THE USE OF SYMBOLIC LOGIC AS A TOOL IN THE TEACHING OF GEOMETRY IN THE SECONDARY SCHOOLS

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PREFACE

This paper is expository in nature and presents teaching techniques developed and used by the writer in a secondary school classroom. It is written primarily for teachers of geometry. The main objective is to demonstrate how the mathematics teacher can incorporate elements of symbolic logic into a deductive Euclidean geometry course in such a manner that (1) much of the traditional formal proof approach can be eliminated without sacrificing rigor and (2) the student obtains discovery techniques that will allow him to anticipate and, in some cases, establish the validity of many geometric theorems before they are introduced in the textbook. The writer has experienced success with techniques that will be described, and he presents them in this paper in the hope that other secondary school geometry teachers will find them useful. The paper has been prepared under the assumption that the reader's exposure to symbolic logic has been minimal.

The writer wishes to express his appreciation to his major advisor, Dr. Vernon Troxel, for his assistance and guidance throughout the preparation of this paper. Gratitude is also extended to other committee members--Dr. Douglas B. Aichele, Dr. Gerald K. Goff, and Dr. Thomas Johnsten--for their efforts in providing comments and suggestions while the paper was being prepared.

Special gratitude is certainly due to my wife, Barbara, and our son, Jeff, for their encouragement and sacrifices throughout the 1971-72 school year. An equal amount of gratitude is also extended to my parents, Mr. and Mrs. Ephraim K. Smith, and to my mother-in-law, Mrs. George P. McCallum, all of whom have been extremely helpful in many ways.

Finally, a note of thanks to my employer, Mr. Frederick F. Clark, and to the trustees of The Cate School, Carpinteria, California, for their generous sabbatical year policy that has allowed me to pursue graduate study.

TABLE OF CONTENTS

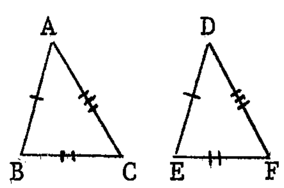
LIST OF TABLES

Table		Page

LIST OF SYMBOLS

$=$	is equal to
$\neq$	is not equal to
$\{a, b, c\}$	the set whose elements are a, b, c
$\emptyset$	empty set
$\cap$	intersection
$\cup$	union
$\subset$	is a subset of
$\not\subset$	is not a subset of
$<$	is less than
$>$	is greater than
$\leq$	is equal to or less than
$\geq$	is equal to or greater than
$\|x\|$	the absolute value of x
AB	the length of the segment joining points A and B
$\overline{AB}$	the segment with end points A and B
$\overleftrightarrow{AB}$	the line containing points A and B
$\overrightarrow{AB}$	the ray with end points A and containing B
$\angle ABC$	angle ABC; the union of the noncollinear rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$
$m\angle ABC$	the measure of $\angle ABC$
$\widehat{AB}$	arc AB; the circular arc with end points A and B
$m\,\widehat{AB}$	the measure of $\widehat{AB}$
$\overline{AB} > \overline{CD}$	$AB > CD$

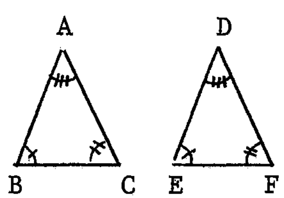
$\angle ABC > \angle DEF$	$m\angle ABC > m\angle DEF$
$\triangle ABC$	the triangle with vertices A, B, and C
$\square ABCD$	the quadrilateral with vertices A, B, C, and D
$\perp$	is perpendicular to
$\not\perp$	is not perpendicular to
$\cong$	is congruent to
$\not\cong$	is not congruent to
$\parallel$	is parallel to
$\not\parallel$	is not parallel to
SAS	The SAS Postulate: Two triangles are congruent if and only if two sides and an included angle of one triangle are congruent to the corresponding parts of the other triangle.
ASA	The ASA Postulate: Two triangles are congruent if and only if two angles and an included side of one triangle are congruent to the corresponding parts of the other triangle.
SSS	The SSS Postulate: Two triangles are congruent if and only if three sides of one triangle are congruent to the corresponding parts of the other triangle.



$AB = DE$ or $\overline{AB} \cong \overline{DE}$

$BC = EF$ or $\overline{BC} \cong \overline{EF}$

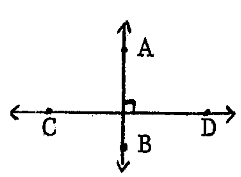
$AC = DF$ or $\overline{AC} \cong \overline{DF}$



$m\angle B = m\angle E$ or $\angle B \cong \angle E$

$m\angle C = m\angle F$ or $\angle C \cong \angle F$

$m\angle A = m\angle D$ or $\angle A \cong \angle D$



$\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$

CHAPTER I

INTRODUCTION

The decade of the sixties was a revolutionary period in mathematics education. Mathematics educators, confronted with the modern theories of educational philosophy and psychology, faced up to the reality that many of their teaching methods, as well as much of their subject matter content, were antediluvian. The historic Cambridge Conference of 1963 and groups such as the Committee on the Undergraduate Program in Mathematics (CUPM), the School Mathematics Study Group (SMSG), the University of Illinois Committee on School Mathematics (UICSM), and others suggested and in many instances implemented many reforms that led to considerable improvement in the quantity and quality of mathematics education.

Despite the numerous improvements there remain unresolved issues. Irving Adler (1), among others, has stated that curriculum revision during the sixties was least successful in secondary school geometry. Despite the fact that reforms suggested by SMSG were almost universally adopted, Adler expressed the feeling that rigor was overemphasized, intuition was neglected, and too much emphasis on proof stood in the way of understanding.

Even as this manuscript is being prepared, diverse opinions about the role of geometry are appearing in print. At one extreme one finds those (15) who feel that there should be a year of work that is primarily geometric, and at the other those (7) who wish to eliminate geometry as a separate course. Another point of view is that geometry should be completely revised so that, by using transformations, it can be integrated with other branches of mathematics (5).

A major point in the arguments of those who seek changes in the geometry curriculum is that students are now being introduced to elements of geometry on an intuitive basis during elementary and junior high school. Hence, the common geometric figures and their basic properties are fairly familiar to the high school sophmore. It would appear, therefore, that high school geometry is an ideal place for the student to experience thoroughly the deductive process of reasoning. His previous exposure to this process has been minimal and to continue on a purely intuitive basis would deny him the opportunity to develop proficiency with this important and extremely useful reasoning process.

A geometry course developed through a deductive process requires a certain amount of rigor. Indeed, the act of deducing a necessary conclusion as a logical consequence of other statements is a rigorous exercise when compared to inducing a probable conclusion from observation or experimentation. With the exception of those who might suggest that the content of sophomore geometry be presented on a purely intuitive basis, none of the diversified opinions concerning the geometry curriculum suggest that the deductive nature of geometry be abandoned. Eccles (5), who desires that geometry be taught using transformations, would retain a substantial block of traditional deductive geometry. When transformations are introduced after perpendicularity, parallelism, and triangle congruence have been covered, they are to be developed deductively. Fehr (7), who wants to discard the present year-long geometry course, would integrate geometry with other courses and have it included in every year beginning in grade seven and continued through grade twelve. His program calls for the developing of the concept of proof and for the proving of theorems. It would appear that strong support exists for retaining the deductive approach to geometry and preserving a degree of rigor. Keeping Adler's views in mind, the question arises as to how a teacher can present geometry in a rigorous manner without overemphasizing rigor and neglecting intuition. The writer directs himself to this question in this paper.

The writer will offer the secondary school geometry teacher some insights into possible uses of symbolic logic in the teaching of geometry. The writer feels that use of some or all of the concepts discussed in this paper may be beneficial in helping students obtain a more complete understanding of the nature of deductive reasoning than would be obtained by strict adherence to material in a textbook. The rigor that is necessary for a deductive development remains. However, as will be shown, the rigor can be used to promote student discovery and intuition.

Following is a brief outline of the logical content of this paper. The concepts of statement, negation, truth table, conjunction, disjunction, conditional, biconditional, equivalence, and tautology are introduced in Chapter II. The teacher is shown how he may introduce and reinforce these concepts by relating them to definitions and postulates that appear in the early chapters of a geometry textbook and by utilizing the basic properties of common geometric figures that students have intuitively induced in earlier grades.

The valid argument forms that are used in a deductive geometric proof are introduced in Chapter III. The writer feels that examination of these argument forms may be beneficial to students as an aid in understanding the nature of deductive proof.

The traditional definitions of converse, inverse, and contrapositive of a statement are introduced in Chapters IV, V, and VI, respectively. Using these definitions as a guide, three new statement forms are constructed, defined to be (1) a partial converse, (2) a partial inverse, and (3) a partial contrapositive. Emphasis is placed on demonstrating many creative uses of these statement forms.

Indirect proof is discussed and used in most secondary school geometry textbooks. The aforementioned concepts of symbolic logic can be used to supplement what is usually a brief textbook presentation of indirect proof. This is the subject of discussion in Chapter VII.

Based on the writer's personal experience in a geometry classroom, he believes that the concepts of symbolic logic presented in this paper can be integrated into a geometry course without sacrificing time from geometric content. A suggested outline for this procedure is found in Chapter VIII.

It is interesting to note that opinions on the teaching of logic in secondary schools are as diverse as those concerning the geometry curriculum. Presented in the May, 1971, issue of <u>The Mathematics Teacher</u> are pro and con articles on the question "Should Mathematical Logic be Taught Formally in Mathematics Classes?" On the pro side, Exner argues that our present school mathematics sequence does not prepare a student very well in the matter of proving things and feels that the introduction of some formal logic "would replace some misguided formalism that is already present." (6, p. 396). On the con side, Hilton expresses a concern that too much logical rigor could be harmful to the student. However, he does feel that "certain logical techniques should be taught explicitly" (6, p. 390) and expands on this by saying:

> For example, the student should know how to negate a proposition involving universal and existential quantifiers and thus how to set about the search for a counterexample to such a proposition....(6, p. 390).

It should be noted that even the Report of the Cambridge Conference of 1963 sheds little light on the issue of teaching logic.

> It will have been observed in section 5, under Logic and Foundations, that the treatment of formal logic is very meagre. We do not know how thorough the treatment of logic should be. Since we do not propose to teach logic as a subject in its own right, the problem is pedagogic and hence pragmatic....(8, p. 47).

Despite an expressed uncertainity about the role of logic in the secondary school classroom, the authors of the 1963 Cambridge Conference Report exhibit a belief that an introduction to logic may well be justified if it can be beneficial and practical in the learning experience of any mathematics class. This writer thus feels that there is justification for using symbolic logic in sophomore geometry classes and in presenting the material contained in this paper. His experiences in the classroom have led him to believe that an exposure to symbolic logic has been a valuable learning experience for his geometry students and that a knowledge of the techniques used may possibly be beneficial to other geometry teachers.

The writer has attempted to reach a compromise position between Exner and Hilton. Though not teaching logic as a subject in itself, he introduces those logical concepts mentioned previously and relates them to material in the geometry textbook being used for the year of study. The writer hopes to demonstrate that his procedure, if carefully developed by the geometry teacher, preserves the rigor that is necessary for a deductive development of geometry without being harmful to students.

In preparing this paper, the writer has attempted to present material that will facilitate application of the techniques to almost any deductive development of geometric structure. This is not to say that the specific examples illustrated can be utilized in every geometry classroom. For instance, no examples are given that relate to transformations, since the writer has taken his examples from five modern textbooks, the structures of which are developed without transformations. The thought underlying this omission is that transformation geometry is not widely used at the present time and to interchange transformation and non-transformation examples would be confusing. However, though most of the examples in this paper are not directly applicable to transformation geometry, the techniques relating to the development of a deductive system certainly are. For that matter, the techniques could be utilized in other branches of mathematics, since the logical derivation of a necessary conclusion from a stated hypothesis does not require that the statements relate specifically to geometry.

The geometry textbooks that have been referenced in the writing of this paper are the following:

<u>Exploring Geometry</u>, by Keedy, Jameson, Smith, and Mould. Holt, Rinehart and Winston, Inc., 1967.

<u>Geometry</u>, by Goodwin, Vannatta, and Crosswhite. Charles E. Merrill Publishing Company, 1970.

<u>Geometry</u>, by Moise and Downs. Addison-Wesley Publishing Company, 1971.

<u>Geometry, A Dimensional Approach</u>, by Rosenberg, Johnson, and Kinsella. The Macmillan Company, 1968.

<u>School Mathematics Geometry</u>, by Anderson, Garon, and Gremillion. Houghton Mifflin Company, 1969.

The writer's analysis of these texts consisted of a thorough study of each book in an effort to extract geometric material common to each text. The writer then considered the concepts of symbolic logic to be introduced in this paper. Finally, he selected topics and examples from the common geometric material that, in his opinion, would best illustrate the use of symbolic logic as a tool in the secondary school geometry classroom.

Each of the texts develop geometric structure by means of an axiomatic deductive system. That is to say, primitive terms, definitions, and postulates are established in early chapters and these are used to prove elementary theorems, which in turn are used to prove more advanced theorems relating to the common geometric figures and the conditions for congruence, parallelism, perpendicularity, etc. For the most part, the texts are patterned after the material produced by SMSG.

The fact that the use of symbolic logic in geometry is the subject of this paper should be put in proper perspective before the reader progresses into the next chapter. Clearly, the authors of the five geometry texts use logic in the sense that they derive necessary (as contrasted to probable) conclusions from existing statements that have been accepted or proved to be true. This writer will exemplify their use of logic to promote discovery techniques and enrichment exercises and to present what he considers to be desirable alternatives to many of their proofs. This is accomplished by the gradual development of some of the concepts of symbolic logic.

Anderson and Moise do not make any use of symbolic logic. In varying degrees, Goodwin (pages 87-115), Keedy (pages 133-137), and Rosenberg (pages 64-71, 173-179) introduce some of the symbols of symbolic logic, but do not make them an essential part of their geometric development. All of the texts identify the converse form of a statement; and three of them (Goodwin, Keedy, and Rosenberg) identify inverse and contrapositive forms, and touch upon truth tables. Again, however, these concepts are not worked into the development of material throughout the texts. None of the texts contains "validity" or "valid argument" in the index, although Goodwin displays valid argument forms on pages 91-92 and Rosenberg defines "valid syllogism" on page 70. Once again, however, the concept of validity is not utilized in the deductive development, and most of the examples of valid arguments do not relate to geometry. A typical example, from Rosenberg's text, is the following:

PREMISES: All boys are giants.
Sam is a boy.

CONCLUSION: Sam is a giant. (19, p. 70).

This writer believes that the five books in the sample are excellent geometry textbooks. The comments that have been made, and similar ones that will be made throughout this paper, should not be considered as criticisms of the texts. Indeed, it would require a text of immense bulk to cover the geometric content in these books and also incorporate a development of symbolic logic within its pages. It is the carefully conceived structure of geometry within these texts that allows one to employ the concepts of symbolic logic presented in this paper.

Four of the five texts adhere to a modern trend and integrate three-dimensional Euclidean geometry with plane Euclidean geometry. Rosenberg is the exception, deferring a study of three-dimensional geometry until Chapter 11. In the examples presented throughout this paper, it is to be assumed that all statements about geometric figures are made relative to three-dimensional Euclidean space, unless otherwise specified. In some instances, it will be specified that statements about a planar figure are made relative to the plane of the figure, as some of the theorems of plane Euclidean geometry are not valid when stated relative to three-dimensional Euclidean space.

Since most of the logical concepts presented can be introduced in the early months of a geometry course, a large portion of the examples in this paper relate to material normally found in the early chapters of a geometry text. However, since the logical concepts are applicable throughout the entire course, this paper does contain examples relating to circles, spheres, and other figures usually found in the later chapters of a textbook.

The reader will find exercises at the end of many sections in this paper. These serve as examples of problems that can be assigned to students or discussed in class to reinforce the logical concepts developed. Solutions for the exercises appear in the Appendix.

CHAPTER II

BASIC CONCEPTS OF LOGIC

Introduction

A deductive course in geometry is concerned with proof. That is to say, the student is taught the science of reaching a necessary conclusion from one or more given statements. Before one can begin to discuss the idea of proof, a foundation must be established by introducing primitive terms (such as point, line, and plane), definitions, and postulates. The time required for this part of a geometry course is considerably less than it was formerly because much of the important information about geometrical objects is taught in the elementary school and many useful conclusions pertaining to these objects are established by the end of grade eight. (7, p. 151). Since the pre-proof period is essentially a review for students, this is the ideal time to introduce basic concepts of logic that will provide students with a powerful discovery tool when a study of proof is undertaken. The logical concepts will undoubtedly represent new material for the students. However, it requires only a few minutes of class time to introduce these concepts and they can continually be reinforced by applying them to material in the early chapters of the geometry text, which, as has been stated, contains material familiar to the students. Many examples will be given in this paper to demonstrate how this can be done.

Statements

Fundamental to logic and geometry is the concept of a statement. A statement is an assertion that is either true or false, but not both. In geometry, postulates are statements that are assumed to be true, and theorems are statements that are proved from other statements. A statement possesses one of two truth values: T (for true) or F (for false).

Example 2.1

The following sentences are statements.

1. $\{a,b\} \subset \{a,b,c\}$.
2. Two distinct lines may intersect in two distinct points.
3. All right angles are congruent.
4. A triangle is a quadrilateral.

A simple statement is a single independent clause--that is, it contains one subject and verb, can stand alone grammatically, and expresses a complete thought. The statements in Example 2.1 are examples of simple statements. One can combine two or more simple statements by using statement connectives such as "and" or "or" or "if ... then ..." to create a compound statement. The statement, "An isosceles triangle may contain a right angle and no triangle contains two right angles," is an example of a compound statement. Each simple statement used in the construction of a compound statement is called a component statement of the compound statement.

A teacher of geometry and his students spend considerable time working with compound statements because virtually all definitions, postulates, and theorems are compound statements. Definitions, postulates, and theorems are true statements and the geometry student should have a clear understanding of what it means to say that a compound statement is true. One also encounters false compound statements in geometry. For instance, in the discussion of converses of theorems, many authors of geometry texts emphasize that the converse of a theorem may be a false statement.

The following sections are devoted to exploring the basic types of compound statements encountered in secondary school geometry, the connectives used in constructing such statements, and the assignment of truth values to the statements. Throughout this paper, statements will often be denoted by lower case letters, such as p, q, and r. The notation "p:" means that p denotes the statement following the colon. For instance,

p: Vertical angles are congruent

means that p denotes the statement "Vertical angles are congruent."

Negation

If p denotes a statement, then $\sim p$ (read "not p") denotes a statement called the <u>negation</u> of p. If p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.

Example 2.2

p: $5 + 2 = 7$. (T).

$\sim p$: $5 + 2 \neq 7$. (F).

Example 2.3

p: A line contains three noncollinear points. (F).

∿p: A line does not contain three noncollinear points. (T).

The statement ∿p may also be written in the following manner.

∿p: It is not true that a line contains three noncollinear points.

The teacher should emphasize that if p and q are negations of one another, then <u>one must be true whenever the other is false, and vice-versa</u>. Consider these statements:

p: Lines L_1 and L_2 are parallel.

q: Lines L_1 and L_2 intersect.

Students are likely to consider q as the negation of p, arguing that if q is true, then p must be false. However, if p is false, it does not follow that q is necessarily true. In three-space, if L_1 and L_2 are not parallel, then they may be skew. (If "skew lines" have not been defined, the teacher can demonstrate two non-parallel lines that do not intersect by picking two appropriate edges in the classroom.) Of course, if one is assuming that L_1 and L_2 are distinct coplanar lines, the p and q are indeed negations of each other. This example points out the fact that one must examine statements relative to the space in which one is working.

As another example, consider statement 2 in Example 2.1. This statement is false in Euclidean three-space. However, if "lines" refer to lines on a sphere, then the statement is true.

Example 2.4

p: No two sides of ΔABC are equal in length.

q: The three sides of ΔABC are equal in length.

In this instance, p and q are not negations of each other, since it is possible for both statements to be false. Let AB = BC = 4 and AC = 6, for instance.

One might reasonably question the use of Example 2.4 in the beginning stages of a geometry course, pointing out the fact that many texts do not define "triangle" until two or three chapters of preliminary material have been introduced. Moise, for example, first defines "triangle" in Chapter 4 of his text. This writer sees nothing wrong with utilizing the intuitive knowledge that students possess to stress concepts that will be beneficial to them when they begin to work with deductive proofs. It can reasonably be assumed that high school sophmores recognize triangles, rectangles, circles, etc., and know some of their basic properties. It can also be noted that authors of geometry texts appeal to the intuitive knowledge of students. Moise refers to triangles in his Chapter 1 problem sets long before formally defining "triangle."

The concept of a quantifier can be emphasized when discussing negation. A teacher can point out in a geometry text numerous examples in which a quantifier is directly used or implied. For instance, the postulate stating: "For any two points there is exactly one line that contains them" (12, p. 34), is a statement about <u>all</u> pairs of two distinct points. A way to negate this statement is to say, "There exists at least one pair of distinct points not contained on exactly one line."

In general, if x represents an element of a specified set and if p denotes a statement, the negation of the statement, "For all x, p," may be written in any one of the following useful forms.

(1) For some x: ∿p.

(2) There exists at least one x such that ∿p.

(3) It is not true that for all x, p.

Example 2.5

p: All triangles are equilateral.

∿p: (1) Some triangles are not equilateral.

(2) There exists at least one triangle that is not equilateral.

(3) It is not true that all triangles are equilateral.

The statement ∿p in Example 2.5 is a statement about <u>some</u> of the elements in a specified set. In this case, the set is the set of all triangles. In geometry one often encounters a statement that is true for a proper subset of a specified set, U, but false when applied to all elements in U. For instance, many true statements about squares are false when applied to quadrilaterals.

In general, if x is an element of a specified set, the negation of the statement, "For some x, p," may be written as follows:

(1) For all x, ∿p.

(2) It is not true that for some x,p.

(3) There does not exist an x such that p.

Example 2.6

p: Some triangles are right triangles.

$\sim$p: (1) All triangles are not right triangles.

(2) It is not true that some triangles are right triangles.

(3) There does not exist a triangle that is a right triangle.

Exercise Set 2a

1. For each statement p, write the statement $\sim$p.

   (a) p: $\frac{8}{-2} = 4$.

   (b) p: The point M is between points A and B.

   (c) p: $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$.

   (d) p: $\angle PQR$ is not a right angle.

   (e) p: Two intersecting lines are not contained in a unique plane.

   (f) p: Every segment has a midpoint.

   (g) p: All pairs of supplementary angles are congruent.

   (h) p: Some pairs of perpendicular lines form right angles.

2. In Example 2.5, can $\sim$p be written "No triangles are equilateral"?

In problems 3-5, explain why p and q are not negations of each other.

3. p: $\angle A$ is acute.

   q: $\angle A$ is obtuse.

4. p: Any set containing three distinct points is collinear.

   q: No set containing three distinct points is collinear.

5. p: There exist two planes that intersect in a line.

   q: There exist two planes that do not intersect in a line.

Conjunction

If p and q denote statements, then the <u>conjunction</u> of p with q is a statement denoted by $p \wedge q$ (read "p and q"). The statement $p \wedge q$ is true when both p and q are true; otherwise it is false. Assignment of a truth value to the compund statement requires knowledge of the truth value of p and the truth value of q. Since p has two possible truth values and q has two possible truth values, there are $2 \cdot 2 = 4$ arrangements for the values of the two statements. Table I illustrates the truth values of $p \wedge q$ for all possible arrangements of the truth values of its component statements. Such a table is called a <u>truth table</u>.

TABLE I

TRUTH TABLE FOR $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.7

p: Angles with the same measure are congruent.

q: Vertical angles are congruent.

$p \wedge q$: Angles with the same measure are congruent and vertical angles are congruent.

The statement $p \wedge q$ is true since both p and q are true.

Example 2.8

p: A triangle is the union of three segments.

q: The union of three segments is a triangle.

$p \wedge q$: A triangle is the union of three segments and the union of three segments is a triangle.

The statement $p \wedge q$ is false since q is false. Note that the truth value of p is irrelevent in this example (See Table I).

Symbols of inclusion, such as parentheses, brackets, and braces, are used to indicate a grouping of component statements within a compound statement. For instance, if p, q, and r denote statements, then the compound statement $(p \wedge q) \wedge r$ represents the conjunction of the statement $p \wedge q$ with the statement r. The truth table for $(p \wedge q) \wedge r$ contains eight rows, since there are $2 \cdot 2 \cdot 2 = 2^3$ possible arrangements for the truth values of p, q, and r. The geometry teacher should observe the systematic pattern of T's and F's in the first three columns of Table II below. Adherence to this pattern saves time and promotes uniformity when one is listing truth value arrangements for three component statements.

TABLE II

TRUTH TABLE FOR $(p \wedge q) \wedge r$

p	q	r	$(p \wedge q)$	$\wedge$	r
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	F	F
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	F	F	T
F	F	F	F	F	F

It should be carefully noted that the compound statement $(p \wedge q) \wedge r$ is true if and only if all of its component statements are true.

One can easily extend Table II to include a column for the truth values of the statement $p \wedge (q \wedge r)$. A result of this exercise is that one can write the statement $(p \wedge q) \wedge r$ as $p \wedge q \wedge r$, since the order in which the component statements are considered has no effect on the truth table of the compound statement.

Exercise Set 2b

1. Let p: A square is a rectangle.

   q: A rectangle is a square.

   Determine the truth value of each statement below.

(a) $p \wedge q$. (b) $p \wedge \sim q$.

(c) $\sim p \wedge q$. (d) $\sim p \wedge \sim q$.

2. Let p: Congruent angles have the same measure.

   q: Supplementary angles are congruent.

   r: Vertical angles are congruent.

   Determine the truth value of each statement below.

   (a) $p \wedge q \wedge r$. (b) $\sim p \wedge q \wedge r$.

   (c) $p \wedge \sim q \wedge r$. (d) $\sim p \wedge \sim q \wedge \sim r$.

3. The definition of "betweenness for points" in Anderson's text is stated as follows: "B is <u>between</u> A and C is (1) A, B, and C are distinct points on the same line and (2) AB + BC = AC." (3, p.50).

   Let p: A, B, and C are distinct points.

   q: A, B, and C are collinear.

   r: AB + BC = AC.

   Then one can say that B is between A and C if $p \wedge q \wedge r$ is a true statement.

   Assume now that it is known that B is <u>not</u> between A and C. What can be said about the relative truth values of p, q, and r?

4. Construct truth tables for (a) $\sim p \wedge \sim q$; (b) $p \wedge q \wedge \sim r$.

5. If p, q, r, and s denote statements, how many rows would appear in a truth table for $p \wedge q \wedge r \wedge s$?

Disjunction

If p and q denote statements, then the <u>disjunction</u> of p with q is a statement denoted by $p \vee q$ (read "p or q"). The statement $p \vee q$ is true if at least one of p and q is true; otherwise it is false.

TABLE III

TRUTH TABLE FOR $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The word "or" is used in the inclusive sense in the definition of disjunction. Hence, if one of the component statements of $p \vee q$ is true, then $p \vee q$ is true irrespective of the truth value of the other component statement.

Example 2.9

p: B is vertex of ΔABC.

q: B is an interior point of ΔABC.

$p \vee q$: B is vertex of ΔABC or B is an interior point of ΔABC.

The statement $p \vee q$ is true since p is true. Furthermore, note that it makes no difference in this example whether or not q is true.

Example 2.10

p: The intersection of two lines must be $\emptyset$.

q: The intersection of two planes may be a point.

$p \vee q$: The intersection of two lines must be $\emptyset$ or the intersection of the two planes may be a point.

The statement $p \vee q$ is false since both p and q are false.

Exercise Set 2c

1. Let p: A line may intersect a plane in exactly one point.

   q: A line not contained in a plane may intersect the plane in more than one point.

   Determine the truth value for each statement below.

   (a) $p \vee q$. (b) $p \vee \sim q$.

   (c) $\sim p \vee q$. (d) $\sim p \vee \sim q$.

2. Construct truth tables for $(p \vee q) \vee r$ and $p \vee (q \vee r)$. Are the symbols of inclusion necessary?

3. Refer to problem 3, Exercise Set 2b. If the statement $p \vee q \vee r$ is true, can one conclude that B is between A and C?

4. Let p: A right triangle may be isosceles.

   q: A right triangle may be equilateral.

   r: A right triangle may contain two right angles.

   Determine the truth value of each statement below.

   (a) $p \vee q \vee r$. (b) $\sim p \vee q \vee r$.

   (c) $p \vee \sim q \vee \sim r$ (d) $\sim p \vee \sim q \vee r$.

Conditional Statements

If one scans the pages of a geometry text, he will find among the definitions, postulates, theorems, and exercises numerous examples of compound statements using the logical connective "if ... then ..." Statements of this type are encountered on a daily basis in a secondary school geometry course.

If p and q denote statements, then the statement denoted by $p \rightarrow q$ (read "If p, then q") is called a <u>conditional statement</u>. The statement $p \rightarrow q$ is true except when p is true and q is false, in which case it is false.

TABLE IV

TRUTH TABLE FOR $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $p \rightarrow q$ may also be read in any one of the following ways:

(1) q, if p.

(2) p only if q.

(3) p is a sufficient condition for q.

(4) q is a necessary condition for p.

Example 2.11

p: Rectangles have four right angles.

q: Squares have four right angles.

$p \rightarrow q$: If rectangles have four right angles, then squares have four right angles.

The condition $p \rightarrow q$ may also be read in any one of the following ways:

(1) Squares have four right angles if rectangles have four right angles.

(2) Rectangles have four right angles only if squares have four right angles.

(3) A sufficient condition for squares having four right angles is that rectangles have four right angles.

(4) A necessary condition for rectangles having four right angles is that squares have four right angles.

The conditional $p \rightarrow q$ is true since it is not possible for q to be false while p is true.

Example 2.12

p: $\angle A$ and $\angle B$ are supplementary.

q: $\angle A$ and $\angle B$ are complementary.

$p \rightarrow q$: If $\angle A$ and $\angle B$ are supplementary, then $\angle A$ and $\angle B$ are complementary.

The conditional $p \rightarrow q$ is false since q is false whenever p is true.

Example 2.13

p: All angles have the same measure.

q: All segments have the same length.

$p \rightarrow q$: If all angles have the same measure, then all segments have the same length.

The statement p is false. Therefore, it is impossible for q to be false while p is true. Hence the conditional $p \rightarrow q$ is true.

It will be emphasized at this point that the conditional $p \to q$ is false only when p is true and q is false (See Table IV). Hence, whenever p is false, the conditional $p \to q$ is true irrespective of the truth value of q. This situation is often confusing to students. If time permits, and if a teacher has able students, he might wish to discuss the underlying reasons for the assignment of truth values to $p \to q$. An excellent reference is Lightstone (14, p. 5). An alternate approach, and perhaps the most desirable in the majority of cases, is for the teacher to explain that the assignment of truth values to $p \to q$ is merely a matter of definition and that most of the work in geometry involving the conditional $p \to q$ will be limited to cases in which p is true. (It can be noted that even in forms of indirect proof, a student begins his reasoning process with a statement assumed to be true.)

Exercise Set 2d

1. Let p: $\angle A$ and $\angle B$ have the same measure.

   q: $\angle A$ and $\angle B$ are congruent.

   (a) Write out in words the conditional $p \to q$.

   (b) Write the conditional in (a) using the phrase "only if."

   (c) Write the conditional in (a) using the word "necessary."

   (d) Write the conditional in (a) using the word "sufficient."

   (e) What is the truth value of each of the following statements?

   (i) $p \to q$. (ii) $q \to p$. (iii) $p \to \sim q$.

2. Follow the directions in problem 1 using the following statements.

   p: $\triangle ABC$ is isosceles.

   q: $\triangle ABC$ is equilateral.

3. Moise presents the following definition:

"If $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays, and $\overrightarrow{AC}$ is any other ray, then $\angle BAC$ and $\angle CAD$ form a linear pair." (17, p. 91).

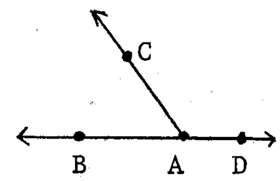


Figure 1

He then states the following postulates:

"If two angles form a linear pair, then they are supplementary." (17, p. 91). Suppose now that it is known that two angles do not form a linear pair. Can one conclude that the two angles are not supplementary?

Biconditional Statements

The conjunction of the conditional $p \to q$ with the conditional $q \to p$ is a statement called a biconditional. The truth values for the biconditional $(p \to q) \wedge (q \to p)$ are shown in Table V.

TABLE V

TRUTH TABLE FOR $(p \to q) \wedge (q \to p)$

p	q	$(p \to q)$	$\wedge$	$(q \to p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

The biconditional $(p \to q) \wedge (q \to p)$ is generally denoted by $p \leftrightarrow q$. One can note from Table V that $p \leftrightarrow q$ is true if and only if p and q have the same truth value; otherwise it is false. The biconditional $p \leftrightarrow q$ is most commonly read in one of the following two ways.

(1) p if and only if q.

(2) p is a necessary and sufficient condition for q.

Example 2.14

p: $\angle A$ is a right angle.

q: $m\angle A = 90$.

$p \leftrightarrow q$: (1) $\angle A$ is a right angle if and only if $m\angle A = 90$.

(2) A necessary and sufficient condition for $m\angle A = 90$ is that $\angle A$ is a right angle.

The statement $p \leftrightarrow q$ is true since p and q cannot have opposite truth values.

Example 2.15

p: L_1 and L_2 are parallel lines.

q: $L_1 \cap L_2 = \emptyset$.

$p \leftrightarrow q$: (1) L_1 and L_2 are parallel lines if and only if $L_1 \cap L_2 = \emptyset$.

(2) A necessary and sufficient condition for L_1 and L_2 to be parallel lines is that $L_1 \cap L_2 = \emptyset$.

The biconditional $p \leftrightarrow q$ is false since q can be true while p is false. (Two skew lines have an empty intersection, but they are not parallel.) Note that if the statements in this example are made relative to the Euclidean plane, then the biconditional $p \leftrightarrow q$ is true.

Exercise Set 2e

1. Let p: $\angle A$ is the complement of $\angle B$.

   q: $\angle B$ is the complement of $\angle A$.

   (a) Write the biconditional $p \leftrightarrow q$ using the phrase "if and only if."

   (b) Write the biconditional $p \leftrightarrow q$ using the phrase "necessary and sufficient."

   (c) Determine the truth value of each statement below.

   (i) $p \leftrightarrow q$. (ii) $\sim p \leftrightarrow q$.

2. Follow the directions in problem 1 using the following statements.

   p: $\triangle ABC$ and $\triangle DEF$ are congruent.

   q: $\triangle ABC$ and $\triangle DEF$ are equilateral.

Tautologies and Equivalent Statements

A compound statement that is always true regardless of the truth values of its component statements is called a <u>tautology</u>. The statement $p \vee \sim p$ is an example of a simple tautology. See Table VI below.

TABLE VI

TRUTH TABLE FOR $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Two compound statements constructed from a set of simple statements are <u>equivalent</u> if the compound statements have identical truth values for all possible arrangements of the truth values of their component statements. If p_1 and p_2 denote equivalent compound statements, then the biconditional $p_1 \leftrightarrow p_2$ is a tautology since p_1 and p_2 cannot have opposite truth values.

Example 2.16

Table VII demonstrates that $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$ is a tautology.

TABLE VII

TRUTH TABLE FOR $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$

p	q	$\sim(p \wedge q)$		$\sim p \vee \sim q$			$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$		
T	T	F	T	F	F	F	F	T	F
T	F	T	F	F	T	T	T	T	T
F	T	T	F	T	T	F	T	T	T
F	F	T	F	T	T	T	T	T	T

It can easily be established with a truth table that $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ is also a tautology. This tautology and the one stated in Example 2.16 are known as <u>DeMorgan's Laws</u>.

Example 2.17

Table VIII demonstrates that $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ is a tautology.

TABLE VIII

TRUTH TABLE FOR $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

p	q	$p \rightarrow q$	$\sim p \vee q$			$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$		
T	T	T	F	T	T	T	T	T
T	F	F	F	F	F	F	T	F
F	T	T	T	T	T	T	T	T
F	F	T	T	T	F	T	T	T

When the concepts of equivalence and tauology are introduced, a point to be emphasized is that the truth value of a compound statement may be obtained by determining the truth value of an equivalent statement. This fact will be of extreme importance in the remaining chapters of this paper. Also, the tautologies developed in this section and those in problems 1 and 2 of Exercise Set 2f will be referenced later in this paper, so they should be noted carefully at this time.

Exercise Set 2f

1. Show that $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tauology.
2. Show that $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$ is a tautology.

3. The negation of a tautology is a statement whose truth values are all false. Such a statement is called a <u>fallacy</u>. Construct a truth table to demonstrate the $(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow \sim q)$ is a fallacy.

Suggestions for Enrichment

An interesting class discussion may be generated by asking students to explain their everyday uses of the logical connectives discussed in this chapter. Does a student use the connective "or" in the inclusive or the exclusive sense? In what context does a student use the "if ... then ..." connective in daily conversation? What does the use of the "if ... then ..." connective convey to the student in a politician's speech or a magazine advertisement? A student reflecting on his everyday use of the connectives discussed in this chapter may well discover that his usage is consistent with the definitions stated in this chapter.

Insurance contracts (life, automobile, etc.) contain many conditional statements. A teacher can give examples of conditional statements from an insurance contract that he may own. He can discuss with his class some of the possible reasons for the many court cases involving an interpretation of a conditional statement in an insurance contract.

A teacher familiar with the computer programming language BASIC (<u>B</u>eginner's <u>A</u>ll-purpose <u>S</u>ymbolic <u>I</u>nstruction <u>C</u>ode) can explain the use of the IF ... THEN ... command as an instruction to the computer. Other programming languages have a similar command.

Summary

It has been the purpose of this chapter to demonstrate how a geometry teacher can use material in the early chapters of a geometry text and a student's intuitive knowledge of basic geometric figures to illustrate the logical concepts of statement, compound statement, and statement connectives. Emphasis was also placed on the determination of the truth values of compound statements. The geometry teacher who has previously studied symbolic logic will realize that this chapter does not include all topics and symbols that one might expect to encounter in a symbolic logic course. Rather, only those concepts that this writer has found useful in the deductive development of geometric structure have been introduced. Since a geometry course involves considerable geometric symbolism, the use of logical symbolism has purposely been kept to a minimum.

A teacher might use the logical concepts introduced to reinforce continually a student's understanding of important mathematical phrases. For instance, a teacher can stress the meaning of the phrase "exactly one" by examining the negation of the postulate stating "For every two points there is exactly one line that contains both points." (17, p. 47).

The essence of deductive geometric proof requires that one be able to logically derive true statements from existing true statements. The following chapter will show how a geometry teacher might use the concepts of symbolic logic developed in this chapter to aid his students in gaining insight into the nature of deductive proof.

CHAPTER III

ARGUMENT FORMS

Proof

Having introduced some of the basic concepts of symbolic logic in the pre-proof stage of a geometry course, the teacher is in a position to use these concepts when it is time to begin a study of proof. It should be kept in mind that a student probably conceives of proof as an argument designed to convince somebody of something. The student undoubtedly has in mind the inductive process (although he may not know it by that name), which is the very heart of the scientific method of proof and involves reaching a conclusion from experimentation or observation.

It is generally not difficult to convince a student that most of his experience with proof in mathematics and in everyday life in general has been inductive in nature. A few examples might be helpful.

Example 3.1

A student has heard it asserted that the sum of the measures of the angles of a triangle is 180. He carefully constructs many triangles and measures the angles with a protractor. Summing the measures of the angles in each triangle and finding that the sum is always in the neighborhood of 180 (small errors in measurement are likely to occur), he concludes that the assertion is true.

Example 3.2

Mary has observed that Mr. Brown is always angry with a student who shows up late for his class. Today, Mary will be late for Mr. Brown's class. She concludes that Mr. Brown will be angry with her.

Example 3.3

Henry substitutes the integers 1, 2, 3, ..., 25 for n in the expression $n^2 - n + 41$ and observes that the value of the expression is a prime number. He concludes that $n^2 - n + 41$ is always a prime number when a positive integer is substituted for n.

The conclusions in the three examples above can only be classified as probable (as contrasted to necessary) conclusions. The conclusion in Example 3.1 is indeed true in Euclidean geometry. However, the student can certainly appreciate that it is not a necessary result of his experiment, for he cannot possibly verify it by summing the measures of the three angles in every triangle. The conclusion in Example 3.2 does not necessarily follow from Mary's observations. Is it not possible that Mr. Brown will be in a happy and forgiving mood on this particular day? The conclusion in Example 3.3 is not a necessary consequence of Henry's experiment, for it is false. This can be demonstrated by letting $n = 41$.

Consider now a set H of statements and let h be the conjunction of statements in H. If c is a statement, then the conditional statement

$$h \rightarrow c$$

is called an argument, or an inference. The statement h is the hypothesis of the argument and c is the conclusion of the argument. The statements in H may consist of definitions, postulates, or statements, the truth values of which have been previously established. If all of the statements in H are true, then h, being the conjunction of these statements, is true. If at least one of the statements in H is false, then h is false.

If the truth of h necessitates the truth of c, then $h \rightarrow c$ is a valid argument. In other words, the argument $h \rightarrow c$ is valid if it is impossible for c to be false while h is true.

Example 3.4

Consider the following argument: If $m\angle A = 30$ and $m\angle B = 150$, then $\angle A$ and $\angle B$ are supplementary angles.

Let p: $m\angle A = 30$.

q: $m\angle B = 150$.

r: $\angle A$ and $\angle B$ are supplementary.

The argument is symbolically represented by

$$\underbrace{(p \wedge q)}_{\text{Hypothesis}} \rightarrow \underbrace{r}_{\text{Conclusion}},$$

The argument is valid since it is impossible for the conclusion to be false while the hypothesis is true.

If, for a given argument, the conclusion may be false while the hypothesis is true, then the argument is <u>invalid</u>. In this case, the conclusion is not a necessary consequence of the hypothesis.

Example 3.5

Consider the following argument: If $m\angle A < 45$ and $m\angle B < 50$, then $\angle A$ and $\angle B$ are not complementary.

Let p_1: $m\angle A < 45$.

p_2: $m\angle B < 50$.

c: $\angle A$ and $\angle B$ are not complementary.

The argument is symbolically represented by

$$\underbrace{(p_1 \wedge p_2)}_{\text{Hypothesis}} \rightarrow \underbrace{c}_{\text{Conclusion}}.$$

The argument is invalid since it is possible for the conclusion to be false while the hypothesis is true. For example, let $m\angle A = 41$ and $m\angle B = 49$.

The process of using valid arguments to obtain a necessary conclusion from a set of given statements is called <u>deductive reasoning</u>. A typical geometric problem presents the student with a conclusion and a hypothesis from which the conclusion is to be obtained by the construction of a series of valid arguments. Some of the fundamental valid argument forms will be introduced in the following sections. The argument forms have various names. The names used in this paper conform to those used in Rosenberg's textbook.

Exercise Set 3a

In each problem below, identify the hypothesis and conclusion of each argument. Then determine whether or not the argument is valid.

1. If $a \neq b$ and $b \neq c$, then $a \neq c$.
2. If $\triangle ABC$ is equilateral, then $\triangle ABC$ is isosceles.
3. If $\angle A$ and $\angle B$ are acute, then they are complementary angles.
4. If P_1 and P_2 are distinct points in plane E, then $\overleftrightarrow{P_1P_2} \subset E$.

The Law of Detachment

A very common form of argument is illustrated by the following:

If a triangle is equilateral, then it is isosceles.

$\triangle ABC$ is equilateral.

Therefore, $\triangle ABC$ is isosceles.

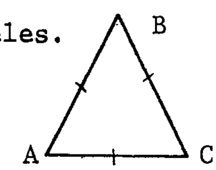


Figure 2

This is a valid argument because it fits the fundamental argument form $[(p \to q) \wedge p] \to q$, which is tautology. Table IX illustrates that the conclusion q cannot be false when the hypothesis $(p \to q) \wedge p$ is true.

TABLE IX

TRUTH TABLE FOR $[(p \to q) \wedge p] \to q$

p	q	$(p \to q) \wedge p$			$[(p \to q) \wedge p] \to q$		
T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	F	F	F	T	T
F	F	T	F	F	F	T	F

This form of argument is called the <u>law of detachment</u>.

In order to emphasize the component statements of the hypothesis and conclusion in an argument form, a common method for presenting an argument will be adopted in this chapter. The component statements of the hypothesis will be written on separate lines and a horizontal segment will separate the hypothesis from the conclusion of the argument. The law of detachment is then presented as follows:

$p \to q$ } The conjunction of these statements is the hypothesis.
$\underline{p}$ }

q Conclusion.

Many of the statements used in initial proofs are postulates and these allow early emphasis on the law of detachment.

Example 3.6

$p \to q$	If two points of a line lie in a plane, then the line lies in the plane. (Postulate).
p	Points A and B of line L lie in plane E.
q	$\overleftrightarrow{AB}$ lies in plane E.

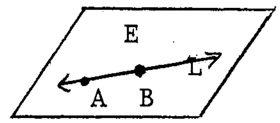


Figure 3

Example 3.7

$p \to q$	If there exists an SAS correspondence between two triangles, then the triangles are congruent. (Postulate).
p	There exists an SAS correspondence between ΔABC and ΔDEF.
q	ΔABC and ΔDEF are congruent.

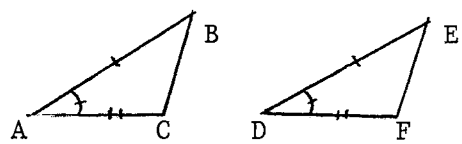


Figure 4

A proven theorem may also be used as part of the hypothesis in the law of detachment.

Example 3.8

$p \to q$	If two sides of a triangle are congruent, then the angles opposite these sides are congruent. (Theorem.)
p	In ΔABC, $\overline{AB} \cong \overline{AC}$.
q	$\angle B \cong \angle C$.

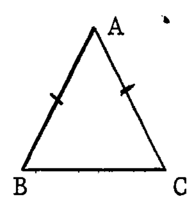


Figure 5

Some theorems are stated in the form of a biconditional. Since $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$, one can apply the law of detachment to biconditional statements.

Example 3.9

$p \leftrightarrow q$: A triangle is equiangular if and only if it is equilateral.

If the validity of $p \leftrightarrow q$ has been established, one can utilize the law of detachment in the following ways.

$p \rightarrow q$	A triangle is equiangular if it is equilateral.
p	$\triangle ABC$ is equilateral.
q	$\triangle ABC$ is equiangular.

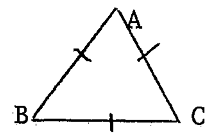


Figure 6

$q \rightarrow p$	A triangle is equilateral if it is equiangular.
q	$\triangle RST$ is equiangular.
p	$\triangle RST$ is equilateral.

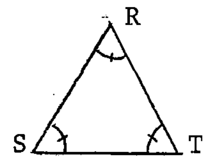


Figure 7

Exercise Set 3b

Use the law of detachment to supply the necessary conclusion.

1. If a triangle has one right angle, then its other angles are acute.

   In $\triangle ABC$, $\angle C$ is a right angle.

   ?

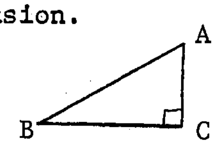


Figure 8

2. The diagonals of a rhombus are perpendicular.

   □ABCD is a rhombus.

   ?

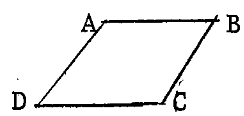


Figure 9

3. A line is the perpendicular bisector of a segment if and only if it is perpendicular to the segment and bisects the segment.

   $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ bisects $\overline{CD}$.

   ?

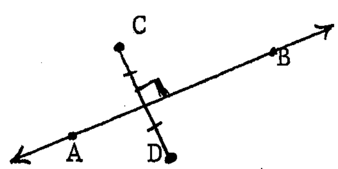


Figure 10

Exercise Set 3c

An invalid argument form that is sometimes confused with the law of detachment is the following:

$$\begin{array}{l} p \rightarrow q \\ \underline{q \quad\quad} \\ p \end{array}$$

1. Construct a truth table to demonstrate that $[(p \rightarrow q) \wedge q] \rightarrow p$ is not a tautology.

In problems 2-3, construct a drawing to demonstrate that the arguments presented are invalid.

2. If a quadrilateral is a square, then it is a rectangle.

   <u>□ABCD is a rectangle.</u>

   □ABCD is a square.

3. If two angles form a linear pair, then they are supplementary.

   <u>∠A and ∠B are supplementary.</u>

   ∠A and ∠B form a linear pair.

Law of Transitivity of Conditionals

Another common valid argument form is the <u>law of transitivity of conditionals</u>. This argument is shown below.

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ p \rightarrow r \end{array}$$

Table X below illustrates that $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.

TABLE X

TRUTH TABLE FOR $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

p	q	r	$(p \rightarrow q) \wedge (q \rightarrow r)$			$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$		
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T	F
T	F	T	F	F	T	F	T	T
T	F	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Example 3.10

$p \rightarrow q$ | If D is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

$q \rightarrow r$ | If $m\angle BAC = m\angle BAD + m\angle DAC$, then $n\angle BAC > m\angle BAD$.

$p \rightarrow r$ | If D is in the interior of $\angle BAC$, then $m\angle BAC > m\angle BAD$.

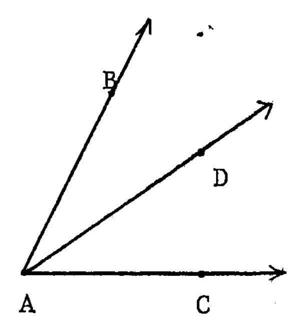


Figure 11

Example 3.11

$p \rightarrow q$	If two lines are perpendicular to the same plane, then they are parallel.
$q \rightarrow r$	If two lines are parallel, then they are coplanar.
$p \rightarrow r$	If two lines are perpendicular to the same plane, then the lines are coplanar.

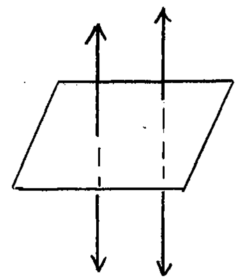


Figure 12

Exercise Set 3d

Use the law of transitivity of conditionals to establish a conclusion for each argument.

1. If two angles are complementary, then each of them is acute.

   If each of two angles is acute, then they are not supplementary.

   ?

2. If $\overline{MR}$ is a median of scalene triangle MNP, then $\overline{MR}$ does not bisect $\angle NMP$.

   If $\overline{MR}$ does not bisect $\angle NMP$, then $\overline{MR}$ is not perpendicular to $\overleftrightarrow{NP}$.

   ?

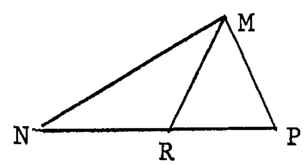


Figure 13

3. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

   The opposite angles of a parallelogram are congruent.

   ?

The Use of Valid Arguments in a Formal Proof

The law of detachment and the law of transitivity of conditionals are used extensively in formal proofs. Identification of these argument forms in some of the introductory formal proofs in a geometry course may be beneficial in helping students appreciate that a deductive proof is a series of valid arguments, each of which has a true hypothesis and, hence, a true conclusion. It is important to note that the validity of an argument does not depend on whether the hypothesis is true. However, if the argument is valid and the hypothesis is true, then the conclusion must be true.

The following formal proof is similar to those that one would expect to encounter in the early stages of geometric proof. The valid argument forms discussed will be identified in the proof.

Given: The figure with $\overline{AC} \cong \overline{CD}$ and $\overline{BC} \cong \overline{CE}$.

Prove: $\angle A \cong \angle D$.

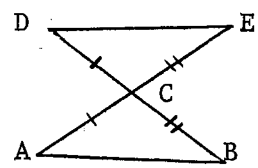


Figure 14

Proof

Statement	Reasons
1. $\overline{AC} \cong \overline{CD}$ and $\overline{BC} \cong \overline{CE}$.	1. Given
2. $\angle ACB \cong \angle DCE$.	2. Vertical angles are congruent.
3. $\triangle ACB \cong \triangle DCE$.	3. SAS.
4. $\angle A \cong \angle D$.	4. Corresponding parts of congruent triangles are congruent.

It should be emphasized that all statements in the Statement column are made relative to Figure 14. This is generally the case in any proof that involves a given figure. Also, while not explicitly stated, one is to assume that the points A, C, and E are distinct and collinear and similarly for B, C, and D. These assumptions are considered as part of Statement 1 and are necessary in the establishing that $\angle ACB$ and $\angle DCE$ are vertical angles. The question arises as to what unstated assumptions should be implicitly made about a given figure to be used in a geometric proof. This writer feels that a firm rule cannot be established and that the context of a specific problem usually indicates the assumptions which should be made. For the sake of brevity, these assumptions are conventionally not presented in a written version of a formal proof.

The geometry teacher can note that all statements in the Statements column of the proof are true. Statement 1 is given as true. Statements 2, 3, and 4 can be established as true using the law of detachment with the corresponding conditional statement represented by the phrase or abbreviation in the Reasons column, as shown below.

> If two angles form a pair of vertical angles, then they are congruent. (Reason 2).
>
> $\angle ACB$ and $\angle DCE$ form a pair of vertical angles. (The truth of this statement follows from the definition of vertical angles applied to the given figure.)
>
> $\angle ACB \cong \angle DCE$. (Statement 2).

> If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of a second triangle, then the triangles are congruent. (Reason 3).
>
> Two sides and an included angle of $\triangle ACB$ are congruent respectively to two sides and an included angle of $\triangle DCE$. (This statement represents the conjunction of Statements 1 and 2.)
>
> $\triangle ACB \cong \triangle DCE$. (Statement 3).

> If two triangles are congruent, then their corresponding parts are congruent. (Reason 4).
>
> $\angle A$ and $\angle D$ are corresponding parts of congruent triangles ACB and DCE, respectively. (The truth of this statement follows from the definition of corresponding parts of congruent triangles.)
>
> $\angle A \cong \angle D$. (Statement 4).

The teacher can now emphasize that the truth of Statement 1 (including the implicit assumptions mentioned) necessitates the truth of Statement 2. Hence

$$(\text{Statement } 1) \rightarrow (\text{Statement } 2)$$

is a valid argument. It has been shown that the truth of Statement 2 necessitates the truth of Statement 3, meaning that

$$(\text{Statement } 2) \rightarrow (\text{Statement } 3)$$

is a valid argument. (It should be noted here that the above argument is valid relative to the truth of Statement 1. That is, Statement 2 necessitates the truth of Statement 3 if one knows that Statement 1 is true.) Similarly,

$$(\text{Statement } 3) \rightarrow (\text{Statement } 4)$$

is a valid argument. One can now establish that

$$(\text{Statement } 1) \rightarrow (\text{Statement } 4)$$

is a valid argument by a double application of the law of transitivity of conditionals, as shown below.

$$\begin{array}{l} (\text{Statement } 1) \rightarrow (\text{Statement } 2) \\ (\text{Statement } 2) \rightarrow (\text{Statement } 3) \\ \hline (\text{Statement } 1) \rightarrow (\text{Statement } 3). \end{array}$$

$$\begin{array}{l} (\text{Statement } 1) \rightarrow (\text{Statement } 3) \\ (\text{Statement } 3) \rightarrow (\text{Statement } 4) \\ \hline (\text{Statement } 1) \rightarrow (\text{Statement } 4) \end{array}$$

Based on his experience in the classroom, the writer feels that a similar analysis of a few elementary formal proofs gives students a deeper insight into the nature of deductive geometric proof. Such an analysis stresses the following points:

(1) The "Given" in the statement of a geometry problem (proof) is a set of true statements. The conjunction of these statements represents the hypothesis, h, of an argument.

(2) The "Prove" in the statement of a geometry problem is a statement that represents the conclusion, c, of the argument.

(3) The objective of the person working on the problem is to show that the conclusion is a necessary consequence of the hypothesis. That is, he is to show that the argument $h \rightarrow c$ is valid.

(4) In a direct proof, the objective in (3) is obtained by creating a series of valid arguments.

$$c_1 \rightarrow c_2$$
$$c_2 \rightarrow c_3$$
$$c_3 \rightarrow c_4$$
$$\cdots$$
$$c_n \rightarrow c_{n+1} \quad \text{(n a positive integer).}$$

(In the series of arguments above, c_1 denotes h and c_{n+1} denotes c. If $2 \leq k \leq n$, the argument $c_k \rightarrow c_{k+1}$ is valid relative to the truth of $c_1, c_2, \ldots, c_{k-1}$.) Repeated use of the law of transitivity of conditionals is then used to establish the validity of $h \rightarrow c$. It should be noted that in direct proof, to be discussed in Chapter VII of this paper, one still creates a series of valid arguments. However, the first argument in the series does not have h as its hypothesis.

There are two other valid argument forms that are often useful in geometric proofs. They will be introduced at this time.

The Law of Contraposition

A third valid argument form is the <u>law of contraposition</u>. This argument form is shown below.

$p \to q$

$\underline{\sim q \quad\quad}$

$\sim p$

Table XI illustrates that $[(p \to q) \wedge \sim q] \to \sim p$ is a tautology.

TABLE XI

TRUTH TABLE FOR $[(p \to q) \wedge \sim q] \to \sim p$

p	q	$(p \to q) \wedge \sim q$			$[(p \to q) \wedge \sim q] \to \sim p$		
T	T	T	F	F	F	T	F
T	F	F	F	T	F	T	F
F	T	T	F	F	F	T	T
F	F	T	T	T	T	T	T

Example 3.12

p→q	If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
∿q	▱ABCD is not a parallelogram.
∿p	$\overline{AC}$ and $\overline{BD}$ do not bisect each other.

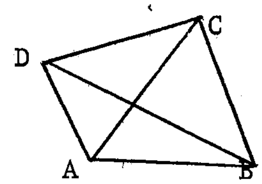


Figure 15

Example 3.13

p↔q A quadrilateral is a rectangle if and only if it has four right angles.

p→q	If a quadrilateral has four right angles, then it is a rectangle.
∿q	▱ABCD is not a rectangle.
∿p	▱ABCD does not have four right angles.

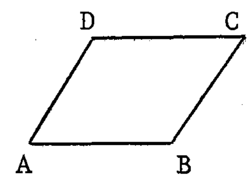


Figure 16

q→p	If a quadrilateral is a rectangle, then it has four right angles.
∿p	▱ABCD does not have four right angles.
∿q	▱ABCD is not a rectangle.

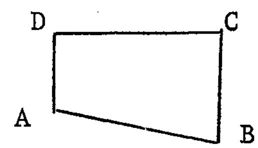


Figure 17

The law of contraposition emphasizes the importance of understanding the concept of negation. Consider the following example.

Example 3.14

p→q	If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
∿p	▱ABCD is not a parallelogram.
∿p	?

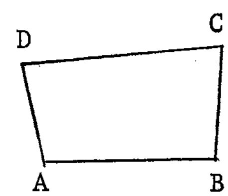


Figure 18

What must be negated is the statement p: $(\overline{AB}\cong\overline{CD})\wedge(\overline{AD}\cong\overline{BC})$. According to one of DeMorgan's Laws (Example 2.16), the statement $\sim p$ must be $(AB\neq CD)\vee(AD\neq BC)$. This allows three possibilities:

(1) $\overline{AB}\ncong\overline{CD}$ and $\overline{AD}\ncong\overline{BC}$. (Figure 19).

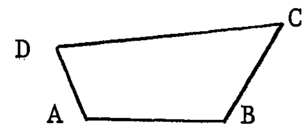


Figure 19

(2) $\overline{AB}\cong\overline{CD}$ and $\overline{AD}\ncong\overline{BC}$. (Figure 20).

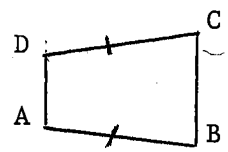


Figure 20

(3) $\overline{AB}\ncong\overline{CD}$ and $\overline{AD}\cong\overline{BC}$. (Figure 21).

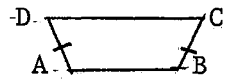


Figure 21

A common error is to accept (1) as $\sim p$. Clearly, (1) alone is not a necessary consequence of the hypothesis. In other words, if one defines statement r to be

$$r:\quad \overline{AB}\ncong\overline{CD} \text{ and } \overline{AD}\ncong\overline{BC},$$

then the following argument is <u>not</u> valid.

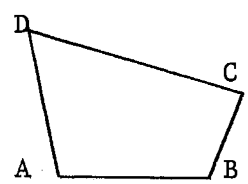


Figure 22

$p\rightarrow q$	If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
$\sim q$	▱ABCD is not a parallelogram.
r	$\overline{AB}\ncong\overline{CD}$ and $\overline{AD}\ncong\overline{BC}$.

Since r is not equivalent to $\sim p$, the argument form is not the law of contraposition. But this alone would not make the argument invalid. Invalidity results from the fact that r is not a necessary consequence of the hypothesis.

Exercise Set 3e

Use the law of contraposition to supply a necessary conclusion for each argument.

1. If two lines are skew, then they do not intersect.

   <u>Lines L_1 and L_2 intersect.</u>

   ?

2. A median of a triangle bisects the side to which it is drawn.

   <u>$\overline{PQ}$ does not bisect $\overline{MN}$.</u>

   ?

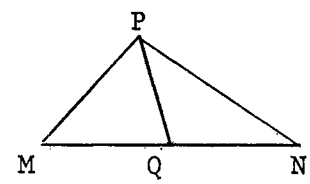


Figure 23

3. In $\triangle ABC$, if $m\angle A > m\angle C$, then $BC > AB$.

   <u>$BC \leq AB$.</u>

   ?

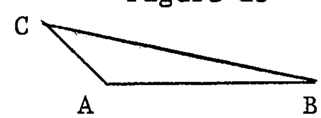


Figure 24

Exercise Set 3f

The following argument form is sometimes confused with the law of contraposition.

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim p} \\ \sim q \end{array}$$

1. Construct a truth table to demonstrate that $[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$ is not a tautology.

In problems 2-3, construct a drawing to demonstrate that the following arguments are invalid.

2. If each of two lines is parallel to a third line, then they are parallel to each other.

L_1, L_2, and L_3 are lines and L_1 is not parallel to L_2.
___

L_1 is not parallel to L_3 and L_1 is not parallel to L_3.

3. If $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$, then $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} \neq \emptyset$.

$\overleftrightarrow{AB}$ is not perpendicular to $\overleftrightarrow{CD}$.
___

$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \emptyset$.

The Law of Elimination

A fourth valid argument form is the law of elimination. This argument form is shown below.

$$\begin{array}{l} p \vee q \\ \underline{\sim p \quad\quad} \\ q \end{array}$$

Table XII below illustrates that $[(p \vee q) \wedge \sim p] \rightarrow q$ is a tautology.

TABLE XII

TRUTH TABLE FOR $[(p \vee q) \wedge \sim p] \rightarrow q$

p	q	$(p \vee q) \wedge \sim p$			$[(p \vee q) \wedge \sim p] \rightarrow q$		
T	T	T	F	F	F	T	T
T	F	T	F	F	F	T	F
F	T	T	T	T	T	T	T
F	F	F	F	T	F	T	F

Example 3.15

$p \vee q$	Two distinct planes intersect in a line or they are parallel.
$\sim p$	Planes E_1 and E_2 do not intersect in a line.
q	E_1 is parallel to E_2.

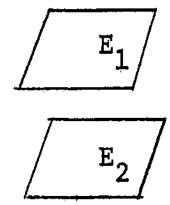


Figure 25

Example 3.16

$p \vee q$	Two similar triangles have the same area or they are not congruent.
$\sim p$	ΔABC and ΔDEF are similar and do not have the same area.
q	ΔABC and ΔDEF are not congruent.

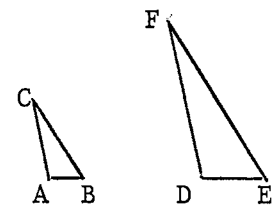


Figure 26

Exercise Set 3g

Use the law of elimination to supply a necessary conclusion for the following arguments.

1. Every triangle is either isosceles or scalene.

   <u>ΔABC is scalene.</u>

   ?

2. $\frac{AD}{BD} \neq \frac{AE}{CE}$ or $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{BC}$.

   <u>$\overleftrightarrow{DE}$ is not parallel to $\overleftrightarrow{BC}$.</u>

   ?

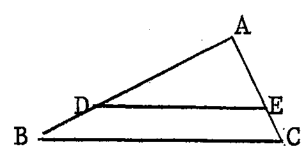


Figure 27

3. ΔABC is not a right triangle or $a^2+b^2 = c^2$.

   <u>$a^2+b^2 \neq c^2$.</u>

   ?

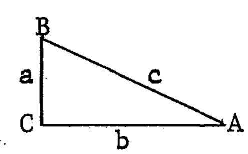


Figure 28

4. What conclusion is needed to make the following argument valid?

$p \lor q \lor r$

$\underline{\sim p}$

?

Truth and Validity

It should be pointed out to students that the validity of an argument depends upon the form of the argument and not upon the truth value of the hypothesis or the truth value of the conclusion. The following three examples illustrate this fact.

Example 3.17

$p \rightarrow q$	If a triangle is isosceles, then it is equilateral.
p	ΔABC is isosceles.
q	ΔABC is equilateral.

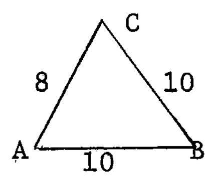


Figure 29

Argument: Valid, since its form is that of the law of detachment.

Hypothesis: False, since $(p \rightarrow q) \land p$ is false.

Conclusion: False.

Example 3.18

$p \rightarrow q$	A square is a triangle.
$q \rightarrow r$	A triangle is a rectangle.
$p \rightarrow r$	A square is a rectangle.

Argument: Valid, since its form is that of the law of transitivity of conditionals.

Hypothesis: False, since $(p \rightarrow q) \land (q \rightarrow r)$ is false.

Conclusion: True.

Example 3.19

p→q	If two angles form a pair of vertical angles, then they are congruent.
q	$\angle BEC$ and $\angle AED$ are congruent.
p	$\angle BEC$ and $\angle AED$ form a pair of vertical angles.

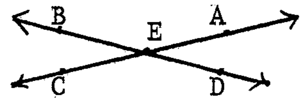


Figure 30

Argument: Invalid (See Exercise Set 3c).

Hypothesis: True, since $(p \to q) \wedge q$ is true.

Conclusion: True.

Exercise Set 3h

In each problem, tell whether the argument is valid or invalid and state the truth value of the hypothesis and conclusion.

1. If $\angle A$ and $\angle B$ are vertical angles, then $\angle A \cong \angle B$.

   $\underline{\angle A \ncong \angle B.}$

   $\angle A$ and $\angle B$ are not vertical angles.

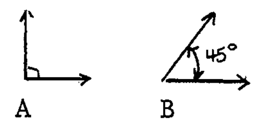


Figure 31

2. If a triangle contains two congruent angles, then all of its angles are congruent.

   <u>ΔDEF contains two congruent angles.</u>

   All of the angles of ΔDEF are congruent.

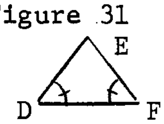


Figure 32

3. If a triangle is equilateral, then it is isosceles.

   <u>ΔABC is not equilateral.</u>

   ΔABC is not isosceles.

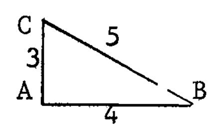


Figure 33

Suggestions for Enrichment

Students can be encouraged to look and listen for the valid argument forms outside of the geometry classroom. Use of the argument forms can often be found in debates (school, political, etc.). For instance, one might encounter an argument similar to the following:

> Anyone who supports the concept of neighborhood schools is a segregationist.
>
> Jones supports the concept of neighborhood schools.
>
> Jones is a segregationist.

The argument is valid since its form is that of the law of detachment. But what is the truth value of the hypothesis? What is the truth value of the conclusion? Will the answers to these questions differ from person to person? Perhaps the most important question is the following: Does the validity of the argument mean that the conclusion is true? If Jones is a candidate for a school board position and if the argument above is presented by Wilson, a candidate for the same position, does Wilson wish to have the public believe that validity necessitates the truth of the conclusion?

One might also uncover arguments similar to the following:

> Anyone who supports the concept of neighborhood schools is a segregationist.
>
> Wilson does not support the concept of neighborhood schools.
>
> Wilson is not a segregationist.

The argument above is invalid (See Exercise Set 3e). Again, one can ask a series of questions concerning the truth values of the hypothesis and conclusion.

Summary

The common argument forms that appear both implicitly and explicitly in geometry textbooks have been introduced and demonstrated in this chapter. Hopefully, the material in this chapter will be useful to the secondary school geometry teacher as he strives to instill in his students an understanding and appreciation of the deductive reasoning process and the distinctions between this process and inductive reasoning.

It has been mentioned that a typical geometry problem presents the student with a stated hypothesis and a stated conclusion. The student's objective is to show with logical arguments that the conclusion is a necessary consequence of the hypothesis. The fact that the hypothesis and conclusion are stated somewhat limits the student to establishing the validity of an argument constructed by another individual. It is the opinion of this writer that the geometry student should be supplied with discovery techniques so that his activity will not be confined to establishing the validity of arguments constructed by authors of geometry textbooks.

Since the process of working within a deductive system is a relatively new experience for the secondary school student, it is reasonable to assume that the student does not possess knowledge of useful techniques for discovering meaningful geometric arguments. The following chapters will attempt to show how a geometry teacher can provide discovery techniques for the student and how these techniques may be used to increase student interest and enthusiasm.

CHAPTER IV

PARTIAL CONVERSES

Introduction to the Concept of a Partial Converse

After an introduction to formal proofs, the authors of the five sample textbooks introduce the concept of "converse of a statement." Though Euclid made no mention of "converse" (or for that matter, "inverse" or "contrapositive," each of which will be discussed in later chapters) in his Elements, authors of modern geometry texts find it a useful concept. It is interesting to note the definitions in the five sample texts listed in Table XIII.

With the exception of Goodwin, the text definitions imply that a theorem or postulate has but one converse. The idea that the statement $q \rightarrow p$ is the converse of $p \rightarrow q$ is certainly satisfactory when one considers theorems similar to the following:

> Theorem 1: If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

The theorem is written in the following manner in order to emphasize the component statements in the hypothesis and conclusion.

Hypothesis: $\triangle ABC$.	Conclusion:
p: $\overline{AC} \cong \overline{BC}$.	q: $\angle A \cong \angle B$.

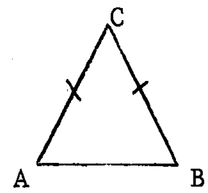


Figure 34

(Note: since the theorem is about triangles, the fact that the figure is a triangle is not considered as a component statement of the hypothesis.)

TABLE XIII

TEXTBOOK DEFINITIONS OF "CONVERSE"

Text	Definition of "converse"
Anderson (p. 238)	"More generally, the converse of a statement expressed in <u>if</u>...<u>then</u> form is the statement obtained by interchanging the <u>if</u> and <u>then</u> parts (with nouns and pronouns also interchanged where appropriate.)"
Goodwin (p. 100)	"If a conditional statement contains multiple distinct conditions and conclusions, converses are obtained by interchanging any number of distinct conditions with an equal number of distinct conclusions of the original implication."
Keedy (p. 126)	"The conditional sentences $A \rightarrow B$ and $B \rightarrow A$ are converses of each other."
Moise (p. 159)	This text does not state a formal definition. It gives examples of statements that are converses of each other.
Rosenberg (p. 179)	"The implication $q \rightarrow p$ is the converse of the implication $p \rightarrow q$."

Theorem 1 has the form $p \rightarrow q$, and all of the sample texts list Theorem 2 below as the converse of Theorem 1. Theorem 2 clearly has the form $q \rightarrow p$.

Theorem 2: If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

Hypothesis: $\triangle ABC$	Conclusion:
q: $\angle A \cong \angle B$.	p: $\overline{AC} \cong \overline{BC}$.

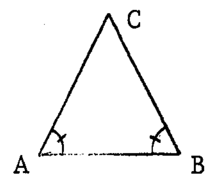


Figure 35

Examination of a geometry text will reveal that the statements of many geometric theorems are more complicated than that of Theorem 1. Consider the following theorem.

Theorem 3: If two sides of one triangle are congruent respectively to two sides of a second triangle, and if the measure of the included angle of the first triangle is greater than the measure of the included angle of the second, then the third side of the first triangle is larger than the third side of the second.

Theorem 3 is commonly called the Hinge Theorem, or the Scissors Theorem. Its hypothesis and conclusion are restated below.

Hypothesis: $\triangle ABC$ and $\triangle DEF$.	Conclusion:
p_1: $\overline{AB} = \overline{DE}$.	q: $\overline{BC} > \overline{EF}$.
p_2: $\overline{AC} = \overline{DF}$.	
p_3: $\angle A > \angle D$.	

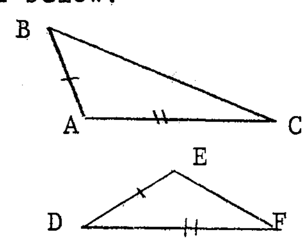


Figure 37

Argument form: $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$.

With the exception of Rosenberg, the sample texts list Theorem 4 below as a converse of Theorem 3. (Anderson and Moise specifically identify Theorem 4 as <u>the</u> Converse Hinge Theorem.)

Theorem 4: If two sides of one triangle are congruent respectively to two sides of a second triangle, and if the third side of the first triangle is longer than the third side of the second, then the included angle of the first triangle is larger than the included angle of the second.

Hypothesis: $\triangle ABC$ and $\triangle DEF$.

p_1: $\overline{AB} \cong \overline{DE}$.

p_2: $\overline{AC} \cong \overline{DF}$.

q: $\overline{BC} > \overline{EF}$.

Argument form: $(p_1 \wedge p_2 \wedge q) \rightarrow p_3$.

Conclusion:

p_3: $\angle A > \angle D$.

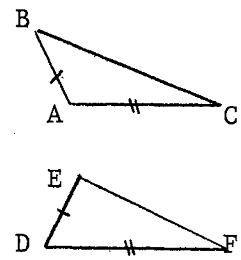


Figure 37

One can observe that a converse of Theorem 3 was obtained by interchanging one component statement of the hypothesis with one component statement of the conclusion. Hence, it would certainly seem that one might also consider

(1) $(p_1 \wedge p_3 \wedge q) \rightarrow p_2$, and

(2) $(p_2 \wedge p_3 \wedge q) \rightarrow p_1$

as converses of Theorem 3, even though these converses have little use in the development of geometric structure. Since determining the validity or non-validity of arguments (1) and (2) is considered unimportant by authors of the texts, the teacher can demonstrate that the arguments are invalid with a drawing. For instance, consider (1).

Hypothesis: $\triangle ABC$ and $\triangle DEF$.

p_1: $\overline{AB} \cong \overline{DE}$.

q: $\overline{BC} > \overline{EF}$.

p_3: $\angle A > \angle D$.

Conclusion:

p_2: $\overline{AC} \cong \overline{DF}$.

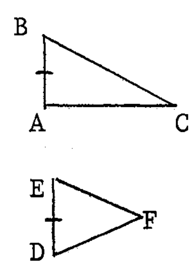


Figure 38

Figure 38 illustrates that the conclusion p_2 is not a necessary consequence of the hypothesis. One can do a similar thing with $(p_2 \wedge p_3 \wedge q) \rightarrow p_1$.

If one adheres strictly to the definitions of Anderson, Keedy, and Rosenberg, it is difficult to justify the classification of Theorem 4 as a converse of Theorem 3. According to any of these definitions, the converse of Theorem 3 would be

$$q \rightarrow (p_1 \wedge p_2 \wedge p_3).$$

It can be noted that the statement of Theorem 4 (a valid argument) and the statement $q \rightarrow (p_1 \wedge p_2 \wedge p_3)$ are not equivalent since one can construct numerous examples to demonstrate that the latter argument is invalid. Moise avoids this logical inconsistency by not stating a formal definition of converse.

In the Anderson, Keedy, and Rosenberg texts, as well as in many other standard treatments, the definitions of converse and some of the examples given are inconsistent, as has been demonstrated with the Hinge Theorem. All examples are consistent with Goodwin's definition. The difference is that Goodwin's definition acknowledges more than one converse, and the others do not. However, Goodwin's definition is inconsistent with many standard treatments of logic, which assume that a conditional statement has exactly one converse.

The fact is that if one considers a conditional geometric statement and examines it in light of Goodwin's definition, one is likely to find many interesting and useful arguments. It is the purpose of this chapter to show how a geometry teacher can use these arguments to promote discovery and intuitive thinking in a deductive geometry course.

Since it is desirable to avoid the logical inconsistencies cited above, the arguments constructed from conditional geometric statements will not be called converses. They will be referred to as partial converses. A formal definition of partial converse will now be stated.

> A partial converse of an argument is an argument that is obtained by interchanging any number of component statements in the hypothesis with an equal number of component statements in the conclusion.

One might reasonably ask why the definition of partial converse requires that an equal number of component statements be interchanged. First, as will shortly be established, that stated definition yields a high percentage of valid (and interesting) arguments. Secondly, there is evidence that an unequal interchange of component statements will not produce similar results. After Lazar examined many theorems and constructed arguments by interchanging a number of component statements in the hypothesis with an unequal number of component statements in the conclusion, he made the following observation:

> No theorem in geometry was found which yielded a true converse by an unequal interchange of hypothesis and conclusion. (13, p. 107).

The number of partial converses of a given argument depends, of course, on the number of component statements in the hypothesis and conclusion. Table XIV illustrates how one can calculate the number of partial converses for some of the common argument forms. The symbol

${}_nC_k$ represents the number of ways one can choose k statements from a set of n statements ($n \geq k$).

TABLE XIV

CALCULATION OF NUMBER OF PARTIAL CONVERSES
FOR COMMON ARGUMENT FORMS

Argument form	Number of partial converses
$p \to q$	${}_1C_1 \cdot {}_1C_1 = 1 \cdot 1 = 1$
$(p_1 \wedge p_2) \to q$	${}_2C_1 \cdot {}_1C_1 = 2 \cdot 1 = 2$
$(p_1 \wedge p_2) \to (q_1 \wedge q_2)$	${}_2C_1 \cdot {}_2C_1 + {}_2C_2 \cdot {}_2C_2 = 2 \cdot 2 + 1 \cdot 1 = 5$
$(p_1 \wedge p_2 \wedge p_3) \to q$	${}_3C_1 \cdot {}_1C_1 = 3 \cdot 1 = 3$
$(p_1 \wedge p_2 \wedge p_3) \to (q_1 \wedge q_2)$	${}_3C_1 \cdot {}_2C_1 + {}_3C_2 \cdot {}_2C_2 = 3 \cdot 2 + 3 \cdot 1 = 9$
$(p_1 \wedge p_2 \wedge p_3) \to (q_1 \wedge q_2 \wedge q_3)$	${}_3C_1 \cdot {}_3C_1 + {}_3C_2 \cdot {}_3C_2 + {}_3C_3 \cdot {}_3C_3 =$ $3 \cdot 3 + 3 \cdot 3 + 1 \cdot 1 = 19$

It can be observed from Table XIV that the argument $p \to q$ has exactly one partial converse, $q \to p$. According to any of the text definitions of converse, the converse of $p \to q$ is identical to the partial converse of $p \to q$.

Use of Partial Converses to Promote Discovery

Once introduced to the concept of a partial converse, the student has at his disposal an extremely useful technique for discovering geometric arguments, some of which may be valid and useful. Now it is certainly not expected that a student will be able to establish immediately the validity or non-validity of every argument that he discovers. It is entirely possible that more material will need to be developed before validity or non-validity of a specific argument can be established. At this point intuition can enter a deductive course in geometry, for students can exercise their intuitive abilities and "guess" whether a discovered argument is valid. In most cases, the right-ness or wrong-ness of the guess can be established at some time during the course when the appropriate material is developed.

Example 4.1

Once a student has proved the following theorem, he can "discover" five other arguments by looking at its partial converses. In this case, all of the partial converses are valid arguments.

Theorem 5: The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.

Hypothesis: $\triangle ABC$.

p_1: $AB=AC$.

p_2: $\overline{AD}$ bisects $\angle BAC$.

Conclusion:

q_1: $BD=DC$.

q_2: $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

Argument form: $(p_1 \wedge p_2) \rightarrow (q_1 \wedge q_2)$.

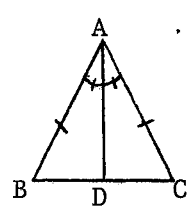


Figure 39

This argument is easily proved valid using SAS and elementary supplementary angle theorems.

Partial Converse 1: $(p_1 \wedge q_1) \rightarrow (p_2 \wedge q_2)$.

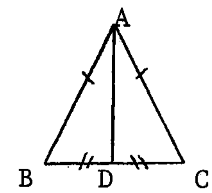


Figure 40

Hypothesis: ΔABC.

p_1: AB=AC.

q_1: BD=DC.

Conclusion:

p_2: $\overline{AD}$ bisects $\angle BAC$.

q_2: $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

The validity of this argument can be established using SSS and elementary supplementary angles theorems.

Partial Converse 2: $(p_1 \wedge q_2) \rightarrow (q_1 \wedge p_2)$.

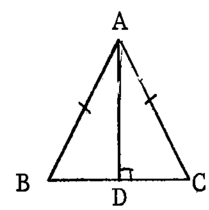


Figure 41

Hypothesis: ΔABC.

p_1: AB=AC.

q_2: $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

Conclusion:

q_1: BD=DC.

p_2: $\overline{AD}$ bisects $\angle BAC$.

The validity of this argument can be established when the hypotenuse-leg triangle congruence theorem is developed.

Partial Converse 3: $(q_1 \wedge p_2) \rightarrow (p_1 \wedge q_2)$.

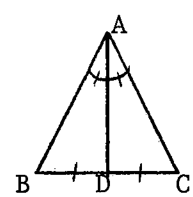


Figure 42

Hypothesis: ΔABC.

q_1: BD=DC.

p_2: $\overline{AD}$ bisects $\angle BAC$.

Conclusion:

p_1: AB=AC.

q_2: $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

This argument can be established as valid by using the theorem that states that the bisector of an angle of a triangle divides the opposite sides into segments that are proportional to the adjacent sides. That is, $\frac{BD}{DC}=\frac{AB}{AC}$. Since BD=DC, one may conclude that AB=AC. The truth of q_2 easily follows.

Partial Converse 4: $(q_2 \wedge p_2) \rightarrow (p_1 \wedge q_1)$.

Hypothesis: $\triangle ABC$.	Conclusion:
q_2: $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.	p_1: $AB=AC$.
p_2: $\overline{AD}$ bisects $\angle BAC$.	q_1: $BD=DC$.

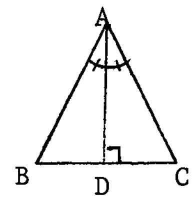


Figure 43

Validity is easily established using ASA.

Partial Converse 5: $(q_1 \wedge q_2) \rightarrow (p_1 \wedge p_2)$.

Hypothesis: $\triangle ABC$.	Conclusion:
q_1: $BD=DC$.	p_1: $AB=AC$.
q_2: $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.	p_2: $\overline{AD}$ bisects $\angle BAC$.

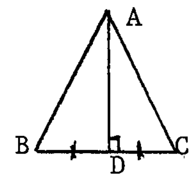


Figure 44

Validity is easily established using ASA.

The teacher can continually reinforce the concept of partial converse by having students examine the partial converses of theorems that are stated in the problem sets in their texts. If such problems are carefully chosen by the teacher, the student can carry out this task without a great expenditure of time. For instance, the problem in Example 4.2 appears in most texts after theorems about parallel lines are introduced. As part of a homework assignment, a student can be asked to establish the validity of the stated argument and to examine its partial converses.

Example 4.2

Problem: $\overline{AD}$ and $\overline{CD}$ bisect each other at E. Prove that $\overleftrightarrow{AD} // \overleftrightarrow{CB}$.

Hypothesis: The figure.	Conclusion:
p_1: $AE=BE$.	q: $\overleftrightarrow{AD} // \overleftrightarrow{CB}$.
p_2: $CE=DE$.	
Argument form: $(p_1 \wedge p_2) \rightarrow q$.	

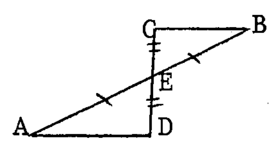


Figure 45

The argument and its two partial converses are easily proved valid using ASA.

Partial Converse 1: $(p_1 \wedge q) \rightarrow p_2$.

Hypothesis: The figure.	Conclusion:
p_1: AE=BE.	p_2: CE=DE.
q: $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$.	

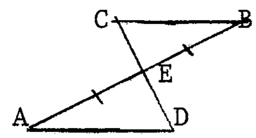


Figure 46

Partial Converse 2: $(p_2 \wedge q) \rightarrow p_1$.

Hypothesis: The figure.	Conclusion:
p_2: CE=DE.	p_1: AE=BE.
q: $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$.	

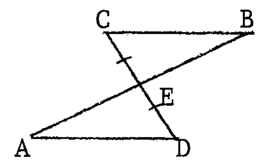


Figure 47

An alert student may notice that the two partial converses in Example 4.2 are not distinct arguments. This fact is certainly worthy of class discussion.

It should be carefully noted that a partial converse of a valid argument is not necessarily a valid argument. For example, two of the three partial converses of Theorem 3 are invalid.

Exercise Set 4a

Examine the partial converses of each theorem and identify those that are invalid arguments.

1. Theorem 6: If two sides of a triangle are not congruent, then the angles opposite these sides are not congruent, and the larger angle is opposite the longer side.

Hypothesis: ΔABC	Conclusion:
p: $\overline{AB} > \overline{AC}$.	q: $\angle C > \angle B$.

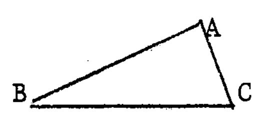


Figure 48

2. Theorem 7: A line containing the center of a circle and perpendicular to a chord of the circle bisects the chord.

Hypothesis: The circle with chord $\overline{AB}$.

p_1: $\overleftrightarrow{CD}$ is a line containing the center of the circle.

p_2: $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$.

Conclusion:

q: $\overleftrightarrow{CD}$ bisects $\overline{AB}$.

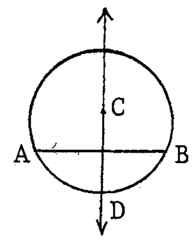


Figure 49

3. Theorem 8: If opposite sides of a quadrilateral are parallel, the opposite sides are congruent and the opposite angles are congruent.

Hypothesis: Quadrilateral ABCD.

p_1: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

p_2: $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

Conclusion:

q_1: $\overline{AB} \cong \overline{CD}$.

q_2: $\overline{AD} \cong \overline{BC}$.

q_3: $\angle A \cong \angle C$.

q_4: $\angle B \cong \angle D$.

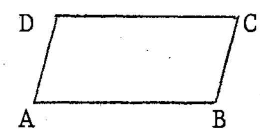


Figure 50

Using Partial Converses to Relate Theorems and Postulates

The concept of a partial converse provides a teacher and his students an opportunity to establish relationships between many theorems and postulates that are treated as unrelated in textbooks. It is certainly not essential that these relationships be established. However, as the geometry teacher can note, the establishment of these relationships stresses the use of definitions and equivalent statements and extends the use of partial converses as a discovery technique.

The definition stated below is taken from Rosenberg's text. The corresponding definitions in the other sample texts are equivalent to this definition.

Definition: $\triangle ABC$ is said to be <u>congruent</u> to $\triangle A'B'C'$ if and only if

$\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{AC} \cong \overline{A'C'}$

$\angle C \cong \angle C'$, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$. (19, p. 153).

This definition will be utilized in the following example.

Example 4.3

The SAS Postulate is usually the first triangle congruence postulate presented in geometry textbooks. This postulate is stated below.

SAS Postulate: If two sides and the included angle of one triangle are congruent respectively to corresponding parts of a second triangle, then the triangles are congruent.

Using the stated definition of congruent triangles, the conclusion of the SAS Postulate is equivalent to saying that corresponding sides and corresponding angles of the two triangles are congruent. However, the hypothesis of the SAS Postulate states that three of these corresponding parts are congruent. Using the definition of congruent triangles and the law of transitivity of conditionals, one can construct an argument equivalent to the SAS Postulate.

Hypothesis: ΔABC and $\Delta A'B'C'$.

p_1: $a=a'$.

p_2: $m\angle C=m\angle C'$.

p_3: $b=b'$.

Conclusion:

q_1: $m\angle A=m\angle A'$.

q_2: $c=c'$.

q_3: $m\angle B=m\angle B'$.

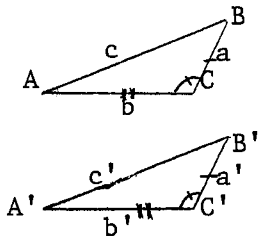


Figure 51

Argument form: $(p_1 \wedge p_2 \wedge p_3) \rightarrow (q_1 \wedge q_2 \wedge q_3)$.

Among the partial converses of this argument one finds the following:

Partial Converse 1: $(q_1 \wedge q_2 \wedge q_3) \rightarrow (p_1 \wedge p_2 \wedge p_3)$.

Hypothesis: ΔABC and $\Delta A'B'C'$.

q_1: $m\angle A=m\angle A'$.

q_2: $c=c'$.

q_3: $m\angle B=m\angle B'$.

Conclusion:

p_1: $a=a'$.

p_2: $m\angle C=m\angle C'$.

p_3: $b=b'$.

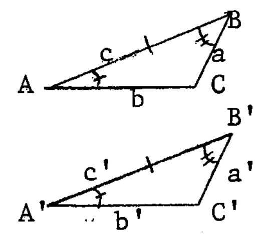


Figure 52

This argument is equivalent to the ASA Postulate, stated below. (In some texts, ASA is established as a theorem.)

> ASA Postulate: If two angles and the included side of one triangle are congruent respectively to the corresponding parts of a second triangle, then the triangles are congruent.

Partial Converse 2: $(p_1 \wedge q_2 \wedge p_3) \rightarrow (q_1 \wedge p_2 \wedge q_3)$.

Hypothesis: ΔABC and $\Delta A'B'C'$.

p_1: $a=a'$.

q_2: $c=c'$.

p_3: $b=b'$.

Conclusion:

q_1: $m\angle A=m\angle A'$.

p_2: $m\angle C=m\angle C'$.

q_3: $m\angle B=m\angle B'$.

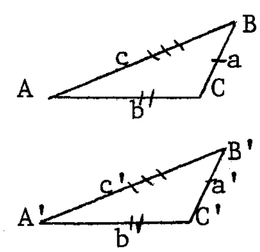


Figure 53

This argument is equivalent to the SSS Postulate. (In some texts, SSS is established as a theorem.)

> SSS Postulate: If three sides of one triangle are congruent to the three sides of a second triangle, then the triangles are congruent.

Partial Converse 3: $(p_1 \wedge q_1 \wedge q_3) \rightarrow (p_2 \wedge q_2 \wedge p_3)$.

Hypothesis: ΔABC and $\Delta A'B'C'$.

p_1: $a = a'$.

q_1: $m\angle A = m\angle A'$.

q_3: $m\angle B = m\angle B'$.

Conclusion:

p_2: $m\angle C = m\angle C'$.

q_2: $c = c'$.

p_3: $b = b'$.

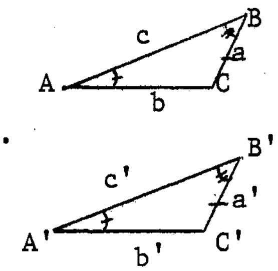


Figure 54

This argument is equivalent to the SAA Theorem.

> SAA Theorem: If two angles and a non-included side of one triangle are congruent respectively to two angles and the corresponding non-included side of a second triangle, then the triangles are congruent.

Example 4.4

In a discussion of similar triangles, some seemingly unrelated theorems in textbooks can be related using the partial converse concept. First, the class must understand a definition of similar polygons, such as the following.

> Definition: Two polygons are <u>similar</u> to each other if their corresponding angles are congruent and their corresponding sides are proportional. (19, p. 311).

The first theorem established in a study of similar triangles is usually the following:

> AA Similarity Theorem: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

After introducing the definition of similar polygons, the teacher can show that the conclusion of the AA Similarity Theorem is equivalent to saying that corresponding sides of the two triangles are proportional and that corresponding angles are congruent. However, the hypothesis of the AA similarity Theorem states that two pairs of corresponding angles are congruent. Hence, it is only necessary to conclude that the corresponding sides are proportional and that the third angles of each triangle are congruent. Using the definition of similar triangles and the law of transitivity for conditionals, the teacher can show that the AA Similarity Theorem is equivalent to the following argument.

Hypothesis: ΔABC and $\Delta A'B'C'$.

p_1: $m\angle A = m\angle A'$.

p_2: $m\angle B = m\angle B'$.

Conclusion:

q_1: $a:a' = b:b'$.

q_2: $a:a' = c:c'$.

q_3: $b:b' = c:c'$.

q_4: $m\angle C = m\angle C'$.

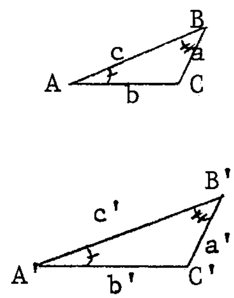


Figure 53

Argument form: $(p_1 \wedge p_2) \rightarrow (q_1 \wedge q_2 \wedge q_3 \wedge q_4)$.

Among the partial converses of this theorem, one finds the following:

Partial Converse 1: $(q_1 \wedge q_2) \rightarrow (p_1 \wedge p_2 \wedge q_3 \wedge q_4)$.

Hypothesis: ΔABC and $\Delta A'B'C'$.

q_1: $a:a' = b:b'$.

q_2: $a:a' = c:c'$.

Conclusion:

p_1: $m\angle A = m\angle A'$.

p_2: $m\angle B = m\angle B'$.

q_3: $b:b' = c:c'$.

q_4: $m\angle C = m\angle C'$.

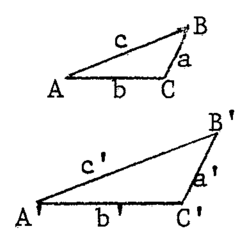


Figure 54

This argument is equivalent to the SSS Similarity Theorem, stated below.

SSS Similarity Theorem: Two triangles are similar if corresponding sides are proportional.

Partial Converse 2: $(q_2 \wedge p_2) \rightarrow (q_1 \wedge p_1 \wedge q_3 \wedge q_4)$.

Hypothesis: $\triangle ABC$ and $\triangle A'B'C'$.

q_2: $a:a' = c:c'$.

p_2: $m\angle B = m\angle B'$.

Conclusion:

q_1: $a:a' = b:b'$.

p_1: $m\angle A = m\angle A'$.

q_3: $b:b' = c:c'$.

q_4: $m\angle C = m\angle C'$.

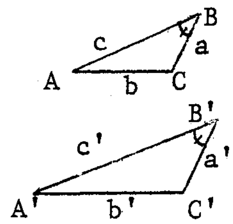


Figure 55

This argument is equivalent to the SAS Similarity Theorem, stated below.

SAS Similarity Theorem: Two triangles are similar if an angle of one is congruent to an angle of the other, and the corresponding sides including the angles are proportional.

This writer has found from past experience that students often confuse definitions with postulates and basic theorems. For instance, when asked for a definition of congruent triangles, a student often states a triangle congruence postulate, such as SAS. Also, when asked for a definition of similar triangles, a student often states a triangle similarity theorem, such as the AA Similarity Theorem. The question "Why can't we use a postulate (or theorem) as a definition?" has been asked by even the most able student. Using a triangle congurence postulate as an example, a teacher can explain that a postulate is a useless statement unless one knows the definitions of all the terms within it. The necessity of this can be exemplified by a teacher as he shows the relationships between theorems and postulates in a manner similar to that demonstrated in Examples 4.3 and 4.4 The definitions of congruent triangles and similar polygons were essential in representing the initial arguments of these examples in symbolic form.

It is interesting to note that authors of geometry texts occasionally contribute to a student's difficulty in understanding the distinction between a definition and a postulate or a theorem. For instance, Moise presents the following theorem without ever having defined the "perpendicular bisecting plane of a segment."

> The perpendicular bisecting plane of a segment is the set of all points equidistant from the end points of the segment. (17, p. 251).

Since a formal definition of "perpendicular bisecting plane of a segment" is not stated, it is not surprising that a student might consider the statement of the above theorem as a definition. An interesting contrast can be made by noting the following theorem presented earlier in Moise's text.

> The perpendicular bisector of a segment, in a plane, is the set of all points of the plane that are equidistant from the end points of the segment. (17, p. 188).

Prior to presenting this theorem, Moise states a formal definition of the "perpendicular bisector of a segment in a plane." It is this writer's opinion that such inconsistencies contribute to the confusion of a student who has not made a clear-cut distinction between a definition and a postulate or theorem.

Exercise Set 4b

1. Consider the theorem stated in problem 3 of Exercise Set 4a. Find partial converses of this theorem that are presented as theorems in your geometry textbook.

2. Show that each of the four theorems below is a partial converse of the other three. The arguments are stated relative to the plane of the circle.

Theorem 9: A line perpendicular to a radius at its outer end is tangent to the circle.

Theorem 10: A tangent to a circle is perpendicular to the radius drawn to the point of contact.

Theorem 11: The perpendicular to a tangent to a circle at its point of contact passes through the center of the circle.

Theorem 12: The perpendicular line from the center of a circle to a tangent meets it at the point of contact.

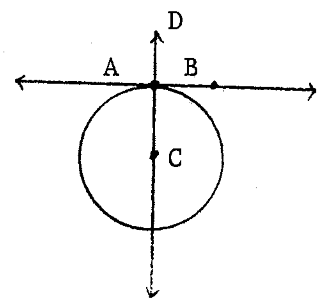


Figure 56

A Partial Converse Theorem

It has been established that a partial converse of a valid argument is not necessarily a valid argument. However, there are instances when the validity of three trichotomy-related arguments necessitates the validity of a partial converse of each of the three arguments, as shown in Theorem 13. The proof of this theorem is not difficult and, as will be demonstrated, the theorem has interesting applications in geometry. Since the proof does involve indirect reasoning, it will be noted at this point that indirect reasoning is discussed or used in the early chapters of each of the sample texts.

Theorem 13: Let p be a statement and a, a', b, b' be real numbers.

If the arguments

(1) $[p\wedge(a=a')]\rightarrow(b=b')$

(2) $[p\wedge(a>a')]\rightarrow(b>b')$

(3) $[p\wedge(a<a')]\rightarrow(b<b')$

are valid, then the arguments

(4) $[p\wedge(b=b')]\rightarrow(a=a')$

(5) $[p\wedge(b>b')]\rightarrow(a>a')$

(6) $[p\wedge(b<b')]\rightarrow(a<a')$

are also valid. Note that arguments (4), (5), and (6) are, respectively, partial converses of the arguments (1), (2), and (3).

The vaildity of argument (4) will now be established. The only way that (4) can be invalid is for the statemet $a=a'$ to be false while $p\wedge(b=b')$ is true. It will be shown that this cannot happen.

Assume that $p\wedge(b=b')$ is true. Then p is true, and $b=b'$ is true. Also, the Trichotomy Property of Real Numbers specifies that exactly one of the statements $a>a'$, $a<a'$, $a=a'$ must be true.

If $a>a'$ is true, then $p\wedge(a>a')$ is true. Therefore, by valid argument (2), $b>b'$ is true. This is impossible since $b=b'$ is true. Therefore, $a>a'$ is false.

If $a<a'$ is true, then $p\wedge(a<a')$ is true. Therefore, by valid argument (3), $b<b'$ is true. This is impossible since $b=b'$ is true. Therefore, $a<a'$ is false.

It has been shown that the truth of $p\wedge(b=b')$ necessitates that $a>a'$ is false and that $a<a'$ is false. Hence, using the Trichotomy Property, $a=a'$ is true. Since the truth of $p\wedge(b=b')$ necessitates the truth of $a=a'$, the argument (4) $[p\wedge(b=b')]\rightarrow(a=a')$ is valid. In similar manner, one can show that arguments (5) and (6) are valid.

Example 4.5

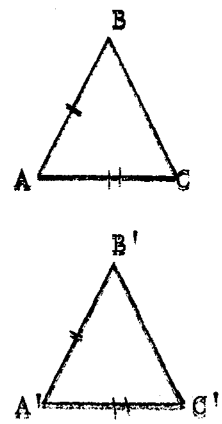


Figure 57

Consider the two triangles, $\triangle ABC$ and $\triangle A'B'C'$, with the properties that $AB=A'B'$ and $AC=A'C'$. Let p: $(AB=A'B')\wedge(AC=A'C')$.

The following valid arguments are useful in geometry, and they are presented in most geometry textbooks.

(1) $[p\wedge(m\angle A=m\angle A')]\rightarrow(BC=B'C')$.

(2) $[p\wedge(m\angle A>m\angle A')]\rightarrow(BC>B'C')$.

(3) $[p\wedge(m\angle A<m\angle A')]\rightarrow(BC<B'C')$.

According to Theorem 13, if the arguments above are established as valid, then the following arguments are also valid.

(4) $[p\wedge(BC=B'C')]\rightarrow(m\angle A=m\angle A')$.

(5) $[p\wedge(BC>B'C')]\rightarrow(m\angle A>m\angle A')$.

(6) $[p\wedge(BC<B'C')]\rightarrow(m\angle A<m\angle A')$.

The arguments (4), (5), and (6) are also useful in geometry and are presented as theorems in most geometry textbooks. However, in the text presentations, the validity of each argument is postulated or established by a formal proof. Theorem 13 renders this process unnecessary if the validity of (1), (2), and (3) is established.

Theorem 13 can be used with other sets of three trichotomy-related arguments. Two such sets appear in Exercise Set 4c.

Exercise Set 4c

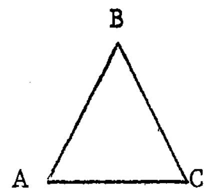


Figure 58

1. Given $\triangle ABC$. If the theorems

   (1) If AB=BC, then $m\angle A=a\angle C$,

   (2) If AB>BC, then $m\angle A>m\angle C$,

   (3) If AB<BC, then $m\angle A<m\angle C$

   are established as valid, use Theorem 13 to list three other valid arguments.

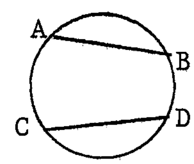


Figure 59

2. Let $\overline{AB}$ and $\overline{CD}$ be chords of a circle. If the theorems

   (1) If AB=CD, then $m\ \widehat{AB}=m\ \widehat{CD}$,

   (2) If AB>CD, then $m\ \widehat{AB}>m\ \widehat{CD}$,

   (3) If AB<CD, then $m\ \widehat{AB}<m\ \widehat{CD}$

   are established as valid, use Theorem 13 to list three other valid arguments.

Suggestions for Enrichment

The writer has found that much enthusiasm is developed in the geometry classroom if a mild competitive atmosphere is occasionally created by pitting various groups (a group might be a set of students in a row of seats) against each other in a contest that involves demonstrating the validity or non-validity of partial converses of valid argument. Much of the excitement is developed when the groups compete to establish that a certain partial converse is invalid. Non-validity is generally demonstrated with a drawing. When an individual creates a drawing that demonstrates the non-validity of a partial converse (that is, a drawing demonstrating that the conclusion is not a necessary consequence of the hypothesis), he raises his hand and then has the opportunity to produce his drawing on the blackboard. If his drawing is acceptable, his group earns a point. If the drawing does not demonstrate non-validity, the second group is given the opportunity to earn the point.

As an example, a partial converse of the SAS Postulate (Example 4.3) is the following:

> Two triangles are congruent if two sides and a non-included angle of one are congruent respectively to two sides and the non-included angle of the other.

An acceptable drawing that demonstrates the non-validity of the above argument is shown in Figure 60. Note that ΔABC and ΔABD satisfy the hypothesis of the theorem, but that they are not congruent. This example is a difficult one.

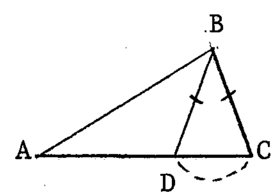


Figure 60

If the students are unable to produce a drawing to demonstrate non-validity, the teacher can create the drawing and give the point to the group finding triangles that satisfy the hypothesis, but not the conclusion, of the argument.

A similar activity consists of examining the partial converses of a postulate or theorem to find related arguments that are presented as theorems or postulates in the textbook. The group finding the largest number of related theorems or postulates is the winner. In this section it has been demonstrated that the SAS Postulate and the AA Similarity Theorem have partial converses that are basic geometric postulates or theorems. Many other valid geometric arguments have this property. For instance, the partial converses of Theorem 8 (Problem 3, Exercise Set 4a) include most of the basic theorems about parallelograms.

Summary

In this chapter the definitions of "converse" presented in the five sample texts were examined. Inconsistencies existing between the stated definitions, their use in the texts, and the basic laws of logic were noted. These inconsistencies presented no major problem in the geometric development in the texts, since it is generally the statement of an argument that is important in a geometry text and not the fact that it is called the converse of another argument.

However, the writer hopes that it has been established that if a teacher eliminates the inconsistencies by introducing the concept of a partial converse to his class, he has given his students a useful technique for discovering other meaningful geometric arguments. He can encourage use of this technique by organizing activities similar to those mentioned in the previous section or by other means that he may devise. It is this writer's opinion that the introduction of a concept of a partial converse enhances the learning experience in a deductive geometry course by offering the student an opportunity to assume an active role in the examination and construction of geometric arguments.

CHAPTER V

PARTIAL INVERSES

The Definition of a Partial Inverse

Secondary school students have been exposed to hundreds of hours of commercials on television and radio, in newspapers, and through other media. They are undoubtedly familiar with the following type of sales pitch:

> If you brand of soap is Nodirto, then you are using a good soap.

Having introduced the students to the argument above, they can be asked to complete the following statement in a manner that would please the manufacturers of Nodirto soap.

> If your brand of soap is not Nodirto, then _____________.
>
> (Answer: you are not using a good soap.)

Students have little difficulty with this kind of exercise and quickly recognize that if they consider the first argument to be $p \rightarrow q$, then the second is clearly $\sim p \rightarrow \sim q$. The manufacturers of Nodirto obviously hope that a listener or reader, upon hearing or seeing the first argument, will subconsciously produce the second one and consider the two arguments as equivalent. The alert student will intuitively induce that the arguments are not equivalent, and can convince himself with a truth table that $(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$ is not a tautology.

The statement $\sim p \rightarrow \sim q$ is generally considered to be in <u>inverse</u> of $p \rightarrow q$, and this is the way Keedy and Rosenberg define "inverse" in their texts. Goodwin says, "The <u>inverse</u> of an implication is written by negating both the antecedent and the consequent." (9, p. 103). Anderson and Moise do not state any definition of "inverse," nor do they use the concept in their respective texts.

It was demonstrated in the previous chapter that meaningful arguments can be constructed by examining partial converses of a geometric argument. Another set of arguments may be constructed from a given argument using the following definition.

> A <u>partial inverse</u> of an argument is an argument formed by negating a number of component statements of the hypothesis and an equal number of component statements in the conclusion.

It can be noted that the single partial inverse of the argument $p \rightarrow q$ is $\sim p \rightarrow \sim q$. In this case, the partial inverse is the same as the inverse as defined by Goodwin, Keedy, and Rosenberg.

Example 5.1

Theorem 14: If two lines are parallel, their slopes are equal.

Hypothesis	Conclusion
Hypothesis: Lines L_1 and L_2.	Conclusion:
p: $L_1 // L_2$.	q: Slope of L_1 = slope of L_2.

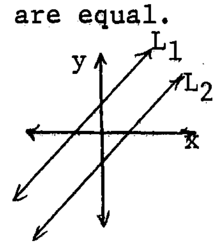


Figure 63

Argument form: $p \rightarrow q$.

The partial inverse $\sim p \rightarrow \sim q$ is easily stated: If two lines are not parallel, their slopes are not equal. In this case, the partial inverse is a valid argument.

Example 5.2

> Theorem 15: If two lines are parallel, their y-intercepts are not equal.

As with Example 5.1, the partial inverse is easily stated: If two lines are not parallel, then their y-intercepts are equal. This argument is invalid, as demonstrated in Figure 64.

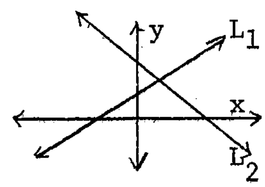


Figure 64

Example 5.3

> Theorem 16: In a plane, if a line intersects one of two parallel lines in exactly one point, then it intersects the other line in exactly one point.

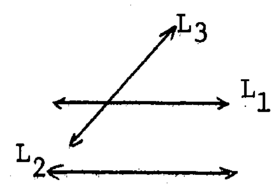


Figure 65

Hypothesis:	Conclusion:
Coplanar lines L_1, L_2, L_3.	
p: $L_1 // L_2$.	q: L_3 and L_2 intersect in exactly one point.
p_2: L_3 and L_1 intersect in exactly one point.	

Argument form: $(p_1 \wedge p_2) \rightarrow q$.

Partial Inverse 1: $(\sim p_1 \wedge p_2) \rightarrow \sim q$.

Hypothesis: Coplanar lines L_1, L_2, L_3.

$\sim p_1$: $L_1 \not\parallel L_2$.

p_2: L_3 and L_1 intersect in exactly one point.

Conclusion:

$\sim q$: L_3 and L_2 do not interesect in exactly one point.

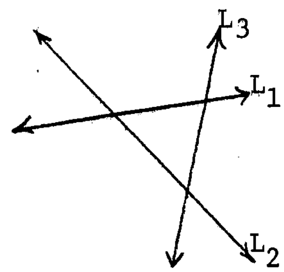


Figure 66

This argument is invalid (See Figure 66).

Partial Inverse 2: $(p_1 \wedge \sim p_2) \rightarrow \sim q$.

Hypothesis: Coplanar lines L_1, L_2, L_3

p_1: $L_1 /\!/ L_2$.

$\sim p_2$: L_3 and L_1 do not intersect in exactly one point.

Conclusion:

$\sim q$: L_3 and L_2 do not intersect in exactly one point.

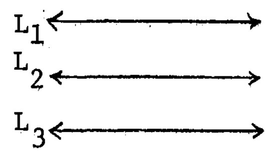


Figure 67

This argument is valid. Since the lines are coplanar, $\sim p_2$ is equivalent to saying that $L_3 /\!/ L_1$ and $\sim q$ is equivalent to saying $L_3 /\!/ L_2$. The validity is established using the theorem stating that two lines parallel to a third line are parallel.

Example 5.4

Theorem 17: In a plane, any point on the perpendicular bisector of a segment is equidistant from the end points of the segment.

Hypothesis: Line L and segment $\overline{AB}$.

p_1: P is a point on L

p_2: $L \perp \overleftrightarrow{AB}$.

p_3: L bisects $\overline{AB}$.

Conclusion:

q: PA=PB.

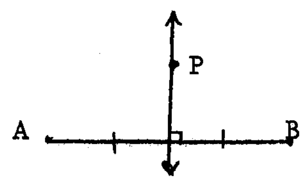


Figure 68

Argument form: $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$.

Without going into elaborate detail on this example, the writer concludes that the partial inverse

$$(\sim p_1 \wedge p_2 \wedge p_3) \rightarrow \sim q$$

is equivalent to saying that any point not on the perpendicular bisector of the segment is not equidistant from the end points of the segment. This partial inverse is valid. The partial inverse

$$(p_1 \wedge \sim p_2 \wedge p_3) \rightarrow \sim q$$

is not valid since $\sim q$ (that is, $PA \neq PB$) is not a necessary consequence of the hypothesis. This can be seen by taking P to be the point of intersection of L and $\overline{AB}$. The partial inverse

$$(p_1 \wedge p_2 \wedge \sim p_3) \rightarrow \sim q$$

can be established as valid with an indirect proof (See Figure 69).

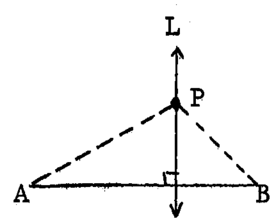


Figure 69

The arguments in the above examples contain exactly one component statement in their respective conclusions. The writer has tried negating more than one statement in the hypothesis and conclusions of many arguments, but has never found any interesting (from an instructional standpoint) and useful arguments by this process. This is not to say that the concept of a partial inverse cannot be meaningfully applied when an argument contains more than one component statement in its conclusion. Any argument, the conclusion of which is the conjunction of n statements, can be written as n arguments, each having one component statement in its conclusion. Interesting arguments can often be obtained by examining partial inverses of the n agruments. As an example, it can be observed in problem 4 of Exercise Set 5 that

$$q_1: \overleftrightarrow{AD} \perp \overleftrightarrow{BC}$$

is a necessary consequence of the given hypothesis. Hence

$$(p_1 \wedge p_2) \rightarrow (q \wedge q_1)$$

is a valid argument. The partial inverse

$$(\sim p_1 \wedge \sim p_2) \rightarrow (\sim q \wedge \sim q_1)$$

is meaningful, but, in the writer's opinion, not interesting from an instructional standpoint. However, the partial inverses of the arguments

$$(p_1 \wedge p_2) \rightarrow q$$

and

$$(p_1 \wedge p_2) \rightarrow q_1$$

include interesting and valid arguments.

Exercise Set 5

In problems 1-3, write the partial inverse of each argument and determine if it is valid.

1. If $\triangle ABC$ is equilateral, then $\triangle ABC$ is isosceles.
2. If $\frac{5}{a} = \frac{7}{b}$, then $\frac{a}{5} = \frac{b}{7}$.
3. If $\triangle ABC$ is congruent to $\triangle DEF$, then $\triangle ABC$ is similar to $\triangle DEF$.

In problems 4-5, examine the partial inverses of each argument and determine their validity or non-validity.

4. Hypothesis: $\triangle ABC$.

   p_1: $AB = AC$.

   p_2: $\overrightarrow{AD}$ bisects $\angle BAC$.

   Conclusion:

   q: $\overrightarrow{AD}$ bisects $\overline{BC}$.

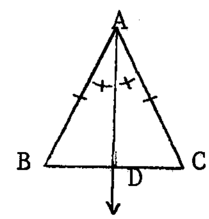


Figure 67

5. Hypothesis: Quadrilateral ABCD.

   p_1: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

   p_2: $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

   p_3: $m\angle A = 90$.

   Conclusion:

   q: $m\angle C = 90$.

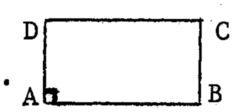


Figure 68

Suggestions for Enrichment

Students can be asked to explore the relationship between the number of partial converses and the number of partial inverses for a given argument form. It is not a difficult task to establish that the numbers are equal.

It can be established with a truth table that

$$(q \rightarrow p) \leftrightarrow (\sim p \rightarrow \sim q)$$

is a tautology. That is, the converse of the argument $p \rightarrow q$ is equivalent to the inverse of the argument. Students can then examine the possibility that a similar result holds for the partial converses and partial inverses of the argument

$$(p_1 \wedge p_2) \to q.$$

It is interesting to note that

$$[(q \wedge p_2) \to p_1] \leftrightarrow [(\sim p_1 \wedge p_2) \to \sim q]$$

is a tautology. However,

$$[(q \wedge p_2) \to p_1] \leftrightarrow [(p_1 \wedge \sim p_2) \to \sim q]$$

is not. In general, it can be established that for each partial converse of an argument, there is exactly one equivalent partial inverse, and vice-versa.

Summary

The purpose of this chapter was to introduce the concept of a partial inverse and demonstrate its use as a discovery technique. However, the major significance of a partial inverse in the deductive development of geometry will be discussed after the concept of "partial contrapositive" is introduced in the following chapter.

CHAPTER VI

PARTIAL CONTRAPOSITIVES

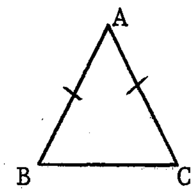
Examination of the Definition of Contrapositive

In previous chapters it was demonstrated how students can discover other theorems by examining the partial converses and partial inverses of postulates and theorems. But it was also demonstrated that partial converses and partial inverses of valid arguments are not necessarily valid arguments. Therefore, after discovering a new and seemingly valid argument by these techniques, the student must establish the validity of the argument before using it in the development of new material. In this chapter the reader will be introduced to an argument form that will automatically produce a valid argument if obtained from a valid argument.

The argument form to be introduced is similar to, and in some instances identical to, the statement form known as a "contrapositive." Three of the sample textbooks define "contrapositive," and two (Anderson and Moise) do not. Goodwin, Keedy, and Rosenberg all state that the argument $\sim q \to \sim p$ is the <u>contrapositive</u> of $p \to q$. It is easy to establish that $\sim q \to \sim p$ is equivalent to $p \to q$ with a truth table (problem 1, Exercise Set 2f). Hence, one can easily create a valid argument from another valid argument that has only one component statement in both hypothesis and conclusion by constructing the contrapositive of the **argument**.

Example 6.1

Theorem 18: If two sides of a triangle are congruent, the angles opposite these sides are congruent.



Hypothesis: Δ ABC.

p: $\overline{AB} \cong \overline{AC}$.

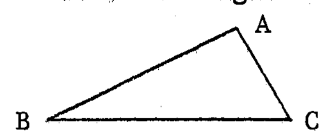
Conclusion:

q: $\angle B \cong \angle C$.

Argument form: $p \to q$.

Figure 72

Contrapositive of Theorem 18: If two angles of a triangle are not congruent, then the sides opposite these angles are not congruent.



Hypothesis: ΔABC.

$\sim q$: $\angle B \ncong \angle C$.

Conclusion:

$\sim p$: $\overline{AB} \ncong \overline{AC}$.

Figure 73

Argument form: $\sim q \to \sim p$.

The validity of $\sim q \to \sim p$ is usually established by indirect proof. However, indirect proof is not needed here. The argument $\sim q \to \sim p$ is valid by contrapositive argument.

Example 6.2

Consider the following argument: If two distinct lines are parallel, their y-intercepts are equal.

This argument is clearly invalid (since the two lines would intersect if their y-intercepts were equal), and hence its contrapositive, which is an equivalent statement, should also be invalid. The contrapositive can be stated in the following manner:

If the y-intercepts of two distinct lines are not equal, then the lines are not parallel.

The non-validity of this argument can be established by considering the lines $y=x$ and $y=x+1$.

Consider now an argument of the form $(p_1 \wedge p_2) \rightarrow q$. The contrapositive of this argument (using the traditional definition) is

$$\sim q \rightarrow \sim (p_1 \wedge p_2).$$

It is easily established with a truth table that

$$[(p_1 \wedge p_2) \rightarrow q] \leftrightarrow [\sim q \rightarrow \sim (p_1 \wedge p_2)]$$

is a tautology. Hence, if $(p_1 \wedge p_2) \rightarrow q$ is valid, then so is $\sim q \rightarrow \sim (p_1 \wedge p_2)$. This means that the truth of $\sim q$ necessitates the truth of $\sim (p_1 \wedge p_2)$, which, by one of DeMorgan's Laws, is equivalent to $\sim p_1 \vee \sim p_2$. One can conclude that the truth of $\sim q$ necessitates that it is impossible for both of $\sim p_1$ and $\sim p_2$ to be false. This leaves three possibilities:

(1) $\sim p_1$ and $\sim p_2$ are both true.

(2) $\sim p_1$ is true and $\sim p_2$ is false.

(3) $\sim p_1$ is false and $\sim p_2$ is true.

Example 6.3

Given quadrilateral ABCD (Figure 74). The validity of the following argument is easily established.

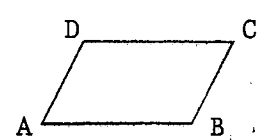


Figure 74

$$[(\overleftrightarrow{AB} // \overleftrightarrow{CD}) \wedge (\angle B \cong \angle D)] \rightarrow (\overleftrightarrow{AD} // \overleftrightarrow{BC}).$$

The contrapositive of this argument is

$$(\overleftrightarrow{AD} \not\parallel \overleftrightarrow{BC}) \rightarrow \sim[(\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}) \wedge (\angle B \cong \angle D)].$$

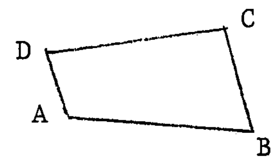


Figure 75

The truth of $\overleftrightarrow{AD} \not\parallel \overleftrightarrow{BC}$ would necessitate the truth of exactly one of the following statements.

(1) $(\overleftrightarrow{AB} \not\parallel \overleftrightarrow{CD}) \wedge (\angle B \not\cong \angle D)$. (See Figure 75.)

(2) $(\overleftrightarrow{AB} \not\parallel \overleftrightarrow{CD}) \wedge (\angle B \cong \angle D)$. (See Figure 76.)

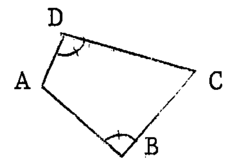


Figure 76

(3) $(\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}) \wedge (\angle B \not\cong \angle D)$. (See Figure 77.)

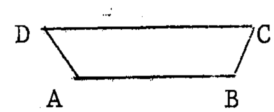


Figure 77

The conclusion of the contrapositive in Example 6.3 is meaningful and informative. However, it does have certain limitations. First, one can only conclude that exactly one of the statements (1), (2), or (3) must be true without being able to specify which one of the three is true. Secondly, in the contrapositive argument, one cannot determine a truth value for either of the statements $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\angle B \cong \angle D$. The point to emphasize is that if the hypothesis of a valid argument contains more than one component statement, one is unable to specify the truth value of these component statements in the contrapositive argument. This fact somewhat limits the practical use of the contrapositive of a valid argument with a compound hypothesis in a deductive geometry course.

The Definition of a Partial Contrapositive

At the beginning of this chapter, the contrapositive of the argument $p \rightarrow q$ was examined. The contrapositive of this argument did not have the aforementioned limitations of the contrapositive of an argument with a compound hypothesis. This section is devoted to constructing a valid argument form from a valid argument with a compound hypothesis such that the constructed argument does not have these limitations. The argument form is described in the following definition.

> A <u>partial contrapositive</u> of an argument is an argument obtained by negating a number of component statements in the hypothesis and an equal number of component statements in the conclusion and then interchanging the negated statements.

It should be noted that the single partial contrapositive of $p \rightarrow q$ is identical to the contrapositive, $\sim q \rightarrow \sim p$.

Attention will now be centered on partial contrapositives of arguments with conclusions containing exactly one component statement. As will be shown, these partial contrapositives are extremely useful in deductive geometry and do not have the previously-mentioned limitation. (It will again be noted that an argument with n component statements in its conclusion can be written as n arguments, each with one component statement in its conclusion.)

If an argument has the form $(p_1 \wedge p_2) \rightarrow q$, then the partial contrapositives of this argument are

$$(p_1 \wedge \sim q) \rightarrow \sim p_2 \qquad \text{and} \qquad (\sim q \wedge p_2) \rightarrow \sim p_1.$$

TABLE XV

TRUTH VALUES OF $(p_1 \wedge p_2) \rightarrow q$ AND ITS PARTIAL CONTRAPOSITIVES

p_1	p_2	q	$p_1 \wedge p_2$	$p_1 \wedge \sim q$	$\sim q \wedge p_2$	$(p_1 \wedge p_2) \rightarrow q$			$(p_1 \wedge \sim q) \rightarrow \sim p_2$			$(\sim q \wedge p_2) \rightarrow \sim p_1$		
T	T	T	T	F	F	T	T	T	F	T	F	F	T	F
T	T	F	T	T	T	T	F	F	T	F	F	T	F	F
T	F	T	F	F	F	F	T	T	F	T	T	F	T	F
T	F	F	F	T	F	F	T	F	T	T	T	F	T	F
F	T	T	F	F	F	F	T	T	F	T	F	F	T	T
F	T	F	F	F	T	F	T	F	F	T	F	T	T	T
F	F	T	F	F	F	F	T	T	F	T	T	F	T	T
F	F	F	F	F	F	F	T	F	F	T	T	F	T	T

Table XV shows that both of these partial contrapositives are equivalent to $(p_1 \wedge p_2) \rightarrow q$.

In a similar manner, one can consider the argument $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ and establish that it is equivalent to its three partial contrapositives

$(p_1 \wedge p_2 \wedge \sim q) \rightarrow \sim p_3$,

$(p_1 \wedge \sim q \wedge p_3) \rightarrow \sim p_2$, and

$(\sim q \wedge p_2 \wedge p_3) \rightarrow \sim p_1$.

In general, the argument $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is equivalent to every one of its n partial contrapositives.

Using any valid argument, the student now has at his disposal a technique for constructing other valid arguments. A point to emphasize is that one can establish the validity of an argument containing a one-component statement conclusion merely by identifying it as a partial contrapositive of a valid argument.

Example 6.4

Theorem 19: The bisector of the vertex angle of an isosceles triangle is perpendicular to the base of the triangle.

Hypothesis: $\triangle ABC$.

p_1: AB=AC.

p_2: $\overrightarrow{AD}$ bisects $\angle BAC$.

Argument form: $(p_1 \wedge p_2) \rightarrow q$.

Conclusion:

q: $\overrightarrow{AD} \perp \overleftrightarrow{BC}$.

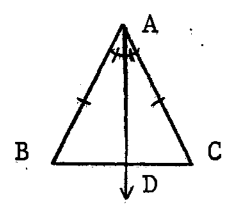


Figure 78

This is a valid argument, and hence its two partial contrapositives are valid.

Partial Contrapositive 1: $(p_1 \wedge \sim q) \rightarrow \sim p_2$.

Hypothesis: $\triangle ABC$.

p_1: $AB = AC$.

$\sim q$: $\overline{AD} \not\perp \overleftrightarrow{BC}$.

Conclusion:

$\sim p_2$: $\overrightarrow{AD}$ does not bisect $\angle BAC$.

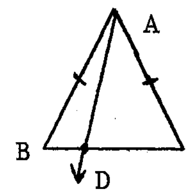


Figure 79

In words: If a ray (line) containing the vertex of an isosceles triangle is not perpendicular to the base, then the ray (line) does not bisect the vertex angle of the triangle.

Partial Contrapositive 2: $(\sim q \wedge p_2) \rightarrow \sim p_1$.

Hypothesis: $\triangle ABC$.

$\sim q$: $\overline{AD} \not\perp \overleftrightarrow{BC}$.

p_2: $\overrightarrow{AD}$ bisects $\angle BAC$.

Conclusion:

$\sim p_1$: $AB \neq AC$.

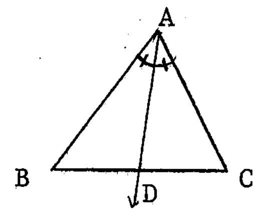


Figure 80

In words: If the bisector of an angle of a triangle is not perpendicular to the opposite side, then the sides of the triangle which include the angle are not congruent.

Example 6.5

Theorem 20: Any point on a perpendicular bisector of a segment is equidistant from the end points of the segment.

Hypothesis: Line L and segment $\overline{AB}$.

p_1: P is a point on L.

p_2: $L \perp \overleftrightarrow{AB}$.

p_3: L bisects $\overline{AB}$.

Conclusion:

q: $PA = PB$.

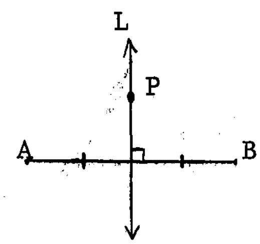


Figure 81

Argument form: $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$.

This argument is valid and hence its three partial contrapositives are valid.

Partial Contrapositive 1: $(p_1 \wedge p_2 \wedge \sim q) \rightarrow \sim p_3$.

Hypothesis: Line L and segment $\overline{AB}$.

p_1: P is a point on L.

p_2: $L \perp \overleftrightarrow{AB}$.

$\sim q$: $PA \neq PB$.

Conclusion:

$\sim p_3$: L does not bisect $\overline{AB}$.

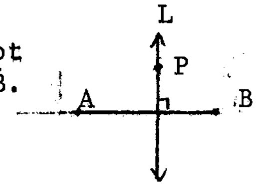


Figure 82

In words: If a line is perpendicular to a segment, and if there exists a point on the line which is not equidistant from the end points of the segment, then the line does not bisect the segment.

Partial Contrapositive 2: $(p_1 \wedge \sim q \wedge p_3) \rightarrow \sim p_2$.

Hypothesis: Line L and segment $\overline{AB}$.

p_1: P is a point on L.

$\sim q$: $PA \neq PB$.

p_3: L bisects $\overline{AB}$.

Conclusion:

$\sim p_2$: $L \not\perp \overleftrightarrow{AB}$.

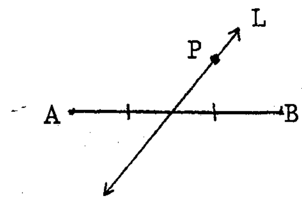


Figure 83

In words: If a line bisects a segment, and if there exists a point on the line which is not equidistant from the end points of the segment, then the line is not perpendicular to the segment.

It is interesting to note that if the hypothesis of partial contrapositive 2 is to be true, then P must not be $\overline{AB} \cap L$. If P is taken to be $\overline{AB} \cap L$, the hypothesis is false (since $\sim q$ would be false). However, the argument is still valid. (Recall that an argument is invalid only if the conclusion can be false when the hypothesis is true.)

Partial Contrapositive 3: $(\sim q \wedge p_2 \wedge p_3) \rightarrow \sim p_1$.

Hypothesis: Line L and segment $\overline{AB}$.	Conclusion:
$\sim q$: PA≠PB.	$\sim p_1$: P is not on L.
p_2: L⊥$\overleftrightarrow{AB}$.	
p_3: L bisects $\overline{AB}$.	

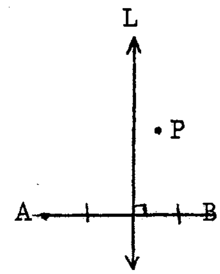


Figure 84

In words: If a line is a perpendicular bisector of a segment, and if a point is not equidistant from the end points of the segment, then the point is not on the line.

Exercise Set 6a

Write the partial contrapositives of each argument.

1. Supplements of congruent angles are congruent.
2. If a triangle is equilateral, then it is isosceles.
3. If two circles have unequal areas, then their radii are unequal.
4. If $a + b = c$ and $b>0$, then $a<c$.
5. If two planes are perpendicular to the same line, the planes are parallel.

Using Partial Contrapositives to Prove General Theorems

Many of the general theorems that appear in geometry textbooks can be proved using partial contrapositives. In many instances, this saves considerable time while preserving rigor, as students do not have to struggle through a complicated formal proof. (A teacher must exercise discretion here, since it is not to be denied that some complicated formal proofs are worthy of exploration by geometry students.)

Example 6.6

Prove Theorem 21: Two lines cut by a transversal are parallel if a pair of alternate interior angles are congruent.

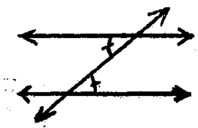


Figure 85

If the Exterior Angle Theorem for triangles has been established, the following theorem is easily proved.

Theorem 22: If two lines are not parallel, then a pair of alternate interior angles are not congruent.

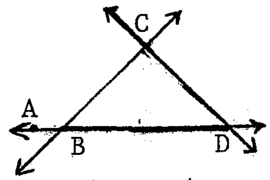


Figure 86

(Of the five sample texts, only Goodwin introduces the Exterior Angle Theorem after parallel line theorems have been established.) By constructing the contrapositive of Theorem 22, one obtains a valid argument, which is Theorem 21.

Example 6.7

Prove Theorem 23: A line joining the midpoints of two sides of a triangle is parallel to the third side.

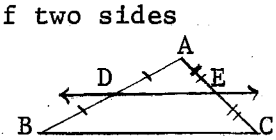


Figure 87

To prove Theorem 23 using partial contrapositives, one needs the following theorem.

Theorem 24: If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on every transversal.

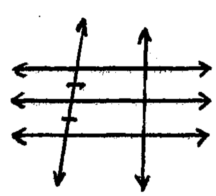


Figure 88

(Three of the five sample texts--Anderson, Goodwin, and Moise--establish Theorem 24 after Theorem 23. However, in each of these texts, Theorem 24 could easily be proved first.)

A corollary of Theorem 24 is the following: If a line is parallel to one side of a triangle and bisects a second side, then it bisects the third side.

Hypothesis: $\triangle ABC$.

p_1: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

p_2: D is the midpoint of $\overline{AB}$.

Conclusion:

q: E is the midpoint of $\overline{AC}$.

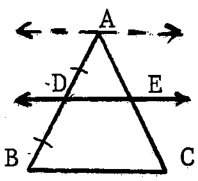


Figure 89

Argument form: $(p_1 \wedge p_2) \rightarrow q$.

A partial inverse of this corollary is $(\sim p_1 \wedge p_2) \rightarrow \sim q$.

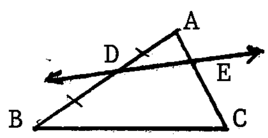


Figure 90

Hypothesis: $\triangle ABC$.

$\sim p_1$: $\overleftrightarrow{DE} \not\parallel \overleftrightarrow{BC}$.

p_2: D is the midpoint of $\overline{AB}$.

Conclusion:

$\sim q$: E is not the midpoint of $\overline{AC}$.

The validity of this partial inverse can be established by introducing the unique line through B parallel to $\overleftrightarrow{DE}$. This line will intersect $\overleftrightarrow{AC}$ at a point F. (See Figure 91.) Then AE = EF by Theorem 24. Since EF $\neq$ EC, one concludes that AE $\neq$ EC, and that E is not the midpoint of $\overline{AC}$.

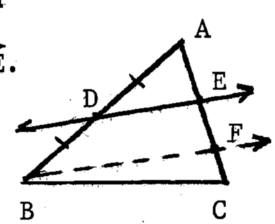


Figure 91

Since $(\sim p_1 \wedge p_2) \rightarrow \sim q$ is valid, a partial contrapositive, $(q \wedge p_2) \rightarrow p_1$ is also valid.

Hypothesis: $\triangle ABC$.

q: E is the midpoint of $\overline{AC}$.

p_2: D is the midpoint of $\overline{AB}$.

Conclusion:

p_1: $\overleftrightarrow{DE} // \overleftrightarrow{BC}$.

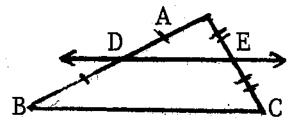


Figure 92

Note that this partial contrapositive is Theorem 23, the argument that was to be proved valid.

These examples illustrate how a partial contrapositive of a valid argument may be used to establish the validity of common geometric theorems. It is, of course, impossible to present illustrations that can be directly applied to all geometry texts, for any two texts may present completely different orders of theorems. However, using any text that presents a deductive development of geometry, the alert teacher and his students can utilize methods of proof similar to those in the examples. For just one instance, problems presented in problem sets are often partial contrapositives of other arguments that have been established as valid.

Exercise Set 6b

In each problem, if the stated argument can be established as valid, what common geometric theorem could be obtained by using partial contrapositives?

1. If a transversal cuts two lines and if a pair of alternate interior angles are not congruent, then the lines are not parallel.
2. If a line is perpendicular to one of two non-perpendicular planes, then the line is not contained in the second plane.

3. If A, M, and C are points on a line L, and if M and A are on opposite sides of any other line that contains C, then M is not between A and C.

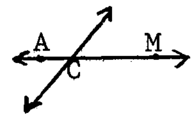


Figure 93

4. In $\triangle ABC$, if $\overleftrightarrow{DE}$ is not parallel to $\overleftrightarrow{BC}$, then $\frac{AD}{DB} \neq \frac{AE}{EC}$.

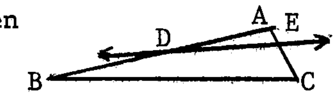


Figure 94

Finding a Necessary Conclusion for a Given Hypothesis

The typical geometry problem involving proof presents the student with a stated hypothesis and a stated conclusion to be obtained from the hypothesis by logical argument. The conclusion generally does not follow immediately from the hypothesis. To create the series of valid arguments referred to in Chapter 3, the student must take the given hypothesis and construct an interim conclusion. He then uses the interim conclusion in conjunction with the hypothesis to construct another conclusion which is either the conclusion of the stated argument or another interim conclusion. This process is continued until the conclusion of the stated argument becomes a necessary consequence of the conjunction of the given hypothesis with obtained interim conclusions. Hence, a good portion of a student's time is spent constructing interim conclusions that can be used to obtain eventually the stated conclusion.

A common error made by students in the construction of a formal proof involves obtaining an interim conclusion that is not a necessary consequence of the stated hypothesis and other interim conclusions. The writer has found that this type of error is very common in proofs involving geometric inequalities. That is, students often have difficulty obtaining necessary conclusions (interim or otherwise) from a hypothesis that contains statements of inequality. The writer has also found that the occurrence of this type of error can be lessened by introducing the partial contrapositive technique demonstrated in the following examples.

Example 6.8

What conclusion is a necessary consequence of the given hypothesis?

Hypothesis: $\triangle ABC$ and $\triangle DEF$.

p_1: $AB=DE$.

p_2: $BC=EF$.

p_3: $m\angle A \neq m\angle D$.

Conclusion:

q: ?

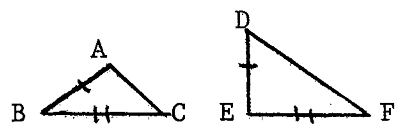


Figure 95

Solution: A method of attack is to examine the partial contrapositive form

$$(p_1 \wedge p_2 \wedge \sim q) \rightarrow \sim p_3.$$

Hypothesis: $\triangle ABC$ and $\triangle DEF$.

p_1: $AB=DE$.

p_2: $BC=EF$.

$\sim q$: ?

Conclusion:

$\sim p_3$: $m\angle A = m\angle D$.

Now if $\sim q$ is defined to be the statement

$$m\angle B = m\angle E,$$

then the partial contrapositive is a valid argument. Hence q: $m\angle B \neq m\angle E$ is a necessary consequence of the given hypothesis. (It should be noted that $\sim q$ can also be defined to the statement $AC=DF$.)

Example 6.9

Supply a necessary conclusion for the given hypothesis.

Hypothesis: ΔABC.

p_1: $\overline{AB}$ is the longest side.

p_2: $m\angle B = 30$.

p_3: $BC \neq \sqrt{3}(AC)$.

Conclusion:

q: ?

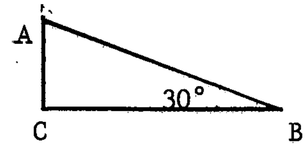


Figure 96

Solution: Proceding as in Example 6.8, one can examine the partial contrapositive form $(p_1 \wedge p_2 \wedge \sim q) \rightarrow \sim p_3$.

Hypothesis: ΔABC.

p_1: $\overline{AB}$ is the longest side.

p_2: $m\angle B = 30$.

$\sim q$: ?

Conclusion:

$\sim p_3$: $BC = \sqrt{3}(AC)$.

If $\sim q$ is defined to be the statement

ΔABC is a right triangle,

then the partial contrapositive is valid. Hence

q: ΔABC is not a right triangle

is a necessary conclusion of the given hypothesis.

Example 6.10

Supply a necessary conclusion for the given hypothesis.

Hypothesis: The circle with external point P.

p_1: $\overline{PA}$ is a tangent segment to the circle

p_2: $PA \neq PB$.

Conclusion:

q: ?

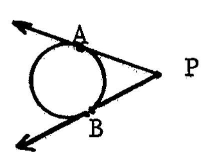


Figure 97

Solution: Examine the partial contrapositive

$$(p_1 \wedge \sim q) \rightarrow \sim p_2.$$

Hypothesis: The circle with external point P.	Conclusion:
p_1: $\overline{PA}$ is a tangent segment to the circle.	$\sim p_2$: PA=PB.
$\sim q$: ?	

If $\sim q$ is defined to be the statement

$\overleftrightarrow{PB}$ is tangent to the circle at B,

then the partial contrapositive is a valid argument. Hence, for the original hypothesis, a necessary conclusion is

q: $\overleftrightarrow{PB}$ is not tangent to the circle at B.

Note that q is not equivalent to saying that $\overleftrightarrow{PB}$ is not tangent to the circle.

Exercise Set 6c

In each problem, supply a necessary conclusion for the stated hypothesis.

1. Hypothesis: Quadrilateral ABCD.	Conclusion:
p_1: E is the midpoint of $\overline{AB}$.	q: ?
p_2: F is the midpoint of $\overline{BC}$.	
p_3: G is the midpoint of $\overline{CD}$.	
p_4: $\overleftrightarrow{GH}$ is not parallel to $\overleftrightarrow{EF}$.	

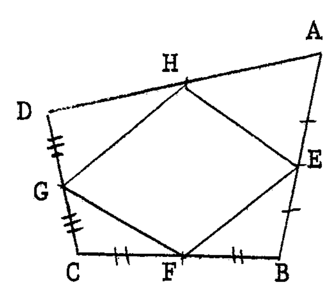


Figure 98

2. Hypothesis: $\triangle ABC$ with a point D between B and C.

p_1: $AB \neq AC$.

p_2: $\angle BAD \cong \angle CAD$.

Conclusion:

q: ?

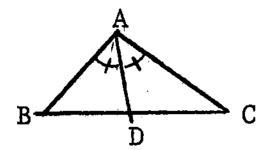


Figure 99

3. Hypothesis: The sphere and plane R.

p_1: R is tangent to the sphere at point P.

p_2: $\overleftrightarrow{OP}$ is not perpendicular to R.

Conclusion:

q: ?

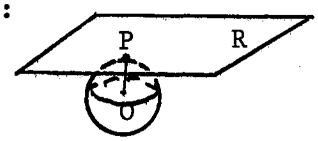


Figure 100

Relating Converses and Inverses

Most geometry textbooks, including those in the sample for this manuscript, contain many characterization (locus) problems. Problems of this type involve proving that a given set, S, consists of <u>all</u> of the points satisfying a given condition. That is, one is required to show that

(1) If P is any point in S, then P satisfies the given condition.

(2) If P is any point satisfying the given condition, then P is in S.

In many characterization problems, statement (1) can be represented in the conditional form $p \rightarrow q$. A simple truth table can be constructed to show that

$$(q \rightarrow p) \leftrightarrow (\sim p \rightarrow \sim q)$$

is a tautology. That is, the converse and the inverse of $p \rightarrow q$ are equivalent statements. (One can also note that $\sim p \rightarrow \sim q$ is the contrapositive of $q \rightarrow p$.) Hence, to show that the converse of $p \rightarrow q$ is a valid argument, it suffices to show that the inverse of $p \to q$ is a valid argument.

Example 6.11

In a plane, if one wishes to prove that the perpendicular bisector of a segment is the set of all points equidistant from the end points of the segment, he must prove that

(1) all points on the perpendicular bisector of the segment are equidistant from the end points of the segment,

and the converse of (1), which is

(2) all points equidistant from the end points of the segment are on the perpendicular bisector of the segment.

However, to prove (2), it suffices to prove the inverse of (1), which is

(2a) all points not on the perpendicular bisector of the segment are not equidistant from the end points of the segment.

The procedure for the total proof is outlined below.

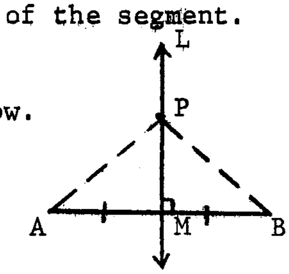


Figure 101

(1) Hypothesis:	Conclusion:
$\overline{AB}$ and its perpendicular bisector L.	
p: P is any point on L.	q: PA=PB.

If P is M, the result follows immediately. If P is not M, the validity of $p \to q$ is easily established using SAS.

(2a) Hypothesis: $\overline{AB}$ and its perpendicular bisector L.

p: P is any point not on L.

Conclusion:

q: $PA \neq PB$.

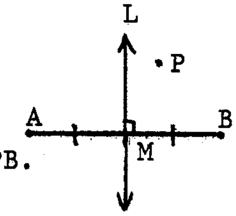


Figure 102

The validity of (2a) can be established using the procedure discussed in the last section. First, one can note that $\overleftrightarrow{PM} \not\perp \overleftrightarrow{AB}$, since $\overleftrightarrow{PM}$ is not L and the perpendicular to $\overleftrightarrow{AB}$ at M is unique. Thus, one can present the hypothesis of (2a) in the following manner in order to seek a necessary conclusion q_1.

Hypothesis: $\overline{AB}$ and its perpendicular bisector L.

p_1: $\overleftrightarrow{PM} \not\perp \overleftrightarrow{AB}$.

p_2: $AM = BM$.

Conclusion:

q_1: ?

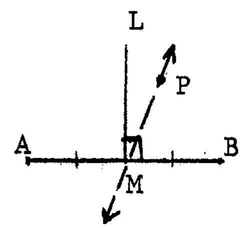


Figure 103

Examining a partial contrapositive of $(p_1 \wedge p_2) \rightarrow q_1$, one obtains the following:

Hypothesis: $\overline{AB}$ and its perpendicular bisector L.

$\sim q_1$: ?

p_2: $AM = BM$.

Conclusion:

$\sim p_1$: $\overleftrightarrow{PM} \perp \overleftrightarrow{AB}$.

Now if $\sim q_1$ is defined to be the statement

$$PA = PB,$$

then $(\sim q_1 \wedge p_2) \rightarrow \sim p_1$ is a valid argument. Hence

$$q_1: \ PA \neq PB$$

is a necessary consequence of $p_1 \wedge p_2$ and the argument (2a) is valid.

Exercise Set 6d

1. To prove that

In a plane, the set of all points equidistant from two parallel lines L_1 and L_2 is a line parallel to L_1 and L_2 and midway between them,

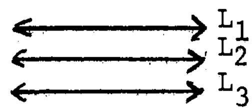


Figure 104

one would prove that

(a) any point on L is ________________________________

and

(b) any point not on L is ________________________________.

2. Follow the outline below to prove that

The set of all points in the interior of an angle that are equidistant from the sides of the angle is the bisecting ray, minus its end point.

(a) Prove that if $\overrightarrow{CD}$ is the bisecting ray of $\angle ACE$ and if P is a point on $\overrightarrow{CD}$ other than C, then P is equidistant from $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

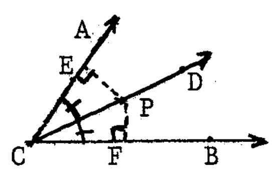


Figure 105

(b) Now prove that if P is in the interior of $\angle ACB$ and not on the bisecting ray $\overrightarrow{CD}$, then P is not equidistant from $\overrightarrow{CA}$ and $\overrightarrow{CB}$. (See following.)

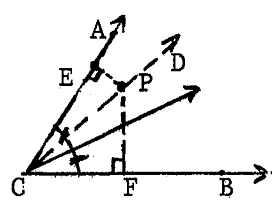


Figure 106

(i) Why is $m\angle ACP \neq m\angle BCP$?

(ii) Fill in the hypothesis and explain why p_1 and p_2 are true.

Hypothesis: ____________	Conclusion:
p_1: PC=PC.	q: ?
p_2: $\angle PEC$ and $\angle PFC$ are right angles.	
p_3: $m\angle ACP \neq m\angle BCP$.	

(iii) A partial contrapositive of $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is $(p_1 \wedge p_2 \wedge \sim q) \rightarrow \sim p_3$. How can one define $\sim q$ to make this partial contrapositive into a valid argument?

Hypothesis: ____________	Conclusion:
p_1: PC=PC.	p_3: $m\angle ACP = m\angle BCP$.
p_2: $\angle PEC$ and $\angle PFC$ are right angles.	
q: ?	

Suggestions for Enrichment

The geometry teacher can use Venn diagrams to illustrate many of the topics discussed in this and previous chapters. Let p denote a statement and let U be a universe in which p has a definite truth value. Let p be that part of the universe U in which p is true. The universe U will represented by a rectangle and its interior, and P will be represented by a circle and its interior (the shaded region in Figure 107). The part of the universe U in which p is false will be denoted by P. Hence P is the non-shaded region in Figure 107.

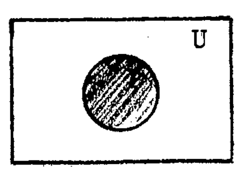


Figure 107

Assume now that $(p_1 \wedge p_2) \rightarrow q$ is a valid argument. This fact may be illustrated by Figure 108. The shaded region in Figure 108 represents the portion of U in which the hypothesis of the argument $(p_1 \wedge p_2) \rightarrow q$ is true. Note that the truth of $p_1 \wedge p_2$ necessitates the truth of q.

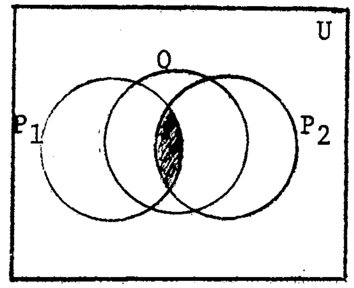


Figure 108

Teachers and students can inductively conclude that $(p_1 \wedge p_2) \rightarrow q$ is a valid argument if and only if $(p_1 \cap p_2) \subset Q$.

Figure 108 can be used to reinforce previously-discussed logical topics. For instance, it can be noted that

(1) $(P_1 \cap \overline{Q}) \subset \overline{P}_2$. This illustrates that the validity of $(p_1 \wedge p_2) \rightarrow q$ necessitates the validity of a partial contrapositive, $(p_1 \wedge \sim q) \rightarrow \sim p_2$.

(2) $(P_1 \cap \overline{P}_2) \not\subset \overline{Q}$. This illustrates that the validity of $(p_1 \wedge p_2) \rightarrow q$ does <u>not</u> necessitate the validity of a partial inverse, $(p_1 \wedge \sim p_2) \rightarrow \sim q$.

(3) $(P_1 \cap Q) \not\subset P_2$. This illustrates that the validity of $(p_1 \wedge p_2) \rightarrow q$ does <u>not</u> necessitate the validity of a partial converse, $(p_1 \wedge q) \rightarrow p_2$.

Figure 108 is the most general illustration of the validity of $(p_1 \wedge p_2) \rightarrow q$. Although there are other diagrams that illustrate the validity of this argument, such as the one shown in Figure 109, it should be noted that Figure 109 indicates

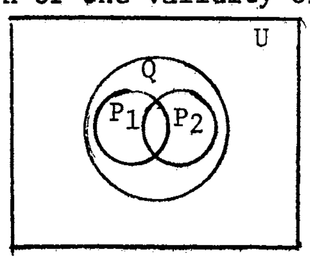


Figure 109

that $p_1 \rightarrow q$ is a valid argument. The validity of $p_1 \rightarrow q$ is not a necessary consequence of the validity of $(p_1 \wedge p_2) \rightarrow q$ (see Figure 108). Hence, Figure 109 does not represent the general case.

It is also possible to construct Venn diagrams to illustrate $(p_1 \wedge p_2) \rightarrow q$ as a invalid argument. One such diagram is shown in Figure 110. Note that $P_1 \cap P_2 \not\subset Q$. This illustrates that the truth of q is not a necessary consequence of the truth of $p_1 \wedge p_2$.

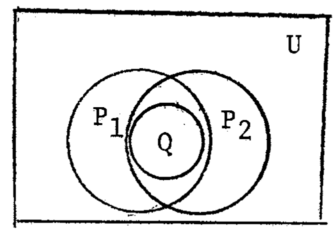


Figure 110

Summary

It has been demonstrated that one can deduce valid arguments by constructing partial contrapositives of previously established valid arguments. But although many such deductions are instructive in the classroom, it should be evident to a geometry teacher that many of the numerous valid arguments that can be deduced in this manner will not be of any use in the future development of a geometric structure that is being created. In other words, it makes little sence to examine every partial contrapositive of every valid argument.

The development of geometric structure will differ from text to text, and if a geometry teacher is to utilize partial contrapositives effectively, he must be thoroughly familiar with his text and emphasize those partial contrapositives that will be beneficial in future work. A clever teacher will allow students to discover useful valid arguments by suggesting that they examine the contrapositives of specific theorems. With careful guidance, students might establish the validity of an important theorem before the class reaches the point in the text where the theorem is presented. Refer to example 6.6 for instance. It is often possible for students to establish the validity of the theorem

> Two lines cut by a transversal are parallel if a pair of alternate interior angles are congruent

before it is introduced in a text. The writer has frequently observed that the feeling of being "one up on the author(s) of a geometry text" is very satisfying to students. In some instances, certain students have made a habit of looking ahead in the text in an attempt to "beat the author(s) to the proof of a theorem."

The wise use of partial contrapositives can also eliminate the need for many time-consuming formal proofs, allowing more class time for some of the enrichment activities that have been suggested. However, as has been mentioned, a teacher should carefully examine a lengthy formal proof for possible educational value before substituting a partial contrapositive proof for it.

CHAPTER VII

RELATING SYMBOLIC LOGIC TO INDIRECT PROOF

Types of Indirect Proof

Most geometry texts, including those in the sample for this paper, discuss the concept of indirect proof and use it, in varying degrees, as a method of proof in their deductive development of geometry. It is the purpose of this chapter to relate this form of proof to some of the concepts of symbolic logic that have been developed.

Anderson states, "To give an indirect proof of a statement, suppose that the statement is false and deduce a contradiction." (3, p. 192). Anderson is describing a common form of indirect proof generally called proof by contradiction, or reductio ad absurdum. A symbolic analysis of the indirect proof process as described by Anderson will now be given.

Let p, q, and r denote statements such that p is true and r is true. If one wishes to show that $p \to q$ is a valid argument using proof by contradiction, he assumes that $p \to q$ is false and shows that this leads to a contradiction of a known fact. This amounts to saying that the validity of $p \to q$ can be established by showing that

(1) $\sim(p \to q) \to \sim r$

is a valid argument. Note carefully that if $\sim(p \to q)$ is true (that is, if $p \to q$ is false), then the validity of (1) necessitates the truth of $\sim r$. This is the contradiction referred to by Anderson, since r is given as true. The basic question to be analyzed is the following: *Why does the validity of (1) necessitate the truth of* q *whenever* p *and* r *are true?*

As a first analysis, it can be noted that the validity of (1) necessitates the validity of its contrapositive

$$r \to (p \to q).$$

Therefore, since r is true, the statement $p \to q$ is true. Since $p \to q$ is true and p is true, this necessitates the truth of q.

As a second analysis, the biconditional

$$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$$

is a tautology (Problem 2, Exercise Set 2f). Hence, (1) is equivalent to

(2) $(p \wedge \sim q) \to \sim r.$

Since (1) is valid, then (2) is also valid. Therefore the argument

(3) $(p \wedge r) \to q$

is valid since it is a partial contrapositive of (2). Since p and r are true, the statement $p \wedge r$ is true. The validity of (3) thus necessitates the truth of q.

Example 7.1

An example of an indirect proof using the contradiction method is shown below. The proof is presented in Moise's text. The Line Postulate referenced in the proof states, "For every two points there is exactly one line containing both points." (17, p. 62).

> If two different lines intersect, their intersection contains only one point.
>
> Proof. If two different lines intersect at two different points P and Q, then there would be two lines containing P and Q. The Line Postulate tells us that this never happens. (17, p. 63).

One can identify in this proof the statements p, q, and r referred to in the discussion on proof by contradiction.

p: Two distinct lines intersect.

q: The intersection of the distinct lines contains only one point.

r: Two distinct points are contained on exactly one line.

A second method of indirect proof is called proof by cases, or proof by elimination. If q_1, q_2, ..., q_n are statements satisfying the condition that exactly one of them must be true in any given instance, and if p is a true statement, then exactly one of the arguments

$p \rightarrow q_1$

$p \rightarrow q_2$

...

$p \rightarrow q_n$

is valid, since only one of them can have a true conclusion for the true hypothesis, p. If $p \rightarrow q_j$ is the valid argument, its validity is established by showing that the arguments

$$p \rightarrow q_k \quad (1 \leq k \leq n,\ j \neq k)$$

are all invalid. This is usually accomplished by the contradiction method. That is, one attempts to establish that the validity of $p \rightarrow q_k$ $(1 \leq k \leq n,\ j \neq k)$ necessitates the truth of a statement known to be false.

Example 7.2

Consider the following theorem: The shortest distance from a point to a line is the length of the perpendicular segment from the point to the line.

Hypothesis: Line L and a point P not on L.

p_1: $\overleftrightarrow{PQ} \perp L$ at Q.

p_2: R is a point on L distinct from Q.

Conclusion:

q_1: $PQ < PR$.

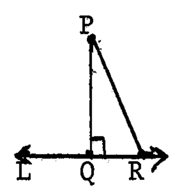


Figure 111

The Trichotomy Property of real numbers assures one that exactly one of the statements

q_1: $PQ < PR$

q_2: $PQ = PR$

q_3: $PQ > PR$

is true for any two segment lengths PQ and PR. Hence, since $p_1 \wedge p_2$ is true, exactly one of the following arguments is valid.

(1) $(p_1 \wedge p_2) \rightarrow q_1$

(2) $(p_1 \wedge p_2) \rightarrow q_2$

(3) $(p_1 \wedge p_2) \rightarrow q_3$.

Outlining the proof, one would hope to establish that $\sim q_1$, which is equivalent to $q_2 \vee q_3$, is false. The way that this is done would vary from text to text, again depending upon the order in which the material is presented. In Keedy, prior to presenting the stated theorem, it is established that

(a) If two sides of a triangle are congruent, the angles opposite these sides are congruent.

(b) If a triangle has one right angle, then its other angles are acute.

(c) If two sides of a triangle are not congruent, then the angles opposite them are not congruent, and the largest angle is opposite the longer side.

If q_3 is true, then by (c), $m\angle PRQ > m\angle PQR = 90$, a contradiction of (b). Hence q_3 is false.

If q_2 is true, then using (a), one obtains $m\angle PRQ = 90$, a contradiction of (b). Hence q_2 is false.

Since q_2 and q_3 are false, it follows that $q_2 \vee q_3$ is false; that is $\sim q_1$ is false. Then q_1 is true and $(p_1 \wedge p_2) \rightarrow q_1$ is the valid argument.

It should be noted that one can look upon the proof by elimination in Example 7.2 as a double application of the law of elimination, previously discussed in Chapter III (see also problem 4, Exercise Set 3g). This fact is demonstrated below.

$$[(p_1 \wedge p_2) \rightarrow q_1] \vee [(p_1 \wedge p_2) \rightarrow q_2] \vee [(p_1 \wedge p_2) \rightarrow q_3]$$
$$\sim [(p_1 \wedge p_2) \rightarrow q_3]$$
$$\overline{[(p_1 \wedge p_2) \rightarrow q_1] \vee [(p_1 \wedge p_2) \rightarrow q_2]}$$

$$[(p_1 \wedge p_2) \rightarrow q_1] \vee [(p_1 \wedge p_2) \rightarrow q_2]$$
$$\sim [(p_1 \wedge p_2) \rightarrow q_2]$$
$$\overline{(p_1 \wedge p_2) \rightarrow q_1}$$

A third method of indirect proof presented in some textbooks is called proof by contraposition. Goodwin, Keedy, and Rosenberg briefly discuss this method of proof. As its name suggests, this method of proof involves establishing the validity of $p \rightarrow q$ by showing that its contrapositive, $\sim q \rightarrow \sim p$ is a valid argument. This method of establishing the validity of arguments was thoroughly discussed in Chapter VI. Though many texts label proof by contraposition as an indirect method of proof, one is certainly justified, in light of previous discussion, in considering it a way of making a direct proof.

Exercise Set 7

1. Consider the theorem: If a line intersects a plane not containing it, then the intersection contains exactly one point.

If one assumes that the intersection of the line and the plane contains two or more points, (a) what conclusion is a necessary consequence of this assumption? (b) what contradiction is obtained?

2. Let p: a, b, and c are the lengths of sides of a triangle.

   q_1: $a + b > c$.

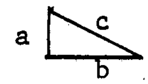


Figure 112

   (a) Assume that one wishes to prove the validity of $p \rightarrow q_1$ by the elimination method. Find statements q_2 and q_3 such that exactly one of q_1, q_2, q_3 is true.

   (b) Having found the statements q_2 and q_3 in part (a), one would then have to prove that the arguments $p \rightarrow q_2$ and $p \rightarrow q_3$ are ________________.

3. Let $\overline{AB}$ be the diameter of a circle and let C be a point on the circle distinct from A and B. If one wished to present a proof by contraposition that $m\angle ACB = 90$, then one would show that $(m\angle ABC \neq 90) \rightarrow$ __________________ is a valid argument.

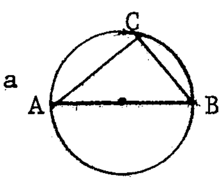


Figure 113

Suggestions for Enrichment

A teacher and his students can discuss daily situations in which a person may use indirect reasoning to arrive at a conclusion. A television repairman may conclude that the trouble in a specific television set is caused by a picture tube, since all other parts of the set have been found to be in satisfactory condition. A doctor may use indirect reasoning to make an educated guess as to the cause of a patient's problem. For instance, the doctor might reason that a patient's stomach pain is caused by gallstones because the patient's appendix has previously been removed and x-rays have shown the absence of ulcers. Students in the teacher's geometry class may recall arriving at an answer on a multiple guess test, not by directly determining the correct answer, but by eliminating all but one of the possible answers because of their apparent incorrectness.

Summary

Experience with many geometry textbooks has led this writer to agree with those (10) who claim that many of the explanations of indirect proof which commonly appear in geometry textbooks are likely to confuse the student, if not actually mislead him. Most of the text explanations are brief, and merely prescribe a procedure for the student to follow and offer one or two examples demonstrating an application of the procedure. It is hoped that this chapter has offered the geometry teacher a deeper insight into the nature of indirect proof and has suggested ways that might make this powerful method of proof more meaningful for his students if symbolic logic is used.

CHAPTER VIII

CONCLUSION

It has been the purpose of this paper to demonstrate how a secondary school geometry teacher can introduce elements of symbolic logic into a geometry course but at the same time, preserve rigor, provide students with discovery techniques and a better understanding of deductive proof, and reduce the time spent on traditional formal proofs. The extent to which a teacher chooses to utilize these concepts of symbolic logic will vary from teacher to teacher, and the way it is done will depend upon the manner in which geometric material is developed in the textbook being used. However, this writer hopes that he has convinced the reader that it is indeed possible to use concepts of symbolic logic constructively in any deductive development of geometry. Now that the reader has seen many possible uses for elements of symbolic logic in the teaching of geometry, a procedure for introducing them without a great expenditure of time will now be summarized.

In the beginning weeks of a geometry course, when definitions, primitive terms, and postulates are being established to be used in a later development of theorems, the teacher can introduce the concepts of statement, negation, conjunction, disjunction, conditional and biconditional statements, tautology, equivalent statements, and truth tables. A few minutes of class discussion per day, combined with statement examples (not all of which have to be related to geometry) will make these concepts familiar to the students. The teacher can utilize the student's intuitive knowledge of basic geometric figures for this purpose. As part of homework assignments, students can construct simple truth tables. In doing so, they might make some interesting discoveries, such as the fact that the converse of a statement is not equivalent to the statement itself. During this period, the teacher may identify the converse, inverse, and contrapositive forms of the argument $p \rightarrow q$. It is not suggested that the definitions of partial converse, partial inverse, and partial contrapositive be introduced during this pre-proof period.

When a discussion of elementary proof is introduced in the textbook, the argument forms discussed in Chapter III of this paper can be introduced. The teacher can point out the valid argument forms that present themselves in the elementary formal proofs that appear in the problem sets of the text. Invalid argument forms (Exercise Sets 3c and 3f) should be discussed, since students should clearly understand the distinction between valid and invalid arguments. The reason for this is that valid arguments can be used over and over again in the process of deducing useful conclusions, while invalid arguments cannot be used in this way. The writer hopes that introduction to the valid argument forms will help the students to understand another important distinction: That between deductive reasoning and inductive reasoning. If students can make such a distinction, they will more clearly appreciate what is expected of them when a geometric proof is required.

The concepts of converse, inverse, and contrapositive should initially be related to the conditional $p \to q$. The following points can be emphasized by constructing truth tables.

(1) $q \to p$ is <u>not</u> equivalent to $p \to q$.

(2) $\sim p \to \sim q$ is <u>not</u> equivalent to $p \to q$.

(3) $\sim q \to \sim p$ is equivalent to $p \to q$.

(4) $q \to p$ is equivalent to $\sim p \to \sim q$.

(5) The inverse of $p \to q$ is the contrapositive of the converse of $p \to q$.

These concepts can be reinforced daily by examining the converses, inverses, and contrapositives of definitions, postulates, and simple theorems. The definitions of partial converse, partial inverse, and partial contrapositive can be introduced when a teacher feels the need for them. Perhaps the ideal time for the partial converse definition would be when the triangle congruence postulates are presented, since (as was demonstrated in this paper) these postulates and the AAS Theorem are partial converses of one another. The writer feels that all of the "partial" definitions should be introduced shortly after the triangle congruence postulates, for they offer the students techniques for discovering other theorems. Also, if these concepts are established prior to the introduction of indirect proof in a textbook, they can be used to supplement what is commonly a very brief explanation of indirect proof.

It should be noted that while the material discussed in this paper can be presented in such a way as to be within the realm of understanding for high school geometry students, its presentation requires a dedicated teacher who truly wants to interact with his students. Dedication is required because the teacher must be willing to do some work outside of the textbook as he prepares his lesson plans. If he successfully develops the logical concepts discussed, he has, in the opinion of this writer, given his students a powerful discovery tool and he must be willing to allow them to display openly their discoveries, thoughts, and ideas. The writer believes that even the most able or energetic student will lose enthusiasm if he is forced to keep his ideas and discoveries to himself. The enrichment suggestions contained in the paper provide opportunities for active student participation. When class members can participate and experience the thrill of discovery, it is this writer's opinion that geometry and the deductive reasoning process can generate interest and enthusiasm while offering an exciting challenge to both teacher and students.

In conclusion, it must be emphasized that this paper contains the thoughts, ideas, and opinions of only one geometry teacher. He has found them to be extremely useful in his geometry classroom. It is certainly not suggested that all, or even some, of these ideas and techniques can be successfully utilized in every geometry classroom. It is the individual geometry teacher who must decide which, if any, of these ideas and techniques are applicable to his teaching situation.

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APPENDIX

SOLUTIONS TO EXERCISES

Exercise Set 2a

1. (a) $\frac{8}{-2} \neq 4$.

   (b) The point M is not between points A and B.

   (c) $\overleftrightarrow{AB}$ does not intersect $\overleftrightarrow{CD}$.

   (d) $\angle PQR$ is a right angle.

   (e) Two intersecting lines are contained in exactly one plane.

   (f) There exists a segment that does not have exactly one midpoint.

   (g) Some pairs of supplementary angles are not congruent.

   (h) All pairs of perpendicular lines do not form right angles.

2. No.

3. If $\angle A$ is a right angle, then both p and q are false.

4. Both p and q are false.

5. Both p and q are true.

Exercise Set 2b

1.	(a) F.	(b) T.	(c) F.	(d) F.
2.	(a) F.	(b) F.	(c) T.	(d) F.

3. At least one of the statements p, q, r is false.

4. (a)

TABLE XVI

TRUTH TABLE FOR $\sim p \wedge \sim q$

p	q	$\sim p$	$\wedge$	$\sim q$
T	T	F	F	F
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

(b)

TABLE XVII

TRUTH TABLE FOR $p \wedge q \wedge \sim r$

p	q	r	$(p \wedge q)$	$\wedge$	$\sim r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	F	T
F	T	T	F	F	F
F	T	F	F	F	T
F	F	T	F	F	F
F	F	F	F	F	T

5. $2^4 = 16.$

Exercise Set 2c

1. (a) T. (b) T. (c) F. (d) T.

2. TABLE XVIII

TRUTH TABLE FOR $(p \vee q) \vee r$ AND $p \vee (q \vee r)$

p	q	r	$(p \vee q) \vee r$			$p \vee (q \vee r)$		
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T	F
F	T	T	T	T	T	F	T	T
F	T	F	T	T	F	F	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	F	F	F	F	F

Symbols of inclusion are not necessary.

3. No.

4. (a) T. (b) F. (c). T. (d) T.

Exercise Set 2d

1. (a) If $\angle A$ and $\angle B$ have the same measure, then $\angle A$ and $\angle B$ are congruent.

   (b) $\angle A$ and $\angle B$ have the same measure only if $\angle A$ and $\angle B$ are congruent.

   (c) A necessary condition for $\angle A$ and $\angle B$ to have the same measure is that $\angle A$ and $\angle B$ are congruent.

   (d) A sufficient condition for $\angle A$ and $\angle B$ to be congruent is that $\angle A$ and $\angle B$ have the same measure.

   (e) (i) T. (ii) T. (iii) F.

2. (a) If ΔABC is isosceles, then ΔABC is equilateral.

   (b) ΔABC is isosceles only if ΔABC is equilateral.

   (c) A necessary condition for ΔABC to be isosceles is that ΔABC is equilateral.

   (d) A sufficient condition for ΔABC to be equilateral is that ΔABC is isosceles.

   (e) (i) F. (ii) T. (iii) F.

3. No.

Exercise Set 2e

1. (a) $\angle A$ is the complement of $\angle B$ if and only if $\angle B$ is the complement of $\angle A$.

   (b) A necessary and sufficient condition for $\angle A$ to be the complement of $\angle B$ is that $\angle B$ is the complement of $\angle A$.

   (c) (i) T. (ii) F.

2. (a) ΔABC and ΔDEF are congruent if and only if ΔABC and ΔDEF are equilateral.

(b) A necessary and sufficient condition for ΔABC to be congruent to ΔDEF is that ΔABC and ΔDEF are equilateral.

(c) (i) F. (ii) F.

Exercise Set 2f

1.

TABLE XIX

TRUTH TABLE FOR $(p \to q) \leftrightarrow (\sim q \to \sim p)$

p	q	$p \to q$	$\sim q \to \sim p$			$(p \to q) \leftrightarrow (\sim q \to \sim p)$		
T	T	T	F	T	F	T	T	T
T	F	F	T	F	F	F	T	F
F	T	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T

2.

TABLE XX

TRUTH TABLE FOR $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$

p	q	$\sim(p \to q)$		$p \wedge \sim q$			$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$		
T	T	F	T	T	F	F	F	T	F
T	F	T	F	T	T	T	T	T	T
F	T	F	T	F	F	F	F	T	F
F	F	F	T	F	F	T	F	T	F

3. TABLE XXI

TRUTH TABLE FOR $(p\leftrightarrow q)\leftrightarrow(p\leftrightarrow\sim q)$

p	q	$p\leftrightarrow q$	$p\leftrightarrow\sim q$			$(p\leftrightarrow q)\leftrightarrow(p\leftrightarrow\sim q)$		
T	T	T	T	F	F	T	F	F
T	F	F	T	T	T	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	F	F	T	T	F	F

Exercise Set 3a

1. Hypothesis: $a \neq b$ and $b \neq c$.

   Conclusion: $a \neq c$.

   Argument is invalid.

2. Hypothesis: $\triangle ABC$ is equilateral.

   Conclusion: $\triangle ABC$ is isosceles.

   Argument is valid.

3. Hypothesis: $\angle A$ and $\angle B$ are acute.

   Conclusion: $\angle A$ and $\angle B$ are complementary.

   Argument is invalid.

4. Hypothesis: P_1 and P_2 are distinct points in plane E.

   Conclusion: $\overleftrightarrow{P_1P_2} \subset E$.

   Argument is valid.

Exercise Set 3b

1. ∠A and ∠B are acute.
2. The diagonals of □ABCD are perpendicular.
3. $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CD}$.

Exercise Set 3c

1. TABLE XXII

TRUTH TABLE FOR $[(p\to q)\wedge q]\to p$

p	q	$(p\to q)\wedge q$	$[(p\to q)\wedge q]\to p$
T	T	T T T	T T T
T	F	F F F	F T T
F	T	T T T	T F F
F	F	T F F	F T F

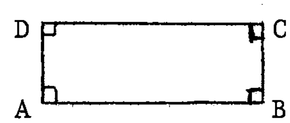
2. 

Figure 114

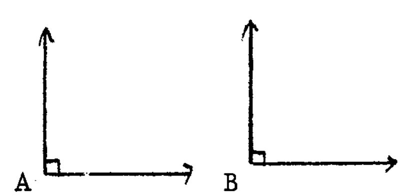
3. 

Figure 115

Exercise Set 3d

1. If two angles are complementary, then they are not supplementary.
2. If $\overline{MR}$ is a median of scalene triangle MNP, then $\overline{MR}$ is not perpendicular to $\overleftrightarrow{NP}$.
3. If the diagonals of a quadrilateral bisect each other, then the opposite angles on the quadrilateral are congruent.

Exercise Set 3e

1. L_1 and L_2 are not skew.
2. $\overline{PQ}$ is not a median of MNP.
3. $m\angle A \leq m\angle C$.

Exercise Set 3f

1.

TABLE XXIII

TRUTH TABLE FOR $[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$

p	q	$(p \rightarrow q) \wedge \sim p$			$[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$		
T	T	T	F	F	F	T	F
T	F	F	F	F	F	T	T
F	T	T	T	T	T	F	F
F	F	T	T	T	T	T	T

2.

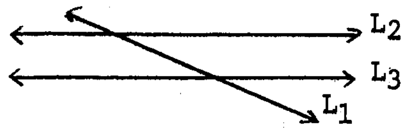


Figure 116

3.

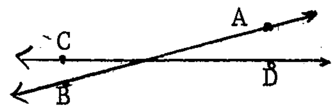


Figure 117

Exercise Set 3g

1. ΔABC is isosceles

2. $\frac{AD}{BD} \neq \frac{AE}{CE}$.

3. ΔABC is not a right triangle.

4. $q \vee r$.

Exercise Set 3h

	Argument	Hypothesis	Conclusion
1.	Valid	True	True
2.	Valid	False	False
3.	Invalid	True	True

Exercise Set 4a

1. There exists one partial converse and it is valid.

2. There are two partial converses. If one is working in the plane of the circle, both partial converses are valid. In space, the partial converse $(p_2 \wedge q) \rightarrow p_1$ is not valid.

3. $(q_1 \wedge q_2) \rightarrow (p_1 \wedge p_2 \wedge q_3 \wedge q_4)$ is valid.
   $(q_1 \wedge q_3) \rightarrow (p_1 \wedge p_2 \wedge q_2 \wedge q_4)$ is invalid.

$(q_1 \wedge q_4) \rightarrow (p_1 \wedge p_2 \wedge q_2 \wedge q_3)$ is invalid.

$(q_2 \wedge q_3) \rightarrow (p_1 \wedge p_2 \wedge q_1 \wedge q_4)$ is invalid.

$(q_2 \wedge q_4) \rightarrow (p_1 \wedge p_2 \wedge q_1 \wedge q_3)$ is invalid.

$(q_3 \wedge q_4) \rightarrow (p_1 \wedge p_2 \wedge q_1 \wedge q_2)$ is valid.

$(p_1 \wedge q_1) \rightarrow (p_2 \wedge q_2 \wedge q_3 \wedge q_4)$ is valid.

$(p_1 \wedge q_2) \rightarrow (p_2 \wedge q_1 \wedge q_3 \wedge q_4)$ is invalid.

$(p_1 \wedge q_3) \rightarrow (p_2 \wedge q_1 \wedge q_2 \wedge q_4)$ is valid.

$(p_1 \wedge q_4) \rightarrow (p_2 \wedge q_1 \wedge q_2 \wedge q_3)$ is valid.

$(p_2 \wedge q_1) \rightarrow (p_1 \wedge q_2 \wedge q_3 \wedge q_4)$ is invalid.

$(p_2 \wedge q_2) \rightarrow (p_1 \wedge q_1 \wedge q_3 \wedge q_4)$ is valid.

$(p_2 \wedge q_3) \rightarrow (p_1 \wedge q_1 \wedge q_2 \wedge q_4)$ is valid.

$(p_2 \wedge q_4) \rightarrow (p_1 \wedge q_1 \wedge q_2 \wedge q_3)$ is valid.

Exercise Set 4b

1. See solution for problem 3, Exercise Set 4a.

2. The component statements in the hypothesis and conclusion of Theorem 9 are shown below. Each of the other three theorems is obtained by interchanging q and one of the component statements of the hypothesis.

Hypothesis: The circle with center at C.	Conclusion:
p_1: $\overleftrightarrow{AD}$ contains the center of the circle.	q: $\overleftrightarrow{AB}$ is tangent to the circle.
p_2: A is a point on the circle.	
p_3: $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ at A.	

Exercise Set 4c

1. If $m\angle A = m\angle C$, then $AB = BC$.

   If $m\angle A > m\angle C$, then $AB > BC$.

   If $m\angle A < m\angle C$, then $AB < BC$.

2. If $m\ \widehat{AB} = m\ \widehat{CD}$, then $AB = CD$.

   If $M\ \widehat{AB} > m\ \widehat{CD}$, then $AB > CD$.

   If $m\ \widehat{AB} < m\ \widehat{CD}$, then $AB < CD$.

Exercise Set 5

1. If ΔABC is not equilateral, then ΔABC is not isosceles. (Invalid).
2. If $\frac{5}{a} \neq \frac{7}{b}$, then $\frac{a}{5} \neq \frac{b}{7}$. (Valid).
3. If ΔABC is not congruent to ΔDEF, then ΔABC is not similar to ΔDEF. (Invalid).
4. There are two partial inverses and both are valid.
5. There are three partial inverses and all are valid.

Exercise Set 6a

1. If the supplement of two angles are not congruent, then the angles are not congruent.
2. If a triangle is not isosceles, then it is not equilateral.
3. If the radii of two circles are equal, the circles have equal areas.
4. Partial Contrapositive 1: If $a + b = c$ and if $a \geq c$, then $b \leq c$.

   Partial Contrapositive 2: If $a \geq c$ and if $b > 0$, then $a + b \neq c$.

5. There are two partial contrapositives and both are equivalent to the following: If two planes are not parallel, and if one of the planes is perpendicular to a line, then the other plane is not perpendicular to that line.

Exercise Set 6b

1. If two parallel lines are cut by a transversal, a pair of alternate interior angles are congruent.
2. If a line is perpendicular to a plane, then every plane containing the line is perpendicular to the given plane.
3. If M is between A and C on line L, then M and A are on the same side of any other line that contains C.
4. If $\frac{AD}{DE} = \frac{AE}{EC}$, then $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{BC}$. (If a line divides two sides of a triangle into segments that are proportional, then it is parallel to the third side.)

Exercise Set 6c

1. H is not the midpoint of $\overline{AD}$.
2. $BD \neq DC$.
3. $\overleftrightarrow{OP}$ does not contain the center of the sphere.

Exercise Set 6d

1. (a) equidistant from L_1 and L_2.

   (b) not equidistant from L_1 and L_2.
2. (a) $\Delta CPE \cong \Delta CPF$ by AAS. Therefore $PE = PF$.

   (b) (i) The bisector of an angle is unique.

(ii) Hypothesis: $\angle ACB$ and bisecting ray $\overrightarrow{CD}$, P is a point not on $\overrightarrow{CD}$.

p_1 is true by identity.

p_2 is true by the definition of distance from a point to a line.

(iii) $\sim q$: PE = PF.

Exercise Set 7

1. (a) The line is contained in the plane.

   (b) Statement (a) contradicts the fact that the line is not contained in the plane.

2. (a) q_1: $a + b = c$. q_2: $a + b < c$.

   (b) invalid.

3. $\overline{AB}$ is not a diameter of the circle.

VITA

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