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SOME RESULTS ON p NEAR-RINGS AND RELATED NEAR-RINGS

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CHAPTER I

INTRODUCTION

1. Notational Convention

A left near-ring is an algebraic system $(N;+,\cdot)$ such that

(a) $(N;+)$ is a group,

(b) $(N;\cdot)$ is a semigroup,

(c) $x\cdot(y + z) = x\cdot y + x\cdot z$ for all $x, y, z \in N$.

All of the near-rings in this paper are left near-rings, so hereafter near-ring will mean left near-ring. We adopt the usual convention of denoting $x\cdot y$ by xy.

The names maximal sub-C-ring and maximal sub-Z-ring found in Berman and Silverman [1] are used. When discussing ideals the proofs will not involve the definition but instead will use the condition established by Blackett [2].

The integers modulo p will be denoted by $(Z_p;+,\cdot)$ or sometimes more simply by Z_p. However occasionally we will deal with $(Z_p;+,')$ where $\cdot$ and $'$ are different multiplications. Then $(Z_p;+,')$ may be denoted by Z_p so some care must be used to see exactly what Z_p means in any given argument.

2. Preview of Results

Clay and Lawver [4] studied a class of Boolean near-rings that were in some sense dependent upon a Boolean ring with identity. Part of this paper extends some of their results. A class of p near-rings, that are in some sense dependent upon a p ring with identity, is studied. When the results are specialized to p = 2 they agree with those of Clay and Lawver. The results needed about p rings may be found in McCoy [9] and [10].

Then results by Ligh [8] were extended from β near-rings to more general near-rings in the last chapter. A decomposition theorem for this more general class of near-rings is established.

Most of the near-rings used as examples in this paper are labeled as they appear in Clay [3].

CHAPTER II

MOTIVATION OF THE DEFINITION

1. General Remarks

Let p be a prime. A near-ring $(N;+,\cdot)$ is a p near-ring iff $px = 0$ and $x^p = x$ for every $x \in N$.

Clay and Lawver [4] did a partial study of a class of Boolean near-rings. They began with a Boolean ring with identity $(B;+,\cdot;1)$ and then defined a multiplication * such that $(B;+,*)$ was a Boolean near-ring. With this in mind we start with a p ring with identity $(N;+,\cdot;1)$. A suitable way to define * such that $(N;+,*)$ would be a p near-ring was not immediately obvious so two particular cases were examined first. The basic plan was that $x * y$ should be a polynomial in x and y with fixed coefficients in N.

2. The 3 Ring Case

Let $(N;+,\cdot;1)$ be a 3 ring with identity. Define $* : N \times N \to N$ by

$$x * y = \gamma x^2 y + \alpha xy + \beta y + ax^2y^2 + bx^2 + cxy^2 + dx + ey^2 + f$$

where $\alpha, \beta, \gamma, a, b, c, d, e, f \in N$. We want $(N;+,*)$ to be a 3 near-ring so in particular $x * 0 = 0$ for all $x \in N$. $0 * 0 = 0$ results in $f = 0$. $1 * 0 = 0$ and $2 * 0 = 0$ produce the following equations.

$$b + d = 0 \tag{1}$$

$$b + 2d = 0 \tag{2}$$

Then (1) and (2) imply that $b = d = 0$. $0 * 0 = 0 * 1 + 0 * 2$ implies that $e = 0$. Continuing in this fashion $1 * 0 = 1 * 1 + 1 * 2$ yields

$$2a + 2c = 0 \tag{3}$$

and $2 * 0 = 2 * 1 + 2 * 2$ gives us

$$2a + c = 0 \ . \tag{4}$$

Equations (3) and (4) imply that $a = c = 0$.

We now demand that the associative law hold in some carefully selected cases. The outcome of $0 * (0 * 1) = (0 * 0) * 1$ is

$$\beta^2 = \beta \ . \tag{5}$$

$1 * (0 * 1) = (1 * 0) * 1$ and $2 * (0 * 1) = (2 * 0) * 1$ result in

$$\beta\gamma + \alpha\beta = 0 \tag{6}$$

and

$$\beta\gamma + 2\alpha\beta = 0 \ . \tag{7}$$

As a consequence of (6) and (7) we find that $\alpha\beta = \beta\gamma = 0$.

$(N;+,*)$ is to be a 3 near-ring so $x * (x * x) = x$ for all $x \in N$. Then $1 * (1 * 1) = 1$ yields

$$2\alpha\gamma + \gamma^2 + \alpha^2 + \beta = 1 \ . \tag{8}$$

Similarly $2 * (2 * 2) = 2$ produces

$$2\alpha\gamma + 2\gamma^2 + 2\alpha^2 + 2\beta = 2 \ . \tag{9}$$

Equations (8) and (9) imply that $\alpha\gamma = 0$. Finally $1 * (1 * 1) = (1 * 1) * 1$

implies that $\gamma^2 = \gamma$ and equation (8) may be written as

$$\gamma = 1 - \alpha^2 - \beta \ . \tag{8'}$$

Since $\alpha\beta = 0$ and $\beta^2 = \beta$ it follows that $\gamma^2 = \gamma$. We now prove the following result.

<u>Theorem 2.1</u>: Let $(N;+,\cdot;1)$ be a 3 ring with identity and $\alpha, \beta \in N$ such that $\alpha\beta = 0$ and $\beta^2 = \beta$. If $x * y = (1 - \alpha^2 - \beta)x^2y + \alpha xy + \beta y$ for all $x, y \in N$ then $(N;+,*)$ is a 3 near-ring.

<u>Proof</u>: For ease of computation let $\gamma = 1 - \alpha^2 - \beta$. $(N;+)$ is known to be a commutative group. We must show that $x * (y * z) = (x * y) * z$, $x * (y + z) = x * y + x * z$ and $x * x * x = x$ for all $x, y, z \in N$.

$$\begin{aligned} x * (y * z) &= x * (\gamma y^2z + \alpha yz + \beta z) = \gamma x^2(\gamma y^2z) + \alpha x(\alpha yz) + \beta(\beta z) \\ &= \gamma x^2y^2z + \alpha^2xyz + \beta z \\ (x * y) * z &= (\gamma x^2y + \alpha xy + \beta y) * z = \gamma(\gamma x^2y)^2z + \alpha(\alpha xy)z + \beta z \\ &= \gamma x^2y^2z + \alpha^2xyz + \beta z \end{aligned}$$

Thus $x * (y * z) = (x * y) * z$.

$$\begin{aligned} x * (y + z) &= \gamma x^2(y + z) + \alpha x(y + z) + \beta(y + z) \\ &= \gamma x^2y + \alpha xy + \beta y + \gamma x^2z + \alpha xz + \beta z = x * y + x * z \\ x * (x * x) &= \gamma(x^2x^2x) + \alpha^2(xxx) + \beta(x) = (\gamma + \alpha^2 + \beta)x = x \end{aligned}$$

Therefore $(N;+,*)$ is a 3 near-ring.

Some examples of this type of near-ring are now given. Begin with $(Z_3;+,\cdot)$. If $\alpha,\beta \in Z_3$ such that $\alpha = \beta = 0$ then $\gamma = 1$ and a 3 near-ring of this type results which is not a ring. But if $\alpha, \beta \in Z_3$ such that $\alpha = 1$ or 2, $\beta = 0$ then $\gamma = 0$ and a 3 near-ring of this type results which is isomorphic to Z_3.

3. The 5 Ring Case

For further motivation let $(N;+,\cdot;1)$ be a 5 ring with identity and define $* : N \times N \to N$ by

$$x * y = ax^4y^4 + bx^4y^3 + cx^4y^2 + dx^4y + ex^4 + fx^3y^4 + gx^3y^3 + hx^3y^2 + ix^3y + jx^3 + kx^2y^4 + \ell x^2y^3 + mx^2y^2 + nx^2y + px^2 + qxy^4 + rxy^3 + sxy^2 + txy + ux + vy^4 + wy^3 + \rho y^2 + \sigma y + \tau.$$

We want $(N;+,*)$ to be a 5 near-ring so let us first impose the condition that $x * 0 = 0$ for all $x \in N$. $0 * 0 = 0$ gives us $\tau = 0$ immediately. For $x = 1, 2, 3, 4$, $x * 0 = 0$ yields the following equations.

$$e + j + p + u = 0 \tag{1}$$

$$e + 3j + 4p + 2u = 0 \tag{2}$$

$$e + 2j + 4p + 3u = 0 \tag{3}$$

$$e + 4j + p + 4u = 0 \tag{4}$$

These equations imply that $e = j = p = u = 0$.

The conditions that $0 = 0 * 1 + 0 * 4$ and $0 = 0 * 2 + 0 * 3$ respectively result in the following equations.

$$2v + 2\rho = 0 \tag{5}$$

$$2v + 3\rho = 0 \tag{6}$$

Thus $v = \rho = 0$. As a consequence of the conditions $0 = 1 * 1 + 1 * 4$, $0 = 1 * 2 + 1 * 3$, $0 = 2 * 1 + 2 * 4$, $0 = 2 * 2 + 2 * 3$, $0 = 3 * 1 + 3 * 4$, $0 = 3 * 2 + 3 * 3$, $0 = 4 * 1 + 4 * 4$ and $0 = 4 * 2 + 4 * 3$ we obtain the following equations respectively.

$$2a + 2c + 2f + 2h + 2k + 2m + 2q + 2s = 0 \qquad (7)$$

$$2a + 3c + 2f + 3h + 2k + 3m + 2q + 3s = 0 \qquad (8)$$

$$2a + 2c + f + h + 3k + 3m + 4q + 4s = 0 \qquad (9)$$

$$2a + 3c + f + 4h + 3k + 2m + 4q + s = 0 \qquad (10)$$

$$2a + 2c + 4f + 4h + 3k + 3m + q + s = 0 \qquad (11)$$

$$2a + 3c + 4f + h + 3k + 2m + q + 4s = 0 \qquad (12)$$

$$2a + 2c + 3f + 3h + 2k + 2m + 3q + 3s = 0 \qquad (13)$$

$$2a + 3c + 3f + 2h + 2k + 3m + 3q + 2s = 0 \qquad (14)$$

These equations imply that $a = c = f = h = k = m = q = s = 0$. Then $0 * 2 = 0 * 1 + 0 * 1$ results in $w = 0$. Conditions $1 * 2 = 1 * 1 + 1 * 1$, $2 * 2 = 2 * 1 + 2 * 1$, $3 * 2 = 3 * 1 + 3 * 1$ and $4 * 2 = 4 * 1 + 4 * 1$ respectively lead to the following equations.

$$b + g + \ell + r = 0 \qquad (15)$$

$$b + 3g + 4\ell + 2r = 0 \qquad (16)$$

$$b + 2g + 4\ell + 3r = 0 \qquad (17)$$

$$b + 4g + \ell + 4r = 0 \qquad (18)$$

From these equations we see that $b = g = \ell = r = 0$. Thus the original expression is reduced to $x * y = dx^4y + ix^3y + nx^2y + txy + \sigma y$.

We now demand that the associative law hold for some selected elements. $(0 * 0) * 1 = 0 * (0 * 1)$ results in $\sigma = \sigma^2$. The conditions $(1 * 0) * 1 = 1 * (0 * 1)$, $(2 * 0) * 1 = 2 * (0 * 1)$, $(3 * 0) * 1 = 3 * (0 * 1)$ and $(4 * 0) * 1 = 4 * (0 * 1)$ result respectively in the following equations.

$$d\sigma + i\sigma + n\sigma + t\sigma = 0 \qquad (19)$$

$$d\sigma + 3i\sigma + 4n\sigma + 2t\sigma = 0 \qquad (20)$$

$$d\sigma + 2i\sigma + 4n\sigma + 3t\sigma = 0 \qquad (21)$$

$$d\sigma + 4i\sigma + n\sigma + 4t\sigma = 0 \qquad (22)$$

These equations imply that $d\sigma = i\sigma = n\sigma = t\sigma = 0$.

We now make the following change in notation. Let $x * y = ax^4y + bx^3y + cx^2y + dxy + ey$ where $ae = be = ce = de = 0$ and $e^2 = e$. With straightforward computation one finds that

$$\begin{aligned} x * (y * z) = (\,&(a^2)x^4y^4 + (ab)x^4y^3 + (ac)x^4y^2 + (ad)x^4y + (ab)x^3y^4 \\ &+ (b^2)x^3y^3 + (bc)x^3y^2 + (bd)x^3y + (ac)x^2y^4 + (bc)x^2y^3 \\ &+ (c^2)x^2y^2 + (cd)x^2y + (ad)xy^4 + (bd)xy^3 + (cd)xy^2 \\ &+ (d^2)xy + e\,)\,z \end{aligned}$$

and

$$\begin{aligned} (x * y) * z = (\,&(a + a^3c^2 + 2a^3bd + ab^4 + 2ab^2cd + ac^4 + ad^4)x^4y^4 \\ &+ (a^3b + ab^2d + 3b^3c + 3abc^2 + 3bcd^2)x^4y^3 + (a^2c \\ &+ 2bcd + c^3)x^4y^2 + (ad)x^4y + (4a^4b + 2a^3cd + 4ab^3c \\ &+ 4ac^3d + 4a^2d^3)x^3y^4 + (3a^2b^2 + abcd + 3b^3d + 3b^2c^2 \\ &+ bd^3)x^3y^3 + (2abc + 2c^2d)x^3y^2 + (bd)x^3y + (4a^4c \\ &+ a^3b^2 + a^3d^2 + 4a^2bcd + 4ab^3d + ab^2c^2 + ab^2d^2 + 4a^2c^3 \\ &+ ac^2d^2 + 4abd^3)x^2y^4 + (3a^2bc + 3ab^3 + b^2cd + bc^3 \\ &+ 3abd^2)x^2y^3 + (2ac^2 + b^2c + cd^2)x^2y^2 + (cd)x^2y \\ &+ (4a^4d + 2a^3bc + 4a^2b^3 + 4abc^3 + 4acd^3)xy^4 + (3a^2bd \\ &+ ab^2c + b^4 + 3bc^2d + 3b^2d^2)xy^3 + (2acd + 2bc^2)xy^2 \\ &+ (d^2)xy + e\,)\,z. \end{aligned}$$

The expressions for x * (y * z) and (x * y) * z must be equal. Furthermore the coefficients of like terms in these two expressions are equal. To see this we first change notation to the following. Let

$$\begin{aligned} x * (y * z) = (&a_1x^4y^4 + a_2x^4y^3 + a_3x^4y^2 + (ad)x^4y + a_4x^3y^4 + a_5x^3y^3 \\ &+ a_6x^3y^2 + (bd)x^3y + a_7x^2y^4 + a_8x^2y^3 + a_9x^2y^2 + (cd)x^2y \\ &+ a_{10}xy^4 + a_{11}xy^3 + a_{12}xy^2 + (d^2)xy + e)\, z \end{aligned}$$

and

$$\begin{aligned} (x * y) * z = (&b_1x^4y^4 + b_2x^4y^3 + b_3x^4y^2 + (ad)x^4y + b_4x^3y^4 + b_5x^3y^3 \\ &+ b_6x^3y^2 + (bd)x^3y + b_7x^2y^4 + b_8x^2y^3 + b_9x^2y^2 + (cd)x^2y \\ &+ b_{10}xy^4 + b_{11}xy^3 + b_{12}xy^2 + (d^2)xy + e)\, z. \end{aligned}$$

If we demand that x * (y * 1) - (x * y) * 1 = 0 for x = 1, 2, 3, 4 and y = 1, 2, 3 then some equations result that yield the desired outcome. Let $c_i = a_i - b_i$. 1 * (1 * 1) - (1 * 1) * 1 = 0 leads to

$$\begin{aligned} &c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} \\ &\qquad + c_{12} = 0\ . \end{aligned} \tag{23}$$

Similarly 2 * (1 * 1) - (2 * 1) * 1 = 0, 3 * (1 * 1) - (3 * 1) * 1 = 0, ···, 3 * (3 * 1) - (3 * 3) * 1 = 0 and 4 * (3 * 1) - (4 * 3) * 1 = 0 result respectively in the following equations.

$$\begin{aligned} &c_1 + c_2 + c_3 + 3c_4 + 3c_5 + 3c_6 + 4c_7 + 4c_8 + 4c_9 + 2c_{10} \\ &\qquad + 2c_{11} + 2c_{12} = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} &c_1 + c_2 + c_3 + 2c_4 + 2c_5 + 2c_6 + 4c_7 + 4c_8 + 4c_9 + 3c_{10} \\ &\qquad + 3c_{11} + 3c_{12} = 0 \end{aligned} \tag{25}$$

$$\begin{aligned} &c_1 + c_2 + c_3 + 4c_4 + 4c_5 + 4c_6 + c_7 + c_8 + c_9 + 4c_{10} \\ &\qquad + 4c_{11} + 4c_{12} = 0 \end{aligned} \tag{26}$$

$$c_1 + 3c_2 + 4c_3 + c_4 + 3c_5 + 4c_6 + c_7 + 3c_8 + 4c_9 + c_{10} + 3c_{11} + 4c_{12} = 0 \tag{27}$$

$$c_1 + 3c_2 + 4c_3 + 3c_4 + 4c_5 + 2c_6 + 4c_7 + 2c_8 + c_9 + 2c_{10} + c_{11} + 3c_{12} = 0 \tag{28}$$

$$c_1 + 3c_2 + 4c_3 + 2c_4 + c_5 + 3c_6 + 4c_7 + 2c_8 + c_9 + 3c_{10} + 4c_{11} + 2c_{12} = 0 \tag{29}$$

$$c_1 + 3c_2 + 4c_3 + 4c_4 + 2c_5 + c_6 + c_7 + 3c_8 + 4c_9 + 4c_{10} + 2c_{11} + c_{12} = 0 \tag{30}$$

$$c_1 + 2c_2 + 4c_3 + c_4 + 2c_5 + 4c_6 + c_7 + 2c_8 + 4c_9 + c_{10} + 2c_{11} + 4c_{12} = 0 \tag{31}$$

$$c_1 + 2c_2 + 4c_3 + 3c_4 + c_5 + 2c_6 + 4c_7 + 3c_8 + c_9 + 2c_{10} + 4c_{11} + 3c_{12} = 0 \tag{32}$$

$$c_1 + 2c_2 + 4c_3 + 2c_4 + 4c_5 + 3c_6 + 4c_7 + 3c_8 + c_9 + 3c_{10} + c_{11} + 2c_{12} = 0 \tag{33}$$

$$c_1 + 2c_2 + 4c_3 + 4c_4 + 3c_5 + c_6 + c_7 + 2c_8 + 4c_9 + 4c_{10} + 3c_{11} + c_{12} = 0 \tag{34}$$

From equations (23),$\cdots$,(34) we see that $c_i = 0$ for $i = 1, 2, 3, \cdots, 12$. Thus we obtain the following equations.

$$a + a^3c^2 + 2a^3bd + ab^4 + 2ab^2cd + ac^4 + ad^4 = a^2 \tag{35}$$

$$a^3b + ab^2d + 3b^3c + 3abc^2 + 3bcd^2 = ab \tag{36}$$

$$a^2c + 2bcd + c^3 = ac \tag{37}$$

$$4a^4b + 2a^3cd + 4ab^3c + 4ac^3d + 4a^2d^3 = ab \tag{38}$$

$$3a^2b^2 + abcd + 3b^3d + 3b^2c^2 + bd^3 = b^2 \tag{39}$$

$$2abc + 2c^2d = bc \tag{40}$$

$$4a^4c + a^3b^2 + a^3d^2 + 4a^2bcd + 4ab^3d + ab^2c^2 + 4a^2c^3 + ab^2d^2 + ac^2d^2 + 4abd^3 = ac \tag{41}$$

$$3a^2bc + 3ab^3 + b^2cd + bc^3 + 3abd^2 = bc \tag{42}$$

$$2ac^2 + b^2c + cd^2 = c^2 \tag{43}$$

$$4a^4d + 2a^3bc + 4a^2b^3 + 4abc^3 + 4acd^3 = ad \tag{44}$$

$$3a^2bd + ab^2c + b^4 + 3bc^2d + 3b^2d^2 = bd \tag{45}$$

$$2acd + 2bc^2 = cd \tag{46}$$

We now demand that $x * (x * (x * (x * x))) = x$. However

$$\begin{aligned} x * (x * (x * (x * x))) = {} & (4a^3b + 2a^2cd + 4ab^2c + 2abc^2 \\ & + 3ab^2d + 4ad^3 + 4b^3c + 3bcd^2 \\ & + 4c^3d + 4bc^2d)x^4 + (a^2b^2 + a^2d^2 \\ & + 4abc^2 + 4ac^3 + 2b^3c + b^2c^2 \\ & + 2b^3d + 2bd^3 + c^2d^2 + 2bcd^2)x^3 \\ & + (4a^3d + 2a^2bc + 4ab^3 + 3abd^2 \\ & + 2ac^2d + 3b^2cd + 4b^2c^2 + 4abcd \\ & + 4bc^3 + 4cd^3)x^2 + (a^4 + 2a^2bc \\ & + a^2c^2 + 2ab^2c + 2acd^2 + b^4 \\ & + 2b^2d^2 + 4b^2cd + 2bc^3 + c^4 \\ & + d^4 + e)x \ . \end{aligned}$$

We now simplify this to $x * (x * (x * (x * x))) = qx^4 + rx^3 + sx^2 + tx$. Then for $x = 1, 2, 3, 4$ we obtain the following equations.

$$q + r + s + t = 1 \tag{47}$$

$$q + 3r + 4s + 2t = 2 \tag{48}$$

$$q + 2r + 4s + 3t = 3 \tag{49}$$

$$q + 4r + s + 4t = 4 \tag{50}$$

These equations imply that $q = r = s = 0$ and $t = 1$. Hence the following equations result.

$$4a^3b + 2a^2cd + 4ab^2c + 2abc^2 + 3ab^2d + 4ad^3 + 4b^3c + 3bcd^2 + 4c^3d + 4bc^2d = 0 \tag{51}$$

$$a^2b^2 + a^2d^2 + 4abc^2 + 4ac^3 + 2b^3c + b^2c^2 + 2b^3d + 2bd^3 + c^2d^2 + 2bcd^2 = 0 \tag{52}$$

$$4a^3d + 2a^2bc + 4ab^3 + 3abd^2 + 2ac^2d + 3b^2cd + 4b^2c^2 + 4abcd + 4bc^3 + 4cd^3 = 0 \tag{53}$$

$$a^4 + 2a^2bc + a^2c^2 + 2ab^2c + 2acd^2 + b^4 + 2b^2d^2 + 4b^2cd + 2bc^3 + c^4 + d^4 + e = 1 \tag{54}$$

From equations (35), $\cdots$,(46) and (51), $\cdots$,(54) we may determine first that $ad = 0$. This implies that $b = c = 0$, $a^2 = a$ and finally that $1 = a + d^4 + e$. If $a = 1 - d^4 - e$ then it is routine to show that $a^2 = a$ since $de = 0$ and $e^2 = e$.

Theorem 3.1: Let $(N;+,\cdot;1)$ be a 5 ring with identity and α, $\beta \in N$ such that $\alpha\beta = 0$ and $\beta^2 = \beta$. If $x * y = (1 - \alpha^4 - \beta)x^4y + \alpha xy + \beta y$ for all x, $y \in N$ then $(N;+,*)$ is a 5 near-ring.

Proof: The proof of this theorem is routine and will be omitted.

There is another reason for omitting the proof of this theorem. In the first section of the next chapter a more complete theorem is given. This more complete theorem is proven there.

Some examples of this type of 5 near-ring are now given. Begin with $(Z_5;+,\cdot)$. If $\alpha, \beta \in Z_5$ such that $\alpha = 0$, $\beta = 1$ then $\gamma = 0$ and a 5 near-ring of this type results which is not a ring. However if $\alpha, \beta \in Z_5$ such that $\alpha = 1, 2, 3, 4$ and $\beta = 0$ then $\gamma = 0$. In this case a 5 near-ring of this type results which is isomorphic to Z_5.

CHAPTER III

(α,β) p NEAR-RINGS

1. A Class of p Near-Rings

We now turn to the more general case where $(N;+,\cdot;1)$ is a p ring with identity. If $a \in N$ and $a \neq 0$ then $a^0 = 1$.

In the following discussion α, $\beta \in N$ such that $\alpha\beta = 0$, $\beta^2 = \beta$ and $x * y = (1 - \alpha^{p-1} - \beta)x^{p-1}y + \alpha xy + \beta y$ for all x, $y \in N$. If $p > 2$ then α is the coefficient of xy. However if $p = 2$ then that is not the case. Then $x * y = (1 - \alpha - \beta)xy + \alpha xy + \beta y = (1 - \beta)xy + \beta y$. In a 2 ring or Boolean ring $-a = a$ so $x * y = (1 + \beta)xy + \beta y$. Thus the coefficient of xy is $1 + \beta$. In certain theorems that follow it will be convenient to refer to the coefficient of xy. However it is very cumbersome to keep repeating the phrase "the coefficient of xy." For this reason we will hereafter regard α as the coefficient of xy and when $p = 2$ it will be understood that $\alpha = 1 + \beta$.

Theorem 1.1: Let $(N;+,\cdot;1)$ be a p ring with identity and α, $\beta \in N$ such that $\alpha\beta = 0$ and $\beta^2 = \beta$. If $x * y = (1 - \alpha^{p-1} - \beta)x^{p-1}y + \alpha xy + \beta y$ for all x, $y \in N$ then $(N;+,*)$ is a p near-ring. Furthermore $(N;+,*)$ is a p ring with identity iff $\alpha^{p-1} = 1$.

Proof: Recall that a near-ring $(A;+,\cdot)$ is a p near-ring iff $x^p = x$ and $px = 0$ for all $x \in A$. It is known that $(N;+)$ is a commutative group and

$px = 0$ for all $x \in N$. For ease of computation let $\gamma = 1 - \alpha^{p-1} - \beta$ then $\alpha\gamma = \beta\gamma = 0$ and $\gamma^2 = \gamma$. Let $x, y, z \in N$. Then

$$\begin{aligned} x * (y * z) &= x * (\gamma y^{p-1}z + \alpha yz + \beta z) = \gamma x^{p-1}(\gamma y^{p-1}z) + \alpha x(\alpha yz) + \beta(\beta z) \\ &= \gamma x^{p-1}y^{p-1}z + \alpha^2 xyz + \beta z \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (\gamma x^{p-1}y + \alpha xy + \beta y) * z = \gamma(\gamma x^{p-1}y)^{p-1}z + \alpha(\alpha xy)z + \beta z \\ &= \gamma(x^{p-1})^{p-1}y^{p-1}z + \alpha^2 xyz + \beta z = \gamma x^{p-1}y^{p-1}z + \alpha^2 xyz + \beta z. \end{aligned}$$

Hence $x * (y * z) = (x * y) * z$ for all $x, y, z \in N$.

$$\begin{aligned} x * (y + z) &= \gamma x^{p-1}(y + z) + \alpha x(y + z) + \beta(y + z) \\ &= \gamma x^{p-1}y + \alpha xy + \beta y + \gamma x^{p-1}z + \alpha xz + \beta z = x * y + x * z \end{aligned}$$

Thus $(N;+,*)$ is a near-ring. If $x \in N$ then x^2 means xx. Then let $x^{(2)} \equiv x * x$ and $x^{(n)} \equiv x * x^{(n-1)}$ if n is an integer and $n \geq 2$. If $m \geq 2$ then $x^{(m)} = \gamma x + \alpha^{m-1}x^m + \beta x$. The proof of this is routine by induction and will be omitted. Then $x^{(p)} = \gamma x + \alpha^{p-1}x^p + \beta x = (\gamma + \alpha^{p-1} + \beta)x = 1x = x$ since $\gamma = 1 - \alpha^{p-1} - \beta$. Hence $(N;+,*)$ is a p near-ring. Now let $\alpha^{p-1} = 1$. If $p = 2$ then $1 + \beta = \alpha = 1$. Thus $\beta = 0$ so $x * y = xy$ which clearly makes $(N;+,*)$ a 2 ring or Boolean ring. If $p > 2$ then $\gamma = \gamma 1 = \gamma\alpha^{p-1} = 0$ and $\beta = \beta 1 = \beta\alpha^{p-1} = 0$ so $x * y = \alpha xy$. Let $x, y, z \in N$ then $(x + y) * z = \alpha(x + y)z = \alpha xz + \alpha yz = x * z + y * z$. Thus $(N;+,*)$ is a p ring. Conversely let $(N;+,*)$ be a p ring. If $p = 2$ then $\alpha = 1 + \beta$ and $x * y = \alpha xy + \beta y$. Then $\beta = 0 * 1 = (1 + 1) * 1 = 1 * 1 + 1 * 1 = 0$ so $\alpha = 1 + 0 = 1$. If $p > 2$ then $\beta = 0 * 1 = (0 + 0) * 1 = 0 * 1 + 0 * 1 = \beta + \beta = 2\beta$. Thus $\beta = 0$. Then $(1 + 1) * 1 = 1 * 1 + 1 * 1$ implies that

$2^{p-1}\gamma + 2\alpha = 2\gamma + 2\alpha$ or $2^{p-1}\gamma = 2\gamma$. Under the ring operations + and $\cdot$ the elements $0, 1, 2, \cdots, p-1$ form a field isomorphic to Z_p. The nonzero elements form a group under $\cdot$ and its order is $p - 1$ so $2^{p-1} = 1$. Thus $\gamma = 2\gamma$ so $\gamma = 0$. Since $\gamma = 1 - \alpha^{p-1} - \beta$ then it follows that $\alpha^{p-1} = 1$. Finally we notice that α^{p-2} is the identity. If $p = 2$ then this is obvious. Now let $p > 2$ and $x \in N$. $x * \alpha^{p-2} = \alpha x \alpha^{p-2} = \alpha^{p-1}x = 1x = x$ and $\alpha^{p-2} * x = \alpha\alpha^{p-2}x = \alpha^{p-1}x = 1x = x$. Thus α^{p-2} is the identity.

A p near-ring $(N;+,*)$ is an <u>(α,β) p near-ring</u> iff there exists a p ring with identity $(N;+,\cdot;1)$ and $\alpha, \beta \in N$ such that $\alpha\beta = 0$, $\beta^2 = \beta$ and $x * y = (1 - \alpha^{p-1} - \beta)x^{p-1}y + \alpha xy + \beta y$ for all $x, y \in N$.

It is perhaps worth mentioning that there are p near-rings that are not (α,β) p near-rings. Let $(N;+,\cdot;1)$ be $(Z_5;+,\cdot;1)$. According to the listing in Clay [3] this 5 ring is one of those in class (10). Let $\alpha, \beta \in Z_5$ such that $\alpha\beta = 0$ and $\beta^2 = \beta$. As before let $\gamma = 1 - \alpha^4 - \beta$. If $\alpha \neq 0$ then $\beta = 0$ since $\alpha\beta = 0$. Also $\alpha\gamma = \alpha(1 - \alpha^4) = \alpha - \alpha^5 = 0$ so $\gamma = 0$. Thus $x * y = \alpha xy$ which is again a 5 ring in class (10). If $\alpha = 0$ then $\beta = 0$ or $\beta \neq 0$. If $\beta \neq 0$ then $\beta = 1$ because $\beta^2 = \beta$. Then $\gamma = 1 - \beta = 0$ so $x * y = y$. Thus class (9) results. Finally if $\beta = 0$ then $\gamma = 1$ so $x * y = x^4y$. If $x = 0$ then $x * y = 0$ but if $x \neq 0$ then $x * y = y$. Hence class (8) results. These are the only classes that occur as (α,β) 5 near-rings. However a simple check shows that class (7) contains three 5 near-rings and as shown above they cannot be (α,β) 5 near-rings.

2. Special p Near-Rings

Now we will consider a sub-class of these (α,β) p near-rings. Let $(N;+,\cdot;1)$ be a p ring with identity and let $a \in N$. If $b = 1 - a^{p-1}$ then $ab = a(1 - a^{p-1}) = a - a^p = 0$ and $b^2 = (1 - a^{p-1})^2 = 1 - 2a^{p-1} + a^{p-1} = 1 - a^{p-1} = b$. Hence if $\alpha \in N$ and $\beta = 1 - \alpha^{p-1}$ then they determine an (α,β) p near-ring $(N;+,*)$. In this case $\gamma = 1 - \alpha^{p-1} - \beta = 0$ so $x * y = \alpha xy + (1 - \alpha^{p-1})y$ for all $x, y \in N$. A special p near-ring is an (α,β) p near-ring such that $\beta = 1 - \alpha^{p-1}$. Note that every (α,β) 2 near-ring is a special 2 near-ring. This special 2 near-ring coincides with what Clay and Lawver [4] called a special Boolean near-ring. Hence results established in this section are generalizations of some of the results of Clay and Lawver.

In the following discussion of special p near-rings when α is mentioned it will be understood that this is the α in the definition of $*$. Let $(N;+,\cdot;1)$ be a p ring with identity and $t \in N$. Then define $P(t) \equiv \{a \in N : at^{p-1} = a\}$. We note that $0, t \in P(t)$. If $t \in N$ and $L \subset N$ then define $L(t) \equiv \{a \in N : a = \ell t^{p-1}$ for some $\ell \in L\}$. We may note that $L(t)$ is empty iff L is empty. The following observations are listed here for future reference.

Theorem 2.1: Let $(N;+,\cdot;1)$ be a p ring with identity, $t \in N$ and $L \subset N$.

(a) $P(t)$ is an ideal of $(N;+,\cdot;1)$ with identity t^{p-1}.

(b) If $(L;+)$ is a subgroup of $(N;+)$ then $(L(t);+)$ is a subgroup of $(N;+)$.

(c) If $(L;+,\cdot)$ is a subring of $(N;+,\cdot;1)$ then $(L(t);+,\cdot)$ is a subring of $(N;+,\cdot;1)$.

(d) $L(t) \subset P(t)$.

Proof: (a) Let a, b ε P(t) so $at^{p-1} = a$ and $bt^{p-1} = b$. Then $(a - b)t^{p-1} = at^{p-1} - bt^{p-1} = a - b$ so $a - b \varepsilon P(t)$. Let $x \varepsilon N$. Then $(xa)t^{p-1} = x(at^{p-1}) = xa$. Hence P(t) is an ideal and clearly t^{p-1} is the identity.
(b) Let a, b ε L(t). Then $a = \ell t^{p-1}$ and $b = \ell' t^{p-1}$ for some ℓ, $\ell' \varepsilon L$. $a - b = (\ell - \ell')t^{p-1} \varepsilon L(t)$ since $\ell - \ell' \varepsilon L$. Thus (L(t);+) is a group.
(c) Let a, b ε L(t). Then $a = \ell t^{p-1}$ and $b = \ell' t^{p-1}$ for some ℓ, $\ell' \varepsilon L$. $ab = (\ell t^{p-1})(\ell' t^{p-1}) = (\ell\ell')(t^{p-1})^2 = (\ell\ell')t^{p-1} \varepsilon L(t)$ since $\ell\ell' \varepsilon L$. Thus $(L(t);+,\cdot)$ is a subring.
(d) Let a ε L(t). Then $a = \ell t^{p-1}$ for some $\ell \varepsilon L$. Then $at^{p-1} = (\ell t^{p-1})t^{p-1} = \ell(t^{p-1})^2 = \ell t^{p-1} = a$ so a ε P(t). Hence $L(t) \subset P(t)$.

Theorem 2.2: Let (N;+,*) be a special p near-ring. Denote the maximal sub-Z-ring of (N;+,*) by N_z and the maximal sub-C-ring of (N;+,*) by N_c. Then $N_z = P(1 - \alpha^{p-1})$ and $N_c = P(\alpha)$.
Proof: Recall that $(1 - \alpha^{p-1})^2 = 1 - \alpha^{p-1}$ so that a ε $P(1 - \alpha^{p-1})$ iff $a(1 - \alpha^{p-1}) = a$. Also recall that $N_z \equiv \{a \varepsilon N : x * a = a \text{ for all } x \varepsilon N\}$ and $N_c \equiv \{a \varepsilon N : 0 * a = 0\}$. Let a ε N_z. Then $x * a = a$ for all $x \varepsilon N$ so in particular $0 * a = a$. Thus $a = 0 * a = \alpha 0 a + (1 - \alpha^{p-1})a = a(1 - \alpha^{p-1})$ so a ε $P(1 - \alpha^{p-1})$. Then $N_z \subset P(1 - \alpha^{p-1})$. Now let a ε $P(1 - \alpha^{p-1})$. Then $a = a(1 - \alpha^{p-1}) = a - a\alpha^{p-1}$ so $a\alpha^{p-1} = 0$. Then $a\alpha = a\alpha^p = a\alpha^{p-1}\alpha = 0\alpha = 0$. Now let x ε N then $x * a = \alpha x a + (1 - \alpha^{p-1})a = 0 + a = a$. Thus a ε N_z so $P(1 - \alpha^{p-1}) \subset N_z$. Therefore $N_z = P(1 - \alpha^{p-1})$.
Let a ε N_c. Then $0 = 0 * a = \alpha 0 a + (1 - \alpha^{p-1})a = 0 + a - a\alpha^{p-1}$ so $a\alpha^{p-1} = a$. Thus a ε $P(\alpha)$ and $N_c \subset P(\alpha)$. Now let a ε $P(\alpha)$. Then $a\alpha^{p-1} = a$ or $a(1 - \alpha^{p-1}) = 0$. Then $0 * a = \alpha 0 a + (1 - \alpha^{p-1})a = 0 + 0 = 0$ so a ε N_c. Thus $P(\alpha) \subset N_c$. Therefore $N_c = P(\alpha)$.

Now we will begin a study of the ideal structure of the special p near-ring. Let $(N;+,\cdot)$ be a near-ring and $L \subset N$. L is a <u>left ideal</u> of $(N;+,\cdot)$ iff $(L;+)$ is a normal subgroup of $(N;+)$ and $x\ell \in L$ for all $x \in N$ and for all $\ell \in L$.

<u>Lemma 2.3</u>: Let $(N;+,*)$ be a special p near-ring. If $t \in N$ then $P(t)$ is a left ideal of $(N;+,*)$.

<u>Proof</u>: By Theorem 2.1 we know that $(P(t);+)$ is a commutative group. Let $x \in N$ and $a \in P(t)$. $(x * a)t^{p-1} = (\alpha xa + (1 - \alpha^{p-1})a)t^{p-1}$ $= (\alpha x + (1 - \alpha^{p-1}))at^{p-1} = (\alpha x + (1 - \alpha^{p-1}))a = \alpha xa + (1 - \alpha^{p-1})a$ $= x * a$. Thus $x * a \in P(t)$ so $P(t)$ is a left ideal of $(N;+,*)$.

<u>Lemma 2.4</u>: Let $(N;+,*)$ be a special p near-ring and $t \in N$. If L is a left ideal of $(N;+,*)$ then $L(t)$ is a left ideal of $(N;+,*)$.

<u>Proof</u>: As noted in Theorem 2.1, $(L(t);+)$ is a group and it is commutative because $(N;+)$ is commutative. Let $x \in N$ and $a \in L(t)$. Then for some $\ell \in L$, $a = \ell t^{p-1}$. Then $x * a = \alpha xa + (1 - \alpha^{p-1})a = \alpha x\ell t^{p-1}$ $+ (1 - \alpha^{p-1})\ell t^{p-1} = (x * \ell)t^{p-1} \in L(t)$ since $x * \ell \in L$.

<u>Theorem 2.5</u>: Let $(N;+,*)$ be a special p near-ring. If L is a left ideal of $(N;+,*)$ then $L = L(1 - \alpha^{p-1}) \oplus L(\alpha)$, a direct sum of left ideals of $(N;+,*)$. Conversely if $R \subset P(1 - \alpha^{p-1})$ and $S \subset P(\alpha)$ are left ideals of $(N;+,*)$ then $R \oplus S$ is a left ideal of $(N;+,*)$.

<u>Proof</u>: Since $(1 - \alpha^{p-1})^2 = 1 - \alpha^{p-1}$ then $a \in L(1 - \alpha^{p-1})$ iff for some ℓ $a = \ell(1 - \alpha^{p-1})$. Let L be a left ideal of $(N;+,*)$. Then, by Lemma 2.4, $L(1 - \alpha^{p-1})$ and $L(\alpha)$ are left ideals of $(N;+,*)$. Let $y \in P(1 - \alpha^{p-1}) \cap P(\alpha)$.

Then $x * y = y$ for all $x \in N$ and $0 * y = 0$ so $y = 0 * y = 0$. Therefore $P(1 - \alpha^{p-1}) \cap P(\alpha) = 0$. Then $L(1 - \alpha^{p-1}) \cap L(\alpha) \subset P(1 - \alpha^{p-1}) \cap P(\alpha) = 0$ so $L(1 - \alpha^{p-1}) \cap L(\alpha) = 0$. Let $a = \ell(1 - \alpha^{p-1}) \in L(1 - \alpha^{p-1})$. Then $a = \ell(1 - \alpha^{p-1}) = 0 * \ell \in L$ since L is a left ideal. Thus $L(1 - \alpha^{p-1}) \subset L$. Let $a = \ell\alpha^{p-1} \in L(\alpha)$. If $\alpha = 0$ then $a \in L$. If $\alpha \neq 0$ then $(1 - \alpha^{p-1})\ell = 0 * \ell \in L$. Also $\alpha^{p-1}\ell + (1 - \alpha^{p-1})\ell = \alpha^{p-2} * \ell \in L$. It follows then that $a = \alpha^{p-2} * \ell - 0 * \ell \in L$. Therefore $L(\alpha) \subset L$. Clearly then $L(1 - \alpha^{p-1}) \oplus L(\alpha) \subset L$. Now let $\ell \in L$. Then $\ell = \ell(1 - \alpha^{p-1}) + \ell\alpha^{p-1}$. $\ell(1 - \alpha^{p-1}) \in L(1 - \alpha^{p-1})$ and $\ell\alpha^{p-1} \in L(\alpha)$ so $L \subset L(1 - \alpha^{p-1}) \oplus L(\alpha)$. Therefore $L = L(1 - \alpha^{p-1}) \oplus L(\alpha)$. Conversely let $R \subset P(1 - \alpha^{p-1})$ and $S \subset P(\alpha)$ be left ideals of $(N;+,*)$. $R \cap S \subset P(1 - \alpha^{p-1}) \cap P(\alpha) = 0$ so $R \oplus S$ is at least a direct sum of left ideals. Now let $x \in N$ and let $y = r + s \in R \oplus S$. Then $x * y = x * (r + s) = x * r + x * s \in R \oplus S$ because R and S are left ideals. Therefore $R \oplus S$ is a left ideal of $(N;+,*)$.

<u>Lemma 2.6</u>: Let $(N;+,*)$ be a special p near-ring and let $L \subset P(\alpha)$ be a left ideal of $(N;+,*)$. If $a \in L$ then $P(a) \subset L$.

<u>Proof</u>: Let $a \in L \subset P(\alpha)$. Then $a = a\alpha^{p-1}$. If $a = 0$ then $P(a) = 0 \subset L$. Let $a \neq 0$ and $x \in P(a)$. Then $x = xa^{p-1}$. Since $y * a \in L$ for all $y \in N$ it follows that $(\alpha^{p-2}xa^{p-2}) * a \in L$. But $(\alpha^{p-2}xa^{p-2}) * a = \alpha^{p-1}xa^{p-2}a = (\alpha^{p-1}a)xa^{p-2} = axa^{p-2} = xa^{p-1} = x$. Thus $x \in L$ so $P(a) \subset L$. Note that for the case $\alpha = 0$ the conclusion still holds.

<u>Theorem 2.7</u>: Let $(N;+,*)$ be a special p near-ring and $L \subset P(\alpha)$. Then L is an ideal of $(N;+,*)$ iff L is a left ideal of $(N;+,*)$.

Proof: If L is an ideal of $(N;+,*)$ then clearly L is a left ideal of $(N;+,*)$. Now let L be a left ideal of $(N;+,*)$. Let $x, y \in N$ and $a \in L$. Then $(x + a) * y - x * y = \alpha(x + a)y + (1 - \alpha^{p-1})y - \alpha xy - (1 - \alpha^{p-1})y = \alpha ay = \alpha ya + 0 = \alpha ya + (1 - \alpha^{p-1})a = y * a \in L$. Therefore L is an ideal of $(N;+,*)$.

Theorem 2.8: Let $(N;+,*)$ be a special p near-ring and let $L \subset P(1 - \alpha^{p-1})$. Then the following are equivalent:

(a) L is an ideal of $(N;+,*)$,

(b) L is a left ideal of $(N;+,*)$,

(c) $(L;+)$ is a subgroup of $(P(1 - \alpha^{p-1});+)$.

Proof: It is clear that (a) implies (b) and (b) implies (c). Now suppose that (c) holds. L is a normal subgroup because addition is commutative. Let $x, y \in N$ and $a \in L$ then $x * a = a$ since $L \subset P(1 - \alpha^{p-1})$ and by Theorem 2.2, $P(1 - \alpha^{p-1}) = N_z$. Thus $x * a \in L$. $(x + a) * y - x * y = \alpha(x + a)y + (1 - \alpha^{p-1})y - \alpha xy - (1 - \alpha^{p-1})y = \alpha ay = \alpha ya = \alpha y(a(1 - \alpha^{p-1})) = 0 \in L$. Thus L is an ideal so (c) implies (a).

Theorem 2.9: Let $(N;+,*)$ be a special p near-ring. Then I is an ideal of $(N;+,*)$ iff I is a left ideal of $(N;+,*)$.

Proof: If I is an ideal of $(N;+,*)$ then it is certainly a left ideal of $(N;+,*)$. Now let I be a left ideal of $(N;+,*)$. Then, by Theorem 2.5, $I = I(1 - \alpha^{p-1}) \oplus I(\alpha)$ where $I(1 - \alpha^{p-1})$ and $I(\alpha)$ are left ideals of $(N;+,*)$ in $P(1 - \alpha^{p-1})$ and $P(\alpha)$ respectively. Then, by Theorem 2.8 and and Theorem 2.7, it follows that $I(1 - \alpha^{p-1})$ and $I(\alpha)$ are ideals. Let $x, y \in N$ and $a + b \in I(1 - \alpha^{p-1}) \oplus I(\alpha) = I$ then $(x + (a + b)) * y - x * y$

$= ((x + b) + a) * y - (x + b) * y + (x + b) * y - x * y \in I(1 - \alpha^{p-1}) \oplus I(\alpha)$ $= I$. Thus I is an ideal.

If $(N;+,\cdot)$ is a near-ring and I is an ideal of $(N;+,\cdot)$ then it is well known that N/I is a near-ring. Clearly if $(N;+,\cdot)$ is a p near-ring then N/I is a p near-ring also. Under certain conditions N/I is a p ring. Now we will investigate these conditions for special p near-rings.

<u>Lemma 2.10</u>: Let $(N;+,*)$ be a special p near-ring. If $a, b, c \in N$ then $(a + b) * c - a * c - b * c = -(1 - \alpha^{p-1})c$.

<u>Proof</u>: Let $a, b, c \in N$. Then $(a + b) * c - a * c - b * c = \alpha(a + b)c + (1 - \alpha^{p-1})c - \alpha ac - (1 - \alpha^{p-1})c - \alpha bc - (1 - \alpha^{p-1})c = \alpha ac + \alpha bc - \alpha ac - \alpha bc - (1 - \alpha^{p-1})c = -(1 - \alpha^{p-1})c$.

<u>Theorem 2.11</u>: Let $(N;+,*)$ be a special p near-ring and I an ideal of $(N;+,*)$. Then N/I is a p ring iff $P(1 - \alpha^{p-1}) \subset I$.

<u>Proof</u>: Let $a, b, c \in N$. The following statements are equivalent:

$$((a + I) + (b + I)) * (c + I) = (a + I) * (c + I) + (b + I) * (c + I)\ , \tag{1}$$

$$((a + b) + I) * (c + I) = (a * c) + I + (b * c) + I\ , \tag{2}$$

$$((a + b) * c) + I = ((a * c) + (b * c)) + I\ , \tag{3}$$

$$(a + b) * c - a * c - b * c \in I\ , \tag{4}$$

$$-(1 - \alpha^{p-1})c \in I. \tag{5}$$

Let N/I be a p ring and let $x \in P(1 - \alpha^{p-1})$. Then $((a + I) + (b + I)) (-x + I) = (a + I) * (-x + I) + (b + I) * (-x + I)$ for any $a, b \in N$. By the above equivalent statements then $-(1 - \alpha^{p-1})(-x) \in I$. However,

$x = x(1 - \alpha^{p-1}) = -(1 - \alpha^{p-1})(-x)$ so $x \in I$ and hence $P(1 - \alpha^{p-1}) \subset I$. Conversely let $P(1 - \alpha^{p-1}) \subset I$. $-(1 - \alpha^{p-1})c \in P(1 - \alpha^{p-1})$ for all $c \in N$ so by the above equivalent statements (1) holds for all $a, b, c \in N$. Hence N/I is a p ring.

In approaching the question as to when an ideal of a special p near-ring $(N;+,*)$ is a direct summand we first establish the following result.

Theorem 2.12: Let $(N;+,\cdot;1)$ be a p ring with identity and let A be an ideal of $(N;+,\cdot;1)$. Then A is a direct summand iff $A = P(a)$ for some $a \in N$.

Proof: Let $A = P(a)$ for $a \in N$. Then $N = P(1 - a^{p-1}) \oplus P(a)$ $= P(1 - a^{p-1}) \oplus A$. Hence A is a direct summand. Conversely let A be a direct summand. Then $N = A \oplus B$ where B is also an ideal of $(N;+,\cdot;1)$. If $x \in A$ and $y \in B$ then $xy \in A \cap B$ since A, B are ideals. Thus $xy = 0$. $1 \in N$ so $1 = a + b$ where $a \in A$ and $b \in B$. Then $a = a(1) = a(a + b)$ $= a^2 + ab = a^2$ so $a^2 = a$. If $x \in A$ then $x = x(1) = x(a + b) = xa + xb = xa$. Thus $xa = x$ so $x \in P(a)$ and $A \subset P(a)$. If $x \in P(a)$ then $x = xa \in A$ since A is an ideal so $P(a) \subset A$. Hence $A = P(a)$.

Let $(N;+,*)$ be a special p near-ring and let A, B be ideals of $(N;+,*)$. Let $N = A \oplus B$. Then, since $C = C(1 - \alpha^{p-1}) \oplus C(\alpha)$ for $C = A$ or B, $N = (A(1 - \alpha^{p-1}) \oplus A(\alpha)) \oplus (B(1 - \alpha^{p-1}) \oplus B(\alpha))$. It follows then that $N = (A(1 - \alpha^{p-1}) \oplus B(1 - \alpha^{p-1})) \oplus (A(\alpha) \oplus B(\alpha))$. Theorem 2.1 implies that $A(1 - \alpha^{p-1}), B(1 - \alpha^{p-1}) \subset P(1 - \alpha^{p-1})$ and $A(\alpha), B(\alpha) \subset P(\alpha)$.

Thus $A(1 - \alpha^{p-1}) \oplus B(1 - \alpha^{p-1}) \subset P(1 - \alpha^{p-1})$ and $A(\alpha) \oplus B(\alpha) \subset P(\alpha)$. Let $x \in P(\alpha)$. Then $x \in N = A \oplus B$ so $x = a + b$ where $a \in A$ and $b \in B$. If $\alpha = 0$ then $a\alpha^{p-1} = 0 \in A$. Let $\alpha \neq 0$. Then $0 * a = (1 - \alpha^{p-1})a \in A$ and $\alpha^{p-2} * a = \alpha^{p-1}a + (1 - \alpha^{p-1})a \in A$ since A is an ideal of $(N;+,*)$. Thus $a\alpha^{p-1} = \alpha^{p-2} * a - 0 * a \in A$ so in either case $a\alpha^{p-1} \in A$. Similarly $b\alpha^{p-1} \in B$. Then $x = x\alpha^{p-1} = (a + b)\alpha^{p-1} = a\alpha^{p-1} + b\alpha^{p-1}$. But $x = a + b$. The uniqueness of representation of elements in $A \oplus B$ implies that $a = a\alpha^{p-1}$ and $b = b\alpha^{p-1}$. Thus $a \in A(\alpha)$ and $b \in B(\alpha)$. Then $x = a + b \in A(\alpha) \oplus B(\alpha)$. Thus $P(\alpha) \subset A(\alpha) \oplus B(\alpha)$. Therefore $P(\alpha) = A(\alpha) \oplus B(\alpha)$. Similarly one can show that $P(1 - \alpha^{p-1}) = A(1 - \alpha^{p-1}) \oplus B(1 - \alpha^{p-1})$.

Before we consider the next result recall the following definitions. A group $(G;+)$ is <u>bounded</u> iff $nG = 0$ for some fixed integer n. A subgroup $(S;+)$ of a group $(G;+)$ is <u>pure</u> iff the equation $mx = a \in S$ is solvable in S whenever it has a solution in G. From Fuchs [7] we have the following result.

<u>Theorem 2.13</u>: A bounded pure subgroup is a direct summand.

The main result about direct summands is the following.

<u>Theorem 2.14</u>: Let $(N;+,*)$ be a special p near-ring and let I be an ideal of $(N;+,*)$. I is a direct summand of N iff $I = P(\delta) \oplus M$ where $P(\delta) \subset P(\alpha)$ is an ideal and $M \subset P(1 - \alpha^{p-1})$ is a subgroup, hence an ideal.

<u>Proof</u>: Let I be a direct summand of N then $N = I \oplus L$ and L is an ideal. As noted earlier $I = I(\alpha) \oplus I(1 - \alpha^{p-1})$ where each is a left ideal respectively contained in $P(\alpha)$ and $P(1 - \alpha^{p-1})$. By Theorem 2.7 and Theorem 2.8 $I(\alpha)$ and $I(1 - \alpha^{p-1})$ are each ideals. By previous work

$P(\alpha) = I(\alpha) \oplus L(\alpha)$. $I(\alpha)$ is a direct summand of $P(\alpha)$. $(P(\alpha);+,\cdot;\alpha^{p-1})$ is a p ring with identity so, by Theorem 2.12, any direct summand of $P(\alpha)$ is of the form $P(\delta)$ for $\delta \in P(\alpha)$. Thus $I(\alpha) = P(\delta)$ for $\delta \in P(\alpha)$. Hence $I = I(\alpha) \oplus I(1 - \alpha^{p-1}) = P(\delta) \oplus I(1 - \alpha^{p-1})$. Then $P(\delta) = I(\alpha) \subset P(\alpha)$ is an ideal and $I(1 - \alpha^{p-1}) \subset P(1 - \alpha^{p-1})$ is an ideal. Conversely let $I = P(\delta) \oplus M$ where $P(\delta) \subset P(\alpha)$ is an ideal and $M \subset P(1 - \alpha^{p-1})$ is a subgroup. Since $(P(\alpha);+,\cdot;\alpha^{p-1})$ is a p ring with identity then, by Theorem 2.12, $P(\delta)$ is a direct summand of $P(\alpha)$. The group M is bounded since $pM = 0$. Let $a \in M$, m be an integer and $mx = a$ have a solution in $P(1 - \alpha^{p-1})$. Thus $mx' = a$ where $x' \in P(1 - \alpha^{p-1})$. Let i be an integer such that $im \equiv 1$ modulo p so $x' = ia \in M$. Hence $mx = a$ has a solution in M so M is pure. Therefore, by Theorem 2.13, M is a direct summand of $P(1 - \alpha^{p-1})$. Thus $P(\alpha) = P(\delta) \oplus A$ and $P(1 - \alpha^{p-1}) = M \oplus B$ where A, B are ideals. Then $N = P(\alpha) \oplus P(1 - \alpha^{p-1}) = (P(\delta) \oplus A) \oplus (M \oplus B) = (P(\delta) \oplus M) \oplus (A \oplus B) = I \oplus (A \oplus B)$. Hence I is a direct summand of N.

As noted previously results in this section generalize some of the results of Clay and Lawver [4]. Now we consider a result in that paper that is incorrect. The statement of Theorem 5.1 [4] in the notation of this chapter would be as follows. Let $(B;+,\cdot;1)$ be a Boolean ring with identity. Let $\sigma,\tau \in B$ define special Boolean near-rings $(B;+,*\sigma)$ and $(B;+,*\tau)$ respectively. Then the following are equivalent:

(a) $(B;+,*\sigma)$ is isomorphic to $(B;+,*\tau)$,

(b) $P(1 + \sigma)$ is isomorphic to $P(1 + \tau)$ as subrings of $(B;+,\cdot;1)$,

(c) $P(\sigma)$ is isomorphic to $P(\tau)$ as subrings of $(B;+,\cdot;1)$,

(d) There exists an automorphism f of $(B;+,\cdot;1)$ such that $f(\sigma) = \tau$.

Consider the following counterexample. Let the index set I be $\{1,2,3,\cdots\}$ and let B be the complete direct sum of B_i for $i \varepsilon I$ where each $B_i = Z_2$. Let $\sigma = (1,0,0,0,\cdots)$ and $\tau = (1,1,0,0,\cdots)$. Then $1 + \sigma = (0,1,1,1,\cdots)$ and $1 + \tau = (0,0,1,1,\cdots)$. By definition $P(1 + \sigma) = \{a \varepsilon B:a(1 + \sigma) = a\}$ or $P(1 + \sigma) = \{(0,x_2,x_3,\cdots):x_i \varepsilon Z_2\}$. Similarly $P(1 + \tau) = \{(0,0,x_3,x_4,\cdots):x_i \varepsilon Z_2\}$. Define $f : P(1 + \sigma) \to P(1 + \tau)$ by $f(0,a_1,a_2,a_3,\cdots) \equiv (0,0,a_1,a_2,a_3,\cdots)$. It is clear that f is an isomorphism of $P(1 + \sigma)$ onto $P(1 + \tau)$. However, it is also clear that $P(\sigma)$ is not isomorphic to $P(\tau)$ since $O(P(\sigma)) = 2$ and $O(P(\tau)) = 4$.

3. Some Results About the (α,β) p Near-Ring

Now let us return to the more general case of an (α,β) p near-ring. When α and β are mentioned it will be understood that they are the α and β in the definition of *. It will be noted that each result in this section has as a corollary a result in section 2. For the special p near-rings the results were much more complete. For this reason they were presented first. Now let $(N;+,\cdot;1)$ be a p ring with identity. If s, t ε N then define $P(s,t) \equiv \{a \varepsilon N:sa = 0 \text{ and } at^{p-1} = a\}$. We may note that $0, (1 - s^{p-1})t \varepsilon P(s,t)$. If s, t ε N and $L \subset N$ then define $L(s,t) \equiv \{a \varepsilon N:sa = 0 \text{ and } a = \ell t^{p-1} \text{ for some } \ell \varepsilon L\}$. It is possible for $L(s,t)$ to be empty but not in the cases we will consider. In particular when $(L;+)$ is a subgroup of $(N;+)$ then $0 \varepsilon L(s,t)$.

Theorem 3.1: Let $(N;+,\cdot;1)$ be a p ring with identity. Let s, t ε L and $L \subset N$. Then

(a) $P(s,t)$ is an ideal of $(N;+,\cdot;1)$ with identity t^{p-1}.

(b) If $(L;+)$ is a subgroup of $(N;+)$ then $(L(s,t);+)$ is a subgroup of $(N;+)$.

(c) If $(L;+,\cdot)$ is a subring of $(N;+,\cdot;1)$ then $(L(s,t);+,\cdot)$ is a subring of $(N;+,\cdot;1)$.

(d) $L(s,t) \subset P(s,t)$.

Proof: (a) Let $a, b \in P(s,t)$. Then $sa = sb = 0$, $at^{p-1} = a$ and $bt^{p-1} = b$. Thus $s(a - b) = sa - sb = 0 - 0 = 0$ and $(a - b)t^{p-1} = at^{p-1} - bt^{p-1} = a - b$ so $a - b \in P(s,t)$. Let $x \in N$. Then $s(xa) = x(sa) = 0$ and $(xa)t^{p-1} = x(at^{p-1}) = xa$ so $xa \in P(s,t)$. Hence $P(s,t)$ is an ideal and clearly t^{p-1} is the identity.

(b) Let $a, b \in L(s,t)$. Then $sa = sb = 0$, $a = \ell t^{p-1}$ and $b = \ell' t^{p-1}$ for some $\ell, \ell' \in L$. Then $a - b = \ell t^{p-1} - \ell' t^{p-1} = (\ell - \ell')t^{p-1}$ where $\ell - \ell' \in L$. Finally $s(a - b) = sa - sb = 0 - 0 = 0$ so $a - b \in L(s,t)$. Hence $(L(s,t);+)$ is a subgroup of $(N;+)$.

(c) Let $a, b \in L(s,t)$. Then $ab = (\ell t^{p-1})(\ell' t^{p-1}) = (\ell\ell')(t^{p-1})^2 = (\ell\ell')t^{p-1}$ and $\ell\ell' \in L$ since L is a ring. Thus $(L(s,t);+,\cdot)$ is a subring.

(d) Let $a \in L(s,t)$. Then $sa = 0$ and $a = \ell t^{p-1}$ for some $\ell \in L$. Thus $at^{p-1} = (\ell t^{p-1})t^{p-1} = \ell(t^{p-1})^2 = \ell t^{p-1} = a$ so $a \in P(s,t)$. Therefore $L(s,t) \subset P(s,t)$.

Theorem 3.2: Let $(N;+,*)$ be an (α,β) p near-ring. Denote the maximal sub-Z-ring by N_z and the maximal sub-C-ring by N_c. Then $N_z = P(\alpha,\beta)$ and $N_c = P(\beta,1 - \beta)$.

Proof: By definition $N_z \equiv \{a \in N: x * a = a \text{ for all } x \in N\}$ and $N_c \equiv \{a \in N: 0 * a = 0\}$. Let $a \in N_z$. Then $x * a = a$ for all $x \in N$. Thus

$0 * a = \gamma 0a + \alpha 0a + \beta a = \beta a$ so $\beta a = a$. $\alpha * a = (1 - \alpha^{p-1} - \beta)\alpha^{p-1}a + \alpha^2 a + \beta a = 0 + \alpha^2 a + a = \alpha^2 a + a$ so $\alpha^2 a = 0$. Then $\alpha a = \alpha^p a = \alpha^{p-2}(\alpha^2 a) = \alpha^{p-2}(0) = 0$ so $a \in P(\alpha,\beta)$. Thus $N_z \subset P(\alpha,\beta)$. Now let $a \in P(\alpha,\beta)$ so $\alpha a = 0$ and $a\beta = a$. Let $x \in N$. Then $x * a = (1 - \alpha^{p-1} - \beta)x^{p-1}a + \alpha xa + \beta a = 0 + 0 + a = a$. Hence $a \in N_z$ so $P(\alpha,\beta) \subset N_z$. Therefore $N_z = P(\alpha,\beta)$. Now let $a \in N_c$ then $0 * a = 0$. But $0 * a = \gamma 0a + \alpha 0a + \beta a = \beta a$ so $\beta a = 0$. $(1 - \beta)^2 = 1 - 2\beta + \beta = 1 - \beta$ so $(1 - \beta)^{p-1} = 1 - \beta$. Thus $x(1 - \beta)^{p-1} = x$ iff $x(1 - \beta) = x$. Then $a(1 - \beta) = a - a\beta = a - 0 = a$ so $a \in P(\beta,1 - \beta)$. Hence $N_c \subset P(\beta,1 - \beta)$. Let $a \in P(\beta,1 - \beta)$. Then $\beta a = 0$ and $a(1 - \beta) = a$. $0 * a = \gamma 0a + \alpha 0a + \beta a = 0 + 0 + \beta a = \beta a = 0$. Thus $a \in N_c$ so $P(\beta,1 - \beta) \subset N_c$. Therefore $N_c = P(\beta,1 - \beta)$. (Observe that $P(\beta,1 - \beta) = P(\beta,1)$.)

We will now examine the ideal structure of the (α,β) p near-ring.

<u>Lemma 3.3</u>: Let $(N;+,*)$ be an (α,β) p near-ring. If $s, t \in N$ then $P(s,t)$ is a left ideal of $(N;+,*)$.

<u>Proof:</u> By Theorem 2.1, $(P(s,t),+)$ is a commutative group. Let $x \in N$ and $a \in P(s,t)$. Then $sa = 0$ and $at^{p-1} = a$. Thus $s(x * a) = s(\gamma x^{p-1}a + \alpha xa + \beta a) = (\gamma x^{p-1} + \alpha x + \beta)sa = (\gamma x^{p-1} + \alpha x + \beta)0 = 0$ and $(x * a)t^{p-1} = (\gamma x^{p-1}a + \alpha xa + \beta a)t^{p-1} = (\gamma x^{p-1} + \alpha x + \beta)\ at^{p-1} = (\gamma x^{p-1} + \alpha x + \beta)a = \gamma x^{p-1}a + \alpha xa + \beta a = x * a$. Hence $P(s,t)$ is a left ideal of $(N;+,*)$.

<u>Lemma 3.4</u>: Let $(N;+,*)$ be an (α,β) p near-ring. If L is a left ideal of $(N;+,*)$ and $s,t \in N$ then $L(s,t)$ is a left ideal of $(N;+,*)$.

<u>Proof</u>: By Theorem 3.1, $(L(s,t);+)$ is a group. $(N;+)$ is a commutative

group. Let $x \in N$ and $a \in L(s,t)$. Then $sa = 0$ and $a = \ell t^{p-1}$ for some $\ell \in L$. From the proof of Lemma 3.3 we have $s(x * a) = 0$. Then $x * a = \gamma x^{p-1}a + \alpha xa + \beta a = \gamma x^{p-1}\ell t^{p-1} + \alpha x\ell t^{p-1} + \beta\ell t^{p-1} = (\gamma x^{p-1}\ell + \alpha x\ell + \beta\ell)t^{p-1} = (x * \ell)t^{p-1}$. However, $x * \ell \in L$ since L is a left ideal so $x * a \in L(s,t)$. Thus $L(s,t)$ is a left ideal of $(N;+,*)$.

<u>Theorem 3.5</u>: Let $(N;+,*)$ be an (α,β) p near-ring. If L is a left ideal of $(N;+,*)$ then $L = L(\alpha,\beta) \oplus L(\beta,1 - \beta)$, a direct sum of left ideals of $(N;+,*)$. Conversely if $R \subset P(\alpha,\beta)$ and $S \subset P(\beta,1 - \beta)$ are left ideals of $(N;+,*)$ then $R \oplus S$ is a left ideal of $(N;+,*)$.

<u>Proof</u>: Let L be a left ideal of $(N;+,*)$. By Lemma 3.4, $L(\alpha,\beta)$ and $L(\beta,1 - \beta)$ are left ideals of $(N;+,*)$. Let $a \in L(\alpha,\beta)$. Then $a\alpha = 0$ and $a = \ell\beta$ for some $\ell \in L$. Thus $a = \ell\beta = 0 * \ell \in L$ so $L(\alpha,\beta) \subset L$. Next let $a \in L(\beta,1 - \beta)$. Then $a\beta = 0$ and $a = \ell(1 - \beta)$. Thus $\ell\beta = 0 * \ell \in L$ and $\ell \in L$ so $a = \ell(1 - \beta) = \ell - \ell\beta \in L$. Hence $L(\beta,1 - \beta) \subset L$. If $y \in P(\alpha,\beta) \cap P(\beta,1 - \beta)$ then $x * y = y$ for all $x \in N$ and $0 * y = 0$. Then $y = 0 * y = 0$ so $P(\alpha,\beta) \cap P(\beta,1 - \beta) = 0$. Hence $L(\alpha,\beta) \cap L(\beta,1 - \beta) \subset P(\alpha,\beta) \cap P(\beta,1 - \beta) = 0$ so $L(\alpha,\beta) \cap L(\beta,1 - \beta) = 0$. It is immediate that $L(\alpha,\beta) \oplus L(\beta,1 - \beta) \subset L$. Now let $\ell \in L$. Then $\ell = \ell\beta + \ell(1 - \beta) \in L(\alpha,\beta) \oplus L(\beta,1 - \beta)$. Thus $L \subset L(\alpha,\beta) \oplus L(\beta,1 - \beta)$. Therefore $L = L(\alpha,\beta) \oplus L(\beta,1 - \beta)$, a direct sum of left ideals of $(N;+,*)$. Conversely let $R \subset P(\alpha,\beta)$ and $S \subset P(\beta,1 - \beta)$ be left ideals of $(N;+,*)$. Since $R \cap S \subset P(\alpha,\beta) \cap P(\beta,1 - \beta) = 0$ then $R \oplus S$ is at least a direct sum of left ideals. Let $x \in N$ and $y = r + s \in R \oplus S$. Then $x * y = x * (r + s) = x * r + x * s \in R \oplus S$ since R and S are left ideals. Thus $R \oplus S$ is a left ideal of $(N;+,*)$.

Theorem 3.6: Let $(N;+,*)$ be an (α,β) p near-ring and let $L \subset P(\alpha,\beta)$. The following are equivalent:

(a) L is an ideal of $(N;+,*)$,

(b) L is a left ideal of $(N;+,*)$,

(c) $(L;+)$ is a subgroup of $(P(\alpha,\beta);+)$.

Proof: Clearly (a) implies (b) and (b) implies (c). Now let $(L;+)$ be a subgroup of $(P(\alpha,\beta);+)$. Let $x, y \in N$, $a \in L \subset P(\alpha,\beta)$. Then $x * a = a \in L$. $a \in P(\alpha,\beta)$ implies that $a\alpha = 0$ and $a\beta = a$. Then $a\gamma = a(1 - \alpha^{p-1} - \beta) = a - 0 - a = 0$. Thus $(x + a) * y - x * y$ $= \gamma(x + a)^{p-1}y + \alpha(x + a)y + \beta y - \gamma x^{p-1}y - \alpha xy - \beta y = \gamma(x + a)^{p-1}y$ $+ \alpha\, ay - \gamma x^{p-1}y = \gamma(x + a)^{p-1}y - \gamma x^{p-1}y = \sum_{i=1}^{p-1} \binom{p-1}{i} x^{p-1-i} a^{i} \gamma = 0.$ Hence $(x + a) * y - x * y \in L$ so L is an ideal of $(N;+,*)$. Therefore (c) implies (a).

4. Further Results About the (α,β) p Near-Rings

The following results concerning p rings are well known. They are found, for example, in McCoy [9].

Theorem 4.1: A finite p ring has p^k elements for some positive integer k. It has a unit element and is isomorphic to the direct sum of k fields Z_p, where Z_p denotes the integers modulo p.

Theorem 4.2: A necessary and sufficient condition that a ring be isomorphic to a subdirect sum of fields Z_p is that it be a p ring.

For a detailed treatment of the direct sum and subdirect sum of rings one source is McCoy [10]. For a similar treatment of near-rings a source is Fain [5]

If $(N;+,\cdot;1)$ is a p ring with identity then N is isomorphic to a subdirect sum of fields N_i for i in some index set I and with each $N_i = Z_p$. An element in this subdirect sum will be of the form $(x_i)_{i \in I}$. If I is finite, say $O(I) = k$, then $(x_i)_{i \in I}$ will be simplified to $(x_1, x_2, \cdots, x_k)$. This isomorphism will be used to identify $x \in N$ with $(x_i)_{i \in I}$ or $(x_1, x_2, \cdots, x_k)$, if $O(I) = k$.

Now let $(N;+,*)$ be an (α,β) p near-ring. Then $\alpha, \beta \in N$ such that $\alpha\beta = 0$ and $\beta^2 = \beta$. $\beta = (b_i)_{i \in I}$ and $\beta^2 = \beta$ implies that $b_i = 0$ or 1 for all $i \in I$. Similarly $\gamma \equiv 1 - \alpha^{p-1} - \beta$ is such that $\gamma^2 = \gamma$. If $\gamma = (c_i)_{i \in I}$ then $c_i = 0$ or 1 for all $i \in I$. The conditions that $\alpha\beta = \beta\gamma = \alpha\gamma = 0$ and $1 = \alpha^{p-1} + \beta + \gamma$ imply that for each $i \in I$ exactly one of a_i, b_i, c_i will be nonzero, where $\alpha = (a_i)_{i \in I}$.

Theorem 4.3: Let $(N;+,\cdot;1)$ be a p ring with identity and $O(N) = p^k$. Then there are $(p + 1)^k$ choices for the pair α and β that will result in an (α,β) p near-ring. Furthermore $(p)^k$ of these result in special p near-rings and $(p - 1)^k$ result in p rings.

Proof: Let β be a k-tuple with i 0's and k - i 1's as its components. There are $\binom{k}{i}$ such elements. The only condition on α is that $\alpha\beta = 0$. Thus α must be 0 in the places where β is 1 and in each of the i places where β is 0 then α can be any element of Z_p. For any such β there are p^i choices for α. The total number of choices for α and β is $\sum_{i=0}^{k} \binom{k}{i} p^i = (p + 1)^k$. To obtain a special p near-ring $\beta = 1 - \alpha^{p-1}$ where α is any element of N. Since $O(N) = p^k$ there are p^k choices for α and then β is determined so there are p^k special p near-rings. Finally we recall that an (α,β) p near-ring is a p ring iff $\alpha^{p-1} = 1$. $1 \in N$ is the k-tuple having each component equal to 1. If $a \in Z_p$ and $a \neq 0$ then $a^{p-1} = 1$. Thus the only condition on α that need be imposed is that α be nonzero in every component. That leaves p-1 choices for each of the k components so the number of these (α,β) p near-rings that are p rings is $(p - 1)^k$.

Consider now an (α,β) p near-ring $(Z_p;+,*)$. Since $\alpha\beta = \alpha\gamma = \beta\gamma = 0$ and $\alpha^{p-1} + \beta + \gamma = 1$ it follows that exactly one of α, β and γ is nonzero.

Lemma 4.4: Let $(Z_p;+,*_1)$ and $(Z_p;+,*_2)$ be (α,β) p near-rings determined by α_1, 0 and α_2, 0 where α_1, α_2 are both nonzero. Then $(Z_p;+,*_1)$ is isomorphic to $(Z_p;+,*_2)$.

Proof: Define $h : Z_p \to Z_p$ by $h(x) = \alpha_1\alpha_2^{-1}x$ for all $x \in Z_p$. Now let $x, y \in Z_p$. $h(x + y) = \alpha_1\alpha_2^{-1}(x + y) = \alpha_1\alpha_2^{-1}x + \alpha_1\alpha_2^{-1}y = h(x) + h(y)$. $h(x *_1 y) = h(\alpha_1 xy) = \alpha_1\alpha_2^{-1}(\alpha_1 xy) = \alpha_1^2\alpha_2^{-1}xy$ and $h(x) *_2 h(y) = \alpha_2(\alpha_1\alpha_2^{-1}x)(\alpha_1\alpha_2^{-1}y) = \alpha_1^2\alpha_2^{-1}xy$. If $y \in Z_p$ then $h(\alpha_1^{-1}\alpha_2 y) = y$. If $h(x) = h(y)$ then $\alpha_1\alpha_2^{-1}x = \alpha_1\alpha_2^{-1}y$ so $x = y$. Thus h is an isomorphism.

The conclusion of the previous lemma was perhaps obvious since each near-ring was actually a p ring and isomorphic to Z_p. The interest in this lemma is in the construction of the isomorphism.

Theorem 4.5: Let $(N;+,\cdot;1)$ be a p ring with identity and $\alpha, \beta \in N$. If $\alpha\beta = 0$, $\beta^2 = \beta$ and $\gamma = 1 - \alpha^{p-1} - \beta$ then

(a) αN, βN, γN are ideals of $(N;+,\cdot;1)$ and

(b) $N = \alpha N \oplus \beta N \oplus \gamma N$.

Proof: (a) Let $\alpha x, \alpha y \in \alpha N$ and $z \in N$. Then $\alpha x - \alpha y = \alpha(x - y) \in \alpha N$.

$z(\alpha x) = \alpha(zx) \in \alpha N$. Thus αN is an ideal of $(N;+,\cdot;1)$. Similarly βN, γN are ideals of $(N;+,\cdot;1)$.

(b) Let $x \in \alpha N \cap \beta N$ then $x = \alpha a$ and $x = \beta b$. Hence $x^2 = (\alpha a)(\beta b)$ $= (\alpha\beta)(ab) = 0(ab) = 0$. Thus $x = 0$ so $\alpha N \cap \beta N = 0$. In a similar way we see that $\alpha N \cap \gamma N = \beta N \cap \gamma N = 0$. Clearly $\alpha N \oplus \beta N \oplus \gamma N \subset N$. Now let $x \in N$. Then $x = 1x = (\alpha^{p-1} + \beta + \gamma)x = \alpha(\alpha^{p-2}x) + \beta x + \gamma x$. Hence $N \subset \alpha N \oplus \beta N \oplus \gamma N$. Therefore $N = \alpha N \oplus \beta N \oplus \gamma N$. If $p = 2$ then this is simplified to $N = \alpha N \oplus \beta N$ where $\alpha = 1 + \beta$.

<u>Lemma 4.6</u>: Let $(N;+,*)$ be an (α,β) p near-ring and let z be a nonzero element of N. Then z is right distributive iff $z \in \alpha N$.

<u>Proof</u>: First let $p = 2$ so $\alpha = 1 + \beta$. Now let $z \in \alpha N = (1 + \beta)N$ and $x, y \in N$. Then $(x + y) * z = \alpha(x + y)z + \beta z = \alpha(x + y)z = \alpha xz + \alpha yz$ $= \alpha xz + \beta z + \alpha yz + \beta z = x * z + y * z$. Thus z is right distributive. Conversely let z be right distributive. Then $(1 + \beta) * z$ $= 1 * z + \beta * z$ or $(1 + \beta)z + \beta z = (1 + \beta)z + \beta z + (1 + \beta)\beta z + \beta z$. Thus $\beta z = 0$ so $z \in (1 + \beta)N = \alpha N$. Now let $p > 2$. Let $z \in \alpha N$ then $z = \alpha z'$ for some $z' \in N$. For $x, y \in N$ we have $(x + y) * z = \alpha(x + y)z$ $= \alpha xz + \alpha yz = x * z + y * z$. Hence z is right distributive. Now let z be right distributive. If $N = \alpha N$ then $z \in \alpha N$. Now let $N \neq \alpha N$. Either $\gamma = 0$ or $\gamma \neq 0$. If $\gamma = 0$ then for $x, y \in N$, $(x + y) * z$ $= x * z + y * z$ implies that $\alpha(x + y)z + \beta z = \alpha xz + \beta z + \alpha yz + \beta z$. Thus $\beta z = 0$. $z = z(1) = z(\alpha^{p-1} + \beta) = \alpha(z\alpha^{p-2}) \in \alpha N$. Finally if $\gamma \neq 0$ then let $x = y = \gamma$. Then $(\gamma + \gamma) * z = \gamma * z + \gamma * z$ implies that $\gamma(2\gamma)^{p-1}z + \beta z = \gamma z + \beta z + \gamma z + \beta z$. However $(2\gamma)^{p-1}\gamma = 2^{p-1}\gamma = \gamma$ so this results in $\gamma z + \beta z = (\gamma + \beta)z = 0$. Then $z = z(1) = z(\alpha^{p-1} + \beta + \gamma)$ $= \alpha(z\alpha^{p-2}) + z(\beta + \gamma) = \alpha(z\alpha^{p-2}) \in \alpha N$.

Lemma 4.7: Let $(N;+,*_1)$ and $(N;+,*_2)$ be (α,β) p near-rings determined by α_1, β_1 and α_2, β_2 respectively and let $0(N) = p^k$. Let α_1 and α_2 have exactly i_1 and i_2 nonzero components and let them occur in the first i_1 and i_2 places of α_1 and α_2 respectively. Let β_1 and β_2 have exactly j_1 and j_2 nonzero components and let them occur in places $i_1 + 1,\cdots,$ $i_1 + j_1$ and $i_2 + 1,\cdots,i_2 + j_2$ of β_1 and β_2 respectively. Then $(N;+,*_1)$ is isomorphic to $(N;+,*_2)$ iff $i_1 = i_2$ and $j_1 = j_2$.

Proof: First let $i_1 = i_2 = i$ and $j_1 = j_2 = j$. Because the nonzero elements of β and γ are 1's it follows that $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. If $i = 0$ then $\alpha_1 = \alpha_2 = 0$. Thus $x *_1 y = \gamma_1 x^{p-1}y + \beta_1 y = \gamma_2 x^{p-1}y + \beta_2 y$ $= x *_2 y$ so clearly $(N;+,*_1)$ is isomorphic to $(N;+,*_2)$. If $i > 0$ then let a_{1r} be the r^{th} component of α_1 and a_{2r} the r^{th} component of α_2 for $1 \leq r \leq i$. Let $g_r : Z_p \to Z_p$ be defined by $g_r(x) \equiv a_{1r}a_{2r}^{-1}x$. If $x \in N$ then $x = (x_1,x_2,\cdots,x_k)$. Define $g : N \to N$ by $g(x) \equiv (g_1(x_1),g_2(x_2),\cdots,$ $g_i(x_i),x_{i+1},\cdots,x_k)$. It is routine to verify that g is 1 - 1, onto and that $g(x + y) = g(x) + g(y)$ for all $x, y \in N$. Then by the nature of β_1, β_2, γ_1, γ_2 and g it follows that $g(\beta_1 y) = \beta_1 y = \beta_2 y = \beta_2 g(y)$ and $g(\gamma_1 x^{p-1}y) = \gamma_1 x^{p-1}y = \gamma_2 x^{p-1}y = \gamma_2 g(x)^{p-1}g(y)$. Now consider $g(\alpha_1 xy)$ and $\alpha_2 g(x)g(y)$. $g(\alpha_1 xy) = g(a_{11}x_1y_1,\cdots,a_{1i}x_iy_i,0,0,\cdots,0)$ $= (g_1(a_{11}x_1y_1),\cdots,g_i(a_{1i}x_iy_i),0,0,\cdots,0) = (a_{11}^2a_{21}^{-1}x_1y_1,\cdots,$ $a_{1i}^2a_{2i}^{-1}x_iy_i,0,0,\cdots,0)$. $\alpha_2 g(x)g(y) = (a_{21}g_1(x_1)g_2(x_2),\cdots,$ $a_{2i}g_i(x_i)g_i(y_i),0,\cdots,0) = (a_{11}^2a_{21}^{-1}x_1y_1,\cdots,a_{1i}^2a_{2i}^{-1}x_iy_i,0,0,\cdots,0)$. Thus $g(\alpha_1 xy) = \alpha_2 g(x)g(y)$. Hence $g(x *_1 y) = g(\gamma_1 x^{p-1}y + \alpha_1 xy + \beta_1 y)$ $= g(\gamma_1 x^{p-1}y) + g(\alpha_1 xy) + g(\beta_1 y) = \gamma_2 g(x)^{p-1}g(y) + \alpha_2 g(x)g(y) + \beta_2 g(y)$ $= g(x) *_2 g(y)$. Therefore g is an isomorphism. Conversely let g be an isomorphism of $(N;+,*_1)$ onto $(N;+,*_2)$. If $x, y \in N$ then

$g(x *_1 y) = g(x) *_2 g(y)$. Therefore $g(\gamma_1 x^{p-1} y) + g(\alpha_1 xy) + g(\beta_1 y)$ $= \gamma_2 g(x)^{p-1} g(y) + \alpha_2 g(x)g(y) + \beta_2 g(y)$. If $x = 0$ then $g(\beta_1 y) = \beta_2 g(y)$ for all $y \in N$. Assume $j_1 \neq j_2$. Then without loss of generality let $j_1 > j_2$. But $g(\beta_1 y) = \beta_2 g(y)$ means that $g(\beta_1 N) \subset \beta_2 N$. However $O(\beta_1 N) > O(\beta_2 N)$ so it is impossible to map $\beta_1 N$ into $\beta_2 N$ with a $1 - 1$ mapping. This contradiction means that $j_1 = j_2$. Now assume $i_1 \neq i_2$ and again without loss of generality let $i_1 > i_2$. Let $x, y \in N$ and $z \in \alpha_1 N$. Then z is right distributive. Hence $(x + y) *_1 z$ $= x *_1 z + y *_1 z$ so $(g(x) + g(y)) *_2 g(z) = g(x) *_2 g(z) + g(y) *_2 g(z)$. Therefore $g(z) \in \alpha_2 N$ and furthermore $g(\alpha_1 N) \subset \alpha_2 N$. However $O(\alpha_1 N) > O(\alpha_2 N)$ so it is impossible to map $\alpha_1 N$ into $\alpha_2 N$ with a $1 - 1$ mapping. Again we have a contradiction and are forced to conclude that $i_1 = i_2$. Note that for $p = 2$ we could have terminated this proof when $j_1 = j_2$ was established. In that case $i_1 = k - j_1 = k - j_2 = i_2$ because of the condition that $\alpha = 1 + \beta$.

Let $(N;+,*)$ be an (α,β) p near-ring and $O(N) = p^k$. Suppose that α and β have exactly i and j nonzero components respectively. Because α and β are never nonzero in the same component there is at least one permutation f of N such that $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$ where α' and β' have the following properties. The i nonzero components of α' occur in the first i places. The j nonzero components of β' occur in places $i + 1,\cdots,i + j$. This will be used in the following theorem.

<u>Theorem 4.8</u>: Let $(N;+,*_1)$ and $(N;+,*_2)$ be (α,β) p near-rings determined by α_1, β_1 and α_2, β_2 respectively and let $O(N) = p^k$. Let α_1 and α_2 have exactly i_1 and i_2 nonzero components respectively. Let β_1 and β_2

have exactly j_1 and j_2 components respectively. Then $(N;+,*_1)$ is isomorphic to $(N;+,*_2)$ iff $i_1 = i_2$ and $j_1 = j_2$.

Proof: For $r = 1, 2$ let f_r be a permutation of N such that $f_r(\alpha_r) = \alpha_r'$ and $f_r(\beta_r) = \beta_r'$ where α_r' is nonzero in its first i_r places and β_r' is nonzero in places $i_r + 1,\cdots,i_r + j_r$. Thus f_r is an isomorphism of $(N;+,*_r)$ onto $(N;+,*_r')$, the (α,β) p near-ring determined by α_r' and β_r'. Let $i_1 = i_2$ and $j_1 = j_2$. Then, by Lemma 4.7, there exists an isomorphism g of $(N;+,*_1')$ onto $(N;+,*_2')$. Then

$(N;+,*_1) \xrightarrow{f_1} (N;+,*_1') \xrightarrow{g} (N;+,*_2') \xrightarrow{f_2^{-1}} (N;+,*_2)$ so $h = f_2^{-1}gf_1$ is an isomorphism of $(N;+,*_1)$ onto $(N;+,*_2)$. Conversely let h be an isomorphism of $(N;+,*_1)$ onto $(N;+,*_2)$. Then

$(N;+,*_1') \xrightarrow{f_1^{-1}} (N;+,*_1) \xrightarrow{h} (N;+,*_2) \xrightarrow{f_2} (N;+,*_2')$ so $g = f_2hf_1^{-1}$ is an isomorphism of $(N;+,*_1')$ onto $(N;+,*_2')$. Hence by Lemma 4.7 we have that $i_1 = i_2$ and $j_1 = j_2$.

It is of interest to know how many distinct classes of isomorphic (α,β) p near-rings are associated with a fixed p ring with identity $(N;+,\cdot;1)$ and how many (α,β) p near-rings belong to each class. To that end we prove the following theorem.

Theorem 4.9: Let $(N;+,\cdot;1)$ be a p ring with identity and let $O(N) = p^k$.

(a) Let $\alpha', \beta' \in N$ determine an (α,β) p near-ring. If α' has exactly i nonzero components and β' has exactly j nonzero components then there are $\binom{k}{i\ \ j\ \ k-i-j}(p-1)^i$ elements in the equivalence class of (α,β) p near-rings isomorphic to the one determined by this α' and β'.

(b) There are $(k + 1)(k + 2)/2$ distinct equivalence classes of (α,β) p near-rings associated with the given p ring.

Proof: The nonzero components of β' are 1's so there are $\binom{k}{j}$ choices for β'. For α' each nonzero component could be any of $p - 1$ elements so there are $\binom{k-j}{i}(p - 1)^i$ choices for α'. There are then $\binom{k}{j}\binom{k-j}{i}(p - 1)^i$ choices for α,β that result in an (α,β) p near-ring isomorphic to the one determined by α', β'. But $\binom{k}{j}\binom{k-j}{i}(p - 1)^i = \binom{k}{i\ \ j\ \ k-i-j}(p - 1)^i$. The number of distinct equivalence classes may be counted by considering $\binom{k}{i\ \ j\ \ k-i-j}$ for all possible i, j. If $0 \leq r \leq k$ and $i = r$ then j could be $0,1,2,\cdots,k - r$. Thus the total number of classes can be found by letting r range from 0 to k and adding the choices for j. Hence the number of equivalence classes is $(k + 1) + k + (k - 1) + \cdots + 2 + 1 = \sum_{s=1}^{k+1} s = (k + 1)(k + 2)/2$.

Theorem 4.10: Let $(N;+,*)$ be an (α,β) p near-ring. Denote $(N;+,*)$ by N. N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i where each N_i is one of the following types:

(a) N_i is $(Z_p;+,\cdot)$, the integers modulo p,

(b) N_i is $(Z_p;+,')$, where $x\ '\ y = y$ for all $x, y \in Z_p$,

(c) N_i is $(Z_p;+,'')$, where $0\ ''\ y = 0$ but $x\ ''\ y = y$ otherwise.

Furthermore if $O(N) = p^k$, α has exactly i nonzero components and β has exactly j nonzero components then N is isomorphic to a direct sum of exactly i near-rings of type (a), j near-rings of type (b) and $k - i - j$ near-rings of type (c).

Proof: Elements in $(N;+,\cdot;1)$ and $(N;+,*)$ have the same representation. That is if $x \in N$ then $x = (x_i)_{i \in I}$ and $x_i \in N_i$ where $N_i = Z_p$. Hence N is isomorphic to a subdirect sum of near-rings N_i for $i \in I$ and each N_i is a near-ring $(Z_p;+,\cdot\cdot)$ where $\cdot\cdot$ is some multiplication determined by $*$. Clearly each of these is subdirectly irreducible. Let $\alpha = (a_i)_{i \in I}$,

$\beta = (b_i)_{i \in I}$ and $\gamma = (c_i)_{i \in I}$. As noted earlier for each $i \in I$ exactly one of a_i, b_i or c_i is nonzero. Let $a_r \neq 0$ then $b_r = c_r = 0$. Now consider $x * y$ for $x, y \in N$. Then $x * y = ((x * y)_i)_{i \in I}$ and $(x * y)_r = a_r x_r y_r$. Hence N_r is $(Z_p;+,\cdot)$. Let $b_s \neq 0$ then $b_s = 1$ and $a_s = c_s = 0$. For $x, y \in N$ we again consider $x * y$. $x * y = ((x * y)_i)_{i \in I}$ and $(x * y)_s = y_s$. Hence N_s is $(Z_p;+,')$ as described in (b). Let $c_t \neq 0$ then $c_t = 1$ and $a_t = b_t = 0$. For $x, y \in N$ we have $x * y = ((x * y)_i)_{i \in I}$ and $(x * y)_t = x_t^{p-1} y_t$. Thus if $x_t = 0$ then $x_t^{p-1} y_t = 0$ but if $x_t \neq 0$ then $x_t^{p-1} y_t = y_t$. Hence N_t is $(Z_p;+,'')$ as described in (c). The remainder of the proof is routine.

A near-ring $(N;+,\cdot)$ is <u>small</u> iff for each $x \in N$ either $xy = y$ for all $y \in N$ or $xy = 0y$ for all $y \in N$.

<u>Corollary 4.11</u>: Let $(N;+,*)$ be an (α,β) p near-ring and denote it by N. Then N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i where N_i is one of the following types:

(a) N_i is Z_p,

(b) N_i is small.

<u>Proof</u>: This is immediate from Theorem 4.10.

<u>Theorem 4.12</u>: Let $(N;+,*)$ be an (α,β) p near-ring. Then $(N;+,*)$ is d.g. iff $(N;+,*)$ is a p ring.

<u>Proof</u>: Let $(N;+,*)$ be a p ring then it is distributive and hence d.g. Conversely let $(N;+,*)$ be d.g. Thus there exists a subset S of N whose elements are right distributive and additively generate N. However, by Lemma 4.6, $S \subset \alpha N$. By Theorem 4.5, αN is an ideal of $(N;+,\cdot;1)$ so

$N \subset \alpha N$. But it is known that $\alpha N \subset N$ so therefore $N = \alpha N$. Then for some $x \in N$, $1 = \alpha x$. Then $\alpha^{p-1} = \alpha^{p-1}1 = \alpha^{p-1}\alpha x = \alpha^p x = \alpha x = 1$. Thus $(N;+,*)$ is a p ring by Theorem 1.1.

CHAPTER IV

OTHER RESULTS

1. Introduction

Let $(N;+,*)$ be an (α,β) p near-ring and $x, y, z \in N$. $x * y * z = \gamma x^{p-1}y^{p-1}z + \alpha^2 xyz + \beta z = \gamma y^{p-1}x^{p-1}z + \alpha^2 yxz + \beta z = y * x * z$. The purpose of this chapter is to study near-rings $(N;+,\cdot)$ with two properties. The first is $xyz = yxz$ for all $x, y, z \in N$. The second is that for each $x \in N$ there exists a positive integer $n(x) > 1$ such that $x^{n(x)} = x$.

2. Weakly Commutative and ν Near-Rings

In the earlier chapters it was important to identify a near-ring by symbols like $(N;+,*)$ because of the presence of a p ring with identity $(N;+,\cdot;1)$ where $\cdot$ and $*$ were, in general, different multiplications. This will not be a problem in this chapter so a near-ring $(N,+,\cdot)$ will be denoted by N. Let N be a near-ring and $x \in N$. Then define $A_x = \{a \in N : xa = 0\}$. For left near-rings the definition of a right ideal is not standard so this is the definition that will be used. Let N be a near-ring and $I \subset N$. Then I is a <u>right ideal</u> of N iff $(I;+)$ is a normal subgroup of $(N;+)$ and $(x + a)y - xy \in I$ for all $x, y \in N$ and for all $a \in I$.

Lemma 2.1: If N is a near-ring and $x \in N$ then A_x is a right ideal of N.

Proof: Let r, $s \in N$ and a, $b \in A_x$. $x(a - b) = xa - xb = 0 - 0 = 0$ so $a - b \in A_x$. $x(-r + a + r) = -xr + xa + xr = -xr + 0 + xr = -xr + xr = 0$ and hence $-r + a + r \in A_x$. Thus $(A_x;+)$ is a normal subgroup of $(N;+)$. $x((r + a)s - rs) = x(r + a)s - xrs = (xr + xa)s - xrs = (xr + 0)s - xrs = xrs - xrs = 0$. Therefore $(r + a)s - rs \in A_x$ so we conclude that A_x is a right ideal of N.

Lemma 2.2: Let N be a near-ring and $e \in N$. If there exists a positive integer $k > 1$ such that $e^k = e$ and $A_e = 0$ then e^{k-1} is a left identity.

Proof: Let $x \in N$. Then $e(e^{k-1}x - x) = ee^{k-1}x - ex = e^kx - ex = ex - ex = 0$. Thus $e^{k-1}x - x = 0$ for all $x \in N$ or $e^{k-1}x = x$ for all $x \in N$.

A near-ring N is weakly commutative iff $xyz = yxz$ for all x, y, $z \in N$. The next result is due to Szeto [12].

Theorem 2.3: If N is a weakly commutative near-ring and $x \in N$ then A_x is an ideal of N.

Let N be a near-ring such that for every $x \in N$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. By convention $n(x)$ will mean the smallest integer greater than 1 such that $x^{n(x)} = x$. By this convention $n(0) = 2$ since $0^2 = 0$. A near-ring N is a ν near-ring iff for every $x \in N$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ and $xyz = yxz$ for all x, y, $z \in N$. Clearly every (α,β) p near-ring is a ν near-ring. Similarly every weakly commutative p near-ring is a ν near-ring.

Theorem 2.4: Let N be a weakly commutative near-ring. Then for every $x, y \in N$ and for every positive integer k, $(xy)^k = x^k y^k$.

Proof: The proof is easy by induction. Let $x, y \in N$. It is certainly true when $k = 1$. Next note that $(xy)^2 = (xy)(xy) = x(yxy) = x(xy^2) = x^2y^2$. Now assume that $(xy)^n = x^n y^n$ for some positive integer n. Then $(xy)^{n+1} = (xy)^n(xy) = (x^n y^n)(xy) = x^n(y^n xy) = x^n(xy^{n+1}) = x^{n+1}y^{n+1}$. Therefore $(xy)^k = x^k y^k$ for every positive integer k and for all $x, y \in N$.

Theorem 2.5: Let N be a near-ring such that for every $x \in N$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. If N has a right identity e then e is an identity.

Proof: Let $x \in N$ such that $ex = 0$. Then $x = x^{n(x)} = (xe)^{n(x)} = (xe)(xe)\ldots(xe)(xe) = x(ex)\ldots(ex)e = x0e = x0 = 0$. Thus $A_e = 0$. $e^2 = e$ so by Lemma 2.2 e is a left identity. Therefore e is an identity.

Theorem 2.6: If N is a small near-ring then N is weakly commutative.

Proof: Let $x, y, z \in N$. There are four possible cases.

(1) x, y are both left identities. Then $xyz = yz = z$ and $yxz = xz = z$.

(2) x is a left identity and $yw = 0w$ for all $w \in N$. Then $xyz = x(yz) = yz$ and $yxz = y(xz) = yz$.

(3) $xw = 0w$ for all $w \in N$ and y is a left identity. Then $xyz = x(yz) = xz$ and $yxz = y(xz) = xz$.

(4) $xw = ow$ and $yw = 0w$ for all $w \in N$. Then $xyz = x(yz) = x(0z) = (x0)z = 0z$ and $yxz = y(xz) = y(0z) = (y0)z = 0z$.

Therefore $xyz = yxz$ for all $x, y, z \in N$ so N is weakly commutative.

Theorem 2.7: Let N be a subdirectly irreducible near-ring such that for every $x \in N$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. If A_x is an ideal for every $x \in N$ then N has a left identity.

Proof: If $N = 0$ then the result follows. Let $N \neq 0$ and define $R = \{x \in N: A_x \neq 0\}$. Also define $A = \bigcap\{A_x: x \in R\}$. If R is empty then $A = N$. If R is not empty then because N is subdirectly irreducible it follows that $A \neq 0$. Let $x \in A$ and $x \neq 0$. Assume that $R = N$. Then $x \in A_y$ for all $y \in N$. Hence $x \in A_{x^{n(x)-1}}$ so $x^{n(x)-1}x = 0$. But $x = x^{n(x)-1}x$ so $x = 0$. This is a contradiction so $R \neq N$. Hence there exists an $e \in N$ such that $A_e = 0$. By Lemma 2.2 then N has a left identity, namely $e^{n(e)-1}$.

Corollary 2.8: If N is a subdirectly irreducible ν near-ring then N has a left identity.

Proof: By Theorem 2.3 A_x is an ideal for every $x \in N$ so the hypotheses of Theorem 2.7 are satisfied. Thus the conclusion follows and N has a left identity.

Theorem 2.9: Let N be a subdirectly irreducible ν near-ring. If $a \in N$, $a \neq 0$ and $A_a \neq 0$ then $ay = 0y$ for all $y \in N$ and $A_a = A_0$.

Proof: Let $R = \{x \in N: A_x \neq 0\}$ and $A = \bigcap\{A_x: x \in R\}$. Note that $a \in R$. Since N is subdirectly irreducible $A \neq 0$. Let $w \in A$ and $w \neq 0$. Then $xw = 0$ for all $x \in R$ and in particular $aw = 0$. Assume $A_w \neq 0$. Then $w \in A \subset A_w$ so $w \in A_w$. Then $w^{n(w)-1} \in A_w$ since A_w is an ideal. Thus $w = ww^{n(w)-1} = 0$ which is a contradiction. Hence $A_w = 0$ and by Lemma 2.2, $w^{n(w)-1}$ is a left identity. If $n(w) = 2$ then $aw^{n(w)-1} = aw = 0$. If $n(w) > 2$ then $aw^{n(w)-1} = aw^{n(w)-2}w = w^{n(w)-2}aw = w^{n(w)-2}0 = 0$. Let $y \in N$. Then $ay = a(w^{n(w)-1}y) = (aw^{n(w)-1})y = 0y$. Finally $ay = 0$ iff

$0y = 0$ so $A_a = A_o$.

Corollary 2.10: Let N be a subdirectly irreducible ν near-ring such that $0N = 0$.

(a) For every nonzero $x \in N$, $A_x = 0$ so $x^{n(x)-1}$ is a left identity.

(b) N has no zero divisors.

Proof: If $N = 0$ then the conclusions follow so now let $N \neq 0$. Let $x \in N$ and $x \neq 0$. Assume $A_x \neq 0$ then by Theorem 2.9, $xy = 0y$ for all $y \in N$. But $0N = 0$ so $0y = 0$ for all $y \in N$. Then $x = xx^{n(x)-1} = 0x^{n(x)-1} = 0$ which is a contradiction. Hence $A_x = 0$ and by Lemma 2.2, $x^{n(x)-1}$ is a left identity. Let $a, b \in N$ such that $ab = 0$. Then $a = 0$ or $a \neq 0$. If $a \neq 0$ then by the preceding $a^{n(a)-1}$ is a left identity. Thus $0 = a^{n(a)-2}0 = a^{n(a)-2}(ab) = a^{n(a)-1}b = b$. Hence N has no zero divisors.

The following theorem is due to Fröhlich [6].

Theorem 2.11: Let N be a d.g. near-ring with identity. Then each of the following conditions is necessary and sufficient for N to be a ring.

(a) N is distributive.

(b) $(N;+)$ is commutative.

Theorem 2.12: Let N be a subdirectly irreducible ν near-ring such that $0N = 0$ and let e be a nonzero element of N such that for every nonzero $x \in N$, $x^{n(x)-1} = e$. Then N is a field.

Proof: Let $x, y \in N$ and $x, y \neq 0$. Then $xy = xy^{n(y)} = xyy^{n(y)-1} = xye = yxe = yxx^{n(x)-1} = yx$. Thus $(N;\cdot)$ is commutative so N is distributive and hence d.g. By Corollary 2.10, N has a left identity e which by commutativity is a right identity. Thus N is a d.g. near-ring with identity which is distributive. Therefore by Theorem 2.11 N is a ring. Let $x \in N$ and $x \neq 0$. If $n(x) = 2$ then $x = e$ so $x^{-1} = e$. If $n(x) > 2$ then $x^{-1} = x^{n(x)-2}$. Thus N is a field.

<u>Corollary 2.13</u>: Let N be a subdirectly irreducible weakly commutative p near-ring such that $0N = 0$. If there exists a nonzero $e \in N$ such that for every nonzero $x \in N$, $x^{p-1} = e$ then N is Z_p.

<u>Proof</u>: It follows from Theorem 2.12 that N is a field. Therefore N is a subdirectly irreducible p ring with identity. The only subdirectly irreducible p ring with identity is Z_p so N is Z_p.

The following theorem is due to Fain [5].

<u>Theorem 2.14</u>: Every near-ring N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i.

Before proceeding further consider the following definition. A near-ring N is <u>almost small</u> iff $\{A_x : x \in N\}$ contains at most two distinct sets. Clearly every small near-ring is almost small. However, there are almost small near-rings that are not small. Examples in the cyclic 4 group as listed in Clay [3] are (3), (7) and (12). Furthermore there are ν near-rings that are not almost small. Examples as listed in Clay [3] are (7) in the Klein 4 group and (27) in the cyclic 6 group. However these are both rings. An example that is not a ring is (53) of the cyclic 6 group. Hence there is some merit to the following theorem.

<u>Theorem 2.15</u>: Every ν near-ring N is isomorphic to a subdirect sum of subdirectly irreducible ν near-rings N_i where each N_i is one of the following types:

(a) N_i is a field,

(b) N_i is almost small.

Proof: By Theorem 2.14 N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i. Each N_i is the homomorphic image of a ν near-ring so each is a ν near-ring.

(1) $ON_i = 0$ and there exists a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x^{n(x)-1} = e$. It follows then, by Theorem 2.12, that N_i is a field.

(2) $ON_i = 0$ and there does not exist a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x^{n(x)-1} = e$. By Corollary 2.10 for every nonzero $x \in N_i$, $A_x = 0$ and furthermore N_i has no zero divisors. Then $A_o = N_i$ and $A_x = 0$ otherwise. Thus N_i is almost small.

(3) $ON_i \neq 0$. Let $x \in N_i$. Then $A_x = 0$ or $A_x \neq 0$. If $A_x \neq 0$ then by Theorem 2.9 $A_x = A_o$. Thus $A_x = 0$ or $A_x = A_o$ so N_i is almost small.

Recall that a ν near-ring N is a β near-ring iff for every $x \in N$, $x^2 = x$ (or $n(x) = 2$). The following result due to Ligh [8] may be obtained now as a corollary.

Corollary 2.16: Every β near-ring N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i where each N_i is one of the following types:

(a) N_i is Z_2,

(b) N_i is small.

Proof: By Theorem 2.14 N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i. Each N_i is the homomorphic image of a β near-ring so each N_i is a β near-ring.

(1) $ON_i = 0$ and there exists a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x = e$. Thus N_i contains only 0 and e. By Theorem 2.12 N_i is a field. Thus N_i is Z_2.

(2) $ON_i = 0$ and there does not exist a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x = e$. Either $N_i = 0$ or $N_i \neq 0$. If $N_i = 0$ it is small. If $N_i \neq 0$ then, by Corollary 2.10, for every nonzero $x \in N_i$, x is a left identity. Thus N_i is small.

(3) $ON_i \neq 0$. Let $x \in N_i$. Then $A_x = 0$ or $A_x \neq 0$. If $A_x = 0$ then, by Lemma 2.2, x is a left identity. If $A_x \neq 0$ then, by Theorem 2.9, $xy = 0y$ for all $y \in N_i$. Thus N_i is small. Hence the conclusion follows.

<u>Corollary 2.17</u>: Every weakly commutative p near-ring N is isomorphic to a subdirect sum of subdirectly irreducible p near-rings N_i where each N_i is one of the following:

(a) N_i is Z_p,

(b) N_i is almost small.

<u>Proof</u>: By Theorem 2.14 N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i. Each N_i is a p near-ring.

(1) $ON_i = 0$ and there exists a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x^{p-1} = e$. Then by Corollary 2.13 N_i is Z_p.

(2) $ON_i = 0$ and there does not exist a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x^{p-1} = e$. By Theorem 2.15 N_i is almost small.

(3) $ON_i \neq 0$. Again by the proof of Theorem 2.15 N_i is almost small.

<u>Theorem 2.18</u>: Let N be a subdirectly irreducible ν near-ring with a nonzero right distributive element $r \in N$. Then N is a field.

Proof: If $A_r \neq 0$ then $ry = 0y$ for all $y \in N$ by Theorem 2.9. Then $r = r^{n(r)} = rr^{n(r)-1} = 0r^{n(r)-1} = 0$ which is a contradiction. Thus $A_r = 0$ so $r^{n(r)-1}$ is a left identity. Now define $L_r = \{a \in N: ar = 0\}$. It is routine to show that L_r is an ideal. Define $R = \{x \in N: A_x \neq 0\}$ and $A = \bigcap\{A_x: x \in R\}$. Since N is subdirectly irreducible $A \neq 0$. Assume $A \cap L_r \neq 0$. Then let $w \in A \cap L_r$ and $w \neq 0$. If $w \in A \cap L_r$ then $w^{n(w)-1} \in A \cap L_r$ because $A \cap L_r$ is an ideal. Either $A_w = 0$ or $A_w \neq 0$. Let $A_w \neq 0$. Then $A \subset A_w$ so $w^{n(w)-1} \in A_w$. Thus $w = ww^{n(w)-1} = 0$ which is a contradiction. If $A_w = 0$ then $w^{n(w)-1}$ is a left identity. Then $r = w^{n(w)-1}r = 0$ because $w^{n(w)-1} \in L_r$. This too is a contradiction so $A \cap L_r = 0$. Therefore $L_r = 0$ so $yr = 0$ iff $y = 0$. Let $x \in N$ then $(xr^{n(r)-1} - x)r = xr^{n(r)} - xr = xr - xr = 0$. Then $xr^{n(r)-1} = x$ for all $x \in N$. Thus $r^{n(r)-1}$ is a right identity. It is known to be a left identity so $r^{n(r)-1}$ is the identity for N. Let x, y $\in$ N. Then $xy = xyr^{n(r)-1} = yxr^{n(r)-1} = yx$ so $(N;\cdot)$ is commutative. Thus N is distributive and hence d.g. By Theorem 2.11 then N is a ring. Thus N is a commutative ring with identity. By Corollary 2.10 for every nonzero $x \in N$, $x^{n(x)-1} = r^{n(r)-1}$. Thus by Theorem 2.12 N is a field.

Corollary 2.19: Let N be a subdirectly irreducible ν near-ring with right identity $e \neq 0$. Then N is a field.

Proof: N has a nonzero right distributive element, namely e. Therefore the conclusion follows by Theorem 2.18.

Theorem 2.20: Let N be a subdirectly irreducible weakly commutative p near-ring with a nonzero right distributive element. Then N is Z_p.

Proof: By Theorem 2.18 N is a field. Thus N is a subdirectly irreducible p ring with identity. Hence N is Z_p.

Corollary 2.21: Let N be a subdirectly irreducible weakly commutative p near-ring with right identity $e \neq 0$. Then N is Z_p.

Proof: N has a nonzero right distributive element, e. Thus the hypotheses of Theorem 2.20 are satisfied so the conclusion must follow. Hence N is Z_p.

Theorem 2.22: Let N be a ν near-ring. N is a commutative ring iff every nonzero homomorphic image of N contains a nonzero right distributive element.

Proof: If $N = 0$ then the conclusion follows. Let $N \neq 0$. If N is a commutative ring then every nonzero homomorphic image of N is commutative. Thus it contains a nonzero right distributive element. Conversely let every nonzero homomorphic image of N contain a nonzero right distributive element. By Theorem 2.14, N is isomorphic to a subdirect sum of subdirectly irreducible ν near-rings N_i. By hypothesis each N_i contains a nonzero right distributive element. By Theorem 2.18, each N_i is a field. The direct sum of the N_i is a commutative ring with identity. However N is a subdirect sum of the N_i. Therefore N is a commutative ring.

Corollary 2.23: Let N be a weakly commutative p near-ring. Then N is a p ring iff every nonzero homomorphic image of N contains a nonzero right distributive element.

Proof: If $N = 0$ then the conclusion follows. Now let $N \neq 0$. Let N be a p ring. Then N is a commutative ring so, by Theorem 2.22, every nonzero homomorphic image of N contains a nonzero right distributive element. Conversely let every nonzero homomorphic image of N contain a nonzero right distributive element. Then, by Theorem 2.22, N is a commutative ring. It is known that $px = 0$ and $x^p = x$ for all $x \in N$. Therefore N is a p ring.

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