y_2, y_3)}{W(y_1, y_2, y_3)}(t),$$

$$z_2(t) = \frac{W(y_1, y_3)}{W(y_1, y_2, y_3)}(t),$$

$$z_3(t) = \frac{W(y_1, y_2)}{W(y_1, y_2, y_3)}(t)$$

form a fundamental set for $(1)^*$.

Theorem 1.18 (Polya [17]). If the functions $u, f_1, f_2, \ldots, f_n \in C^{n-1}(I)$ on some interval I, then

(i) $W(uf_1, \ldots, uf_n) = u^n W(f_1, \ldots, f_n)$

(ii) $W(W(f_1, f_2), W(f_1, f_3), W(f_2, f_3)) = W^2(f_1, f_2, f_3)$.

Remark 1.19 Barrett [4] introduced the canonical form

$$L_3[y] = \{r_2(t)[y'' + q_1(t)y]\}' + q_2(t)y' = 0, \quad t \in [a, \infty), \tag{6}$$

where $r_2(t) > 0$, and $r_2(t), q_1(t), q_2(t) \in C[a, \infty)$. Equation (6) is equivalent to the system

$$\alpha' = B(t)\alpha, \tag{7}$$

where

$$B(t) = \begin{pmatrix} 0 & 1 & 0 \\ -q_1(t) & 0 & \frac{1}{r_2(t)} \\ 0 & q_2(t) & 0 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix}.$$

That is, if $y(t)$ is a solution of (6), then

$$\alpha(t) = \begin{pmatrix} y\ (t) \\ y_1(t) \\ y_2(t) \end{pmatrix}$$

is a solution of (7), where $y_1 = y'$ and $y_2 = r_2[y'' + q_1 y]$. Conversely, if

$$\alpha(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix}$$

is a solution of (7), then $y(t) = a_1(t)$ is a solution of (6). Consequently, the standard uniqueness and existence theorems for (6) follow from system (7).

Theorem 1.20 (Rabenstein [19]). If y_1, y_2 and y_3 are solutions of equation (1), then

$$W[y_1, y_2, y_3](t) = W[y_1, y_2, y_3](a) \exp\left(-\int_a^t P(s)ds\right).$$

The above relation is sometimes referred to as Abel's formula.

CHAPTER II

OSCILLATION

The purpose of this chapter is to investigate oscillatory properties of fundamental sets for equation (1) and subspaces of the solution space Ω. The coefficients in equation (1) are continuous on some half ray $[a, \infty)$. Hence, Ω is a vector space of dimension three. The case where p, q, and r are constants suggests that every fundamental set contains a nonoscillatory solution. However, this is not true as will be shown in Chapter IV; in fact, it will be shown that Ω can be strongly oscillatory. The following example shows that there are examples of equation (1) which have fundamental sets which contain 0, 1, 2, or 3 nonoscillatory solutions.

Example 2.1 Let $y_1(t) = e^t$, $y_2(t) = e^t \sin t$, and $y_3(t) = e^t \cos t$. Then $\{y_1, y_2, y_3\}$ is a fundamental set for the third order linear differential equation given by

$$W(y, y_1(t), y_2(t), y_3(t)) = 0 , \qquad (8)$$

The following are fundamental sets for equation (8) with 0, 1, 2, 3 nonoscillatory solutions respectively:

$$B_0 = \{e^t(1+\cos t),\ e^t\cos t,\ e^t\sin t\} ,$$

$$B_1 = \{e^t,\ e^t\sin t,\ e^t\cos t\} ,$$

$$B_2 = \{e^t,\ e^t(2+\cos t),\ e^t \sin t\}\ ,$$

$$B_3 = \{e^t,\ e^t(2+\cos t),\ e^t(2+\sin t)\}\ .$$

Although B_0 contains three oscillatory solutions, equation (8) has a nonoscillatory solution $y_1(t) = e^t$. The following theorem due to Utz [23] provides a generalization of this example. The proof given below is provided by the author.

Theorem 2.2 Recall equation (2) given by

$$z''' + P(t)z' + Q(t)z = 0\ ,$$

where $P(t),\ Q(t) \in C[a,\infty)$. If

(a) $P(t) \leq 0,\ Q(t) > 0$,

(b) $\dfrac{2\,P(t)}{Q(t)} + \dfrac{d^2}{dt^2}\left(\dfrac{1}{Q(t)}\right) \leq 0$,

and some solution of (2) oscillates, then there are three linearly independent oscillatory solutions of (2); yet some nontrivial solution is nonoscillatory.

Proof: As $P(t) \leq 0,\ Q(t) > 0$, (2) has a nontrivial nonoscillatory solution $z(t)$ such that $\lim_{t\to\infty} z(t) = c$ by Theorem 1.14. Conditions (a) and (b) imply the existence of two linearly independent oscillatory solutions of (2) (Theorem 1.15) say, $z_1(t)$ and $z_2(t)$ such that z, z_1, z_2 are linearly independent. Then $z_1(t) + z(t)$, $z_2(t) + z(t)$ oscillate by Theorem 1.15. Since z, z_1, z_2 are linearly independent, $z + z_1,\ z + z_2,\ z_1$ (or z_2) are the required solutions and the proof is complete.

Utz [23] states that $\hat{S}$vec [21] has shown that if $P(t) \equiv 0$ and $Q(t) > 0$ in equation (2), then equation (2) has an oscillatory solution. The author was not able to find such a fact in Svec's paper. In fact, the statement is false as shown by the following example.

<u>Example 2.3</u> The differential equation

$$y''' + \frac{3}{8t^3} y = 0, \quad t \in [1, \infty), \tag{9}$$

has as a fundamental set $\{t^{3/2}, t^{(3+\sqrt{13})/4}, t^{(3-\sqrt{13})/4}\}$. Now, $y(t) = t^{3/2}$ is a nonoscillatory solution of (9) and $W(t^{3/2}, t^{(3+\sqrt{13})/4}) \neq 0$ on $[a, \infty)$. Hence, by Theorem 1.4, equation (9) has no oscillatory solutions.

Utz [23] claims that the following follows from Theorem 2.2. If $P(t) \equiv 0$, $Q(t) > 0$ and $\frac{d^2}{dt^2}\left(\frac{1}{Q(t)}\right) \leq 0$, then the conclusion of Theorem 2.2 follows. However, Utz's proof requires that $Q(t) > 0$ and $P(t) \equiv 0$ implies that equation (2) has an oscillatory solution. But, Example 2.3 is not a counterexample as $\frac{d^2}{dt^2}\left(\frac{1}{Q(t)}\right) > 0$. Thus, Utz's claim seems to be an open question.

The properties RO and RN describe the structure of Ω. For example, if equation (1) has property RO, then it follows from the definition of RO that there exist a two dimensional subspace of Ω such that each oscillatory solution of (1) is contained in this subspace. The following theorems, due to Ahmad [1], give a characterization of properties RO and RN.

<u>Theorem 2.4</u> If equation (1) has solutions $y_1(t)$, $y_2(t)$, and $y(t)$ such that $y(t)$ is non-vanishing on $[a, \infty)$, $y_1(t)$ and $y_2(t)$ are

oscillatory and $W(y_1, y_2)(t)$ is non-vanishing on $[a, \infty)$, then equation (1) has property RO if and only if

$$\lim_{t\to\infty} \frac{y_1(t)}{y(t)} = \lim_{t\to\infty} \frac{y_2(t)}{y(t)} = 0 . \tag{10}$$

Proof: Assume that condition (1) holds. To show that (1) has property RO, it is sufficient to show that

$$z(t) = c_1\, y(t) + c_2\, y_1(t) + c_3\, y_2(t)$$

is oscillatory only if $c_1 = 0$. As $y(t) \neq 0$ on $[a, \infty)$, $z(t)$ can be written

$$z(t) = y(t)\left(c_1 + c_2 \frac{y_1(t)}{y(t)} + c_3 \frac{y_2(t)}{y(t)}\right) .$$

Therefore, $z(t)$ is oscillatory only if $c_1 = 0$. Hence, (1) has property RO.

Next, assume that equation (1) has property RO. In order to establish (10), it must be shown that given $\varepsilon > 0$ there exist T_i such that $\left|\frac{y_i(t)}{y(t)}\right| < \varepsilon$ for $t \geq T_i$, $i = 1, 2$.

Suppose that there exists an $\varepsilon > 0$ such that the above does not hold. Hence, there exists a sequence $\{t_{in}\}$ such that $t_{in} \to \infty$ and

$$\frac{y_i(t_{in})}{y(t_{in})} = \varepsilon \quad \text{or} \quad \frac{y_i(t_{in})}{y(t)} = -\varepsilon , \quad i = 1, 2 .$$

This is possible as $\frac{y_i(t)}{y(t)}$ is oscillatory. If the first equality holds, then $y_i(t) - \varepsilon\, y(t)$ is an oscillatory solution, contradicting the fact that (1) has property RO. A similar contradiction is reached if the second equality holds. Hence, condition (1) must hold and the proof is complete.

Theorem 2.5 Suppose that equation (1) has solutions $y_1(t)$, $y_2(t)$ and $v(t)$ such that $v(t)$ does not vanish on $[a, \infty)$ and $y_1(t)$, $y_2(t)$ are oscillatory with $W(y_1, y_2)(t)$ non-vanishing on $[a, \infty)$. Then equation (1) has property RN if and only if every nontrivial linear combination of $\frac{y_1(t)}{v(t)}$ and $\frac{y_2(t)}{v(t)}$ is unbounded above and below.

Proof: First, assume that

$$z(t) = c_2 \frac{y_1(t)}{v(t)} + c_3 \frac{y_2(t)}{v(t)}, \quad c_2^2 + c_3^2 > 0,$$

is unbounded above and below. Then, given $c_1 > 0$, there exists a sequence $\{t_n\}$ such that $t_n < t_{n+1}$, $t_n \to \infty$ and

$$z(t_{2i-1}) > c_1, \; z(t_{2i}) < -c_1, \quad i = 1, 2, \ldots,$$

So, there exist a sequence $\{\overline{t}_n\}$ such that $\overline{t}_n \to \infty$ and $z(\overline{t}_n) = c_1$ for all n. Thus,

$$\begin{aligned} y(t) &= -c_1 v(t) + c_2 y_1(t) + c_3 y(t) \\ &= v(t)(-c_1 + z(t)) \end{aligned}$$

is a nonoscillatory solution only if $z(t) \equiv 0$, that is $c_2 = c_3 = 0$. Hence, equation (1) has property RN.

Next, assume that equation (1) has property RN. Suppose that $z(t)$ is bounded above. So, there exists a number M such that $z(t) < M$ for all $t \in [a, \infty)$. Then

$$y(t) = v(t)[-M + z(t)]$$

is a nonoscillatory solution which is not a constant multiple of $v(t)$, contradicting the fact that equation (1) has property RN. Hence, $z(t)$ is unbounded above. Similarly, $z(t)$ is unbounded below and the proof is complete.

The following results concerning property RO were obtained by Benharbit [5]. The proof of his first lemma employs a technique which has been widely used in the current literature.

<u>Lemma 2.6</u> In equation (2), let $P(t) < 0$, $Q(t) < 0$ on $[a, \infty)$. If $P \in C^1[a, \infty)$ with $P'(t) \leq 0$, then all oscillatory solutions of (2), if any, are bounded on $[a, \infty)$.

Proof: Let $y(t)$ be an oscillatory solution of (2), t_1 a fixed zero of $y'(t)$, and t_2 any other zero of $y'(t)$ such that $t_2 > t_1$. These points exist by Rolle's theorem as $y(t)$ oscillates.

Let the maximum of $y^2(t)$ on $[t_1, t_2]$ occur at $\bar{t}$. Then $y'(\bar{t}) = 0$ as $y'(t_1) = y'(t_2) = 0$. Define

$$F[y(t)] = [y'(t)]^2 - 2y(t)y''(t) - P(t)y^2(t). \tag{11}$$

It can be verified by differentiation that

$$F[y(t)] = F[y(t_1)] - \int_{t_1}^{t} P'(s)y^2(s)ds$$
$$+ 2\int_{t_1}^{t} Q(s)y^2(s)ds .$$

If $\bar{t} = t_1$, then the maximum of $y^2(t)$ on $[t_1, t_2]$ is given by $y^2(t_1)$. Now, if $\bar{t} > t_1$

$$F[y(\bar{t})] = F[y(t_1)] - \int_{t_1}^{\bar{t}} P'(s) y^2(s) ds + 2 \int_{t_1}^{\bar{t}} Q(s) y^2(s) ds$$

$$\leq F[y(t_1)] - y^2(\bar{t}) \int_{t_1}^{\bar{t}} P'(s) ds, \tag{12}$$

$$= F[y(t_1)] - y^2(\bar{t})(P(\bar{t}) - P(t_1)).$$

Using the fact that $y'(\bar{t}) = 0$ and equation (11), it follows that

$$F[y(\bar{t})] = -2y(\bar{t})y''(\bar{t}) - P(\bar{t})y^2(\bar{t}). \tag{13}$$

From (12) and (13),

$$-P(t_1)y^2(\bar{t}) - 2y(\bar{t})y''(\bar{t}) \leq F[y(t_1)].$$

Now $y(\bar{t})y''(\bar{t}) < 0$. Otherwise, by Theorem 1.16, $y(t)$ would be nonoscillatory which is a contradiction. So,

$$P(t_1)y^2(\bar{t}) \leq F[y(t_1)]$$

or

$$y^2(\bar{t}) \leq \frac{F[y(t_1)]}{P(t_1)}.$$

Therefore,

$$\max_{t \in [t_1, t_2]} y^2(t) \leq y^2(t_1) + \frac{F[y(t_1)]}{P(t_1)},$$

(Recall that two cases were considered $\bar{t} = t_1$, $\bar{t} > t_1$) and t_1 is fixed. Hence, $y(t)$ is bounded.

Theorem 2.6 Assume the hypothesis of the preceeding lemma. If equation (2) is oscillatory, then it has property RO.

Proof: By Theorem 2 of [2] (which will be presented in Chapter III) there exist two linearly independent oscillatory solutions of equation (2), say $y_1(t)$ and $y_2(t)$, such that any nontrivial linear combination of $y_1(t)$ and $y_2(t)$ is oscillatory and the zeros of $y_1(t)$ and $y_2(t)$ separate.

If $W(y_1, y_2)(\bar{t}) = 0$, then there exist constants c_1 and c_2 such that

$$z(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1^2 + c_2^2 > 0,$$

satisfies

$$z(\bar{t}) = z'(\bar{t}) = 0, \quad z''(\bar{t}) > 0.$$

Hence, by Theorem 1.16, $z(t)$ is nonoscillatory which is a contradiction. Therefore, $W(y_1, y_2)(t) \neq 0$ on $[a, \infty)$.

Let $y_3(t)$ be the solution of (2) defined by

$$y_3(a) = y_3'(a) = 0, \quad y_3''(a) = 1.$$

By Theorem 1.16, $y_3(t)$ is a nonoscillatory solution. So $\{y_1, y_2, y_3\}$ is a fundamental set for equation (2). By Theorem 2 of [2], $\lim_{t\to\infty} y_3(t) = \infty$. Let $y(t)$ be any solution of (2), then

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t).$$

By the preceeding lemma, $y_1(t)$ and $y_2(t)$ are bounded. Hence, $y(t)$ is unbounded if $c_3 \neq 0$ as $y_3(t)$ is unbounded. Thus, $y_3(t)$

can be assumed to be any nonoscillatory solution. So, y is oscillatory if and only if $c_3 = 0$. Thus, equation (2) has property RO, and the proof is complete.

The following results were obtained by Gustafson [11].

Theorem 2.7 A necessary and sufficient condition for every two dimensional subspace of the solution space of

$$(P(t)u'')' + Q(t)u' = 0 , \tag{14}$$

where $P, Q \in C[a, \infty)$ and $P(t) \neq 0$ on $[a, \infty)$, to contain both an oscillatory and a nonosillatory solution is that the following condition be satisfied: there exists a fundamental set $\{1, u(t), v(t)\}$ with $u(t)$ and $v(t)$ oscillatory satisfying

(i) $u(t) + a$, $v(t) + b$ nonoscillatory implies $u(t) + a + c(u(t) + b)$ is oscillatory for some c,

(ii) $u(t) + a$, $v(t) + b$ oscillatory implies that $(u(t) + a) + c(v(t) + b)$ is nonoscillatory for some c.

(iii) $u(t) + kv(t)$ nonoscillatory implies $u(t) + kv(t) + c$ is oscillatory for some c.

Proof: Suppose that every two dimensional subspace of the solution space of (11) contains oscillatory and nonoscillatory solutions. Let $\{\bar{u}, \bar{v}, 1\}$ be any fundamental set for (11). The two dimensional generated by $\{1, \bar{v}\}$ contains an oscillatory solution $v = c_1 \bar{v} + c_2$ by hypothesis. Similarly, there exist constants c_3 and c_4 such that $u = c_3 u + c_4$ is oscillatory. Therefore, $\{1, u, v\}$ is a fundamental set with u and v being oscillatory.

To show property (i) holds for $\{1, u, v\}$, suppose that $v+a$ and $u+b$ are nonoscillatory. The two dimensional subspace generated by $\{v+a, u+b\}$ contains an oscillatory solution

$$w(t) = \lambda(u(t)+a) + \mu(u(t)+b)$$

by hypothesis. Also, $\lambda\mu \neq 0$ as $u+a$, and $v+b$ are nonoscillatory. Hence, $z(t) = \frac{1}{\lambda} w(t)$ is the required solution. Properties (ii) and (iii) can be verified in a similar manner.

Suppose there exists a fundamental set $\{1, u, v\}$ for (14) with u and v oscillatory and conditions (i), (ii), (iii) satisfied. Let X be any two dimensional subspace of the solution space of (14) with basis $\{x_1(t), x_2(t)\}$. Now

$$x_i = a_i u + b_i v + c_i, \quad i = 1, 2. \tag{15}$$

Assume that X is nonoscillatory. It follows from (15) that if $c_1b_2 - c_2b_1 = 0$, then $c_1x_2 - c_2x_1$ is a constant multiple of u. But X is nonoscillatory so $c_1x_2 - c_2x_1 \equiv 0$. Hence, $c_1 = c_2 = 0$ as x_1 and x_2 are linearly independent. Thus, $x_i = a_iu + b_iv \in X$ and $a_i \neq 0$, $b_i \neq 0$, $i = 1, 2$ as X is nonoscillatory. Now,

$$a_1x_2 - a_2x_1 = (a_1b_2 - a_2b_1)v \in X,$$

But v is oscillatory, so $a_1 = a_2 = 0$. But this is not possible. Therefore, $c_1b_2 - c_2b_1 \neq 0$. Now,

$$c_1x_2 - c_2x_1 = (c_1a_2 - c_2a_1)u + (c_1b_2 - c_2b_1)v$$

is in X. As $c_1b_2 - c_2b_1 \neq 0$, $c_1a_2 - c_2a_1 \neq 0$ as X is nonoscillatory. Thus,

$$a_1x_1 - a_2x_1 = (a_1b_2 - a_2b_1)v + (a_1c_2 - a_2c_1) \not\equiv 0$$

is in X. Thus, $u + kv \in X$. Two cases shall be considered:

Case I: $1 \in X$. Condition (iii) implies there exists a constant c such that $u + kv + c$ is oscillatory. But, $u + kv + c \in X$ which is a contradiction.

Case II: $1 \notin X$. Then the solution space of equation (14) can be expressed as the direct sum of the space generated by $\{1\}$ and X. Hence, $u + a$ and $v + b$ must be in X for some constants a and b. Otherwise, u or v would be in X which is not possible as X is nonoscillatory. But, then by condition (i) there exists a constant c such that $(u + a) + c(v + b)$ is oscillatory which is a contradiction. Therefore, X cannot be nonoscillatory.

Next, assume that X is strongly oscillatory. If $a_1b_2 - a_2b_1 = 0$, then

$$a_1x_2 - a_2x_1 = a_1c_2 - c_1a_2$$

is in X. But X is strongly oscillatory, so $a_1 = a_2 = 0$. Thus, $u + a$ and $v + b \in X$ for some a, b and $u + a$, $v + b$ are oscillatory as X is strongly oscillatory. But condition (ii) implies there exist a nontrivial linear combination of $u + a$ and $v + b$ which is non-oscillatory. This is a contradiction. Therefore, $c = a_1b_2 - a_2b_1 \neq 0$ which implies that

$$\frac{1}{c}(a_1x_2 - a_2x_1) = v + a$$

and

$$\frac{1}{c}(b_1x_2 - b_2x_1) = u + b$$

are in X and condition (ii) leads to a contradiction. Therefore, X is not strongly oscillatory and the proof is complete.

Theorem 2.8 If every two dimensional subspace of the solution space of equation (14) contains both oscillatory and nonoscillatory solutions, then there exists a fundamental set $\{1, u, v\}$ of (14) such that

(i)' $u \geq 0$, $v \leq 0$ for large t,

(ii)' $u + a(v+b)$ oscillates for some $a \leq 0 (b \geq 0)$.

Proof: By Theorem 2.7, there exists a fundamental set $\{1, u_0, v_0\}$ such that u_0 and v_0 are oscillatory and conditions (i), (ii) and (iii) are satisfied.

If u_0 does not satisfy (i)', select k such that $u_1 = u_0 + kv_0$ is nonoscillatory. By Theorem 2.7, there exists a constant c such that $u_1 + c$ oscillates. Without loss of generality, c may be assumed to be positive. Let

$$c_1 = \inf\{c : u_1 + c \text{ oscillates}\} .$$

Suppose $u_1 + c$, $u_1 + d$ oscillate, $c < r < d$. Then

$$u_1 + c < u_1 + r < u_1 + d .$$

So, $u_1 + r$ oscillates. Thus,

$$\{c : u_1 + c \text{ oscillates}\}$$

is connected. Therefore, $u_1 + d$ is eventually of constant sign for each $d < c_1$. Let $u = \pm(u_1 + d)$.

Now, $u = \pm(u_0 + kv_0 + d)$, hence $\{1, u, v_0\}$ is a fundamental set. Choose a such that $u + a$ is oscillatory to get a solution $u^* = \pm(u + a + kv_0 + d)$ with the same properties as u. Then $\{1, u, u^*\}$ is the required fundamental set, and the proof is complete.

CHAPTER III

ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF SOLUTIONS

The purpose of this chapter is to study the oscillatory and asymptotic behavior of equation (1) given by

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0 ,$$

where $p(t)$, $q(t)$, and $r(t) \in C[a, \infty)$. The following results were obtained by Ahmad and Lazer [2].

<u>Lemma 3.1</u> If in equation (1) $p(t) \leq 0$, $q(t) \leq 0$, $r(t) < 0$, where $t \in [a, \infty)$ and $y(t)$ is a solution of (1) with

$$y(t_0) \geq 0 , \quad y'(t_0) \geq 0 \quad \text{and} \quad y''(t_0) > 0$$

for some $t_0 \in [a, \infty)$, then

$$y(t) > 0 , \; y'(t) > 0 , \; y''(t) > 0 \quad \text{for} \quad t > t_0$$

and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \infty .$$

Proof: First it will be shown that for $t > t_0$ $y''(t) > 0$. To do this consider

$$w(t) = y(t)\, y'(t)\, y''(t) .$$

If $y''(t) = 0$ for some $t > t_0$, then there exists a smallest number $t_1 > t_0$ such that $y''(t_1) = 0$. Otherwise $y''(t_0) = 0$ which is a contradiction. Since $y''(t) > 0$ for all $t \in (t_0, t_1)$, $y(t_0) > 0$ and $y'(t_0) > 0$, $y(t) > 0$ and $y'(t) > 0$ for all $t \in (t_0, t_1)$. Also, $y'''(t) > 0$ for all $t \in (t_0, t_1)$ as $p(t) \leq 0$, $q(t) \leq 0$ and $r(t) < 0$. Thus,

$$w'(t) = y'^2(t)\,y''(t) + y(t)\,y''^2(t) + y(t)\,y'(t)\,y'''(t) > 0$$

for all $t \in (t_0, t_1)$. Integrating $w'(t)$ from t_0 to t_1 yields

$$0 = w(t_0) + \int_{t_0}^{t_1} w'(s)ds .$$

Hence, $y''(t) > 0$ for $t \geq t_0$. Thus, $y'(t) > 0$, $y(t) > 0$ for $t > t$. It follows directly that $y'''(t) > 0$ for $t > t_0$. Clearly,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \infty.$$

and the proof is complete.

Lemma 3.2 If in equation (1) $p(t) \leq 0$, $q(t) \leq 0$, $r(t) < 0$ and $y(t)$ is a nontrivial nonoscillatory solution of (1), then there exists a number $t_1 \geq a$ such that $y'(t) \neq 0$ for $t \geq t_1$.

Proof: Since $y(t)$ is nonoscillatory, it may be assumed that there exists a number $t_1 \geq a$ such that $y(t) > 0$ for all $t \geq t_1$. Let $T \geq t_1$ be a number such that $y'(T) = 0$ and $y''(T) \geq 0$. By Lemma 3.1, $y'(t) > 0$ for all $t > T$. If $y'(T) = 0$ implies that $y''(T) < 0$ then $y'(t) = 0$ for at most one t and the proof is complete.

Theorem 3.3 In equation (1), if $p(t) \leq 0$, $q(t) \leq 0$, $r(t) < 0$ then the following conditions are equivalent:

A. There exists an oscillatory solution of (1).

B. If $w(t)$ is a nontrivial nonoscillatory solution of equation (1), then there exists a number $t_0 \geq a$, such that $w(t)w'(t)w''(t) \neq 0$ for $t \geq t_0$, and $\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) = \operatorname{sgn} w''(t)$ for $t \geq t_0$.

Proof: Suppose condition A holds. Let $u(t)$ be an oscillatory solution of (1). If $w(t)$ is a nonoscillatory solution of (1), then, by Lemma 3.2, there exists a number $t_1 \geq a$ such that

$$w(t)w'(t) \neq 0 \quad \text{for} \quad t \geq t_1 . \tag{16}$$

As $u(t)$ is oscillatory and $w(t)$ is nonoscillatory, there exists a number $s \geq t_1$ such that $W(u, w)(s) = 0$. Otherwise, by Theorem 1.4, equation (1) would be nonoscillatory. Hence, there exist numbers c_1 and c_2 such that

$$c_1 u(s) + c_2 w(s) = 0 ,$$

$$c_1 u'(s) + c_2 w'(s) = 0 ,$$

$$c_1^2 + c_2^2 = 1 .$$

Let $z(t) = c_1 u(t) + c_2 w(t)$. Since $u(t)$ is oscillatory and $w(t)$ is nonoscillatory, $u(t)$ and $v(t)$ are linearly independent. Hence, $z''(s) \neq 0$. Without loss of generality, assume that $z''(s) > 0$. Since $z(s) = z'(s) = 0$,

$$z(t) > 0 , \; z'(t) > 0 , \; z''(t) > 0 \quad \text{for} \quad t > s ,$$

by Lemma 3.1. Furthermore,

$$z'''(t) = -p(t)z''(t) - q(t)z'(t) - r(t)z(t) > 0$$

by the conditions on p, q and r. So,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = \infty . \tag{17}$$

From (16), either $w(t)w'(t) < 0$ or $w(t)w'(t) > 0$. If $w(t)w'(t) < 0$ for $t \geq t_1$, then $w(t)$ is bounded on $[t_1, \infty)$. Thus from (17)

$$\lim_{t\to\infty} c_1 u(t) = \lim_{t\to\infty} [z(t) - c_2 w(t)] = \infty .$$

But this contradicts the fact that $u(t)$ is oscillatory. Hence, $\operatorname{sgn} w(t) = \operatorname{sgn} w'(t)$ for $t \geq t_1$. If $w''(t_2) = 0$, then

$$\begin{aligned} \operatorname{sgn} w'''(t_2) &= \operatorname{sgn}[q(t_2)w'(t_2) + r(t_2)w(t_2)] \\ &= -\operatorname{sgn} w(t_2) . \end{aligned}$$

So, $w''(t)$ can have at most one zero on $[t_1, \infty)$. Therefore, there exists a number t_3 such that

$$w(t)w'(t)w''(t) \neq 0 \quad \text{for} \quad t \geq t_3 .$$

Suppose that $w(t)'w''(t) < 0$, $t \geq t_3$. This implies that $w'(t)$ is bounded. Then by (17)

$$\lim_{t\to\infty} c_1 u'(t) = \lim_{t\to\infty} [z'(t) - c_2 w'(t)] = \infty ,$$

This contradicts the fact that $u(t)$, hence $u'(t)$, is oscillatory. Thus,

$$\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) = \operatorname{sgn} w''(t) \quad \text{for} \quad t \geq t_3$$

and condition B holds.

Suppose statement B holds. Let $w(t)$ be a nonoscillatory solution satisfying the conditions of statement B. The following technique is worth noting. Let $z_0(t)$, $z_1(t)$, $z_2(t)$ be solutions of (1) defined by

$$z_k^{(j)}(a) = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}, \quad j, k = 0, 1, 2. \tag{18}$$

For each positive integer $n > a$, let a_{0n}, a_{2n}, b_{1n}, b_{2n} be numbers such that

$$a_{0n} z_0(n) + a_{2n} z_2(n) = 0, \tag{19}$$

$$b_{1n} z_1(n) + b_{2n} z_2(n) = 0, \tag{20}$$

$$a_{0n}^2 + a_{2n}^2 = b_{1n}^2 + b_{2n}^2 = 1, \tag{21}$$

Define for each $n > a$ solutions of (1) by

$$u_n(t) = a_{0n} z_0(t) + a_{2n} z_2(t),$$

$$v_n(t) = b_{1n} z_1(t) + b_{2n} z_2(t).$$

As the unit ball is compact, there exists a sequence of integers $\{n_k\}$ such that

$$\lim_{n_k \to \infty} a_{0n_k} = a_0, \ \lim_{n_k \to \infty} a_{2n_k} = a_1$$

$$\lim_{n_k \to \infty} b_{1n_k} = b_1 , \quad \lim_{n_k \to \infty} b_{2n_k} = b_2$$

and

$$a_0^2 + a_2^2 = b_1^2 + b_2^2 = 1 . \tag{23}$$

Let

$$\left. \begin{aligned} u(t) &= a_0 z_0(t) + a_2 z_2(t) , \\ v(t) &= b_1 z_1(t) + b_2 z_2(t) \end{aligned} \right\} . \tag{24}$$

Then $u(t)$ and $v(t)$ are solutions of equation (1) and

$$u_{n_k}^{(j)}(t) \to u^{(j)}(t) , \quad v_n^{(j)}(t) \to v^{(j)}(t) , \qquad \text{for} \quad j = 0, 1, 2 . \tag{25}$$

To see this consider the solution space Ω with the norm

$$\| y(t) \|_a = |y(a)| + |y'(a)| + |y''(a)| ,$$

Now, (25) clearly holds in the $\| \ \|_a$ norm. But in a finite dimensional space all norms are equivalent. Hence, (25) is valid.

Suppose $u(t)$ is nonoscillatory. It follows from (21) and the independence of $z_0(t)$ and $z_1(t)$ that $u(t)$ is a nontrivial solution. Condition B guarantees the existence of a number $t_0 \geq a$ such that

$$u(t_0) u'(t_0) u''(t_0) \neq 0$$

and

$$\operatorname{sgn} u(t_0) = \operatorname{sgn} u'(t_0) = \operatorname{sgn} u''(t_0) .$$

From (25) it follows that there exists an integer N such that $n_k \geq N$ implies

$$u_{n_k}(t_0)\, u'_{n_k}(t_0)\, u''_{n_k}(t_0) \neq 0$$

and

$$\operatorname{sgn} u_{k_n}(t_0) = \operatorname{sgn} u'_{n_k}(t_0) = \operatorname{sgn} u''_{n_k}(t_0) .$$

By Lemma 3, 1, $u_{n_k}(t) \neq 0$ for $t \geq t_0$ and $n_k \geq N$. However, for all $n_k > \max [N, t_0\}$, $u_{n_k}(n_k) = 0$ by (19). Hence, $u(t)$ must be oscillatory. Similarly $v(t)$ is oscillatory, and the proof is complete.

Theorem 3.4 The solutions $u(t)$ and $v(t)$ in Theorem 3.3 have the following properties:

(a) $u(t)$ and $v(t)$ are linearly independent,

(b) any nontrivial linear combination of $u(t)$ and $v(t)$ is oscillatory, and

(c) if $y_1(t)$ and $y_2(t)$ are two linear combinations of $u(t)$ and $v(t)$ which are linearly independent, then the zeros of $y_1(t)$ and $y_2(t)$ separate.

Proof: Suppose that there exist constants c_1 and c_2 such that

$$c_1 u(t) + c_2 v(t) \equiv 0 .$$

Then, by (23) and (24),

$$c_1 a_0 z_0(t) + b_1 c_2 z_1(t) + (c_1 a_2 + c_2 b_2)\, z_2(t) \equiv 0 ,$$

Now, $z_0(t)$, $z_1(t)$, $z_2(t)$ are linearly independent, If $c_1 \neq 0$, then $a_0 = 0$. So, $a_2 = \pm 1$ and $u(t) = \pm z_2(t)$. However, $z_2(t)$ is non-oscillatory by Lemma 3. 1. This is a contradiction to the assumption that $u(t)$ is oscillatory. Therefore, $c_1 = 0$. Similarly $c_2 = 0$. Hence, $u(t)$ and $v(t)$ are linearly independent.

Let $y(t) = d_1 u(t) + d_2 v(t)$ with $d_1^2 + d_2^2 \neq 0$. Since $u(t)$ and $v(t)$ are linearly independent, $y(t)$ is a nontrivial solution of (1). If $y(t)$ is nonoscillatory, then by Theorem 3.3 there exists a number $t_0 \geq a$ such that

$$y(t_0)\, y'(t_0)\, y''(t_0) \neq 0$$

and

$$\operatorname{sgn} y(t_0) = \operatorname{sgn} y'(t_0) = \operatorname{sgn} y''(t_0) .$$

Set

$$y_{n_k}(t) = d_1 u_{n_k}(t) + d_2 v_{n_k}(t),$$

where $u_{k_n}(t)$ and $v_{k_n}(t)$ are defined as in Theorem 3.3. Then $y_{n_k}(t)$ converges to $y(t)$ as in Theorem 3.3. Also, the solution $y(t)$ is oscillatory as in Theorem 3.3.

We wish to show that $W(u, v)(t) \neq 0$ for $t \in [0, \infty)$. Suppose to the contrary that there exists $s \in [a, \infty)$ such that $W(u, v)(s) = 0$. Then there exist constants c_1 and c_2 such that

$$c_1 u(s) + c_2 v(s) = 0 ,$$

$$c_1 u'(s) + c_2 u'(s) = 0 ,$$

$$c_1^2 + c_2^2 \neq 0 ,$$

Set $y(t) = c_1 u(t) + c_2 v(t)$. Then $y(s) = y'(s) = 0$. But $u(t)$ and $v(t)$ are linearly independent. Hence, $y''(s) \neq 0$. Thus, by Lemma 3.1,

$y(t)$ is nonoscillatory which is a contradiction to part (b). Therefore, $W(u(t), v(t)) \neq 0$ on $[a, \infty)$.

If $y(t)$ is any linear combination of $u(t)$ and $v(t)$, then $W(y, u, v)(t) \equiv 0$ on $[a, \infty)$. If $H(t) = u'(t)v''(t) - v'(t)u''(t)$, then

$$W(u, v)(t)\, y''(t) - W'(u, v)(t)\, y'(t) + H(t)\, y(t) = 0 .$$

Thus, $y(t)$ is a solution of the nonsingular linear second order equation

$$\left[\frac{y'}{W(u, v)(t)}\right]' + \frac{H(t)}{W^2(u, v)(t)}\, y = 0 , \tag{26}$$

By the Sturm separation theorem [22, p. 5], the zeros of linearly independent solutions of equation (26) separate, and part (c) follows.

Corollary 3.5 If the conditions of Theorem 3.3 hold, then a necessary and sufficient condition for equation (1) to have no oscillatory solutions is that there exists a solution $z(t)$ of (1) such that

$$z(t)\, z'(t) < 0 \qquad t \geq T \tag{27}$$

or

$$z'(t)\, z''(t) < 0 \qquad t \geq T \tag{28}$$

for some $T \in [a, \infty)$.

Proof: Suppose there exists a solution satisfying (27) or (28). Hence, condition B of Theorem 3.3 is not satisfied. Thus, equation (1) has no oscillatory solution.

Suppose equation (1) has no oscillatory solutions. Hence, there exists a nonoscillatory solution $z(t)$ which does not satisfy property

B of Theorem 3.3. By Lemma 3.2, there exists a number $t_0 \geq a$ such that $z(t)z'(t) \neq 0$ for $t \geq t_0$. If $z(t)z'(t) > 0$ for $t \geq t_0$, then as in Theorem 3.3 there exists $T \geq t_0$ such that $z'(t)z''(t) \neq 0$ for $t \geq T$. Therefore, $z'(t)z''(t) < 0$ for $t \geq t_0$ as condition B is not satisfied, and the proof is complete.

Theorem 3.6 Consider equation (2) given by

$$y''' + P(t)y' + Q(t)y = 0 ,$$

where $P(t)$, $Q(t) \in C[a, \infty)$ and $P(t) \leq 0$, $Q(t) < 0$, $2Q(t) - P'(t) \leq 0$ on $[a, \infty)$. If equation (2) is oscillatory, then there exist two linearly independent oscillatory solutions $u(t)$ and $v(t)$ of (2) whose zeros separate. Furthermore, a solution of (2) is oscillatory if and only if it is a nontrivial linear combination of $u(t)$ and $v(t)$. If $w(t)$ is a nontrivial solution of (2) which is not a linear combination of $u(t)$ and $v(t)$, then

$$\lim_{t\to\infty} |w(t)| = \lim_{t\to\infty} |w'(t)| = \infty .$$

Proof: By hypothesis (2) has an oscillatory solution. The conditions of Theorem 3.4 are satisfied. Hence, equation (2) has two linearly independent solutions, say $u(t)$ and $v(t)$, whose zeros separate and any nontrivial linear combination of $u(t)$ and $v(t)$ is also oscillatory. Furthermore, by Lemma 1.13, $u'(t)$ and $v'(t)$ are bounded.

Let $z(t)$ be the solution of (2) defined by $z(a) = z'(a) = 0$, $z''(a) = 1$. By Lemma 3.1, $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ for $t > a$. Hence,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = \infty,$$

Since $z(t)$ is nonoscillatory, $\{u(t), v(t), z(t)\}$ is a fundamental set for (2). Suppose

$$w(t) = c_1 u(t) + c_2 v(t) + c_3 z(t), \quad c_3 \neq 0.$$

Since $c_1 u(t) + c_2 u(t)$ is oscillatory and $\lim_{t\to\infty} z(t) = \infty$, $w(t)$ is unbounded. Now, $\lim_{t\to\infty} w'(t) = \infty$ as $u'(t)$ and $v'(t)$ are bounded and $\lim_{t\to\infty} z'(t) = \infty$. Hence, $w(t)$ is oscillatory if and only if $c_3 = 0$. And the proof is complete.

The following results concerning properties R, RO and RN were obtained by Ahmad [1].

<u>Theorem 3.7</u> If equation (1) is (2, 1) disconjugate on $[a, \infty)$, then it has property R on $[a, \infty)$.

Proof: Let $z_1(t)$, $z_2(t)$ and $z_3(t)$ be the solutions of (1) defined by

$$z_k^{(j)}(a) = \delta_{jk}, \quad j, k = 1, 2, 3.$$

For each $n > a$, let $\alpha = (a_{1n}, a_{2n}, a_{3n})$ and $\beta = (b_{1n}, b_{2n}, b_{3n})$ be such that

$$a_{1n}^2 + a_{2n}^2 + a_{3n}^2 = b_{1n}^2 + b_{2n}^2 + b_{3n}^2 = 1,$$

$$\alpha \cdot \beta = 0 \quad \text{and} \quad \alpha \cdot z_n = 0,$$

where $z_n = (z_1(n), z_2(n), z_3(n))$. That is, α and β are orthogonal unit vectors which are both orthogonal to z_n. Let

$$u_{1n}(t) = a_{1n} z_1(t) + a_{2n} z_2(t) + a_{3n} z_3(t) ,$$

$$u_{2n}(t) = b_{1n} z_1(t) + b_{2n} z_2(t) + b_{3n} z_3(t) .$$

Now, $W(u_{1n}, u_{2n})(t) \neq 0$ for $a \leq t < n$. Suppose to the contrary there exists $t_0 \in [a, n)$ such that $W(u_{1n}, u_{2n})(t_0) = 0$. Then there exist constants c_1 and c_2 such that

$$c_1 u_{1n}(t_0) + c_2 u_{2n}(t_0) = 0 ,$$

$$c_1 u'_{1n}(t_0) + c_2 u'_{2n}(t_0) = 0 ,$$

$$c_1^2 + c_2^2 \neq 0 .$$

Then $u(t) = c_1 u_{1n}(t) + c_2 u_{2n}(t)$ satisfies $u(t_0) = u'(t_0) = 0$ and $u(n) = 0$. This contradicts (2, 1) disconjugacy. Hence, $W(u_{1n}, u_{2n})(t) \neq 0$ for $a \leq t < n$.

Without loss of generality, assume that $W(u_{1n}, u_{2n})(t) > 0$, $a \leq t < n$. As in the proof of Theorem 3.3, there exist a sequence $\{n_k\}$ and numbers $a_1, a_2, a_3, b_1, b_2, b_3$ such that

$$\lim_{k\to\infty} (a_{1n_k}, a_{2n_k}, a_{3n_k}) = (a_1, a_2, a_3) ,$$

$$\lim_{k\to\infty} (b_{1n_k}, b_{2n_k}, b_{3n_k}) = (b_1, b_2, b_3) ,$$

and

$$\sum_1^3 a_i^2 = \sum_1^3 b_i^2 = 1 , \quad \sum_1^3 a_i b_i = 0 .$$

Let

$$u_1 = a_1 z_1 + a_2 z_2 + a_3 z_3 ,$$

$$u_2 = b_1 z_1 + b_2 z_2 + b_3 z_3 .$$

Now, $c_1 u_1 + c_2 u_2 = 0$ implies that $c_1(a_1, a_2, a_3) = c_2(b_1, b_2, b_3)$. It follows that $c_1 = c_2 = 0$ because $\sum_1^3 a_i b_i = 0$ and $\Sigma a_i^2 = \Sigma b_i^2 = 1$.

The next objective is to show that $W(u_1, u_2)(t) \neq 0$ for $t \in [a, \infty)$. To do this it will be shown that if $W(u_1, u_2)(T) = 0$, then $u_1(T) = u_2(T) = 0$.

Suppose that $W(u_1, u_2)(T) = 0$, $T \in [a, \infty)$. For each $t \in [a, \infty)$

$$W(u_{1n_k}, u_{2n_k})(t) \to W(u_1, u_2)(t)$$

and

$$W(u_{1n_k}, u_{2n_k})(t) > 0 .$$

Hence, $W(u_1, u_2)(t) \geq 0$. Therefore, $W(u_1, u_2)$ has a relative minimum at T. Hence, $W'(u_1, u_2)(T) = 0$. Thus,

$$\begin{aligned} u_1(T)\, u_2'(T) - u_2(T)\, u_1'(T) &= 0 , \\ u_1(T)\, u_2''(T) - u_2(T)\, u_1''(T) &= 0 , \end{aligned} \tag{29}$$

If $u_1(T) \neq 0$ or $u_2(T) \neq 0$, then it follows from (29) that the matrix

$$M = \begin{bmatrix} u_1(T) & u_2(T) \\ u_1'(T) & u_2'(T) \\ u_1''(T) & u_2''(T) \end{bmatrix}$$

has rank 1. This implies that there exist constants c_1 and c_2 such that

$$c_1 u_1(T) + c_2 u_2(T) = 0 ,$$

$$c_1 u_1'(T) + c_2 u_2'(T) = 0 ,$$

$$c_1 u_1''(T) + c_2 u_2''(T) = 0 ,$$

$$c_1^2 + c_2^2 \neq 0 .$$

Hence, $c_1 u_1 + c_2 y_2 \equiv 0$; contradicting the linear independence of $u_1(t)$ and $u_2(t)$. So, $u_1(T) = u_2(T) = 0$.

Suppose that $W(u_1, u_2)(\sigma) = 0$, $\sigma > T$. By the above argument, $u_1(\sigma) = u_2(\sigma) = 0$. Thus, $u_1'(T) \neq 0$ and $u_2'(T) \neq 0$ as (1) is (2, 1) disconjugate. Let

$$y(t) = u_2'(T) u_1(t) - u_1'(T) u_2(t) .$$

Then $y(T) = y'(T) = 0$ and $y(\sigma) = 0$. This contradicts the assumption that (2) is (2, 1) disconjugate. So, it has been shown that $W(u_1, u_2)(t) \neq 0$ for $t > T$.

Since $W(u_1, u_2)(t) \neq 0$ for $t > T$, $u_1(t)$ and $u_2(t)$ must be oscillatory. Otherwise, by Theorem 1.4, equation (1) would be non-oscillatory. Equation (1) is (2, 1) disconjugate. Hence $u_1'(T) \neq 0$ and $u_2'(T) \neq 0$. Thus,

$$y(t) = u_2'(T) u_1(t) - u_1'(T) u_2(t)$$

is a nontrivial solution of (1). Since $W(u_1, u_2)(t) \neq 0$ for $t > T$, $u_1(t)$ and $u_2(t)$ are solutions of a nonsingular second order linear differential equation as in Theorem 3.4. Hence, $y(t)$ is also a solution and oscillates by the Sturm separation theorem [22, p. 5]. Now, $y(T) = y'(T) = 0$ since $u_1(T) = u_2(T) = 0$ and $W(u_1, u_2)(T) = 0$. Because equation (1) is $(2, 1)$ disconjugate, it follows that

$$W(u_1, u_2)(t) \neq 0, \quad t \in [a, \infty),$$

As equation (1) is $(2, 1)$ disconjugate, the solution defined by $v(a) = v'(a) = 0$, $v''(a) \neq 0$ is nonoscillatory. Therefore, equation (1) has property R, and the proof is complete.

Corollary 3.8 If equation (1) is oscillatory and $p(t) \leq 0$, $q(t) \leq 0$, $r(t) < 0$ for $t \in [a, \infty)$, then equation (1) has property R.

Proof: Equation (1) is $(2, 1)$ disconjugate by Lemma 3.1. Thus the result follows directly from Theorem 3.7.

Example 3.9 In Example 1.9, it was shown that the equation

$$y''' + y' = 0 \qquad (3)$$

has property R. However, the solution $y(t) = 1 - \sin t$ satisfies $y(\frac{\pi}{2} + 2\pi n) = y'(\frac{\pi}{2} + 2\pi n) = 0 \quad n = 1, 2, \ldots$. Hence, (3) is not $(2, 1)$ disconjugate. So, property R does not imply $(2, 1)$ disconjugacy.

The following results were obtained by Benharbit [3].

Theorem 3.10 If equation (2) has an oscillatory solution $y(t)$ and

$$P(t) > 0, \quad Q(t) < 0, \quad 2Q(t) - P'(t) < 0 \quad \text{on} \quad [a, \infty),$$

then equation (2) has another oscillatory solution $z(t)$ such that $y(t)$ and $z(t)$ are linearly independent.

Proof: Let $y(t)$ be an oscillatory solution of (2), b a number such that $y(b) \neq 0$, and $\{t_i\}$ an increasing sequence of zeros of $y(t)$ with $t_1 > b$. (This is enough to insure that $\lim_{t\to\infty} t_i = \infty$). Let $y_1(t)$, $y_2(t)$, $y_3(t)$ be the solutions of (2) defined by

$$y_k^{(j)} = \delta_{kj}, \quad k, j = 1, 2, 3 ,$$

Thus, $\{y_1, y_2, y_3\}$ is a fundamental set for (2). For each ingeger n, let

$$z_n(t) = c_{1n} y_1(t) + c_{2n} y_2(t) + c_{3n} y_3(t)$$

be a solution of (2) satisfying the boundary condition

$$z_n(b) = z_n(t_n) = 0 .$$

Since, $y_2(b) = y_3(b) = z_n(b) = 0$, it follows that $c_{1n} = 0$. Hence,

$$z_n(t) = c_{2n} y_2(t) + c_{3n} y_3(t) ,$$

where $c_{2n}^2 + c_{3n}^2 = 1$. As $\{(c_{2n}, c_{3n})\}$ is a sequence of points on the unit ball, there exists a subsequence $\{c_{2n_k}, c_{3n_k}\}$ converging to (c_2, c_3) such that $c_2^2 + c_3^2 = 1$. Define a solution of (1) by

$$z(t) = c_2 z_2(t) + c_3 z_3(t) .$$

As in Theorem 3.3, $\{c_{2n_k} z_2^{(j)}(t), c_{3n_k} z_3^{(j)}(t)\}$ converges to $\{c_2 z_2^{(j)}(t), c_3 z_2^{(j)}(t)\}$, $j = 0, 1, 2$. Hence, $c_{2n_k} z_2^{(j)} + c_{2n_k} z_3^{(j)}$ converges to $z^{(j)}(t)$, $j = 0, 1, 2$. The solution $z(t)$ is nontrivial as

$c_2^2 + c_3^2 = 1$ and $z_2(t)$ and $z_3(t)$ are linearly independent. The solutions $z_{n_k}(t)$ and $y(t)$ are linearly independent as $y(b) \neq 0$ and $z_{n_k}(b) = 0$. Also, $z_{n_k}(t)$ and $y(t)$ have a common zero at t_{n_k}. By Theorem 1.12, equation (2) is (2, 1) disconjugate. Hence, by Theorem 1.11, the zeros of $y(t)$ and $z_{n_k}(t)$ separate in $[a, t_{n_k}]$.

Let c and d be consecutive zeros of $y(t)$ such that $c < d$. Select n_k such that $n_k > d$. Then, as above, the zeros of $z_{n_j}(t)$ and $y(t)$ separate for $n_j \geq n_k$ on $[a, x_{n_j})$; hence on $[c, d]$. So $z_{n_j}(t)$ has a zero say t_j in $[c, d]$ for all $n_j \geq n_k$. The t_j have an accumulation point $t_0 \in [c, d]$ as $[c, d]$ is compact. Hence, $z(t_0) = \lim_{n_k \to \infty} z_{n_k}(t_{n_k}) = 0$, and $z(t)$ is oscillatory. Furthermore, $z(t)$ and $y(t)$ are linearly independent as $z(b) = 0$ and $y(b) \neq 0$, and the proof is complete.

<u>Theorem 3.11</u> Every linear combination of $y(t)$ and $z(t)$ in the above theorem is oscillatory and $W(y, z)(t) \neq 0$ on $[b, \infty)$.

Proof: Let $v(t) = c_1 y(t) + c_2 z(t)$. If $c_1 = 0$ or $c_2 = 0$, then $v(t)$ is oscillatory as $y(t)$ and $z(t)$ are oscillatory. Suppose that $c_1 c_2 \neq 0$ and $v(t)$ is nonoscillatory. Without loss of generality, assume that $v(t) > 0$ for all $t \geq b$. As equation (2) is (2, 1) disconjugate, all zeros of (2) are simple. But $c_2 z(t) > -c_1 y(t)$, $y(t)$ having simple zeros implies there exists an interval (t_1, t_2) such that $c_2 z(t) > 0$ on (t_1, t_2). This contradicts the results of Theorem 3.10. Therefore, every nontrivial linear combination of $y(t)$ and $z(t)$ oscillates.

Suppose that $W(y, z)(t_0) = 0$ for some $t_0 \in [b, \infty)$. Then there exist constants c_1 and c_2 such that

$$c_1 y(t_0) + c_2 z(t_0) = 0 ,$$

$$c_1 y'(t_0) + c_2 z'(t_0) = 0 ,$$

$$c_1^2 + c_2^2 = 0 .$$

So, $w(t) = c_1 y(t) + c_2 z(t)$ is an oscillatory solution with a double zero at t_0. This contradicts the fact that equation (2) is (2, 1) disconjugate. Therefore, $W(y, v)(t) \neq 0$ on $[b, \infty)$ and the proof is complete.

<u>Lemma 3.12</u> Let $P(t) \geq 0$, $2Q(t) - P'(t) < 0$ for $t \in [a, \infty)$. If $y(t)$ is a solution of equation (2), then $F[y(t)] > 0$ for all $t \in [a, \infty)$, where

$$F[y(t)] = [y'(t)]^2 - 2y(t)y''(t) - P(t)y^2(t) .$$

Proof: Consider

$$F_1[y(t)] = F[y(a)] + \int_a^t [2Q(s) - P'(s)]y^2(s)ds .$$

Now,

$$F_1'[y(t)] = [2Q(t) - P'(t)]y^2(t) ,$$

However,

$$\begin{aligned} F'[y(t)] &= 2y'(t)y''(t) - 2y'(t)y''(t) \\ &\quad - 2y(t)y'''(t) - P'(t)y^2(t) - 2P(t)y(t)y'(t) \\ &= -2y(t)[-P(t)y' - Q(t)y(t)] - P'(t)y^2(t) \\ &\quad - 2P(t)y(t)y'(t) \\ &= y^2(t)[2Q(t) - P'(t)] , \end{aligned} \tag{30}$$

Therefore, $F'[y(t)] = F_1'[y(t)]$. It follows that

$$F[y(t)] = F[y(a)] + \int_a^t [2Q(s) - P'(s)]y^2(s)ds \ .$$

Also, from (30), $F'(y(t)) < 0$ except at the zeros of $y(t)$. Hence, $F[y(t)]$ is strictly decreasing.

Let $\{t_i\}$ be an increasing sequence of zeros of $y(t)$. Then $\lim_{i\to\infty} t_i = \infty$ and

$$\begin{aligned} F[y(t_i)] &= [y'(t_i)]^2 - 2y(t_i)\,y''(t_i) - P(t_i)\,y^2(t_i) \\ &= y'(t_i) \geq 0 \ . \end{aligned}$$

However, $F[y(t)]$ is strictly decreasing, Therefore, $F[y(t)] > 0$ for all $t \in [a, \infty)$.

<u>Theorem 3.13</u> If in equation (2) $P(t) > 0$, $2Q(t) - P'(t) < 0$ and $y(t)$ is an oscillatory solution of (2), then the zeros of $y(t)$ and $y'(t)$ separate.

Proof: By the preceeding lemma and its proof, $F[y(t)] > 0$ and is strictly decreasing on $[a, \infty)$. Hence, $\lim_{t\to\infty} F[y(t)]$ exists and is nonnegative.

The solution $y(t)$ oscillates. Thus, between any two consecutive zeros of $y(t)$ there exists a zero of $y'(t)$. Let t_k be a zero of $y'(t)$, then

$$\begin{aligned} F[y(t_k)] &= [y'(t_k)]^2 - 2y(t_k)\,y''(t_k)] \\ &\quad - P(t_k)\,y^2(t_k) \\ &= -2y(t_k)\,y''(t_k) - P(t_k)\,y^2(t_k) \ . \end{aligned}$$

Since $F[y(t)] > 0$, $P(t) > 0$, it follows that

$$y(t_k)\, y''(t_k) < 0 .$$

Let a_1 and a_2 be consecutive zeros of $y(t)$. Then $y'(t)$ can have at most a finite number of zeros on $[a_1, a_2]$. If not, then there exists $t^* \in (a_1 a_2)$ such that $y'(t^*) = y''(t^*) = 0$. This implies that

$$F[y(t^*)] = -P(t^*)\, y^2(t^*) < 0 ,$$

as $y(t^*) \neq 0$ since $y(t) \not\equiv 0$. This is a contradiction. So, $y'(t)$ has at most a finite number of zeros in $[a_1, a_2]$. Let t_1 and t_2 be two consecutive zeros of $y'(t)$ in $[a_1, a_2]$. These are simple zeros of $y'(t)$. Otherwise, $F[y(t)] < 0$ on $[a_1, a_2]$, and the proof is complete.

The results following Theorem 3.14 below were obtained by Pfeiffer [18]. These results use integrability criterion to guarantee the existence of oscillatory and nonoscillatory solutions of equation (2).

<u>Theorem 3.14</u> (Hinton [13]). If $r(t) > 0$ on $[a, \infty)$ and

$$\frac{r''(t)}{r^{1+1/n}(t)} ,$$

for $n = 1, 2, \ldots$ is in $L(a, \infty)$ {functions which are Lebesgue integrable on (a, ∞)}, then

(i) $r^{1/n}(t) \notin L(a, \infty)$

(ii) $\dfrac{r'(t)}{r^{1+1/n}(t)} \in L(a, \infty)$ and

(iii) $\left(\frac{r'(t)}{r^{1+1/2n}(t)}\right)^2 \in L(a,\infty)$.

Lemma 3.15 If $r(t) \neq 0$, $\frac{r''(t)}{r^{4/3}(t)} \in L(a,\infty)$, α is any real number such that $\alpha r^{1/3}(t) > 0$ on $[a,\infty)$ and $\beta = \pm\frac{1}{3}$, then

$$\lim_{t\to\infty} r^{\beta}(t)\exp\left(-\alpha\int_a^t r^{1/3}(s)ds\right) = 0 .$$

Proof: Without loss of generality, assume that $r(t) > 0$ on $[a,\infty)$. Then $\alpha > 0$. First it will be shown that

$$\lim_{t\to\infty} \frac{r'(t)}{r(t)^{4/3}} = 0 .$$

By Theorem 3.14 (ii), as $\frac{r''(t)}{r^{4/3}(t)} \in L(a,\infty)$, $\left(\frac{r'(t)}{r^{4/3}(t)}\right)' \in L(a,\infty)$, $\lim_{t\to\infty}\frac{r'(t)}{r^{4/3}(t)}$ exists and equals some constant c. So,

$$\lim_{t\to\infty}\left[\frac{r'(t)}{r^{4/3}(t)}\right]^2 = c^2 .$$

Suppose that $c \neq 0$. Then there exists a number $t_0 \geq a$ such that $t \geq t_0$ implies

$$\frac{c^2}{2} < \left(\frac{r'(t)}{r^{4/3}(t)}\right)^2 < \frac{3}{2}c^2 .$$

Hence,

$$\frac{c^2}{2} r^{1/3}(t) < \left(\frac{r'(t)}{r^{7/6}(t)}\right)^2 < \frac{3}{2}c^2 r^{1/3}(t) .$$

By Theorem 3.14 (iii), $\left(\frac{r'(t)}{r^{7/6}(t)}\right)^2 \in L(a,\infty)$. This implies that $r^{1/3}(t) \in L(a,\infty)$. This is a contradiction to part (i) of Theorem 3.14. Therefore, $\lim_{t\to\infty} \frac{r'(t)}{r^{4/3}(t)} = 0$. Hence, there exists a number $a_1 \geq a$ such that for $t \geq a_1$

$$\beta\left(\frac{\alpha\, r'(t)}{\alpha\, r(t)}\right) < \frac{\alpha}{2}\, r^{1/3}(t) .$$

Integrating the above from a_1 to t gives

$$\log\left(\frac{\alpha\, r(t)}{\alpha\, r(a_1)}\right)^{\beta} < \frac{\alpha}{2}\int_{a_1}^{t} r^{1/3}(s)ds .$$

Thus,

$$0 < \frac{\alpha\, r(t)}{\alpha\, r(a_1)} < \frac{\alpha}{2}\int_{a_1}^{t} r^{1/3}(s)ds$$

and

$$0 < \left(\frac{\alpha\, r(t)}{\alpha\, r(a_1)}\right)^{\beta} \exp\left(-\alpha\int_{a_1}^{t} r^{1/3}(s)ds\right) < \exp\left(-\frac{\alpha}{2}\int_{a}^{t} r^{1/3}(s)ds\right).$$

As $\lim_{t\to\infty}\int_{a_1}^{t} r^{1/3}(s)ds = \infty$,

$$\lim_{t\to\infty}\left(\frac{\alpha\, r(t)}{\alpha\, r(a_1)}\right)^{\beta} \exp\left(-\alpha\int_{a_1}^{t} r^{1/3}(s)ds\right) = 0$$

and the proof is complete.

Theorem 3.16 If $Q(t) \neq 0$ on $[a,\infty)$ and

$$\frac{Q''(t)}{Q^{4/3}(t)}, \quad \frac{P(t)}{Q^{1/3}(t)}$$

are in $L(a, \infty)$, then there are three linearly independent solutions of equation (2) such that one solution $y_1(t)$ is nonoscillatory on $[a, \infty)$ and the other two solutions $y_2(t)$ and $y_3(t)$ are oscillatory on $[a, \infty)$.

Proof: The hypothesis of corollary 5 of [13] are satisfied. So, there exist solutions $z_1(t)$, $z_2(t)$, $z_3(t)$ of equation (2) and a number $t_0 \geq a$ such that for $t \geq t_0$ and $k = 1, 2, 3$

$$\left.\begin{aligned} z_k(t) &= Q^{-1/3}(t) \exp \mu_k \int_{t_0}^{t} Q^{1/3}(s)ds \ (1+0(1)) , \\ z_k'(t) &= \mu_k \exp \mu_k \int_{t_0}^{t} r^{1/3}(s)ds \ (1+0(1)) , \\ z_k''(t) &= \mu_k^2 \exp \mu_k \int_{t_0}^{t} r^{1/3}(s)ds \ (1+0(1)) , \end{aligned}\right\} \qquad (31)$$

where $\mu_1 = 1$, $\mu_2 = \frac{1+\sqrt{3}\, i}{2}$, $\mu_3 = \frac{1-\sqrt{3}\, i}{2}$, and $\lim_{t\to\infty} (1+0(1)) = 1$.

Let $y_1(t) = z_1(t)$. Since $Q(t) > 0$ for all $t \in [a, \infty)$, $y_1(t)$ does not oscillate. For $k = 2, 3$ let

$$z_k = \theta_k(t) \exp(i\ \Psi_k(t)) , \qquad (32)$$

where $\theta_k(t)$ and $\Psi_k(t)$ are real-valued functions on $[a, \infty)$. Define

$$y_2(t) = \theta_2(t) \cos \Psi_2(t) , \quad y_3(t) = \theta_3(t) \sin \Psi_2(t) .$$

If

$$\lim_{t\to\infty} |\Psi_2(t)| = \lim_{t\to\infty} |\Psi_3(t)| = \infty ,$$

then $y_2(t)$ and $y_3(t)$ will be oscillatory. From (31) and (32) for

$k = 2,3$

$$\lim_{t\to\infty} \frac{\theta_k(t)\exp(i\,\Psi_k(t))}{Q^{-1/3}(t)\exp\left(\mu_k \int_{t_0}^{t} Q^{1/3}(s)ds\right)} = \lim_{t\to\infty} \frac{z_k(t)(1+0(1))}{z_k(t)} = 1 ,$$

Hence, as $|\exp(i\,\Psi_k(t))| = 1$,

$$\lim_{t\to\infty} \frac{\theta_k(t)}{Q^{-1/3}(t)\,\exp\left(\frac{1}{2}\int_{t_0}^{t} Q^{1/3}(s)ds\right)} = 1 .$$

Also,

$$\lim_{t\to\infty} \arg \frac{z_k}{z_k(1+0(1))} = 0, \quad k = 1,2,3 ,$$

Thus,

$$\begin{cases} \lim\limits_{t\to\infty} \left\{\Psi_2(t) - \frac{\sqrt{3}}{2}\int_{t_0}^{t} Q^{1/3}(s)ds + 2k_1\pi\right\} = 0, & k_1 \text{ a fixed integer,} \\ \lim\limits_{t\to\infty} \left\{\Psi_3(t) + \frac{\sqrt{3}}{2}\int_{t_0}^{t} Q^{1/3}(s)ds + 2k_2\pi\right\} = 0, & k_2 \text{ a fixed integer.} \end{cases} \tag{33}$$

From Theorem 3.14 (i),

$$\left|\int_{t_0}^{t} Q^{1/3}(s)ds\right| = \infty .$$

Hence, it follows from (33) that

$$\lim_{t\to\infty} |\Psi_k(t)| = \infty, \quad \text{for} \quad k = 2,3 .$$

Therefore, $y_1(t)$ and $y_2(t)$ are oscillatory on $[t_0, \infty)$. By computing $W(y_1, y_2, y_3)(t_0)$, the solutions $y_1(t)$, $y_2(t)$, $y_3(t)$ may be shown to be linearly independent, and the proof is complete.

Example 3.17 Consider the differential equation

$$y''' + (\frac{1}{t}) \sin t \, y' + t^3 y = 0, \quad t \in (1, \infty). \tag{34}$$

Now, $Q(t) = t^3 > 0$ on $(1, \infty)$,

$$\frac{Q''(t)}{Q^{4/3}(t)} = \frac{3}{t^3} \in L(1, \infty),$$

and

$$\frac{P(t)}{Q^{1/3}(t)} = \frac{\sin t}{t^3} \in L(1, \infty).$$

Hence, equation (34) has a fundamental set consisting of one nonoscillatory and two oscillatory solutions by Theorem 3.17.

Theorem 3.18 If $P(t) > 0$ on $[a, \infty)$ and monotonic on $[a_1, \infty)$ for some $a_1 \geq a$, $\frac{P''(t)}{P^{3/2}(t)}$ and $\frac{Q(t)}{P(t)} \in L(a, \infty)$, then there is a fundamental set for equation (2) consisting of one nonoscillatory solution and two oscillatory solutions on $[a, \infty)$.

Proof: Similar to the proof of Theorem 3.16.

Example 3.19 Consider the differential equation

$$y''' + t^3 y' - t \sin t \, y = 0, \quad t \in (1, \infty). \tag{35}$$

Now,

$$P(t) = t^3 > 0,$$

$$\frac{P''(t)}{P(t)} = \frac{3t}{t^{9/2}} = \frac{3}{t^{7/2}} \in L(1, \infty) ,$$

and

$$\frac{Q(t)}{P(t)} = \frac{\sin t}{t^2} \in L(1, \infty) .$$

Hence, by Theorem 3. 18, equation (35) has a fundamental set consisting of one nonoscillatory solution and two oscillatory solutions.

CHAPTER IV

THE ADJOINT EQUATION

The purpose of this chapter is to investigate the relations, if any, with regard to oscillatory behavior of solutions of equation (1) given by

$$y''' + p(t)y''' + q(t)y' + r(t)y = 0 ,$$

and its formal adjoint $(1)^*$ given by

$$y''' - (p(t)y)'' + (q(t)y)' - r(t)y = 0 .$$

In particular, the relationships between equation (1) and its formal adjoint $(1)^*$ with respect to the properties R, RO, RN, oscillation and weak oscillation will be studied.

The following results were obtained by Ahmad [1].

Theorem 4.0 If equation (1) has property R on $[a, \infty)$, then its formal adjoint $(1)^*$ has property R on $[b, \infty)$, where $b \geq a$. Conversely, if equation $(1)^*$ has property R on $[b, \infty)$, then equation (1) has property R on $[c, \infty)$, where $c \geq b$.

Proof: As the adjoint of equation $(1)^*$ is equation (1), the second part of the theorem will follow directly from the first part. Assume equation (1) has property R. Then, by the definition of property R, (1) has a fundamental set $\{u_1(t), u_2(t), v(t)\}$ such that $u_1(t)$ and

$u_2(t)$ are oscillatory with $W(u_1, u_2)(t) > 0$ for all $t \in [a, \infty)$ and $v(t)$ is a nonoscillatory solution. Let

$$F(t) = \exp \int_a^t p(s)ds \ .$$

Then $U_1(t) = F(t)\,W(u_1, v)(t)$, $U_2(t) = F(t)\,W(u_2, v)(t)$ and $V(t) = F(t)\,W(u_1, u_2)(t)$ are solutions of $(1)^*$ by Theorem 1.7. The solution $U_1(t)$ is oscillatory. Otherwise, there exists $t_0 \geq a$ such that $W(u_1, v)(t) \neq 0$ for $t \geq t_0$. But $v(t)$ is nonoscillatory. This implies that (1) is nonoscillatory by Theorem 1.4 which is a contradiction. Hence, $U_1(t)$ is oscillatory. Similarly, $U_2(t)$ is oscillatory. As $F(t) \neq 0$ and $W(u_1(t), u_2(t)) \neq 0$, $t \in [a, \infty)$, $V(t)$ is a nonoscillatory solution. Using Wronskian identities from Polya's paper [17],

$$\begin{aligned} W(U_1, U_2)(t) &= W(F\,W(u_1, v),\ F\,W(u_2, v))(t) \\ &= F^2(t)\,W(W(u_1, v),\ W(u_2, v))(t) \\ &= F^2(t)\,v(t)\,W(v, u_1, u_2)(t) \ . \end{aligned}$$

Thus, $W(U_1, U_2)(t) \neq 0$ for all $t \geq b$ as $F(t) \neq 0$, $W(v, u_1, u_2)(t) \neq 0$ and $v(t)$ is nonoscillatory. Therefore, $(1)^*$ has property R.

Theorem 4.1 If equation (1) has property RO on some interval $[a, \infty)$, then its adjoint $(1)^*$ has property RN on some interval $[b, \infty)$, $b \geq a$.

Proof: Suppose equation (1) has property RO. Then, (1) has linearly independent solutions $u_1(t)$, $u_2(t)$, $v(t)$ such that $v(t)$ is nonoscillatory, $u_1(t)$ and $u_2(t)$ are oscillatory. Furthermore, a solution of (1) is oscillatory if and only if it is a linear combination of $u_1(t)$ and $u_2(t)$. Also,

$$W(u_1, u_2)(t) \neq 0, \quad t \in [a, \infty).$$

Let $U_1(t)$, $U_2(t)$, $V(t)$ be solutions of $(1)^*$ as in Theorem 4.0. Then $U_1(t)$, $U_2(t)$ are oscillatory while $V(t)$, $W(U_1, U_2)(t)$ are nonoscillatory. Suppose $(1)^*$ has a nonoscillatory solution

$$z(t) = c_1 v(t) + z_2 u_1(t) + c_3 u_2(t), \quad \text{where} \quad c_2^2 + c_2^2 \neq 0. \tag{36}$$

Assume $c_2 \neq 0$. Then $W(u_1, z)(t)$ is oscillatory as in the proof of the previous theorem. Also,

$$W(U_1, z(t)) = c_1 W(u_1, v)(t) + c_2 W(U_2, V) + c_3 W(u_1, v). \tag{37}$$

In Theorem 4.0,

$$W(U_1, U_2)(t) = F^2(t) v(t) W(v, u_1, u_2)(t),$$

$$W(U_1, V)(t) = F^2(t) u_1(t) W(u_1, v, u_2)(t),$$

and

$$W(U_2, V)(t) = F^2(t) u_2(t) W(u_1, v, u_2)(t).$$

However, $W(v, u_1, u_2)(t) = -W(u_1, v, u_2)(t)$. Hence,

$$W(U_1, Z)(t) = F^2(t) W(u_1, v, u_2)(t) [c_1 u(t) - c_2 u_2(t) + c_3 v(t)].$$

Now, $W(U_1, Z)(t)$ oscillates, but $F(t)$ and $W(u_1, v, u_2)(t)$ are nonoscillatory. Thus,

$$y(t) = c_1 u_1(t) + c_2 u_2(t) + c_3 v(t)$$

is an oscillatory solution of (1). This contradicts the assumption that (1) has property RO. So, $c_2^2 + c_3^2 = 0$. This implies that $(1)^*$ has property RN, and the proof is complete.

Remark 4.2 The converse of Theorem 4.1 does not hold as shown by Benharbit [3].

Corollary 4.3 Consider equation (2) where $P(t)$ is differentiable, $P(t) \leq 0$, $P'(t) - Q(t) \leq 0$ and $P'(t) - 2Q(t) \leq 0$ on $[a, \infty)$. If the adjoint of (2) is oscillatory, then equation (2) has property RN on $[b, \infty)$ where $b \geq a$.

Proof: By Theorem 2.6, the adjoint of (2), given by

$$y''' + P(t)'y + (P'(t) - Q(t))y, \tag{38}$$

has property RO. It follows from Theorem 4.1 that equation (2) has property RN on some interval $[b, \infty)$, $b \geq a$.

The following theorem was proven by Gustafson [10].

Theorem 4.4 Given an integer $n > 2$, there exists an nth order linear differential equation $Ly = 0$, such that Ly is nonoscillatory on $[a, \infty)$ while its formal adjoint $L^*y = 0$ is strongly oscillatory on $[a, \infty)$.

Although the above theorem does provide an example where equation (1) is nonoscillatory while $(1)^*$ is strongly oscillatory, the functions p, q, r are not elementary functions. This example seems artificial and one might wonder if a simpler example might exist.

The following results are due to Dolan [8]. Before considering his results some notations must be introduced. Dolan uses a canonical form which was introduced by Barrett [6]. Equations (1) and $(1)^*$ take the form

$$L_3(y) = \{\ell[y'' + q_1 y]\}' + q_2 y = 0 , \tag{6}$$

$$L_3^*(y) = \{(\ell y')' + q_2 y\}' + q_1(\ell z') = 0 , \tag{6*}$$

where

$$\left.\begin{aligned} \ell(t) &= \exp\int_a^t p(s)ds , \\ q_1(t) &= \frac{1}{\ell(t)} \int_a^t \ell(s)\, r(s) ds , \\ q_2(t) &= \ell(t)[p(t) - q_1(t)] , \end{aligned}\right\} \tag{39}$$

for $t \geq a$.

If $y(t)$ and $z(t)$ are in $C^{(3)}[a, \infty)$, then Lagranges equation has the form

$$\{y ; z\}' = z\, Ly + y\, L^* z ,$$

where

$$\{y ; z\} = \sum_{k=0}^{2} (-1)^k D_k y D_{2-k}^* z \tag{40}$$

and

$$\left.\begin{aligned} D_0 y &= y & D_0^* z &= z , \\ D_1 y &= y' & D_1^* z &= \ell z' , \\ D_2 y &= \ell(y'' + q_1 y) & D_2^* z &= (\ell z')' + q_2 z . \end{aligned}\right\} \tag{41}$$

For $y_i(t) \in \Omega$, $z_j(t) \in \Omega^*$, $i,j = 1,2,3$. Let

$$m(y_1, y_2) = \mathrm{Det}(D_i y_j) \qquad i = 0,1, \; j = 1,2 ; \tag{42}$$

$$m^*(z_1, z_2) = \mathrm{Det}(D_i^* z_j) \qquad i = 0,1, \; j = 1,2 ; \tag{43}$$

$$M(y_1, y_2, y_3) = \mathrm{Det}(D_i y_j) \qquad i = 0,1,2, \; j = 1,2,3 ; \tag{44}$$

$$M^*(z_1, z_2, z_3) = \mathrm{Det}(D_i^* z_j) \qquad i = 0,1,2, \; j = 1,2,3 . \tag{45}$$

It follows from equation (1) and $(1)^*$, (44), (45), and Theorem 1.20 that $M(y_1, y_2, y_3)(t)$ and $M^*(z_1, z_2, z_3)(t)$ are constant if $y_i \in \Omega$, $z_i \in \Omega^*$, $i = 1,2,3$. Furthermore, $M(y_1, y_2, y_3)(t) \neq 0$ $\{M'(z_1, z_2, z_3)(t) \neq 0\}$ if and only if $\{y_1, y_2, y_3\}$ $(\{z_1, z_2, z_3\})$ is a fundamental set for equation (1) $\{(1)^*\}$.

From (40), (42), (43), (44) and (45), it follows that

$$\left.\begin{aligned} \{y_1 ; m(y_2, y_3)\} &= M(y_1, y_2, y_3) \text{ if } y_i \in \Omega, \; i = 1,2,3 ; \\ \{w^*(z_1, z_2) ; z_3\} &= M^*(z_1, z_2, z_3) \text{ if } z_i \in \Omega^*, \; i = 1,2,3 . \end{aligned}\right\} \tag{46}$$

The following lemma was stated by Dolan [8]. However, the proof as given by Dolan is incorrect. The author offers the following proof.

<u>Lemma 4.5</u> If $y(t) \in \Omega$ $\{z(t) \in \Omega^*\}$, then there exist solutions $z_i(t) \in \Omega^*$ $\{y_i(t) \in \Omega\}$, $i = 1,2$ such that

$$y(t) = m^*(z_1, z_2)(t) \{z(t) = m(y_1, y_2)(t)\}$$

Proof: Let $y(t) \in \Omega$. If $y(t) \equiv 0$ on $[a, \infty)$, let $z_1(t) \equiv 0$ and $z_2(t)$ be any solution of $(1)^*$.

Suppose $y(t) \not\equiv 0$ on $[a, \infty)$. Let $y_1(t)$ and $y_2(t) \in \Omega$ such that $W(y, y_1, y_2)(a) = 1$. Let

$$g(t) = \exp\left(-\int_a^t p(s)ds\right) = \frac{1}{W(y_1, y_2, y_3)(t)} ,$$

Then

$$z_1(t) = g(t)\, W(y_1, y_2)(t) , \quad z_2 = g(t)\, W(y_2, y)(t) ,$$

$$\text{and} \qquad z_3 = g(t)\, W(y_1, y_2)(t)$$

are solutions of $(1)^*$ by Theorem 1. 17.

Let

$$\ell(t) = \frac{1}{g(t)} , \tag{47}$$

Then $\ell(t)\, W(z_1, z_2)$ is a solution of (1) by Theorem 1, 17. From Theorem 1, 18

$$\begin{aligned} \ell(t)\, W(z_1, z_2)(t) &= \ell(t)\, W(g\, W(y_1, y),\ g\, W(y_2, y))(t) \\ &= \ell(t)\, g^2(t)\, W(W(y_1, y),\ W(y_2, y))(t) \\ &= g(t)\, y(t)\, W(y, y_1, y_2) \\ &= y(t) . \end{aligned}$$

But, $m^*(z_1, z_2)(t) = \ell(t)\, W(z_1, z_2)(t)$ and the proof is complete.

<u>Lemma 4. 6</u> If $y_i(t) \in \Omega$ $(z_1(t) \in \Omega^*)$ $i = 1, 2$, then $y_1(t)$ and $y_2(t)$ are linearly dependent if and only if $m(y_1, y_2)(t) \equiv 0$ $(m^*(z_1, z_2)(t) \equiv 0)$ on $[a, \infty)$.

Proof: If $y_i(t) \in \Omega$ $i = 1, 2$, $y_1(t)$ and $y_2(t)$ are linearly dependent, then as $W(y_1, y_2)(t) \equiv 0$ on $[a, \infty)$ it follows that

$$m(y_1, y_2)(t) = \ell(t) W(y_1, y_2)(t) \equiv 0 \quad \text{on} \quad [a, \infty) .$$

Now, suppose that $m(y_1, y_2) \equiv 0$ on $[a, \infty)$. Let $y(t) \in \Omega - [y_1(t), y_2(t)]$ ($[y_1(t), y_2(t)]$ the subspace spanned by $y_1(t)$ and $y_2(t)$). Then from (40), as $m(y_2, y_3)(t) \equiv 0$,

$$\begin{aligned} \ell(t) W(y, y_1, y_2)(t) &= M(y, y_1, y_2)(t) \\ &= \{y ; m(y_2, y_3)\} \\ &\equiv 0. \end{aligned}$$

But, $\ell(t) \neq 0$ on $[a, \infty)$. Hence, $W(y, y_1, y_2)(t) \equiv 0$ on $[a, \infty)$ which implies that $y(t)$, $y_1(t)$, $y_2(t)$ are linearly dependent, However, $y \in \Omega - [y_1(t), y_2(t)]$ thus $y_1(t)$ and $y_2(t)$ are linearly dependent, and the proof is complete.

In equation (1) make the substitution $\overline{y} = \rho u$, where $\overline{y}(t)$ and $\rho(t)$ are in Ω. Then $u(t)$ satisfies the differential equation

$$L_\rho(t) = \{R_2(\rho)[R_1(\rho)y']'\}' + Q(\rho)[R_1(\rho)y'] = 0 , \tag{47}$$

where

$$\left.\begin{aligned} R_1(\rho) &= \rho^2 , \\ R_2(\rho) &= \frac{\ell}{\rho} , \\ Q(\rho) &= \frac{1}{\rho^2}[D_2\rho + q_2\rho] , \end{aligned}\right\} \tag{48}$$

and $\ell(t)$, $q_2(t)$ are given by (39).

Similarly, if $\overline{z}$ and $\rho^* \in \Omega^*$, make the substitution $\overline{z} = \rho^* v$ in equation $(1)^*$. Then $v(t)$ satisfies the differential equation

$$L^*_{\rho^*}(z) = \{R^*_2(\rho^*)[R^*_1(\rho^*)z']'\}' + Q^*(\rho^*)[R^*_1(\rho^*)z'] = 0 ,$$

where

$$R_1^*(\rho^*) = \frac{1}{\rho^*},$$

$$R_2^*(\rho^*) = \ell \rho^{*2},$$

$$Q^*(\rho^*) = \frac{1}{\rho^{*2}} [D_2^* \rho^* + \ell q_1 \rho^*],$$

and $\ell(t)$, $q_1(t)$ are as in (39).

If $\rho(t) \in \mathfrak{L}$ and H_ρ is the solution space of equation (47), then

$$\mathfrak{L} = [\rho] H_\rho . \tag{51}$$

Furthermore, if $\rho(t)$ is nonoscillatory and $\bar{z} \in \mathfrak{L}^*$, then it follows from (40) and (41) that $\bar{z}$ satisfies the nonsingular linear second order differential equation

$$(R_2(\rho)y')' + Q(\rho)y = \frac{\{\rho;\bar{z}\}}{R_1(\rho)}, \tag{52}$$

on any interval where $\rho(t)$ does not have a zero.

<u>Definition 4.7</u> If $y_0(t) \in \mathfrak{L}$ $\{z_0(t)$ $\mathfrak{L}^*\}$, let

(i) $\mathfrak{L}^*_{y_0} = \{z(t) \in \mathfrak{L}^* : \{y;z\} \equiv 0\}$,

(ii) $\mathfrak{L}_{z_0} = \{y(t) \in \mathfrak{L} : \{y_0;z\} \equiv 0\}$.

<u>Lemma 4.8</u> If $y(t)$ $\{z(t)\}$ is a nontrivial solution of equation (1) $\{(1)^*\}$, then $z_0(t) \in \mathfrak{L}^*_y \{y_0(t) \in \mathfrak{L}_z\}$ if and only if

$$z_0(t) = m(y, \bar{y})(t) \{y_0(t) = m^*(z, \bar{z})(t)\}$$

for some $\bar{y}(t) \in \mathfrak{L}$ $\{\bar{z}(t) \in \mathfrak{L}^*\}$.

Proof: Suppose $z_0(t) \in \Omega_y^*$. It follows from Definition 4.7 that $\{y\,;z_0\}(t) \equiv 0$. By Lemma 4.5, there exist solutions $y_1(t)$ and $y_2(t)$ in Ω such that

$$z_0(t) = m(y_1, y_2)(t) .$$

Hence,

$$\{y\,;z_0\}(t) = \{y\,;m(y_1, y_2)\}(t) \equiv 0$$

on $[a, \infty)$. However,

$$\{y\,;m(y_1, y_2)\}(t) = M(y, y_1, y_2)(t)$$

$$= \ell(t)\, W(y, y_1, y_2)(t) .$$

This implies that $W(y, y_1, y_2)(t) \equiv 0$ on $[a, \infty)$. Therefore, $y(t)$, $y_1(t)$, and $y_2(t)$ are linearly dependent. So, there exist constants c, c_1, and c_2 such that

$$c\, y(t) + c_1 y_1(t) + c_2 y_2(t) \equiv 0 \quad \text{on} \quad [a, \infty)$$

and $c^2 + c_1^2 + c_2^2 > 0$.

As $y(t)$ is a nontrivial solution, either c_1 or c_2 is nonzero, say c_1. Then

$$y_1(t) = -\frac{1}{c_1}(c\, y(t) + c_2 y_2(t)) .$$

Thus,

$$z_0(t) = m(y_1, y_2)(t) = m(-\frac{1}{c_1}(c\, y + c_2\, y), y_2)(t)$$

$$= -\frac{c}{c_2}\, m(y, y_2)(t) = m(y, -\frac{c}{c_2}\, y_2)(t)$$

$$= m(y, \overline{y})(t) ,$$

where $\overline{y}(t) = -\frac{c}{c_2} y_2(t)$.

Suppose that $z_0(t) = m(y, \overline{y})(t)$. Then

$$\begin{aligned}\{z_0; y\}(t) &= \{m(y, \overline{y}); y\} = M(y, \overline{y}, y)(t) \\ &= \ell(t) W(y, \overline{y}, y)(t) \\ &\equiv 0 .\end{aligned}$$

Therefore, $z_0(t) \in \Omega_y$, and the proof is complete.

Lemma 4.9 If $y(t)$ $\{z(t)\}$ is a nontrivial solution of (1) $\{(1)^*\}$, then $\Omega_y^*\{\Omega_z\}$ has dimension 2.

Proof: Let $y_1(t)$ and $y_2(t) \in \Omega$ be such that $\{y(t), y_1(t), y_2(t)\}$ is a fundamental set for Ω. By Theorem 1.7,

$$z_1(t) = m(y, y_1)(t), \quad z_2(t) = m(y, y_2)(t)$$

$$z_3(t) = m(y_1, y_2)(t)$$

form a fundamental set for Ω^*. If $z_0(t) \in \Omega_y^*$, then, by Lemma 4.8, $z_0(t) = m(y; \overline{y})(t)$, where $\overline{y}(t) \in \Omega$. Let

$$\overline{y}(t) = c\, y(t) + c_1 y_1(t) + c_2 y_2(t) ,$$

Then

$$z_0(t) = m(y, \overline{y})(t) = c_1 m(y, y_1)(t) + c_2 m(y, y_2)(t) .$$

Hence, Ω_y^* has dimension 2 and the proof is complete.

Lemma 4.10 If $y_0(t)$ $\{z_0(t)\}$ is a nontrivial solution of (1) $\{(1)^*\}$ and $\overline{z}(t) \in \Omega^* - \Omega_{y_0}^*$ $\{\overline{y}(t) \in \Omega - \Omega_{z_0}\}$, then

$$\mathfrak{L} = [y_0] \oplus \mathfrak{L}_{\overline{z}} \{\mathfrak{L}^* = [z_0] \oplus \mathfrak{L}^*_{\overline{y}}\} .$$

Proof: Let $\overline{z}(t)$ $\mathfrak{L}^* - \mathfrak{L}^*_{y_0}$. This implies that $\{y_0; \overline{z}\}(t) \not\equiv 0$. Hence, $y_0(t) \notin \mathfrak{L}_{\overline{z}}$ and $[y_0] \cap \mathfrak{L}_{\overline{z}} = [0]$. $\mathfrak{L}_{\overline{z}}$ has dimension 2 by Lemma 4.9. Hence $\mathfrak{L} = [y_0] \oplus \mathfrak{L}_{\overline{z}}$.

Lemma 4.11 If $\mathfrak{L}_1\{\mathfrak{L}_1^*\}$ is a two dimensional subspace of $\mathfrak{L}\{\mathfrak{L}^*\}$, then there exists a nontrivial solution z_0 of $(1)^*$ $\{y_0$ of $(1)\}$ such that

$$\mathfrak{L}_1 = \mathfrak{L}_{z_0} \{\mathfrak{L}_1^* = \mathfrak{L}_{y_0}\} .$$

Proof: Let $\mathfrak{L}_1$ be a subspace of $\mathfrak{L}$ of dimension 2 with basis $\{y_1(t), y_2(t)\}$. If $z_0(t) = m(y_1, y_2)(t)$, then $z_0(t) \in \mathfrak{L}^*$ by Theorem 1.17. Also,

$$\{z_0; y_i\}(t) = M(y_1, y_2, y_i)(t) \equiv 0 \quad i = 1, 2 ,$$

So, $y_1(t)$ and $y_2(t)$ are in $\mathfrak{L}_{z_0}$. It follows that $\mathfrak{L}_1 = \mathfrak{L}_{z_0}$ as $\mathfrak{L}_{z_0}$ has dimension 2.

Lemma 4.12 If $y_i(t) \in \mathfrak{L}$ $\{z_i(t) \in \mathfrak{L}^*\}$ $i = 1, 2$ are linearly independent, then

$$\mathfrak{L}^*_{y_1} \cap \mathfrak{L}^*_{y_2} = [m(y_1, y_2)] \{\mathfrak{L}_{z_1} \cap \mathfrak{L}_{z_2} = [m(z_1, z_2)]\} .$$

Proof: If $z_0(t) \in \mathfrak{L}^*_{y_1} \cap \mathfrak{L}^*_{y_2}$, then $\{y_i; z_0\} = 0$, $i = 1, 2$ by Definition 4.7. There exists $y(t) \in \mathfrak{L}$ such that $z_0(t) = m(y_1, y)(t)$ by Lemma 4.8. So,

$$\begin{aligned} M(y_2, y_1, y)(t) &= \{y_2; m(y_1, y)\}(t) \\ &= \{y_2; z_0\}(t) = 0 . \end{aligned}$$

Hence, there exist constants c, c_1, c_2 not all zero such that

$$c\,y(t) + c_1 y_1(t) + c_2 y_2(t) \equiv 0 .$$

Now, $c \neq 0$ as $y_1(t)$ and $y_2(t)$ are linearly independent. So,

$$y(t) = -\frac{1}{c}\,(c_1 y_1(t) + c_2 y_2(t)) .$$

Therefore,

$$\begin{aligned} z_0(t) &= m(y_1, y) \\ &= -\frac{c_2}{c}\, m(y_1, y_2) \in [m(y_1, y_2)] . \end{aligned}$$

Conversely, by (46),

$$\begin{aligned} \{y_i ; c\, m(y_1, y_2)\} &= c\, M(y_i, y_1, y_2) \\ &= 0, \quad i = 1, 2 . \end{aligned}$$

It follows that $\Omega^*_{y_1} \cap \Omega^*_{y_2} = [m(y_1, y_2)]$.

<u>Lemma 4.13</u> If $\rho_0(t)$ $\{\rho_0^*\}$ is a nonoscillatory solution of equation (1) $\{(1)^*\}$, then the subspace $\Omega^*_{\rho_0}$ $\{\Omega_{\rho_0^*}\}$ is either nonoscillatory or strongly oscillatory. Moreover, $\Omega^*_{\rho_0}$ $\{\Omega_{\rho_0^*}\}$ is strongly oscillatory if and only if for each nonoscillatory solution $\rho(t) \in \Omega\{\rho^*(t) \in \Omega^*\}$, $\Omega^*_{\rho}\{\Omega_{\rho^*}\}$ is strongly oscillatory. Furthermore, the zeros of oscillatory solutions of (1) separate each other and eventually become simple.

Proof: Let $\rho_0(t)$ be a nonoscillatory solution of (1). Then by (52), the subspace $\Omega^*_{\rho_0}$ of Ω^* is the solution space of the linear second order differential equation

$$[R_2(\rho_0)y']' + Q(\rho_0)y = 0 \tag{53}$$

on $[b, \infty)$ if $\rho_0(t) \neq 0$ on $[b, \infty)$, $b \geq a$.

So, equation (53) is either strongly oscillatory or nonoscillatory by the Sturm comparison theorem [22, p. 5] and the zeros of linearly independent solutions separate. Also, nontrivial solutions of (53) can have at most simple zeros.

If $\rho_1(t)$ and $\rho_2(t)$ are nonoscillatory solutions of (1), then the above shows that $\Omega^*_{\rho_1}$ and $\Omega^*_{\rho_2}$ are either strongly oscillatory or nonoscillatory. However, by Lemma 4.12, $m(\rho_1, \rho_2) \in \Omega^*_{\rho_1} \cap \Omega^*_{\rho_2}$. Therefore, $\Omega^*_{\rho_1}$ is strongly oscillatory if and only if $\Omega^*_{\rho_2}$ is strongly oscillatory, and the proof is complete.

Lemma 4.14 If every two dimensional subspace of $\Omega\{\Omega^*\}$ is weakly oscillatory, then $\Omega^*\{\Omega\}$ is strongly oscillatory.

Proof: From Definition 4.7, if $y(t) \in \Omega$, then $y(t) \in \Omega_z$ if and only if $z(t) \in \Omega^*_y$. Let $y_1(t)$ and $y_2(t)$ be linearly independent solutions of (1) which are in Ω_z. From Lemma 4.12

$$z(t) \in \Omega^*_{y_1} \cap \Omega^*_{y_2} = [m(y_1, y_2)] .$$

If $y_1(t)$ and $y_2(t)$ are linearly dependent, then $m(y_1, y_2) \equiv 0$ by Lemma 4.6. Hence,

$$m : \Omega_z \times \Omega_z \to [z] .$$

As every two dimensional subspace of Ω is weakly oscillatory, it follows from Lemma 4.9 that Ω_z is weakly oscillatory. Thus, there exist solutions of (1), $y_1(t)$ and $y_2(t)$ in $\mathfrak{L}_z$ which are oscillatory and nonoscillatory respectively. Now, $z(t) = c\,m(y_1, y_2)(t)$. If $z(t)$ is nonoscillatory, then $m(y_1, y_2)(t) \neq 0$ on $[b, \infty)$, $b \geq a$. But this implies that $W(y_1, y_2)(t) \neq 0$, which says that $\mathfrak{L}$ is nonoscillatory as $y_2(t)$ is nonoscillatory. (This follows from Theorem 1.4.) This is a contradiction. Hence, $z(t)$ is oscillatory. As $z(t)$ was arbitrary, $\mathfrak{L}^*$ is strongly oscillatory and the proof is complete.

<u>Theorem 4.15</u> If equation (1) $\{(1)^*\}$ is weakly oscillatory, then equation $(1)^*$ $\{(1)\}$ is oscillatory.

Proof: As equation (1) is weakly oscillatory by hypothesis, there exists a nonoscillatory solution $\rho(t)$ of (1). By (51), $\mathfrak{L} = [\rho]H_\rho$. Hence, H_ρ must be oscillatory. Let $u(t)$ be an oscillatory solution in H_ρ, then $u'(t)$ is also oscillatory. Now, $u(t)$ satisfies

$$L_p(u) = \{R_2(\rho)[R_1(\rho)u']'\}' + Q(\rho)[R_1(\rho)u'] = 0 ,$$

$R_1(\rho) = \rho^2$, thus

$$[R_2(\rho^2(t)\,u'(t))']' + Q(\rho)(\rho^2(t)\,u'(t)) = 0 .$$

So, $\rho^2(t)\,u'(t)$ satisfies (53) and consequently $\rho^2(t)\,u'(t) \in \mathfrak{L}_\rho^*$. Now, $\mathfrak{L}_\rho^* \subset \mathfrak{L}^*$, and $u'(t)$ is oscillatory. Hence, $\mathfrak{L}^*$ is oscillatory and the proof is complete.

<u>Corollary 4.16</u> If equation (1) $\{(1)^*\}$ is weakly oscillatory, then equation $(1)^*$ $\{(1)\}$ has a strongly oscillatory two dimensional subspace.

Proof: Let $\rho(t)$ be a nonoscillatory solution of (1). Then, as in the proof of Theorem 4.15, $\mathfrak{L}_\rho^*$ is oscillatory. Hence, $\mathfrak{L}_\rho^*$ is strongly oscillatory by Lemma 4.13.

Corollary 4.17 If equation (1) $\{(1)^*\}$ is nonoscillatory, then equation $(1)^*$ $\{(1)\}$ is nonoscillatory or strongly oscillatory.

Proof: Contrapositive of Theorem 4.15.

Theorem 4.18 If $\mathfrak{L}\{\mathfrak{L}^*\}$ contains a nonoscillatory two dimensional subspace, then $\mathfrak{L}^*\{\mathfrak{L}\}$ is either nonoscillatory or strongly oscillatory. And if $z_i(t)$ $\{y_i(t)\}$ $i = 1, 2$ are linearly independent solutions of $\mathfrak{L}^*(\mathfrak{L})$, then there is a number $t_0 \geq a$ such that if $z_1(t)\{y_1(t)\}$ has at least three zeros on some interval $I \subset [t_0, \infty)$, then $z_2(t)\{y_2(t)\}$ has at least one zero on I.

Proof: Let $\mathfrak{L}_1$ be a nonoscillatory subspace of $\mathfrak{L}$. By Lemma 4.11, there exists a solution $z_0(t) \in \mathfrak{L}^*$ such that $\mathfrak{L}_1 = \mathfrak{L}_{z_0}$. Suppose $y_1(t)$ and $y_2(t)$ are a basis for $\mathfrak{L}_1$, $z(t) \in \mathfrak{L}^*$. Now, $\mathfrak{L}_z \cap \mathfrak{L}_1 \neq [0]$ by Lemma 4.12. So, there exist constants c_1 and c_2 such that

$$\{c_1 y_1 + c_2 y_2 ; z\} = 0, \quad c_1^2 + c_2^2 > 0 .$$

Therefore,

$$z(t) \in \mathfrak{L}_{\bar{y}}^* ,$$

where $\bar{y}(t) = c_1 y_1(t) + c_2 y_2(t)$. But $z(t)$ was arbitrary. Hence,

$$\mathfrak{L}^* = \bigcup_{\bar{y} \in \mathfrak{L}_1} \mathfrak{L}_{\bar{y}}^* .$$

From Lemma 4.13, Ω^* is nonoscillatory or strongly oscillatory depending on whether $z_0(t)$ is nonoscillatory or oscillatory as $z_0(t) \in \Omega^*_{\bar{y}}$ for all $\bar{y}(t) \in \Omega_1$.

Suppose that Ω^* is strongly oscillatory. Then $z_1(t)$ and $z_2(t)$ are linearly independent solutions of $(1)^*$, and

$$z_i(t) \in \Omega^*_{\bar{y}_i}, \quad i = 1, 2 \quad \text{for some} \quad \bar{y}_i(t) \in \Omega_1 .$$

By Lemma 4.12,

$$m(\bar{y}_1, \bar{y}_2)(t) \in \Omega^*_{\bar{y}_1} \cap \Omega^*_{\bar{y}_2} .$$

From Lemma 4.3, the zeros of $m(\bar{y}_1, \bar{y}_2)(t)$ eventually separate the zeros of $z_1(t)$ and $z_2(t)$ and conversely. So, if $z_1(t)$ has three zeros on I, $m(\bar{y}_1, \bar{y}_2)(t)$ has two zeros on I. Therefore, $z_2(t)$ has at least one zero on I, and the proof is complete.

<u>Lemma 4.19</u> If $\Omega\{\Omega^*\}$ contains a nonoscillatory solution $\rho(t)\{\rho^*(t)\}$ that has no zeros on $[b, \infty)$ for some number $b \geq a$ and $y(t) \in \Omega - [\rho]$ $\{z(t) \in \Omega^* - [\rho^*]\}$, then either

(i) There exist a solutions $z_0(t) \in \Omega^* - \Omega^*_\rho$ $\{y_0 \in \Omega - \Omega_{\rho^*}\}$ such that

$$y - \frac{\{y ; z_0\}}{\{\rho ; z_0\}} \rho \left\{ z - \frac{\{y_0 ; z\}}{\{y_0 ; \rho^*\}} \rho^* \right\}$$

is oscillatory, or

(ii) $\lim_{t\to\infty} \frac{y(t)}{\rho^*(t)}$ $\left\{ \lim_{t\to\infty} \frac{z(t)}{\rho^*(t)} \right\}$ exists, and $\Omega^*\{\Omega\}$ is either nonoscillatory or strongly oscillatory.

Proof: If $y(t) \in \Omega - [\rho]$, then $y(t)$ and $\rho(t)$ are linearly independent. Let λ be a real number. Then, by Lemma 4.12,

$$\Omega^*_{y-\lambda\rho} \cap \Omega^*_\rho = [m(y - \lambda\rho , \rho)]$$

as $y(t) - \lambda\rho(t)$, $\rho(t)$ are linearly independent. Now, $\Omega^*_{y-\lambda\rho}$ is of dimension 2. Hence, there exists $\bar{z}(t) \in \Omega^*_{y-\lambda\rho} \cap \Omega^* - \Omega^*_\rho$. So,

$$\{y - \lambda\rho ; \bar{z}\} = 0, \quad \text{but} \quad \{\rho ; \bar{z}\} \neq 0 .$$

Hence,

$$\lambda = \frac{\{y ; \bar{z}\}}{\{\rho ; \bar{z}\}} ,$$

Let $u(t) = \frac{y(t)}{\rho(t)}$ on $[b, \infty)$ and

$$\underline{\alpha} = \lim \inf u(t) , \tag{54}$$

$$\overline{\alpha} = \lim \sup u(t) . \tag{55}$$

Now, $u(t)$ is a solution of equation (47) as $\Omega = [\rho]H_\rho$. Also, $v(t) = 1$ is a solution of (47). Hence, if $\lambda \in (\underline{\alpha}, \overline{\alpha})$, then $u(t) - \lambda$ is in H_ρ. Now, $u(t) - \lambda$ is oscillatory as there exist sequences $\{t_i\}$ and $\{t_j\}$ of numbers in $[a, \infty)$ such that

$$\lim_{i\to\infty} t_i = \lim t_j = \infty,$$

$$\lim_{i\to\infty} u(t_i) = \underline{\alpha} ,$$

and

$$\lim_{j\to\infty} u(t_j) = \overline{\alpha} .$$

It was proven above that there exists $z_0(t) \in \Omega^* - \Omega^*_\rho$ such that

$$\lambda = \frac{\{y\,;z_0\}}{\{\rho\,;z_0\}}\ .$$

So,

$$y(t) - \frac{\{y\,;z_0\}}{\{\rho_0\,;z_0\}}\,\rho$$

is oscillatory. Thus, if $y(t) - \lambda\,\rho(t)$ is nonoscillatory for each real λ then $\underline{\alpha} = \overline{\alpha}$, that is, $\lim_{t\to\infty} \frac{y(t)}{\rho(t)}$ exist.

Suppose that $y(t) - \lambda\,\rho(t)$ is nonoscillatory for each real λ. Then $[y(t), \rho(t)]$ is nonoscillatory as $\rho(t)$ is nonoscillatory and

$$w(t) = c_1\,y(t) + c_2\,\rho(t) = c_1(y(t) + \frac{c_2}{c_1}\,\rho(t)\,,\quad c_1 \neq 0$$

is nonoscillatory. Therefore, by Theorem 4.18, Ω^* is either non-oscillatory or strongly oscillatory, and the proof is complete.

Remark 4.20 Assume the hypothesis of Lemma 4.19. Let $y \in \Omega - [\rho]$ $\{z \in \Omega^* - [\rho^*]\}$. It follows, as in the proof of Lemma 4.19, that if there exists a unique number $\lambda\{\lambda^*\}$ such that

$$y(t) - \lambda\,\rho(t)\ \ \{z(t) - \lambda^*\,\rho^*(t)\}$$

is oscillatory, then

$$\lim_{t\to\infty} \frac{y(t)}{\rho(t)}\ \left\{\lim_{t\to\infty} \frac{z(t)}{\rho^*(t)}\right\}$$

exists. Conversely, if

$$\lim_{t\to\infty} \frac{y(t)}{\rho(t)}\ \left\{\lim_{t\to\infty} \frac{z(t)}{\rho^*(t)}\right\}$$

exists, then there is at most one number $\lambda_0(\lambda_0^*)$ such that

$$y(t) - \lambda_0 \rho(t) \{z(t) - \lambda_0^* \rho^*(t)\}$$

is oscillatory.

Definition 4.21 If $y_i(t) \in \Omega\{z_i(t) \in \Omega^*\}$ $i = 1, 2$, then

(i) $y_1 = 0(y_2)$ $\{z_1 = 0(z_2)\}$ means that $y_2(t)\{z_2(t)\}$ is nonoscillatory and

$$\lim_{t\to\infty} \frac{y_1(t)}{y_2(t)} = 0 \left\{ \lim_{t\to\infty} \frac{z_1(t)}{z_2(t)} = 0 \right\} ;$$

(ii) $y_1 \sim y_2 \{z_1 \sim z_2\}$ means that $y_2(t)\{z_2(t)\}$ is nonoscillatory and

$$\lim_{t\to\infty} \frac{y_1(t)}{y_2(t)} \left\{ \lim_{t\to\infty} \frac{z_1(t)}{z_2(t)} \right\}$$

exists and is a finite number. In this case, $y_1(t)\{z_1(t)\}$ is said to be asymptotic to $y_2(t)\{z_2(t)\}$.

Remark 4.22 From Definition 4.21 it follows directly that $y_1 = 0(y_2)\{z_1 = 0(z_2)\}$ if and only if $(y_1 - \lambda y_2) \sim y_2$, $\lambda \neq 0$, $\{(z_1 - \lambda^* z_2) \sim z_2, \lambda^* \neq 0\}$. Also, $y_1 \sim y_2 \{z_1 \sim z_2\}$ if and only if $(y_1 - \lambda y_2) = 0(y_1)\{(z_1 - \lambda^* z_2) = 0(z_2)\}$, where

$$\lambda = \lim_{t\to\infty} \frac{y_1(t)}{y_2(t)} \left\{ \lambda^* = \lim_{t\to\infty} \frac{z_1(t)}{z_2(t)} \right\} .$$

Theorem 4.23 If $\Omega\{\Omega^*\}$ contains a nonoscillatory solution, then there exists a nonoscillatory solution $\rho_0(t)\{\rho^*(t)\}$ of (1) $\{(1)^*\}$

such that for each solution

$$y(t) \in \Omega - [\rho_0] \quad \{z \in \Omega^* - [\rho_0^*]\}$$

at least one of the following holds:

(i) There are distinct numbers λ_1 and λ_2 $(\lambda_1^*$ and $\lambda_2^*)$ such that the solutions $y(t) - \lambda_i \rho_0(t) \{z(t) - \lambda_i^* \rho_0^*(t)\}$ $i = 1, 2$ are oscillatory;

(ii) $y = O(\rho_0) \{z = O(\rho_0^*)\}$; or

(iii) $y \sim \rho_0 \{z \sim \rho_0^*\}$.

Proof: Let $\rho_1(t)$ be a nonoscillatory solution in Ω. Then, by Lemma 4. 19 and Remark 4. 20, either there are distinct numbers λ_1 and λ_2 such that the solutions

$$y(t) - \lambda_i \rho_1(t), \quad i = 1, 2$$

are oscillatory, or

$$\lim_{t \to \infty} \frac{y(t)}{\rho_1(t)} \tag{56}$$

exists.

Suppose for each $y \in \Omega - [\rho_1]$, the limit in (56) is finite. Then let $\rho_0(t) = \rho_1(t)$ and the theorem is proven. Suppose there exists a solution $y_1(t) \in \Omega - [\rho_1]$ such that

$$\lim_{t \to \infty} \left| \frac{y_1(t)}{\rho_1(t)} \right| = \infty . \tag{57}$$

This implies that $y_1(t)$ is nonoscillatory. So, $\lim_{t \to \infty} \frac{\rho_1(t)}{y_1(t)}$ exists and the conclusion of the theorem holds for all $y(t) \in [y_1(t), \rho_1(t)]$ with

$\rho_0(t) = y_1(t)$. As $\rho_0(t)$ and $\rho_1(t)$ are in $\Omega_{m(\rho_0, \rho_1)}$ and $\Omega_{m(\rho_0, \rho_1)}$ has dimension 2, $[\rho_0, \rho_1] = \Omega_{m(\rho_0, \rho_1)}$.

If the conclusions of the theorem are not satisfied for all $y(t) \in \Omega$, let $\rho_0(t) = \rho_2(t)$. Then there exists a solution $y_2(t) \in \Omega - \Omega_{m(\rho_1, \rho_2)}$ such that

$$\lim_{t\to\infty} \left| \frac{y_2(t)}{\rho_2(t)} \right| = \infty .$$

Set $\rho_0(t) = y_2(t)$. Then

$$\lim_{t\to\infty} \frac{\rho_i(t)}{\rho_0(t)} = \lim_{t\to\infty} \frac{\rho_i(t)}{\rho_2(t)} \lim_{t\to\infty} \frac{\rho_2(t)}{\rho_0(t)}$$

is finite for $i = 1, 2$.

As $\rho_1(t)$, $\rho_2(t)$ are linearly independent and $\rho_0(t) \in \Omega - [\rho_1(t), \rho_2(t)]$, $\{\rho_0(t), \rho_1(t), \rho_2(t)\}$ forms a fundamental set for (1). Thus, if $y(t) \in \Omega$, then

$$y(t) = c_0\rho_0(t) + c_1\rho_1(t) + c_2\rho_2(t) ,$$

Then by (54) and (55)

$$\lim_{t\to\infty} \frac{y(t)}{\rho_0(t)} = c_0 .$$

Therefore, $y = O(\rho_0)$ or $y \sim \rho_0$, and the proof is complete.

The following examples [8] make use of the above theory.

Example 4.24 Consider the nonsingular second-order differential equation

$$r_1(t)y'' + p_1(t)y' + q_1(t)y = 0 \quad t \in [a, \infty), \tag{58}$$

where

$$r_1(t) = 1 + (\varepsilon + \frac{1}{t^2}) \sin t + \frac{1}{t}(\frac{\varepsilon}{t} + \cos t) > 0$$

and

$$p_1(t) = \frac{1}{t}(1 + \frac{2}{t^2}) \sin t + (\frac{2}{t^3} - \cos t)\varepsilon ;$$

$$q_1(t) = 1 + \frac{1}{t^2} \sin t - \frac{2}{t^2} \cos t, \quad 0 < \varepsilon < 1, \quad t \geq a = \frac{4}{1-\varepsilon^2} .$$

Equation (58) has a fundamental set consisting of

$$y_1(t) = \sin t + \varepsilon ,$$

and

$$y_2(t) = \cos t + \frac{1}{t} .$$

Let

$$r(t) \equiv 0 ,$$

$$p(t) = \frac{p_1(t)}{r_1(t)} ,$$

$$q(t) = \frac{q_1(t)}{r_1(t)} , \quad \text{on} \quad [a, \infty) .$$

Then, the solutions of (1) have the form

$$y(t) = c_1(-\cos t + \varepsilon t) + c_2(\sin t + \ln t) + c_3 , \quad t \geq a .$$

Now $u_1(t) = -\cos t + \varepsilon t$ is nonoscillatory and $W(u_1, (t), u_2(t)) > 0$ where $u_2(t) = \sin t + \ln t$. Hence, (1) is nonoscillatory by Theorem 1.4.

Using Theorem 1.17, the solutions of $(1)^*$ can be expressed as

$$z(t) = c_1(\sin t + \varepsilon) + c_2(\cos t + \frac{1}{t}) + c_3 w(t), \quad t \geq a,$$

where $w(t) = (-\cos t + \varepsilon t)(\cos t - \frac{1}{t}) - (\sin t + \ln t)(\sin t + \varepsilon)$, $t \geq a$. Now, Ω^* is oscillatory as $\sin t + \varepsilon$ is oscillatory. So, Ω^* is strongly oscillatory by Corollary 4.17.

The following is an example of equation (1) which is weakly oscillatory while equation $(1)^*$ is strongly oscillatory.

Example 4.25 Consider the nonsingular linear second-order equation

$$(r(t)y')' + r^2(t)y = 0,$$

where $r(t) = [1 + \sqrt{2}\,\epsilon \sin(t + \frac{\pi}{4})]^{-1} > 0$, $0 < \varepsilon < \frac{1}{\sqrt{2}}$, $t \geq 0$, which has as a fundamental set consisting of

$$y_1(t) = \sin t + \varepsilon$$

and

$$y_2(t) = \cos t + \varepsilon .$$

Let

$$p(t) = \frac{r'(t)}{r(t)} ,$$

$$q(t) = r(t) ,$$

and

$$r(t) = 0 \quad \text{on} \quad [0, \infty) .$$

Then equation (1) has as a fundamental set consisting of

$$\rho(t) = 1 ,$$

$$y_1(t) = \cos t - \varepsilon t ,$$

$$y_2(t) = \sin t + \varepsilon t .$$

Thus, Ω is weakly oscillatory as $\rho(t)$ is nonoscillatory, while $y_1(t) + y_2(t)$ is an oscillatory solution. For each real number λ,

$$y_1(t) + \lambda \rho(t)$$

is a nonoscillatory solution of (1). Hence, Ω^* is strongly oscillatory or nonoscillatory by Theorem 4.18. However,

$$z(t) = W^{-1}[\rho, y_1, y_2](t)(\sin t + \varepsilon)$$

is an oscillatory solution by Theorem 1.4. Hence, Ω^* is strongly oscillatory.

Dolan [8] asks if there exist examples of equation (1) such that

(i) Ω and Ω^* are both strongly oscillatory,

(ii) Every two dimensional subspace of Ω is weakly oscillatory.

Gustafson [11] provides examples of both (i) and (ii). To do this, he proves the following.

Theorem 4.26 Let $u(t)$ be a nontrivial solution of

$$u^{(n)} + a_{n-1}(t)u^{(n-1)} + \ldots + a_0(t)u = 0, \quad t \in [a,b] \tag{59}$$

with $u^{(n-1)}(a)u^{(n-1)}(b) < 0$, and a zero of order $n-1$ at each of a and b. If $v(t)$ is a solution of the adjoint equation of (59), then $v(t)$ has at least one zero in $[a,b]$.

Proof: Let $\{u_1(t), \ldots, u_n(t)\}$ be a basis for the solution space of equation (59) with $u_n = u$. Define

$$W(t) = W[u_1(t), \ldots, u_n(t)] ,$$

$$v_j(t) = W[u_1(t), \ldots, \hat{u}_j(t), \ldots, u_n(t)] ,$$

where "$\wedge$" is used to indicate that the jth column is deleted. Then

$$w_j(t) = \frac{v_j(t)}{W(t)} , \quad j = 1, 2, \ldots, n$$

form a basis for the adjoint of equation (59) by Birkoff [6]. If $z(t)$ is a solution of the adjoing of equation (59), then

$$z(t) = c_1 w_1(t) + \ldots + c_n w_n(t) .$$

As $u_n(t)$ has a zero of order $n-1$ at a and b, $z(a) = c_n w_n(a)$ and $z(b) = c_n w_n(b)$. But

$$W(a) = w_n(a) u^{(n-1)}(a)$$

and

$$W(b) = w_n(b) u^{(n-1)}(b)$$

as $u(t)$ has a zero of order $n-1$ at a and b. So

$$z(a) z(b) = c_n^2 w_n(a) w_n(b)$$

$$= \frac{v_n(a)}{u^{(n-1)}(a) v_n(a)} \quad \frac{v_n(b)}{u^{(n-1)}(b) v_n(b)} \; c^2{}_n$$

$$= c_n^2 \; \frac{1}{u^{(n-1)}(a) \, u^{(n-1)}(b)} \quad .$$

If $c_n = 0$, then either $z(a)$ or $z(b)$ is zero. If not, then $z(a)\,z(b) < 0$ as $u^{(n-1)}(a)\,u^{(n-1)}(b) < 0$. Hence, there exist $c \in (a, b)$ such that $z(c) = 0$ and the proof is complete.

Definition 4.27 Equation (59) is said to be a separator on $[t_1, \infty)$ if for each $[t_2, \infty) \subseteq [t_1, \infty)$ there exist a solution $u(t) = u(t, t_2)$ and two points $a, b \in [t_2, \infty)$ with $u^{(j)}(a) = u^{(j)}(b) = 0$ for $j = 1, 2, \ldots, n-2$ and $u^{(n-1)}(a)\,u^{(n-1)}(b) < 0$.

Corollary 4.28 (to Theorem 4.26). The adjoint of a separator is strongly oscillatory.

Example 4.29 As,

$$W[1, \sin 2t \cos t, \sin 2t \sin t] = 8 - 3 \sin 2t \sin t > 0 \quad \text{for all } t,$$

$\{1, \sin 2t \cos t, \sin 2t \sin t\}$ are a basis for the solution space of $E^*y = W[y, 1, \sin 2t \cos t, \sin 2t \sin t] = 0$. Let $y(t) = \sin 2t \sin t$, then

$$y^{(j)}(\pi + 2k\pi) = y^{(j)}(2k\pi) = 0, \quad j = 0, 1, \quad k = 1, 2, \ldots,$$

But

$$y^{(2)}(2k\pi) = -y^{(2)}(\pi + 2k\pi) = 4,$$

Hence, $E^*y = 0$ is a spearator. It follows from Corollary 4.28 that Ey is strongly oscillatory.

Example 4.30 The following is an example of equation (1) such that both $\mathfrak{L}$, $\mathfrak{L}^*$ are strongly oscillatory. To construct (1), let $L = E$ on the intervals $[4n\pi, (4n+1)\pi]$. $L = E^*$ on the intervals $[(4n+2)\pi, (4n+3)\pi]$ $n = 0, 1, 2, 3, \ldots$, Define $p(t)$, $q(t)$ and $r(t)$ on the complements of the above intervals such that p, q, and r have continuous third order derivatives. So, both L and L^* are separators as in Example 4.29. Hence, both $\mathfrak{L}$ and $\mathfrak{L}^*$ are strongly oscillstory by Corollary 4.28.

The example of equation (1) satisfying condition (ii) above does not involve elementary functions. The construction makes use of Theorem 2.7.

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this thesis is to collect and present the current research in oscillation theory of the third order linear differential equation in a readable and compact form. Proofs are included not only for completeness, but to display the techniques which were used to obtain these results.

Chapter I gives a brief history of the development of oscillation theory of the third order equation. Also included in Chapter I are definitions and some well-known preliminary results which are necessary to read this thesis.

In Chapter II, the oscillatory nature of fundamental sets is studied. In this chapter, characterizations of properties RO and RN are given.

Chapter III is devoted to the study of asymptotic and oscillatory behavior of solutions. Conditions which guarantee the existence of oscillatory and nonoscillatory solutions are given. The question of when a linear combination of oscillatory solutions oscillates is also considered. Integrability of coefficient functions is used to show that oscillatory and nonoscillatory solutions exist.

The relationships between the third order linear differential equation and its adjoint are studied in Chapter IV. It is shown that if equation (1) has property RO then its adjoint has property RN.

However, the converse does not hold in general. Also, if equation (1) is weakly oscillatory, it is shown that its adjoint is oscillatory. In addition, if equation (1) is nonoscillatory it is shown that its adjoint is strongly oscillatory or nonoscillatory.

Several questions are suggested by this thesis. Is it possible to determine the dimension of the strongly oscillatory subspace of a given differential equation? This appears to be a difficult problem in general. Are there reasonable conditions which may be placed on equation (1) such that if equation (1) has property RN, then its adjoint has property RO.

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