# SEQUENTIAL PROPERTIES IN BANACH SPACES

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# PREFACE

This dissertation is concerned with demonstrating the interrelationships among sequential properties in Banach spaces. Specifically, the relationship between the Dieudonne property and property V is established. An examination of uc and wcc operators, as well as the sets K(X) and N(X), reveal important characterizations of the Dieudonne property and property V. In addition, several characterizations of uc and wcc operators have been discovered. Such findings provide insight into the underlying structure surrounding Banach spaces with property V and the Dieudonne property. Finally, the roles of WCG Banach spaces, Quasi-Reflexive Banach spaces, and Banach spaces with H(X) separable are shown to be decisive in delineating the relationships under study.

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# CHAPTER I

# INTRODUCTION

## Historical Survey

In 1953, A. Grothendieck (8) studied locally convex topological vector spaces X in which every continuous linear operator on X that maps weak Cauchy sequences into weakly convergent sequences is a weakly compact operator. He called this property the Dieudonne property. He ascertains that the space C(K), K a compact Hausdorff space, has the Dieudonne property.

A. Pelczynski (19) in 1962 studied Banach spaces X in which every unconditionally converging operator on X is weakly compact (property V). Property V was generalized to locally convex topological vector spaces by J. Howard (9) in 1968. Pelczynski showed that the space C(K), K a compact Hausdorff space, has property V and provides a relationship between a Banach space having property V and having an unconditional basis. In Howard's study, detailed information of permanence of property V, unconditionally converging operators and the relationship of property V to dual spaces are found.

Prior to introducing property V, Pelczynski (18) in 1958 introduced the concept of property (u). His objective was to study the connection between weakly unconditional convergence and weak completeness. As is later seen, this property for Banach spaces provides a close relationship between Grothendieck's work on the Dieudonne property and

Pelczynski's work on property V. However, Pelczynski shows that $C(K)$, $K$ a compact Hausdorff space, does not have property (u).

Beginning in 1962, R. D. McWilliams published a series of papers on weak sequential convergence. In these papers he introduces the set $K(X)$, which plays an important role in the study of the Dieudonne property.

## Preliminaries

This section is devoted to the stating of basic facts, definitions, and notation. However, definitions and facts found in most Functional Analysis textbooks are excluded. Any undefined notation is the notation used in (23).

Since this paper consists mainly of a study of Banach spaces, the symbols $X$ and $Y$ are assumed to represent Banach spaces. The reader should note that in this paper a Banach space means an <u>infinite</u> dimensional complete real or complex normed linear space.

If $X$ is a Banach space, $X'$ and $X''$ denote the dual and second dual of $X$, respectively. Given $X$, $J$ denotes the canonical isometry embedding $X$ in $X''$.

All linear operators are assumed to be continuous and $L(X,Y)$ denotes the set of all continuous linear operators from $X$ to $Y$.

Let $t$ be a topology on $X$. If $\{x_n\}$ is a sequence in $X$, then $t - \lim_n x_n$ will denote the limit of the sequence $\{x_n\}$ in the $t$ topology. For example, if $t = \sigma(X,X')$, then $\sigma(X,X') - \lim_n x_n$ denotes the limit of the sequence $\{x_n\}$ in the weak topology, $\sigma(X,X')$.

When reference to the unit disk of a Banach space $X$ is made, it is assumed to be the <u>closed</u> unit disk, i.e. $\{x \in X: \|x\| \leq 1\}$.

Definition 2.1. A series $\Sigma x_i$ in $X$ is unconditionally convergent (uc) if for each subseries $\Sigma x_{k_i}$, there is an element $x \in X$ such that $x = \sigma(X,X') - \lim_n \sum_{i=1}^{n} x_{k_i}$.

Several other conditions are known to be equivalent to this definition of a uc series, but for the work of this paper, this definition seems most appropriate. Equivalent conditions are found in (9), (14), and (22).

Definition 2.2. A series $\Sigma x_i$ in $X$ is weakly unconditionally convergent (wuc) if $\Sigma|f(x_i)| < \infty$ for every $f$ in $X'$.

Every uc series is a wuc series, but not conversely. Theorem 2.10 in Chapter II shows a condition under which the converse is true.

Definition 2.3. Let $T \in L(X,Y)$. Then $T$ is said to be:

(a) Weakly compact if $T$ sends bounded sequences into sequences which have a weakly convergent subsequence.

(b) Unconditionally convergent (uc) if $T$ sends wuc series into uc series.

(c) Weakly completely continuous (wcc) if $T$ sends weak Cauchy sequences into weakly convergent sequences.

(d) Completely continuous if $T$ sends weak Cauchy sequences into norm convergent sequences.

(e) Weak Cauchy if $T$ sends bounded sequences into sequences which have weak Cauchy subsequence.

(f) <u>Weak* sequentially compact</u> if $T$ sends bounded sequences into sequences which have a weak* convergent subsequence, where $Y$ is a conjugate space.

The abbreviation uc is used for both unconditionally convergent series and unconditionally convergent operators. However, confusion should not arise since it is evident from the context whether a series or an operator is intended.

$K(A)$ denotes the set

$$\{F \varepsilon X' : \text{there exists a sequence } \{a_n\} \text{ in } A \text{ such that } F = \sigma(X',X) - \lim_n a_n\},$$

where $A$ is a subset of a conjugate Banach space $X'$. $K(JX)$ is simply denoted by $K(X)$.

$N(X)$ denotes the set

$$\{F \varepsilon X'' : \text{there exists a wuc series } \Sigma x_n \text{ in } X \text{ such that } F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Jx_i\}.$$

The sets $K(X)$ and $N(X)$ are decisive in the study of the Dieudonne property and property V which is shown in Chapter II and Chapter III. These sets provide a characterization of wcc and uc operators.

<u>Definition</u> 2.4. A Banach space is said to have the <u>Dieudonne property</u> if every wcc operator on $X$ is weakly compact.

<u>Definition</u> 2.5. A Banach space $X$ is said to have <u>property V</u> if every uc operator on $X$ is weakly compact.

Definition 2.6. A Banach space $X$ is said to be almost reflexive if every bounded sequence in $X$ has a weak Cauchy subsequence.

## Second Adjoint Characterization of Linear Operators

Since a Banach space $X$ can be embedded in its second dual $X''$ in a natural way, the second adjoint of an operator $T$ on $X$ is related to $T$ by $T''|X = T$. In this section certain classes of operators are characterized by the action of their first and second adjoints. The section ends with a generalization of a theorem by A. Grothendieck.

If $A$ is a linear subspace of $X''$ and $JX \subseteq A \subseteq X''$, then $\sigma(X',A)$ defines a linear topology on $X'$, since $JX \subseteq A$ and $JX$ is total over $X'$, (6, p. 418).

Proposition 3.1. If either $X$ or $Y$ is reflexive and $JX \subseteq A \subseteq X''$ ($A$ a subspace of $X''$) then $T''A \subseteq JY$ for all $T \in L(X,Y)$.

Proof: Let $T \in L(X,Y)$ and assume $Y$ is reflexive. Then $T''A \subseteq Y'' = JY$. If $X$ is reflexive, then $JX = A$ and $T''JX \subseteq JY$, so $T''A \subseteq JY$.

Definition 3.2. The uniform operator topology on $L(X,Y)$ is the metric topology of $L(X,Y)$ induced by its norm $\|T\| = \sup\{|Tx|: \|x\| \leq 1\}$.

Notation 3.3. $D(A,B) = \{T \in L(X,Y): T''A \subseteq B\}$, where it is assumed that $A$ is a subspace of $X''$, $JX \subseteq A$, and $B$ is norm closed in $Y''$. Note also that $D(A,B) \subseteq L(X,Y)$ will imply $A \subseteq X''$ and $B \subseteq Y''$. $D(A,B)$ is assumed to be a subset of $L(X,Y)$ unless otherwise specifically stated.

<u>Proposition</u> 3.4. $D(A,B)$ is closed in the uniform operator topology on $L(X,Y)$.

Proof: If $\{T_n\}$ is a sequence in $D(A,B)$ such that $\{T_n\}$ converges to $T$ in $L(X,Y)$, then $\{T_n''\}$ converges to $T''$ in $L(X'',Y'')$, (6, p. 478). If $a \varepsilon A$, then $T_n''a \varepsilon B$ and since $B$ is norm closed in $Y''$, $T''a \varepsilon B$ which implies that $T''A \subseteq B$, hence $T \varepsilon D(A,B)$.

Since $JY$ is norm closed in $Y''$, from Proposition 3.4 the following result ensues at once.

<u>Corollary</u> 3.5. $D(A,JY)$ is closed in the uniform operator topology on $L(X,Y)$.

<u>Proposition</u> 3.6. $D(A,B)$ is a linear subspace of $L(X,Y)$.

Proof: If $T,U \varepsilon D(A,B)$, and if $\alpha$ and $\beta$ are scalars, then $(\alpha T + \beta U)''A = (\alpha T'' + \beta U'')A \subseteq B$.

<u>Proposition</u> 3.7. If $T \varepsilon D(A,B)$ and $W \varepsilon L(Y,Z)$, then $WT \varepsilon D(A,W''B) \subseteq L(X,Z)$.

Proof: $(WT)''A = W''T''A \subseteq W''B$ which implies $WT \varepsilon D(A,W''B)$.

<u>Corollary</u> 3.8. If $T \varepsilon D(A,JY)$ and $W \varepsilon L(Y,Z)$, then $WT \varepsilon D(A,JZ) \subsetneq L(X,Z)$.

Proof: By Proposition 3.7 $WT \varepsilon D(A,W''JY) \subseteq D(A,JZ)$ since $W''JY \subseteq JZ$.

<u>Proposition</u> 3.9. If $W \in D(A,B) \subseteq L(Y,Z)$ and $T \in L(X,Y)$, then $WT \in D(C,B) \subseteq L(X,Z)$ for all $C$ such that $T''C \subseteq A$.

Proof: $(WT)''C = W''T''C \subseteq W''A \subseteq B$, hence $WT \in D(C,B)$. From Proposition 3.6 and Corollary 3.8 the following result is immediate.

<u>Corollary</u> 3.10. $D(A,JX)$ is a left ideal in $L(X,X)$.

Proposition 3.1 through Corollary 3.10 provide a description of the algebraic and topological properties of the class of operators $D(A,B)$. Next a study of the relationship of an operator in $D(A,B)$ to its adjoints is conducted. The study begins with a characterization of an operator by the continuity of its adjoint.

<u>Theorem</u> 3.11. $T \in D(A,B)$ if and only if

$$T': (Y',\sigma(Y',B)) \to (X',\sigma(X',A))$$

is continuous.

Proof: Let $\{y'_\alpha\}$ be a net in $Y'$ such that $y' = \sigma(Y',B) - \lim_\alpha y'_\alpha$. Suppose $T \in D(A,B)$. If $a \in A$, then $T''a \in B$ so,

$$(T''a)y' = \lim_\alpha (T''a)y'_\alpha,$$

$$a(T'y') = \lim_\alpha a(T'y'_\alpha).$$

Thus,

$$T'y' = \sigma(Y',B) - \lim_\alpha T'y'_\alpha$$

and $T'$ is $\sigma(Y',B) - \sigma(X',A)$ continuous.

Conversely, if $T'$ is $\sigma(Y',B) - \sigma(X',A)$ continuous then,

$$T'y' = \sigma(X',A) - \lim_\alpha T'y'_\alpha,$$

$$a(T'y') = \lim_\alpha a(T'y'_\alpha) \text{ for all } a \in A,$$

$$(T''a)y' = \lim_\alpha (T''a)y'_\alpha \text{ for all } a \in A,$$

so $T''a$ is a $\sigma(Y',B)$ continuous linear functional for all $a \in A$; thus, $T''a \in B$ for all $a \in A$; hence, $T''A \subseteq B$.

Definition 3.12. Let $T \in L(X,Y)$ and let $t$ be a linear topology for $Y$. $T$ is said to be a t-compact operator if the $t$ closure of $TS$ is t-compact where $S$ is the unit disk in $X$.

The above definition is a generalization of the notion of compact and weakly compact operators. A compact operator is norm-compact and a weakly compact operator is a $\sigma(Y,Y')$-compact operator.

Corollary 3.13. If $T \in D(A,JY) \subseteq L(X,Y)$, then $T'$ is a $\sigma(X',A)$-compact operator.

Proof: If $S'$ is the unit disk in $Y'$, then $S'$ is $\sigma(Y',JY)$-compact, so $T'$ is a $\sigma(X',A)$-compact operator.

A linear operator $T \in L(X,Y)$ is weakly compact if and only if $T''X'' \subseteq JY$, (6, p. 482). $D(X'',JY)$ is the set of all weakly compact operators; hence, the following are consequential:

(a) If either $X$ or $Y$ is reflexive, every operator in $L(X,Y)$ is weakly compact.

(b) The set of weakly compact operators is closed in the uniform operator topology on $L(X,Y)$.

(c) Linear combinations of weakly compact operators are weakly compact. The product of a weakly compact linear operator and a continuous linear operator is weakly compact.

(d) In the uniform operator topology of $L(X,X)$, the weakly compact operators form a closed two-sided ideal.

(e) An operator in $L(X,Y)$ is weakly compact if and only if its adjoint is continuous with respect to the weak and weak* topologies on $X'$ and $Y'$ respectively.

The next theorem completely characterizes the class $D(A,JY)$ in terms of continuity of their first and second adjoints in addition to compactness of its first adjoint in terms of Definition 3.12. This theorem provides rather complete information about the class $D(A,JY)$.

<u>Theorem</u> 3.14. Let $A$ be a linear subspace of $X'', JX \subseteq A$, and $T \in L(X,Y)$. The following conditions are equivalent.

(a) $T''A \subseteq JY$.

(b) $T'$ is $\sigma(Y',JY) - \sigma(X',A)$ continuous.

(c) $T'$ is a $\sigma(X',A)$-compact operator.

(d) $T''$ is $\tau(A,X')$-norm continuous.

Proof: (a) implies (b) by Theorem 3.11. (b) implies (c) by Corollary 3.13. (c) implies (d): The norm topology on $Y''$ is the linear topology generated by the set of polars of bounded sets in $Y'$, (23, p. 247). Let $N$ be a neighborhood of $0$ in $Y''$. Then there exists a bounded set $B \subseteq X'$ such that $B^0 \subseteq N$. Let $D$ be the balanced convex hull of B. D is then bounded, (23, p. 178). Since T is a $\sigma(X',A)$-compact operator, $\overline{TD}$ is $\sigma(X',A)$ compact. $\overline{TD}$ is convex and balanced, so $\overline{TD}^0 \in \tau(X,X')$. Since $B \subseteq D$, $D^0 \subseteq B^0 \subseteq N$. $TD \subseteq \overline{TD}$, so $(\overline{TD})^0 \subseteq (TD)^0 = (T')^{-1}D^0 \subseteq (T')^{-1}N$. Thus, $(\overline{TD})^0 \subseteq (T')^{-1}N$, which implies that $(T')^{-1}N$ is a neighborhood of 0 in A with the $\tau(A,X')$ topology. It then follows that T' is $\tau(A,X')$-norm continuous. (d) implies (a): A and X' are in duality since each is total over the other, and $\tau(A,X')$ is compatible with the duality. Since $X^{\perp\perp} = A$ and JX is a linear subspace of A, we have $(JX)^{00} = (JX)^{\perp\perp} = A$, so by (23, p. 238, Theorem 1), $(JX)^{00}$ is the $\tau(A,X')$ closure of JX in A; hence, it follows that JX is $\tau(A,X')$ dense in A. Since it is always true that $T''JX \subseteq JY$, and since JX is $\tau(A,X')$ dense in A, it follows by hypothesis that $T''A \subseteq JY$.

From Theorem 3.14, the following well-known properties of weakly compact operators can be deduced. If $T \in L(X,Y)$, the following are equivalent.

(a) T is weakly compact.

(b) $T''X'' \subseteq JY$.

(c) T' is $\sigma(Y',JY) - \sigma(X',X'')$ continuous.

(d) T' is a $\sigma(X',X'')$-compact operator (T' is weakly compact).

(e) T'' is $\tau(X'',X')$-norm continuous.

The sets K(X) and N(X) are linear subspaces of X'' and both contain JX, so the set A in Theorem 2.14 can be replaced by either K(X) or N(X), and similar results about wcc and uc operators are established in the next chapter.

The next theorem relates the $\sigma(X',A)$ compactness of a bounded set $C$ in $X'$ to the existence of a certain $\sigma(X',A)$-compact operator, where $A$ is a subspace of $X''$ containing $X$. Besides Theorem 3.17, another application of this fact is made in Chapter III where the wuc-limited sets studied by Pelczynski (19) are related to compact sets in the $\sigma(X',N(X))$ topology. With the above in mind, let

$$S = \{x \in X: |a(x)| \leq 1 \text{ for all } a \in C\}.$$

Then $S$ is a closed subset in $X$, (23, p. 238, Fact (X)). If $p$ is the gauge of $S$, then $p'$ is a norm on $X/\text{Ker } p$, where $p'(\hat{x}) = p(x)$, $\hat{x} = x + \text{Ker } p$. Let $T$ be the natural map from $X$ to $Y = X/\text{Ker } p$.

With the above notation, the following is evident.

<u>Theorem</u> 3.15. $C$ is $\sigma(X',A)$ compact if and only if $T'$ is a $\sigma(X',A)$-compact operator.

Proof: Assume that $C$ is $\sigma(X',A)$ compact. If $B$ is the unit disk of $Y'$, then $T'B = C$. Since $C$ is $\sigma(X',A)$ compact, it follows that $T'$ is a $\sigma(X',A)$-compact operator.

Conversely, assume that $T'$ is a $\sigma(X',A)$-compact operator. Then if $B$ is the unit disk in $Y'$, $T'B = C$. Thus, by Theorem 3.14 $C$ is $\sigma(X',A)$ compact since $T'$ is $\sigma(Y',Y) - \sigma(X',A)$ continuous and $B$ is $\sigma(Y',Y)$ compact.

Again, as before, let $A = X''$ in Theorem 3.15.

<u>Corollary</u> 3.16. $C$ is $\sigma(X',X'')$ compact if and only if $T'$ is a $\sigma(X',X'')$-compact operator.

The primary theorem of this chapter is presented next. This theorem generalizes a theorem by A. Grothendieck (8). The theorem is used in Chapter III to give characterizations of property V and the Dieudonne property.

Theorem 3.17. Let $A_1$ and $A_2$ be subspaces of $X''$ such that $JX \subseteq A_1 \subseteq A_2 \subseteq X''$. Then the following two conditions are equivalent.

(a) Any $T \in L(X,Y)$ such that $T''A_1 \subseteq JY$ satisfies $T''A_2 \subseteq JY$.

(b) Any $\sigma(X',A_1)$ compact set in $X'$ is also $\sigma(X',A_2)$ compact.

Proof: Assume condition (a), and let $C$ be a $\sigma(X',A_1)$ compact set. Then in the terminology of Theorem 3.15, $T'$ is a $\sigma(X',A_1)$-compact operator. Thus by Theorem 3.14, $T''A_1 \subseteq JY$. By condition (a), $T''A_2 \subseteq JY$. Applying Theorem 3.14, $T'$ is a $\sigma(X',A_2)$-compact operator and thus Theorem 3.15 implies that $C$ is $\sigma(X',A_2)$ compact.

Conversely, assume condition (b), and let $T \in L(X,Y)$ be such that $T''A_1 \subseteq JY$. By Theorem 3.14 $T'$ is $\sigma(Y',Y) - \sigma(X',A_1)$ continuous. If $B$ is the unit disk in $Y'$, then $T'B$ is $\sigma(X',A_1)$ compact. Condition (b) implies $T'B$ is $\sigma(X',A_2)$ compact and Theorem 3.14 then implies that $T''A_2 \subseteq JY$. Hence, condition (a) is satisfied.

# CHAPTER II

## WCC AND UC OPERATORS

### WCC Operators

Recall that a weakly completely continuous operator (wcc) maps weak Cauchy sequences into weakly convergent sequences. The intent is to characterize wcc operators and to investigate the set of all wcc operators in $L(X,Y)$.

<u>Theorem</u> 1.1. $T \in L(X,Y)$ is wcc if and only if $T''K(X) \subseteq JY$.

Proof: Let $F \in K(X)$. There exists a weak Cauchy sequence $\{x_n\}$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. Since $T$ is wcc, $\{Tx_n\}$ is a weak convergent sequence; hence, there exists a $y \in Y$ such that $y = \sigma(Y,Y') - \lim_n Tx_n$, so $Jy = \sigma(Y'',Y') - \lim_n JTx_n$. But since $JTx_n = T''Jx_n$ it follows that $Jy = \sigma(Y'',Y') - \lim_n T''Jx_n$. Recall, $F = \sigma(X'',X') - \lim_n Jx_n$, so $T''F = Jy$; hence, $T''K(X) \subseteq JY$.

Conversely, assume $T''K(X) \subseteq JY$, and let $\{x_n\}$ be a weak Cauchy sequence in $X$. Then there exists an $F \in K(X)$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. Now $T''$ is weak* continuous, so $Jy = \sigma(Y'',Y') - \lim_n T''Jx_n$ for some $y \in Y$, since $T''K(X) \subseteq JY$. But $T''Jx_n = JTx_n$; thus, $Jy = \sigma(Y'',Y') - \lim_n JTx_n$ or $y = \sigma(Y,Y') - \lim_n Tx_n$. Hence, $T$ maps weak Cauchy sequences into weak convergent sequences and $T$ is, therefore, wcc.

The wcc operators are now seen to be a particular class of those in section three of Chapter I. Hence from Proposition 3.1 and 3.4 in Chapter I, we have the following:

<u>Proposition</u> 1.2. (a) If either $X$ or $Y$ is reflexive, every operator in $L(X,Y)$ is wcc. (b) The set of wcc operators in $L(X,Y)$ is closed in the uniform operator topology.

The algebraic properties for the set of wcc operators in $L(X,Y)$ are also readily obtainable. If $V \in L(X,Y)$ and $F \in K(X)$, then there exists a sequence $\{x_n\}$ in $X$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. Since $V''$ is weak* continuous, $V''F = \sigma(Y'',Y') - \lim_n V''Jx_n$ which in turn equals $\sigma(Y'',Y') - \lim_n JVx_n$, so $V''F \in K(Y)$; thus, $V''K(X) \subseteq K(Y)$. Consequently from Proposition 3.6, Corollary 3.8, and Proposition 3.9 in Chapter I, the next proposition follows.

<u>Proposition</u> 1.3. (a) Linear combinations of wcc operators are wcc. (b) The product of a wcc operator and a continuous linear operator is wcc.

In particular, considering only maps from $X$ to $X$, Proposition 1.2 and Proposition 1.3 imply the following corollary.

<u>Corollary</u> 1.4. In the uniform operator topology on $L(X,X)$ the wcc operators form a closed two-sided ideal.

Since a wcc operator $T \in L(X,Y)$ has been characterized by $T''K(X) \subseteq JY$, Theorem 3.14 in Chapter I can be applied; the following results about continuity of the first and second adjoints of a wcc operator can be obtained.

Theorem 1.5. Let T L(X,Y). The following are equivalent:

(a) T is a wcc operator.

(b) $T''K(X) \subseteq JY$.

(c) $T'$ is $\sigma(Y',JY) - \sigma(X',K(X))$ continuous.

(d) $T'$ is a $\sigma(X',K(X))$-compact operator.

(e) $T''$ is $\tau(K(X),X')$-norm continuous.

The above indicative theorem deduces a relationship between the second adjoint and the operator with respect to their being wcc.

Corollary 1.6. Let $T \varepsilon L(X,Y)$. If $T''$ is a wcc operator, then T is a wcc operator.

Proof: By Theorem 1.5, $T'''$ is a $\sigma(X''',K(X''))$-compact operator, so $T'''|Y'$ is a $\sigma(X',K(X''))$-compact operator. Since every $\sigma(X',K(X''))$ compact set in $X'$ is $\sigma(X',K(X))$ compact, $T'''|Y' = T'$ is a $\sigma(X',K(X))$-compact operator. Theorem 1.5 then implies that T is a wcc operator.

By Theorem 1.5, $T \varepsilon L(X,Y)$ is a wcc operator if and only if $T'$ is a $\sigma(X',K(X))$-compact operator. However, an operator T being wcc ($T'$ a $\sigma(X',K(X))$-compact operator) is neither a necessary nor a sufficient condition for $T'$ to be a wcc operator. Consider the identity map $i: c_0 \to c_0$. Since $c_0$ is not weakly complete, i is not a wcc operator; but $i': \ell_1 \to \ell_1$ is a wcc operator since $\ell_1$ is weakly complete. Thus, it is seen that $T'$ wcc does not imply that T is wcc. On the other hand, let i be the identity operator on $\ell_1$, then i is wcc, but $i': m \to m$ is not wcc since m is not weakly complete. Hence, it is seen that T wcc does not imply that $T'$ is wcc.

Using the second adjoint characterization of a weakly compact operator, $T''X'' \subseteq JY$, the second adjoint characterization of a wcc operator, $T''K(X) \subseteq JY$, and the fact that $K(X) \subseteq X''$, the following is apparent.

Proposition 1.7. If $T \in L(X,Y)$ is weakly compact, then $T$ is wcc.

Proposition 1.2 showed that if either $X$ or $Y$ is reflexive then every $T \in L(X,Y)$ is wcc. The next proposition yields the same result if either $X$ or $Y$ is weakly complete.

Proposition 1.8. If $T \in L(X,Y)$ and either $X$ or $Y$ is weakly complete, then $T$ is wcc.

Proof: Suppose that $Y$ is weakly complete and $\{x_n\}$ is a weak Cauchy sequence, then $\{Tx_n\} \subseteq Y$ is a weak Cauchy sequence; hence, weakly convergent. If $X$ is weakly complete, then $\{x_n\}$ is weakly convergent to an $x \in X$. So $x = \sigma(X,X') - \lim_n x_n$, and $x = \sigma(Y,Y') - \lim_n Tx_n$.

The relationship of wcc operator to other operators in products and adjoints is a natural question, and the next objective is to study these relationships.

Proposition 1.9. If $T$ is wcc and $V$ is weak Cauchy, then $TV$, $V'T'$ and $T''V''$ are weakly compact.

Proof: If $\{x_n\}$ is a bounded sequence, then $\{Vx_n\}$ has a weak Cauchy sequence $\{Vx_{n_i}\}$, and hence, $\{TVx_{n_i}\}$ converges weakly. Thus, $TV$ is weakly compact, and so are $V'T'$ and $T''V''$ by Gantmacher's theorem (6, p. 485).

It appears that the weak Cauchy operators are not a special class of the type discussed in Chapter I. However, the second adjoint of a weak Cauchy operator does characterize the operator.

<u>Proposition</u> 1.10. Let $T \in L(X,Y)$. $T$ is weak Cauchy if and only if $T''JS$ is weak* sequentially compact where $S$ is the unit disk of $X$.

Proof: If $T$ is weak Cauchy and $\{G_n\}$ is a sequence in $T''JS$, then $G_n = T''Jx_n$ for some sequence $\{x_n\}$ in $S$, so $G_n = T''Jx_n = JTx_n$. $\{x_n\}$ is bounded and $T$ is weak Cauchy so $\{x_n\}$ has a subsequence $\{z_n\}$ such that $\{Tz_n\}$ is weak Cauchy, so $\{JTz_n\}$ is, therefore, weak* Cauchy which implies $G = \sigma(Y'',Y') - \lim_n JTz_n$ exists. $\{JTz_n\}$ is a subsequence of $\{G_n\}$; hence, $\{G_n\}$ has a subsequence that converges weak* to an element $G \in Y''$; thus, $T''JS$ is weak* sequentially compact.

Conversely, assume $T''JS$ is weak* sequentially compact. If $\{x_n\}$ is a sequence in $S$, then $\{T''Jx_n\}$ is a sequence in $T''JS$. Hence, $\{T''Jx_n\}$ has a subsequence $\{T''Jz_n\}$ which converges weak* in $Y''$. But $T''Jz_n = JTz_n$, so $\{Tz_n\}$ is weak Cauchy; hence, $T$ is a weak Cauchy operator.

If $T \in L(X,Y)$, and $S$ and $S''$ are the unit disks in $X$ and $X''$ respectively, then $T''JS \subseteq T''S''$. So if $T$ is a weak* compact operator, then $T''S''$ will be weak* sequentially compact. Consequently, $T''JS$ is also weak* sequentially compact. Thus, Proposition 1.10 implies the following result.

<u>Corollary</u> 1.11. Let $T \in L(X,Y)$. If $T''$ is weak* compact, then $T$ is weak Cauchy.

The converse to the above corollary is not true. Let $K = [0,2\pi]$ and let $i$ be the identity operator on $c_0(K)$. Since $c_0(K)$ is almost reflexive every bounded sequence will have a weak Cauchy subsequence, so $i$ will be a weak Cauchy operator. However, $i'': m(K) \to m(K)$ is not weak* compact since the unit disk of $m(K)$ is not weak* sequentially compact. Indeed, define a sequence $\{x_n\}$ in $m(K)$ by $x_n(\alpha) = \sin n\alpha$ for all $\alpha \in [0,2\pi]$ and for $n = 1,2,3, \ldots$ . Suppose $\{x_n\}$ has a weak* convergent subsequence $\{x_{n_i}\}$. If $\{x_{n_i}\}$ converges in the weak* topology, then it must converge pointwise on $K$. $\lim_i \sin n_i\alpha$ must exist for each $\alpha \in K$ . However, this is not possible.

Let $T \in L(X,Y)$ and let $S''$ be the unit disk in $X''$. If $T''$ is weak Cauchy, then any sequence in $T''S''$ will have a weak Cauchy subsequence. Thus, the subsequence will be weak* Cauchy and, therefore, weak* convergent. So $T''S''$ is weak* sequentially compact and, hence, $T''$ is weak* compact. Thus, another immediate result is the following.

<u>Corollary</u> 1.12. Let $T \in L(X,Y)$. If $T''$ is weak Cauchy, then $T''$ is weak* compact; in particular, $T$ is weak Cauchy.

The above corollary also shows that if $T \in L(X,Y)$ and $Y$ is a conjugate Banach space, then $T$ weak Cauchy implies $T$ is weak* compact. The converse, however, is not true. The identity map on $\ell_1$ is weak* compact since $c_0$ is separable, but it is not weak Cauchy since $\ell_1$ is weakly complete and not reflexive.

Corollary 1.11 can now be used to obtain a result about the product of a weak* compact operator and a wcc operator. If $T \in L(X,Y)$ and $T''$ is weak* compact, then by Corollary 1.11 $T$ is weak Cauchy.

Proposition 1.9 then implies that $V''T''$ is weakly compact for any wcc operator $V \varepsilon (Y,Z)$.

<u>Corollary</u> 1.13. If $T'' \varepsilon L(X'',Y'')$ is weak* compact and $V \varepsilon L(Y,Z)$ is wcc, then $V''T''$ is weakly compact operator.

For completeness, the following results concerning weak Cauchy and weak* compact operators and their adjoints are included.

If $T \varepsilon L(X,Y)$ is weak Cauchy, then $T''$ is not necessarily weak Cauchy. Since if $T''$ were weak Cauchy, then by Corollary 1.12 $T''$ is weak* compact. The example following Corollary 1.11 shows that this is not true.

If $T \varepsilon L(X,Y)$ is weak Cauchy, then $T'$ is not necessarily weak Cauchy. The identity operator on $c_0$ is weak Cauchy since $c_0$ is almost reflexive, but the identity operator on $\ell_1$ is not weak Cauchy since $\ell_1$ is not almost reflexive.

If $T \varepsilon L(X,Y)$, then $T''$ weak* compact does imply $T$ is weak* compact, assuming the range space $Y$ is a conjugate space. $T''$ weak* compact implies $T$ is weak Cauchy; hence, $T$ is weak* compact. However, the converse is not true. Consider the identity operator $i: \ell_1 \to \ell_1$. $i$ is weak* compact since $c_0$ separable implies the unit disk of $\ell_1$ is weak* sequentially compact. But $i''$ is not weak* compact since in $m'$ weak* convergence is equivalent to weak convergence; hence, the unit disk in $m'$ is not weak* sequentially compact. Thus, $T$ weak* compact does not imply $T''$ is weak* compact. In particular, $T$ weak* compact does not imply $T'$ is weak* compact since the identity operator on $m$ is weak* compact.

The following questions remain unanswered:

(a) If $T'$ is weak* compact, is $T$ weak* compact?

(b) If $T'$ is weak Cauchy, is $T$ weak Cauchy?

## UC Operators

A continuous linear operator is said to be unconditionally converging (uc) if it maps wuc series into uc series. The intent is to show that the uc operators are a particular class of those operators given in section three of Chapter I.

<u>Theorem</u> 2.1. An operator $T \in L(X,Y)$ is uc if and only if $T''N(X) \subseteq JY$.

Proof: Let $F \in N(X)$. There exists a wuc series $\Sigma x_i$ such that $F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Jx_i$. Since $T$ is uc, $\Sigma Tx_i$ is a uc series and, therefore, there exist a $y \in Y$ such that

$$y = \sigma(Y,Y') - \lim_n \sum_{i=1}^{n} Tx_i$$

and

$$Jy = \sigma(Y'',Y') - \lim_n \sum_{i=1}^{n} JTx_i .$$

$$F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Jx_i$$

$$T''F = \sigma(Y'',Y') - \lim_n \sum_{i=1}^{n} T''Jx_i$$

$$T''F = \sigma(Y'',Y') - \lim_n \sum_{i=1}^{n} JTx_i$$

$$T''F = Jy$$

hence,

$$T''N(X) \subseteq JY.$$

Conversely, suppose $T''N(X) \subseteq JY$ and let $\Sigma x_i$ be a wuc series. Let $\Sigma z_i$ be a subseries of $\Sigma x_i$. Since $\Sigma z_i$ is wuc, there exist $F \in N(X)$ such that $F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Jz_i$. Since $T''N(X) \subseteq JY$, there exists a $y \in Y$ such that $T''F = Jy$; thus,

$$Jy = \sigma(Y'',Y') - \lim_n \sum_{i=1}^{n} T''Jz_i$$

and, therefore,

$$y = \sigma(Y,Y') - \lim_n \sum_{i=1}^{n} Tz_i.$$

This shows that $\Sigma Tx_i$ is subseries convergent in the $\sigma(Y,Y')$ topology, hence uc.

The following facts are known, (9), but they also follow from Chapter I and Theorem 2.1.

(a) If either $X$ or $Y$ is reflexive, every operator in $L(X,Y)$ is uc.

(b) The set of uc operators is closed in the uniform operator topology on $L(X,Y)$.

(c) Linear combinations of uc operators are uc operators. The product of a uc operator and a continuous linear operator is uc.

(d) In the uniform operator topology on $L(X,X)$, the uc operators form a closed two-sided ideal.

Similar to wcc operators, the following result about continuity of the first and second adjoints of a uc operator can be obtained.

Theorem 2.2. Let $T \in L(X,Y)$. The following are equivalent:

(a) $T$ is a uc operator.

(b) $T''N(X) \subseteq JY$.

(c) $T'$ is $\sigma(Y',JY) - \sigma(X',N(X))$ continuous.

(d) $T'$ is a $\sigma(X',N(X))$-compact operator.

(e) $T''$ is $\tau(N(X),X')$-norm continuous.

Proof: This follows from Theorem 3.18 in Chapter I and Theorem 2.1.

The following corollary was first proven by Howard (9) using an entirely different method.

Corollary 2.3. Let $T \in L(X,Y)$. If $T''$ is a uc operator then $T$ is a uc operator.

Proof: By Theorem 2.2, $T'''$ is a $\sigma(X''',N(X''))$-compact operator, so $T'''|Y'$ is a $\sigma(X',N(X''))$-compact operator. Since every $\sigma(X',N(X''))$ compact set in $X'$ is $\sigma(X',N(X))$ compact $T'''|Y' = T'$ is a $\sigma(X',N(X))$-compact operator. Theorem 2.2 then implies that $T$ is a uc operator.

Another application of Theorem 2.2 can be deduced directly from (9, Corollary 3.2.4).

Corollary 2.4. Let $T \in L(X,Y)$. $T''$ is a $\sigma(Y'',N(Y'))$-compact operator if and only if there do not exist epimorphisms $h_1 \in L(X,\ell_1)$ and $h_2 \in L(Y,\ell_1)$ such that $h_1 = h_2T$.

A substantial amount of work has been done recently in representing uc operators with domain in $C(S)$, $S$ a compact Hausdorff space. However, the representation of uc operators with range in $C(S)$ remains untouched. This is probably due to the fact that a suitable topology for the conjugate of a Banach space was missing.

Theorem 2.5. (6) Let $S$ be a compact Hausdorff space and let $T$ be a continuous linear operator from $X$ to $C(S)$. Then there exists a mapping $v: S \to X'$ which is continuous with the $\sigma(X',X)$ topology such that:

(a) $Tx(s) = v(s)x, \quad x \in X, \quad s \in S.$

(b) $\|T\| = \sup_{s \in S} |v(s)|.$

Conversely, if given such a map $v$, then the operator $T$ defined by (a) is a continuous linear operator from $X$ to $C(S)$ with norm given by (b). $T$ is weakly compact if and only if $v$ is $\sigma(X',X'')$ continuous. $T$ is compact if and only if $v$ is continuous with respect to the norm topology on $X'$.

It is also known that $T$ is completely continuous (maps weak Cauchy sequences into norm convergent sequences) if and only if $v$ is $\tau(X',X)$ continuous (Mackey topology).

The objective is to show:

(a) $T$ is uc if and only if $v$ is $\sigma(X',N(X))$ continuous.

(b) $T$ is wdc if and only if $v$ is $\sigma(X',K(X))$ continuous.

Only (a) will be proven. The proof of (b) is similar and simpler.

$v$ is defined by $v = T'\pi$ where $\pi: S \to C'(S)$ is given by $\pi(s)(f) = f(s)$, $f \in C(S)$, $s \in S$. $\pi$ is a homeomorphism of $S$ into a compact subset of $C'(S)$ with the $\sigma(C'(S),C(S))$ topology.

<u>Theorem</u> 2.6. $T$ is uc if and only if $v$ is $\sigma(X',N(X))$ continuous.

Proof: If $T$ is uc, then $T'$ is $\sigma(C'(S),C(S)) - \sigma(X',N(X))$ continuous; thus, $v = T'\pi$ will be $\sigma(X',N(X))$ continuous.

Conversely, suppose $v$ is $\sigma(X',N(X))$ continuous. Let $\Sigma y_n$ be a wuc series in $X$ and $\Sigma x_n$ be an arbitrary subseries of $\Sigma y_n$. It suffices to show that $B = \{\sum_{i=1}^{n} Tx_i \mid n = 1,2, \ldots\}$ is $\sigma(C(S),C'(S))$ conditionally compact, since then $T$ would be uc (i.e. $\Sigma Ty_n$ would be $\sigma(C(S),C'(S))$ subseries convergent; hence, uc). $B \subseteq C(S)$, so by (6, p. 269, Theorem 14) $B$ is $\sigma(C(S),C'(S))$ conditionally compact if and only if $B$ is bounded and quasi-equicontinuous. But $B = TA$, where $A = \{\sum_{i=1}^{n} x_i \mid n = 1,2, \ldots\}$, and $A$ is bounded, so $B$ is bounded.

To see that $B$ is quasi-equicontinuous, let $s_\alpha \to s$ in $S$, let $\epsilon > 0$, and let $\alpha_0$ be given. Since $v$ is $\sigma(X',N(X))$ continuous $v(s_\alpha) \to v(s)$ in $\sigma(X',N(X))$. Since $\Sigma x_n$ is wuc, $\sum_{i=1}^{n} Jx_i \xrightarrow[n]{w^*} F \in N(X)$. Thus, $A_1 = A \cup \{F\}$ is a $\sigma(X'',X')$ compact set and $v(s_\alpha)$ and $v(s)$ are in $C(A_1)$. Since $v(s) \in C(A_1)$, then by Arzela's theorem (6, p. 268, Theorem 11) the convergence $v(s_\alpha) \to v(s)$ is quasi-uniform on $A_1$; hence, quasi-uniform on $A$. Thus, there exist a finite set of indices $\alpha_1, \ldots, \alpha_n \geq \alpha_0$ such that for each $a \in A$,

$$\min_{1 \leq i \leq n} |v(s_{\alpha_i})a - v(s)a| < \epsilon.$$

$$\min_{1\le i\le n} \left|v(s_{\alpha_i}) \sum_{j=1}^{k} x_j - v(s) \sum_{j=1}^{k} x_j\right| < \epsilon \quad \text{for each } k.$$

$$\min_{1\le i\le n} \left|\sum_{j=1}^{k} Tx_j(s_{\alpha_i}) - \sum_{j=1}^{k} Tx_j(s)\right| < \epsilon \quad \text{for each } k.$$

$$\min_{1\le i\le n} \left|f(s_{\alpha_i}) - f(s)\right| < \epsilon \quad \text{for each } f \in B.$$

Thus, $B$ is quasi-equicontinuous; and it, therefore, follows that $B$ is $\sigma(C(S),C'(S))$ conditionally compact; and, hence, $T$ is uc.

It is natural to ask what relationship, if any, exists between uc and wcc operators? To answer this, the following result is needed.

<u>Lemma</u> 2.7. $JX \subseteq N(X) \subseteq K(X) \subseteq X''$.

Proof: The inclusions $JX \subseteq N(X)$ and $K(X) \subseteq X''$ are clear. If $F \in N(X)$, then $F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Jx_i$ for some wuc series $\Sigma x_i$ in $X$. But notice $\sum_{i=1}^{n} Jx_i = Jz_n$ for some $z_n \in X$. Hence, $F = \sigma(X'',X') - \lim_n Jz_n$, which implies $F \in K(X)$ and it follows that $N(X) \subseteq K(X)$.

The next result shows that the class of wcc operators in $L(X,Y)$ is a subclass of the class of uc operators in $L(X,Y)$.

<u>Proposition</u> 2.8. If $T \in L(X,Y)$ is a wcc operator, then $T$ is a uc operator.

Proof: By Lemma 2.7 $N(X) \subseteq K(X)$; thus, $T''N(X) \subseteq T''K(X)$. Since $T$ is wcc $T''K(X) \subseteq JY$ and, therefore, $T''N(X) \subseteq JY$. By Theorem 2.1 $T$ is a uc operator.

It is of interest to know when either $K(X) = N(X)$ or $N(X) = JX$. See Chapter III and also the next theorem. A condition under which both conditions are satisfied is now stated.

<u>Lemma</u> 2.9. If $X$ is weakly complete then $K(X) = N(X) = JX$.

Proof: If $F \varepsilon K(X)$ there exists a sequence $\{x_n\}$ in $X$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. Since $X$ is weakly complete and $\{x_n\}$ is weak Cauchy, there exists an $x \varepsilon X$ such that $x = \sigma(X,X') - \lim_n x_n$. $J$ is continuous with respect to the weak and weak* topologies so $Jx = F$ which implies that $F \varepsilon JX$. $K(X) \subseteq JX$ and Lemma 2.7 then implies $K(X) = N(X) = JX$.

Conditions when a Banach space $X$ has no subspace isomorphic to $c_0$ has been shown to be useful (9). Equivalent conditions are: $X''$ has no subspace isomorphic to $m$, and the identity operator on $X$ is a uc operator (9). It is now shown that $N(X) = JX$ can be added to the list.

<u>Theorem</u> 2.10. The following are equivalent:

(a) $i: X \to X$ is a uc operator ($i$ is the identity operator).

(b) $N(X) = JX$.

(c) $X$ has no subspace isomorphic to $c_0$.

(d) $X''$ has no subspace isomorphic to $m$.

Proof: Suppose $i: X \to X$ is a uc operator. By Theorem 2.1, $N(X) \subseteq JX$ and Lemma 2.7 then implies that $N(X) = JX$. Conversely, if $N(X) = JX$, then $i''N(X) \subseteq JX$ and Theorem 2.1 implies $i: X \to X$ is a uc operator.

The next lemma can be proven using a technique due to A. Pelczynski, (22, p. 446). A proof will not be provided here since it is extremely long and easily verified by using Pelczynski's technique.

Lemma 2.11. Let $i$ be the identity map of a subspace $Y$ into $X$. If $F \in N(X)$ and $G \in K(Y)$ are such that $i''G = F$, then $G \in N(Y)$.

In (16) Theorem 2.12 is proven. It is stated without proof in this paper, then a similar result is established for $N(X)$ and $N(Y)$. In the next two theorems let $Y$ be a subspace of a Banach space $X$ and let $i$ be the identity map of $Y$ into $X$.

Theorem 2.12. (16) $i''K(Y) = (i''Y'') \cap K(X)$.

Theorem 2.13. $i''N(Y) = (i''Y'') \cap N(X)$.

Proof: First observe that $i''$ is an isometry from $Y''$ into $X''$ and if $F = i''G$ and $G \in Y''$, then $F(f) = G(f|Y)$ for all $f \in X'$. Suppose $F \in i''N(Y)$, then there exists a wuc series $\Sigma z_i$ in $Y$ such that if $y_n = \sum_{i=1}^{n} z_i$ then $G = (Y'',Y') - \lim_n Jy_n$ and $i''G = F$. So $F(f) = G(f|Y) = \lim_n Jy_n (f|Y) = \lim_n f(y_n)$ for every $f \in X'$ and, hence, $F \in (i''Y'') \cap N(X)$. It then follows that $i''N(Y) \subseteq (i''Y'') \cap N(X)$.

Conversely, if $F \in (i''Y'') \cap N(X)$ then there exists a $G$ in $Y''$ such that $i''G = F$ and also $F \in N(X)$. Since $N(X) \subseteq K(X)$, $(i''Y'') \cap N(X) \subseteq (i''Y'') \cap K(X) = i''K(Y)$. So $F \in i''K(Y)$, i.e. $G \in K(Y)$. $F \in N(X)$, $G \in K(Y)$, and $i''G = F$ imply by Lemma 2.11 that $G \in N(Y)$; hence, $F \in i''N(Y)$ and it follows that $(i''Y'') \cap N(X) \subseteq i''N(Y)$.

Corollary 2.14. Let $Y$ be a subspace of $X$. If $N(X) = JX$, then $N(Y) = JY$.

Proof: $i''N(Y) = (i''Y'') \cap N(X) = (i''Y'') \cap JX = i''JY$.

The next easily verified result shows that it is sufficient to consider only separable subspaces to obtain the converse.

Corollary 2.15. $N(X) = JX$ if and only if $N(Y) = JY$ for each separable subspace $Y$ of $X$.

If $T \in L(X,Y)$ is a uc operator, then $T''N(X) \subseteq JY$. Hence, if $N(X) = X''$, then $T$ is weakly compact. The next corollary describes this behavior with respect to subspaces.

Corollary 2.16. If $X$ is a Banach space such that $N(X) = X''$, then $N(Y) = Y''$ for all subspaces $Y$ of $X$.

Proof: $i''N(Y) = (i''Y'') \cap N(X) = (i''Y'') \cap X'' = i''Y''$.

If $N(X) = X''$, then $X$ has property V. The above corollary shows that if $N(X) = X''$, every subspace of $X$ has property V; although in general not every subspace of a space with property V has property V. Two examples are now given, one where $N(X) \neq X''$ and a non-reflexive one where $N(X) = X''$.

Consider the space $C[0,1]$. In this case $N(C[0,1]) \neq C[0,1]''$. $\ell_1$ is isomorphic to a subspace of $C[0,1]$, so if $N(C[0,1])$ equals $C[0,1]''$, then $N(\ell_1) = \ell_1''$; thus, $\ell_1$ would then have property V which it does not. Note that $C[0,1]$ does have property V; hence, property V does not imply $N(X) = X''$.

Since $K(c_0) = c_0''$, and $K(c_0) = N(c_0)$ as will be seen in Chapter III, it follows that $N(c_0) = c_0''$.

If $N(X)$ can be shown to be weak* sequentially closed, the next two corollaries provide useful information concerning the space $X$.

Corollary 2.17. Let $X$ be a Banach space, $N(X)$ is weak* sequentially closed in $X''$ if and only if $N(Y)$ is weak* sequentially closed in $Y''$ for each separable subspace $Y$ of $X$.

Proof: Let $X_1 = N(X)$, $X_2$ be the weak* sequential closure of $X_1$ in $X''$, $Y_1 = N(Y)$, and $Y_2$ be the weak* sequential closure of $Y$ in $Y''$. Since $i''Y_2 \subseteq i''Y''$ and $i''Y_2 \subseteq X_2$, it follows that $i''Y_2 \subseteq (i''Y'') \cap X_2$. Suppose $N(X)$ is weak* sequentially closed in $X''$, then $X_1 = X_2$. For an arbitrary subspace $Y$ of $X$ by Theorem 2.13, $i''Y_2 \subseteq (i''Y'') \cap X_2 = (i''Y'') \cap X = i''Y_1$, so $Y_1 = Y_2$ which implies that $N(Y)$ is weak* sequentially closed in $Y''$.

Conversely, suppose that $Y_1 = Y_2$ for each separable $Y$ in $X$. Let $F = \sigma(X'',X') - \lim_n F_n$ where $\{F_n\} \subseteq X$. For each $n$ there is a wuc series $\Sigma y_{nk}$ with partial sum sequence $\{x_{nk}\}_{k=1}^{\infty}$ in $X$ such that $F_n = \sigma(X'',X') - \lim_k Jx_{nk}$. If $Z$ is the subspace of $X$ spanned by $\{y_{nk}: n,k = 1,2, \dots\}$, and if $i$ is the identity map from $Z$ into $X$, then there exists $G, G_1, G_2, \dots$ in $Z''$ such that $G_n = \sigma(Z'',Z') - \lim_k Jx_{nk}$ and $G = \sigma(Z'',Z') - \lim_n G_n$. Now each $G_n \in N(Z)$ since $\Sigma y_{nk}$ is wuc in $X$, so by the Hahn-Banach theorem $\Sigma y_{nk}$ is wuc in $Z$. Thus, $G_n = (i'')^{-1} F_n \in Z_1$ and $G = (i'')^{-1}F$. Since $Z$ is separable, by hypothesis $Z_1 = Z_2$ so it follows that there exists a wuc series $\Sigma t_n$ in $Z$ with partial sum sequence $\{z_k\}$ such that $G = \sigma(Z'',Z') - \lim_k Jz_k$; hence, $F(f) = G(f|Z) = \lim_k f(z_k)$ for every $f \in X'$ so $F = \sigma(X'',X') - \lim_k Jz_k$ and thus $F \in X_1$. Hence, $X_1 = X_2$.

Chapter III will show the usefulness of the condition $K(X) = N(X)$, and McWilliams (16) has studied the implications of $K(X)$ being weak* sequentially closed. With these two facts in mind, the following is given.

Corollary 2.18. Let $X$ be a Banach space. $N(X)$ is weak* sequentially closed if and only if $K(X) = N(X)$ and $K(X)$ is weak* sequentially closed.

Proof: If $N(X)$ is weak* sequentially closed, then since $JX \subseteq N(X)$ and $K(X)$ is the weak* sequential closure of $JX$, it follows that $K(X) \subseteq N(X) \subseteq K(X)$. Hence, $K(X) = N(X)$. If $N(X)$ is weak* sequentially closed, then $K(X)$ will be weak* sequentially closed since $K(X) = N(X)$. The converse is obvious.

# CHAPTER III

## PROPERTY V AND THE DIEUDONNE PROPERTY

Operators defined on a Banach space appropriately give some characterizations and/or properties of the respective Banach space. Two such properties of interest (in this chapter) are property V and the Dieudonne property. The purpose, therefore, is to closely scrutinize these properties: conditions for a space to possess or to lack this property, permanence of the property, and other characteristics.

### Property (u)

During an investigation of the Dieudonne property and property V, it was natural to inquire whether or not there was a connection between the two. As is shown, property (u) provides a connecting link. Thus, property (u) merits consideration.

<u>Definition</u> 1.1. A Banach space $X$ is said to have property (u) if for every weak Cauchy sequence $\{x_n\}$ there exists a wuc series $\Sigma u_i$ such that $\{x_n - \sum_{i=1}^{n} u_i\}$ converges weakly to $0$.

A. Pelczynski (18) introduced the concept of a space possessing property (u) and gave several results without proof. Simple proofs are provided in this paper. First a characterization of property (u) is provided by using the sets $K(X)$ and $N(X)$.

Theorem 1.2. A Banach space $X$ has property (u) if and only if $K(X) = N(X)$.

Proof: Assume $X$ has property (u). If $F \in K(X)$ there exists a weak Cauchy sequence $\{x_n\}$ in $X$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. Since $X$ has property (u) there exists a wuc series $\Sigma u_i$ in $X$ such that $\{x_n - \sum_{i=1}^{n} u_i\}$ converges weakly to 0. Hence, $F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Ju_i$ and, therefore, $F \in N(X)$. $K(X) \subseteq N(X)$ implies $K(X) = N(X)$ since $N(X)$ is always a subset of $K(X)$.

Conversely, assume $K(X) = N(X)$ and let $\{x_n\}$ be a weak Cauchy sequence in $X$. There exists an $F \in K(X)$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. Since $K(X) = N(X)$, $F \in N(X)$ which implies there exists a wuc series $\Sigma u_i$ in $X$ such that $F = \sigma(X'',X') - \lim_n \sum_{i=1}^{n} Ju_i$; thus, $\{X_n - \sum_{i=1}^{n} u_i\}$ converges weakly to 0.

Weak completeness and property (u) are closely related as is illustrated by the following.

Corollary 1.3. (18) Every weakly complete Banach space has property (u). In particular, every reflexive space has property (u).

Proof: Recall $JX \subseteq N(X) \subseteq K(X)$. If $X$ is weakly complete $JX = K(X)$, so $N(X) = K(X)$.

Corollary 1.4. (18) If in a space $X$ having property (u) wuc series are uc series then $X$ is weakly complete.

Proof: By Theorem 2.8 in Chapter II, $N(X) = JX$ since all wuc series are uc series. Since $X$ has property (u), $N(X) = K(X)$. Thus, $K(X) = JX$, i.e. $X$ is weakly complete.

_Corollary_ 1.5. (18) If $X$ has property (u), then $X$ is weakly complete if and only if no subspace of $X$ is isomorphic to $c_0$.

Proof: If $X$ is weakly complete, $K(X) = N(X) = JX$; thus, $N(X) = JX$. By Theorem 2.8 in Chapter II, no subspace of $X$ is isomorphic to $c_0$. Conversely, if $N(X) = JX$ and $N(X) = K(X)$, then $JX = K(X)$.

Property (u) can be considered as an inherited property.

_Theorem_ 1.6. (18) If $X$ has property (u), then every subspace $Y$ has property (u).

Proof: If $X$ has property (u), then $K(X) = N(X)$. By Theorem 2.9 and Theorem 2.10 in Chapter II,

$$i''N(Y) = (i''Y'') \cap N(X) = (i''Y'') \cap K(X) = i''K(Y),$$

so $N(Y) = K(Y)$; thus, $Y$ has property (u).

_Corollary_ 1.7. (18) The space $X$ has property (u) if and only if every separable subspace has property (u).

Proof: Let $Y$ be a separable subspace. If $N(Y) = K(Y)$, it follows from Theorem 2.9 and Theorem 2.10 in Chapter II that

$$(i''Y'') \cap K(X) = i''K(Y) = i''N(Y) = (i''Y'') \cap N(X),$$

so $(i''Y'') \cap K(x) = (i''Y'') \cap N(X)$ for every separable subspace $Y$. If $F \varepsilon K(X)$, then there exists a sequence $\{x_n\}$ in $X$ such that $F = \sigma(X'',X') - \lim_n Jx_n$. If $X$ is the separable subspace generated by $\{x_n\}$, then $F \varepsilon i''Y''$; so $F \varepsilon (i''Y'') \cap K(X) = (i''Y'') \cap N(X)$, and it follows that $F \varepsilon N(X)$. Thus, $K(X) = N(X)$. The converse follows from Theorem 1.6.

When does a space with a basis possess property (u)? To partially answer this, the following definition is needed.

Definition 1.8. A basis $\{x_n\}$ of a Banach space is said to be unconditional if every convergent series of the form $\Sigma\alpha_i x_i$ is unconditionally convergent.

If a Banach space $X$ has an unconditional basis, then $X$ has property (u) (22, p. 445). This fact yields many examples of spaces with property (u). In addition, with Theorem 1.6 the following result is obtained.

Corollary 1.9. (18) Every subspace of a space with an unconditional bases has property (u).

The natural bases for $c_0$ and $\ell^p$, $p \geq 1$, are unconditional bases so $c_0$ and $\ell^p$, $p \geq 1$, have property (u). The Haar basis of $L^p([0,1])$, $1 < p < \infty$, is an unconditional basis so $L^p([0,1])$, $1 < p < \infty$, has property (u). $L^1([0,1])$ has property (u) since it is weakly complete, but $L^1([0,1])$ does not have an unconditional basis, (22, p. 441, Theorem 15.3). Thus, property (u) does not imply the existence of an unconditional basis.

Certain spaces have been seen to have property (u). What spaces do not have property (u)? The following is an aid in answering this question.

<u>Proposition</u> 1.10. Let $X$ be a non-reflexive Banach space such that $X''$ is separable. Then $X$ does not have property (u).

Proof: Since $X''$ is separable, $X$ contains no subspace isomorphic to $c_0$; hence, $N(X) = JX$. Suppose $X$ has property (u), then $K(X) = N(X) = JX$ which implies that $X$ is weakly complete. $X'$ separable and $X$ weakly complete imply that $X$ is reflexive, (4, p. 58), a contradiction. So $X$ does not have property (u).

Any of James' spaces (11), $B_n^{(m)}$, $n \geq 3$, $m \leq n - 2$, do not have property (u).

$C[0,1]$ does not have property (u). Thus, $C[0,1]$ can not be embedded in a space with property (u). In particular, $C[0,1]$ can not be embedded in a Banach space with an unconditional basis. Since every separable Banach space is isometrically isomorphic to a subspace of $C[0,1]$, $C[0,1]$ contains a subspace isometrically isomorphic to $B_3$. If $C[0,1]$ did have property (u), then $B_3$ would have property (u) which it does not.

Many of the classical Banach spaces have been shown to possess property (u). In view of Theorem 3.6, their possession of property V is equivalent to their possession of the Dieudonne property. However, other classical Banach spaces, such as $C[0,1]$, separate inquiries into their possession or lack of property V and the Dieudonne property must be conducted.

An explication of the Dieudonne property and property V are the ensuing topics.

## The Dieudonne Property

A Banach space X is said to have the Dieudonne property if every wcc operator $T \in L(X,Y)$ is weakly compact where Y is an arbitrary Banach space. The following theorem provides an equivalent condition for a space X to have the Dieudonne property independent of any reference to operators and their range space. A. Grothendieck proved a similar result for locally convex spaces (8); however, condition (c) in this paper does not require the set to be convex and balanced as does Grothendieck's theorem. In any case, it follows directly from Theorem 1.1 in Chapter II and Theorem 3.17 in Chapter I.

Theorem 2.1. (8) The following conditions on X are equivalent:

(a) X has the Dieudonne property.

(b) Any $T \in L(X,Y)$ such that $T''K(X) \subseteq JY$ satisfies $T''X'' \subseteq JY$.

(c) Any $\sigma(X',K(X))$ compact set in $X'$ is also $\sigma(X',X'')$ compact.

When a Banach space X is reflexive, every continuous linear operator on X is weakly compact. Thus, the next observation by A. Grothendieck (8) follows.

Lemma 2.2. (8) Every reflexive space has the Dieudonne property.

Proposition 2.3. (8) A weakly complete Banach space X has the Dieudonne property if and only if it is reflexive.

Proof: If X has the Dieudonne property and i is the identity map from X to X, then i is a wcc operator and, therefore, weakly compact. If i is weakly compact, then X has a weakly compact unit disk and, hence, reflexive. The converse is Lemma 2.2.

The Banach space $\ell_1$ does not have the Dieudonne property since it is weakly complete and not reflexive. $L^1$ is also weakly complete; hence, it will have the Dieudonne property only when it is reflexive, i.e. when it is finite dimensional.

The natural question of the permanence of the Dieudonne property under inductive limits, projective limits, direct products, direct sums, quotient spaces, and subspaces arises. A. Grothendieck (8) stated without proof that for Banach spaces the Dieudonne property is preserved for direct factors, products, and quotients. Not only will these results be shown, but an extensive inspection of the permanence properties is conducted.

If E is the inductive limit or projective limit of Banach spaces, the space E is not necessarily a Banach space. However, E would be a separated locally convex topological vector space (LCTVS). Hence, the following definition would be appropriate in dealing with such situations.

Definition 2.4. A LCTVS E is said to have the Dieudonne property if every wcc operator $T \in L(E,Y)$ is weakly compact where Y is an arbitrary Banach space.

Theorem 2.1 holds for LCTVS with the additional assumption that the set in condition (c) is also equicontinuous, convex, and balanced (8).

Proposition 2.5. If a space E has (lacks) the Dieudonne property with a compatible topology, then E with any other compatible topology has (lacks) the Dieudonne property.

Proof: Condition (c) of Theorem 2.1 is a condition on $E'$, and all compatible topologies for E have the same conjugate.

Since $\ell_1$ with the norm topology does not have the Dieudonne property, $\ell_1$ with the weak topology does not have the Dieudonne property.

Definition 2.6. Let $\{(E_\alpha,f_\alpha)\}$ be a family of LCTVS E and $f_\alpha$ be linear maps from $E_\alpha$ into a space E such that $\cup f_\alpha(E_\alpha)$ spans E. Furnish E with the weakest convex topology so that all the $f_\alpha$'s are continuous. E with this topology is said to be the inductive limit of the $E_\alpha$'s. E is said to be the regular inductive limit if for every bounded set B in E there exists a $\beta$ such that B is bounded in $E_\beta$.

Proposition 2.7. Suppose E is the regular inductive limit of $\{(E_\alpha,f_\alpha)\}$. If each $E_\alpha$ has the Dieudonne property, then E has the Dieudonne property.

Proof: Let T be a wcc operator from E into a Banach space Y and let B be a bounded set in E. There exist a $\beta$ such that $f_\beta^{-1}(B)$ is bounded in $E_\beta$. Let $T_\beta = T \circ f_\beta$. T is wcc so $T_\beta$ is wcc, and since $E_\beta$ has the Dieudonne property $T_\beta$ is weakly compact. Hence, $T_\beta(f_\beta^{-1}B)$ is weakly relatively compact, but $T_\beta = T \circ f_\beta$, so $T_\beta(f_\beta^{-1}B) = T(B)$ which implies that T is weakly compact. It then follows that E has the Dieudonne property.

For LCTVS the Dieudonne property is not necessarily preserved for quotient spaces; in particular, the property is not necessarily preserved for inductive limits. In (21, p. 195, problem 20) there is given a Montel space E which has a quotient space isomorphic to $\ell_1$. Since E is a Montel space, E is reflexive and, hence, has the Dieudonne property. However, $\ell_1$ does not have the Dieudonne property. The natural map T from E onto the quotient space isomorphic to $\ell_1$ defines $\ell_1$ as the inductive limit of E.

Proposition 2.8. If a Banach space X is the inductive limit of a finite number of Banach spaces $X_n$ with the Dieudonne property, then X has the Dieudonne property.

Proof: Let T be a wcc operator from X to a Banach space Y. Define $T_n = T \circ f_n$ where $f_n$ maps $X_n$ into X. T is wcc so $T_n$ is wcc and, therefore, $T_n$ is weakly compact. Let B be a bounded set in X; then $f_n^{-1}(B)$ is relatively weakly compact since $T_n$ is weakly compact and the union is finite. But $\cup T_n(f_n^{-1}B) = TB$, so T is weakly compact and, thus, X has the Dieudonne property.

Each quotient space is the finite inductive limit of the space.

Corollary 2.9. (8) Every quotient space of a Banach space with the Dieudonne property has the Dieudonne property.

Banach space may be replaced by normed linear space in Proposition 2.8 and Corollary 2.9 since the completeness is not used in the proof.

Definition 2.10. Let $E$ be a vector space. For each $\alpha$ in some index set, let $f_\alpha$ be a linear map from $E$ to a LCTVS $E_\alpha$, such that $\cap f_\alpha^{-1}(0) = \{0\}$. Furnish $E$ with the weakest topology such that each $f_\alpha$ is continuous. Then $E$ is said to be the projective limit of the family $\{(E_\alpha, f_\alpha)\}$.

If $A = \{f_\alpha\}$, then in the terminology of (23) the topology on $E$ is $\sigma(E,A)$.

A subspace $E$ of a LCTVS $F$ is a special case of a projective limit. Consider the natural embedding map $i$ from $E$ into $F$; then the relative topology on $E$ is the weakest topology making $i$ continuous; hence, $E$ is the projective limit of $F$.

The Dieudonne property is not necessarily preserved for subspaces. In particular, projective limits do not necessarily preserve the Dieudonne property. $\ell_1$ is linearly isometric to a subspace of $C(S)$, $S$ a compact Hausdorff space. $C(S)$ has the Dieudonne property, but $\ell_1$ does not have the Dieudonne property.

Proposition 2.11. (8) Suppose $E$ is the direct product of a family $\{E_\alpha\}$ of LCTVS. If each $E_\alpha$ has the Dieudonne property, then $E$ has the Dieudonne property.

Proof: Let $T$ map $E$ into a Banach space $Y$ be a continuous operator, and let $h_\alpha$ be the natural map of $E_\alpha$ into $E$. Then $T_\alpha = T \circ h_\alpha$ is continuous from $E_\alpha$ into $Y$. Since $Y$ is a Banach space, the continuity of $T$ entails that $T_\alpha = 0$ for all but a finite set of indices $\alpha$. It, therefore, suffices to handle the case of a finite product. It is easy to see that for a finite set of LCTVS

$F_1, \ldots, F_n$, $E = F_1 x \ldots x F_n$, that T mapping F into a Banach space Y is weakly compact (wcc) if and only if $T_k = T \circ h_k$ is weakly compact (wcc) for each $k = 1, \ldots, n$. It then follows that the Dieudonne property is preserved under direct products.

The direct sum of spaces with the Dieudonne property has the Dieudonne property. The proof is analogous to Proposition 2.11. It can also be seen that, although the Dieudonne property is not preserved for subspaces, a space E has the Dieudonne property if and only if every complemented subspace has the Dieudonne property.

The question of whether a Banach space has the Dieudonne property can be converted to a question of whether a space of continuous functions has the Dieudonne property. The space C(S), S a compact Hausdorff space, has the Dieudonne property (8). Let C(S;X) denote the set of continuous functions on a compact Hausdorff space S with values in X. In (19) it is shown that C(S;X), for reflexive X, has property V. In the next section it is shown that property V implies the Dieudonne property; hence, for reflexive X, C(S;X) has the Dieudonne property if and only if X has the Dieudonne property. The following theorem provides a partial answer for arbitrary X.

Theorem 2.12. If C(S;X) has the Dieudonne property, then so does X.

Proof: Define $T_s \in L(C(S;X),X)$ by $T_s(f) = f(s)$. $T_s$ is an onto map, so C(S;X) has a quotient space isomorphic to X and, thus, by Corollary 2.9, X has the Dieudonne property.

It is an open question if the converse to the previous theorem is true. However, in a special case it is true. The following definition is needed to obtain a characterization of the Dieudonne property in this special case.

Definition 2.13. Denote by $c_X$ the Banach space of all $X$ valued, convergent sequences, $\{x_n\}$ equipped with the norm $\|\{x_n\}\| = \sup \{\|x_n\| : n \geq 1\}$.

$c_X$ is the space $C(S;X)$ where $S$ is the one-point compactification of the positive integers. A continuous linear operator $T$ mapping $c_X$ into $Y$ has a unique representation in the form

$$y'(T\zeta) = (T_0(\lim x_n))(y') + \sum_{n=1}^{\infty} y'(T_n x_n) \tag{1}$$

where $\zeta = \{x_n\} \in c_X$, $y' \in Y'$, $T_0$ maps $X$ into $Y''$, and $T_n$ maps $X$ into $Y$ are continuous linear operators, and the series on the right hand side of equation (1) satisfies $\sum_{n=1}^{\infty} \|y'(T_n)\| < \infty$, (7, p. 738).

Theorem 2.14. $T \in L(c_X, Y)$ is wcc if and only if:

(a) each $T_i$ in equation (1) is wcc, and

(b) the series $\Sigma T_i$ is such that $\Sigma T_i x_i$ converges for each sequence $\{x_i\} \subseteq X$, $\|x_i\| \leq 1$.

Proof: Suppose that $T$ is wcc. Define a continuous linear operator $P_i$ $(i \geq 1)$ from $X$ into $c_X$ by $P_i x = \{\zeta_{in} x\}_{n=1}^{\infty}$. Let $\{x_i\} \subseteq X$ be such that $\|x_i\| \leq 1$. $\{\sum_{i=1}^{n} P_i x_i\}$ is weak Cauchy, (2, Lemma 2.11). Since $T$ is wcc,

$$\{\sum_{i=1}^{n} TP_i x_i\}$$

is weakly convergent. But $TP_i x_i = T_i x_i$ by equation (1) so $\{\sum_{i=1}^{n} T_i x_i\}$ is weakly convergent. Thus, $\Sigma T_i x_i$ is uc and condition (b) is satisfied.

For $i \geq 1$ and $x \in X$, equation (1) yields $TP_i x = T_i x$. Since $T$ is wcc, $T_i$ is wcc.

Define $Q$ mapping $X$ into $c_X$ by $Qx = (x,x,x, \ldots)$. Condition (b) and equation (1) together imply $T_0 x = TQx - \sum_{i=1}^{\infty} T_i x$ for all $x \in X$, i.e. $T \in L(X,Y)$. Condition (b) and (1, Theorem 2) show that $T$ is the limit in the uniform operator topology on $L(X,Y)$ of the sequence of wcc operators $\{TQ - \sum_{i=1}^{n} T_i\}$. By Proposition 1.3 in Chapter II, $T$ is wcc.

Conversely, suppose $T$ satisfies conditions (a) and (b). For each $n$, let $S_n$ mapping $c_X$ into $Y$ be defined by

$$S_n \zeta = T_0(\lim x_n) + \sum_{i=1}^{n} T_i x_i, \quad \zeta = \{x_n\}.$$

$T_0$ is wcc, and lim is a continuous function so $T_0 \circ \lim$ is wcc by Proposition 1.2 in Chapter II. So each $S_n$ is wcc since $T_0 \circ \lim, T_1, \ldots, T_n$ are all wcc operators. Applying (1, Theorem 2) and condition (b), it follows that $T$ is the limit of $\{S_n\}$ in the uniform operator topology on $L(c_X,Y)$; hence, $T$ is a wcc operator.

Corollary 2.15. A Banach space $X$ has the Dieudonne property if and only if $c_X$ has the Dieudonne property.

Proof: If $c_X$ has the Dieudonne property, then by Theorem 2.12, $X$ has the Dieudonne property.

Conversely, suppose $X$ has the Dieudonne property and $T \varepsilon L(c_X,Y)$ is wcc. Each $T_i$ in equation (1) is wcc by Theorem 2.14. $X$ has the Dieudonne property so each $T_i$ is weakly compact; hence, by (1, Corollary 1) $T$ is weakly compact.

The following theorem will list some of the conditions that imply that a Banach space has the Dieudonne property. It is not intended to be exclusive.

Theorem 2.16. $X$ has the Dieudonne property if any of the following conditions are satisfied:

(a) $X$ has a norm closed subspace $Y$ such that $Y''$ is separable and $X/Y$ is reflexive.

(b) $X''/JX$ is separable.

(c) $K(X) = X''$.

(d) The unit disk of $X''$ is weak* sequentially compact.

(e) $X$ is almost reflexive.

(f) $X'$ is WCG (weakly compactly generated).

(g) $X$ has property V.

(h) $C(S;X)$ has the Dieudonne property.

Proof: (a) implies (b) implies (c): By (15). (c) implies the Dieudonne property follows from Theorem 1.1 in Chapter II. (d) implies (e): If $\{x_n\}$ is a bounded sequence in $X$, then $\{Jx_n\}$ is a bounded sequence in $X''$; thus, it has a weak* convergent subsequence $\{Jx_{n_i}\}$; hence $\{x_{n_i}\}$ is a weak Cauchy subsequence. (f) implies (d): By (20). (g) implies the Dieudonne property is shown in the next section. (h) implies the Dieudonne property is Theorem 2.12. The demonstration (e) implies the Dieudonne property will complete the theorem. Let $\{x_n\}$ be a bounded sequence in X and let $T \varepsilon L(X,Y)$ be wcc. Since X is almost reflexive there exists a weak Cauchy subsequence $\{y_n\}$ of $\{x_n\}$. T wcc implies $\{Ty_n\}$ is weakly convergent. Hence, $Tx_n$ has a weakly convergent subsequence; thus, T is weakly compact.

## Property V

A Banach space X is said to have property V if every uc operator $T \varepsilon L(X,Y)$ is weakly compact where Y is an arbitrary Banach space. The definition of the set N(X) and the characterization of uc operators allows a new manner by which to view this property. Theorem 3.17 in Chapter I with Theorem 2.1 in Chapter II yield the following important characterization of property V in terms of compact sets rather than in terms of operators.

Theorem 3.1. The following conditions on X are equivalent:

(a) X has property V.

(b) Any $T \varepsilon L(X,Y)$ such that $T''N(X) \subseteq JY$ satisfies $T''X'' \subseteq JY$.

(c) Any $\sigma(X',N(X))$ compact set in X' is also $\sigma(X',X'')$ compact.

A discussion of the permanence of property V is given by Howard (9). An application of Theorem 3.1 is included, but first the following definitions are needed.

Definition 3.2. Let T be a separated locally compact space. $C_0(T)$ is the space of continuous functions x on T such that given $\epsilon > 0$, the set $\{t \in T: |x(t)| \geq \epsilon\}$ is relatively compact in $T$. $C_0(T)$ is a Banach space with norm $\|x\| = \sup\{|x(t)|: t \in T\}$.

Definition 3.3. $M(T)$ is the Banach space of bounded Radon measures on $T$, the norm being $\|\mu\| = \int d|\mu|$ where $T$ is a separated locally compact space.

If $T$ is compact then $C_0(T)$ is the space $C(T)$. Pelczynski (19) showed that $C(T)$, $T$ a compact Hausdorff space, has property V. A simpler alternative proof is presented. Recall that the dual of $C_0(T)$ may be identified with $M(T)$ by associating with each $\mu \in M(T)$ the linear form $x \to \int_T x \, d\mu$ on $C_0(T)$.

Theorem 3.4. (19) For any separated locally compact space $T$, $C_0(T)$ has property V.

Proof: Let $u: C_0(T) \to F$ be a uc operator; and let $F$ be an arbitrary Banach space. Grothendieck (8, Theorem 6) proved that $u$ is weakly compact if and only if $u$ transforms any bounded monotone increasing sequence in $C_0(T)$ into a sequence converging weakly in $F$. If $\{x_n\}$ is a bounded monotone increasing sequence in $C_0(T)$, it suffices to show that $x = \sigma(M(T)', M(T)) - \lim_n x_n$ is in $N(C_0(T))$. Since then $u$ being a uc operator would imply $u''(x) \in JF$ and, hence, $u(x_n)$ converges weakly to some $y \in F$. Define $y_1 = x_1$, $y_2 = x_2 - x_1, \ldots, y_n = x_n - x_{n-1}, \ldots$ . Then $\Sigma y_n$ is a series in $C_0(T)$. If $\mu \in M(T)$, then $\mu(x_n - \sum_{i=1}^{n} y_i) = \mu(0) = 0$; hence, $\{x_n - \sum_{i=1}^{n} y_i\}$ converges weakly to 0. Since $\{x_n\}$ is a weak Cauchy sequence, $\lim_n \mu(x_n) < \infty$ for each $\mu \in M(T)$. To show $\Sigma y_n$ is a wuc series, it suffices to only consider positive Radon measures, so let

$\mu$ be an arbitrary positive Radon measure. Since $x_n(t) - x_{n-1}(t) \geq 0$ for all $t \in T$, $|\mu(y_n)| = \mu(y_n)$ and, thus,

$$\lim_n \sum_{i=1}^{n} |\mu(y_i)| = \lim_n \sum_{i=1}^{n} \mu(y_i) = \lim_n \sum_{i=1}^{n} \int_T (x_i - x_{i-1})d\mu$$
$$= \lim_n \mu(x_n) < \infty.$$

Hence, $\Sigma y_n$ is indeed a wuc series. Now since $\{x_n - \sum_{i=1}^{n} y_i\}$ converges weakly to 0, the weak limit point of $\{x_n\}$ is in $N(C_0(T))$.

The relationship between property V and the Dieudonne property is given in the next two propositions.

Proposition 3.5. A Banach space $X$ with property V has the Dieudonne property.

Proof: If $T$ is a wcc operator on $X$, then by Proposition 2.8 in Chapter II, $T$ is a uc operator. Since $X$ has property V, $T$ is weakly compact, and it follows that $X$ has the Dieudonne property.

The converse of Proposition 3.5 is not true. James (11) defined a Banach space $B_3$ such that $B_3$, $B_3'$, and $B_3''$ are separable, but $B_3'''$ is not separable, and $B_3'' = B_3 \oplus \ell_1$. Since $B_3''$ is separable, $B_3'$ will be almost reflexive and, hence, $B_3'$ has the Dieudonne property. But the identity map $i$ on $B_3'$ is a uc operator since if $i$ were not a uc operator, then by Theorem 2.10 in Chapter II $B_3'$ would contain a subspace isomorphic to $c_0$. But $B_3'$ is a conjugate Banach space so $B_3'$ would then contain a subspace isomorphic to $m$ which would imply that $B_3'$ is not separable; a contradiction. If $i$ were weakly compact, then the unit disk of $B_3'$ would be weakly compact; hence, $B_3'$ would be reflexive which it is not. Thus, $B_3'$ does not have property V.

The above example also shows that the converse to Proposition 2.8 in Chapter II is not true. $i$ as above is a uc operator, but not a wcc operator.

<u>Proposition</u> 3.6. If a Banach space $X$ has property (u), then the following are equivalent:

(a) $X$ has property V.

(b) $X$ has the Dieudonne property.

Proof: (a) always implies (b) by Proposition 3.5. Assume (b) and let $T \in L(X,Y)$ be a uc operator. Thus, $T''N(X) \subseteq JY$, but $X$ has property (u) so $K(X) = N(X)$ and, therefore, $T''K(X) \subseteq JY$; hence, $T$ is a wcc operator. Since $X$ has the Dieudonne property, $T$ is weakly compact; hence, $X$ has property V.

A Banach space may have the Dieudonne property and property V and not have property (u). $C(S)$, $S$ a compact Hausdorff space, is an example of such a space.

The basic characterization used by Pelczynski (19) to study property V is " $X$ has property V if and only if every wuc-limited set in $X$ is weakly compact."

<u>Definition</u> 3.7. $A \subseteq X'$ is wuc-limited if $\lim_n \sup_A x'x_n = 0$ for every wuc series $\Sigma x_n$ in $X$.

Theorem 3.9 answers the question of what topology on $X'$ will induce compactness on wuc-limited sets. But first the following proposition is needed.

<u>Proposition</u> 3.8. $T \in L(X,Y)$ is uc if and only if $T'$ maps bounded sets into wuc-limited sets.

Proof: Let $T$ be uc, $\Sigma x_n$ be wuc, and $A$ a bounded set in $Y'$. Since $T$ is uc, $\Sigma Tx_n$ is uc. Hence, $Tx_n \to 0$ and, thus,

$$\lim_n \sup_A y'(Tx_n) = \lim_n \sup_{T'A} T'y'(x_n) = 0;$$

therefore, $T'A$ is wuc-limited.

Conversely, suppose $A$ is bounded in $Y'$, $\Sigma x_n$ is wuc, and $\lim_n \sup_{T'A} T'y'(x_n) = 0$. Then $\lim_n \sup_A y'(Tx_n) = 0$, and it follows that $\Sigma Tx_n$ is a uc series by a result of McArthur (13, Condition (H)). Thus, $T$ is a uc operator.

Let $K$ be a bounded set in $X'$. Let $p$ be the gauge of the set $\{x \in X: |a(x)| \leq 1$ for all $a \in K\}$. Then $p'$ is a norm on $X/\text{Ker } p$ where $p'(\hat{x}) = p(x)$ and $\hat{x} = x + \text{Ker } p$. Let $T$ be the natural map from $X$ to $Y = X/\text{Ker } p$. As in Theorem 3.15 in Chapter I, it can be shown that $K$ is wuc-limited if and only if $T'$ maps bounded sets into wuc-limited sets. Since the proof is almost identical, it is omitted. With the above notation and observations, the problem is solved.

<u>Theorem</u> 3.9. The following are equivalent.

(a) $K$ is wuc-limited.

(b) $T'$ maps bounded sets into wuc-limited sets.

(c) $T$ is uc.

(d) $T'$ is a $\sigma(X',N(X))$-compact operator.

(e) $K$ is $\sigma(X',N(X))$ compact.

Proof: (a) if and only if (b) follows by the remark preceding the theorem. (b) if and only if (c) follows by Proposition 3.8. (c) if and only if (d) follows by Theorem 2.2 in Chapter II. (d) if and only if (e) follows by Theorem 3.15 in Chapter I.

Thus, it is seen that $K \subseteq X'$ is wuc-limited if and only if it is $\sigma(X',N(X))$ compact.

# CHAPTER IV

## CONJUGATE BANACH SPACES WITH WEAK* SEQUENTIALLY COMPACT UNIT DISKS

It is well known that a Banach space is finite dimensional if and only if its unit disk is norm compact and is reflexive if and only if its unit disk is compact in the weak topology. Also the unit disk of a conjugate Banach space is always compact in the weak* topology. Since conditional compactness and sequential compactness are not equivalent in the weak* topology, an inviting question would be: can weak* sequential compactness of the unit disk characterize or be characterized in a Banach space? A result of Banach is that if $X$ is separable, then the unit disk of $X'$ is sequentially compact in the weak* topology. An up-dated study of this is certainly in order. These results provide further conditions for a Banach space to have the Dieudonne property.

### Banach Spaces With H(X) Separable

$H(X)$ denotes the quotient space $X''/JX$.

When $H(X)$ is separable, it is shown that $X$ and $X'$ are almost reflexive, and $X''$ and $X'''$ have weak* sequentially compact unit disks. This implies that $X$ and $X'$ have the Dieudonne property.

Weak* sequential compactness is closely related to the concept of a Banach space being almost reflexive. If a Banach space $X$ is such that its second conjugate has a weak* sequentially compact unit disk, then $X$ is almost reflexive.

For a conjugate Banach space $X$, almost reflexivity is a stronger condition than the weak* sequential compactness of its unit disk. If $X'$ is almost reflexive, then a sequence $\{x'_n\}$ in the unit disk has a $\sigma(X',X'')$ Cauchy subsequence $\{x'_{n_i}\}$; thus, $\{x'_{n_i}\}$ is $\sigma(X',JX)$ Cauchy; hence, $\sigma(X',JX)$ convergent.

<u>Theorem</u> 1.1. (3) If $Y$ is a closed subspace of a Banach space $X$, then $JX + Y^{\perp\perp}$ is a closed subspace of $X''$ and

$$H(X) \cong (JX + Y^{\perp\perp})/JX$$

$$H(X/Y) \cong X''/(JX + Y^{\perp\perp})$$

where $\cong$ means linearly homeomorphic.

It should also be observed that $H(X/Y)$ is linearly homeomorphic to $H(X)/H(Y)$.

<u>Lemma</u> 1.2. $[H(X)]' \cong H(X')$.

Proof: By (5, Theorem 15), $H(X') \cong (JX)^{\perp}$. Since $JX$ is a norm closed subspace of $X''$, it follows that $[H(X)]' \cong (JX)^{\perp}$. So $[H(X)]' \cong (JX)^{\perp} \cong H(X')$; thus, $[H(X)]' \cong H(X')$.

If $H(X)$ is separable, the unit disk of $[H(X)]'$ is weak* sequentially compact. But $[H(X)]' \cong H(X')$, hence, Corollary 1.3.

<u>Corollary</u> 1.3. If $X$ is a Banach space such that $H(X)$ is separable, then the unit disk in $H(X')$ is weak* sequentially compact.

Lacey and Whitley (12) show the usefulness of the almost reflexive property. The following lemma characterizes this property by the use of weak* sequential compactness.

Lemma 1.4. $X$ is almost reflexive if and only if $JS$ is weak* sequentially compact in $X''$ where $S$ is the unit disk in $X$.

Proof: Assume $X$ is almost reflexive. If $\{Jx_n\}$ is a sequence in $JS$, then $\{x_n\}$ is bounded and, therefore, has a weak Cauchy subsequence $\{x_{n_i}\}$. Thus, $\{Jx_{n_i}\}$ is a weak* Cauchy subsequence and by the Banach Steinhaus closure theorem the sequence $\{Jx_{n_i}\}$ converges in the $\sigma(X'',X')$ topology to an element of $X''$.

Conversely, assume $JS$ is weak* sequentially compact in $X''$. If $\{x_n\}$ is a sequence in $S$, then $\{Jx_n\}$ is a sequence in $JS$; hence, there exists a subsequence $\{Jx_{n_i}\}$ which converges in the $\sigma(X'',X')$ topology, and it follows that $\{x_{n_i}\}$ is a weak Cauchy subsequence of $\{x_n\}$.

Many Banach spaces are complemented in their second conjugates (23, p. 214, Problem 29), (5). For those spaces, a characterization is given in the next lemma.

Lemma 1.5. Let a Banach space $X$ be complemented in $X''$. $X''$ has a weak* sequentially compact unit disk if and only if

(a) $X$ is almost reflexive, and

(b) $H(X)$ has a $\sigma(X'',X')$ sequentially compact unit disk.

Proof: Let $\{x''_n\}$ be a bounded sequence in $X''$. Each $x''_n$ is of the form $x''_n = Jx_n + h_n$ where $x_n \in X$ and $h_n \in H(X)$. Since $\{x''_n\}$ is bounded, both sequences $\{x_n\}$ and $\{h_n\}$ are bounded. Thus, there exists a subsequence $\{Jy_n\}$ of $\{Jx_n\}$ which converges in the $\sigma(X'',X')$ topology. If $\{w_n\}$ is the corresponding subsequence obtained from $\{h_n\}$, it also has a $\sigma(X'',X')$ convergent subsequence $\{m_n\}$. Let $\{Jz_n\}$ be the corresponding subsequence of $\{Jy_n\}$. Then $\{Jz_n + m_n\}$ is a $\sigma(X'',X')$ convergent subsequence of $\{Jx_n + h_n\}$, so $X''$ has a weak* sequentially compact unit disk.

Conversely, assume $X''$ has a weak* sequentially compact unit disk. Let $\{x_n\}$ be a bounded sequence in $X$. Then $\{Jx_n\}$ is a bounded sequence in $X''$ so $\{Jx_n\}$ has a $\sigma(X'',X')$ convergent subsequence $\{Jy_n\}$. Thus, $\{y_n\}$ is a weak Cauchy subsequence of $\{x_n\}$ and $X$ is, therefore, almost reflexive. If $\{h_n\}$ is a bounded sequence in $H(X)$, then $\{h_n\}$ is a bounded sequence in $X''$. $\{h_n\}$ has a $\sigma(X'',X')$ convergent subsequence since $X''$ has a weak* sequentially compact unit disk.

A conjugate Banach space is complemented in its second dual (5).

<u>Corollary</u> 1.6. If $X$ is a conjugate Banach space, then the conclusion of Lemma 1.5 holds.

As can easily be seen, properties of $H(X)$ are very useful. For example, one has the following.

<u>Theorem</u> 1.7. If $H(X)$ is separable, then $X'$ is almost reflexive.

Proof: Let $\{f_n\}$ be a bounded sequence in $X'$ and let $M$ be the closed linear span of the sequence $\{f_n\}$. $M$ is separable so $M'$ is separable (15). $M'$ separable implies that $M''$ has a weak* sequentially compact unit disk which in turn implies that $M$ is almost reflexive.

Hence, the bounded sequence $\{f_n\}$ in $M$ has a subsequence $\{g_i\}$ such that $\{g_i\}$ is $\sigma(M,M')$ Cauchy. For each $G \varepsilon X''$, $G|M \varepsilon M'$ so $\{G(g_i)\} = \{(G|M)(g_i)\}$ since $g_i \varepsilon M$. This implies that $\{G(g_i)\}$ is a Cauchy sequence for all $G \varepsilon X''$, and therefore, $\{g_i\}$ is a $\sigma(X',X'')$ Cauchy sequence. Consequently, $X'$ is almost reflexive.

Although characterizations of weak* sequential compactness of the unit disk for an arbitrary conjugate space are sought, the following assists with the delineation of this problem.

Theorem 1.8. If $H(X)$ is separable, then $X'''$ has a weak* sequentially compact unit disk.

Proof: If $H(X)$ is separable, then by Theorem 1.7 $X'$ is almost reflexive. Since $H(X)$ is separable, Corollary 1.3 implies $H(X')$ has a weak* sequentially compact unit disk. $X'$ almost reflexive and the unit disk of $H(X')$ weak* sequentially compact together imply by Corollary 1.6 that $X'''$ has a weak* sequentially compact unit disk.

For $H(X)$ to be separable is very useful. For instance, if $H(X)$ is separable, then $X$ is almost reflexive, $X'$ is almost reflexive, and $X'''$ has a weak* sequentially compact unit disk. It is now shown that $X''$ has a weak* sequentially compact unit disk.

Theorem 1.9. If $H(X)$ is separable, then the unit disk of $X''$ is weak* sequentially compact.

Proof: Let $U''$ be the unit disk in $X''$ and let $U$ be the unit disk of $X$. McWilliams (15) has shown that $H(X)$ separable implies $K(U) = U''$. Assume $\{F_n\}$ is a sequence in $U'' = K(U)$. For each $F_n$

there exist a sequence $\{Jx_{ni}\}_{i=1}^{\infty}$ such that $F_n = \sigma(X'',X') - \lim_i Jx_{ni}$. Let $M$ be the closed linear span of $\{x_{ni}\}$, $n,i = 1,2, \ldots$ . $M$ is a closed separable subspace of $X$, and $H(X)$ is separable so it follows that $H(M)$ is separable (15). Now $H(M)$ and $M$ separable imply that $M''$ is separable (15). So $S''$, the unit disk of $M''$, is weak* sequentially compact since $M''$ is separable. By construction $\{Jx_{ni}\}_{i=1}^{\infty}$ is a sequence in $S''$, so it has a $\sigma(M'',M')$ convergent subsequence $\{Jy_{ni}\}_{i=1}^{\infty}$. Let $G_n = \sigma(M'',M') - \lim_i Jy_{ni}$ and note that each $G_n \in S''$. Recall

$$F_n = \sigma(X'',X') - \lim_i Jx_{ni}$$

so

$$F_n = \sigma(X'',X') - \lim_i Jy_{ni}.$$

If $f \in X'$, then $f|M \in M'$ and since

$$G_n(g) = \lim_i \, [Jy_{ni}](g) \quad \text{for all } g \in M'$$

$$F_n(g) = \lim_i \, [Jy_{ni}](f) \quad \text{for all } f \in X',$$

it follows that $F_n = G_n|X'$. The sequence $\{G_n\}$ is a sequence in $S''$, and $S''$ is weak* sequentially compact so there exists a subsequence $\{G_{n_i}\}$ such that

$$G = \sigma(M'',M') - \lim_i G_{n_i};$$

thus,

$$G|X' = \sigma(X'',X') - \lim_i G_{n_i}|X',$$

or

$$F = \sigma(X'',X') - \lim_i F_{n_i}$$

where $F = G|X'$ and $F_{n_i} = G_{n_i}|X'$; hence, $\{f_n\}$ has a weak* convergent subsequence and, therefore, $U''$ is weak* sequentially compact.

If $H(X)$ is separable, then $X$ and $X'$ are almost reflexive; and $X''$ and $X'''$ have weak* sequentially compact unit disks. Furthermore, letting $X^n$ denote the $n^{th}$ conjugate, McWilliams (15) showed that if $H(X^n)$ is separable, then $H(X^{n-1})$ is separable, and it then follows that $X^i$ for $i = 1, \ldots, n+1$ are almost reflexive and $X^i$, $i = n+2, n+3,$ have weak* sequentially compact unit disks. Combining this result with Theorem 2.20 in Chapter III, the following corollary is obtained.

Corollary 1.10. If $H(X^n)$ is separable, then $X^i$, $i = 1,2, \ldots, n+1$, have the Dieudonne property.

The almost reflexive property is useful in working with the sets $K(X)$ which are considered throughout this paper. This theorem can be used to determine whether a space possesses or lacks the almost reflexive property. The examples which follow the theorem demonstrate this.

Theorem 1.11. If $Y$ is a closed subspace of $X$ and $\emptyset$ is the natural mapping from $X$ onto $Z = X/Y$, then

$$JZ = \emptyset''JX \subseteq \emptyset''K(X) \subseteq K(Z),$$

and

$$\emptyset''K(X) = K(Z)$$

if $X$ is almost reflexive.

Proof: Civin and Yood (3) showed $JZ = \emptyset''JX$, and since $JX \subseteq K(X)$ it follows that $\emptyset''JX \subseteq \emptyset''K(X)$. If $F \in K(X)$, then there exists a sequence $\{x_n\}$ in $X$ such that $F = \sigma(X'',X') - \lim_n Jx_n$.

$$\emptyset''F = \sigma(Z'',Z') - \lim_n \emptyset''Jx_n = \sigma(Z'',Z') - \lim_n J\emptyset''x_n$$

so $\emptyset''F \in K(Z)$ and, therefore, $\emptyset''K(X) \subseteq K(Z)$ is established.

Assume $X$ is almost reflexive and let $H \in K(Z)$. There exists a sequence $\{z_n\}$ in $Z$ such that $H = \sigma(Z'',Z') - \lim_n Jz_n$. There exist a bounded sequence $\{x_n\}$ in $X$ such that $\emptyset x_n = z_n$. Since $X$ is almost reflexive, there exists a $\sigma(X,X')$ Cauchy sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $F = \sigma(X'',X') - \lim_n Jx_{n_i}$.

$$\emptyset''F = \sigma(Z'',Z') - \lim_i \emptyset''Jx_{n_i} = \sigma(Z'',Z') - \lim_i J\emptyset x_{n_i}$$

$$= \sigma(Z'',Z') - \lim_i Jz_{n_i} = H,$$

so $\emptyset''F = H$. Thus, given $H \in K(Z)$ there exists an $F \in K(X)$ such that $\emptyset''F = H$, i.e. $K(Z) \subseteq \emptyset''K(X)$.

Consider the space $\ell_1$. There exists a closed subspace $Y$ of $\ell_1$ such that $\ell_1/Y \cong c_0$. Let $\emptyset$ be the natural map from $\ell_1$ to $c_0$. Since $\ell_1$ is weakly complete, $K(\ell_1) = J\ell_1$ and $K(c_0) \neq Jc_0$ since $c_0$ is not weakly complete. So

$$\emptyset''K(\ell_1) = \emptyset''J\ell_1 = Jc_0 \neq K(c_0);$$

thus, $\emptyset''K(\ell_1) \neq K(c_0)$, and it follows from Theorem 1.11 that $\ell_1$ is not almost reflexive. Lemma 1.5 then implies that $m'$ does not have a weak* sequentially compact unit disk. Theorem 1.10 implies that $H(\ell_1)$ is not separable.

The converse to Theorem 1.9 is not true. Consider the space $c_0$. Since $\ell_1$ is separable, $c_0''$ has a weak* sequentially compact unit disk. However, if $H(c_0)$ were separable, it would follow that $c_0''$ is separable which it is not (15).

## Quasi-Reflexive Banach Spaces

Civin and Yood (3) introduced the concept of a quasi-reflexive space. A study of the relationship between quasi-reflexive Banach spaces and weak* sequential compactness is the theme treated in this section.

Definition 2.1. A real Banach space $X$ is quasi-reflexive of order $n$ if $H(X)$ has dimension $n$.

Immediately it is observed that if $X$ is quasi-reflexive then $H(X)$ is separable; hence, $X$ is almost reflexive. Indeed, since $[H(X)]' \cong H(X')$, the dimension of $H(X)$ is $n$ if and only if the dimension of $H(X')$ is $n$. Therefore, if $X$ is quasi-reflexive of order $n$, then $X^i$ is quasi-reflexive of order $n$ for $i = 1,2, \ldots$ . It is clear that if $X$ is quasi-reflexive of order $n$, then $X$ and all its conjugate spaces are almost reflexive. Therefore, all quasi-reflexive spaces and their conjugates have the Dieudonne property.

Definition 2.2. The Banach space $X$ is said to have property $P_n$ if every norm closed subspace $S$ of $X'$ has codimension $\leq n$ in its weak* sequential closure $K(S)$ in $X'$.

This property was introduced and used by McWilliams (17). By using almost reflexive and the $P_n$ property, one has necessary and sufficient conditions for quasi-reflexivity of order $\leq n$.

Theorem 2.3. A real Banach space $X$ is quasi-reflexivity of order $\leq n$ if and only if $X'$ is almost reflexive and $X$ has property $P_n$.

Proof: If $X$ is quasi-reflexive of order $\leq n$, then $X'$ is almost reflexive. McWilliams (17) has shown that if $X$ is quasi-reflexive of order $\leq n$, then $X$ has property $P_n$.

Conversely, let $X'$ be almost reflexive and let $X$ have property $P_n$. Let $Y$ be a norm closed subspace of $X$ and let $i$ be the identity mapping of $Y$ into $X$. It will be shown that $Y$ has property $P_n$. It then follows that $X$ is quasi-reflexive of order $\leq n$ (17, Theorem 3).

To show $Y$ has property $P_n$, let $S$ be a norm closed subspace of $Y'$. $(i')^{-1}(S)$ is then norm closed in $X'$, and since $X$ has property $P_n$, the dimension of $K((i')^{-1}(S))/(i')^{-1}(S)$ is $\leq n$. Let $T = i'[K((i')^{-1}(S))]$. $T$ is a subspace of $Y'$ containing $S$, and the dimension of $T/S$ is $\leq n$. It is now shown that $T = K(S)$ and, hence, the dimension of $K(S)/S$ is $\leq n$. $Y$ then has property $P_n$. If $g \in T$, then $g = i'f$ for some $f \in K((i')^{-1}(S))$. There then exists a sequence $\{f_n\}$ in $(i')^{-1}(S)$ such that $f = \sigma(X',X) - \lim_n f_n$. Clearly, $i'f_i \in S$, and since $i'$ is weak* continuous $i'f = \sigma(Y',Y) - \lim_n i'f_n$, but $i'f = g$ so $g = \sigma(Y',Y) - \lim_n i'f_n$ which implies $g \in K(S)$ since all the $i'f_n$ 's are in $S$. Consequently, $T \subseteq K(S)$.

To show $K(S) \subseteq T$, let $g \in K(S)$. There exists a sequence $\{g_k\}$ in $S$ such that $g = \sigma(Y',Y) - \lim_k g_k$. $\{g_k\}$ is bounded so there is a bounded sequence $\{h_k\}$ in $X'$ such that $i'h_k = g_k$ for each $k$. $X'$ is almost reflexive so $\{h_k\}$ has a $\sigma(X',X'')$ Cauchy subsequence $\{h_{k_\ell}\}$; thus, there exists an $h \in X'$ so that $h = \sigma(X',X) - \lim_\ell h_{k_\ell}$. $h_{k_\ell}$ is contained in $(i')^{-1}(S)$ so $h \in K((i')^{-1}(S))$; thus, $i'h \in T$. Since

$$h = \sigma(X',X) - \lim_\ell h_{k_\ell},$$

$$i'h = \sigma(Y',Y) - \lim_\ell i'h_{k_\ell}$$

and

$$i'h = \sigma(Y',Y) - \lim_\ell g_{k_\ell} = g.$$

The $\sigma(Y',Y)$ topology is Hausdorff so $g = i'h \in T$. Consequently, $K(S) \subseteq T$.

In the proof of Theorem 2.3 it suffices to require $X'$ to have a weak* sequentially compact unit disk rather than the stronger condition that $X'$ be almost reflexive.

<u>Corollary</u> 2.4. A real Banach space $X$ is quasi-reflexive of order $\leq n$ if and only if $X'$ has a weak* sequentially compact unit disk and $X$ has property $P_n$.

The next theorem summarizes results derived in this paper and those of McWilliams (15).

<u>Theorem</u> 2.5. If $X$ is a real Banach space with property $P_n$, then the following are equivalent.

(a) $H(X)$ is separable.

(b) $K(JX') = X'''$.

(c) $X'$ is almost reflexive.

(d) $X'$ has a weak* sequentially compact unit disk.

(e) $X$ is quasi-reflexive of order $\leq n$.

A Banach space $X$ is said to be a Grothendieck space if each $\sigma(X',X)$ convergent sequence in $X'$ is $\sigma(X',X'')$ convergent. $m$ is such a space.

<u>Corollary</u> 2.6. A nonreflexive real Grothendieck space does not have any of the properties listed in Theorem 2.5.

Proof: If $X$ is a nonreflexive Grothendieck space, then $X$ has property $P_0$ (15). Thus, all five properties in Theorem 2.5 are equivalent. But for a nonreflexive Grothendieck space $X$, $X'$ can not have a weak* sequentially compact unit disk (10).

## WCG Spaces

A Banach space $X$ is said to be <u>weakly compactly generated</u> (<u>WCG</u>) if there exists a weakly compact set such that $X$ is the closed linear span of that set.

A WCG Banach space is a generalization of reflexive and separable Banach spaces. Indeed, if $X$ is separable then $X$ is norm compactly generated, hence WCG; and if $X$ is reflexive the unit disk of $X$ is weakly compact, hence $X$ is WCG.

The class of WCG Banach spaces is included in the class of all Banach spaces whose conjugates have a weak* sequentially compact unit disk. If $X'$ is a WCG conjugate Banach space then $X''$ has a weak* sequentially compact unit disk (20), and it follows that $X$ is almost reflexive. Thus, X' WCG implies that X has the Dieudonne property. A condition that seems weaker than X' being WCG will also imply that X has the Dieudonne property. If X' is isomorphic to a subspace of a WCG Banach space, then the unit disk of X" will be weak* sequentially compact (20). X would then be almost reflexive. It is an open question if every closed subspace of a WCG Banach space is itself WCG.

A well-known property which many authors find useful is that known as the Dunford-Pettis property.

Definition 3.1. A Banach space X is said to have the Dunford-Pettis (DP) property if every weakly compact operator on X is completely continuous.

The following, which is of interest in this paper, illustrates a characteristic of Banach spaces by using the DP property.

Theorem 3.2. (20) Let the Banach space X satisfy the DP property. Then if X (X') is isomorphic to a subspace of a WCG conjugate Banach space (WCG Banach space), every weak Cauchy sequence in X (X') converges in the norm topology of X (X').

By bringing several properties together which have previously been considered, we obtain the following.

Theorem 3.3. If X is a WCG conjugate Banach space, then X has no subspace isomorphic to an almost reflexive Banach space with the DP property. In particular, X has no subspace isomorphic to $c_0$.

Proof: Let $Y$ be an almost reflexive Banach space with the DP property. If $Y$ is isomorphic to a subspace of $X$, then by Theorem 3.2 every weak Cauchy sequence in $Y$ converges in the norm topology on $Y$. If $\{y_n\}$ is a sequence in the unit disk of $Y$, then $\{y_n\}$ has a weak Cauchy subsequence $\{y_{n_i}\}$ since $Y$ is almost reflexive. Thus, $\{y_{n_i}\}$ converges in the norm topology, and it follows that the unit disk of $Y$ is compact; hence, $Y$ is finite dimensional, a contradiction.

$c_0$ is almost reflexive and also has the DP property so in particular $X$ has no subspace isomorphic to $c_0$.

Since "$X$ has no subspace isomorphic to $c_0$" is equivalent to $N(X) = JX$, the next corollary is immediate.

<u>Corollary</u> 3.4. If $X$ is a WCG conjugate Banach space, then $N(X) = JX$.

<u>Corollary</u> 3.5. If $X$ is a WCG conjugate Banach space, then $X$ has property V if and only if $X$ is reflexive.

Proof; If $X$ is reflexive then $X$ has property V. Conversely, since $X$ is a WCG conjugate Banach space, $X$ has no subspace isomorphic to $c_0$. Pelczynski (19) has shown that if $X$ has no subspace isomorphic to $c_0$ and has property V, then $X$ is reflexive.

It can be observed from Corollary 3.5 that no nonreflexive separable conjugate Banach space has property V. $\ell_1$ is nonreflexive, separable and a conjugate Banach space, so as already observed $\ell_1$ does not have property V. If $X''$ is nonreflexive and separable, then $X'$

does not have property V, has the Dieudonne property and does not have property (u).

Corollary 3.6. If $X'$ is a WCG conjugate Banach space with the DP property, then $X''$ is not WCG.

Proof: If $X''$ is WCG, then $X'''$ has a weak* sequentially compact unit disk and, therefore, $X'$ would be almost reflexive. This contradicts Theorem 3.3.

The next corollary eliminates many Banach spaces from having the Dieudonne property.

Corollary 3.7. A WCG conjugate Banach space $X$ with the DP property does not have the Dieudonne property.

Proof: Let $i$ be the identity operator on $X$. By Theorem 3.2 every weak Cauchy sequence in $X$ converges in the norm topology of $X$. Thus, $i$ is a wcc operator. If $X$ had the Dieudonne property, then $i$ would be weakly compact. $X$ would be reflexive. A contradiction, since no space with the DP property is reflexive.

The Dieudonne property is a necessary condition for a Banach space to have property V.

Corollary 3.8. A WCG conjugate Banach space $X$ with the DP property does not have property V.

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