INNER-PRODUCT OPTIMIZATION OF SYSTEMS LINEAR IN CONTROL

By

SRIDHAR MITTA

Bachelor of Engineering
Andhra University
Waltair, India
1967

Master of Technology
Indian Institute of Technology
Kharagpur, India
1969

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1973

INNER-PRODUCT OPTIMIZATION OF

SYSTEMS LINEAR IN CONTROL

Thesis Approved:

R. P. Rhoten
Thesis Adviser

Charles M. Bacon

Henry R. Sebesta

Paul A. McCollum

D. D. Durham
Dean of the Graduate College

ACKNOWLEDGMENTS

I would like to express sincere gratitude to my thesis adviser Dr. Ronald P. Rhoten for his viewpoints and guidance gleaned from many enlightening discussions on this dissertation. My association with him has been extremely valuable, for not only is he an outstanding teacher, but his approach to technical problems is one that I particularly admire.

I wish to thank the members of my doctoral committee, Dr. Charles M. Bacon, Dr. Henry R. Sebesta and Professor Paul A. McCollum for their pertinent questions and fruitful suggestions. Grateful thanks are also due to Dr. R. J. Mulholland for his immense help and encouragement during my graduate studies. I should acknowledge the profitable discussions with Dr. J. P. Chandler and Dr. C. S. Sims as well as the personal counsel of Dr. R. Ramakumar.

I am thankful to my friends Mr. A. Y. Lee, Mr. R. A. Bailey and Mr. J. D. Ledbetter for their helpful comments.

The financial assistance from the Center for Systems Science of Oklahoma State University and the Watumull Foundation of Hawaii are thankfully acknowledged.

In addition, I thank Mrs. Jaquita Icenhower for her typing excellence.

Finally, I would like to express my deep appreciation to all my brothers whose understanding, encouragement and sacrifice were indeed instrumental for my successful academic career.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
	1.1 Optimal Control Problem Formulation	1
	1.2 Techniques for Solution	2
	1.3 Suboptimal Control	7
	1.4 Systems Linear in Control	8
	1.5 Research Objectives	9
	1.6 Organization	10
II.	INNER-PRODUCT PROBLEM FORMULATION	12
	2.1 General Cost Functional	12
	2.2 Linear Self-Adjoint Systems	14
	2.3 Nonlinear Norm-Invariant Systems	17
III.	OPTIMAL INNER-PRODUCT CONTROL OF SYSTEMS LINEAR IN CONTROL	20
	3.1 Introduction	20
	3.2 Fundamental Control Equality	20
	3.3 Evaluation of the Constant of Integration	30
	3.4 Systems Linear in Control	31
	3.5 Control of an Electrodrive Circuit	34
IV.	THE CONTINUOUS STIRRED TANK REACTOR	41
	4.1 Introduction	41
	4.2 The CSTR Dynamical Model	41
	4.3 Steady State Values	43
	4.4 Control of Steady State	44
	4.5 Simulation Results	47
V.	RELATED TOPICS	61
	5.1 Performance Measure Interpretation	61
	5.2 Inverse Problem	65
	5.3 Controllability	65
	5.4 The Epsilon Method	68

Chapter Page

VI. SUMMARY AND CONCLUSIONS 71

6.1 Summary 71
6.2 Conclusions 72

BIBLIOGRAPHY 74

APPENDIX - FINITE TERMINAL-TIME PROBLEM 77

LIST OF TABLES

Table | Page

I. Summary of Pontryagin's Minimum Principle for Solving the Optimal Control Problem 3

II. Summary of Bellman's Dynamic Programming for Solving the Optimal Control Problem 5

III. Normalized Constants 46

IV. Physical Constants 48

V. Program to Evaluate the Unknown Constant 81

LIST OF FIGURES

Figure | Page

1. Inner-Product Controller Structure 16
2. Nonlinear Transducer Characteristic 28
3. Nonlinear Transducer Surface 29
4. State Trajectories for Electrodrive Circuit: Without Switching 37
5. State Trajectories for Electrodrive Circuit: Switching 39
6. Concentration Versus Time 50
7. Temperature Versus Time 51
8. Control Trajectories for the CSTR. 53
9. Control-Cost Trajectories for the CSTR 56
10. State Trajectories for the CSTR: Switching 58
11. State Trajectories for the CSTR: Different Performance Indicies 59
12. Block Diagram of the Computational Algorithm 80

CHAPTER I

INTRODUCTION

1.1 Optimal Control Problem Formulation

Over the past two decades, since control systems design was first formulated as an optimization problem, constant intensive research has produced analytical and numerical design procedures which give great insight into the nature of efficient control systems. Before Wiener (1), control systems design was largely an art, rather than an engineering science. Since then, the field of optimal control theory has attempted to evolve a radically different approach to the design problem. The engineer must formulate the problem accurately, develop mathematical models for the system to be controlled and know the nature of the required measurements. All the various control objectives must be combined into one analytical expression of the cost of operating the system. It is the goal of modern control theory to assist the engineer at this point by producing explicitly the control systems design which will minimize the cost (2). A general procedure for nonlinear systems and arbitrary cost functions perhaps cannot be found, but for certain special problems, excellent design procedures are available.

The optimal control problem can be formulated as follows: Given the dynamical model

$$\dot{x}(t) = f(x(t),u(t),t) \tag{1.1}$$

where x(t) is the state n-vector and u(t) the control m-vector constrained to the set U, it is desired to find the optimal control which minimizes, over the set of admissible controls U, the scalar cost functional or performance measure

$$J = \int_{t_0}^{t_f} g(x(t),u(t),t)\, dt \tag{1.2}$$

subject to initial (and possibly final) conditions. Here t represents the independent variable time and t_0 and t_f the initial and final times respectively. The dot indicates the time derivative. Also, the vector function f(·) and scalar function g(·) are assumed to be continuous in all the variables.

1.2 Techniques for Solution

The two main theoretical approaches to the optimal control problem are Pontryagin's minimum principle and Bellman's dynamic programming. Basically, the minimum principle provides a set of local necessary conditions for optimality in the form of a nonlinear two-point boundary value problem. The entire procedure for solving the optimal control problem by Pontryagin's minimum principle is summarized in Table I (3).

Although the formulation of the solution procedure is quite easy, the actual computational problems are very difficult because of the resultant two-point boundary value problem (4). Also, the minimum principle gives necessary but not sufficient conditions. Finally, the optimal control will be obtained as a function of time (open-loop) rather than as a function of states (closed-loop).

TABLE I

SUMMARY OF PONTRYAGIN'S MINIMUM PRINCIPLE FOR SOLVING THE OPTIMAL CONTROL PROBLEM

Step 1	Form the Hamiltonian H, $H(x,u,\lambda,t) = g(x,u,t) + \lambda^T f(x,u,t)$.
Step 2	Minimize $H(x,u,\lambda,t)$ with respect to all admissible control vectors to find $u^*(x,\lambda,t)$ and obtain the optimal H, $H^*(x,\lambda,t) = H(x,u^*(x,\lambda,t),\lambda,t) = \min_{u \in U} H(x,u,\lambda,t)$.
Step 3	Solve the set of differential equations $\frac{dx}{dt} = \frac{\partial H^*(x,\lambda,t)}{\partial \lambda}$ $\frac{d\lambda}{dt} = -\frac{\partial H^*(x,\lambda,t)}{\partial x}$ with the given initial conditions and terminal boundary conditions and the generalized boundary condition $\left(H^*(x,\lambda,t)\, dt - \lambda^T dx \right) \Big|_{t_f} = 0$.
Step 4	Substitute the results of Step 3 into the expression for $u^*(x,\lambda,t)$ to obtain the optimal control $u^*(t)$.

An alternative approach, which answers many objections to Pontryagin's method, is Bellman's dynamic programming. This method is based on the principle of optimality (6), which states:

> An optimal policy has the property that, whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The complete procedure for the use of this approach is summarized in Table II (3).

Dynamic programming yields optimal control laws in closed-loop form in contrast to Pontryagin's minimum principle. Another advantage of dynamic programming is that it can be used to obtain both numerical and analytical results (6). The main difficulty with this approach is the necessity of solving the nonlinear partial differential equation. In fact, the solution of this equation is so difficult that it has been accomplished only for a few special cases.

There is one class of problems for which it is possible to solve the Hamilton-Jacobi-Bellman equation or the two-point boundary value problem in a reasonably simple manner. This problem is often termed the linear regulator problem. The pioneering work in this area was done by Kalman (7).

The linear regulator problem considers linear plants whose performance measure can be expressed by quadratic functions of state and control. The basic result for this problem is as follows (8): Given a linear and completely controllable system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.3}$$

and the cost functional

TABLE II

SUMMARY OF BELLMAN'S DYNAMIC PROGRAMMING FOR SOLVING THE OPTIMAL CONTROL PROBLEM

Step 1	Form the Hamiltonian H, with λ replaced by ∇V, $H(x,u,\nabla V,t) = g(x,u,t) + \nabla V^T f(x,u,t)$.
Step 2	Minimize $H(x,u,\nabla V,t)$ with respect to all admissible control vectors to find $u^*(x,\nabla V,t)$ and obtain the optimal H, $H^*(x,\nabla V,t) = H(x,u^*(x,\nabla V,t),\nabla V,t) = \min_{u \in U} H(x,u,\nabla V,t)$.
Step 3	Solve the partial differential equation $H^*(x,\nabla V,t) + \frac{\partial V}{\partial t} = 0$ with the appropriate boundary condition to obtain $V(x,t)$. This equation is called the Hamilton-Jacobi-Bellman equation for the control problem.
Step 4	Substitute the results of Step 3 into the expression for $u^*(x,\nabla V,t)$ to obtain the optimal control law $u^*(x,t)$.

$$J = x^T(t_f)\Lambda x(t_f) + \int_{t_0}^{t_f} \{ x^T(t)\hat{Q}(t)x(t) + u^T(t)R(t)u(t) \} dt, \quad (1.4)$$

where u(t) is unconstrained, $\hat{Q}(t)$ and Λ are positive semi-definite matrices and R(t) is a positive definite matrix, then the optimal control law is given by

$$\begin{aligned} u^*(t) &= K(t)x(t) \\ &= -R^{-1}(t)B^T(t)P(t)x(t) , \end{aligned} \quad (1.5)$$

where P(t) is a symmetric and positive definite matrix which is the solution of the matrix Riccati differential equation

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)\, B(t)R^{-1}(t)B^T(t)P(t) - \hat{Q}(t) \quad (1.6)$$

subject to the boundary condition

$$P(t_f) = \Lambda. \quad (1.7)$$

The optimal control law is thus linear with time-varying state feedback.

If the matrices A(t), B(t), R(t) and $\hat{Q}(t)$ are all constant matrices, $\Lambda = [0]$ and $t_f = \infty$, the optimal control is given by the equation

$$u(t) = -R^{-1} B^T \bar{P} x(t), \quad (1.8)$$

where $\bar{P}$ is a constant positive definite matrix which is the solution of the nonlinear matrix algebraic Riccati equation

$$-\bar{P} A - A^T \bar{P} + \bar{P} B R^{-1} B^T \bar{P} - \hat{Q} = [0] . \quad (1.9)$$

The optimal control law is thus seen to be linear with constant feedback.

The selection of weighting matrices in the quadratic cost functional is not a simple matter. Usually they are selected by the designer on the basis of engineering experience coupled with simulation runs for different trial values. In general, it is not possible to solve analytically for the gain matrix K(t). Rather, its determination requires the solution of the Riccati equation by one of several numerical techniques. The computational problems arising in the solution by different methods are described by Fath (9).

Two solution procedures for a rather general optimal control problem have been indicated. These approaches rely upon the solution of either a two-point boundary value problem or a nonlinear partial differential equation, which are amenable only to extensive machine computation. Even a linear plant with quadratic performance measure requires a considerable amount of precomputation and storage of gain matrix values. Hence, there is a need for a different approach to the optimal control problem which applies to nonlinear systems and a general cost functional.

1.3 Suboptimal Control

In spite of the mathematical simplicity of the formulation of the solution by Pontryagin's and Bellman's techniques, there are certain shortcomings associated with the implementation of the solution to the optimal control problem. The application of these procedures to general systems represents a computationally difficult and cumbersome task. In practice, the determination of optimal controls is limited to problems in which the state space is not too large, because the rapid access memory requirements of the computer grow exponentially with the number of state variables. Another serious limitation may be the unavailability of all the states for measurement and feedback.

The inaccessible states can be estimated by using a Luenberger observer (10), but this increases the cost and complexity of the system. Hence, there is a need for designing approximately optimal or suboptimal controls.

Different suboptimal control algorithms have been evolved as a compromise between computational effort and a desire to incorporate realistic implementation. One approach is to solve a restricted problem in which the form of the controller that will be allowed is postulated. The problem then remaining is to choose the values of the controller parameters to achieve optimality within these constraints. This is the so-called specific optimal control problem (11).

In all suboptimal controller synthesis techniques, knowledge of the exact solution to the optimal control problem would be of great benefit. A comparison could be made between the optimal value of the system cost and the value of the cost obtained by using the suboptimal controller. In this manner a judgment could be made as to the acceptability of the suboptimal controller designed. Unfortunately, it is rarely possible to evaluate the exact solution.

The new approach to the optimal control problem to be presented in subsequent chapters has the advantage that the value of the optimal cost will be specified once the form of the performance measure is chosen. This optimal performance value can be calculated even when the plant is nonlinear or time-varying.

1.4 Systems Linear in Control

This dissertation is concerned with the development of a new optimal control synthesis procedure and its application to a particular class of nonlinear systems, those which are linear in control. Mohler (12) has studied the application of such mathematical models to processes in socio-economics, ecology and physiology. Systems linear in control can be described by the mathematical model

$$\dot{x}(t) = a(x(t)) + B(x(t))u(t) \ , \tag{1,10}$$

where $a(\cdot)$ is an n-dimensional vector function of $x(t)$ and $B(\cdot)$ is an $n \times m$ matrix function of $x(t)$. If $a(\cdot)$ and $B(\cdot)$ are linear functions of $x(t)$, the systems are called bilinear. A stationary bilinear system can be represented by

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k u_k(t)x(t) + Cu(t) \ , \tag{1,11}$$

where A, B_k $(k=1,2,\ldots,m)$ and C are appropriate constant matrices and $u_k(t)$ is the k^{th} component of the control vector $u(t)$. Many physical and biological processes have natural models which are bilinear. As examples, nuclear reactor kinematics (13), attitude control of satellites (14), and control of population of species (15) may be mentioned.

It has been shown that bilinear systems are more controllable and in many cases, provide more accurate models than linear systems (16). Hofer (17) provides a survey on optimization of bilinear systems, specifically discussing the time-optimal and quadratic cost problems. In contrast to linear systems optimization, no general analytical results have been obtained, and the optimal control of bilinear systems is largely an untouched field.

1.5 Research Objectives

In general, given a mathematical model of the process, a cost functional and sufficient computational capabilities, Pontryagin's minimum principle or Bellman's dynamic programming can be used to obtain the optimal control law by minimizing the cost functional subject to the dynamic constraints imposed by the system. Recently, however, Rhoten and Mulholland (18,19) presented an approach which potentially offers large reductions in computational requirements. First, a general cost functional is formulated. Next, the optimal trajectories are found without considering the system dynamics. Then, the controller structure is designed to exactly track the optimum trajectories. This technique has been used to select the optimal control law for linear oscillatory plants (18) and for a class of nonlinear plants which are called norm-invariant (19). The intent of this research is to extend this technique to nonlinear systems which are not norm-invariant, and particularly to systems linear in control.

The new synthesis procedure for the optimal control problem has several merits. Once the form of the performance measure is chosen, the optimal control is obtained in closed-loop form. For an infinite-time problem, it is not necessary to solve either a two-point boundary value problem or a nonlinear partial differential equation. A finite-time problem requires the solution of a first order parameter identification problem irrespective of the order of the system to be controlled.

1.6 Organization

The remainder of this dissertation is arranged in the following manner.

The formulation of the inner-product control problem is presented in Chapter II. For the sake of completeness, the primary results of

(18) and (19) are included, with optimal control laws obtained for linear self-adjoint and nonlinear norm-invariant systems.

The primary theoretical results representing an extension of the results of Chapter II to the synthesis of optimal controls for systems linear in control, are found in Chapter III. A fundamental control equality is derived for the more general problem formulation. The non-uniqueness of the optimal control laws and its consequences are discussed with the help of an electrodrive circuit example.

The problem of designing an optimal inner-product controller for a continuous stirred tank reactor is considered in Chapter IV. The response of the chemical reactor to the inner-product controller and to a suboptimal controller obtained by a repeated linearization technique are compared.

The physical interpretation of the cost functional and other related topics are briefly discussed in Chapter V.

Finally, Chapter VI contains a summary of results and conclusions.

CHAPTER II

INNER-PRODUCT PROBLEM FORMULATION

2.1 General Cost Functional

The calculation of optimal controls is in general a very difficult problem, especially the problem of synthesizing control laws realizable in feedback form. The previous chapter has presented a solution technique which yields optimal closed-loop control laws, but only for linear plants with a quadratic performance measure. In many practical control problems the cost functional may be best described by a nonquadratic form or the plant model may be nonlinear. In such cases, the general solutions lead to either a two-point boundary value problem resulting in an open-loop control (8) or a nonlinear partial differential equation (20).

It has been shown, however, that if the cost functional is judiciously selected, the optimal control laws for a wide variety of plant descriptions can be obtained in closed-loop form without actually solving a two-point boundary value problem (18,19). The problem formulation and a brief summary of results of (18,19) will be presented in this chapter for the sake of completeness.

The structure of the cost functional to be considered is consistent with the objective of driving the state vector to zero in norm. If $x(t)$ represents the state vector, a primary error signal $\rho(t)$ is defined by

$$\rho(t) = x^T(t)x(t)$$
$$= ||x(t)||^2 , \tag{2.1}$$

where $\rho(t)$ is the inner-product of the state with itself and is the square of the Euclidean distance to the origin in state space. To enable the error to be penalized in as general a manner as possible, an error penalty function $h(\rho(t))$ will appear in the cost functional integrand, where $h(\rho)$ is restricted only by

$$h(\rho) > 0 \quad \text{if} \quad \rho \neq 0 ,$$

and (2.2)

$$h(0) = 0.$$

To penalize for excessive control inputs, it is noted that the time derivative of $\rho(t)$, denoted by $\dot{\rho}(t)$, provides at least an indirect measure of the power input at any instant of time. As will be shown in later chapters, a slightly more general form of $\rho(t)$ will enable $\dot{\rho}(t)$ to be identical to power input for certain problems. In contrast, the normal quadratic penalty function for control input will not be a direct power measure. The integral of a non-negative function of $\dot{\rho}(t)$ will thus provide a measure of the total control energy input. This discussion leads directly to a general cost functional

$$J(\rho) = \int_{t_0}^{t_f} \{h(\rho(t)) + \dot{\rho}^2(t)\} \, dt. \tag{2.3}$$

Since the initial problems considered in (18) and (19) were cast as infinite-time regulators, t_f was assumed to be infinite.

2.2 Linear Self-Adjoint Systems

The optimal control of linear self-adjoint systems for which the control is bounded in norm is examined by Athans et al. (21). Rhoten and Mulholland (18) considered the problem of selecting optimal inner-product controllers for linear self-adjoint systems. The precise system under consideration is

$$\dot{x}(t) = A(t)x(t) + u(t), \quad x(t_0) = x_0, \tag{2.4}$$

where x(t) is the state n-vector, u(t) the control n-vector and A(t) is an n x n matrix. It is assumed that the system is self-adjoint, or

$$A(t) + A^T(t) = [0] \quad \text{for all } t \geq t_0. \tag{2.5}$$

The optimal control law which minimizes the performance measure Equation (2.3) will be derived now by classical variational techniques. The procedure is somewhat unusual in that no consideration will be given to the plant dynamics until the optimal trajectories have been determined. After the optimal trajectories have been found, the controller structure will be designed to exactly track the optimal trajectories.

A curve minimizing Equation (2.3) (with $t_f = \infty$) must satisfy the Euler-Lagrange equation and the associated boundary conditions:

$$2\,\frac{d^2\rho(t)}{dt^2} = \frac{dh(\rho(t))}{d\rho(t)} \tag{2.6}$$

$$\begin{aligned} \rho(t_0) &= \rho_0 \\ &= x_0^T\, x_0 \end{aligned} \tag{2.7}$$

$$\lim_{t\to\infty} (\dot{\rho}(t)) = 0. \tag{2.8}$$

Multiplying Equation (2.6) by $\dot{\rho}(t)$, and integrating once, yields

$$\dot{\rho}^2(t) = h(\rho(t)) + C \ , \tag{2.9}$$

where C is the constant of integration. Since the final time is allowed to approach infinity, the constant of integration can be shown to be zero by the use of final boundary condition Equation (2.8), and the initial condition Equation (2.7) still holds. Equation (2.9) can also be written as

$$\dot{\rho}(t) = - \sqrt{h(\rho(t))} \tag{2.10}$$

for the infinite-time case. The sign of the square-root must be negative to yield stable trajectories.

Examining now the plant dynamics, with Equation (2.4) premultiplied by $x^T(t)$, there results

$$x^T(t)\dot{x}(t) \ = \ x^T(t)A(t)x(t) + x^T(t)u(t). \tag{2.11}$$

From the definition of $\rho(t)$ and $\dot{\rho}(t)$, and using the fact that A(t) is skew-symmetric, Equation (2.11) reduces to

$$\begin{aligned} \dot{\rho}(t) &= 2\ x^T(t)\dot{x}(t) \\ &= 2\ x^T(t)u(t). \end{aligned} \tag{2.12}$$

The inner-product nonlinear feedback controller structure of Figure 1 is now hypothesized. This structure indicates that

$$u(t) = \psi(\rho)x(t), \tag{2.13}$$

where $\psi(\rho)$ is a nonlinear scalar function of ρ. Then, the Equation (2.12), which is merely a description of the plant trajectories (in norm), becomes

$$\begin{aligned} \dot{\rho}(t) &= 2\ \psi(\rho)x^T(t)x(t) \\ &= 2\ \psi(\rho)\ \rho(t). \end{aligned} \tag{2.14}$$

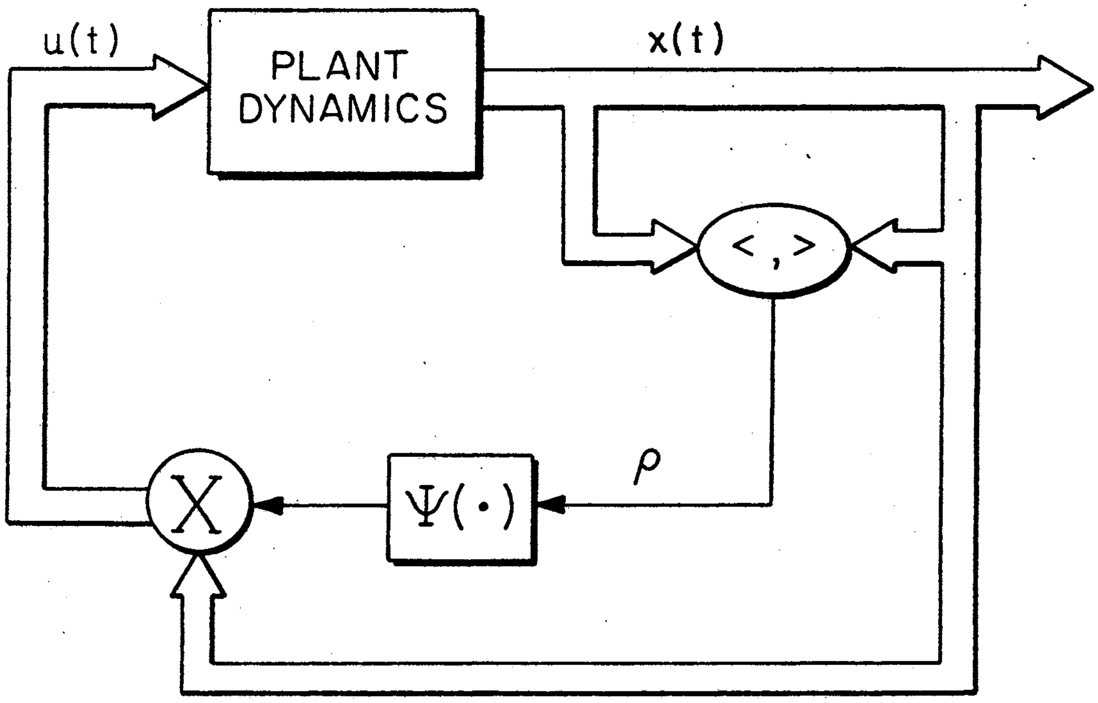

Figure 1. Inner-Product Controller Structure

The objective now is to select the nonlinearity $\psi(\cdot)$ such that the solution of Equation (2.14) also satisfies Equation (2.10), which describes the optimal trajectories. Combining Equations (2.10) and (2.14), and solving for $\psi(\rho)$, there results

$$\begin{aligned}\dot{\rho}(t) &= - \sqrt{h(\rho(t))} \\ &= 2\,\psi(\rho)\rho(t),\end{aligned} \tag{2.15}$$

and

$$\psi(\rho) = \frac{-\sqrt{h(\rho)}}{2\rho}. \tag{2.16}$$

So, the optimal controller is specified by

$$u(t) = -\frac{\sqrt{h(\rho)}}{2\rho}\,x(t). \tag{2.17}$$

Thus, an analytical expression for the optimal control law for a linear self-adjoint system has been obtained which minimizes a general performance measure. While only one controller structure was examined, the development in succeeding chapters removes this constraint of fixed configuration.

2.3 Nonlinear Norm-Invariant Systems

This section represents a generalization of the previous section for a class of nonlinear systems termed norm-invariant. Norm-invariant systems have the property that in the absence of control the norm of the state vector remains constant. The property of norm-invariance for a physical system is often a consequence of the conservation of momentum. A physical system which falls in this class is an asymmetrical body spinning in space. Athans et al. (22) examined the problems of minimum-time, minimum-fuel and minimum-energy control for such norm-invariant systems. Mulholland and Rhoten (19) obtained an explicit solution of control laws which minimize the general cost functional of Equation (2.3).

A system described by the ordinary differential equation

$$\dot{x}(t) = a(x(t),t) + u(t), \quad x(t_0) = x_0 \tag{2.18}$$

where $a(\cdot)$ is a nonlinear vector function of state and time, is norm-invariant if the solution of

$$\dot{x}(t) = a(x(t),t) \tag{2.19}$$

has the property that

$$||x(t)|| = ||x(t_0)|| \tag{2.20}$$

for all $x(t_0)$ and all $t \geq t_0$. But, for any $x(t)$,

$$\frac{d||x(t)||}{dt} = \frac{d}{dt}[x^T(t)x(t)]^{1/2}$$

$$= \{\frac{1}{2}[x^T(t)x(t)]^{-1/2}\}\{2[x^T(t)\dot{x}(t)]\}$$

$$= x^T(t)\dot{x}(t) / ||x(t)||. \tag{2.21}$$

Using Equations (2.21) and (2.19) it can be seen that Equation (2.20) is equivalent to

$$x^T(t)a(x(t),t) = 0, \tag{2.22}$$

for all $x(t)$ and all $t \geq t_0$. Proceeding as before, it can be easily concluded that the optimal control law Equation (2.17),

$$u(t) = \frac{-\sqrt{h(\rho)}}{2\rho}\, x(t), \tag{2.23}$$

still holds for the norm-invariant systems described by Equation (2.18).

An optimization technique has thus been presented which permits the explicit solution of control laws for linear self-adjoint and nonlinear norm-invariant systems which minimize a general performance criterion. The controller structure uses the square of the Euclidean distance to the origin in state space as the primary error signal and requires one nonlinear transducer in the feedback loop.

While the optimization approach presented in this chapter is new, the class of systems for which it is applicable is rather restricted. Very few systems satisfy the norm-invariance criterion, and the dimension of the control is usually less than that of the state. Moreover, some applications may require a finite terminal time. Extension of the results presented here to include a more general class of nonlinear systems, especially systems linear in control, will be presented in the next chapter and form the primary theoretical contribution of this dissertation.

CHAPTER III

OPTIMAL INNER-PRODUCT CONTROL OF SYSTEMS LINEAR IN CONTROL

3.1 Introduction

In this chapter, which represents a generalization of the results of Rhoten and Mulholland (18,19), the problem of optimal inner-product control is examined for a general class of nonlinear systems. A fundamental control equality is derived for the more general problem formulation, and the optimal control laws are obtained as a function of the state vector. Both infinite and finite final-time problems are considered. Finally, the non-uniqueness of the optimal control law and its consequences are discussed.

3.2 Fundamental Control Equality

For a plant described by the differential equation

$$\dot{x}(t) = f(x(t),u(t),t), \quad x(t_0) = x_0 \tag{3.1}$$

where $x(t)$ represents the state and $u(t)$ the control, it is desired to select a control $u(t)$ such that the cost functional

$$\begin{aligned} J &= \int_{t_0}^{t_f} L(\rho(t),\dot{\rho}(t))\, dt \\ &= \int_{t_0}^{t_f} \{h(\rho(t)) + \dot{\rho}(t)^2\}\, dt \end{aligned} \tag{3.2}$$

is minimized when evaluated along the solution trajectories of Equation (3.1). Here, $\rho(t)$ represents the primary error signal and is defined by the relation

$$\rho(t) = x^T(t)Q(t)x(t), \tag{3.3}$$

where $Q(t)$ is a positive definite symmetric weighting matrix and $h(\cdot)$ is a positive definite real-valued function with $h(0) = 0$. The philosophy of the new approach is to evaluate the solution trajectories (in norm) which minimize the chosen cost functional Equation (3.2), and then to select the controller structure such that the norm of the solution of Equation (3.1) exactly tracks the predetermined trajectories.

A necessary condition for the cost functional Equation (3.2) to have a relative extremum is provided by the Euler-Lagrange equation and the associated transversality conditions,

$$2\frac{d^2\rho(t)}{dt^2} = \frac{dh(\rho(t))}{d\rho(t)} \tag{3.4}$$

$$\rho(t_0) = \rho_0$$

$$= x_0^T Q(t_0) x_0 \tag{3.5}$$

and

$$\lim_{t\to\infty} \rho(t) = 0 \quad \text{for variable terminal point} \tag{3.6}$$

or

$$\rho(t_f) = \rho_f \quad \text{for fixed terminal point,} \tag{3.7}$$

where ρ_f is the final value of the inner-product of the state vector. It should be noted that ρ_f does not determine a unique combination of final states. If a specific point in state space is the desired target set, rather than a sphere of certain radius a simple transformation of variables is required to transform the target set to the origin.

Sufficient conditions for a weak minimum of the functional Equation (3.2) require (20)

$$\frac{\partial^2 L(\rho,\dot{\rho})}{\partial \rho^2} \geq 0 \ , \tag{3.8}$$

and

$$\frac{\partial^2 L(\rho,\dot{\rho})}{\partial \rho^2} - \frac{d}{dt}\left[\frac{\partial^2 L(\rho,\dot{\rho})}{\partial \rho \partial \dot{\rho}}\right] \geq 0 \ . \tag{3.9}$$

It can be easily verified that Equation (3.2) always satisfies Equation (3.8), while Equation (3.9) requires that

$$\frac{\partial^2 h(\rho)}{\partial \rho^2} \geq 0 \ . \tag{3.10}$$

For example, if

$$h(\rho) = \rho^{2^n} \ , \ n=1,2,\ldots,$$

Equation (3.10) is satisfied and a weak minimum is guaranteed. It is possible, of course, to have a weak minimum even if $h(\rho)$ is not an even power of ρ and Equation (3.10) cannot be satisfied.

Multiplying Equation (3.4) by $\dot{\rho}(t)$ and integrating once yields

$$\dot{\rho}^2(t) = h(\rho(t)) + C \ , \tag{3.11}$$

where C is the constant of integration to be evaluated using the final boundary condition Equations (3.6) or (3.7), and the initial condition Equation (3.5) still holds. Equation (3.11) can be written as

$$\dot{\rho}(t) = \pm\sqrt{h(\rho(t))} + C \ . \tag{3.12}$$

The selection of the sign of the square-root is not a major problem, as the correct choice is usually obvious from the boundary values. For regulator-type problems, where the final value of $\rho(t)$ is smaller than the initial value of $\rho(t)$, the negative sign holds. On the other hand, for problems requiring increase in the state norm, the positive sign of the square-root is the correct choice.

As an example consider the problem of minimizing the functional

$$J = \int_0^{\ln 2} [\rho^2(t) + \dot{\rho}^2(t)]\, dt \qquad (3.13)$$

such that

$$\rho(0) = 5$$

and (3.14)

$$\rho(\ln 2) = 1.$$

The Euler-Lagrange equation for this problem is therefore

$$\ddot{\rho}(t) = \rho(t), \qquad (3.15)$$

the solution of which can be written as

$$\rho(t) = c_1\, e^{-t} + c_2\, e^{t}. \qquad (3.16)$$

The constants c_1 and c_2 are determined by applying the boundary condition Equation (3.14), and Equation (3.16) becomes

$$\rho(t) = 6\, e^{-t} - e^{t}. \qquad (3.17)$$

It is simple to demonstrate that a minimum is indeed obtained since the sufficient condition Equation (3.10) is satisfied.

Multiplying Equation (3.15) by $\dot{\rho}(t)$ and integrating once yields

$$\dot{\rho}^2(t) = \rho^2(t) + C, \qquad (3.18)$$

where C is the constant of integration. Since the minimizing trajectories are already known, the value of C can be easily obtained,

$$\begin{aligned} C &= \dot{\rho}^2(0) - \rho^2(0) \\ &= 49 - 25 \\ &= 24. \end{aligned}$$

Equation (3.18) can be written as

$$\dot{\rho}(t) = \pm\sqrt{\rho^2(t) + 24} \ , \tag{3.19}$$

and a simple substitution of $\rho(t)$ and $\dot{\rho}(t)$ in Equation (3.19) will reveal that the negative sign of the square-root holds. Alternatively, if

$$\rho(0) = 1$$

and (3.20)

$$\rho(\ln 2) = 5,$$

the minimizing solution is

$$\rho(t) = -2\ e^{-t} + 3\ e^{t} \tag{3.21}$$

and the value of C is still 24. For this set of boundary values, only the positive sign of the square-root satisfies Equation (3.19). Since the problems considered in this work are cast as regulators, the sign of the square-root in Equation (3.12) will be taken as negative.

To begin the solution of $u(t)$, the plant dynamics are considered. If Equation (3.1) is premultiplied by $x^T(t)Q(t)$, it can be demonstrated that

$$\frac{1}{2}\dot{\rho}(t) = \frac{1}{2}x^T(t)\dot{Q}(t)x(t) + x^T(t)Q(t)f(x(t),u(t),t) \ , \tag{3.22}$$

where it is noted that

$$\frac{1}{2}\dot{\rho}(t) = \frac{1}{2}x^T(t)\dot{Q}(t)x(t) + x^T(t)Q(t)\dot{x}(t) \tag{3.23}$$

from the definition of $\rho(t)$. In order for Equation (3.22), which is merely a description of the plant norm trajectories, to describe optimal trajectories, Equation (3.12) must also be satisfied. Thus, the selection of the optimal control $u(t)$ has been reduced to the solution of the scalar fundamental control equality

$$-\frac{1}{2}\sqrt{h(\rho(t)) + C} = \frac{1}{2}x^T(t)\dot{Q}(t)x(t) + x^T(t)Q(t)f(x(t),u(t),t). \quad (3.24)$$

At this point it would perhaps be well to relate the fundamental control equality Equation (3.24) to the optimal control laws of the preceeding chapter. If $Q(t) = I$, $t_f \to \infty$ and the plant is described by

$$\dot{x}(t) = a(x(t)) + u(t), \quad (3.25)$$

where $a(x(t))$ satisfies the norm-invariance property of Equation (2.22), Equation (3.24) reduces to

$$-\frac{1}{2}\sqrt{h(\rho(t))} = x^T(t)\ u(t), \quad (3.26)$$

and the results of Chapter II follow.

Suppose, however, that while $Q(t) = I$ and $t_f \to \infty$, the system is not norm-invariant. The control equality Equation (3.24) then becomes

$$-\frac{1}{2}\sqrt{h(\rho(t))} = x^T(t)\ a(x(t)) + x^T(t)\ u(t). \quad (3.27)$$

Even if the controller structure of Figure 1 is assumed, thus expressing the optimal control as a scalar nonlinear transducer which is closed form in $\rho(t)$ multiplying the state vector, the transducer characteristic cannot usually be evaluated. This fact is not unexpected, since nonlinear differential equations are not linear with respect to initial conditions.

Nevertheless, if it is desirable to implement the controller structure of Figure 1, this can be accomplished. For known initial conditions, the optimal trajectories are simulated using the control

$$u(t) = -\frac{\frac{1}{2}\sqrt{h(x^T(t)\ x(t))} + x^T(t)\ a(x(t))}{x^T(t)\ x(t)}\ x(t), \quad (3.28)$$

where it is seen that Equation (3.28) satisfies the fundamental control equality Equation (3.24). Then, the many-to-one relationship between $x(t)$ and $\rho(t)$ is evaluated along the optimal trajectories and the appropriate nonlinearity can be reconstructed. Of course, different initial conditions would yield different trajectories and hence different transducer characteristics.

As an example, suppose $u_1(t)$ and $u_2(t)$ are to be selected to minimize the performance measure

$$J = \int_0^\infty [\rho^2(t) + \dot{\rho}^2(t)]\, dt \tag{3.29}$$

along the solution of

$$\dot{x}_1(t) = 2\, x_2(t) + e^{-t}[x_1(t) + x_2(t)]^2 + u_1(t) \tag{3.30}$$

$$\dot{x}_2(t) = - x_1^2(t) - x_2(t) + u_2(t), \tag{3.31}$$

with

$$\rho(t) = x^T(t)\, x(t). \tag{3.32}$$

Then the solution of Equation (3.24) for $u_1(t)$ and $u_2(t)$ yields

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \psi(x^T(t)\, x(t)) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3.33}$$

where

$$\psi(x^T(t)\, x(t)) = - \left\{ \frac{1}{2} + \frac{[x_1(t)\ x_2(t)] \begin{bmatrix} 2x_2(t) + e^{-t}(x_1(t)+x_2(t))^2 \\ -x_1^2(t) - x_2(t) \end{bmatrix}}{x_1{}^2(t) + x_2{}^2(t)} \right\}, \tag{3.34}$$

The optimal trajectories, simulated on a digital computer for the system of Equations (3.30) and (3.31) using the initial conditions $x_1(0) = x_2(0) = 5$, yield the characteristic presented in

Figure 2. Since different initial conditions would yield different characteristics, an optimal transducer would be represented by a surface, and an example is presented in Figure 3 for the above problem with $x_1(0) = 5$, $-5 \le x_2(0) \le 5$.

The fundamental control equality is valid for a general error function $h(\rho)$, but the primary error signal $\rho(t)$ is somewhat constrained. However, several useful possibilities for $\rho(t)$ cause no difficulties. As an example, if a performance measure were desired which would more nearly correspond to the quadratic form, $\rho(t)$ could be defined by

$$\rho(t) = \sqrt{x^T(t)Q(t)x(t)}. \tag{3.35}$$

Here, $\rho(t)$ represents the Euclidean distance in state space rather than the square of distance as defined by Equation (3.3). By now setting $h(\rho) = \rho^2$, the performance measure of Equation (3.2) is seen to closely resemble the familiar quadratic cost functional. All of the aforementioned analysis can be performed for this $\rho(t)$, and the fundamental control equality becomes

$$-\rho(t)\sqrt{h(\rho(t)) + C} = \frac{1}{2}x^T(t)\dot{Q}(t)x(t) + x^T(t)Q(t)f(x(t),u(t),t). \tag{3.36}$$

To recapitulate, the optimal control problem has been reduced to the evaluation of a constant of integration, to be discussed in the next section, and the solution of the algebraic Equation (3.24) (or Equation (3.36)) for $u(t)$. Specifically, neither the $2n^{th}$ order set of differential equations with split boundary conditions of the minimum principle nor the nonlinear partial differential equation of dynamic programming need be solved. Of course, the difficulties to be encountered in the solution of Equation (3.24) or Equation (3.36) depend

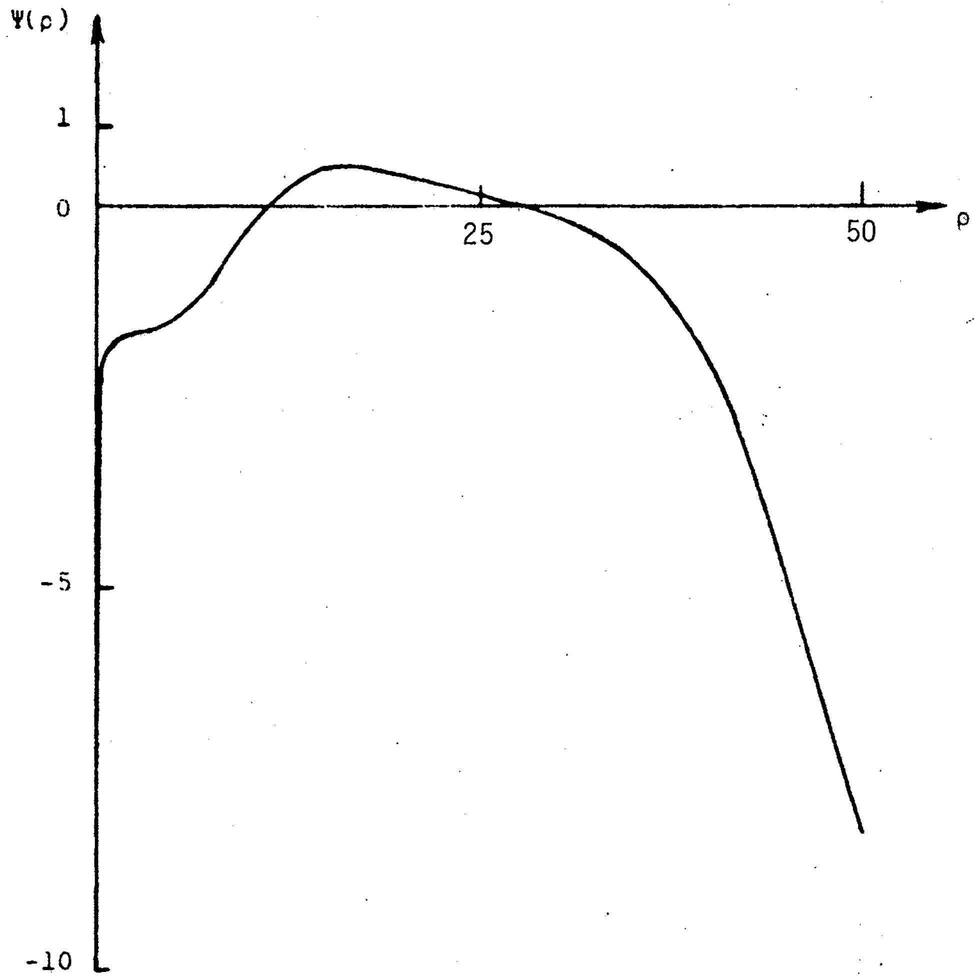

Figure 2. Nonlinear Transducer Characteristic

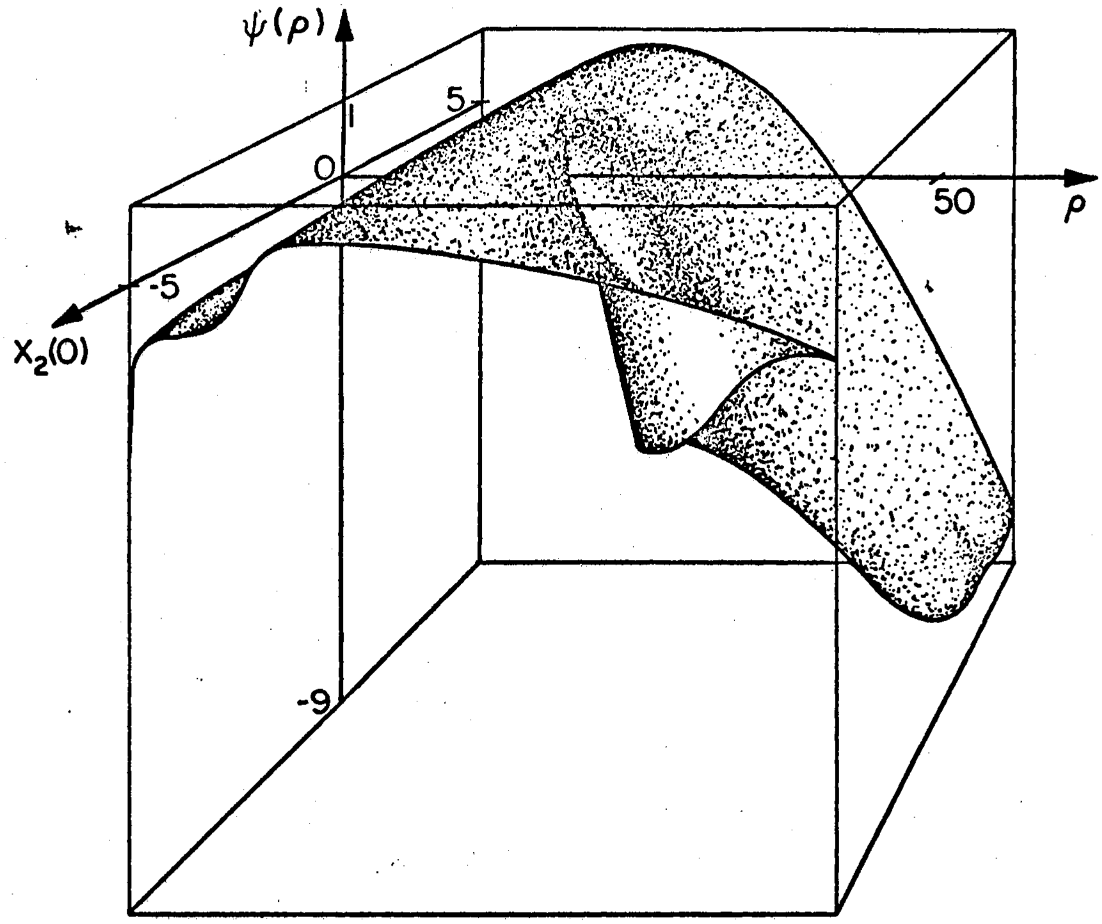

Figure 3. Nonlinear Transducer Surface

inherently on the plant dynamics. As the results of the previous chapter has indicated, the solution is indeed simple for linear self-adjoint and nonlinear norm-invariant systems. Succeeding sections will illustrate solution techniques for systems linear in control.

3.3 Evaluation of the Constant of Integration

For the asymptotic control problems in which the final time approaches infinity, the constant of integration, C, in Equation (3.11) can easily be shown to be zero by the final condition Equation (3.6). While many applications of performance measures with infinite final time are known, others require a finite terminal time. For example, air-to-air and surface-to-air missiles must have their performance indicated with respect to a finite, and often quite small, intercept time. For such applications the numerical evaluation of the constant of integration, C, in Equation (3.11) is required, resulting in a second order two-point boundary value problem. The order is emphasized, since the primary restriction to the use of the minimum principle is the inability to accurately solve the required two-point boundary value problem when the system dimension becomes quite large. Here, only a second order problem must be solved, independent of the plant order. Additionally, this problem can be reformulated as a parameter identification problem which is always first order in $\rho(t)$. This procedure and an appropriate numerical algorithm are presented in the Appendix.

The final advantage of this technique is concerned with the evaluation of a constant rather than a time trajectory. In general, if the plant is nonlinear or the performance measure is non-quadratic, the optimal control law derived from the minimum principle will be in open-loop form, requiring the entire trajectory solution of the two-point boundary value problem. The resultant control law must be stored as a function of time, often requiring excessive computer memory capacity. The goal of this approach is to require only the precomputation and storage of the constant C, with the remaining control law evaluations accomplished in real time and implemented in closed-loop form.

3.4 Systems Linear in Control

The optimal inner-product controller will now be designed for a specific plant structure in which the control enters linearly. The system equation is given by

$$\dot{x}(t) = a(x(t)) + B(x(t))u(t) \tag{3.37}$$

where x(t) is the state n-vector, u(t) is the control m-vector, B(x(t)) is an n x m matrix function of the state and a(x(t)) is an n-vector function of the state. The optimal inner-product control of such systems when the control matrix is not a function of the states has been considered by Leeper (23). For systems described Equation (3.37), the fundamental control equality Equation (3.24) reduces to (time arguments are suppressed for notational simplicity)

$$-\frac{1}{2}\sqrt{h(\rho)+C} = \frac{1}{2}x^T\dot{Q}x + x^TQa(x) + x^TQB(x)u. \tag{3.38}$$

Defining

$$\phi(x) = -[\frac{1}{2}x^T\dot{Q}x + \frac{1}{2}\sqrt{h(\rho)+C} + x^TQ\,a(x)] , \tag{3.39}$$

Equation (3.38) can be written as

$$\phi(x) = x^TQ\,B(x)u. \tag{3.40}$$

While it is true that Equation (3.40) does not determine a unique control u, all the controls which satisfy Equation (3.40) force the system to track the optimum trajectories (in norm). To illustrate the non-uniqueness of the optimal control, it is assumed for the moment that the state and control vectors are of same dimension. Then,

$$u = \frac{\phi(x)}{x^T Q\, B(x)x}\; x \tag{3.41}$$

satisfies the fundamental control equality. Notice, though, that this controller becomes unbounded whenever

$$x^T Q\, B(x)x = 0. \tag{3.42}$$

While this does not invalidate the optimality of the solution, it obviously implies that implementation is not possible. In general, it is not possible to know in advance whether or not the solution of the differential Equation (3.37) with the control described by Equation (3.41) will satisfy Equation (3.42). Furthermore, the choice of the weighting matrix Q also influences when Equation (3.42) is satisfied.

Another solution which satisfies the fundamental control equality Equation (3.40) is given by

$$u = \frac{\phi(x)}{x^T Q x}\; B^{-1}(x)\; x, \tag{3.43}$$

requiring that

$$\det\,(B(x)) \neq 0 \tag{3.44}$$

along the optimal state trajectories. It is often possible to determine the region in the state space where Equation (3.44) is not satisfied from the knowledge of the system dynamics alone. Systems which do satisfy Equation (3.44) are termed as directly controllable (24). The control action, in such systems, can affect the derivatives of each of the components of the state vector directly and independently.

A more general solution of Equation (3.40), applicable when the dimension of the state is not equal to the dimension of the control, can be written as

$$u = \frac{\phi(x)}{x^T Q\ B(x)\ M(x)x}\ M(x)\ x, \tag{3.45}$$

where $M(x)$ is an arbitrary $m \times n$ matrix function of the state vector. Equations (3.41) and (3.43) can be obtained by setting $M(x)$ equal to the identity matrix and $B^{-1}(x)$ respectively. In general, the components of $M(x)$ can be constants or functions of the states. It is important to note, of course, that Equation (3.45) does provide a closed-loop system.

The general control law of Equation (3.45) can still become unbounded if

$$x^T\ Q\ B(x)\ M(x)\ x = 0. \tag{3.46}$$

But, since $M(x)$ is arbitrary, it may be possible to select its entries so that Equation (3.46) is never satisfied. Unfortunately, Equation (3.46) must be evaluated along the possible system trajectories. So, extensive simulation may be required in the selection of an appropriate $M(x)$. Another approach is to provide several gain matrices, using one until $x^T Q\ B(x)M(x)x$ becomes smaller than some preselected value, at which time an alternate $M(x)$ is used.

It may be that neither of the above approaches can succeed. That is, $B(x)$ may be such that for some value of x, no $M(x)$ exists which will not satisfy Equation (3.46). If such is the case, control must be provided by some suboptimal scheme. However, if a suboptimal control law is required, the values of the cost functional for both the unobtainable optimal solution and the suboptimal solution can be found by simulation. Thus, the exact degradation in system performance can be found, and various suboptimal control laws compared.

While it is true that the non-uniqueness of the controller structure might be interpreted as a source of some concern, it does allow greater design flexibility in selecting the optimal controller. In addition to changes in M(x) required for finite control signals, simulations may indicate changes which will reduce the chances of unexpected problems such as control magnitude and/or rate saturations.

Some of these different aspects will now be illustrated with the help of a simple example.

3.5 Control of an Electrodrive Circuit

The model of a simplified electrodrive circuit which was proposed by Feldbaum (25) can be written as

$$\dot{x}_1 = x_2 u_1 \tag{3.47}$$

$$\dot{x}_2 = u_2. \tag{3.48}$$

The time-optimal and quadratic cost problems for this circuit have been investigated by Hofer (17). In this study, it was shown that extensive computations were required to evaluate the optimal control laws, and that the resultant controllers were rather complex to implement. Moreover, for the time-optimal case, the optimal trajectories were not unique.

In order to design an optimal inner-product controller for the electrodrive circuit, the performance measure was chosen as

$$J = \int_0^\infty [\rho^2(t) + \dot{\rho}^2(t)]\, dt, \tag{3.49}$$

where

$$\rho(t) = x^T(t)\, x(t). \tag{3.50}$$

Then, the general control law can be written as

$$u = \frac{\frac{1}{2} x^T x}{x^T \begin{bmatrix} x_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} x} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} x, \tag{3.51}$$

where m_{ij} (i,j=1,2) are the entries of the arbitrary matrix M(x) and can be either constants or functions of x(t).

In order to illustrate the capabilities as well as the limiations of selecting M(x), three choices will be considered;

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.52}$$

$$M_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \tag{3.53}$$

and

$$M_3 = \begin{bmatrix} 1 & 0 \\ 0 & x_2 \end{bmatrix}. \tag{3.54}$$

For all three of these cases, the optimal control becomes unbounded as $x_2 \to 0$. Indeed, a brief examination of Equation (3.41) shows that there is no M(x) that will yield a bounded control for $x_2 = 0$ and $x_1 \neq 0$. Even if $m_{11} = m_{21} = 0$ (an attempt to make the numerator small for $x_1 \neq 0$), the control is given by

$$u = \frac{-\frac{1}{2}(x_1^2 + x_2^2)}{x_2(m_{12}\, x_1\, x_2 + m_{22}\, x_2)} \begin{bmatrix} m_{12} & x_2 \\ m_{22} & x_2 \end{bmatrix}, \qquad (3.55)$$

and $x_2\ (m_{12}\ x_1\ x_2 + m_{22}\ x_2) \to 0$ with the square of x_2, while the numerator for each component of u approaches zero only as $x_2 \to 0$. In addition to $x_2 = 0$, M_1 and M_2 yield additional lines of infinite u. These are easily shown to be

$$x_2 = -\ x_1^2 \qquad (M(x) = M_1), \qquad (3.56)$$

and

$$x_2 = -\ 4\ x_1^2 \quad (M(x) = M_2). \qquad (3.57)$$

To first examine the problem of selecting a single M(x), the circuit model was simulated on a digital computer with $M(x) = M_1$ and an initial state (3,2). The resulting state trajectory is shown in Figure 4 as a-c, with the lines of infinite u also indicated for each M(x). When the time reaches 4.4 seconds x_2 approaches zero, resulting in a simulation halt or giving results which are not meaningful, depending on the type of integration routine used. In general, a fixed step routine may "miss" the time at which $x_2 = 0$ while a variable step size algorithm, in an attempt to reduce error, will decrease the step size until simulation is halted.

Since u tends to infinity as x_2 approaches zero for all of the M's, one must be chosen such that x_1 approaches zero more rapidly than x_2 does. The choice of M_2 accomplishes this, with the resultant trajectory a-0. If, however, the initial state is changed to (3,-2), neither M_1 nor M_2 will yield bounded controls, with the sample trajectory d-f shown for $M(x) = M_1$. A selection of M_3 does provide an acceptable solution, with trajectory d-0 indicating a control bounded

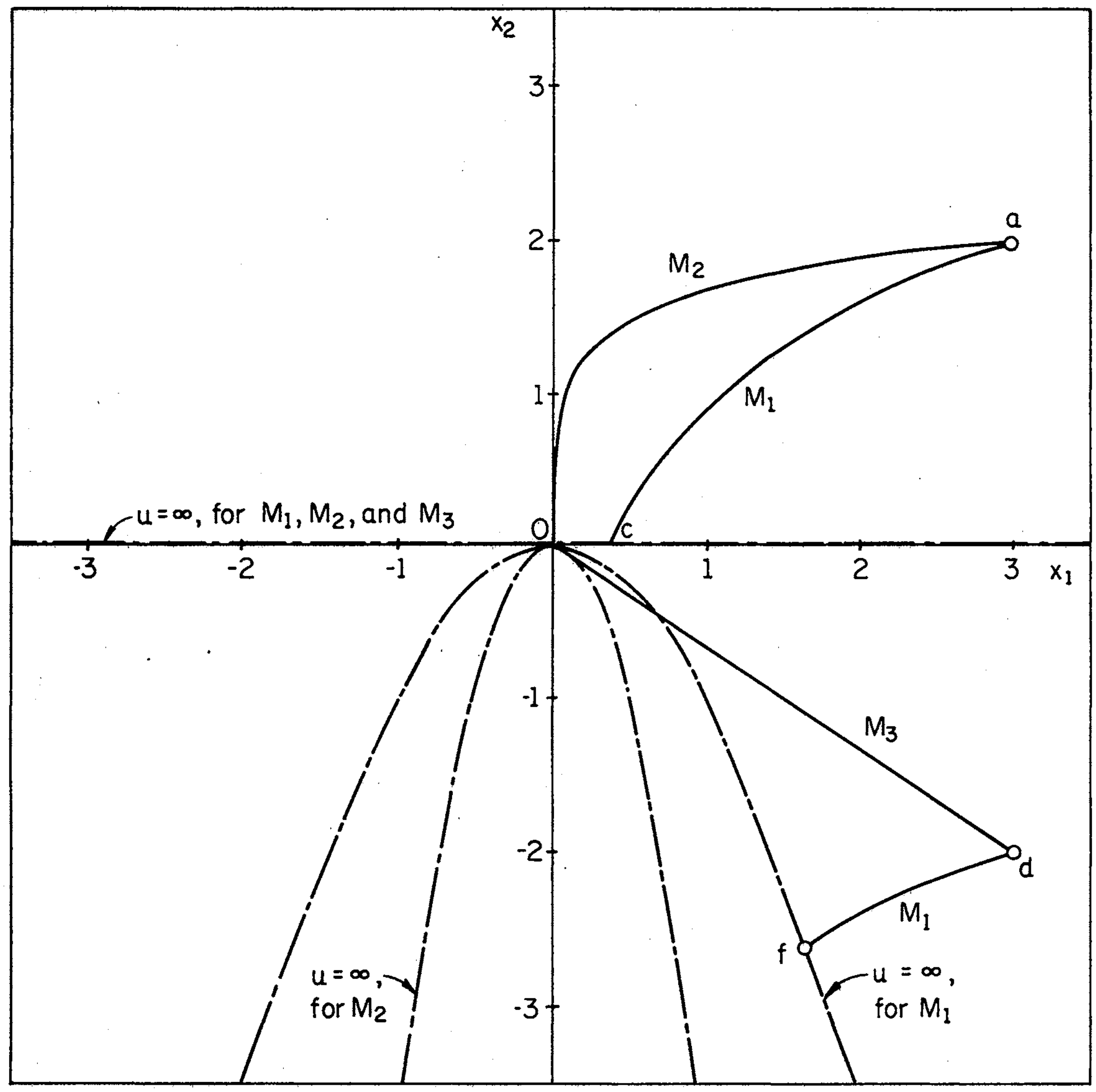

Figure 4. State Trajectories for Electrodrive Circuit: Without Switching

for all t. This change in character of solutions due to changes in initial conditions is a manifestation of the nonlinear character of the problem.

The preceeding example has indicated how simulation may enable the design engineer to select an appropriate M(x). Remaining problems arise if simulation is not possible or if $x_2(0) = 0$. If the former is the case, it might be desirable to use any reasonable M(x) until the control became too large, and then change to a new gain matrix. Such an approach is shown in Figure 5. With the initial state (3,2), M_1 is used until x_2 becomes close to zero, at which time the gains are changed to correspond to M_3. The resultant trajectory is shown as a-b-0.

Although a change in control law has been made while the system was in operation, the solution is still optimal, though not uniquely so. The inner-product time history will remain the same no matter when the change from M_1 to M_3 occurs; only the individual state trajectories will differ. Also shown in Figure 5 is the trajectory d-e-0 resulting from an initial state (3,-2), an initial $M(x) = M_1$, and a change to $M(x) = M_3$.

Considering now the problem of $x_2(0) = 0$, it might at first be thought that the proposed solution is of no value, since u will initially be unbounded for any M(x). However, if <u>any</u> control which will drive x_2 away from zero is momentarily applied, and then an optimal controller used, the exact degradation in performance can be found. Since the optimal norm trajectory is known to satisfy

$$\dot{\rho}(t) = -\rho(t), \tag{3.58}$$

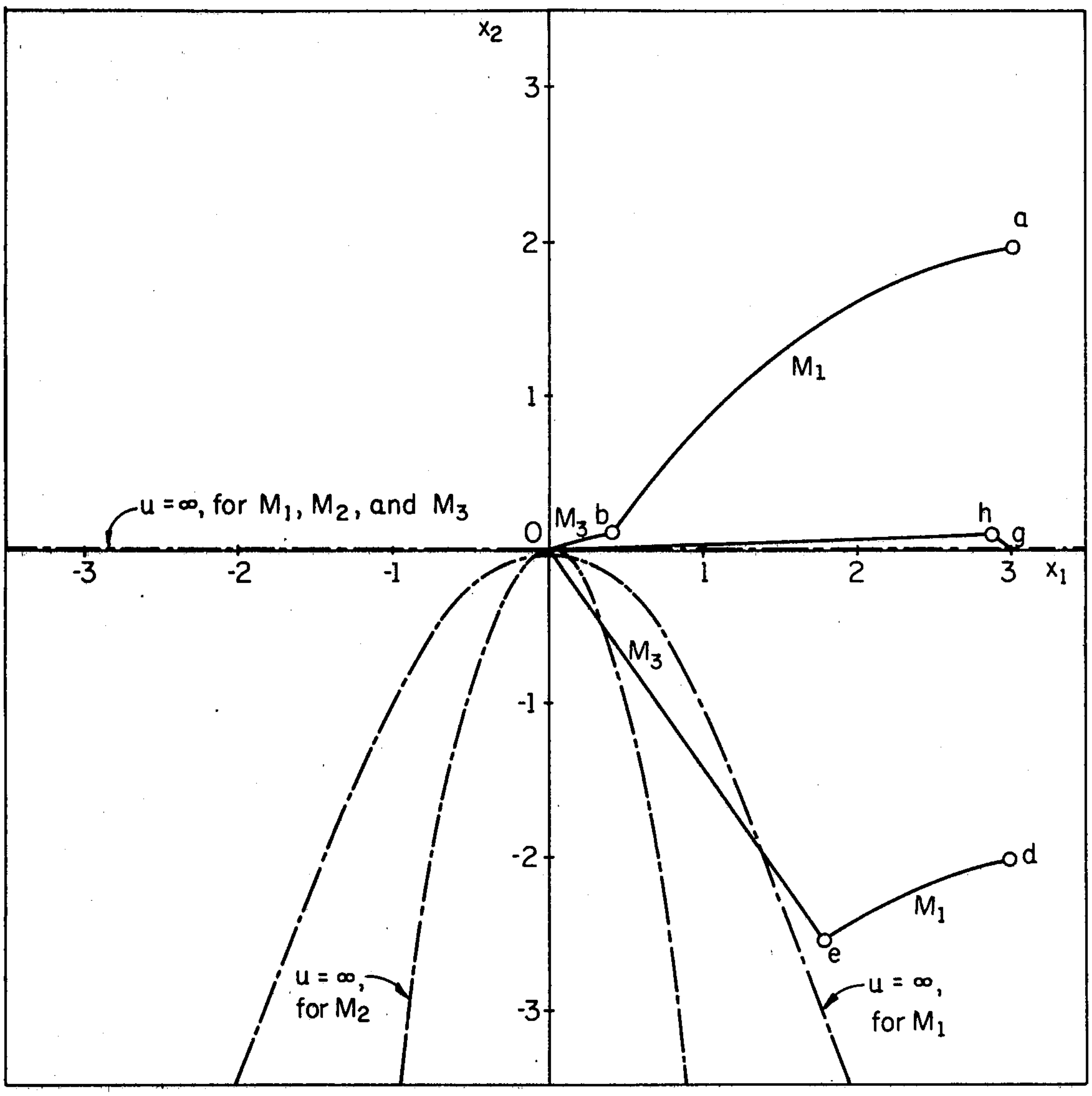

Figure 5. State Trajectories for Electrodrive Circuit: Switching

the performance measure reduces to an elementary quadrature. Thus, the optimal, though unattainable, value is found to be

$$J^* = x_1^4(0). \tag{3.59}$$

A simulation of the actual trajectory will allow calculation of the actual J obtained.

As an example, if $x_1(0) = 3$ and $x_2(0) = 0$, the optimal cost is 81. If a linear control

$$u = \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} x \tag{3.60}$$

is applied for 0.5 seconds, and then the optimal control used with $M(x) = M_3$, the solution trajectory g-h-0 of Figure 5 results in an index value of 84.52, an increase of less than 5%.

To summarize, the nonuniqueness of control laws allows the design engineer great flexibility. If numerous simulations are possible, a single M(x) may be found to be acceptable. If not, alternate M's may be used as conditions warrant. Finally, if no optimal control is implementable, a suboptimal controller may be briefly used to drive the system to a point in state space where an optimal controller is bounded, and the resultant performance degradation calculated exactly.

CHAPTER IV

THE CONTINUOUS STIRRED TANK REACTOR

4.1 Introduction

This chapter considers the problem of designing an optimum inner-product controller for the continuous stirred tank reactor (CSTR). The CSTR is used extensively in the organic chemical industry for a wide range of reactions. Its virtues are its simplicity of construction and ease of temperature control. The study of the CSTR as a control process has received considerable attention in chemical engineering literature (26-32). Fournier and Groves (26) applied parameter search techniques to determine an approximate control algorithm, and Weber and Lapidus (27) presented a suboptimal controller design with a quadratic measure of performance. Fournier et al. (28) demonstrated that a hybrid controller can be used to implement the suboptimal control law.

In this chapter, the CSTR dynamical model is presented and an optimal inner-product controller is obtained. The response of the CSTR to this optimal inner-product controller and to a suboptimal controller obtained by a repeated linearization technique are compared. Finally, a discussion of the results is provided.

4.2 The CSTR Dynamical Model

The CSTR consists of a tank of volume v, into which there is injected a continuous flow of reacting material at a feed rate w. The reacted material passes continuously from the tank at the same rate. The reactor is well stirred so that the concentration, C_A, and the temperature, T, of the reactants are constant throughout the volume. It is assumed that a second order irreversible reaction of the form $2\,A \rightarrow B$ is taking place, and that the reaction rate is given by kC_A^2, where k is the reaction rate constant.

Under these assumptions, a mass balance gives

$$v \frac{dC_A}{d\tau} = \frac{w}{\rho_d} (C_{Af} - C_A) - kC_A^2 , \tag{4.1}$$

where τ is the time, ρ_d is the density and C_{Af} is the inlet concentration. The reaction rate constant can be expressed as a function of temperature by using the Arrheneous expression

$$k = k_0 \exp(-a/T) , \tag{4.2}$$

where k_0 is the frequency factor and a is a constant. An energy balance gives

$$v\rho_d C_p \frac{dT}{d\tau} = \text{total heat generated - total heat removed}$$

$$= kvC_A (-\Delta H) + C_p w (T_f - T) + q , \tag{4.3}$$

where ΔH is the heat of reaction, T_f the feed temperature, C_p the average heat capacity of the reactor, and q the heat added.

It is simple to assume that the amount of heat added is directly manipulated and enough heat transfer is available so that no saturation occurs. But in practice, q is a function of temperature of the reactants and the form of cooling chosen. If the temperature of the reactor is controlled by a jacketed pot through which a coolant flows at a sufficiently high rate to maintain a uniform coolant temperature, T_c, then,

$$q = h\ A_c\ (T - T_c)\ , \tag{4.4}$$

where h is the overall heat transfer coefficient and A_c is the area of the cooling surface.

4.3 Steady State Values

The accepted procedure for the operation of a CSTR is to design the chemical reactor to operate at a steady state condition. This approach is based on the implicit assumption that some steady state system will always correspond to the most profitable plant. Douglas (29) presented a detailed account of optimum steady state design. At steady state the rate of change of concentration and temperature are zero. This gives,

$$\frac{w_s}{\rho_d}\,(C_{Af} - C_A) - v\ k_s\ C_{As}^2 = 0 \tag{4.5}$$

$$C_p\ w_s\ (T_f - T_s) + q_s - (\Delta H)\ v\ k_s\ C_{As}^2 = 0\ , \tag{4.6}$$

where the subscript 's' denotes steady state values. In general, any solution to Equation (4.5) and Equation (4.6) is a steady state solution, and due to the nonlinear nature of these equations, multiple solutions are possible. Perlmutter (30) discussed how unique solutions to Equations (4.5) and (4.6) can be obtained by establishing ranges of the system parameters. Since there are two equations with six variables, four of them must be chosen while the other two variables can then be calculated.

If feed conditions and heating mechanisms are specified, there are normally three equalibrium points. Rajagopalan and Seshadri (31) presented a computer algorithm to find the equalibrium states. An alternative technique is to specify the concentration and temperature of both the feed and the product. Then, the steady state values of feed rate and heat added can be calculated easily:

$$k_s = k_0 \exp(-a/T_s) \tag{4.7}$$

$$w_s = v\, k_s\, C_{As}^2\, \rho_d \,/\, (C_{Af} - C_{As}) \tag{4.8}$$

$$q_s = (\Delta H)\, v\, k_s\, C_{As}^2 - C_p\, w_s\, (T_f - T_s). \tag{4.9}$$

A stability analysis of possible steady states can be found in Perlmutter (30).

4.4 Control of Steady State

If the desired steady state solution is unstable, or if perturbations die away too slowly, it is necessary to dynamically control the reactor. The simplest control system measures the deviations of concentration and temperature from their steady state values and varies the feed rate and amount of heat added (i.e. coolant flow rate). Usually, the control law is determined from among an admissible set of controls such that some suitable performance measure is minimized. Since the reactor dynamics are nonlinear, the evaluation of such controls poses a complex computational task in the form of solving either a two-point boundary value problem or a nonlinear partial differential equation.

By selecting the performance measure in the general form as discussed in section 3.1, it is possible to obtain an analytical expression for the optimal inner-product controller rather easily.

It is convenient to first introduce the normalized dimensionless variables

$$x_1 = \frac{C_A - C_{As}}{C_{As}} \quad (a) \qquad x_2 = \frac{T - T_s}{T_s} \quad (b)$$

$$u_1 = \frac{w - w_s}{w_s} \quad (c) \qquad u_2 = \frac{q - q_s}{q_s} \quad (d)$$

$$t = \frac{\tau w_s}{V\rho_d} . \quad (e) \qquad (4.10)$$

Substituting the relations Equations (4.10a) - (4.10e) and (4.2) into the reactor Equations (4.1) and (4.3), there results

$$\dot{x}_1 = c_1(x_1+1)^2 \exp(-\alpha/x_2+1) + c_2x_1+c_2x_1u_1+c_3u_1+c_3 \qquad (4.11)$$

$$\dot{x}_2 = c_4(x_1+1)^2 \exp(-\alpha/x_2+1) + c_5x_2+c_5x_2u_1+c_6u_1+c_7u_2+c_6+c_7 \qquad (4.12)$$

where definitions of the c's and α are shown in Table III. Equations (4.11) and (4.12) may be written compactly as

$$\dot{x} = a(x) + B(x)u , \qquad (4.13)$$

where

$$x = [x_1 \quad x_2]^T$$

$$u = [u_1 \quad u_2]^T$$

$$a(x) = \begin{bmatrix} c_1(x_1+1)^2 \exp(-\alpha/x_2+1) + c_2x_1+c_3 \\ c_4(x_1+1)^2 \exp(-\alpha/x_2+1) + c_5x_2+c_6+c_7 \end{bmatrix} \qquad (4.14)$$

and

$$B(x) = \begin{bmatrix} c_2 x_1 + c_3 & 0 \\ c_5 x_2 + c_6 & c_7 \end{bmatrix} .$$

So, the reactor dynamics Equation (4.13) are seen to be linear in control and of the form as discussed in section 3.4, and the results of that section apply. Hence the optimal control law which minimizes the performance measure

TABLE III

NORMALIZED CONSTANTS

Constant	Definition	Value
c_1	$\frac{-v\, k_o\, \rho_d\, C_{As}}{w_s}$	-1.630×10^{9}
c_2	-1	-1.000
c_3	$\frac{C_{Af} - C_{As}}{C_{As}}$	1.000
c_4	$\frac{-\Delta H\, v\, k_o\, C_{As}^2}{C_p\, T_s\, w_s}$	1.078×10^{8}
c_5	-1	1.000
c_6	$\frac{T_f - T_s}{T_s}$	1.515×10^{-1}
c_7	$\frac{q_s}{C_p\, T_s\, w_s}$	8.540×10^{-2}
α	$\frac{a}{T_s}$	21.120

$$J = \int_{0}^{\infty} [\, h(\rho(t)) + \dot{\rho}^2(t) \,]\, dt \tag{4.15}$$

where

$$\rho(t) = x^T(t)Q(t)x(t), \tag{4.16}$$

is given by

$$u = -\left[\frac{\frac{1}{2}\sqrt{h(\rho)} + \frac{1}{2}\, x^T\dot{Q}\, x + x^T Q\, a(x)}{x^T Q\, B(x)\, M(x)\, x}\right] M(x)\, x. \tag{4.17}$$

Alternatively, if $\rho(t)$ is defined by

$$\rho(t) = \sqrt{x^T(t)\, Q(t)\, x(t)}, \tag{4.18}$$

the optimal control becomes

$$u = -\left[\frac{\rho\sqrt{h(\rho)} + \frac{1}{2}\, x^T\dot{Q}\, x + x^T Q\, a(x)}{x^T\, Q\, B(x)\, M(x)\, x}\right] M(x)\, x. \tag{4.19}$$

The matrix M(x) in Equations (4.17) and (4.19) is an arbitrary matrix to be selected.

4.5 Simulation Results

The values of physical constants which have been chosen to simulate a realistic situation are presented in Table IV, with the corresponding values of normalized constants given in Table III. The feed temperature and feed concentration are adjusted to coincide with the initial conditions.

The first problem of the design process is to select a suitable performance measure. To initially provide a performance measure somewhat similar to the familiar quadratic cost functional,

TABLE IV

PHYSICAL CONSTANTS

Constant	Value	Units
v	13.38	ft^3
ρ_d	55.00	lb/ft^3
c_p	1.00	Btu/lb^0R
ΔH	-12.00×10^3	Btu/lb mole
a	14.00×10^3	^0R
k_o	83.33×10^7	ft^3/lb mole min
C_{Af}	0.40	lb mole/ft^3
T_f	560.00	^0R
C_{As}	0.20	lb mole/ft^3
T_s	660.00	^0R
k_s	0.51	ft^3/lb mole min
w_s	75.20	lb/min
q_s	42.38×10^2	Btu/min

$$J = \int_0^\infty [\, \eta^2 \rho^2(t) + \dot{\rho}^2(t) \,] \, dt \tag{4.20}$$

will be minimized, where

$$\rho(t) = \sqrt{x^T(t)\; Q(t)\; x(t)} \tag{4.21}$$

and η is a constant. Since precise concentration levels are usually considered of more importance than temperature variations, the weighting matrix is selected to be

$$Q(t) = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix} . \tag{4.22}$$

Further, letting the arbitrary matrix M(x) be the unit matrix in Equation (4.19), the optimal control law becomes

$$u = -\left[\frac{\eta\; x^T\; Q\; x + x^T\; Q\; a(x)}{x^T\; Q\; B(x)\; x}\right] x, \tag{4.23}$$

where a(x), B(x) and Q are given by Equations (4.14) and (4.22) respectively. The reactor dynamics, with the control of Equation (4.23), were simulated on a digital computer and the resulting state trajectories are shown in Figures 6 and 7 for two values of η. It is clear that as η increases, errors are penalized more heavily than error derivatives and the controls increased to more rapidly drive the state variables to the desired steady state values.

To provide comparitive solutions, a repeated linearization technique is used. The nonlinear system of CSTR equations can be linearized about the assumed trajectories $x(t) = x_0$ and $u(t) = 0$, to obtain

$$\dot{\hat{x}}(t) = A\; \hat{x}(t) + B\; \hat{u}(t). \tag{4.24}$$

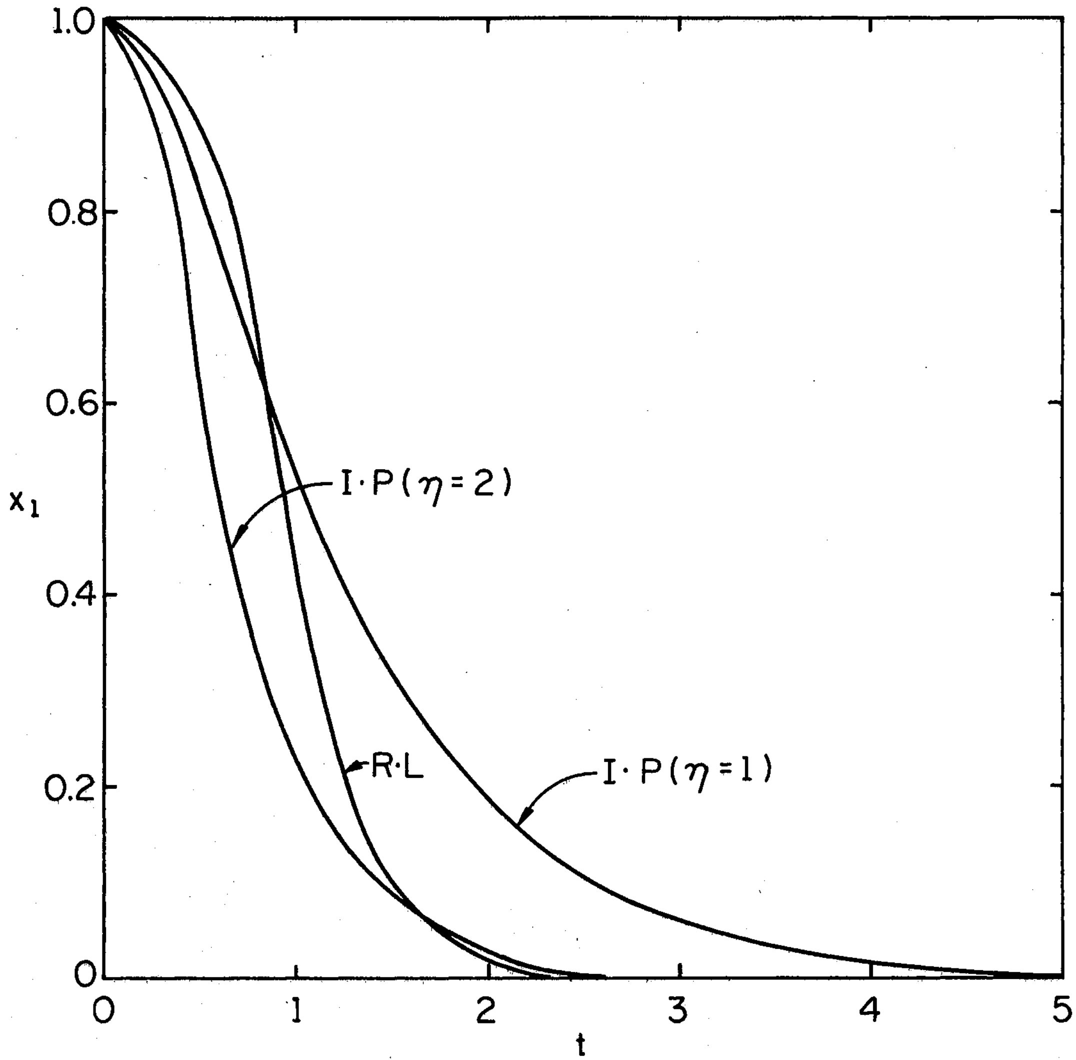

Figure 6. Concentration Versus Time

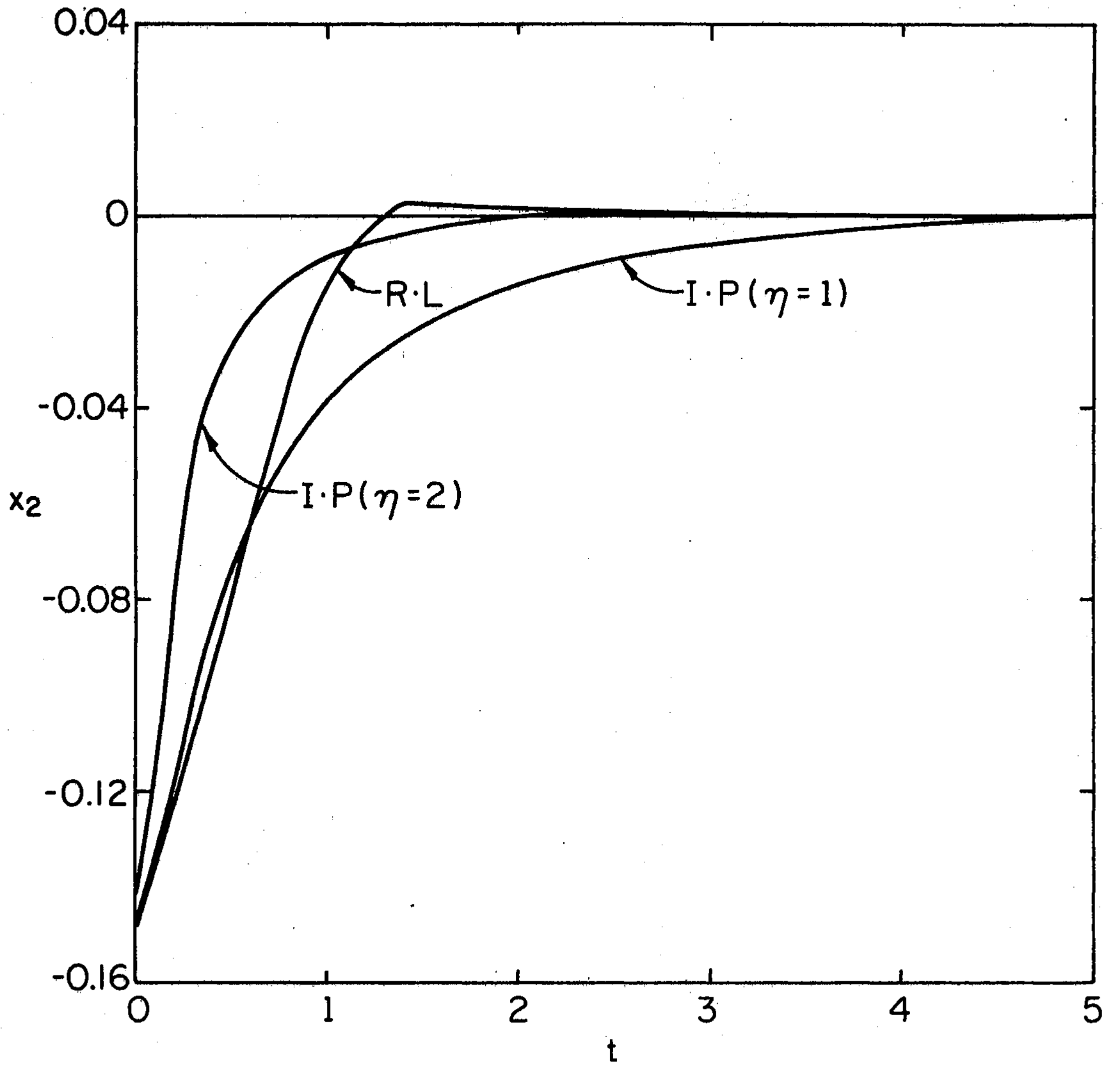

Figure 7. Temperature Versus Time

Selecting a quadratic cost functional

$$J = \int_0^\infty [\ \hat{x}^T(t)\ \hat{Q}(t)\ \hat{x}(t) + \hat{u}^T(t)\ \hat{u}(t)\ ]\ dt, \qquad (4.25)$$

the control law which minimizes Equation (4.25) can be seen from section 1.2 to be of the form

$$\hat{u}(t) = K\ \hat{x}(t). \qquad (4.26)$$

This control, obtained for the linearized system, is applied to the actual nonlinear system as long as the state trajectories remain within certain prescribed limits about x_0, with the limits selected to insure a valid linearization. Suppose t_1 is the time when these limits are exceeded, and let $x(t_1)$ and $u(t_1)$ be the state and control values at that instant. The CSTR equations are then linearized about $x(t_1)$ and $u(t_1)$, and a new value for K found as before. The new control is applied, and the process repeated. Curves resulting from this approach are also presented in Figures 6 and 7. It can be easily observed that two rather distinct optimization techniques have resulted in significantly different controller structures having solution curves which are not entirely dissimilar.

In Chapter III it was demonstrated that the arbitrary matrix M(x) in the optimal control law could usually be selected to yield bounded controls; it may also be possible to choose M(x) to give not only finite but also desirable control laws. The control trajectories for the above example are shown in Figure 8 with $\eta = 2$. Although throughout the synthesis it was tacitly assumed that the control vector was unconstrained, it is clear that the controls as shown could be improved with respect to their relative magnitudes. That is, a reduction in the normalized flow rate u_2 might be desirable even if the

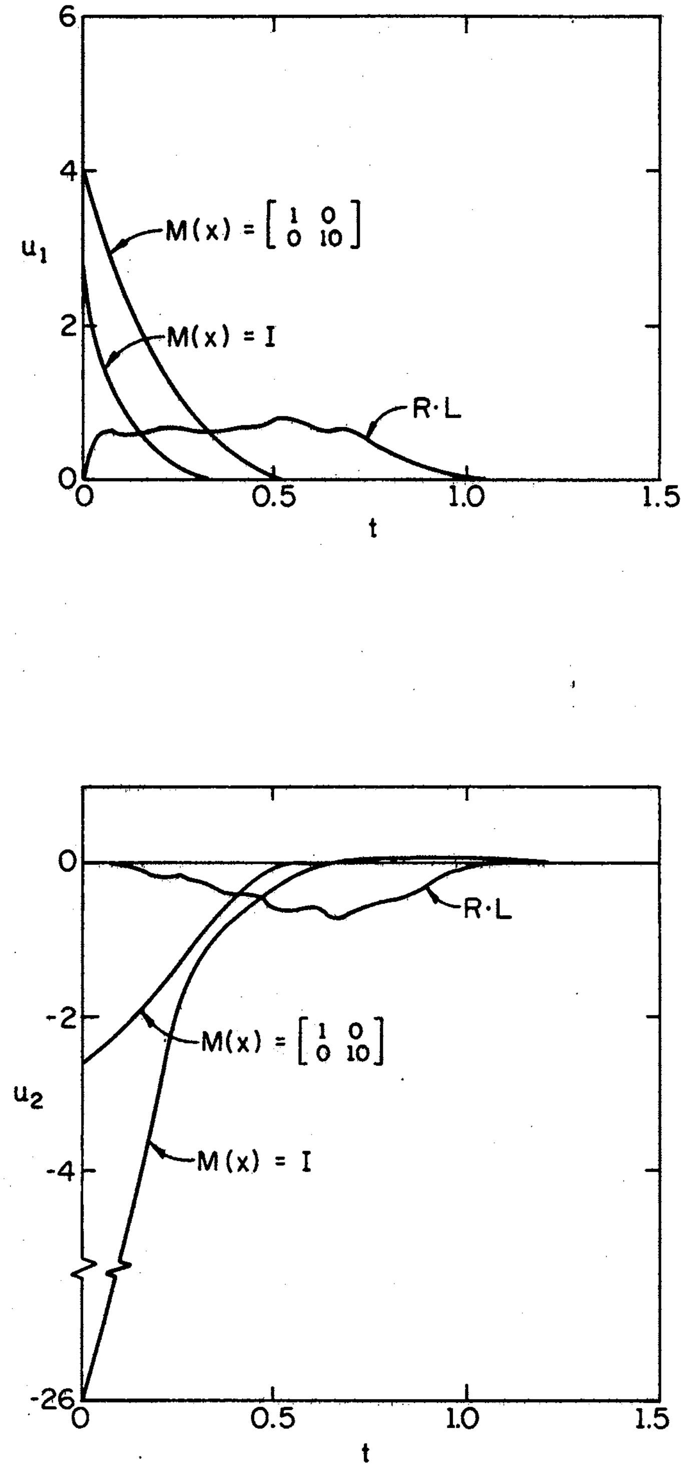

Figure 8. Control Trajectories for the CSTR

normalized feed rate, u_1, were increased.

By setting

$$M(x) = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \quad (4.27)$$

in Equation (4.19), the control trajectories are changed considerably, and are also shown in Figure 8. The new control curves are certainly more balanced in terms of maximum magnitudes, and the state trajectories changed very little. However, even if the individual state trajectories were changed in a significant manner, the performance measure would not increase, as system optimality is independent of the choice of M(x).

The control signals resulting from the repeated linearization technique are also shown in Figure 8. It is of interest to note that such different control trajectories can yield solution trajectories which are quite similar. It is impossible, however, to really compare the two solution processes. The repeated linearization scheme is not suitable for on-line implementation, since the Riccati equation must be solved at each time of linearization. While this would be possible if the dynamical system being controlled were quite slowly varying and a large computer were available in an on-line mode, both of those conditions are rarely satisfied.

The closed-loop implementation of the inner-product controller might also be considered somewhat complex, and would indeed be so if the control signals were to be generated in an analog process. However, a very small digital machine would be quite capable of performing the indicated operations.

In order to demonstrate the intuitive idea that

$$E_1(t) = \int_0^t \dot{\rho}^2(\tau)\, d\tau \tag{4.28}$$

provides an indirect measure of control cost, normally indicated by

$$E_2(t) = \int_0^t u^T(\tau)\, u(\tau)\, d\tau, \tag{4.29}$$

the time history of $E_1(t)$ and $E_2(t)$ are plotted in Figure 9 for the above example with M(x) given by Equation (4.27). The plots of $E_1(t)$ and $E_2(t)$ clearly indicate the close relationship for this particular example.

While the preceeding controllers were designed with a constant matrix M, it is certainly acceptable to select the entries of M(x) as functions of the state variables. Indeed, a rather obvious choice for M(x) would be $B^{-1}(x)$ with the optimal control Equation (4.23) then becoming

$$u = -\left[\frac{\eta x^T\, Q\, x + x^T\, Q\, a(x)}{x^T\, Q\, x}\right] B^{-1}(x)\, x. \tag{4.30}$$

The necessary condition for $B^{-1}(x)$ to exist for all $t \geq t_o$ is

$$\det\,(\,B(x)\,) \neq 0 \quad \text{for all } x(t). \tag{4.31}$$

From the definition of B(x) and the c's, it can be noted that Equation (4.31) is equivalent to

$$\begin{aligned} x_1(t) &\neq -\, c_3/c_2 \\ &\neq x_1(t_o), \end{aligned} \tag{4.32}$$

and the control provided by Equation (4.30) becomes singular at the initial time t_o. To overcome this difficulty, the control of Equation

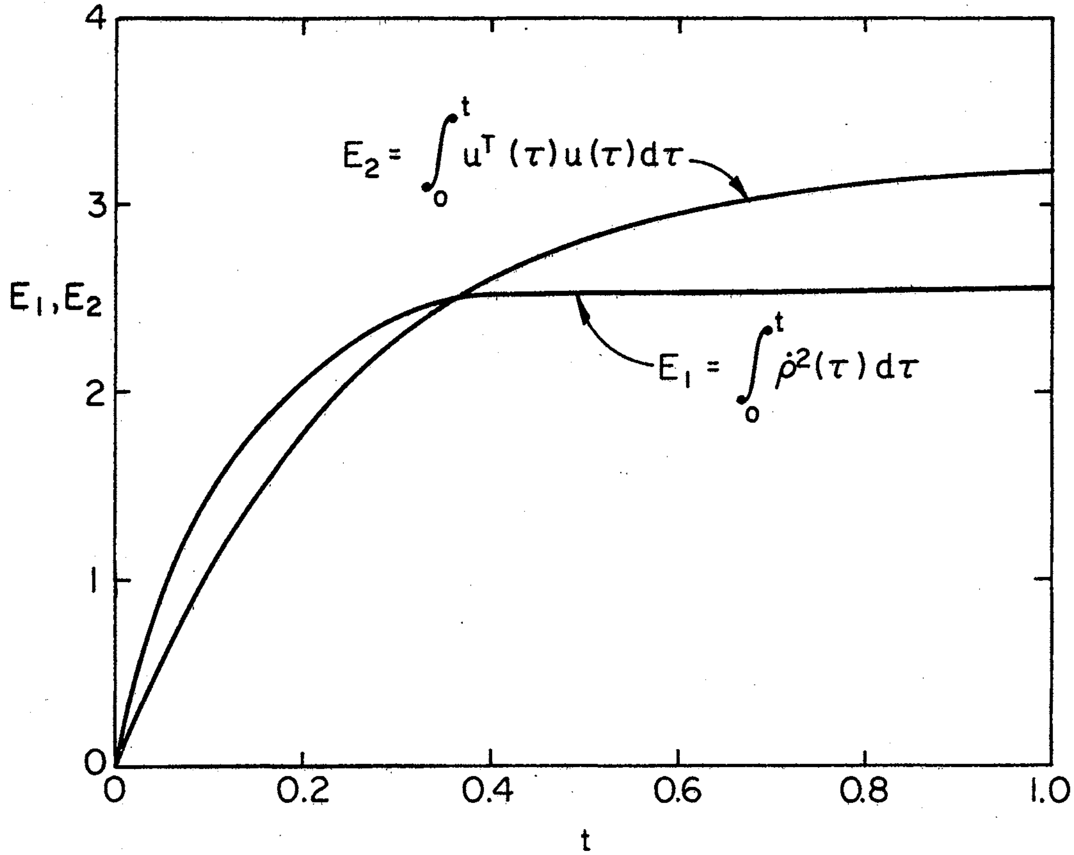

Figure 9. Control-Cost Trajectories for the CSTR

(4.19) with

$$M(x) = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \tag{4.33}$$

is applied for a short time (0.25 seconds) and then the control law is switched to the expression given by Equation (4.30). The resulting state trajectories are shown in Figure 10.

To illustrate the effect of different selections of $h(\rho)$, the state trajectories resulting from minimizing the performance measures

$$J_1 = \int_0^\infty [\rho^4(t) + \dot{\rho}^2(t)]\, dt \tag{4.34}$$

and

$$J_2 = \int_0^\infty [\rho^4(t) + \rho^2(t) + \dot{\rho}^2(t)]\, dt \tag{4.35}$$

with

$$\rho(t) = x^T(t)\, Q(t)\, x(t), \tag{4.36}$$

$$Q(t) = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}, \tag{4.37}$$

and

$$M(x) = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \tag{4.38}$$

are provided in Figure 11. If it were desired to heavily penalize large error signals, an integrand containing a ρ^4 error term would be an appropriate performance measure. However, such an integrand will simultaneously penalize small errors hardly at all, leading to a solution which "drifts" as soon as the error norm becomes less than one.

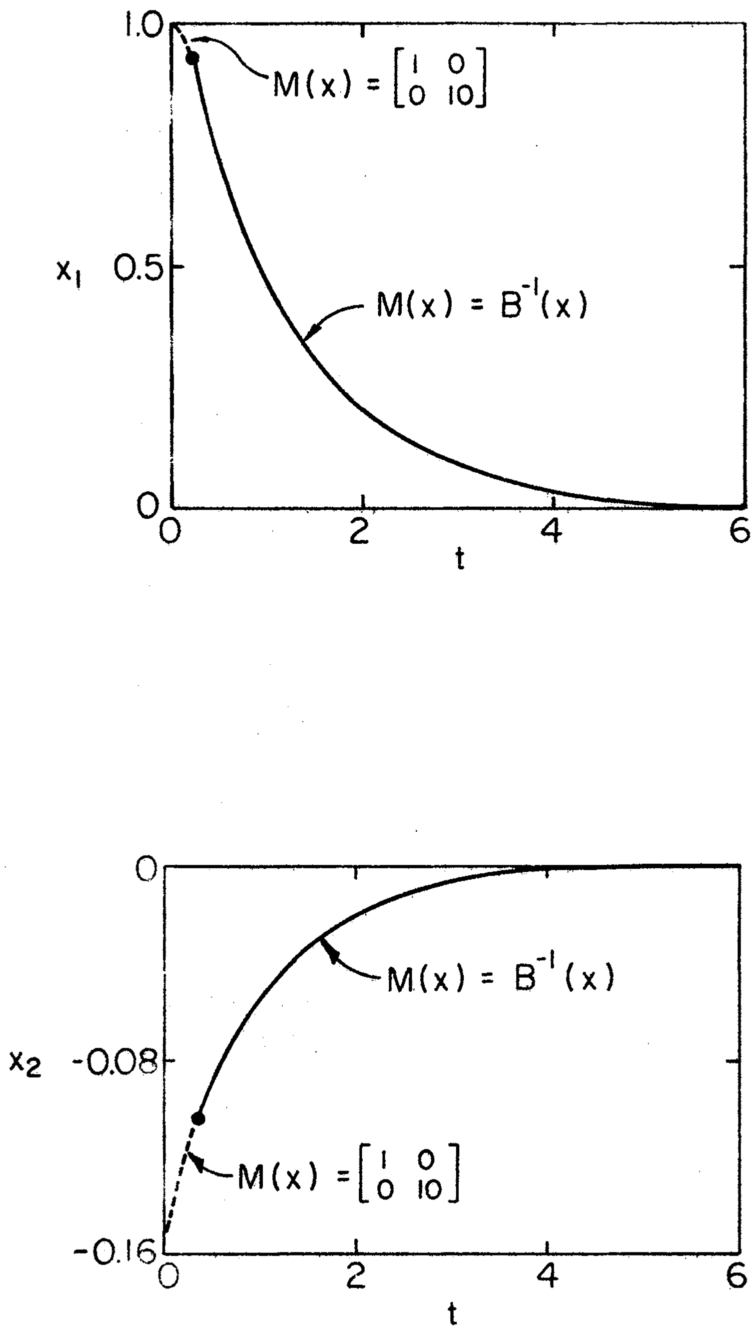

Figure 10. State Trajectories for the CSTR: Switching

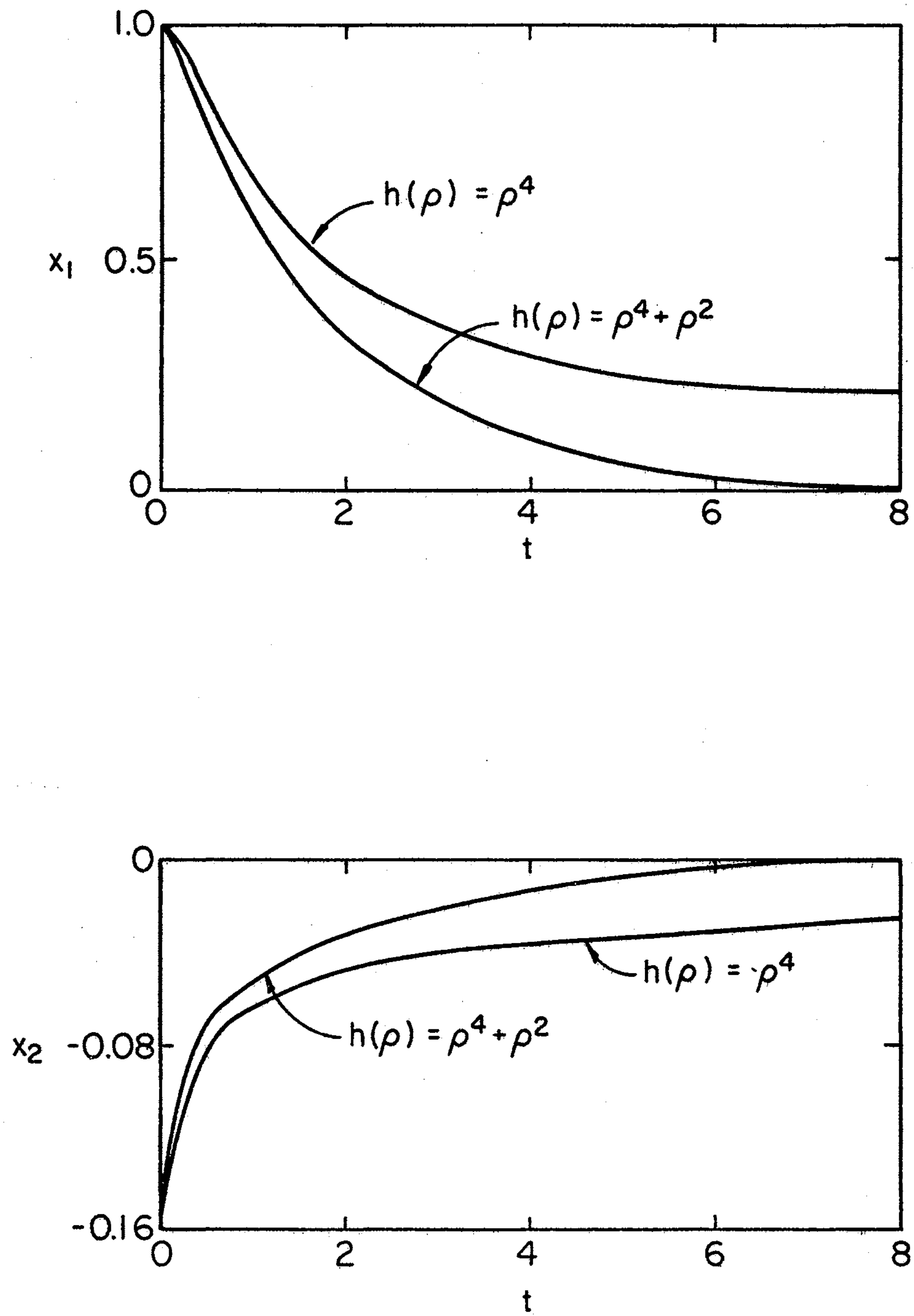

Figure 11. State Trajectories for the CSTR: Different Performance Indicies

If such a solution is undesirable, an additional term ρ^2 could be added to the integrand. This error term is dominated by ρ^4 for $\rho >> 1$, yet dominates ρ^4 for $\rho << 1$. The generality of the error function $h(\rho)$ is thus seen to provide an additional design tool not available from control solutions with restrictive measures of system performance.

CHAPTER V

RELATED TOPICS

This investigation has led to several new and unanswered questions concerning the optimal control of nonlinear systems. While they are interesting and of importance, the nonlinear nature of the problem precludes a general analysis, and extensive treatments of specific examples lie beyond the scope of this dissertation. However, several of these questions will be briefly examined for the sake of completeness.

5.1 Performance Measure Interpretation

Since the design procedure presented herein is intimately related to the form of the performance measure chosen, the physical interpretation of the general performance measure

$$J = \int_{t_0}^{t_f} [h(\rho) + \dot{\rho}^2] \, dt \qquad (5.1)$$

should be examined (time arguments are again suppressed for notational simplicity). While it is clear that the term $h(\rho)$ in the integral Equation (5.1) is an error penalty function, it is not at first clear how the term $\dot{\rho}^2$ is related to the control power inputs. The following example will illustrate that $\dot{\rho}^2$ may indeed be a power measure which is superior to the standard $u^T Ru$. It will also show that a plant description which at first does not appear to be norm-invariant may, through a suitable transformation of state variables, possess this desirable property.

Consider a body spinning in free space and let 1,2,3 denote the body-fixed principal axes through the center of mass. Let I_k and y_k (k=1,2,3) represent the moments of inertia and angular velocities about the principal axes respectively. It is well known (8) that in the absence of external torques, the differential equations satisfied by the three angular velocities are

$$\begin{aligned} I_1 \dot{y}_1 &= (I_2 - I_3)\, y_2\, y_3 \\ I_2 \dot{y}_2 &= (I_3 - I_1)\, y_3\, y_1 \\ I_3 \dot{y}_3 &= (I_1 - I_2)\, y_1\, y_2 \,. \end{aligned} \tag{5.2}$$

Computing the rate of change of the magnitude of the velocity vector y yields

$$\begin{aligned} \frac{d}{dt} ||y|| &= \frac{d}{dt} \sqrt{y_1^2 + y_2^2 + y_3^2} \\ &= \left\{ \frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3} \right\} \frac{y_1\, y_2\, y_3}{||y||} \\ &\neq 0 \,. \end{aligned} \tag{5.3}$$

It is thus clear that the differential equations describing the angular velocities y are not norm-invariant. Suppose, however, that it is desired to write the system equations in terms of angular momenta x instead of angular velocities y. Then x is defined by the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{5.4}$$

and Equation (5.2) yield

$$\begin{aligned} \dot{x}_1 &= \frac{I_2 - I_3}{I_2 I_3} x_2 x_3 \\ \dot{x}_2 &= \frac{I_3 - I_1}{I_3 I_1} x_3 x_1 \\ \dot{x}_3 &= \frac{I_1 - I_2}{I_1 I_2} x_1 x_2 . \end{aligned} \tag{5.5}$$

It can now be easily shown that

$$\begin{aligned} \frac{d}{dt}||x|| &= \frac{d}{dt} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ &= 0 , \end{aligned} \tag{5.6}$$

and the system Equation (5.5) is norm-invariant.

If u represents a vector of control torques, the equations of motion become

$$\begin{aligned} \dot{x}_1 &= \frac{I_2 - I_3}{I_2 I_3} x_2 x_3 + u_1 \\ \dot{x}_2 &= \frac{I_3 - I_1}{I_3 I_1} x_3 x_1 + u_2 \\ \dot{x}_3 &= \frac{I_1 - I_2}{I_1 I_2} x_1 x_2 + u_3 . \end{aligned} \tag{5.7}$$

Since the objective of controller design is to reduce to zero each component of angular velocity (or, correspondingly, each component of angular momentum), an appropriate primary error can be defined by

$$\rho = x^T Q x \tag{5.8}$$

where Q is any positive definite symmetric matrix. If a particular choice of Q is made, given by relation,

$$Q = \frac{1}{2}\begin{bmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{bmatrix}, \tag{5.9}$$

it is seen that

$$\begin{aligned}\dot{\rho} &= 2x^T Q \dot{x} \\ &= 2x^T Q u \\ &= \sum_{k=1}^{3} \frac{1}{I_k} x_k u_k \\ &= \sum_{k=1}^{3} (\text{angular velocity})_k (\text{torque})_k \\ &= \text{Total Power} .\end{aligned} \tag{5.10}$$

Thus, it is seen that ρ is an appropriate error signal for a wide variety of Q's, and by proper choice of Q, $\dot{\rho}^2$ can be made equal to the square of the total power.

The designer who wishes to obtain a minimum-energy controller usually tries to minimize the integral

$$E = \int_{t_0}^{t_f} u^T R u \, dt, \tag{5.11}$$

where R is a positive definite weighting matrix. However, as has been clearly noted in (8), Equation (5.11) may only be an indicator of total energy and not proportional to the energy irrespective of the choice of R. Thus, for a body spinning in free space, the integral of $\dot{\rho}^2$ is related to the total energy while Equation (5.11) is not.

5.2 Inverse Problem

The inverse problem of optimal control can be stated loosely as follows: "Given a dynamical system and a known control law, find the performance criteria (if any) for which this control is optimum." Kalman (7) considered a precise formulation of this problem for linear non-autonomous systems, and Thau (33) investigated the inverse problem for certain nonlinear control systems. Debs and Athans (34) examined the problem of reducing the angular momenta of a space vehicle to zero, with the method of solution based on the inverse problem of optimal control.

It is easy to show that an inverse problem has also been solved for the norm-invariant systems of section 2.3, in that if the control law is known, a performance measure which is being minimized can be found in closed form. That is, if the optimal control is specified by

$$u = \psi(\rho)\, x, \tag{5.12}$$

$h(\rho)$, in the performance criterion

$$J = \int_{t_0}^{t_f} [h(\rho) + \dot{\rho}^2]\, dt \tag{5.13}$$

which is being minimized, can be written as

$$h(\rho) = [2\psi(\rho)\rho]^2 . \tag{5.14}$$

5.3 Controllability

The concept of controllability of linear systems was introduced by Kalman (7), and recently extended to nonlinear systems (35-37). A state x_0 is said to be controllable at time t_0 if there exists a control function $u(\cdot)$, depending on x_0 and t_0, and defined over some finite closed interval $[t_0, t_f]$, such that $x(t_f) = 0$. If this is true for every state x_0, then the system is said to be completely controllable.

Lee and Markus (35) applied this concept of controllability to autonomous nonlinear systems represented by

$$\dot{x} = f(x, u), \tag{5.15}$$

where $f(\cdot)$ is a n-vector function of state x and control u. It was further assumed that the system Equation (5.15) is sufficiently smooth in a neighborhood of the origin and $f(0,0) = 0$. Letting

$$A = \frac{\partial f}{\partial x}(0,0) \tag{5.16}$$

and

$$H = \frac{\partial f}{\partial x}(0,0) \tag{5.17}$$

it was shown that if the linear system

$$\dot{x} = A\,x + H\,u \tag{5.18}$$

is completely controllable, then the set of points from which the origin can be reached in finite time by Equation (5.15) is an open connected set containing the origin. Of course, this is only a local controllability condition.

Hermes (36) extended the concept of complete controllability to systems linear in control using the geometric interpretation of the nonintegrability of Pfaffians. The system under consideration is

$$\dot{x} = a(x,t) + B(x,t)\,u \tag{5.19}$$

where $B(x,t)$ is a continuous n x m matrix function of state and time. It is also assumed that $1 \leq m \leq n$. Let $D(x,t)$ be a continuous (n-m) x n matrix function of state and time such that

$$D(x,t)\ B(x,t) = 0 \tag{5.20}$$

in some domain of interest. The Pfaffian system associated with Equation (5.19) is then given by

$$D(x,t)\ dx - D(x,t)\ a(x,t)\ dt = 0. \tag{5.21}$$

The system Equation (5.19) is completely controllable at (x_0,t_0) if the associated Pfaffian Equation (5.21) is not integrable at (x_0,t_0). On the other hand, if the Pfaffian is integrable, then the system Equation (5.19) is not completely controllable. As an illustration, consider the example treated by Geshwin and Jacobson (37). The system is

$$\dot{x}_1 = - x_1 + (2\ x_1\ x_2 + 1)\ u \tag{5.22}$$

$$\dot{x}_2 = x_2 - x_2^2\ u\ . \tag{5.23}$$

Let the matrix D(x,t) be chosen as

$$D(x,t) = (x_2^2 \quad 2x_1x_2+1)\ . \tag{5.24}$$

The associated Pfaffian equation is

$$x_2^2\ dx_1 + (2x_1x_2+1)\ dx_2 - (x_2+x_1x_2^2)\ dt = 0. \tag{5.25}$$

Letting

$$r = (x_1 \quad x_2 \quad t) \tag{5.26}$$

and

$$Z(r) = (x_2^2 \quad 2x_1x_2+1 \quad -x_2-x_1x_2^2)\ , \tag{5.27}$$

Equation (5.25) can be written as

$$Z(r) \cdot dr = 0. \tag{5.28}$$

The necessary and sufficient condition that the Pfaffian Equation (5.28) is integrable at a point (x_0,t_0) is that

$$Z(r) \cdot \text{curl}\ Z(r) = 0, \tag{5.29}$$

in a neighborhood of (x_0,t_0). A simple computation will reveal that

Equation (5.29) is satisfied by Equation (5.27) and hence the Pfaffian Equation (5.25) is integrable. Thus the system of Equations (5.22) and (5.23) is not completely controllable.

For higher order nonlinear systems it is not easy to use the Pfaffian approach. However, a few results have been obtained for certain special cases. Geshwin and Jacobson (37) presented only sufficient conditions for complete controllability for systems of the form Equation (5.19). Their development was motivated by Lyapunov stability and optimal control theory.

For successful control, it is normally necessary that systems be completely controllable. It is rather easy to determine the controllability of linear systems. The controllability conditions for general nonlinear systems either do not exist or are extremely difficult to apply even in special cases. So, the questions concerning the existence of optimal controls for nonlinear systems cannot be answered completely. Hence, in this dissertation, optimal controls are characterized assuming that they do exist.

5.4 The Epsilon Method

If, as is occasionally the case, no bounded optimal inner-product controls exist, a suboptimal control must be utilized, at least momentarily. The question, then, is how much computing time should be devoted to the design of the suboptimal controller? To illustrate, a modern suboptimal synthesis procedure, the epsilon method (38), will be briefly described. Let the dynamical system and the cost functional be given by

$$\dot{x} = f(x,u,t), \tag{5.30}$$
$$x(t_o) = x_o,$$
$$x(t_f) = x_f,$$

and

$$J = \int_{t_0}^{t_f} [h(\rho) + \dot{\rho}^2]\, dt, \tag{5.31}$$

where

$$\rho = x^T\, Q\, x. \tag{5.32}$$

This method seeks to minimize the epsilon functional

$$J(u,\varepsilon) = \int_{t_0}^{t_f} \frac{1}{2\varepsilon}||\dot{x} - f(x,u,t)||^2\, dt + \int_{t_0}^{t_f} [h(\rho) + \dot{\rho}^2]\, dt \tag{5.33}$$

as epsilon approaches zero.

Taylor and Constantinides (39) have discussed the essential points of this approach. The epsilon method provides a non-dynamical formulation, since no dynamic equations are explicitly solved. As epsilon approaches zero, Equation (5.33) provides a sequence of trajectories and controls that can be made to be arbitrarily close to the optimum value.

The functional minimization can be transformed to an ordinary minimization using the Rayleigh-Ritz expansion. One of the most obvious functions to use in such an expansion is the function $\sin(i\pi t/t_f)$. Thus, the state and control variables can be written as

$$y = \begin{bmatrix} x \\ u \end{bmatrix} = y(t_o) + \frac{t}{t_f}\,[y(t_f)-y(t_o)] + D_m \begin{bmatrix} \text{Sin}(\pi t/t_f) \\ \cdots \\ \text{Sin}(m\pi t/t_f) \end{bmatrix}, \tag{5.34}$$

where D_m is an l x m matrix of the parameters d_{ij} to be optimized. Here l is the sum of the dimensions of state and control vectors and m is the number of terms in Rayleigh-Ritz expansion. The derivative of y can be written as

$$\dot{y} = \frac{1}{t_f}\left[y(t_o)-y(t_o)\right] + D_m \begin{bmatrix} \frac{\pi}{t_f}\,\mathrm{Cos}(\pi t/t_f) \\ \cdots \\ \frac{m\pi}{t_f}\,\mathrm{Cos}(m\pi t/t_f) \end{bmatrix} \tag{5.35}$$

and the norm in Equation (5.33) will thus not be identically zero if Equations (5.34) and (5.35) are used for $\dot{x}$, x and u. However, the dynamic error will approach zero as the optimum parameter matrix D_m is obtained. A modified Newton-Raphson method for minimizing the epsilon functional was used in (39) to obtain an iterative sequence which converges to the optimum solution.

It is clear that such an involved procedure calls for extensive off-line computing. If a simple controller can be momentarily applied to the system, and cause only a small amount of performance degradation, it would appear that complex suboptimal schemes are not required. In the electrodrive circuit example of section 3.5, a simple linear control caused less than a 5% increase in system performance cost, a probably acceptable increase considering ease of both design and implementation.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary

A comprehensive treatment of the optimal control of a general class of nonlinear systems which are linear in control has been presented. The design procedure is somewhat unusual in that a general performance measure is formulated with the objective of driving the state vector to zero in norm without extreme error derivatives. The optimal trajectories which minimize the chosen cost functional are then determined and the controller structure finally designed to exactly track the optimal trajectories in norm.

The selection of the optimal control has been reduced to the solution of the scalar fundamental equality. While the constant of integration in this equality has been shown to be zero for asymptotic control problems, finite final-time problems require the evaluation of the constant resulting in a parameter identification problem which is always first order in the state inner-product.

The optimal inner-product controllers have been designed for systems linear in control. These control laws are not unique and could become unbounded at some points in the state space. However, the non-uniqueness of the controller structure allows greater design flexibility and could be used to obtain bounded controls. Some of these aspects have been illustrated with the help of a simple electrodrive circuit problem.

The new synthesis procedure developed in this dissertation should be applicable to a wide variety of engineering problems. Specific application of the results presented has been made to the problem of optimal regulation of a continuous stirred tank reactor. To provide comparative solutions, a well known technique employing repeated linearizations has also been used. The two distinct optimization techniques have resulted in significantly different controller structures having state trajectories which are not entirely dissimilar.

Finally, the physical interpretation of the inner-product performance measure, a discussion of complete controllability and a brief description of a suboptimal technique using the epsilon technique have been presented.

6.2 Conclusions

Optimal control laws synthesized by the design procedure presented herein are closed-loop in structure and superior from an engineering point of view, to open-loop solutions. The complex computational problems of solving either a two-point boundary value problem or a nonlinear partial differential equation have been avoided. This technique requires the evaluation of just a constant rather than a time trajectory with the remaining control law evaluations accomplished in real-time. Because the controls do not require on-board storage of computed signals, the rapid access memory requirements for large scale systems have been greatly reduced.

The primary disadvantage of this new procedure is that it may require unbounded control inputs at isolated points in time. As indicated by the example of the electrodrive circuit, it may or may not be possible to change, as those points are approached, to an alternative optimal, yet bounded, controller. If this is not possible, a suboptimal controller is required; however, the values of the performance for both the optimal solution and the suboptimal solution can be evaluated. Thus, the exact reduction of the system performance can be calculated and various suboptimal schemes compared.

The inner-product controller approach offers a number of areas for further research. It would be desirable to have a systematic procedure to select the gain matrix in the optimal control law. The sensitivity analysis of the control solution may provide some clues to this procedure. It is also felt that the selection of weighting matrix Q in the inner-product will dictate whether or not the optimal bounded inner-product controls exists for a given performance index.

BIBLIOGRAPHY

(1) Wiener, N. Extrapolation, Interpolation and Smoothing of Stationary Time Series. Cambridge, Mass.: MIT Press, 1966.

(2) Widnall, W. S. Applications of Optimal Control Theory to Computer Controlled Design. Cambridge, Mass.: MIT Press, 1968.

(3) Schultz, D. G., and J. L. Melsa. State Functions and Linear Control Systems. New York: McGraw-Hill, 1967.

(4) Bellman, R., and R. E. Kalaba. Quasalinearization and Nonlinear Boundary Value Problems. New York: American Elsevier, 1965.

(5) Bellman, R. Adaptive Control Process: A Guided Tour. Princeton, N.J.: Princeton University Press, 1961.

(6) Bellman, R. Dynamic Programming. Princeton, N. J.: Princeton University Press, 1957.

(7) Kalman, R. E. "Contributions to the Theory of Optimal Control." Bol. Soc. Mat. Mex., Vol. 5, 1960, pp. 102-119.

(8) Athans, M., and P. L. Falb. Optimal Control. New York: McGraw-Hill, 1966.

(9) Fath, A. F. "Computational Aspects of the Linear Regulator Problem." Proc. JACC, 1969, pp. 44-49.

(10) Luengerger, D. G. "Observers for Multivariable Systems." IEEE Trans. Automatic Control, Vol. AC-11, 1966, pp. 190-197.

(11) Sims, C. S., and J. L. Melsa. "A Survey of Specific Optimal Techniques in Control and Estimation." International Journal of Control, Vol. 14, 1971, pp. 299-308.

(12) Mohler, R. R. "A Common Model in Socio-Economics, Ecology and Physiology." IEEE Systems, Man and Cybernetics Group Annual Symposium Record, 1971, pp. 389-294.

(13) Mohler, R. R., and C. N. Shen. Optimal Control of Nuclear Reactors. New York: Academic Press, 1970.

(14) Bryson, A. E. and Y. C. Ho. Applied Optimal Control. Waltham, Mass.: Blaisdell, 1969.

(15) Mohler, R. R. Chicom Resurgence. Canoga Park, Calif.: Xyzyx Information Corporation, 1971.

(16) Rink, R. E., and R. R. Mohler. "Completely Controllable Bilinear Systems." SIAM Journal of Control, Vol. 6, 1968, pp. 477-486.

(17) Hofer, E. P. "Optimal Control of Bilinear Systems with Application to an Electric Circuit." Fourth Asilomar Conference on Circuits and Systems Proceedings, 1970. pp. 231-235.

(18) Rhoten, R. P., and R. J. Mulholland. "Selection of Optimal Inner-Product Controllers for Oscillatory Plants." International Journal of Control, Vol. 14, 1971, pp. 929-935.

(19) Mulholland, R. J., and R. P. Rhoten. "Optimal Inner-Product Angular Momentum Controllers." IEEE Trans. Automatic Control, Vol. AC-17, 1972, pp. 95-97.

(20) Sage, A. P. Optimum Control Systems. Englewood Cliffs, N.J.: Prentice Hall, 1968.

(21) Athans, M., P. L. Falb, and R. T. Lacoss. "Optimal Control of Self-Adjoint Systems." IEEE Trans. Applied Industry, Vol. 83, 1964, pp. 161-166.

(22) Athans, M., P. L. Falb, and R. T. Lacoss. "Time-, Fuel-, and Energy-Optimal Control of Nonlinear Norm-Invariant Systems." IEEE Trans. Automatic Control, Vol. AC-8, 1963, pp. 196-202.

(23) Leeper, J. L. "Asymptotic Inner-Product Feedback Control." Ph.D. Dissertation, Stillwater, Oklahoma: Oklahoma State University, 1972.

(24) Kipiniak, W. "A Note on Norm-Invariant Systems." IEEE Trans. Automatic Control, Vol. AC-9, 1964, pp. 188-189.

(25) Boltyanskii, V. G. "Sufficient Conditions for Optimality and the Justification of the Dynamic Programming Method." SIAM Journal of Control, Vol. 4, 1966, pp. 326-361.

(26) Fournier, C. D., and F. R. Groves. "Approximate Optimal Closed-Loop Control of Nonlinear Systems via Parameter Search." Proc. JACC, 1970, pp. 542-548.

(27) Weber, A. P. J., and L. Lapidus. "Supoptimal Control of Nonlinear Systems: Unconstrained." A.I.Ch.E. Journal, Vol. 17, 1971, pp. 641-648.

(28) Fournier, C. D., F. R. Groves, and A. B. Corripio. "A Hybrid Controller for Pseudo-Optimal Feedback Control of Nonlinear Systems." Simulation, Vol. 17, 1971, pp. 168-176.

(29) Douglas, J. M. Process Dynamics and Control: Volume I. Englewood Cliffs, N.J.: Prentice Hall, 1972.

(30) Perlmutter, D. D. Stability of Chemical Reactors. Englewood Cliffs, N.J.: Prentice Hall, 1972.

(31) Rajagopalan, T., and V. Seshadri. "Analysis of Continuous Stirred Tank Reactor as a Multivariable Process and Algorithms for Computer Determination of the Equilibrium States." International Journal of Control, Vol. 15, 1972, pp. 497-507.

(32) Sridhar, M., and R. P. Rhoten. "Optimal Inner-Product Control of the Continuous Stirred Tank Reactor." SWIEEECO Record, 1972, pp. 275-278.

(33) Thau, F. E. "On the Inverse Optimum Control Problem for a Class of Nonlinear Autonomous Systems." IEEE Trans. Automatic Control, Vol. AC-12, 1967, pp. 674-681.

(34) Debs, A. S., and M. Athans. "On the Optimal Angular Velocity Control of Asymmetrical Space Vehicles." IEEE Trans. Automatic Control, Vol. AC-14, 1969, pp. 80-83.

(35) Lee, E. B., and L. Markus. "Optimal Control for Nonlinear Processes." Arch. Ration. Mech. Anal., Vol. 8, 1961, pp. 36-58.

(36) Hermes, H. "Controllability and the Singular Problem." SIAM Journal of Control, Vol. 2, 1965, pp. 241-260.

(37) Gershwin, S. B., and D. H. Jacobson. "A Controllability Theory for Nonlinear Systems." IEEE Trans. Automatic Control, Vol. AC-16, 1971, pp. 37-46.

(38) Balakrishnan, A. V. "A Computational Approach to the Maximum Principle." Journal of Computer and Systems Science, Vol. 5, 1971, pp. 163-191.

(39) Taylor, J. M., and C. T. Constantinides. "Optimization: Application of the Epsilon Method." IEEE Trans. Automatic Control, Vol. AC-17, 1972, pp. 128-131.

(40) Unruh, D. R. "Determination of Optimal Parameters for Dynamical Systems." Ph.D. Dissertation, Stillwater, Oklahoma: Oklahoma State University, 1970.

APPENDIX

FINITE TERMINAL-TIME PROBLEM

The finite terminal-time optimal inner-product control problem requires the evaluation of the unknown constant C in the differential equation

$$\dot{\rho}(t) = -\sqrt{h(\rho(t)) + C}, \quad \rho(t_o) = \rho_o, \tag{A.1}$$

such that $\rho(t_f) = \rho_f = 0$. In Equation (A.1), $\rho(t) \varepsilon R^1$ and $h(\cdot)$ is a continuous scalar function of $\rho(t)$. Here t represents the independent variable time and t_o and t_f the initial and final times respectively.

This problem can be viewed as a parameter identification problem, and an iterative method termed "The Method of Seeking Principal Planes" can be used to determine the unknown constant (40). This method utilizes the performance index

$$PI = (\rho(t_f) - \rho_f)^2 , \tag{A.2}$$

and requires the determination of the values of PI, the gradient of PI (GPI) and the second partial of PI (SPI). Taking the partial derivatives of Equation (A.2) with respect to the unknown constant C twice, the following is obtained:

$$GPI = \frac{\partial PI}{\partial C} = 2 \, (\rho(t_f) - \rho_f) \, \frac{\partial \rho(t_f)}{\partial C} \tag{A.3}$$

$$SPI = \frac{\partial^2 PI}{\partial C^2} = 2 \, (\rho(t_f) - \rho_f) \, \frac{\partial^2 \rho(t_f)}{\partial C^2} + 2 \, \left(\frac{\partial \rho(t_f)}{\partial C} \right)^2 . \tag{A.4}$$

Next, the values of $\partial\rho(t_f)/\partial C$ and $\partial^2\rho(t_f)/\partial C^2$ must be obtained.

Taking partial derivatives of Equation (A.1) with respect to C twice and reversing the order of partial derivatives yields

$$\frac{d}{dt}\left(\frac{\partial\rho}{\partial C}\right) = -\frac{1}{2}\,(h(\rho) + C)^{-\frac{1}{2}}\left(\frac{dh(\rho)}{d\rho}\frac{\partial\rho}{\partial C} + 1\right) \qquad (A.5)$$

$$\frac{d}{dt}\left(\frac{\partial^2\rho}{\partial C^2}\right) = -\frac{1}{2}\,(h(\rho)+C)^{-\frac{1}{2}}\left(\frac{dh(\rho)}{d\rho}\frac{\partial^2\rho}{\partial C^2} + \frac{d^2h(\rho)}{d\rho^2}\left(\frac{\partial\rho}{\partial C}\right)^2\right) +$$

$$\frac{1}{4}\,(h(\rho)+C)^{-3/2}\left(\frac{dh(\rho)}{d\rho}\frac{\partial\rho}{\partial C} + 1\right). \qquad (A.6)$$

Since the initial condition of $\rho(t)$ is independent of the parameter guess,

$$\frac{\partial\rho(t_0)}{\partial C} = 0 \qquad \text{and} \qquad \frac{\partial^2(t_0)}{\partial C^2} = 0. \qquad (A.7)$$

If a guess is made for the unknown parameter, Equations (A.1), (A.5) and (A.6) can be integrated from t_0 to t_f and will yield all the information needed for evaluating PI, GPI and SPI.

The philosophy to be employed for obtaining the next guess of the parameter is to move in the negative gradient direction far enough so that the PI is reduced to zero. The block diagram of the algorithm is given in Figure 12.

The computational algorithm for this problem is composed of three subroutines, FINCON, DERFUN and RKINT. The listings for these subroutines are given at the end of this appendix. The subroutine RKINT is called by FINCON, and its purpose is to provide the values of $\rho(t_f)$, $\partial\rho(t_f)/\partial C$, and $\partial^2\rho(t_f)/\partial C^2$. Subroutine DERFUN provides the values of derivative functions for RKINT. In all the subroutines, the following definitions are assumed:

$$Y(1) = \rho(t)$$

$$Y(2) = \frac{\partial \rho}{\partial C}$$

$$Y(3) = \frac{\partial^2 \rho}{\partial C^2} \quad .$$

In order to use the computational algorithm, the user must supply the cards defining the derivatives YD(k) of Y(k) for k = 1, 2 and 3. These cards are placed in the subroutine DERFUN between the COMMON and RETURN cards.

The data card to be supplied by the user has Format (6F10.0, E15.5, I5) and contains

Column	1 - 10	Initial value of ρ(RO∅)
Column	11 - 20	Final value of ρ(ROF)
Column	21 - 30	Initial guess for C
Column	31 - 40	Initial time (T∅)
Column	41 - 50	Final time (TF)
Column	51 - 60	Integration interval (TSTEP)
Column	61 - 75	Tolerance on PI (EPS)
Column	76 - 80	Integer number corresponding to the maximum number of iterations (NMAX).

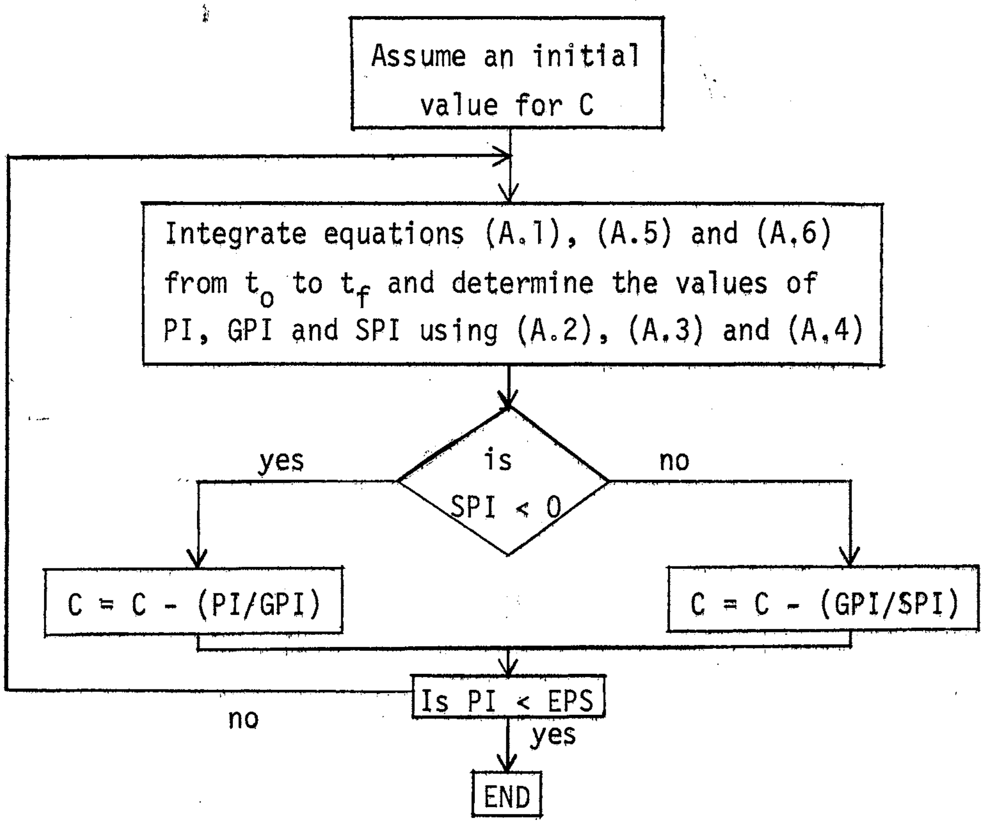

Figure 12. Block Diagram of the Computational Algorithm

TABLE V.

PROGRAM TO EVALUATE THE UNKNOWN CONSTANT

```
      COMMON Y(3),YD(3),TIME,TSTEP,TFINAL,NSYS,IPRINT,C ,KR,KW
      KR = 5
      KW = 6
      CALL FINCON
      STOP
      END
      SUBROUTINE FINCON
C
C     ****************************************************************
C
C        THIS PROGRAM FINDS THE CONSTANT C OF THE DIFFERENTIAL
C
C        EQUATION
C
C                  RODOT =-SQRT(H(RO)+C),  RO(0)=ROO
C
C        SUCH THAT RO(TFINAL)=ROF. AN ITERATIVE METHOD CALLED
C
C        'THE METHOD OF SEEKING PRINCIPAL PLANES' IS USED.
C
C        REF: DETERMINATION OF OPTIMAL PARAMETERS FOR DYNAMICAL
C
C        SYSTEMS BY D.R.UNRUH,  DOCTORAL THESIS, OKLA. ST. UNIV.
C
C        STILLWATER, 1970.  IN THIS PROGRAM
C
C           Y(1)= RO
C           Y(2)= FIRST  PARTIAL OF RO W.R.T C
C           Y(3)= SECOND PARTIAL OF RO W.R.T C
C
C        * SUBROUTINE DERFUN DEFINES THE DERIVATIVE FUNCTIONS
C        YD(K) OF Y(K), K=1,2,3.
C
C        *  SUBROUTINE RKINT  IS A FORTH ORDER RUNGE-KUTTE
C        INTEGRATION ROUTINE
C
C        *  USER PROVIDES THE FOLLOWING QUANTITIES (6F10.0, E15.0, I5)
C
C           ROO    = INITIAL VALUE OF RO
C           ROF    = FINAL   VALUE OF RO
C           C      = INITIAL GUESS FOR C
C           TO     = INITIAL TIME
C           TFINAL = FINAL TIME
C           TSTEP  = INTEGRATION INTERVAL
C           EPS    = TOLERENCE
C           NMAX   = MAX. NO. OF ITERATIONS
C
C        *  CONVERGENCE IS ACHIEVED WHEN PI=(RO(TFINAL)-ROF)**2
C        IS LESS THAN EPS
C
C        *  KR IS THE READING UNIT NO. AND KW IS THE WRITING UNIT NO.
C
C     ****************************************************************
C
      COMMON Y(3),YD(3),TIME,TSTEP,TFINAL,NSYS,IPRINT,C ,KR,KW
    1 FORMAT(6F10.0,E15.0,I5)
    2 FORMAT(1H1)
    3 FORMAT(/, 9X,8H RO(0)= ,E12.4, 5X,12H RO(TFINAL)= , E12.4,  5X,
     $      4H TO= ,E12.4,5X,8H TFINAL= ,E12.4,//,9X,8H EPS=   ,E12.4,
     *      5X, 6H NMAX= , I3,//)
    4 FORMAT(/,  9X, 22H INITIAL GUESS OF C = ,E12.4,//)
    5 FORMAT(/,10X,5HNITER,13X,2H C,19X,2HPI,18X,3HGPI,18X,3HSPI,/)
    6 FORMAT( 8X, I5, 5(1X,E20.4))
    7 FORMAT(//, 9X, 23H CONVERGED VALUE OF C = ,E13.6,8X,4H PI=,E13.6,
     $      //)
    8 FORMAT(//, 9X, 34H ** TAKES TOO MANY ITERATIONS **   ,//)
    9 FORMAT( /, 9X, 95(1H+),//)
      READ(KR,1) ROO,ROF,C,TO,TFINAL,TSTEP,EPS,NMAX
      WRITE(KW,2)
      WRITE(KW,9)
      WRITE(KW,3) ROO,ROF,TO,TFINAL,EPS,NMAX
      WRITE(KW,4) C
      WRITE(KW,5)
      IPRINT = 0
      NSYS=3
      NITER = 0
   10 Y(1) = ROO
      Y(2) = 0.0
      Y(3) = 0.0
      TIME = TO
C-----
C        CALCULATE PI, GRADIENT OF PI AND SECOND PARTIAL OF PI
C-----
      CALL RKINT
      DUM = Y(1) - ROF
      PI = DUM*DUM
      GPI = 2.0*DUM*Y(2)
      SPI = 2.0*DUM*Y(3) + 2.0*Y(2)*Y(2)
C-----
C        DETERMINE THE NEXT VALUE OF C
C-----
      IF(SPI .LE. 0.0) GO TO 20
      C = C - (GPI/SPI)
      GO TO 30
   20 C = C - (PI/GPI)
   30 NITER = NITER + 1
C-----
C        TEST FOR CONVERGENCE
C-----
      IF (PI .LE. EPS) GO TO 40
      WRITE(KW,6) NITER,C,PI,GPI,SPI
C-----
C        TEST FOR MAX. NO. OF ITERATIONS
C-----
      IF(NITER .GT. NMAX) GO TO 50
      GO TO 10
   40 IPRINT = 1
      Y(1) = ROO
      Y(2) = 0.0
      Y(3) = 0.0
      TIME = TO
      CALL RKINT
      WRITE(KW,7) C,PI
      GO TO 60
   50 WRITE(KW,8)
   60 WRITE(KW,9)
      WRITE(KW,2)
      RETURN
```

TABLE V. (Continued)

PROGRAM TO EVALUATE THE UNKNOWN CONSTANT

```
      END
      SUBROUTINE DERFUN
C
C     ****************************************************************
C
C        USER PROVIDES THE EQUATIONS DEFINING THE DERIVATIVES
C
C        OF RO,   FIRST PARTIAL OF RO AND SECOND PARTIAL OF RO
C
C     ****************************************************************
C
      COMMON Y(3),YD(3),TIME,TSTEP,TFINAL,NSYS,IPRINT,C ,KR,KW
      YD(1)=-SQRT(Y(1)*Y(1)+C)
      YD(2) = (Y(1)*Y(2)+0.5)/YD(1)
      YD(3) = (Y(1)*Y(3)+Y(2)*Y(2)-YD(2)*YD(2))/YD(1)
      RETURN
      END
      SUBROUTINE RKINT
C
C     ****************************************************************
C
C                FOURTH-ORDER RUNGE-KUTTE INTEGRATION ROUTINE
C
C     ****************************************************************
C
      COMMON Y(3),YD(3),TIME,TSTEP,TFINAL,NSYS,IPRINT,C ,KR,KW
      DIMENSION YU(3),DELA(3),DELB(3),DELC(3),DELD(3)
    1 FORMAT (///,12X,5H TIME, 15X, 3H RO, /)
    2 FORMAT (8X, E12.4, 8X, E12.4)
      STPBY2 = TSTEP/2.0
      NTIMES = TFINAL/TSTEP + 0.5
      IF (IPRINT .EQ. 0) GO TO 70
      WRITE(KW,1)
      WRITE(KW,2) TIME,Y(1)
   70 CONTINUE
      DO 60 I=1,NSYS
   60 YU(I) = Y(I)
      CALL DERFUN
      DO 100 IT=1,NTIMES
      DO 10 I=1,NSYS
      DELA(I) = TSTEP*YD(I)
   10 Y(I) = YU(I) + 0.5*DELA(I)
      TIME = TIME + STPBY2
      CALL DERFUN
      DO 20 I=1,NSYS
      DELB(I) = TSTEP*YD(I)
   20 Y(I) = YU(I) + 0.5*DELB(I)
      CALL DERFUN
      DO 30 I=1,NSYS
      DELC(I) = TSTEP*YD(I)
   30 Y(I) = YU(I) + DELC(I)
      TIME = TIME + STPBY2
      CALL DERFUN
      DO 40 I=1,NSYS
   40 DELD(I) = TSTEP*YD(I)
      DO 50 I=1,NSYS
      DEL = (DELA(I) + 2.0*DELB(I) + 2.0*DELC(I) + DELD(I))/6.0
      YU(I) = YU(I) + DEL
   50 Y(I) = YU(I)
```

```
      CALL DERFUN
      IF(IPRINT.EQ.0) GO TO 100
      WRITE(KW,2) TIME,Y(1)
  100 CONTINUE
      RETURN
      END
```

VITA

Sridhar Mitta

Candidate for the Degree of

Doctor of Philosophy

Thesis: INNER-PRODUCT OPTIMIZATION OF SYSTEMS LINEAR IN CONTROL

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Chittoor, Andhra Pradesh, India, January 10, 1946, the son of Mr. and Mrs. Mitta Venkatachelam.

Education: Graduated from Zilla Parishad Multipurpose School, Chittoor, India, in June, 1962; attended Sri Venkateswara University College of Engineering, Tirupati, India, from 1962 to 1964 and College of Engineering, Kakinada, India, from 1964 to 1967; received the Bachelor of Engineering degree in Electronics and Communications from Andhra University, Waltair, India, in December 1967; received the Master of Technology degree in Industrial Electronics from the Indian Institute of Technology, Kharagpur, India, in August, 1969; completed the requirements for the degree of Doctor of Philosophy at Oklahoma State Universtiy, Stillwater, Oklahoma, in May 1973.

Professional Experience: Graduate Trainee, Electronics and Radar Research Establishment, Bangalore, India, Summer 1969; Research Assistant, Center for Systems Science, Oklahoma State University, Stillwater, Oklahoma, from February 1970 to October 1972.

Professional Organizations: Member of Sigma Xi, Eta Kappa Nu and I.E.E.E.