72-3433

ROSSA, Robert Frank, 1942-
LOWER RADICAL AND RELATED CLASSES FOR NOT NECESSARILY ASSOCIATIVE RINGS.

The University of Oklahoma, Ph.D., 1971
Mathematics

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

LOWER RADICAL AND RELATED CLASSES FOR NOT NECESSARILY ASSOCIATIVE RINGS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

ROBERT FRANK ROSSA

Norman, Oklahoma

1971

LOWER RADICAL AND RELATED CLASSES FOR NOT NECESSARILY ASSOCIATIVE RINGS

APPROVED BY

Gene Levy

John G. Brixey

Kirby C. Smith

George M. Ewing

DISSERTATION COMMITTEE

ACKNOWLEDGEMENT

The author wishes to express his sincere gratitude and appreciation to Professor Gene Levy for his advice and encouragement during the preparation of this dissertation.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION AND PREREQUISITES	1
II.	APPLICATIONS OF THE EXTENSION-UNION CONSTRUCTION	12
III.	STRONG SUBRING HEREDITY AND RELATED PROPERTIES	28
IV.	STRONG RIGHT HEREDITY AND RELATED TOPICS	41
BIBLIOGRAPHY		58

LOWER RADICAL AND RELATED CLASSES FOR NOT NECESSARILY ASSOCIATIVE RINGS

CHAPTER I

INTRODUCTION AND PREREQUISITES

1.1. Introduction. We are concerned with continuing the investigations of general radical theory initiated by A. G. Kurosh [18] and S. A. Amitsur [1, 2, 3]. More recent contributions have been made by V. A. Andrunakievic [7, 8], T. Anderson, N. Divinsky, A. Sulinski [5, 6], E. P. Armendariz [9] and A. E. Hoffman [13]. Kurosh introduced the concept of the lower radical class LM determined by a class M of rings and gave a construction for it which has been modified by Anderson, Divinsky, and Sulinski [6]; their construction is usually referred to as the A. D. S. construction. Other constructions of LM have been given by W. G. Leavitt, Y. L. Lee [20], R. Tangeman and D. Kreiling [17]. The lower radical LM is the minimal radical class containing M; much of our work centers around the construction of LM given by Tangeman and Kreiling, which we call the extension-union construction. The definitions and previous results which are used throughout are presented in Section 1.2.

It is natural to ask how placing various conditions on a class may affect its lower radical. One property that occurs frequently in radical theory is the hereditary property; a class of rings has this property provided every ideal of a ring in the class is also in the class. Hoffman and Leavitt [14], using the A. D. S. construction, showed that if M is hereditary, then LM is also hereditary. We begin Chapter II by proving several generalizations of this result by considering the analogous hereditary properties for one-sided ideals and subrings; these results will prove useful in Chapters III and IV. Our proofs employ the extension-union construction. We generalize the result of Hoffman and Leavitt more directly by showing that if every ideal of each ring of M is in LM, then LM is hereditary. Hoffman and Leavitt also considered hereditary classes of hereditarily idempotent rings, that is, rings in which every ideal is idempotent; we give a new proof of their result using the extension-union construction. We also consider heredity to large ideals and show that this implies ordinary heredity.

The remainder of Chapter II is concerned with applying the extension-union construction to classes M which satisfy a variety of properties. We consider briefly classes of rings which fail to satisfy the ascending or descending chain conditions. Our result for the ascending chain condition is quite general, but we require more restrictive conditions in the corresponding theorem concerning the descending chain condition; in particular, we require associativity in the latter. Next we turn to the examination of a property introduced for the study of strong heredity by W. G. Leavitt [19]; analogous properties will be studied at length in Chapters III and IV. We place rather strong conditions on the centers of rings in M and show that these, too, are preserved by passage to the lower radical. We give a new proof of a result of Hoffman [13] on the lower radical construction in the presence of two universal classes and conclude Chapter II by touching on upper radical classes.

Chapter III is devoted to the study of strongly subring hereditary radical classes, that is, radical classes P for which $P(I) = I \cap P(R)$ for each subring I of every ring R in a suitable type of universal class. We begin by adopting a setting sufficiently large for our investigations, one in which every subring of the rings under consideration will be available. We introduce a property, property (b), which is equivalent to strong heredity for radical classes in the presence of subring heredity. We construct for each class M a minimal radical class containing M which satisfies property (b) as well as a minimal strongly subring hereditary radical class containing M. A characterization of the semisimple classes whose radical classes satisfy property (b) generalizes a result of Armendariz [9]. We conclude by addressing, briefly, the classical situation of hypernilpotent radicals in the class of associative rings and show that a strongly subring hereditary radical class of this type contains all fields.

Chapter IV begins with the study of strong right heredity, the analogue for right ideals of strong subring heredity and strong heredity. As in Chapter III, we first insure that we are provided with a type of universal class large enough to encompass our activities. The property, called property (p), which we introduce is equivalent to strong right heredity in the presence of right heredity. We obtain equivalent formulations of this property and of strong right heredity and show that none of the radical classes encountered in the classical situation can have property (p). At this point we seem to be close to an unanswered question of Koethe [16]; it is unknown whether a nil right ideal of a ring generates a nil two-sided ideal. We construct for each class M a minimal radical class containing M which satisfies property (p) as well as a minimal strongly right hereditary radical class containing M. We characterize the semisimple classes whose radical classes satisfy property (p) and those whose radical classes are strongly right hereditary, again generalizing results of Armendariz. Similar strong radical properties have been studied recently by N. Divinsky, J. Krempa and A. Sulinski [12].

The rest of Chapter IV is devoted to the study of a new radical class $R^0(M)$ obtained from a simple modification of the extension-union construction. For each limit ordinal β, in the construction of LM, we admit a ring R if R is the union of a chain of its ideals contained in the previously obtained classes. At the corresponding point in the construction of $R^0(M)$, we admit a ring R if it is the union of a chain of its right ideals contained in the previously obtained classes. The radical class $R^0(M)$ seems worthy of our attention for two reasons. First, LM and $R^0(M)$ are identical if M has property (p) and is homomorphically closed. Second, $R^0(M)$ is contained in every radical class P containing M such that, for each R in the universal class under consideration, every ideal I of R which is a sum of P-right ideals of R is contained in P(R). We remark that various properties which may be possessed by a class M are preserved by passage to $R^0(M)$. Finally we note that a similar class may be defined using left ideals in place of right ideals and note that even in the associative case these two classes need not be identical.

<u>1.2</u>. <u>General</u> <u>Radical</u> <u>Theory</u>. In this section we present the basic definitions and results of general radical theory for not necessarily associative rings that bear directly upon this dissertation. We will use the notation and terminology of A. G. Kurosh [18]. Some familiarity with the basic concepts of ring theory, as set forth, for instance, in [23], will be assumed. The term <u>ring</u> will indicate a not necessarily associative ring; when we need associativity, we will state the requirement explicitly. The term <u>ideal</u> used without modification will mean two-sided ideal. The isomorphism theorems for associative rings remain valid for nonasso-

ciative rings. On the other hand, if R is a ring and I is an ideal of R, then I^2 need not be an ideal of R. We will say that a ring R is nilpotent if and only if there exists a positive integer n such that every product of n elements of R in any association is zero.

A class M of rings is said to be <u>homomorphically closed</u> if every homomorphic image of each ring of M is also in M. If M is any class of rings, let $H(M) = \{S : S = R\phi$ for some $R \in M$ and some homomorphism $\phi\}$. Then H(M) is homomorphically closed, and any homomorphically closed class of rings which contains M must also contain H(M) [13, page 20, Proposition 4.1]. H(M) is called the <u>homomorphic closure</u> of M. A class M of rings is said to be <u>hereditary</u> if every ideal of each ring in M is also in M. A class W of rings is a <u>universal</u> class if it is both homomorphically closed and hereditary. In the following discussion, W denotes an arbitrary universal class.

<u>Definition 1.1</u>. A class $P \subseteq W$ is said to be a radical class in W if it has the following two properties:

(R1) P is homomorphically closed.

(R2) If $R \in W$ is not in P, then R has a nonzero homomorphic image which has no nonzero ideals in P.

A ring R in a radical class P may be called a <u>P-ring</u>. If I is an ideal of a ring R and I is a P-ring, we say that I is a <u>P-ideal</u> of R. Our first two theorems provide useful characterizations of radical classes.

<u>Theorem 1.1</u>. [18, page 16]. The class $P \subseteq W$ is a radical class in W if and only if P is homomorphically closed and every ring $R \in W$ contains a P-ideal J which satisfies the following two conditions:

(1) J contains every P-ideal of R.

(2) R/J contains no nonzero P-ideals.

If P is a radical class in W and $R \in W$, the ideal of R which satisfies the conditions of Theorem 1.1 will be called the <u>P-radical</u> of R and will be denoted by P(R). A ring is said to be <u>P-semisimple</u> if it has no nonzero P-ideals.

<u>Theorem 1.2</u>. [2, page 105]. The class $P \subseteq W$ is a radical class if and only if it satisfies the following three conditions:

(R1) P is homomorphically closed.

(R3) P is <u>extension closed</u>, that is, if I is a P-ideal of a ring $R \in W$ and R/I is a P-ring, then R is a P-ring.

(R4) If $\{I_\gamma : \gamma \in \Gamma\}$ is a chain of P-ideals of a ring $R \in W$, then $\bigcup_{\gamma \in \Gamma} I_\gamma$ is a P-ideal of R.

If P is a radical class in W, we denote the class of P-semisimple rings in W by SP; we may refer to SP as the <u>semisimple class of P</u>.

<u>Definition 1.2</u>. A class $Q \subseteq W$ is said to be a <u>semisimple class</u> if it satisfies the following two properties:

(S1) Every nonzero ideal of each ring in Q has a nonzero homomorphic image in Q.

(S2) If every nonzero ideal of $R \in W$ has a nonzero homomorphic image in Q, then R is in Q.

<u>Definition 1.3</u>. Let Q be a class satisfying (S1). Then $UQ = \{R \in W :$ R has no nonzero homomorphic image in Q$\}$.

Our next theorem shows the significance of UQ.

<u>Theorem 1.3</u>. [18, page 19]. If Q is a class satisfying property (S1), then UQ is a radical class such that $Q \subseteq SUQ$; moreover, UQ is the largest radical class whose semisimple class contains Q.

UQ is called the upper radical class determined by Q in W. The relationship between the operators S and U is clarified by the next theorem.

Theorem 1.4. [18, page 17]. If P is a radical class, then SP is a semisimple class and USP = P. If Q is a semisimple class, then SUQ = Q. Thus Q $\subseteq$ W is a semisimple class for a radical class in W if and only if Q satisfies conditions (S1) and (S2).

E. P. Armendariz has characterized semisimple classes in a class W of associative rings in the following way.

Theorem 1.5. [9]. A class Q is semisimple in an associative class W if and only if it has the following four properties:

(1) Q is hereditary.

(2) Any subdirect sum of rings in Q is also in Q.

(3) Q is extension closed.

(4) If I is an ideal of R $\in$ W and $0 \neq I/J \in Q$ for some ideal J of I, then there exists an ideal K of R contained in I such that $0 \neq I/K \in Q$.

A ring R is alternative if $x^2y = x(xy)$ and $xy^2 = (xy)y$ for all x,y in R; thus every associative ring is alternative. Although in any class W of alternative rings every semisimple class is hereditary [5], this is not true in general [10]. Y. L. Lee showed [22] that in the universal class of associative rings every class M determines an upper radical class, that is, a radical U maximal with respect to the property that all rings in M are U-semisimple. Jenkins and Kreiling [15] extended this result to alternative rings but showed that in general a class M in an arbitrary universal class W need not determine an upper radical class.

Every class M, however, in an arbitrary universal class W, determines a minimal radical class in W containing M, the lower radical $L_W M$ of Kurosh. We will present two constructions of this class, the modification of Kurosh's construction given by Anderson, Divinsky and Sulinski [6], which we shall refer to as the A. D. S. construction, and the extension-union construction of Tangeman and Kreiling [17]. In both constructions the definition is accomplished by means of transfinite induction.

In the A. D. S. construction, a class M_β is defined for each ordinal number β in the following way. Let $M_1 = H(M)$ and for any ordinal number $\beta > 1$ let $M_\beta = \{R \in W$: every nonzero image $R\phi$ contains a nonzero ideal $I \in M_\alpha$ for some $\alpha < \beta\}$. Then $L_W M$, the lower radical of M relative to W, is defined to be $\bigcup_\beta M_\beta$.

Theorem 1.6. [6]. $L_W M$ is a radical class which is minimal among radical classes in W containing M.

The extension-union construction proceeds as follows. For any class M, define $R_1(M)$ to be the homomorphic closure of M. Let $\beta > 1$ be an ordinal number and suppose the classes $R_\alpha(M)$ have been defined for all $\alpha < \beta$. If β is not a limit ordinal, so that $\beta - 1$ exists, admit R to $R_\beta(M)$ if and only if $R \in W$ and there exist rings $S,T \in R_{\beta-1}(M)$ such that R contains an ideal I isomorphic to S and R/I is isomorphic to T. If, on the other hand, β is a limit ordinal, admit R to $R_\beta(M)$ if and only if $R \in W$ and R contains a chain $\{I_\gamma : \gamma \in \Gamma\}$ of ideals such that each I_γ is isomorphic to some member of $\bigcup_{\alpha<\beta} R_\alpha(M)$ and $R = \bigcup_{\gamma\in\Gamma} I_\gamma$. Finally, let $R(M) = \bigcup_\beta R_\beta(M)$.

Lemma 1.1. [17]. If α and β are ordinal numbers with $\alpha \leq \beta$, then $R_\alpha(M) \subseteq R_\beta(M)$.

Lemma 1.2. [17]. For every ordinal $\beta \geq 1$, $R_\beta(M)$ is homomorphically closed. Hence R(M) is homomorphically closed.

With the assistance of Theorem 1.2, one then has

Theorem 1.7. [17]. $R(M) = L_W M$.

We shall employ this construction routinely throughout this dissertation, and it will be advantageous to streamline it a bit. Since the classes $R_\alpha(M)$ are all homomorphically closed, we may describe the construction of $R_\beta(M)$ when β is not a limit ordinal as follows. We admit a ring R to $R_\beta(M)$ if and only if there exists an ideal I of R such that I and R/I are both in $R_{\beta-1}(M)$. When β is a limit ordinal, we include R in $R_\beta(M)$ if and only if R is the union of a chain $\{I_\gamma\}$ of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. We reserve the subscript γ for this construction and will understand without explicit mention that the indices γ are members of some index set Γ.

We point out two results on lower radical classes:

Theorem 1.8. [14]. If M is a hereditary class, then LM is hereditary.

Theorem 1.9. [18]. Let R be a simple ring and suppose M is a homomorphically closed class. Then $R \in LM$ if and only if $R \in M$.

Tangeman and Kreiling [17] have given a proof of Theorem 1.8 using the extension-union construction whose proof establishes the following lemma, which we use in Theorem 2.5.

Lemma 1.3. [17]. If M is a hereditary class, then each of the classes $R_\alpha(M)$ is hereditary.

If P_1 and P_2 are radical classes, we say that $P_1 \leq P_2$ if every ring in P_1 is also in P_2; equivalently, $SP_2 \leq SP_1$.

Radicals have found their most profound applications in the structure theory of associative rings, and we shall have occasion to mention some of the radical classes encountered therein, including the nil radical class N and the Jacobson radical class J. In this theory one is most interested in the radical classes P which agree with J (and N) on right artinian rings in the sense that for each right artinian ring R, P(R) = J(R). Let $\mathcal{J}$ denote the lower radical class determined by the class of all zero simple rings and let $\mathcal{T}$ denote the upper radical class determined by the class of all matrix rings over division rings. Then, for every such radical class P, $\mathcal{Y} \leq P \leq \mathcal{T}$ [11, page 40]. One such radical class is the Baer lower radical class β; this is the lower radical class LZ determined in the universal class of associative rings by the class Z of rings with zero multiplication. A radical class P in a universal class W is said to be hypernilpotent if it contains all rings with zero multiplication in W and hence all nilpotent rings in W. Thus the Baer lower radical is the smallest hypernilpotent radical in the universal class of associative rings.

An elementary and useful result in the theory of associative rings is due to V. A. Andrunakievic.

Lemma 1.4. [7, Lemma 4]. Let R be an associative ring and let B be an ideal of an ideal A of R. Let B' denote the ideal of R generated by B. Then $(B')^3 \subseteq B$.

A radical class P in a universal class W such that $P(I) = I \cap P(R)$ for every ideal I of each ring $R \in W$ is said to be strongly hereditary; this terminology was introduced by W. G. Leavitt [19]. In a universal class of alternative rings if I is an ideal of a ring R, then the radical P(I) is also an ideal of R for any radical class P [5]; as a consequence, $P(I) \subseteq I \cap P(R)$. Kurosh [18] showed that in the universal class of associative rings a radical class P is hereditary if and only if P is strongly hereditary. This equivalence has been established for alternative rings as well, but in general a hereditary radical class need not be strongly hereditary [9].

Theorem 1.10. [19]. A hereditary radical class P in a universal class W is strongly hereditary if and only if it has property:

(a) If $J \in P$ is an ideal of an ideal I of some ring $R \in W$ then $J' \in P$ where J' is the ideal of R generated by J.

Finally we mention three properties which may be possessed by a class M of rings. M is said to be <u>right hereditary</u> if every right ideal of each ring in M is also in M, <u>left hereditary</u> if every left ideal of each ring in M is also in M, and <u>subring hereditary</u> provided every subring of each ring in M is also in M.

CHAPTER II

APPLICATIONS OF THE EXTENSION-UNION CONSTRUCTION

The unifying theme of this chapter is the utilization of the extension-union construction to establish that various properties which may be possessed by a class M of not necessarily associative rings in a universal class W are also possessed under suitable conditions by the lower radical class LM. Although some of our results have been demonstrated previously using the A. D. S. construction of LM, all of our proofs will be new. We begin with several generalizations of Theorem 1.8.

<u>Theorem 2.1</u>. Let $M \subseteq W$, where W is a universal class. If M is right hereditary, then LM is right hereditary.

<u>Proof</u>. The class $R_1(M)$ is the homomorphic closure of M. Thus if $R \in R_1(M)$, then $R \cong K/J$ where $K \in M$ and J is an ideal of K. If I is any right ideal of R, then there exists a right ideal I' of K containing J such that $I'/J \cong I$. Since M is right hereditary, $I' \in M$. Therefore I, being a homomorphic image of I', is in $R_1(M)$. Hence $R_1(M)$ is right hereditary.

Thus suppose β is an ordinal number greater than 1 and that the classes $R_\alpha(M)$ are right hereditary for all $\alpha < \beta$. Let $R \in R_\beta(M)$ and suppose I is a right ideal of R. If β is a limit ordinal, then $R = \cup I_\gamma$ where $\{I_\gamma\}$ is a chain of ideals of R each belonging to one of the classes $R_\alpha(M)$ with $\alpha < \beta$. Now $I = \cup (I_\gamma \cap I)$; since for each γ the ring $I_\gamma \cap I$ is a right ideal of I_γ and each of the classes $R_\alpha(M)$ is right hereditary, $\{I_\gamma \cap I\}$ is a chain of ideals of I contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. Therefore $I \in R_\beta(M)$.

If β is not a limit ordinal, there exists an ideal J of R such that J, R/J $\in R_{\beta-1}(M)$. Since $R_{\beta-1}(M)$ is right hereditary, $I \cap J$ and $I/I \cap J \cong (I + J)/J$ both belong to $R_{\beta-1}(M)$. Therefore $I \in R_{\beta}(M)$. Thus, in either case, $R_{\beta}(M)$ is right hereditary, so that the theorem is established by transfinite induction, for since each $R_{\beta}(M)$ is right hereditary, LM = $\bigcup_{\beta} R_{\beta}(M)$ is right hereditary.

Because the proof of the following result is practically identical with the preceding argument, we will omit it.

Theorem 2.2. Let $M \subseteq W$, where W is a universal class. If M is left hereditary, then LM is left hereditary.

Theorem 2.3. Let $M \subseteq W$, where W is a universal class. If M is subring hereditary, then LM is subring hereditary.

Proof. If $R \in R_1(M)$, then R is a homomorphic image $K\phi$ of some ring $K \in M$. Then if I is any subring of R, $I = J\phi$ for some subring J of K. Now $J \in M$ since M is subring hereditary, so that $I \in R_1(M)$. Thus $R_1(M)$ is subring hereditary.

Now assume that $\beta > 1$ and that $R_{\alpha}(M)$ is subring hereditary for all $\alpha < \beta$. Let $R \in R_{\beta}(M)$ and suppose I is any subring of R. If β is a limit ordinal, then $R = \bigcup I_{\gamma}$, where $\{I_{\gamma}\}$ is a chain of ideals of R each belonging to one of the subring hereditary classes $R_{\alpha}(M)$ with $\alpha < \beta$. Then $I = \bigcup (I \cap I_{\gamma})$. Since each $I \cap I_{\gamma}$ is a subring of I_{γ}, each $I \cap I_{\gamma} \in \bigcup_{\alpha<\beta} R_{\alpha}(M)$. Therefore $I \in R_{\beta}(M)$, as each $I \cap I_{\gamma}$ is an ideal of I.

If β is not a limit ordinal, there exists an ideal J of R such that J, R/J $\in R_{\beta-1}(M)$. Since $R_{\beta-1}(M)$ is subring hereditary by the inductive hypothesis, $I \cap J$ and $I/I \cap J \cong (I + J)/J$ are both in $R_{\beta-1}(M)$. Thus $I \in R_{\beta}(M)$. In either case we have shown that $R_{\beta}(M)$ is subring hereditary, and so, by transfinite induction, this is true for every ordinal number β. Therefore $LM = \bigcup_\beta R_\beta(M)$ is subring hereditary.

We can generalize Theorem 1.8 in a different direction as follows.

Theorem 2.4. Let M be a class of rings with the property that if $R \in M$, then every ideal of R is in LM. Then LM is hereditary.

Proof. If $R \in R_1(M)$ and I is an ideal of R, then I is a homomorphic image of an ideal J of some ring in M. But $J \in LM$ by hypothesis, so that $I \in LM$ because LM is homomorphically closed. Thus suppose that for each ring $R \in \bigcup_{\alpha<\beta} R_\alpha(M)$ every ideal of R is in LM. Let $R \in R_\beta(M)$ and let I be an ideal of R. If β is a limit ordinal, then $R = \bigcup I_\gamma$, where $\{I_\gamma\}$ is a chain of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. Then $I = \bigcup (I \cap I_\gamma)$. For each γ, $I \cap I_\gamma$ is an ideal of I_γ and hence is in LM by hypothesis. Now $I \cap I_\gamma$ is a chain of ideals of I so that $I \in LM$ by Theorem 1.2, condition (R4). On the other hand, if $\beta - 1$ exists, then there exists an ideal J of R such that J, $R/J \in R_{\beta-1}(M)$. By the inductive hypothesis, $I/(I \cap J) \cong (I+J)/J \in LM$ and $I \cap J \in LM$. Therefore $I \in LM$ by Theorem 1.2 condition (R3).

Corollary 2.1. Suppose M is a class of rings with the property that if $R \in M$, then every principal ideal of R is in LM. Then LM is hereditary.

Proof. Every ideal of R is a sum of principal ideals of R for each $R \in M$. Therefore, by condition (1) of Theorem 1.1, every ideal of R is in LM.

The reader will notice the Theorems 2.1, 2.2, and 2.3 may be generalized in a similar way. For instance, suppose every right ideal of each ring of M is in LM; then LM is right hereditary.

A ring is hereditarily idempotent provided each of its ideals is idempotent. The following result was proved by Leavitt and Hoffman using the A. D. S. construction of LM; we present a new proof employing the extension-union construction.

<u>Theorem 2.5</u>. [14]. Let M be a hereditary class of hereditarily idempotent rings. Then so is LM.

<u>Proof</u>. If M is hereditary, each of the classes $R_\beta(M)$ is hereditary by Lemma 1.3 and LM is hereditary by Theorem 1.8. Thus it is sufficient to show that for each ordinal number $\beta \geq 1$, $R_\beta(M)$ contains only idempotent rings. Since each homomorphic image of an idempotent ring is idempotent, $R_1(M)$ contains only idempotent rings. Thus suppose that each of the classes $R_\alpha(M)$, $\alpha < \beta$, contains only idempotent rings. First suppose β is a limit ordinal and that $R \in R_\beta(M)$, so that R is the union of a chain $\{I_\gamma\}$ of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. If $x \in R$, then $x \in I_\gamma$ for some index γ, so that $x \in I_\gamma^2 \subseteq R^2$. Hence $R \subseteq R^2$, so that R is idempotent. If, on the other hand, β is not a limit ordinal, then there exists an ideal J of R such that J, $R/J \in R_{\beta-1}(M)$. If $x \in R$, then $x - \sum_{i=1}^{n} r_i s_i \in J$ for some $r_i, s_i \in R$ and some integer n since R/J is idempotent. Hence $x - \sum_{i=1}^{n} r_i s_i = \sum_{j=1}^{m} t_i u_i$ for some $t_i, u_i \in J$ and some integer m since J is idempotent. Therefore $x = \sum_{i=1}^{n} r_i s_i + \sum_{j=1}^{m} t_i u_i \in R^2$, so that we have $R \subseteq R^2$ and hence R is idempotent in this case as well. Thus by transfinite induction $R_\beta(M)$ is a class of idempotent rings for each ordinal number β, so that $LM = \bigcup_\beta R_\beta(M)$ is a class of hereditarily idempotent rings.

An ideal L of a ring R is said to be a <u>large</u> ideal of R if, for every ideal I of R, $L \cap I = 0$ implies $I = 0$. The next proposition shows that heredity to large ideals is equivalent to ordinary heredity.

<u>Proposition 2.1</u>. Let M be a homomorphically closed class with the property that if $R \in M$ and L is a large ideal of R, then $L \in M$. Then M is hereditary.

Proof. Let $R \in M$ and let K be an ideal of R. By Zorn's Lemma we assert the existence of an ideal J of R maximal with respect to the property that $J \cap K = 0$. Then $J + K$, we claim, is a large ideal of R. If I is an ideal of R and $(J + K) \cap I = 0$, let $k \in K \cap (I + J)$. Then $k = x + y$ with $x \in I$ and $y \in J$, so that $k - y = x \in I \cap (J + K) = 0$. Hence $k = y \in J \cap K = 0$. Therefore $K \cap (I + J) = 0$, so that by the maximality of J we must have $I = 0$. Hence $J + K \in M$ by hypothesis, so that $K \cong (J + K)/J \in M$ by the assumption that M is homomorphically closed.

Proposition 2.2. Let P be a hypernilpotent, hereditary radical class in an associative universal class W. Suppose $R \in W$ has a large P-ideal X and suppose I is a nonzero ideal of R. Then I has a large P-ideal.

Proof. $X \cap I$ is a P-ideal of I by the heredity of P, so that if $X \cap I$ is a large ideal of I we are done. Otherwise, let J be an ideal of I maximal with respect to the property that $X \cap I \cap J = 0$ and let J' be the ideal of R generated by J. Then by Lemma 1.4 $J'^3 \subseteq J$ and we have $X \cap J'^3 = 0$. However, since X is a large ideal of R, $J'^3 = 0$ so that $J' \in P$, for P by hypothesis contains all nilpotent rings in W. By heredity, $J \in P$. Thus $(X \cap I) + J \in P$. Let K be an ideal of I such that $[(X \cap I) + J] \cap K = 0$. Then we have $(X \cap I) \cap (J + K) = 0$, for if $x \in X \cap I$ and $x = j + k$ with $j \in J$ and $k \in K$, then $k = x - j \in [(X \cap I) + J] \cap K$ and so $k = 0$, whence $x = j \in X \cap I \cap J = 0$. By the maximality of J, we have $K = 0$. Hence $(X \cap I) + J$ is a large P-ideal of I.

Theorem 2.6. Suppose M is homomorphically closed and that no ring in M has an identity. Then no ring in LM has an identity.

Proof. Since $M = R_1(M)$, $R_1(M)$ has no rings with identity. Thus suppose $\beta > 1$ is an ordinal number and that no ring in the class $\bigcup_{\alpha<\beta} R_\alpha(M)$ has an identity. Let $R \in R_\beta(M)$. If β is a limit ordinal, then $R = \cup I_\gamma$ where $\{I_\gamma\}$ is a chain of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. If R has an identity 1, then $1 \in I_\gamma$ for some index γ, so that I_γ has an identity in contradiction with the inductive hypothesis. If $\beta - 1$ exists, then R has an ideal J such that $J, R/J \in R_{\beta-1}(M)$. If R has an identity 1, then $1 \notin J$ since J does not have an identity. But then $1 + J$ is an identity for R/J so that again we have a contradiction.

<u>Theorem 2.7</u>. Suppose M is a homomorphically closed class with the property that no nonzero ring in R has a. c. c. on right ideals. Then no nonzero ring in LM has a. c. c. on right ideals.

<u>Proof</u>. We have assumed that $R_1(M) = M$ contains no nonzero rings with a. c. c. Thus suppose R is a nonzero ring in $R_\beta(M)$ and that no nonzero ring in $\bigcup_{\alpha<\beta} R_\alpha(M)$ has a. c. c. on right ideals. If $\beta - 1$ exists, there exists an ideal J of R such that $J, R/J \in R_{\beta-1}(M)$. If R had a. c. c. on right ideals, then so would R/J, so that necessarily $R/J = 0$ by the inductive hypothesis. But then $J = R$ would have a. c. c. on right ideals and $J \in R_{\beta-1}(M)$, which is a contradiction. If, on the other hand, β is a limit ordinal, then $R = \cup I_\gamma$ where $\{I_\gamma\}$ is a chain of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. If R had a. c. c. on right ideals, then $\{I_\gamma\}$ would have a maximal element $I_{\gamma'}$, so that necessarily $I_{\gamma'} = R$ which again contradicts the inductive hypothesis. The theorem is therefore proved by transfinite induction.

The hypothesis that M be homomorphically closed is essential in this theorem, for let R be the zero ring which is the direct sum of a countable collection of cyclic groups of order two. Then if $M = \{R\}$, $R_1(M)$ contains noetherian rings, in particular cyclic groups of order two, although R is not noetherian.

We remark that if M is a homomorphically closed class with the property that every ring in M is noetherian, LM may contain rings which are not noetherian. For let W be the universal class of zero rings and let M be the class of zero rings whose additive groups are cyclic p-groups. Then the zero ring on $Z_p(\infty)$, being the union of a chain of cyclic p-groups, is in LM, but $Z_p(\infty)$ is not noetherian (see, for instance, [11, page 14] for a discussion of $Z_p(\infty)$).

We can draw much the same conclusion if we turn our attention to artinian rings. Let W be the universal class of zero rings and let M = {R,0} where R is the zero ring of order two, so that M is a homomorphically closed class of artinian rings. Then the direct sum of an infinite number of copies of R is in LM and fails to have d. c. c. on ideals.

The following result of T. Szele will be required in the proof of Theorem 2.9.

Theorem 2.8. [24]. A group G is the additive group of a nilpotent artinian ring if and only if the minimum condition is satisfied by the subgroups of G.

We have been unable to prove the following theorem for arbitrary universal classes.

Theorem 2.9. Suppose M is a homomorphically closed class of associative rings such that no nonzero ring in M has d. c. c. on right ideals. Then, in the universal class W of associative rings, LM has no nonzero rings with d. c. c. on right ideals.

Proof. We have assumed that $M = R_1(M)$. Thus suppose $\beta > 1$ and assume $R_\alpha(M)$ contains no nonzero right artinian rings for each $\alpha < \beta$. Let R be a nonzero ring in $R_\beta(M)$. If $\beta - 1$ exists, then R has an ideal I such that $I, R/I \in R_{\beta-1}(M)$. If R has d. c. c. on right ideals, then so does R/I, so that $R/I = 0$ by the inductive hypothesis. But then $R = I$ has d. c. c. on right ideals, which is a contradiction. Therefore in this case R cannot be right artinian.

If β is a limit ordinal, then $R = \bigcup I_\gamma$ where $\{I_\gamma\}$ is a chain of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. Suppose R is right artinian and let N denote the classical radical of R. If $R \neq N$, then we have $R/N = \bigcup ((I_\gamma + N)/N)$ and for each index γ, $(I_\gamma + N)/N \cong I_\gamma/(I_\gamma \cap N) \in \bigcup_{\alpha<\beta} R_\alpha(M)$ as this class is homomorphically closed by Lemma 1.2. Then R/N is a semi-simple right artinian ring which is the union of a chain of ideals which are not right artinian; this by the Weddenburn-Artin Theorem [23, Theorem 5.59] is a contradiction, for R/N is isomorphic to the direct sum of a finite number of complete matrix rings of finite order over division rings. Therefore $R = N$. Since a nil ring with d. c. c. on right ideals is nilpotent [23, Theorem 4.30] R has the minimum condition on subgroups by Theorem 2.8. Thus the additive group of each I_γ must have minimum condition on subgroups, so that each I_γ must have minimum condition on right ideals, so that again we have a contradiction.

The proof of the following result is due to R. Tangeman (unpublished) who has also obtained a simpler proof using the A. D. S. construction. The theorem is concerned with the following property, which was introduced in Theorem 1.10:

(a) If $J \in M$ and J is an ideal of an ideal I of a ring $R \in W$, then the ideal J' of R generated by J is also in M.

<u>Theorem 2.10</u>. If M is a homomorphically closed class satisfying property (a), then LM satisfies property (a).

Proof. By hypothesis, $R_1(M) = M$ satisfies property (a). Suppose $R_\alpha(M)$ has property (a) for each $\alpha < \beta$ and let J be an ideal of an ideal I of $R \in W$ such that $J \in R_\beta(M)$. Let J' be the ideal of R generated by J. We must show that J' is also in $R_\beta(M)$.

If β is a limit ordinal, then $J = \cup J_\gamma$ where $\{J_\gamma\}$ is a chain of ideals of J contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. For each γ let J'_γ be the ideal of I generated by J_γ; then since $J_\gamma \subseteq J'_\gamma \subseteq J$, $J = \cup J'_\gamma$. By property (a), since, for each γ, $J_\gamma \in R_\alpha(M)$ for some $\alpha < \beta$, we have $J'_\gamma \in R_\alpha(M)$. For each γ let J''_γ be the ideal of R generated by J'_γ. Again by the inductive hypothesis $J''_\gamma \in \bigcup_{\alpha<\beta} R_\alpha(M)$. Since $J = \cup J'_\gamma$, we have $J' = (\cup J'_\gamma)'$. Since $\cup J''_\gamma$ is an ideal of R containing J'_γ for each γ, $(\cup J'_\gamma)' \subseteq \cup J''_\gamma$. Let $x \in \cup J''_\gamma$; then $x \in J''_\gamma$ for some γ and hence x is in every ideal of R which contains J'_γ, in particular $(\cup J'_\gamma)'$. Thus $\cup J''_\gamma \subseteq (\cup J'_\gamma)'$. Hence $J' = \cup J''_\gamma$, where $\{J''_\gamma\}$ is a chain of ideals of J' contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. Therefore $J' \in R_\beta(M)$.

If β is not a limit ordinal, then J has an ideal K such that K, $J/K \in R_{\beta-1}(M)$. Let K' be the ideal of I generated by K, so that $K' \in R_{\beta-1}(M)$ by the inductive hypothesis. Now $K' \subseteq J$ and $J/K' \cong (J/K)/(K'/K)$ so that $J/K' \in R_{\beta-1}(M)$ by Lemma 1.2. Thus we may as well assume that K is an ideal of I. Since $J' \subseteq I$, K is an ideal of J'. Let K^* be the ideal of R generated by K. Since K is an ideal of I and $K \in R_{\beta-1}(M)$, the inductive hypothesis implies that $K^* \in R_{\beta-1}(M)$. Now R/K^* has the ideal I/K^* which in turn has the ideal $(J + K^*)/K^* \cong J/(J \cap K^*)$. Since $J + K^*$ is an ideal of J', $(J + K^*)/K^*$ is an ideal of J'/K^*. Now J'/K^* is an ideal of R/K^* containing $(J + K^*)/K^*$. If U/K^* is any ideal of R/K^* containing $(J + K^*)/K^*$, then U is an ideal of R containing $J + K^*$ so that U certainly contains J. Thus $J' \subseteq U$ and so $J'/K^* \subseteq U/K^*$. Therefore J'/K^* is the ideal of R/K^* generated by $(J + K^*)/K^*$. Now $J/J \cap K^* \cong (J/K)/(J \cap K^*/K)$, so that by Lemma 1.2 $(J + K^*)/K^* \in R_{\beta-1}(M)$. Hence, using property (a), we have $J'/K^* \in R_{\beta-1}(M)$. Therefore $J' \in R_\beta(M)$.

Thus by transfinite induction each of the classes $R_\beta(M)$ satisfies property (a) so that $LM = \bigcup_\beta R_\beta(M)$ must satisfy property (a). This completes the proof.

As a consequence, we may note that if a class M with property (a) is homomorphically closed and hereditary, then LM is strongly hereditary. For LM has Property (a) by Theorem 2.10 and is hereditary by Theorem 1.8 and hence is strongly hereditary by Theorem 1.10.

<u>Theorem 2.11</u>. Let W be a universal class and let $M \subseteq W$ be a homomorphically closed class such that the center of every ring in M is contained in a subring hereditary radical subclass P of the class C of commutative rings. Then the center of every ring in LM is contained in P.

<u>Proof</u>. For each ring R, we let Z(R) denote the center of R. We have assumed that every ring in $R_1(M) = M$ has center in P. Thus suppose that for each $\alpha < \beta$ and each $R \in R_\alpha(M)$ we have $Z(R) \in P$. Let $R \in R_\beta(M)$. If β is a limit ordinal, then R is the union of a chain $\{I_\gamma\}$ of ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. Since $Z(R) \cap I_\gamma \subseteq Z(I_\gamma)$ for each γ and P is subring hereditary, $Z(R) \cap I_\gamma \subseteq P$ for each γ. Thus $Z(R) = \bigcup (Z(R) \cap I_\gamma)$ is the union of a chain of its ideals contained in P. Since P is a radical class, $Z(R) \in P$ by condition (R4) of Theorem 1.2.

If $\beta - 1$ exists, then R has an ideal J such that $J, R/J \in R_{\beta-1}(M)$. Now $Z(R)/(Z(R) \cap J) \cong (Z(R) + J)/J \subseteq Z(R/J)$ so that $Z(R)/Z(R) \cap J \in P$ as P is subring hereditary and homomorphically closed. Also $Z(R) \cap J \subseteq Z(J) \in P$ so that $Z(R) \cap J \in P$ again because P is subring hereditary. Then by Theorem 1.2, condition (R3), since P is a radical class, $Z(R) \in P$.

We make several observations concerning Theorem 2.11 and note some of its consequences. If M is homomorphically closed and every ring in M has zero center, then every ring in LM has zero center. The rings with zero center in the universal class of associative rings do not form a homomorphically closed class. For consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a,b \text{ real} \right\}$. If $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in Z(R)$, then $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = 0$ so that $a = 0$; moreover, $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, while $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ so that $b = 0$. Thus $Z(R) = 0$. Now R has the ideal $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \text{ real} \right\}$ and R/I is isomorphic to the field of real numbers. Thus the hypothesis that M is homomorphically closed cannot be omitted in the statement of Theorem 2.11.

If the center of every ring in M is nilpotent, then LM may contain rings without nilpotent centers. For example, the ring of [11, Example 3, page 19] is a commutative Baer lower radical ring which is not nilpotent. We can say, however, that if the center of every ring in M is nil, then the center of every ring in LM is nil.

We may specialize Theorem 2.11 as follows. Suppose $M \subseteq W$ is homomorphically closed and satisfies the following conditions:

(1) If $R \in M$, then $Z(R) \in M$.

(2) Every subring of a commutative ring in M is also in M.

Then the center of every ring in LM is also in LM. We would like to be able to weaken the rather stringent condition (2) so as to be able to say, for instance, that if M is hereditary and satisfies condition (1), then the center of every ring in LM is in LM. The non-limit ordinal case has been intractable.

Proposition 2.3. Let M be a hereditary class of associative rings such that if $R \in M$, then $Z(R) \in M$. Suppose S is the union of a chain $\{I_\gamma\}$ of ideals of S contained in M. Then Z(S) is the union of a chain of ideals of Z(S) contained in M.

Proof. We have $Z(S) = \bigcup (Z(S) \cap I_\gamma)$. Now for each index γ, $Z(S) \cap I_\gamma \subseteq Z(I_\gamma)$. We claim that in fact $Z(S) \cap I_\gamma$ is an ideal of $Z(I_\gamma)$. Thus let $r \in S$, $c \in Z(S) \cap I_\gamma$ and $x \in Z(I_\gamma)$. Then

$$\begin{aligned} r(xc) &= (rx)c && \text{(by associativity)} \\ &= c(rx) && \text{(as } c \in Z(R)) \\ &= (cr)x && \text{(by associativity)} \\ &= x(cr) && \text{(as } cr \in I_\gamma \text{ and } x \in Z(I_\gamma)) \\ &= (xc)r && \text{(by associativity again).} \end{aligned}$$

Thus $xc \in Z(S) \cap I_\gamma$. Hence, since M is hereditary, $Z(S) \cap I_\gamma \in M$ for all γ.

Let us note one more consequence of Theorem 2.11. If the center of every ring in M is a p-ring, then the center of every ring in LM is a p-ring.

We mention two results whose proofs are fairly easy using the extension-union construction; we therefore omit the proofs.

Proposition 2.4. If each ring in M contains no nonzero nilpotent elements, then each ring in LM contains no nonzero nilpotent elements.

Proposition 2.5. If every nonzero ideal of each ring in M contains a nonzero idempotent, then LM has the same property.

Next we show how to construct, for any universal class W and any class $M \subseteq W$, a minimal hypernilpotent radical class in W containing M. Let $M_1 = M \cup \{0\}$ and let $M^* = \{R \in W : R^2 \in M_1\}$.

Theorem 2.12. LM^* is the unique minimal hypernilpotent class in W containing M.

Proof. LM^* contains M^* which by definition contains every zero-ring in W, so that LM^* is a hypernilpotent radical class in W containing M. Let P be any hypernilpotent radical class in W containing M and let $R \in M^*$. Then R/R^2 is a zero ring and $R^2 \in M_1$, so that R/R^2, $R^2 \in P$. By Theorem 1.2, condition (R3), we have $R \in P$. Hence $M^* \subseteq P$ so that $LM^* \subseteq P$ by Theorem 1.6.

We may remark that $LM^* = L(M \cup Z)$ where Z is the class of zero rings, for on the one hand $L(M \cup Z)$ is a hypernilpotent radical class containing M so that $LM^* \subseteq L(M \cup Z)$ by Theorem 2.12, while on the other hand $M \cup Z \subseteq M^*$ so that $L(M \cup Z) \subseteq LM^*$. Thus intuitively the properties that LM^* inherits from M are the properties, inherited by the lower radical, which M shares with Z. For example, if M is hereditary, then $M \cup Z$ is hereditary and so $LM^* = L(M \cup Z)$ is hereditary by Theorem 1.8.

We now turn to a consideration of the lower radical construction in the presence of two universal classes. Theorem 2.13 is due to Hoffman, who accomplished its proof with the A. D. S. construction, and we present a new proof which employs the extension-union construction. First we require a lemma, whose proof we omit.

Lemma 2.1. [13, page 54]. Let P be a radical class in a universal class W. If W' is some other universal class, then $W \cap W'$ is a universal class and $P \cap W'$ is a radical class in $W \cap W'$.

Theorem 2.13. [13, page 58]. Let X and W be universal classes and let $M \subseteq W$ be homomorphically closed. Then $X \cap L_W(M) = L_{X\cap W}(X \cap M)$.

Proof. By the Lemma, $X \cap L_W(M)$ is a radical class in $X \cap W$. Since $M \subseteq L_W(M)$ we have $X \cap M \subseteq X \cap L_W(M)$, so that, inasmuch as $L_{X\cap W}(X \cap M)$ is minimal among radical classes in $X \cap W$ containing $X \cap M$, we have $L_{X\cap W}(X \cap M) \subseteq X \cap L_W(M)$.

We now fix the notation to be used in the remainder of the proof. For each ordinal number β, $R_\beta(M)$ will be the class obtained in the construction of $L_W(M)$, while $R_\beta(X \cap M)$ will be the class obtained in the construction of $L_{X\cap W}(X \cap M)$, at the β^{th} step in the extension-union construction. Since M is homomorphically closed, $X \cap R_1(M) = R_1(X \cap M) = X \cap M$. Thus let $\beta > 1$ be an ordinal number and assume for all $\alpha < \beta$ that $X \cap R_\alpha(M) \subseteq R_\alpha(X \cap M)$. Let $K \in X \cap R_\beta(M)$. If β is a limit ordinal, then $K = \cup K_\gamma$, where $\{K_\gamma\}$ is a chain of ideals of K contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. Moreover, each $K_\gamma \in X$, for X being a universal class is hereditary. Thus each $K_\gamma \in X \cap R_\alpha(M)$ for some $\alpha < \beta$ and hence each $K_\gamma \in R_\alpha(X \cap M)$. Thus $K \in R_\beta(X \cap M)$. If β is not a limit ordinal, then K has an ideal J such that K/J, $J \in R_{\beta-1}(M)$. Since X is hereditary and homomorphically closed, K/J and J are both in $X \cap R_{\beta-1}(M)$. Hence K/J, $J \in R_{\beta-1}(X \cap M)$ by the inductive hypothesis, so that $K \in R_\beta(X \cap M)$. Thus $X \cap R_\beta(M) \subseteq R_\beta(X \cap M)$ for all β, so that $X \cap L_W(M) \subseteq L_{X\cap W}(X \cap M)$.

It is interesting to note that the hypothesis that M be homomorphically closed cannot be removed in the theorem, as Hoffman has shown in an example [13, page 57] which can be summarized as follows. Let R be a ring containing exactly one proper ideal K such that R/K and K are not isomorphic. Let $W = \{R,K,R/K,0\}$, $X = \{R/K,0\}$ and $M = \{R,K,0\}$; then W and X are universal classes, $M \subseteq W$ and M is not homomorphically closed. Now $L_{X\cap W}(X \cap M) = L_X(X \cap M) = L_X(0) = 0$, while since $R/K \in R_1(M)$, $L_W(M) = W$. Thus $X \cap L_W(M) = X \cap W = X$ so that $L_{X\cap W}(X \cap M)$ is properly contained in $X \cap L_W(M)$.

We shall be interested in whether some of the other minimal radical classes we encounter have the property noted for LM in Theorem 2.13: not all of them will. However for the radical class of Theorem 2.12 we have the following result.

Corollary 2.2. Denote by $L^*_W(M)$ the unique minimal hypernilpotent radical class in the universal class W containing the homomorphically closed class M and let X be a universal class. Then $L^*_{X\cap W}(X \cap M) = X \cap L^*_W(M)$.

Proof. Let Z_W stand for the class of zero rings in the universal class W. By the remarks below Theorem 2.12,

$$\begin{aligned} X \cap L^*_W(M) &= X \cap L_W(M \cup Z_W) \\ &= L_{X\cap W}(X \cap (M \cup Z_W)) \qquad \text{(Theorem 2.13)} \\ &= L_{X\cap W}((X \cap M) \cup (X \cap Z_W)) \\ &= L_{X\cap W}((X \cap M) \cup Z_{X\cap W}) \\ &= L^*_{X\cap W}(X \cap M). \end{aligned}$$

Finally, we use the extension-union construction to determine a property of a class Q which satisfies condition (S1) of Definition 1.2; as a corollary, we obtain a characterization of UQ.

Theorem 2.14. If M is a homomorphically closed class and Q is a class satisfying (S1) such that $M \cap Q = 0$, then $LM \cap Q = 0$.

Proof. We have assumed that $R_1(M) \cap Q = 0$. Let $\beta > 1$ and assume $R_\alpha(M) \cap Q = 0$ for all $\alpha < \beta$. Suppose $0 \neq B \in R_\beta(M) \cap Q$. Either B is a union of ideals from the classes $R_\alpha(M)$, $\alpha < \beta$, or B is an extension of an ideal in $R_{\beta-1}(M)$, so that in either case B contains a nonzero ideal I in $R_\alpha(M)$ for some $\alpha < \beta$. By (S1), I has a nonzero image $I\phi \in Q$, but since $R_\alpha(M)$ is homomorphically closed, we have $I\phi \in R_\alpha(M) \cap Q$, a contradiction.

Corollary 2.3. If Q satisfies (S1), then UQ is the union of all homomorphically closed classes whose intersection with Q is zero.

Proof. Let M be the union of all homomorphically closed classes whose intersection with Q is zero. Then M is homomorphically closed and $M \cap Q = 0$. Hence $LM \cap Q = 0$ and, since LM is homomorphically closed, $M = LM$. Now UQ is a radical maximal with respect to having zero intersection with Q, so that $M \subseteq UQ$. Again by construction of M we have $M = UQ$.

CHAPTER III

STRONG SUBRING HEREDITY AND RELATED PROPERTIES

This chapter is devoted to a study of a concept similar to strong heredity, which we have discussed briefly in Chapter I, and certain closely related properties. The concept we introduce, strong subring heredity, is the analogue of strong heredity for subrings instead of ideals. We will formalize our terminology after making some preliminary observations. One of our objectives will be to characterize in various ways the radical classes which are strongly subring hereditary. To accomplish this, we adopt the approach of W. G. Leavitt in his work on strongly hereditary radical classes [19], introducing a modification of property (a) of Theorem 1.10. We will construct minimal radical classes having the properties we are concerned with and will comment on the semisimple classes of some of the radical classes we encounter.

We begin by introducing a type of universal class sufficiently extensive for the ensuing development.

Lemma 3.1. Given any class M of rings, there is a minimal subring hereditary class ΛM containing M.

Proof. Let $\Lambda M = \{R : R$ is a subring of some ring $S \in M\}$. Then ΛM is subring hereditary, for if I is a subring of $R \in \Lambda M$, then R is a subring of $S \in M$ and so I is a subring of S. Hence $I \in \Lambda M$. On the other hand, any subring hereditary class containing M must contain ΛM.

The following should now be clear, and we omit the proof.

Proposition 3.1. For any class M of rings, $H\Lambda M$, the homomorphic closure of ΛM, is the minimal subring hereditary, homomorphically closed class containing M.

We will call a subring hereditary, homomorphically closed class __s-universal__. When we discuss subring heredity, we will understand without explicit mention that we are working within an s-universal class W. We remark that if M is a class of rings contained in an s-universal class W, then LAM is the minimal subring hereditary radical class in W containing M by Theorem 2.3.

__Definition 3.1__. A radical class P in an s-universal class W is called __strongly subring hereditary__ if for all $R \in W$ we have $P(I) = I \cap P(R)$ for each subring I of R.

__Theorem 3.1__. A subring hereditary radical class P of an s-universal class W is strongly subring hereditary if and only if P has the following property (b).

(b) If I is a subring of $R \in W$ and $I \in P$, then the ideal of R generated by I is also in P.

__Proof__. Suppose first that P is strongly subring hereditary. Let $I \in P$ be a subring of $R \in W$ and let J be the ideal of R generated by I. Now $P(J) = J \cap P(R)$ is an ideal of R. Since $I \in P$ and P is strongly subring hereditary, $P(I) = I = I \cap P(R)$ and so $I \subseteq P(R)$. Thus $I \subseteq P(J)$ so that P(J) must be the ideal of R generated by I. Thus $J = P(J)$ and therefore P has property (b).

On the other hand, suppose P is a radical class having property (b). Then if I is a subring of $R \in W$, let $J \in P$ be an ideal of I. Let J' be the ideal of R generated by J, so that $J' \in P$ by property (b). Thus $J' \subseteq P(R)$, so that $J \subseteq P(R)$. In particular $P(I) \subseteq I \cap P(R)$. For the reverse inclusion, by subring heredity $P(I) \supseteq I \cap P(R)$ as $I \cap P(R)$ is a subring of P(R) and is thus in P. Therefore P is strongly subring hereditary.

We remark that in any s-universal class W, if P is a strongly subring hereditary radical class, then P is subring hereditary. For let $R \in P$ and let I be a subring of R. Then $P(I) = I \cap P(R) = I \cap R = I$, so that $I \in P$. The following example shows that the converse is not true. Let M be the class consisting of the ring Z of integers and all its subrings. Then M is certainly subring hereditary and hence so is LM by Theorem 2.3. Let R be the field of real numbers, which contains Z as a subring. R is LM-semisimple by Theorem 1.9 since R is simple and $R \notin M$. Now $LM(Z) = Z \neq LM(R) \cap Z = 0$. Hence, even in the associative case, subring heredity does not imply strong subring heredity.

Lemma 3.2. If P has property (b), then SP is subring hereditary.

Proof. If $R \in SP$ has a subring $I \notin SP$, then I has an ideal $J \neq 0$ in P. But J is in turn a subring of R, so that the ideal J' of R generated by J must be in P by property (b). This is a contradiction, for $J' \neq 0$.

Proposition 3.2. Let P be any radical class in W and let N be any homomorphically closed, hereditary class; every N-subring of each ring in SP is also in SP if and only if for each N-subring I of each ring $R \in W$ we have $P(I) \subseteq P(R)$.

Proof. First suppose every N-subring of each ring in SP is also in SP. Let $R \in W$ and let I be an N-subring of R. If $P(I) \not\subseteq P(R)$, then $0 \neq [P(I) + P(R)]/P(R) \cong P(I)/[P(I) \cap P(R)] \in N \cap P$ since $N \cap P$ is homomorphically closed. Then $[P(I) + P(R)]/P(R)$ is an N-subring of $R/P(R)$ and hence is in SP, but $P \cap SP = 0$, so that we have a contradiction. Conversely, suppose for each N-subring I of each ring $R \in W$ we have $P(I) \subseteq P(R)$. If $R \in SP$, then for any N-subring I of R, $P(I) \subseteq P(R) = 0$, so that I is also in SP.

The following special case of Proposition 3.2 is worth pointing out.

Corollary 3.1. Let P be any radical class in W. Then SP is subring hereditary if and only if for each $R \in W$ and each subring I of R we have $P(I) \subseteq P(R)$.

Lemma 3.3. Let P be any radical class in W. Then P is strongly subring hereditary if and only if both P and SP are subring hereditary.

Proof. If P is strongly subring hereditary, then $P(I) = I \cap P(R)$ for each subring I of each ring $R \in W$, so that both P and SP are subring hereditary. For the converse, suppose P and SP are subring hereditary and let I be a subring of R. By Corollary 3.1, $P(I) \subseteq I \cap P(R)$. Since $P(R) \in P$ and P is subring hereditary, $I \cap P(R) \in P$. Since $I \cap P(R)$ is an ideal of I, $I \cap P(R) \subseteq P(I)$. Thus $I \cap P(R) = P(I)$.

Lemma 3.4. If P is a subring hereditary radical class, then P satisfies property (b) if and only if SP is subring hereditary.

Proof. This follows from Lemma 3.2, Lemma 3.3 and Theorem 3.1.

We remark that property (b) implies property (a) of Theorem 1.10. The following result, which we prove using the extension-union construction, will be used in the construction of a minimal radical class in W containing a given class M and satisfying property (b).

Theorem 3.2. If $M \subseteq W$ is homomorphically closed and satisfies property (b), then LM satisfies property (b).

Proof. $R_1(M)$ has property (b) by hypothesis. Suppose $R_\alpha(M)$ has (b) for all $\alpha < \beta$ and let I be a subring of a ring R with $I \in R_\beta(M)$.

First suppose β is a limit ordinal. Then $I = \cup I_\gamma$ where each $I_\gamma \in \bigcup_{\alpha<\beta} R_\alpha(M)$ and each I_γ is an ideal of I. Let J be the ideal of R generated by I and for each γ let J_γ be the ideal of R generated by I_γ.

Then we claim that $J = \cup J_\gamma$. For $\cup J_\gamma$ is an ideal of R containing $\cup I_\gamma = I$, so that $\cup J_\gamma \supseteq J$, while J is an ideal of R containing every I_γ, hence every J_γ, so that $J \supseteq \cup J_\gamma$. Thus $J \in R_\beta(M)$.

If β is not a limit ordinal, let $I \in R_\beta(M)$ be a subring of R. Let J be the ideal of R generated by I. I has an ideal K with K, $I/K \in R_{\beta-1}(M)$. Now K generates an ideal $P \subseteq J$ of R such that $P \in R_{\beta-1}(M)$. Consider J/P; we claim it is the ideal of R/P generated by $(I + P)/P$. For suppose S/P is an ideal of R/P containing $(I + P)/P$. Then S is an ideal of R containing $I + P$, hence I. Thus S contains J, and so $S/P \supseteq J/P$. Then J/P is minimal among the ideals of R/P containing $(I + P)/P$. Now $(I + P)/P \cong I/(I \cap P)$. Since $K \subseteq I \cap P$, we have a natural map from I/K onto $I/(I \cap P)$. Since all the classes $R_\alpha(M)$ are homomorphically closed by Lemma 1.2, $I/(I \cap P) \in R_{\beta-1}(M)$ and hence $(I + P)/P \in R_{\beta-1}(M)$. Since $R_{\beta-1}(M)$ has property (b), $J/P \in R_{\beta-1}(M)$. Since we have shown P, $J/P \in R_{\beta-1}(M)$, we have $J \in R_\beta(M)$.

The theorem follows by transfinite induction.

We now direct our attention to the construction of minimal radical classes of certain types. First we need a definition.

<u>Definition 3.2</u>. For any class $M \subseteq W$, define $F(M) = \{J' : J \in M$, J is a subring of $R \in W$, and J' is the ideal of R generated by J$\}$.

<u>Theorem 3.3</u>. If W is any s-universal class and $M \subseteq W$, then there is a unique minimal radical class in W containing M which satisfies property (b).

<u>Proof</u>. Let M_1 be the homomorphic closure of M, $M_2 = F(M_1)$, and proceed inductively. If $n > 2$ is odd and M_k has been defined for all $k < n$, let M_n be the homomorphic closure of M_{n-1}. If n is even, let $M_n = F(M_{n-1})$.

Clearly $m \leq k$ implies $M_m \subseteq M_k$. Let $M^* = \cup M_n$, the union being taken over all positive integers n. M^* is clearly homomorphically closed. Also if J is a subring of $R \in W$ and $J \in M^*$, then $J \in M_n$ for some odd n so that J', the ideal of R generated by J, is in $M_{n+1} \subseteq M^*$. Thus M^* has property (b).

If N is any homomorphically closed class containing M and satisfying property (b), then we claim $M^* \subseteq N$. For $M_1 \subseteq N$ since N is homomorphically closed. Suppose $M_n \subseteq N$. If n is odd, M_{n+1} is the homomorphic closure of M_n, so that $M_{n+1} \subseteq N$ since N is homomorphically closed. If n is even, let $J' \in M_{n+1}$. Then there exists $J \in M_n$ such that J' is the ideal of some $R \in W$ generated by $J \subseteq R$. Then $J' \in N$ since N has property (b). Thus $M_{n+1} \subseteq N$ in either case, so that by induction $M^* \subseteq N$. If N is a radical class, then $LM^* \subseteq N$. Since LM^* satisfies (b) by Theorem 3.2, LM^* is therefore the minimal radical class in W containing M and satisfying property (b).

<u>Example 3.1</u>. The class $F(M_1)$ need not be hereditary when M_1 is hereditary. For let K be the field of two elements and let $R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a,b \in K\right\}$. We work within the class W of associative rings, and we identify isomorphic rings. Now $K \cong \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in K\right\}$ and the ideal this subring generates in R is R itself. R has the ideal $I = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in K\right\}$, a zero-ring of order two. Let $M = M_1 = \{K,0\}$; then M is a hereditary class. Now $R \in F(M_1)$ but I is not in $F(M_1)$ since I is a simple abelian group.

Continuing the example, we let $S \in M_2 = F(M_1)$. Then there is a ring X with $K \subseteq S \subseteq X$ such that S is the ideal of X generated by K. Suppose T is an ideal of S with $K \subseteq T$. Let $x \in X$ and let 1 denote the identity element of K; we note that 1 need not be an identity for any of the other rings we have mentioned. Now $x1 \in S$ as S is an ideal of X and so $x1 = x1 \cdot 1 \in T$ (here we have used the associativity). Thus $XK \subseteq T$. Similarly KX and $XKX \subseteq T$. Thus $K + XK + KX + XKX$, the ideal of X generated by K, is contained in T, so that necessarily $S = T$. Hence $F(M_1) = M_2 = \{S : K \subseteq S$ and there is no ideal T of S with $K \subsetneq T \subsetneq S\} \cup \{0\}$.

We have noted previously that $M_2 \subseteq M_3$. Let $X \in M_3$. Then $X \cong S/T$ where $S \in M_2$. Now if $S \neq 0$ then S is generated by K, and as K is simple, S/T is generated by $(K + T)/T \cong K$. Hence $S/T \in M_2$, so that $X \in M_2$. Therefore $M_3 = M_2$.

Now let $U \in M_4$. Then if $U \neq 0$ there exists a ring $S \in M_3 = M_2$ such that $S \subseteq U \subseteq X$, $X \in W$, and U is the ideal of X generated by X. Now $K \subseteq S$ and the ideal K' of X generated by K must contain S by the characterization of M_2 given above. Hence K' is the ideal of X generated by S, so that $K' = U$, i.e., $U \in M_3$.

This shows that $M^* = M_2$, for M_2 is closed homomorphically and $F(M_2) = M_2$. In particular $I \notin M^*$, so that M^* is not hereditary.

Now I is a simple ring, so that LM^* does not contain I by Theorem 1.9. Thus we have shown that, if M is hereditary, the minimal radical class in W containing M which satisfies property (b) need not be hereditary.

Indeed, it seems rather difficult to discover properties that LM^* does inherit from M. For instance, if M is homomorphically closed and has no rings with identity, F(M) may contain rings with identity, as we see from the preceding example.

Remark. If W is an s-universal class, we will for the moment let $L'_W(M)$

denote the minimal radical class in W containing M and satisfying property (b). We may ask whether the analogue for this construction of Theorem 2.13 is true; that is, if X is a second s-universal class, does $X \cap L'_W(M) = L'_{X\cap W}(X \cap M)$?

We can show that $L'_{X\cap W}(X \cap M) \subseteq X \cap L'_W(M)$. Let $J \in X \cap L'_W(M)$ and suppose J is a subring of $R \in X \cap W$. Let J' be the ideal of R generated by J. Then $J' \in L'_W(M)$ since the latter radical class has property (b), and $J' \in X$ since X is subring hereditary. Thus $J' \in X \cap L'_W(M)$, so that $X \cap L'_W(M)$, a radical class in $X \cap W$ by Lemma 2.1, has property (b) and contains $X \cap M$. This completes the proof since $L'_{X\cap W}(X \cap M)$ is the minimal such class.

The reverse inclusion is not true, however, and to see this we use an example from [10]. Let R be the algebra generated over the field F of two elements by the non-associative symbols u, v, w, subject to the relations $u^2 = 0$, $uv = vu = u$, $uw = wu = vw = wv = v^2 = v$ and $w^2 = w$. Then $I = \{0,u,v,u + v\}$ is the only proper ideal of R and $J = \{0,u\}$ is the only proper ideal of I. As usual we identify isomorphic rings. Let $X = \{F,0\}$, $W = \{R,I,J,F,0\}$ and $M = \{J,0\}$. Then X, W and M are all s-universal classes. The ideal of R generated by J is I and $I/J = F$. In $\{w,0\}$ we have a subring of R isomorphic to F, and the ideal of R generated by $\{w,0\}$ is R itself. Hence $L'_W(M) = W$. Now $X \cap L'_W(M) = X$, while on the other hand $L'_{X\cap W}(X \cap M) = L'_{X\cap W}(\{0\}) = \{0\}$.

<u>Theorem 3.4</u>. Let M be any homomorphically closed class of rings in the class W of associative rings and suppose M contains the class Z of rings with zero multiplication. Let LM^* be the unique minimal radical class in W containing M and satisfying property (b). If $R \in W$ has no nonzero subrings in M, then $R \notin LM^*$.

Proof. We shall continue to use the notation of Theorem 3.3. By hypothesis neither R nor any of its subrings is in $M = M_1 = H(M)$. Moreover $R \notin M_2 = F(M_1)$ because R has no proper subrings in M_1, and for the same reason, no nonzero subring of R is in M_2. Let $n \geq 2$ and suppose that no nonzero subring of R is in any of the classes M_k with $k \leq n$. If n is odd, $M_{n+1} = F(M_n)$. Again since R has no proper subrings in M_n, R cannot be in M_{n+1}. The same is true of each nonzero subring of R. If n is even $M_{n+1} = H(M_n)$. If $S/I \cong R$ where $S \in M_n$ and I is an ideal of S, then S has a subring $J \in M_{n-1}$ such that S is the ideal of some ring T generated by J. Since we are working in the class of associative rings, Lemma 1.4 is available; hence, if I' is the ideal of T generated by I, $I'^3 \subseteq I$. But S/I can have no nilpotent subrings, so that $I' \subseteq I$ and hence I is an ideal of T. If $J \subseteq I$ we contradict the fact that J generates the ideal S in T. Hence $J \not\subseteq I$. Now $(J + I)/I$ is a subring of S/I, which by the inductive hypothesis has no nonzero subrings in M_{n-1}. But $0 \neq (I+J)/I \cong J/(I \cap J) \in M_{n-1}$ as M_{n-1} is homomorphically closed and so we have a contradiction. Thus $R \notin M_{n+1}$. Likewise no nonzero subring of R can be in M_{n+1}. The fact that $R \notin M^*$ now follows by induction.

To complete the proof we will use the A. D. S. construction to show that $R \notin LM^*$. In the following M^*_β will denote the class obtained at the β^{th} step in the A. D. S. construction of LM^*. Now $R \notin M_1{}^* = M^*$. Let $\beta > 1$ be an ordinal number and suppose that no nonzero subring of R is in $\bigcup_{\alpha<\beta} M^*_\alpha$. Then since R has no nonzero ideals in $\bigcup_{\alpha<\beta} M^*_\alpha$, R cannot be in M^*_β; by the same reasoning, no nonzero subring of R can be in M^*_β. Therefore $R \notin LM^*$.

Corollary 3.2. With the conditions of Theorem 3.4 on M, a ring $R \in W$ is is LM^*-semisimple if and only if R has no nonzero subrings in M.

Proof. Suppose R has no nonzero subrings in M and let I be an ideal of R. Then I has no nonzero subrings in M so that $I \notin LM^*$ by Theorem 3.4. Thus R is LM^*-semisimple. On the other hand, if R is LM^*-semisimple, let I be an LM^*-subring of R. Since LM^* satisfies property (b), the ideal I' of R generated by I is in LM^*. Hence I' = 0, so that I = 0; in particular, R has no nonzero subrings in M.

Lemma 3.5. Let W be a universal class and let $M \subseteq W$. Then there is a unique minimal class $M' \supseteq M$ which is homomorphically closed, subring hereditary and satisfies property (b).

Proof. Define the classes M_n as follows. Let M_1 be the homomorphic closure H(M) of M, $M_2 = \Lambda(M_1)$, the subring closure of M_1, and $M_3 = F(M_2)$. If $n > 3$ and the classes M_k have been defined for all $k < n$, let M_n be either the homomorphic closure of M_{n-1}, $\Lambda(M_{n-1})$ or $F(M_{n-1})$ according as n is congruent to one, two or zero modulo three. Finally, let $M' = \cup M_n$.

Since $M_k \subseteq M_m$ if $k \leq m$, it is easy to see that M' is homomorphically closed, subring hereditary and satisfies property (b). On the other hand, a straightforward induction like that in Theorem 3.3 shows that any class with these three properties which contain M must also contain M'.

Theorem 3.5. If M is any class of rings in an s-universal class W, then LM' is the unique minimal strongly subring hereditary radical class in W containing M.

Proof. Since $M \subseteq M'$, $M \subseteq LM'$. Since M' is homomorphically closed, subring hereditary and satisfies property (b), LM' has the same properties by Theorem 2.3 and Theorem 3.2. Let $M \subseteq N$ where N is a strongly hereditary radical class. Then N is homomorphically closed, subring hereditary and satisfies property (b) by Theorem 3.1, so that $M' \subseteq N$. Therefore $LM' \subseteq N$ by Theorem 1.6. This completes the proof.

The example given in the remark preceding Theorem 3.4 also shows that $X \cap L_W M' \neq L_{X \cap W}(X \cap M)'$ in general.

To characterize semisimple classes for radical classes having property (b) requires only a slight modification of Theorem 1.5. Here W is not required to be associative.

Theorem 3.6. $Q \subseteq W$ is a semisimple class for a radical class $P \subseteq W$ satisfying property (b) if and only if Q satisfies the following four properties:

(1) Q is subring hereditary.

(2) Any subdirect sum of rings in Q is also in Q.

(3) Q is extension closed.

(4) If I is an ideal of $R \in W$ and $0 \neq I/B \in Q$ for some ideal B of I, then there exists an ideal A of R with $A \subseteq I$ such that $0 \neq I/A \in Q$.

Proof. Suppose Q is a semisimple class for a radical class P which satisfies property (b). Then $SP = Q$ is subring hereditary by Lemma 3.2. Properties (2) and (3) follow as in [9]. To show (4), let I be an ideal of $R \in W$ and suppose $0 \neq I/B \in Q$ for some ideal B of I, so that $P(I) \neq I$. Let A be the ideal of R generated by P(I). Then $A \in P$ by property (b) and $A \subseteq I$, so that in fact $A = P(I)$. Hence $0 \neq I/A \in Q$.

Conversely, as in [9], Q is a semisimple class for some radical class P. Suppose $I \in P$ is a subring of R and let I' be the ideal of R generated by I. Then if $I' \notin P$ we have $0 \neq I'/P(I') \in Q$. By (4), there exists an ideal A of R with $A \subseteq I'$ and $0 \neq I'/A \in Q$. If $I \not\subseteq A$, then $(I + A)/A \cong I/(I \cap A)$ is a nonzero ring in P by the fact that P is homomorphically closed. But Q by (1) is subring hereditary so that this is impossible. Thus $I \subseteq A$, so that again we have a contradiction as I' is the subring of R generated by I.

We now introduce the following condition which may be satisfied by a radical class P:

(λ) If I is a subring of $R \in W$ with the property that $I \cap A \neq 0$ for every nonzero ideal A of R and $I \in SP$, then $R \in SP$.

<u>Proposition 3.3</u>. Let P be a radical class satisfying property (b). If P is subring hereditary, so that by Theorem 3.1 P is strongly subring hereditary, then P has property (λ).

<u>Proof</u>. If P is subring hereditary, let $R \in W$ have a subring $I \in SP$ such that $I \cap A \neq 0$ for every nonzero ideal A of R. Then $P(I) = I \cap P(R)$. Since SP is subring hereditary by Lemma 3.2, $I \cap P(R) \in P \cap SP = 0$, which means $P(R) = 0$ since P(R) is an ideal of R. Therefore $R \in SP$.

To conclude this chapter, we comment briefly on the meaning of strong subring heredity for hypernilpotent, associative radical classes.

<u>Proposition 3.4</u>. Let W be the class of associative rings and P a strongly subring hereditary, hypernilpotent radical class in W. Then P contains every complete matrix ring of finite order over any division ring.

<u>Proof</u>. Let F be any division ring and consider the two-by-two matrix ring F_2 over F. The subring $R = \left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} : a \in F\right\}$ is nilpotent. By property (b), which P possesses by Theorem 3.3, we have $F_2 \in P$, for F_2 is the ideal of F_2 generated by R. Then because $F \cong \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in F\right\}$ is a subring of F_2, F is in P. Now each of the simple rings F_n has a subring isomorphic to F, so that by property (b), each is in P.

Theorem 3.7. [4, Theorem 4]. Every associative ring is a homomorphic image of a subdirect sum of complete matrix rings of finite order over the ring of integers.

Theorem 3.8. Let W be the class of associative rings and let P be a strongly subring hereditary, hypernilpotent radical class in W. Then P = W if and only if P is closed under direct products.

Proof. By Proposition 3.4, P contains every complete matrix ring of finite order over the field of rational numbers. The matrix ring Z_n of order n over the integers is a subring of the matrix ring of order n over the rational numbers, so that $Z_n \in P$. If P is closed under direct products, then every direct product of complete matrix rings of finite order over the integers is in P, and hence every subdirect product of such rings is in P since P is subring hereditary. Since P is homomorphically closed, every associative ring is in P by Theorem 3.7. The converse is evident.

We note, finally, the following simple consequence of Proposition 3.4.

Corollary 3.3. The Baer lower radical class β does not have property (b).

Proof. As usual let Z denote the class of zero rings. Then $\beta = LZ$ is subring hereditary by Theorem 2.3. If β had property (b), it would be strongly subring hereditary by Theorem 3.1. But this is impossible by Proposition 3.4, as β contains no fields.

CHAPTER IV

STRONG RIGHT HEREDITY AND RELATED TOPICS

One can define an analogue of strong heredity using right ideals, and we begin this chapter by adapting the approach of Leavitt [19] to the study of the concept, strong right heredity, we introduce. Our intentions are to characterize strong right heredity, for which purpose we employ a modification of property (a) of Theorem 1.10, to construct minimal radical classes of the types we encounter containing a given class and to say as much as possible about the semisimple classes of the radicals we study. We conclude the chapter with an investigation of certain radical classes obtained by altering the extension-union construction at the limit ordinal steps.

We wish to work within a type of universal class which includes all right ideals of the rings we consider; this requires a few preliminary remarks. Let C be an arbitrary class of rings and let $C = G_0(C)$. Proceeding inductively, define $G_j(C) = \{I : I$ is a right ideal of some ring $R \in G_{j-1}(C)\}$ for $j = 1,2,\ldots$ and define $G(C) = \bigcup_{j=0}^{\infty} G_j(C)$. Of course we have $C \subseteq G(C)$ and $0 \in G(C)$.

Lemma 4.1. $G(C)$ is right hereditary and is minimal among right hereditary classes containing C.

Proof. If $R \in G(C)$, then $R \in G_j(C)$ for some integer $j \geq 0$. Thus any right ideal I of R is a member of $G_{j+1}(C) \subseteq G(C)$; hence, $G(C)$ is right hereditary. Also, if $C \subseteq D$ where D is right hereditary, assume $G_{j-1}(C) \subseteq D$. Then if $R \in G_j(C)$, R is a right ideal of some ring $K \in G_{j-1}(C)$, so that $K \in D$. Since D is right hereditary $R \in D$. Thus $G_j(C) \subseteq D$ and the proof is completed by induction.

In a similar way one can prove the following, taking one-sided ideals at each step in the construction above instead of right ideals alone.

Lemma 4.2. Given any class M of rings, there is a minimal class ΨM containing M such that if $R \in \Psi M$, then every one-sided ideal of R is also in ΨM.

Then $H(\Psi M)$, the homomorphic closure of ΨM, is the minimal homomorphically closed class containing M which is both left and right hereditary. We call such a class osi-universal ("osi" standing for one-sided ideal) and in this chapter we will understand that our activity takes place entirely within osi-universal classes W.

Proposition 4.1. Every class M of rings contained in an osi-universal class W is contained in a unique minimal right hereditary radical class in W.

Proof. Let $G(M)$ be the right hereditary closure of M given in Lemma 4.1; clearly $G(M) \subseteq W$, because W is right hereditary and contains M. Then $LG(M)$ is a radical class in W minimal with respect to the inclusion of $G(M)$. Since $G(M)$ is right hereditary, so is $LG(M)$ by Theorem 1.1; moreover, if P is a right hereditary radical class in W containing M, then P contains $G(M)$ and hence $LG(M)$ by Lemma 4.1 and Theorem 1.6.

Theorem 4.1. Suppose $M \subseteq W$ is homomorphically closed and that M has the following property:

(p) If $J \in M$ is an ideal of a right ideal I of $R \in W$, then the ideal J' of R generated by J is also in M. Then LM also satisfies property (p).

Proof. By hypothesis $R_1(M) = M$ has property (p). Let $\beta > 1$ be an ordinal number and let J be an ideal of a right ideal I of a ring $R \in W$ with $J \in R_\beta(M)$. Let J' denote the ideal of R generated by J, and suppose the classes $R_\alpha(M)$ have property (p) for all $\alpha < \beta$.

First suppose β is a limit ordinal. Then $J = \bigcup J_\gamma$, where $\{J_\gamma\}$ is a chain of ideals of J contained in $\bigcup_{\alpha<\beta} R_\alpha(M)$. For each index γ, let K_γ be the ideal of I generated by J_γ. We claim that $J = \bigcup K_\gamma$, for $\bigcup K_\gamma \supseteq \bigcup J_\gamma = J$, while on the other hand, as J is an ideal of I and $J \supseteq J_\gamma$ for each γ, we have $J \supseteq K_\gamma$ for all γ, so that $J \supseteq \bigcup K_\gamma$. Each $K_\gamma \in \bigcup_{\alpha<\beta} R_\alpha(M)$ by property (a) of Theorem 1.10, which is implied by property (p). For each γ, let K'_γ be the ideal of R generated by K. By property (p) each $K'_\gamma \in \bigcup_{\alpha<\beta} R_\alpha(M)$, and we claim that $J' = \bigcup K'_\gamma$. For J' contains K_γ for each index γ, so that J', being an ideal of R, contains all the K'_γ and hence their union. Conversely, $\bigcup K'_\gamma$ is an ideal of R containing $\bigcup K_\gamma = J$ and hence $\bigcup K'_\gamma \supseteq J'$. Thus $J' \in R_\beta(M)$.

If β is not a limit ordinal, then J has an ideal K with $K, J/K \in R_{\beta-1}(M)$. Now if $P \subseteq J$ is the ideal of I generated by K, then $P \in R_{\beta-1}(M)$ by property (a). Moreover, $J/P \in R_{\beta-1}(M)$ because J/P is a homomorphic image of J/K and $R_{\beta-1}(M)$ is homomorphically closed by Lemma 1.2. Now P generates an ideal Q of R and $Q \in R_{\beta-1}(M)$ by the inductive hypothesis. Consider J'/Q; we claim that this is the ideal of R/Q generated by $(J + Q)/Q$. For suppose S/Q is an ideal of R/Q containing $(J + Q)/Q$; then S contains J and hence J', so that S/Q contains J'/Q. Conversely, J'/Q is an ideal of R/Q. We have $(J + Q)/Q \cong J/(J \cap Q)$ and since $P \subseteq J \cap Q$, $J/(J \cap Q)$ is a homomorphic image of $J/P \in R_{\beta-1}(M)$. Since $R_{\beta-1}(M)$ is homomorphically closed, $(J + Q)/Q \in R_{\beta-1}(M)$; hence, $J'/Q \in R_{\beta-1}(M)$ by property (p), since $(J + Q)/Q$ is an ideal of the right ideal $(I + Q)/Q$ of R/Q. Since Q, $J'/Q \in R_{\beta-1}(M)$, we have $J' \in R_\beta(M)$, and the theorem now follows by transfinite induction.

<u>Definition 4.1</u>. A radical class $P \subseteq W$ is called <u>strongly right heredi-</u>

tary if for each right ideal I of each ring $R \in W$ we have $P(I) = I \cap P(R)$.

Theorem 4.2. A right hereditary radical class $P \subseteq W$ is strongly right hereditary if and only if it has property (p).

Proof. Suppose P is strongly right hereditary. Let $J \in P$ be an ideal of a right ideal I of a ring $R \in W$ and let J' be the ideal of R generated by J. Now $J \subseteq P(I) = I \cap P(R)$ so that $J \subseteq P(R)$. Thus $J \subseteq J' \cap P(R)$, which is an ideal of R. Since J' is the ideal of R generated by J, $J' = J' \cap P(R) = P(J')$. Conversely, suppose P has property (p) and let I be a right ideal of $R \in W$. If $J \in P$ is an ideal of I, then $J' \in P$ by (p). Thus $J' \subseteq P(R)$ so that $J \subseteq P(R)$; in particular, $P(I) \subseteq I \cap P(R)$. On the other hand, by right hereditary, $I \cap P(R)$ is a P-ideal of I, so that $I \cap P(R) \subseteq P(I)$. Hence $P(I) = I \cap P(R)$.

Lemma 4.3. Let P be any radical class in W. Then SP is right hereditary if and only if for each $R \in W$ and each right ideal I of R we have $P(I) \subseteq P(R)$.

Proof. If SP is right hereditary, $R \in W$ and I is a right ideal of R, then suppose $P(I) \not\subseteq P(R)$. Then $0 \neq [P(I) + P(R)]/P(R) \cong P(I)/P(I) \cap P(R) \in P$ since P is homomorphically closed. Now $[P(I) + P(R)]/P(R)$ is a right ideal of $R/P(R) \in SP$. This is a contradiction since $SP \cap P = 0$. For the converse, suppose that for each $R \in W$ and each right ideal I of R we have $P(I) \subseteq P(R)$. If $S \in SP$ then for any right ideal J of S, $P(J) \subseteq P(S) = 0$. Hence SP is right hereditary.

Lemma 4.4. Let P be any radical class. Then P is strongly right hereditary if and only if both P and SP are right hereditary.

Proof. If P is strongly right hereditary, then $P(J) = I \cap P(R)$ for each right ideal I of each ring $R \in W$, so that both P and SP are right hereditary. Conversely, suppose both P and SP are right hereditary and let I be a right ideal of R. By Lemma 4.3, $P(I) \subseteq I \cap P(R)$. Since $P(R) \in P$ and P is right hereditary, $I \cap P(R) \in P$. Since $I \cap P(R)$ is a P-ideal of I, $I \cap P(R) \subseteq P(I)$. Thus $I \cap P(R) = P(I)$.

Lemma 4.5. If P has property (p), then SP is right hereditary.

Proof. If $R \in SP$ has a right ideal $I \notin SP$, then I has an ideal $0 \neq J \in P$. Then the ideal J' of R generated by J is in P by property (p), so we have a contradiction.

Lemma 4.6. If P is a right hereditary radical class, then P satisfies property (p) if and only if SP is right hereditary.

Proof. This follows from Lemma 4.5, Lemma 4.4 and Theorem 4.2.

We note that the example given below Theorem 3.3 also shows that right heredity does not imply strong right heredity in the associative case.

Proposition 4.2. If P is a strongly right hereditary radical class, then for each $R \in W$ and each $I \in G(\{R\})$ (see Lemma 4.1), we have $P(I) = I \cap P(R)$.

Proof. If this result holds for all $I \in G_{j-1}(\{R\})$ with $j > 1$, let $J \in G_j(\{R\}) = G_1(G_{j-1}(\{R\}))$. Then J is a right ideal of some $I \in G_{j-1}(\{R\})$, so $P(J) = J \cap P(I) = J \cap I \cap P(R) = J \cap P(R)$. The result follows by induction.

Theorem 4.3. Let M be a class of rings satisfying property (p). For all $R \in W$, if $I \in M$ is in $G(\{R\})$, then the ideal I' of R generated by I is also in M.

Proof. Let $R \in W$ and let $I \in M \cap G_2(\{R\})$. Then I is a right ideal of a right ideal J of R. Let I^* be the ideal of J generated by I, so that $I^* \in M$ by property (p). Now let $I^{*\prime}$ be the ideal of R generated by I^*;

then $I^{*'}$ is also in M by property (p). Now $I^{*'}$ is an ideal of R containing I, but clearly also any ideal of R containing I must contain I^* and hence $I^{*'}$. Thus $I' = I^{*'}$ is in M.

Now assume that for all $R \in W$ and all $I \in M \cap G_n(\{R\})$ $(n \geq 2)$ that $I' \in M$, where I' is the ideal of R generated by I. Suppose $K \in M \cap G_{n+1}(\{R\})$ for some $R \in W$. Then K is in $M \cap G_2(\{J\})$ for some $J \in G_{n-1}(\{R\})$ and, if K^* is the ideal of J generated by K, $K^* \in M$ by the inductive hypothesis. Now $K^* \in M \cap G_n(\{R\})$ so that if $K^{*'}$ is the ideal of R generated by K^*, then $K^{*'} \in M$. But, as before, $K^{*'} = K'$, so that $K' \in M$.

Corollary 4.1. Property (p) is equivalent to the following property (p').

(p') If $J \in M$ is a right ideal of a right ideal I of $R \in W$, then the ideal J' of R generated by J is also in M.

Corollary 4.2. Let P be a radical class which contains all the zero simple rings and whose semisimple class contains all matrix rings over division rings, i.e., $\mathcal{J} \leq P \leq \mathcal{T}$ in the notation of [11, page 40]; then P cannot have property (p).

Proof. Let F be any of the fields $Z/(p)$, p prime. Let R be the 2×2 matrix ring over F. Then R has the right ideal $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a,b \in F\right\}$ which in turn has the right ideal $I = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F\right\}$ which is a simple zero ring with p elements and hence is in P. Now $I' = R \notin P$, so that P cannot satisfy property (p).

This seems to be related to an unsolved problem of Koethe [16]. We see from Corollary 4.2 that in the special case of the nil radical class, there must exist a right ideal I of a ring R and a nil ideal J of I such that the ideal J' of R generated by J is not nil. Koethe has asked whether a nil right ideal of a ring must generate a nil two-sided ideal. The following property (q) would seem to be of interest in this connection:

(q) If $J \in M$ is a right ideal of R, then the ideal J' of R generated by J is also in M.

We have been unable to determine whether this property will also hold for LM.

For any class M, let $E(M) = \{J' : J$ is an ideal of a right ideal I of a ring $R \in W$, $J \in M$, and J' is the ideal of R generated by $J\}$.

Theorem 4.4. If W is an osi-universal class and $M \subseteq W$, then there exists a unique minimal radical class in W containing M which satisfies property (p).

Proof. Let $M_1 = EH(M)$ and, proceeding inductively, $M_n = EH(M_{n-1})$ for all $n > 1$. Clearly $m \leq k$ implies $M_m \subseteq M_k$. Let $M^* = \cup M_n$, where the union is taken over all positive integers n; then $H(M^*) = M^*$. If J is an ideal of a right ideal I of $R \in W$ and $J \in M^*$, then $J \in M_n$ for some n. Hence $J \in H(M_n)$ so that $J' \in EH(M_n) = M_{n+1} \subseteq M^*$, where J' is the ideal of R generated by J'. Thus M^* is a homomorphically closed class satisfying property (p), so that LM^* has property (p) by Theorem 4.1. Now if A is any homomorphically closed class containing M and satisfying property (p), then $H(M) \subseteq A$ and $EH(M) = M_1 \subseteq A$. If $M_n \subseteq A$, then $H(M_n) \subseteq A$ since A is homomorphically closed and $EH(M_n) \subseteq A$ since A satisfies property (p). By induction $M^* \subseteq A$. If A is a radical class, then $LM^* \subseteq A$. This completes the proof.

For the moment let $L^*_W(M)$ denote the minimal radical class in W containing M which satisfies property (p). Then as in the remark preceding

Example 3.3 we can assert that if X is a second osi-universal class, then $X \cap L^*_W(M) \neq L^*_{X\cap W}(X \cap M)$ in general, although we do have $L^*_{X\cap W}(X \cap M) \subseteq X \cap L^*_W(M)$.

We now construct a minimal strongly right hereditary radical class containing a class M. Let G(C) denote the minimal right hereditary class containing C introduced in Lemma 4.1.

Lemma 4.7. Let W be an osi-universal class and let $M \subseteq W$. Then there is a unique minimal class M' containing M which is homomorphically closed, right hereditary and satisfying property (p).

Proof. Define $M_1 = EGH(M)$ and, inductively, $M_n = EGH(M_{n-1})$ for all $n > 1$. Let $M' = \cup M_n$. Then the lemma may be established by an induction argument as in Theorem 4.3.

Theorem 4.5. If M is any class of rings in W, then LM' is the unique minimal strongly right hereditary radical class containing M.

Proof. Since $M \subseteq M'$, $M \subseteq LM'$. Since M' is homomorphically closed, right hereditary and satisfies property (p), LM' has the same properties by Theorem 2.1 and Theorem 4.1. Therefore LM' is strongly right hereditary by Theorem 4.2. If $M \subseteq P$ where P is a strongly right hereditary radical class, then P is homomorphically closed and right hereditary and hence has property (p). Thus $M' \subseteq P$ and so $LM' \subseteq P$.

We now turn to the problem of characterizing the semisimple classes of radical classes having some of the properties mentioned in this chapter.

Theorem 4.6. Q is a semisimple class for a radical class P with property (p) if and only if Q has the following four properties:

(1) Q is right hereditary.

(2) Q is closed under subdirect sums.

(3) Q is extension closed.

(4) If I is an ideal of R and $0 \neq I/B \in Q$ for some ideal B of I, then there is an ideal A of R with $A \subseteq I$ and $0 \neq I/A \in Q$.

Proof. Suppose that Q is a semisimple class for a radical class P satisfying property (p). Then SP = Q is right hereditary by Lemma 4.5. Properties (2) and (3) follow as in [9]. If I is an ideal of R and $0 \neq I/B \in Q$ for some ideal B of I, then $P(I) \neq I$. Let A be the ideal of R generated by P(I); then $A \in P$ by property (p) and $A \subseteq I$ so that $A = P(I)$. Hence $0 \neq I/A \in Q$.

Conversely, again as in [9], Q is a semisimple class for some radical class P. Suppose $J \in P$ is an ideal of a right ideal I of R, let J' be the ideal of R generated by J and suppose $J' \in P$. Then $J'/P(J') \in Q$. By (4), there is an ideal A of R with $A \subseteq J'$ such that $0 \neq J'/A \in Q$. If $J \not\subseteq A$, $(J + A)/A \cong J/(J \cap A) \neq 0$, and $J/(J \cap A) \in P$ since P is homomorphically closed, so that we have a contradiction. Thus $J \subseteq A$, so that again we have a contradiction, for J' is the ideal of R generated by J, while $J' \neq A$.

Proposition 4.3. Let Q be a semisimple class for a strongly right hereditary radical P. Let I be a right ideal of $R \in W$ and suppose $0 \neq I/B \in Q$. Then there exists a right ideal A of R which is also an ideal of I such that $0 \neq I/A \in Q$.

Proof. $I \notin P$ so that $I/P(I) \in Q$, where $P(I) = I \cap P(R)$.

Theorem 4.7. Q is a semisimple class for a strongly right hereditary radical P = UQ if and only if Q satisfies, in addition to properties (1), (2), (3) and (4) of Theorem 4.6, the following property (5).

(5) If I is a right ideal of R and $I \cap A \neq 0$ for every nonzero ideal A of R, then $I \notin P$ implies $R \notin P$.

Proof. Suppose Q is a semisimple class for a strongly right hereditary radical P. Then P satisfies property (p) by Theorem 4.2 and so Q satisfies (1), (2), (3) and (4) by Theorem 4.6. Now if I is a right ideal of $R \in W$ such that $I \cap A \neq 0$ for every nonzero ideal A of R, and $I \notin P$, we have $P(I) = I \cap P(R) \neq I$. Thus $I \not\subseteq P(R)$ so that $P(R) \neq R$, whence (5).

On the other hand, suppose Q satisfies (1), (2), (3), (4) and (5). Then Q is a semisimple class for a radical class P satisfying (p) by Theorem 4.6. Suppose P is not right hereditary, so that there must exist a right ideal $I \notin P$ of some P-ring $R \in W$. Let $H = \{A : A \text{ is an ideal of } R \text{ and } A \cap I = 0\}$. Then $0 \in H$ and so $H \neq \emptyset$. By Zorn's Lemma, H has a maximal element M. In $R/M \neq 0$ we have $(I + M)/M \cong I \notin P$. If $A/M \neq 0$ is an ideal of $R/M \neq 0$ then $M \subseteq A$ and $M \neq A$, so that by the maximality of M, $A \cap I \neq 0$ in R. Thus $(A/M) \cap (I + M)/M \neq 0$. Hence, by (5), $R/M \notin P$. But P is a homomorphically closed class, so that this contradicts $R \in P$. Therefore P is right hereditary; by Theorem 4.2, it is strongly right hereditary.

Example 4.1. As before we let G(M) denote the minimal right hereditary class containing the class M. $\mathcal{J}$ will denote the Jacobson radical class; we work entirely within the class W of associative rings.

Define the class Q as follows: $R \in Q$ if and only if $G(\{R\}) \cap \mathcal{J} = 0$. Then Q is clearly right hereditary.

Suppose $R/B_i \in Q$ with $\bigcap_{i \in I} B_i = 0$ where I is an index set. Suppose $K \in G(\{R\}) \cap \mathcal{J}$. Then, for each $i \in I$, $(K + B_i)/B_i \cong K/K \cap B_i \in \mathcal{J}$ since $\mathcal{J}$ is homomorphically closed. Thus $(K + B_i)/B_i \in G(\{R/B_i\}) \cap \mathcal{J} = 0$ and so, for all $i \in I$, $K \subseteq B_i$. Thus $K \subseteq \cap B_i = 0$. Therefore Q is closed under the taking of subdirect sums.

Suppose $J, R/J \in Q$. Suppose $I \in G(\{R\}) \cap \mathcal{J}$. Then $(I + J)/J \cong I/(I \cap J) \in \mathcal{J} \cap G(\{R/J\}) = 0$, so $I \subseteq J$. But then $I \in G(\{J\}) \cap \mathcal{J} = 0$ so that $I = 0$. Hence Q is extension closed.

To show property (4) of Theorem 4.6, suppose I is an ideal of R with $0 \neq I/B \in Q$, B an ideal of I. Let B' be the ideal of R generated by B, so that $B' \subseteq I$. Then $B'^3 \subseteq B$ by Lemma 1.4, so that B'/B is nilpotent. Thus $B'/B \in G(\{I/B\}) \cap \mathcal{J} = 0$, so $B' = B$. Thus B is an ideal of R.

Thus Q is the semisimple class for a radical class P with property (p). We claim $P \neq \mathcal{J}$, for $S\mathcal{J}$ does not have property (1). To see this, let R be the 2×2 matrix ring over the field F of two elements, so that $R \in S\mathcal{J}$. R has the right ideal $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a,b \in F\right\}$ which, in turn, has the right ideal $\left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in F\right\}$ which, being nilpotent, is in $\mathcal{J}$. $S\mathcal{J}$ does however have properties (2), (3) and (4).

If Q' is a semisimple class for a radical class P which has property (p) and contains $\mathcal{J}$, then we wish to show that $Q' \subseteq Q$. Let $R \in Q'$. Then as Q' must be right hereditary, $G(\{R\}) \subseteq Q'$. Now $Q' \cap \mathcal{J} = 0$ as $P \supseteq \mathcal{J}$ and $Q \cap P = 0$, so that $G(\{R\}) \cap \mathcal{J} = 0$. Hence $R \in Q$. Therefore UQ is the minimal radical class containing $\mathcal{J}$ which has property (p). We remark that UQ contains all $n \times n$ matrix rings over fields for $n \geq 2$ but contains no fields, so that the situation in the classical case is not so drastic as in Chapter 3.

We may abstract from this example the following:

<u>Proposition 4.4</u>. Let W be the class of associative rings and let $Q \subseteq W$

be any class satisfying properties (1), (2) and (3) of Theorem 4.6 such that $Q \cap Z = 0$, where Z is the class of zero rings. Then Q is a semisimple class for a radical class satisfying property (p).

<u>Proof</u>. We need only show (4) of Theorem 4.6. If I is an ideal of R and $0 \neq I/B \in Q$, where B is an ideal of I, let B' denote the ideal of R generated by B. Then $B'^3 \subseteq B$ by Lemma 1.4 so that $B'/B \in Q$ is nilpotent. Thus $B' = B$, whence B is an ideal of R.

This is related to a result of S. E. Dickson [9, Theorem 4.2] to the effect that a class Q of associative rings is a semisimple class for a radical class which contains all nilpotent rings if and only if $Q \cap Z = 0$ and Q has properties (1), (2) and (3) of Theorem 1.5.

<u>Example 4.2</u>. Let $\mathcal{N}$ be the class of nil rings in the class of associative rings. Define Q_1 by saying that $R \in Q_1$ if and only if $G(\{R\}) \cap \mathcal{N} = 0$. As in the discussion involving the Jacobson radical, Q_1 is the semisimple class such that UQ_1 is the minimal radical class satisfying property (p) and containing $\mathcal{N}$. Let R be the ring of rational numbers of the form $2x/(2y + 1)$, x, y integers (cf. [11, p. 103]). Then $G(\{R\}) \cap \mathcal{N} = 0$ so that $R \in Q_1$; however, R is Jacobson radical so that $R \in UQ$. Clearly $R \in Q$ implies $R \in Q_1$, so that we have shown $UQ_1 < UQ$.

We now turn to the consideration of a radical class obtained by modifying the extension-union construction. For any class M contained in an osi-universal class W, we construct a class $R^0(M)$ as follows. Let $R_1{}^0(M)$ be the homomorphic closure of M. We proceed inductively to define a class $R_\beta{}^0(M)$ for each ordinal number β. If $\beta - 1$ exists, let $R_\beta{}^0(M) = \{R \in W :$ R has an ideal J such that $J, R/J \in R_{\beta-1}{}^0(M)\}$. If β is a limit ordinal, we define $R \in R_\beta{}^0(M)$ if and only if R is the union of a chain $\{I_\gamma\}$ of <u>right</u>

ideals of R such that $I_\gamma \in \bigcup_{\alpha<\beta} R_\alpha^0(M)$. Finally, let $R^0(M) = \bigcup_\beta R_\beta^0(M)$.

Lemma 4.8. Each class $R_\beta^0(M)$ is homomorphically closed. Hence $R^0(M)$ is homomorphically closed.

Proof. $R_1^0(M)$ is homomorphically closed by definition. Let $\beta > 1$ be an ordinal number and suppose $R_\alpha^0(M)$ is homomorphically closed for all $\alpha < \beta$. Let $R \in R_\beta^0(M)$ and let I be an ideal of R. If β is a limit ordinal, there is a chain $\{I_\gamma\}$ of right ideals of R such that each I_γ belongs to $\bigcup_{\alpha<\beta} R_\alpha^0(M)$ and $R = \bigcup I_\gamma$. Now R/I is the union of the chain $\{(I + I_\gamma)/I\}$ of its right ideals; furthermore, each $(I + I_\gamma)/I \cong I_\gamma/(I \cap I_\gamma)$ so that, by the inductive hypothesis, each $(I + I_\gamma)/I \in \bigcup_{\alpha<\beta} R_\alpha^0(M)$. Therefore $R/I \in R_\beta^0(M)$.

If $\beta - 1$ exists, then R contains an ideal J such that $J, R/J \in R_{\beta-1}^0(M)$. By the inductive hypothesis, $(J + I)/I$ and $R/(I + J)$ both belong to $R_{\beta-1}^0(M)$ since the former is isomorphic to $J/(I \cap J)$ and the latter is a homomorphic image of R/J. Since $[R/I)/[(J + I)/I] \cong R/(J + I)$, we have $R/I \in R_\beta^0(M)$. Thus by transfinite induction $R_\beta^0(A)$ is homomorphically closed for all β.

Corollary 4.3. $R^0(M)$ is a radical class.

Proof. This follows readily from Lemma 4.8 and Theorem 1.2.

Theorem 4.8. If M is homomorphically closed and has property (p), then $LM = R^0(M)$.

Proof. By definition $R_\beta(M) \subseteq R_\beta^0(M)$ for each ordinal β, where $R_\beta(M)$ is the class obtained at ordinal number β in the extension-union construction of LM. Note that $R_1(M) = R_1^0(M)$. Suppose that $R_\alpha^0(M) \subseteq R_\alpha(M)$ for all $\alpha < \beta$ and let $R \in R_\beta^0(M)$. If $\beta - 1$ exists, then there exists an ideal J of R such that $J, R/J \in R_{\beta-1}^0(M) \subseteq R_{\beta-1}(M)$; hence, $R \in R_\beta(M)$. If β is a limit ordinal, R is the union of a chain $\{I_\gamma\}$ of right ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha^{\,0}(M)$. Let I_γ' be the ideal of R generated by I. We have shown in the proof of Theorem 4.1 that each class $R_\alpha(M)$ has property (p), so that each $I_\gamma' \in \bigcup_{\alpha<\beta} R_\alpha(M)$; moreover, $\{I_\gamma'\}$ is a chain of ideals of R. Hence $R = \bigcup I_\gamma' \in R_\beta(M)$. Thus $R_\beta^{\,0}(M) = R_\beta(M)$ for all β, and therefore $LM = R^0(M)$.

Property (p) cannot be omitted from the statement of Theorem 4.8. To see this, let Z be the ring of linear transformations of finite rank of a vector space V of countable dimension over a division ring D; this ring is discussed in [11, example 11]. Z is, it is important to remark, a simple ring. If we represent the elements of Z by infinite matrices, these are the matrices which have only a finite number of columns with non-zero entries. Let L_n denote the left ideal of Z consisting of all elements of Z represented by matrices with non-zero entries in at most the first n columns. Then $Z = \bigcup L_n$. Since we have been working with right ideals, we will consider a ring Z^* anti-isomorphic to Z. Under the anti-isomorphism, each left-ideal L_n is mapped anti-isomorphically onto a right ideal L_n^* of Z^*, and $Z^* = \bigcup L_n^*$. Let $M = \{L_n^*\}$. Then Z^*, being a simple ring and failing to be in the homomorphic closure M_1 of M, is not in LM by Theorem 1.9; however, Z^* is in $R_\omega^{\,0}(M_1) \subseteq R^0(M)$.

We may also remark that the condition that M have property (p) is not a necessary condition that $LM = R^0(M)$. For let M be the Jacobson radical class, so that $LM = M$. On the other hand, the union of any chain of Jacobson radical rings is still Jacobson radical, so $R^0(M) = M$. We have observed in Example 4.1 that M does not have property (p).

The position of $R^0(M)$ in the radical scheme of things may be clarified by noticing that if P is a radical class containing a class M such that, for all $R \in W$, every ideal I of R which is a sum of P-right ideals of R is contained in P(R), then $R^0(M) \subseteq P$. We may demonstrate this using transfinite induction; suppose to this end that $R_\alpha{}^0(M) \subseteq P \ \forall \alpha < \beta$ and let $R \in R_\beta{}^0(M)$. If $\beta - 1$ exists, there is an ideal J of R such that J, $R/J \in R_{\beta-1}{}^0(M) \subseteq P$. Since P is a radical class, it is closed under factor extensions, so that $R \in P$. If β is a limit ordinal, then R is the union of a chain $\{I_\gamma\}$ of its right ideals contained in $\bigcup_{\alpha<\beta} R_\alpha{}^0(M) \subseteq P$, so that R is a sum of P-right ideals. Hence $R \in P$. Unfortunately, this does not characterize $R^0(M)$, for let R be a 2×2 matrix ring over a field F and let M be the set of proper right ideals (i.e., "rows") of R. Since M is simple and is not the union of a chain of its right ideals, $R \notin R^0(M)$, although R is the sum of its right ideals. In fact, $R \in SR^0(M)$, which demonstrates that even in the associative case $SR^0(M)$ need not be right hereditary.

Nor for that matter need $R^0(M)$ be right hereditary. Let M consist of the single simple 2×2 matrix ring R over a field F. Then the right ideals of R are clearly not in $R^0(M)$.

We omit the proof of the following propositions, as we need only follow the corresponding proofs in Chapter 2.

Proposition 4.5. If M is hereditary [right hereditary, left hereditary, subring hereditary], then so is $R^0(M)$.

Proposition 4.6. If M is a hereditary class of hereditarily idempotent rings, then so is $R^0(M)$.

Proposition 4.7. If M is homomorphically closed and satisfies property (a) of Theorem 1.10, then so does $R^0(M)$.

We may, however, illustrate the technique of proof with the following:

Proposition 4.8. Suppose M is homomorphically closed and that no ring in M contains a nonzero idempotent. Then no ring in $R^0(M)$ contains a non-zero idempotent.

Proof. By hypothesis, no ring in $R_1{}^0(M)$ contains a non-zero idempotent; suppose this is true of $R_\alpha{}^0(M)$ for all $\alpha < \beta$ and let $R \in R_\alpha{}^0(M)$. Let e be an idempotent in R. If $\beta - 1$ exists, there is an ideal J of R such that R/J, $J \in R_{\beta-1}{}^0(M)$. Then $e + J$ is an idempotent element of R/J, so that $e + J = J$, that is, $e \in J$. But since e is an idempotent of J, $e = 0$. If β is a limit ordinal, then $R = \cup I_\gamma$, where $\{I_\gamma\}$ is a chain of right ideals of R contained in $\bigcup_{\alpha<\beta} R_\alpha{}^0(M)$. Thus $e \in I_\gamma$ for some γ so that $e = 0$.

Remark. If $M \subseteq N$, then $R^0(M) \subseteq R^0(N)$.

Proposition 4.9. $R^0(M) = R^0(LM)$.

Proof. Since $M \subseteq LM$, $R^0(M) \subseteq R^0(LM)$ by the remark. On the other hand, since $R^0(M)$ is a radical class containing M, $LM \subseteq R^0(M)$ so that $R^0(LM) \subseteq R^0(R^0(M)) = R^0(M)$.

We may construct a radical class in the same way using left ideals rather than right ideals. Explicitly, we define $L_1{}^0(M)$ to be the homomorphic closure of M. If $\beta - 1$ exists, we define $L_\beta{}^0(M) = \{R :$ there exists an ideal I of R such that I, $R/I \in L_{\beta-1}{}^0(M)\}$. If β is a limit ordinal, let $R \in L_\beta{}^0(M)$ if and only if R is the union of a chain of left ideals of R contained in $\bigcup_{\alpha<\beta} L_\alpha{}^0(M)$. Finally, we define $L^0(M) = \bigcup_\beta L_\beta{}^0(M)$.

We now present an example to show that even in the associative case $L^0(M)$ need not be equal to $R^0(M)$. Let R be the associative algebra over the field of rational numbers generated by a countable number of symbols $\{x_i : i = 1,2,\ldots\}$ subject to the relations $x_i x_j = x_j$ for all natural numbers i, j. Then R has no proper right ideals. For each natural number n let I_n be the left ideal $\left\{\sum_{i=1}^{n} \alpha_i x_i : \alpha_i \text{ rational}\right\}$. Then $\{I_n\}$ is a chain of left ideals of R and $R = \bigcup I_n$. Let $M = \{I_n\}$. Then $R \in L_\omega^0(M)$, but since R has no proper right ideals, $R \notin R^0(M)$.

BIBLIOGRAPHY

1. Amitsur, S. A., A general theory of radicals, I. Radicals in complete lattices, Am. J. Math. 74 (1952), 774-786.

2. Amitsur, S. A., A general theory of radicals, II. Radicals in rings and bicategories, Am. J. Math. 76 (1954), 100-125.

3. Amitsur, S. A., A general theory of radicals, III. Applications, Am. J. Math. 76 (1954), 126-136.

4. Amitsur, S. A., The identities of PI-rings, Proc. Amer. Math. Soc. 4 (1953), 27-34.

5. Anderson, T., Divinsky, N., and Sulinski, A., Hereditary radicals in associative and alternative rings, Can. Jour. Math. 17 (1965), 594-603.

6. Anderson, T., Divinsky, N., and Sulinski, A., Lower radical properties for associative and alternative rings, J. London Math. Soc. 41 (1966), 417-424.

7. Andrunakievic, V. A., Radicals of associative rings, I (Russian), Mat. Sb. 44 (1958), 179-212.

8. Andrunakievic, V. A., Radicals of associative rings, II (Russian), Mat. Sb. 55 (1961), 329-346.

9. Armendariz, E. P., Closure properties in radical theory, Pacific J. Math. 26 (1968), 1-7.

10. Armendariz, E. P., and Leavitt, W. G., Nonhereditary semisimple classes, Proc. Amer. Math. Soc. 18 (1967), 1114-1117.

11. Divinsky, N. J., Rings and Radicals. London: George Allen and Unwin, 1965.

12. Divinsky, N., Krempa, J., and Sulinski, A., Strong radical properties of alternative and associative rings, J. Algebra 17 (1971), 369-388.

13. Hoffman, A. E., The Constructions of the General Theory of Radicals, Ph. D. Thesis, University of Nebraska, 1966.

14. Hoffman, A. E. and Leavitt, W. G., Properties inherited by the lower radical, Port. Math. 27 (1968), 63-66.

15. Jenkins, T., and Kreiling, D., Semisimple classes and upper-type radical classes of narings, Proc. Amer. Math. Soc. 24 (1970), 378-382.

16. Koethe, G., Schiefkörper unendlichen Ranges über den Zentrum, Math. Ann. 105 (1931), 15-39.

17. Kreiling, D., and Tangeman, R., Lower radicals in non-associative rings, J. Austral. Math. Soc. (to appear).

18. Kurosh, A. G., Radicals of rings and algebras (Russian), Mat. Sb. 33 (1953), 13-26.

19. Leavitt, W. G., Strongly hereditary radicals, Proc. Amer. Math. Soc. 21 (1969), 703-705.

20. Leavitt, W. G., and Lee, Y. L., A radical coinciding with the lower radical in associative and alternative rings, Pacific J. Math. 30 (1969), 459-461.

21. Lee, Y. L., On the construction of lower radical properties, Pacific J. Math. 28 (1969), 393-395.

22. Lee, Y. L., On the construction of upper radical properties, Proc. Amer. Math. Soc. 19 (1968), 1165-1166.

23. McCoy, N. H., The Theory of Rings. New York: Macmillan, 1964.

24. Szele, T., Nilpotent artinian rings, Publ. Math. Debrecen 4 (1955), 71-78.