CHARACTERIZATIONS OF TREES AND GENERALIZED TREES

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CHAPTER I

INTRODUCTION

This paper is concerned with characterizations of continua which have the property that every two points of the continuum are separated by a third point. A continuum with this property is called a tree. For the reader who is familiar with dendrites, several of the characterizations will sound familiar. There has been a great deal written on the properties of dendrites. It should be pointed out that a dendrite is a metric tree. The concept of trees is a generalization of dendrites since it is not necessary for the space to be metric.

The first part of this paper will deal with the concept of partially ordered topological spaces. Chapter II will introduce partially ordered topological spaces and includes the tools needed to characterize trees in terms of a partially ordered topological space. The reader need not be familiar with partially ordered topological spaces. Chapter II includes all of the basic concepts needed. If the reader would like a more detailed study and history of ordered topological spaces, he may refer to Nachbin [12].

The concept of a tree has been contained in the literature for many years. However, the characterization in terms of a partially ordered topological space is fairly new. It first appeared in a paper by Ward [24] in 1954. Not only is this characterization new, but the concept of a partially ordered topological space is new. For many years the study of topological spaces and ordered spaces was carried on as two separate topics. Nachbin [12] in 1947 began his research on spaces which were equipped with a topological structure and an order structure. From his efforts he developed the concept of partially ordered topological spaces.

In Chapter IV the idea of generalized trees is presented. The definition of a generalized tree results from weakening the conditions on the order contained in the characterization of trees in Chapter III. Several characterizations of generalized trees are given and the weak cutpoint ordering is introduced.

There seems to be a great amount of material on the fixed point property. We have added to that amount in Chapter V. Included are several results of Ward [28], Wallace [18] and Smithson [14]. The only proof included will be the proof of the fixed point theorem for generalized trees. This proof is included because it is in terms of an ordered space.

We have required in this paper that the continua be compact. There has been research done with continua which are non-compact with the property that each two points can be separated by a third point. There have been efforts made to determine the conditions necessary for such a continuum to admit a nontrivial continuous partial order. We have not included this topic in the paper because it could very well be a paper in itself.

The paper is self contained to the extent that the reader does not need any knowledge of ordered topological spaces. However, the paper is written at a level that expects the reader to be familiar with the basic concepts of a topological space. A semester course in general topology should be adequate preparation.

CHAPTER II

PARTIALLY ORDERED TOPOLOGICAL SPACES

We begin with the basic definitions and results for partially ordered topological spaces that will be used throughout the paper. The reader who would like a more complete discussion of partially ordered topological spaces may refer to Nachbin [12]. Nachbin's book was one of the first and more complete of the books containing results on the relationships between topological and order structures. One may also refer to Ward [22] for a more complete coverage of partially ordered topological spaces.

<u>Definition 2.1</u> By a <u>quasi order</u> on a set X, we mean a reflexive, transitive binary relation, denoted by, $\leq$.

<u>Definition 2.2</u> If a quasi order is also anti-symmetric, it is a <u>partial order</u>.

<u>Definition 2.3</u> If a quasi order satisfies the following linearity law

$$\text{if } x, y \in X, \text{ then } x \leq y \text{ or } y \leq x,$$

then it is said to be a <u>linear</u> quasi order.

In other words, if in a quasi order all elements are related, then it is a linear quasi order.

If $x \leq y$ and $x \neq y$, we will denote this by $x < y$, and we will often talk about the set of predecessors or the set of successors of a point or of a set. We will use the following notation to express these ideas.

<u>Definition 2.4</u> $L(A) = \{y \in X : y \leq x \text{ for some } x \in A\}$.

<u>Definition 2.5</u> $M(A) = \{y \in X : x \leq y \text{ for some } x \in A\}$.

<u>Definition 2.6</u> $E(A) = L(A) \cap M(A)$.

It is clear from the above definitions that $A \subset E(A)$. If we let X be the set of real numbers with the natural order and $A = [-2, -1] \cup [1, 2]$, then we have $E(A) = [-2, 2]$. Therefore, $A \neq E(A)$, for all A. However, if we let $A = [0, 1]$, then $A = E(A)$.

<u>Definition 2.7</u> If $A = L(A)$, we say that A is <u>monotone decreasing</u> or simply decreasing.

<u>Definition 2.8</u> If $A = M(A)$, we say that A is <u>monotone increasing</u> or simply increasing.

Let X be the set of real numbers with the natural order. If $A = (-\infty, 0]$ then $A = L(A)$ and is monotone decreasing. If $A = [0, \infty)$ then $A = M(A)$ and A is monotone increasing. The only subset of X that is both increasing and decreasing is X.

It is possible to have a set which is both increasing and decreasing. Consider a set X such that there exist an element $x \in X$, such that x is not related to any of the other elements of X. Then $L(x) = \{x\}$ and $M(x) = \{x\}$. Therefore $\{x\}$ is an increasing and decreasing set.

Suppose that X is a topological space endowed with a quasi order. We make the following definitions.

<u>Definition 2.9</u> The quasi order is <u>lower semicontinuous</u> provided, whenever $a \not\leq b$ in X, there is an open set U, with $a \in U$, such that if $x \in U$ then $x \not\leq b$.

<u>Definition 2.10</u> The quasi order is <u>upper semicontinuous</u> provided, whenever $b \not\leq a$ in X, there is an open set U, with $a \in U$, such that if $x \in U$ then $b \not\leq x$.

<u>Example 2.11</u> Let $X = \{a, b, c, d\}$ with a basis consisting of $\{a\}$, $\{a, b\}$, $\emptyset$, $\{a, b, c, d\}$. Let the quasi order on X be given by the direction of the arrows in the following:

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr] & & d \arrow[dl] \\ & c & \end{tikzcd}$$

The quasi order is lower semicontinuous, but not upper semicontinuous. To show the order is not upper semicontinuous, consider the two points b and d. $b \not\leq d$, but any open set containing d will contain a, and $b \leq a$.

<u>Example 2.12</u> Using the same set X and order given in Example 2.11, we can construct a topology that gives an upper semicontinuous order which is not lower semicontinuous. By changing the basis to the sets $\{c\}$, $\{c, a\}$, $\{c, b\}$, $\{c, d\}$, $\emptyset$, and $\{a, b, c, d\}$, we have the

result we want. Again we have $b \not\leq d$ and every open set containing b contains c with $c \leq d$.

<u>Definition 2.13</u> A quasi order is <u>semicontinuous</u> if it is both upper and lower semicontinuous.

<u>Definition 2.14</u> A quasi order is <u>continuous</u> provided, whenever $a \not\leq b$ in X, there are open sets U and V, $a \in U$ and $b \in V$, such that if $x \in U$ and $y \in V$ then $x \not\leq y$.

<u>Definition 2.15</u> A quasi order is <u>strongly continuous</u> provided;

(i) if $a < b$, then there exist open sets U and V such that $a \in U$ and $b \in V$ and if $x \in U$ and $y \in V$ then $x < y$.

(ii) if a and b are not related then there exist open sets U and V such that $a \in U$ and $b \in V$ and, if $x \in U$ and $y \in V$, then x and y are not related.

It is clear from the above definitions that if a quasi order is strongly continuous then the order is also continuous and semicontinuous. It also follows from the definitions that a continuous quasi order is a semicontinuous order. However the converse is not true. There exist semicontinuous quasi orders which are not continuous quasi orders and continuous quasi orders that are not strongly continuous. The following examples show that the converse statements are not true.

<u>Example 2.16</u> Let $X = \bigcup_{n=0}^{\infty} A_n$, where

$$A_0 = \{(0, y) \mid 0 \leq y \leq 1\}, \text{ and}$$

$$A_n = \{(\tfrac{1}{n}, y) \mid 0 \leq y \leq 1\} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Let X have the usual topology of the plane. Define an order on X as follows:

$$(\tfrac{1}{n}, b) \leq (\tfrac{1}{n}, d) \quad \text{if} \quad d \leq b \quad \text{and} \quad n = 1, 2, 3, \ldots$$

$$(0, b) \leq (0, d) \quad \text{if} \quad b \leq d .$$

It is easy to verify that the order defined is semicontinuous. However, the order is not continuous. This can be shown by considering points of the form $(0, b)$ and $(0, d)$, with $b < d$. If we take open sets U and V such that $(0, b) \in U$, $(0, d) \in V$, there exist $x \in U$, $y \in V$ such that $y < x$. This is a contradiction to the definition of a continuous quasi order. Therefore we have an example of a semi-continuous quasi order that is not a continuous quasi order.

Example 2.17 Let X be the unit square of the plane. Let X have the following order,

$$(a, b) \leq (c, d) \quad \text{if} \quad a = c, \; b \leq d$$

$$(a, b) \leq (c, d) \quad \text{if} \quad a \leq c, \; b = 0$$

$$(a, b) \leq (c, d) \quad \text{if} \quad a = 0, \; d = 1 .$$

By a direct application of the definitions, it can be verified that the order is semicontinuous and continuous. However, the order is not strongly continuous. This can be shown by considering points of the form $(a, 0)$ and (a, d) where $d > 0$. Then we have $(a, 0) < (a, d)$, but any open set containing (a, d) will contain points that are not related to $(a, 0)$. Therefore the order is not strongly continuous.

Example 2.18 For an example of a strongly continuous quasi order, we can use the set of real numbers with the natural order.

If a space has an order that is not semicontinuous, continuous or strongly continuous it does not follow that such orders on the space do not exist. In Example 2.16 the defined order was not continuous. However, we can define another order on X that is continuous. We define a new order on the space X in Example 2.16 as follows:

$$\left(\frac{1}{n}, b\right) \leq \left(\frac{1}{n}, d\right) \quad \text{if} \quad b \leq d \quad \text{and} \quad n = 1, 2, 3, \ldots,$$

$$(0, b) \leq (0, d) \quad \text{if} \quad b \leq d.$$

With this order we no longer have the problem as in Example 2.16 with points of the form $(0, b)$ and this new order is continuous. Thus a space may possess both a continuous quasi order and a non-continuous quasi order. The same statements may be made about semicontinuous and strongly continuous orders.

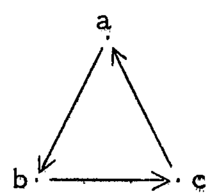
Our next definitions will relate the concepts of ordered sets and topological spaces.

Definition 2.19 A quasi ordered topological space is a topological space together with a semicontinuous quasi order. We will use the notation QOTS for a quasi ordered topological space.

Definition 2.20 A partially ordered topological space is a topological space together with a semicontinuous partial order. We will use the notation POTS for a partially ordered topological space.

Any of the topological spaces with the orders given in Example 2.16, 2.17 and 2.18 are examples of POTS. From the definition of a POTS it is clear that a POTS is also a QOTS. The converse is not true, and the following example verifies this.

Example 2.21 Let $X = \{a, b, c\}$ be provided with the discrete topology. Define an order on X by the directions of the arrows in the following:



The order as defined is not a partial order, but it is a quasi order. Then X is a QOTS that is not a POTS.

As often happens, it is not always easy to prove something directly from a definition. Therefore our first theorem will be a characterization for a QOTS. The proof is simple, but gives us a very useful tool.

Theorem 2.22 X is a QOTS if and only if $L(x)$ and $M(x)$ are closed sets for each $x \in X$.

Proof. Let X be a QOTS and $y \in X$. Suppose $L(y)$ is not closed. Then there exist $z \notin L(y)$ such that z is a limit point of $L(y)$. If $z \notin L(y)$ then $z \not\leq y$ and, by the definition of a QOTS, there exist an open set U such that $z \in U$ and, for all $x \in U$, $x \not\leq y$. Therefore, $U \cap L(y) = \emptyset$, and this contradicts z being a limit point of

$L(y)$. Hence $L(y)$ is closed. By supposing $M(y)$ is not closed, a similar argument holds. Therefore both $M(y)$ and $L(y)$ are closed.

Suppose $L(x)$ and $M(x)$ are closed sets for each $x \in X$. Suppose there exist $a, b \in X$ such that $a \not\leq b$. Then define U to be $X - L(b)$, which is open. Then $a \in U$ and, for all $y \in U$, $y \not\leq b$. Therefore $\leq$ is lower semicontinuous. Now define $V = X - M(a)$, which will be open. Then $b \in V$ and, for all $z \in V$, $a \not\leq z$. Therefore $\leq$ is upper semicontinuous. Then $\leq$ is both upper and lower semicontinuous and, from the definition of semicontinuous, $\leq$ is semicontinuous. So, X is topological space with a semicontinuous order or a QOTS. Q.E.D.

In Chapter III and IV much of the results will involve QOTS and in many cases we will want to show that the quasi order is a continuous quasi order. The following theorem gives two characterizations to use in showing that a quasi order is continuous. One of the characterizations is given in terms of the graph of an order. By the graph of an order we mean the following: Given a set X with an order $\leq$, the graph of $\leq$ is the subset of $X \times X$ formed by the points (x, y), where $x, y \in X$ and $x \leq y$. In the case of the natural order of the real numbers, this graph is the half-plane situated above the bisector of the first and third quadrants.

Theorem 2.23 If X is a topological space with a quasi order, then the following statements are equivalent:

(1) the quasi order is continuous,

(2) the graph of the quasi order is a closed set in $X \times X$,

(3) if $a \not\leq b$ in X, then there are neighborhoods N and N' of a and b, respectively, such that N is increasing, N' is decreasing and $N \cap N' = \emptyset$.

Proof. Let $\leq$ be a continuous quasi order on X. Denote the graph of $\leq$ in $X \times X$ by G. Suppose $(x, y) \in X \times X$ such that $(x, y) \notin G$. Since (x, y) is not an element of the graph of $\leq$, this implies $x \not\leq y$. Since $\leq$ is continuous, there exist open sets U and V such that $x \in U$ and $y \in V$ and for all $a \in U$ and $b \in V$ such that $a \not\leq b$. The definition of a product space implies that $U \times V$ is open in $X \times X$ and that $(x, y) \in U \times V$. Then, for all $(a, b) \in U \times V$, (a, b) is not an element of G. Thus $(U \times V) \cap G = \emptyset$ and this implies (x, y) is not a limit point of G. Therefore G contains all of its limit points and is a closed set. We then have statement (1) implying statement (2).

Suppose the graph of the quasi order is a closed set in $X \times X$. Let $a, b \in X$ such that $a \not\leq b$. Then, from the definition of the graph, (a, b) does not belong to the graph of $\leq$. Since the graph of $\leq$ is closed, (a, b) is not a limit point of the graph. Then there exist an open set of the form $U \times V$ where U and V are open sets in X such that $(a, b) \in U \times V$ and $(U \times V) \cap G = \emptyset$. Then $a \in U$, $b \in V$ and, since $(U \times V) \cap G = \emptyset$, it follows that for $x \in U$ and $y \in V$, $x \not\leq y$ and $y \not\leq x$. Hence, the quasi order is continuous, by the definition, and we have statement (2) implying statement (1).

Let $\leq$ be a continuous quasi order on X with $a, b \in X$ such that $a \not\leq b$. By the definition of continuous, there exist open sets U, V such that for all $x \in U$ and $y \in V$, $x \not\leq y$. Define $N = M(U)$

and $N' = L(V)$. Then N and N' are neighborhoods of a and b respectively.

Suppose $N \cap N' \neq \emptyset$. This implies there exist a $z \in N \cap N'$. From the definitions of N and N', there exist $x \in U$ and $y \in V$ such that $x \leq z$ and $z \leq y$. Then, from the transitive property of $\leq$, we have $x \leq y$. This contradicts the relation of the points in U and V. Hence, we have $N \cap N' = \emptyset$. By the way N and N' were defined, we have N as increasing and N' as decreasing. Then N and N' satisfy the conditions of statement (3). Therefore statement (1) implies statement (3).

Suppose there exist neighborhoods N and N' which satisfy the conditions of statement (3). Let $a, b \in X$ such that $a \not\leq b$. Then there exist open sets U and V such that $a \in U \subset N$ and $b \in V \subset N'$. Since N is increasing and N' is decreasing for all $x \in N$, and $y \in N'$, then $x \not\leq y$. Thus $(a, b) \in U \times V$ and, for all $(x, y) \in U \times V$, $x \not\leq y$. Hence (x, y) is not an element of the graph of $\leq$ and consequently $(U \times V) \cap G = \emptyset$. Therefore (a, b) is not a limit point of the graph of $\leq$. Hence, the graph is closed and we have statement (3) implying statement (2).

We can now assume any of the three statements to be true and show that the other two statements are also true. This completes the proof that the three statements are equivalent. Q.E.D.

Theorem 2.24 A POTS with continuous partial order is a Hausdorff space.

Proof. Let $x, y \in X$ such that $x \neq y$. By the definition of continuous order, there exist open sets U, V such that $x \in U$,

$y \in V$ and $U \cap V = \emptyset$. Therefore X is a Hausdorff space. Q.E.D.

Theorem 2.25 If X is a topological space with a linear quasi order, then continuity and semicontinuity of the quasi order are equivalent properties of it.

Proof. As pointed out earlier, the definition of continuous order implies semicontinuous order. To complete the proof of the theorem we need to show that semicontinuity implies continuity under the conditions stated in the hypothesis. If $a \not\leq b$ then $b < a$, since $\leq$ is linear. If there exists c such that $b \leq c \leq a$, then let $U = X - M(c)$ and $V = X - L(c)$. Then U and V are open sets such that $b \in U$, $a \in V$ and $U \cap V = \emptyset$. If there does not exist such a c, then let $U = X - L(a)$ and $V = X - M(b)$. Then $b \in U$, $a \in V$ and, since $\leq$ is linear, $U \cap V = \emptyset$, and for all $x \in U$ and $y \in V$ we have $x \not\leq y$. Therefore $\leq$ is continuous. Q.E.D.

As can be seen from some of the previous examples, that it is not necessary for all of the elements of a set to be related. In Chapter III and IV we will be concerned with subsets of the space such that all of the elements of the subset are related. In other words, the order on the subset is a linear quasi order. The following definitions formalize this idea.

Definition 2.26 A chain is a subset of a quasi ordered set which is linear with respect to the quasi order.

Definition 2.27 A maximal chain is a chain which is properly contained in no other chain.

In Example 2.16 the subset $\{(\frac{1}{n}, y) \mid \frac{1}{2} \leq y \leq 1\}$ is a chain, but is not a maximal chain. The subset is contained in the maximal chain $\{(\frac{1}{n}, y) \mid 0 \leq y \leq 1\}$. In this space there exist an infinite number of maximal chains, However, in some spaces there may exist only one maximal chain. As an example of this situation, consider the set of real numbers with the natural order. The only maximal chain is the space itself. On the other hand, by applying Zorn's lemma, we are assured of the existence of maximal chains in any quasi ordered set. The following result is due to Wallace [18].

Theorem 2.28 Every maximal chain in a QOTS is a closed set.

Proof. Let C be a maximal chain in a QOTS. We can express C by

$$C = \bigcap \{L(x) \cup M(x) \mid x \in C\},$$

From Theorem 2.22, $L(x)$ and $M(x)$ are closed sets. Then $L(x) \cup M(x)$ is closed and $\bigcap \{L(x) \cup M(x) \mid x \in C\}$ is closed. Q. E. D.

Definition 2.29 An element y in a quasi ordered set X is minimal whenever $x \leq y$ in X implies $y \leq x$.

Definition 2.30 An element y in a quasi ordered set X is maximal whenever $y \leq x$ in X implies $x \leq y$.

In Example 2.16, the point $(0, 0)$ is a minimal element, as are all of the points of the form $(\frac{1}{n}, 1)$. In view of this, it is wrong to conclude that there exists only one minimal element. There may be one, any finite number, or an infinite number of minimal elements.

Also, by considering the set of real numbers with the natural order, we can see that there may not be a minimal element. The same is true when considering maximal elements.

In Chapter III when we develop the characterization of trees, we will be working with compact connected spaces. The next few theorems will involve compact connected spaces. The next few theorems will involve compact connected spaces and will give us some results we will need in Chapters III and IV.

Theorem 2.31 A non-null compact space endowed with a lower (upper) semicontinuous quasi order has a minimal (maximal) element.

Proof. Let $L = \{L(x) | x \in X\}$, where X is a compact space with a lower semicontinuous quasi order. We can partially order L by set inclusion. Then, by the Hausdorff Maximal Principle (Kelley [8], p. 32), there exists $M \subset L$ such that M is a maximal chain with respect to the set inclusion relation. Since $L(x)$ is closed and X is compact, $\{L(x) \mid L(x) \in M\}$ has the finite intersection property. Then there exists $y \in \bigcap \{L(x) \mid L(x) \in M\}$ (Kelley [8], p. 136). We assert that y is a minimal element. For suppose there exists $z \in X$ such that $z < y$. Then $y \notin L(z)$, which implies $y \notin \bigcap \{L(x) \mid L(x) \in M\}$. This is contradiction to the way y was defined. Therefore y is a minimal element.

A similar argument can be used when the quasi order is upper semicontinuous. Q. E. D.

Definition 2.32 A partially ordered set X is dense in the sense of order, or, more simply, order dense provided, whenever $x < y$ in X, there exists $z \in X$ such that $x < z < y$.

As might be expected, not all partially ordered sets are order dense. Any partially ordered set that contains only a finite number of elements will not be order dense. An example of an order dense set that the reader should be familiar with is the set of real numbers with the natural order. It is possible for a proper subset to be order dense. The subset of rational numbers is order dense since there always exists a rational number between any two rational numbers.

In the next three theorems we will investigate conditions sufficient to ensure the connectedness of a POTS and the maximal chains of a POTS. Only the results we will need in Chapter III have been included. For further results the reader may refer to Ward [23] and Eilenberg [4].

Theorem 2.33 A connected chain in a POTS, X, is order dense. If X has compact maximal chains, then any order dense maximal chain is connected.

Proof. Let C be a connected chain in X. Suppose that C is not order dense. Then there exist $x, y \in C$ such that $x < y$ and $M(x) \cap L(y) = x \cup y$. This implies that $C \subset L(x) \cup M(y)$. Since there does not exist an element between x and y, we have $L(x) \cap M(y) = \emptyset$. This implies that $L(x) \cup M(y)$ is not connected. Since $x, y \in C$ then $C \cap L(x) \neq \emptyset$ and $C \cap M(y) \neq \emptyset$. Therefore C is not connected. This contradicts C being connected and, therefore, C is order dense.

Suppose X has compact maximal chains and that C is a non-connected maximal chain of X. Since C is non-connected there exist separated sets P, Q such that $C = P \cup Q$. Then by Theorem 2.31, C contains a maximal element u. Suppose $u \in Q$. Since P is compact, P contains a maximal element p and $p < u$. Define $P' = L(p) \cap C$ and $Q' = C - P'$. Q' is non-empty since $u \in Q'$. Then P' and Q' are separated sets and, by Theorem 2.13, Q' contains a minimal element q. Then $M(p) \cap L(q) = p \cup q$, which implies that C is not order dense. Q.E.D.

To show that compactness is necessary to ensure that any order dense maximal chain is connected, we offer the following example.

Example 2.34 Let $X = [-\infty, 0) \cup (0, \infty)]$, with the natural order. The only maximal chain of X is X. However, X is not compact and is not connected. The order on X is order dense. Therefore, there exists an order dense maximal chain that is not connected. Hence, the condition of compactness is necessary in the proof of the last theorem.

Theorem 2.35 Let X be an order dense POTS with compact maximal chains and suppose that either the set of maximal elements or the set of minimal elements of X is connected. Then X is connected.

Proof. Suppose the set of maximal elements of X is connected and that $X = P \cup Q$, where P and Q are separated sets. Then the set of maximal elements is contained in P or in Q. Without loss of generality, suppose that the set of maximal elements belongs to Q. Then there exists a maximal chain C meeting P. Since C contains a maximal element, by Theorem 2.31, $C \cap Q \neq \emptyset$. Then $C = (C \cap P) \cup (C \cap Q)$, which is a separation of C. This gives a contradiction to Theorem 2.33. Therefore X is connected. A similar argument holds if the set of minimal elements of X is connected. Q.E.D.

<u>Theorem 2.36</u> Let X be a POTS with compact maximal chains. Then a necessary and sufficient condition that every maximal chain be connected is that $L(x) \cap M(y)$ be connected for every pair of elements $x, y \in X$.

<u>Proof.</u> Suppose $L(x) \cap M(y)$ is connected for every pair of elements $x, y \in X$ and that X contains a non-connected maximal chain C. Then $C = P \cup Q$, where P and Q are separated sets. Using the same argument as in Theorem 2.33, we can find points p and q such that $M(p) \cap L(q) = p \cup q$. This contradicts the assumption that $L(x) \cap M(y)$ is connected for every pair of elements $x, y \in X$. Therefore X does not contain a non-connected maximal chain.

Suppose now that there exist $p, q \in X$ such that $L(p) \cap M(q)$ is not connected. Let C be a maximal chain containing p and q. Since $L(p) \cap M(q)$ is not connected, then there exist separated sets P and Q such that $L(p) \cap M(q) = P \cup Q$. Since $L(p) \cap M(q)$ is closed, it is also compact. Then, by Theorem 2.31, $L(p) \cap M(q)$ has a maximal element u, Without loss of generality, suppose $u \in Q$, P is compact and, by Theorem 2.31, P has a maximal element. Define P' and Q' as follows:

$$P' = L(u) \cap [L(p) \cap M(q)]$$

and

$$Q' = [L(p) \cap M(q)] - P' .$$

Q' is non-empty since $u \in Q'$. Then $L(p) \cap M(q) = P' \cup Q'$, where P', Q' are separated sets. By Theorem 2.31, there exists a minimal element $z \in Q'$. Then P' contains a maximal element v and Q' contains a minimal element z. Therefore $M(v) \cap L(z) = v \cup z$. Hence, $L(p) \cup M(y)$ is not order dense which implies C is not order dense. Then, by Theorem 2.33, C is not connected. Therefore, if every maximal chain is connected, then $L(x) \cap M(y)$ is connected for every pair of points $x, y \in X$. Q.E.D.

Definition 2.37 If points p and q are not separated by any point, we write $p \sim q$. If two sets A and B are separated sets, we will denote this fact by writing $A|B$.

The main result of this chapter is a method to construct a partial order that will give us a POTS. We want the order to be such that it will characterize a tree. Up to this point we have not developed a method to construct an order that will give us the desired results. The method is not difficult to develop, but we will need three more definitions and three theorems. Ward [22] was the first to apply the results to characterize trees in terms of a partial order.

In the three theorems to follow, the space we will work with is a locally connected space. This is necessary since a tree is locally connected. It is not obvious from the definition of a tree that a tree is locally connected. Some definitions do assume a tree to be locally connected, but we chose not to include this assumption in our definition. However, our first efforts in Chapter III will be that of showing that a tree is locally connected. Therefore, starting from any of the common definitions we have a tree being locally connected and it is necessary to include this fact in the hypotheses of our next theorems.

Definition 2.38 A point e of a topological space is an endpoint if, whenever $e \in U$, an open set, there is an open set V such that $e \in V \subset \overline{V} \subset U$, and $\overline{V} - V$ is a single point.

Definition 2.39 A prime chain is a continuum which is either an endpoint, a cutpoint, or a nondegenerate set E containing distinct elements a and b with $a \sim b$, and representable as

$$E = \{x : a \sim x \text{ and } x \sim b\}.$$

Definition 2.40 An endelement is a prime chain E with the property that if $E \subset U$, an open set, then there is an open set V such that $E \subset \overline{V} \subset U$ and $\overline{V} - V$ is a single point.

Example 2.41 Let $X = [0, 1]$, with the usual topology. The points 0 and 1 are endpoints and all other points are not endpoints. The space is not a prime chain since for every pair of points there exist a third point which separates the two given points. Each point of the space would be a prime chain since each point is a cutpoint or endpoint.

Example 2.42 Let X be the following space,

$$X = \{(x, y) | x^2 + y^2 = 1\} \cup \{(x, y) | (x - a)^2 + y^2 = 1, \ a = \pm 3\}$$
$$\cup \{(x, 0) | 1 \leq x \leq 2 \text{ or } -2 \leq x \leq -1\}$$

with the usual topology of the plane. In this space there are no endpoints. There are cutpoints, however, all points of the form $\{(x, 0) | 1 \leq x \leq 2 \text{ or } -2 \leq x \leq -1\}$ are cutpoints. The sets $\{(x, y) | (x - a)^2 + y^2 = 1, \ a = \pm 3\}$ and $\{(x, y) | x^2 + y^2 = 1\}$ are prime chains. The set $\{(x, y) | x^2 + y^2 = 1\}$ is not an endelement, for any open set containing $\{(x, y) | x^2 + y^2 = 1\}$ is such that $\overline{V} - V$ will contain at least two points. This gives us an example of a prime chain that is not an endelement.

<u>Theorem 2.43</u> Let X be a connected, locally connected Hausdorff space. If E is an endelement of X, then E contains at most one cutpoint of X.

<u>Proof.</u> Suppose E is an endelement of X such that it contains two distinct cutpoints x_0 and x_1. Then we have

$$X - x_0 = A_0 \cup B_0, \quad A_0 | B_0 \quad \text{and} \quad E - x_0 \subset A_0,$$

$$X - x_1 = A_1 \cup B_1, \quad A_1 | B_1 \quad \text{and} \quad E - x_1 \subset A_1.$$

Since X is locally connected we can let A_0 and A_1 be the components that contain $E - x_0$ and $E - x_0$ respectively. Now if $B_0 - B_1 = \emptyset$ then $B_0 \subset B_1$ and $X = (A_1 \cup A_0) \cup B_0$ with $A_0 \cup A_1 | B_0$. This contradicts that X is connected. Therefore, $B_0 - B_1 \neq \emptyset$. A similar argument shows that $B_1 - B_0 \neq \emptyset$. Then there exist $y_0 \in B_0 - B_1$ and $y_1 \in B_1 - B_0$. Let C_i be the component of y_i in $X - x_i$. Then we have

$$y_0 \in C_0 \subset X - x_0, \quad C_0 \subset B_0$$

and

$$y_1 \in C_1 \subset X - x_1, \quad C_1 \subset B_1.$$

Suppose $C_0 \cap C_1 \neq \emptyset$. Since C_0 and C_1 are connected, then $C_0 \cap C_1$ is connected. Since $x_1 \notin C_0$, $C_0 \cup C_1 \subset B_1$ and it follows that $C_0 \subset B_1$. If $y_0 \in C_0$ then $y_0 \in B_1$, which contradicts the fact that $y_0 \in B_0 - B_1$. Therefore, we have $C_0 \cap C_1 = \emptyset$.

The way we picked y_0 and y_1 gives us that $y_0, y_1 \notin E$. If we let U be the open set defined by $U = X - y_0 - y_1$, then U is an open set such that $E \subset U$, $U \cap C_0 \neq \emptyset$, $U \cap C_1 \neq \emptyset$ and U contains neither C_0 or C_1. Since E is an endelement, there exist an open set V such that $E \subset \overline{V} \subset U$ and $\overline{V} - V = \{p\}$. Since $\overline{V} - V$ is a singleton set, either x_0 or x_1 or both are in V,

If both x_0 and x_1 are in V, then $V \cap C_0 \neq \emptyset$ and $V \cap C_1 \neq \emptyset$, since C_0 and C_1 are components of $X - x_0$ and $X - x_1$ with $x_0 \in \overline{C}_0$ and $x_1 \in \overline{C}_1$. Now $(\overline{V} - V) \cap C_0 \neq \emptyset$, for if not, $C_0 = (C_0 - \overline{V}) \cup (C_0 \cap V)$ with $C_0 - \overline{V}$ and $C \cap V$ non-void open sets since $C_0 \cap V \neq \emptyset$ and $y_0 \in C_0 - \overline{V}$. Therefore, C_0 is not connected, a contradiction to C_0 being a component. This gives us that $(\overline{V} - V) \cap C_0 \neq \emptyset$ or $(\overline{V} - V) \cap C_0 = \{z_0\}$. Same type of argument gives $(\overline{V} - V) \cap C_1 = \{z_1\}$, since $C_0 \cap C_1 = \emptyset$ and $z_0 \neq z_1$. This contradicts that $\overline{V} - V$ is a singleton set. Therefore, the assumption that both x_0 and x_1 are both in V is false. Hence, x_0 or x_1 is not in V. Without loss of generality, suppose $x_1 \notin V$. Then $x_1 \in E \subset \overline{V}$ or $x_1 \in \overline{V} - V$. Suppose $(\overline{V} - V) \cap C_1 = \emptyset$. Then $C_1 = (C_1 - \overline{V}) \cup (C_1 \cap V)$, $C_1 - \overline{V} \neq \emptyset$, $C_1 \cap V \neq \emptyset$ and $C_1 - \overline{V} | C_1 \cap V$. This is a contradiction to C_1 being a component, Hence, $(\overline{V} - V) \cap C_1 \neq \emptyset$. A similar argument shows that $(\overline{V} - V) \cap C_0 \neq \emptyset$. Therefore, $x_1 \in C_0$ and $x_1 \in C_1$ which contradicts $C_0 \cap C_1 = \emptyset$. Hence E contains at most one cutpoint of X. Q.E.D.

Theorem 2.44 If X is a connected, locally connected Hausdorff space and E is an endelement of X containing a cutpoint x of X, then $E - x$ and $X - E$ are separated sets.

Proof. In a locally connected space a component of an open set is an open set. Therefore, it is sufficient to show that the component of $X - x$ containing $E - x$ is $E - x$.

Let C_0 be the component containing $E - x$. Suppose there exist a $y \in C_0 - E$. Since x separates X then $C_0 \neq X - x$ and there exist a component C distinct from C_0. X is a locally connected Hausdorff space, so there exist a connected open set U such that $x \in U$, $y \notin U$ and $C - U \neq \emptyset$. Then $(C_0 \cup U) - y$ is an open set containing E. Since X is locally connected there exists a connected open set V such that $E \subset \overline{V} \subset (C_0 \cup U) - y$ and $\overline{V} - V = \{p\}$. Suppose $p \notin C_0$. Then $C_0 = (C_0 - \overline{V}) \cup (C_0 \cap V)$ with $(C_0 - \overline{V}) \mid (C_0 \cap V)$. This contradicts that C_0 is a component. Therefore, $p \in C_0$ and $p \neq x$. This implies $x \in V$ and $C \cap V \neq \emptyset$. Suppose $p \notin C$. Then $C = (C \cap V) \cup (C - \overline{V})$ with $(C \cap V) \mid (C - \overline{V})$. This is a contradiction to C being a component. Then $p \in C$ and $p \in C_0$ which contradicts $C_0 \cap C = \emptyset$. Therefore the assumption that there exists $y \in C_0 - E$ is false. Then the component of $X - x$ containing $E - x$ is $E - x$. Therefore $E - x$ and $X - E$ are separated sets. Q.E.D.

We are now ready to present the main theorem of this chapter. This theorem gives us a method to induce a quasi order on a locally connected continuum with an endelement. The following definitions establish some of the additional terminology that will be used.

Definition 2.45 Let X be a locally connected continuum with an endelement E. Define a relation, $\leq$, in X by $x \leq y$ if, and only if, $x \in E$, or $x = y$, or x separates E and y in X. We will refer to this relation as the cut-point ordering.

Theorem 2.46 The relation $\leq$ defined in Definition 2.45 is a semi-continuous quasi order. If E is a single point, then $\leq$ is a partial order.

Proof. To show that $\leq$ is a quasi order, we need to consider three cases to show that the relation is transitive. It follows from the definition that the relation is reflexive.

Let $a \leq b$ and $b \leq c$.

(i) If $a = b$ or $b = c$ then the definition asserts $a \leq c$.

(ii) Assume $a \neq b \neq c$ and that $c \in E$. Then $c \leq b$ and, since $b \leq c$, b must belong to E. A similar argument implies that $a \in E$ and it follows that $a \leq c$. Now assume that $a \neq b \neq c$ and $b \in E$. Then $b \leq a$ and this implies that a must belong to E. Hence, $a \leq c$. If $a \neq b \neq c$ and $a \in E$ then $a \leq c$.

(iii) Assume that $a \neq b \neq c$ and that $a, b, c \notin E$. Then from the definition of $\leq$, it follows that

$$X - a = A \cup B, \quad A \mid B, \quad E \subset A, \quad b \in B, \text{ and}$$
$$X - b = A_1 \cup B_1, \quad A_1 \mid B_1, \quad E \subset A, \quad c \in B.$$

Now suppose $c \in A$. Then a is an element of A_1 or B_1. If $a \in A_1$ then

$X = (A \cap B_1) \cup (A \cup B)$ and $(A \cap B_1) | (A \cup B)$.

This is a contradiction to the fact that X is connected.

If $a \in B_1$ then

$$X - a = (A \cap B_1) \cup (A_1 \cup B),$$

with $c \in A \cap B$, and $E \subset A_1 \cup B$. Then $a \leq c$.

Therefore, we have $a \leq c$ for all possible arrangements of a, b, c. Thus the relation $\leq$ is a quasi order.

To show that the relation $\leq$ is semicontinuous, we use Theorem 2.22 and show that $L(x)$ and $M(x)$ are closed, for each $x \in X$. To show that $L(x)$ and $M(x)$ are closed, we consider the two cases, $x \in E$ or $x \in X - E$.

(i) Suppose $x \in E$. Then from the definition of $\leq$, $L(x) = E$ and $M(x) = X$. E and X are both closed, so we have $L(x)$ and $M(x)$ closed if $x \in E$.

(ii) Suppose $x \notin E$. Then

$$L(x) = \{x\} \cup E \cup \{y | y \text{ separates } E \text{ and } x\}.$$

Since the points of E are not separated by any point, we have

$$\{y | y \text{ separates } E \text{ and } x\} = \{y | y \text{ separates } a \text{ and } x, a \in E\}.$$

If a and x are two points in a connected locally connected Hausdorff space X, then the set of cut points separating a and x is closed (Hocking and Young [6], p. 110, Th 3.8). Then $L(x)$ is the union of a finite number of closed sets, which implies that $L(x)$ is closed.

Suppose $x \in X - E$. Then

$$M(x) = \{x\} \cup \{y \mid x \text{ separates } E \text{ and } y \text{ in } X\}.$$

Again we can represent $\{y \mid x \text{ separates } E \text{ and } y\}$ by $\{y \mid x \text{ separates } a \text{ and } y,\ a \in E\}$. By an argument similar to that above, this set is closed. Then $M(x)$ is expressed as the union of two closed sets, which implies $M(x)$ is a closed set.

Therefore, we have shown that the relation $\leq$ is a semicontinuous quasi order. We now show that the quasi order is a partial order if E is a singleton set.

If E is a single point and $a \leq b$, $b \leq a$ then $a = b$. For if $a \neq b$ then, either a or b does not belong to E. Suppose $a \in E$. Then $a \leq b$, but b cannot separate a from E and $b \not\leq a$, Then both a and b do not belong to E. Then we have

$$X - a = A \cup B, \quad A \mid B, \quad E \subset A, \ b \in B$$

and

$$X - b = A_1 \cup B_1, \ A_1 \mid B_1, \ E \subset A, \ a \in B_1.$$

This implies that the following holds,

$$X = (A \cap A_1) \cup (B \cup B_1), \ (A \cap A_1) \mid (B \cup B_1).$$

This contradicts X being connected. Therefore $\leq$ is anti-symmetric, Hence $\leq$ is a partial order. Q.E.D.

To show that it is necessary for the space to be locally connected we give the following example.

<u>Example 2.47</u> Let $X = \bigcup_{i=-1}^{\infty} A_i$ where the A_i are defined as follows:

$$A_{-1} = \{(x, 0) \mid 0 \leq x \leq 1\},$$

$$A_0 = \{(0, y) \mid 0 \leq y \leq 1\}, \text{ and}$$

$$A_n = \{(x, y) \mid x = \frac{1}{n}, 0 \leq y \leq 1\}, \quad n = 1, 2, 3, \ldots .$$

We can choose the point $(1, 1)$ as the endelement in Definition 2.45. Now consider the points $a = (0, 0)$ and $b = (0, \frac{1}{2})$. We then have $a \not\leq b$. Every open set U containing a will also contain a point of the form $(x, 0)$, $x \neq 0$. But all points of the form $(x, 0)$, $x \neq 0$ precede b. Thus the order is not semicontinuous. Therefore, the condition of local connectedness is necessary for the order to be a quasi order.

CHAPTER III

CHARACTERIZATIONS OF TREES

In this chapter we develop several characterizations of trees, some of which are commonly found in textbooks. The main idea presented here is the characterization of trees in terms of POTS, a characterization which is fairly new and seldom found in textbooks.

Before further discussion, we will introduce the formal definition for a tree. First, we will agree that when the term continuum is used, we shall usually mean a compact connected Hausdorff space.

Definition 3.1 A tree is a continuum in which every pair of distinct points is separated by a third point.

Example 3.2 Let $A = \bigcup_{n=0}^{2} A_n$, where A_n is defined as follows:

$$A_0 = \{(x, y) \mid y = 0, \ -1 \leq x \leq 1\},$$

$$A_1 = \{(x, y) \mid x = 1, \ -1 \leq y \leq 1\},$$

and

$$A_2 = \{(x, y) \mid x = -1, \ -1 \leq y \leq 1\}.$$

Then X is a continuum such that every two points are separated by a third point. Therefore X is an example of a tree.

Example 3.3 There exist a space such that the space is a connected Hausdorff space with the property that every pair of points is separated by a third point. An obvious such example is the space of real numbers. However, the set of real numbers is not compact and consequently is not a tree.

There are three things which sould be emphasized about the definition of a tree. First, we have not assumed the continuum to be locally connected. This is assumed in many definitions. However, one of the first results in this chapter is to prove that a tree is locally connected.

Second, we point out that the continuum is compact. There has been some effort to find conditions necessary in order that a connected, locally connected space X, with the property that each two points can be separated in X by the omission of some third point, admit a non-trivial continuous partial order. Exact conditions necessary for a non-compact space to admit such an order are not known, Ward [25]. However, by adding compactness we can get the desired results. This will be our characterization. It has been shown by Wallace [19] that there does exist a non-compact space that does not admit a nontrivial continuous partial order.

Third, the space is not necessarily a metric space. If the space is metric then a tree is called a dendrite and much has been developed in the study of dendrites or metric trees. For a complete coverage of the study of dendrites one may refer to Whyburn [29], one of the first books to contain the concept of dendrites. Even though it was one of the first, it does contain the major part of what is known about dendrites. An even later reference on this topic is Kuratowski [10].

In the last few years there has been some effort directed at finding conditions for a tree to be metrizable [3]. In other words, given a tree, under what conditions is it a dendrite?

In Chapter II, Theorem 2.45, we developed a method of inducing a partial order on a locally connected continuum. In this chapter we will use this theorem to induce a partial order on a tree. Before using the theorem we must show that a tree is locally connected.

Lemma 3.4 A tree is locally connected.

Proof. Let X be a tree. X is regular if, for every point $p \in X$ and every point q of X distinct from p, q is separated from p by a finite set, (Moore [11], p. 129). We then have that every tree is regular in this sense. Then by (Moore [11], p. 129, Theorem 78) if X is a regular continuum then for all points $p \in X$, every domain U such that $p \in U$ contains a domain V containing p and V is bounded by a finite subset of U. Then by (Whyburn [29], p. 19, Theorem 13.1) it follows that X is locally connected. Q.E.D.

Our main characterization of a tree is in terms of a POTS and, as might be expected, we would like to use the results developed in Chapter II. We can use Theorem 2.46 if a tree contains an endelement. Also, we would like the endelement to be a point so the order will be a partial order. We can get the desired results from a theorem contained in a paper by Wallace [21].

Theorem 3.5 If X is a continuum that contains a cutpoint then it contains an endelement.

Proof. See Wallace [21]. Q.E.D.

Thus, by Theorem 3.5, every tree contains an endelement. But the only prime chains of a tree are cutpoints and endpoints. Consequently an endelement must be a single point. We now have the tools to prove the characterization of a tree in terms of a POTS.

Theorem 3.6 Let X be a compact Hausdorff space. A necessary and sufficient condition that X be a tree is that X admit a partial order, $\leq$, satisfying

(i) $\leq$ is semicontinuous

(ii) $\leq$ is order dense

(iii) for $x \in X$, $y \in X$, it follows that $L(x) \cap L(y)$ is a non-null chain

(iv) $M(x) - x$ is an open set, for each $x \in X$.

Proof. Let X be a tree and choose $e \in X$, such that e is an endelement and let X have the semicontinuous partial order defined in Definition 2.45. By this definition, (i) holds. To show that $\leq$ is order dense, we consider any two distinct points $x, y \in X$. We must show that there exist a point z such that $x < z < y$. From the definition of a tree there exists a point z which separates x and y. So we have

$$X - z = A \cup B, \text{ where } A \mid B,\ x \in A, \text{ and } y \in B.$$

If $x = e$, then from the definition of $\leq$, we have $x < z < y$. If $x \neq e$, then

$$X - x = A_1 \cup B_1, \text{ where } A_1 \mid B_1,\ e \in A_1, \text{ and } y \in B_1,$$

Suppose $z \in A_1$. Then

$$X = (A \cup A_1) \cup (B \cap B_1), \text{ where } (A \cup A_1) | (B \cap B_1),$$

which contradicts X being connected. Therefore, $x < z$. Suppose $e \in B$. Then

$$X = (A \cup B_1) \cup (A_1 \cap B), \text{ where } (A \cup B_1) | (A_1 \cap B),$$

which is a contradiction to X being connected. Therefore, $e \in A$ and $z < y$. Hence, $x < z < y$ and consequently $\leq$ is order dense and therefore condition (ii) holds.

To show (iii), we note that $e \in L(x)$, for all $x \in X$. Then $L(x) \cap L(y) \neq \emptyset$, for all $x, y \in X$. Now we must show that $L(x)$ is a chain. Let $x_1, x_2 \in L(x)$. If $x_1 = e$, then we have $x_1 < x_2 < x$. Suppose that $x_1 \neq e$ and $x_2 \neq e$. Then from the definition of $\leq$, we have

$$X - x_1 = A_1 \cup B_1, \text{ where } A_1 | B_1, \ e \in A_1, \text{ and } x \in B_1$$

and

$$X - x_2 = A_2 \cup B_2, \text{ where } A_2 | B_2, \ e \in A_2, \text{ and } x \in B_2 .$$

Suppose $x_1 \not< x_2$ and $x_2 \not< x_1$. Then $x_2 \in A_1$ and $x_1 \in A_2$. This implies that

$$X = (A_1 \cup A_2) \cup (B_1 \cap B_2), \text{ where } (A_1 \cup A_2) | (B_1 \cap B_2) .$$

This implies that X is not connected, a contradiction. Therefore, $x_1 < x_2$ or $x_2 < x_1$ and this implies that $L(x)$ is a chain. Then any subset of $L(x)$ is a chain and, therefore, $L(x) \cap L(y)$ is a non-null chain. Thus condition (iii) holds.

To show (iv), we consider two cases: $x = e$ and $x \neq e$. If $x = e$ then $M(x) - x = X - x$ and $X - x$ is open. If $x \neq e$ then

$$M(x) - x = \{B_\alpha | X - x = A_\alpha \cup B_\alpha,\ A_\alpha | B_\alpha,\ e \in A_\alpha\}.$$

For all α, B_α is open and the union of an arbitrary number of open sets is open. Therefore, $M(x) - x$ is an open set. This implies condition (iv) holds. We have now shown that if X is a tree, it admits a partial order $\leq$, satisfying conditions (i) - (iv).

Now let X be a space that admits a partial order satisfying conditions (i) - (iv). Suppose there exist two distinct minimal elements x and y. Then $L(x) \cap L(y)$ is a non-null chain, by condition (iii). But $L(x) = x$ and $L(y) = y$ since x and y are minimal elements. Therefore, $x = y$, a contradiction that x and y are distinct elements. Hence X has a unique minimal element e and, therefore, the set of minimal elements is connected. Thus, by Theorem 2.35, X is connected and is, consequently, a continuum.

Let x and y be distinct elements of X. If $x < y$ then, by (ii), there exists $z \in X$ such that $x < z < y$. By (i), $M(z)$ is closed and, by (iv), $M(z) - z$ is open. Hence

$$X - z = (M(z) - z) \cup (X - M(z)), \text{ where } (M(z) - z) | X - M(z),$$

$y \in M(z) - z$, and $x \in X - M(z)$. Therefore z separates x and y. If x and y are not comparable, then, by (iii), $L(x) \cap L(y)$ is a non-null chain. By Theorem 2.31, there exists a maximal element z of $L(x) \cap L(y)$. There exists a t such that $z < t < x$. Then $x \in M(t) - t$, which is open, and $y \in X - M(t)$, which is open. Hence, we have

$X - t = (M(t) - t) \cup (X - M(t))$, where $(M(t) - t) | (X - M(t))$, $x \in M(t) - t$, and $y \in X - M(t)$. Therefore t separates x and y. Hence X is a continuum such that every pair of distinct points is separated by a third point. Q.E.D.

Example 3.7 Let X be the space given in Example 3.2. As an endelement of X we choose the point $(1, 1)$. We then have a partial order on X, by Definition 2.45. The following relations hold: $(1, 1) \leq (a, b)$, for all $a, b \in X$, $(1, 0) \leq (1, -1)$ and $(1, 0) \leq (0, 0)$.

We could have chosen the point $(1, -1)$ as the endelement in order to obtain a partial order. If $(1, -1)$ is used, we then have defined the following relations: $(1, -1) \leq (a, b)$, for all $a, b \in X$, $(1, -1) \leq (1, 0)$, and $(1, 0) \leq (0, 0)$. Direct comparison of this relation with the above reveals that they are different. Therefore, the partial order given by Definition 2.45 may not be unique. In the case of a tree, there will be at least two distinct partial orders. This is true because there exist at least two endpoints of a tree.

Since X is compact and $L(x)$ is closed, Theorem 3.6 could be stated with condition (iii) replaced by

(iii') if $x \in X$ and $y \in X$ then $L(x) \cap L(y)$ is a non-empty compact chain.

Definition 3.8 A Hausdorff space X is said to be dendritic if and only if it is connected, locally connected, and has the property that each two points can be separated in X by the omission of some third point.

The reader should not confuse a dendritic space with a dendrite. As mentioned earlier a dendrite is a metric tree. One should notice that in the above definition the space is not required to be compact while a compact dendritic space is a tree. We mentioned earlier in this chapter that it is not known just how nice a space must be in order to admit a nontrivial continuous partial order. However, Ward [25] has stated conditions for a semicontinuous order. The conditions are those stated above, using condition (iii') instead of condition (iii).

Theorem 3.9 A necessary and sufficient condition that a locally connected Hausdorff space be dendritic is that it admit a partial order satisfying (i), (ii), (iii') and (iv).

Proof. See Ward [25]. Q.E.D.

It is natural to seek, at this point, conditions under which a compactification of a dendritic space results in a tree. Ward [25] has found some conditions that imply that a compactification of a dendritic space is a tree.

Definition 3.10 A space X is convex if the sets $L(x)$ and $M(x)$ constitute a subbasis for the closed sets of X, that is, if every closed set of X is the intersection of some family of sets, each of which is the union of a finite family of sets of the form $L(x)$ or $M(x)$.

Theorem 3.11 A convex dendritic space admits a compactification as a tree.

Proof. See Ward [25]. Q.E.D.

The condition that the space be convex in Theorem 3.11 is necessary. For an example of a dendritic space which admits no compactification as a tree see Ward [25].

In the last two theorems we have wandered from our main theme in this chapter. We have included these important related results in hope that it may inspire the reader to further investigate this area and at this point we return to our main efforts of this chapter.

Theorem 3.12 If X is a tree then $\leq$ is continuous.

Proof. To show that $\leq$ is continuous it is necessary to show that, if $x \not\leq y$, then there exist open sets U and V, with $x \in U$ and $y \in V$, such that $a \not\leq b$ whenever $a \in U$ and $b \in V$. Since $\leq$ is order dense, by (ii), we may choose $t \in X$ such that $t < x$ and $t \not\leq y$. We then can choose $U = M(t) - t$ and $V = X - M(t)$. U and V are open sets with the desired properties, so $\leq$ is continuous. Q.E.D.

The order which was developed in Chapter II is often referred to as the cutpoint ordering and we will adopt this terminology in the rest of the paper. It should be pointed out that later on we will introduce another type of ordering which will be referred to as the weak cutpoint ordering.

In Theorem 3.6, the cutpoint ordering was used to get a characterization of trees. The next theorem uses the cutpoint ordering to characterize trees, but with fewer conditions.

Theorem 3.13 Let X be a locally connected continuum. A necessary and sufficient condition that X be a tree is that the cutpoint ordering be order-dense.

Proof. Suppose the cutpoint ordering is order dense. Let x and y be distinct elements of X. If $x < y$ in the cutpoint ordering then $x = e$ or

$$X - x = A \cup B, \text{ where } A \mid B, \ e \in A, \text{ and } y \in B.$$

Since X is locally connected we can pick A such that A is a component, The cutpoint ordering is order dense so there exists a point p such that $x < p < y$. If $x = e$ then, by the definition of the ordering, p separates x and y. If $x \neq e$ then

$$X - p = C \cup D, \text{ where } C \mid D, \ e \in C, \text{ and } y \in D.$$

Now $p \in B$, since $x < p$. Since A is connected, $A \subset C$. Then $x \in C$, since $A \cup \{x\}$ is connected. This implies that $x \in C$ and $y \in D$ and therefore p separates x from y.

If x and y are not comparable, let z be a maximal element in $L(x) \cap L(y)$. Since the order is dense, there exist a p such that $z < p < y$. Thus we have

$$X - p = A \cup B, \text{ where } A \mid B, \ e \in A, \text{ and } y \in B$$

$$X - z = C \cup D, \text{ where } C \mid D, \ e \in C, \text{ and } p \in D.$$

Since X is locally connected we can pick C such that it is connected. Since $C \cup \{z\}$ is connected, we have $C \subset A$. Thus $M(p) = B \cup \{p\}$ and $p \not\leq x$. Hence, $x \in A$ and $y \in B$ and, therefore, p separates x and y. Thus, if x and y are distinct elements, there exist a point separating x and y. Therefore, if the cutpoint is order dense, then X is a tree. If X is a tree then, by Theorem 3.6, the cutpoint ordering is order dense. Q.E.D.

Definition 3.14 A property P of a space is hereditary if and only if each subspace of a space with P also has P.

In this paper the spaces under consideration are continua and because of this we will use the term hereditary to mean the following: a property P of a continuum is hereditary if and only if each subcontinuum also has property P.

The property of a continuum being a tree is hereditary, as is stated in the next theorem. This proof is fairly obvious, but it is included here because we will need this result to prove a later theorem.

Theorem 3.15 Every subcontinuum of a tree is a tree.

Proof. Let K be a subcontinuum of a tree X. If x and y are distinct points of K, then there exists a point p of X such that p separates x and y. Suppose $p \notin K$. Then

$$K = (K \cap A) \cup (K \cap B)$$

where

$$X - p = A \cup B, \quad A \mid B, \quad x \in A, \text{ and } y \in B.$$

Hence $(K \cap A) \mid (K \cap B)$ and K is not a subcontinuum of X. This contradicts the fact that K is a subcontinuum of X. Therefore $p \in K$ and p separates x from y. Hence K is a tree. Q.E.D.

Definition 3.16 A continuum C is unicoherent provided that, if $C = H \cup K$, where H and K are subcontinua, then $H \cap K$ is connected. A continuum is hereditarily unicoherent if every subcontinuum is unicoherent.

<u>Example 3.17</u> Let X be the unit circle, $H = \{(x, y):(x, y) \in X, x < \frac{1}{2}\}$ and $K = \{(x, y):(x, y) \in X, x \geq -\frac{1}{2}\}$. H and K are subcontinua of X and $H \cap K \neq \emptyset$. $H \cap K = \{(x, y): (x, y) \in X, -\frac{1}{2} \leq x \leq \frac{1}{2}\}$, which is not connected. Therefore X is an example of a continuum which is not unicoherent. The real number line is an example of a non-compact continuum that is unicoherent and hereditarily unicoherent.

The next characterization is in terms of a hereditarily unicoherent locally connected continuum. This is one of the more standard characterizations, and it is included because the proof illustrates a use of the cutpoint ordering. Part of the proof depends on what will be developed in Chapter IV, but care has been taken to avoid a circular argument.

<u>Theorem 3.18</u> A necessary and sufficient condition that X be a tree is that X be a hereditarily unicoherent locally connected continuum.

<u>Proof.</u> By Lemma 3.4, a tree is locally connected. In Chapter IV we introduce the concept of a generalized tree and it is shown that every tree is a generalized tree and that every generalized tree is hereditarily unicoherent. Using these results of Chapter IV, it follows that every tree is a hereditarily unicoherent locally connected continuum.

Let $e \in X$ and $x \in X$ such that $x \neq e$. Let A_α be the collection of all continua containing e and x. This collection is non-empty since $X \in A_\alpha$. Define $U(x) = \bigcap A_\alpha$. Since X is hereditarily unicoherent, $U(x)$ is a continuum containing e and x which is also irreducible about e and x. Now define $x \leq y$, if $U(x) \subset U(y)$.

To show that X is a tree, it is necessary to show that $\leq$ is the cutpoint ordering and is order dense. It will follow, by Theorem 3.13, that X is a tree.

Let $p \in X$ and let

$$X - p = A \cup B, \text{ where } A \mid B \text{ and } e \in A.$$

Since X is locally connected, we can choose A such that A is connected. Since A is connected it is not possible to find a smaller connected set containing e in $X - p$. If there did exist a smaller open connected set containing e then it would follow that

$$X - p = C \cup D, \text{ where } C \mid D, \ A \cap C \neq \emptyset \text{ and } A \cap D \neq \emptyset.$$

This contradicts the fact that A is connected. Therefore, $A \cup \{p\} = U(p)$ and $U(p) \subset U(x)$ if and only if $x \in B$. Hence if x is greater than p in the cutpoint order, then $x \in B$ and x is greater than p in the order $\leq$. Also, if x is greater than p in the order $\leq$, then $x \in B$ and x is greater than p in the cutpoint order. Consequently $\leq$ and the cutpoint ordering are the same.

Also $\leq$ is order dense. For suppose not. Then there exist points x and y such that $x < y$ or $U(x) \subset U(y)$ and there does not exist a point p such that $x < p < y$. Hence $U(x) \cup \{y\} = U(y)$ where both $U(x)$ and $\{y\}$ are closed. But a connected set cannot be written as the union of two disjoint closed sets. Therefore, $U(y) \neq U(x) \cup \{y\}$ and there must exist $p \in U(x) - U(y)$ such that $U(x) \subset U(p) \subset U(y)$ or $x < p < y$. Therefore $\leq$ is order dense and, by Theorem 3.13, X is a tree. Q.E.D.

In this chapter we have stressed the characterization of trees in terms of a POTS. In the next chapter we will introduce a new concept by weakening the conditions in Theorem 3.6. As usual, when conditions are replaced by weaker conditions, certain properties are lost. As will be seen, the condition required in Lemma 3.4, that of local connectedness, will no longer have to hold.

CHAPTER IV

GENERALIZED TREES

In Chapter III, a characterization of trees, in terms of a partial order was given in Theorem 3.6, In this chapter some modifications are made on the conditions stated in Theorem 3.6 and the result is a generalization of the concept of a tree. This generalization was first developed by Ward [28]. In this chapter we have included results of some of Ward's earlier efforts and several characterizations developed by others at later times.

The first results in this chapter establish the fact that all trees are generalized trees. As one might expect, several of the characterizations of generalized trees are very similar to what was developed in Chapter III, but one of the main properties of trees that does not necessarily carry over to generalized trees is that of being locally connected. We will include an example of a generalized tree that is not locally connected.

Before further discussion, we formally state the definition of a generalized tree.

Definition 4.1 A zero of a partially ordered set is an element which precedes all other elements of the set.

Definition 4.2 A compact Hausdorff space X is said to be a generalized tree if and only if X admits a partial order satisfying:

(i') $\leq$ is continuous,

(ii) $\leq$ is order dense,

(iii) for $x \in X$, $y \in X$, it follows that $L(x) \cap L(y)$ is a non-null chain, and

(iv') if Y is a closed and connected subset of X, then Y contains a zero.

There are two important requirements apparently missing in the definition of a generalized tree. The first is that of local connectedness, as has already been pointed out, and the second is that X be a continuum.

Our next example will show that local connectedness is not necessary. This example will also show that there exist generalized trees which are not trees. We will then show that all trees are generalized trees, which will establish the fact that generalized trees are indeed a generalization of the concept of trees.

Example 4.3 Let $X = \bigcup_{n=-1}^{\infty} \{A_n\}$, where A_n is defined as follows:

$$A_{-1} = \{(x, 0) : 0 \leq x \leq 1\},$$

$$A_0 = \{(0, y) : 0 \leq y \leq 1\}, \text{ and}$$

$$A_n = \{(\tfrac{1}{n}, y) : 0 \leq y \leq 1\}, \quad n = 1, 2, 3, \ldots .$$

Define $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 = 0$, or $x_1 = x_2$ and $y_1 \leq y_2$. One can show that this is a partial order satisfying the four conditions stated in the definition of a generalized tree. By considering any point of $A_0 - (0, 0)$, it can be seen that X is not locally connected. Hence X is not a tree.

Since X is not a tree, then the partial order defined must fail to satisfy one of the conditions stated in Theorem 3.3. Condition (iv) of Theorem 3.3 states that if $x \in X$, then $M(x) - x$ is an open set. If we let $x = (0, \frac{1}{2})$, then $M(x) - x = \{(0, y) : \frac{1}{2} < y < 1\}$. However, $M(x) - x$ is not open. If we take any open set containing a point of $M(x) - x$, it will contain points outside of $M(x) - x$. It follows that condition (iv) of Theorem 3.6 is stronger than condition (iv) of Definition 4.2.

Theorem 4.4 If X is a tree then X is a generalized tree.

Proof. It is sufficient to show that the order given in Theorem 3.6 satisfies the conditions given in Definition 4.2. Conditions (ii) and (iii) of Theorem 3.6 are exactly the same as conditions (ii) and (iii) of Definition 4.2. From Theorem 3.12, the order of Theorem 3.6 is continuous. We then have condition (i') holding. By Theorem 3.15, the property of being a tree is hereditary with respect to subcontinua. Since every tree has a zero, every subcontinua also has a zero, and, therefore, every closed connected subset of X contains a zero. Consequently, condition (iv') holds for the order of Theorem 3.6, and every tree satisfies the four conditions of Definition 4.2. Hence, every tree is a generalized tree. Q.E.D.

As pointed out earlier, it was not stated in Definition 4.2 that X was a continuum. However, it follows from the definition that X is a continuum, as the next theorem demonstrates. One will notice that this theorem is much like Theorem 3.18 and that, again, one condition missing, that of local connectedness.

Theorem 4.5. A generalized tree is a hereditarily unicoherent continuum. Conversely, a hereditarily unicoherent continuum which admits a partial order, with zero, satisfying (i') and (ii) is a generalized tree.

Proof. Condition (ii) of Definition 4.2 implies that X is order-dense. By Theorem 2.33, all of the maximal chains of X are connected. All of the points of X will be in one of the maximal chains. Since X has a zero element, all of the maximal chains will intersect. Thus X is the union of connected chains having non-empty intersection and is, therefore, a continuum.

To show that X is hereditarily unicoherent, we first show that, if a and b are elements of a subcontinuum A such that $a < b$, then $M(a) \cap L(b) \subset A$. If this is not true, then we can choose a and b such that $M(a) \cap L(b) - \{a, b\} \cap A = \emptyset$. Now, since X is order-dense, there exists p such that $a < p < b$. Let $U = X - M(c)$, where $a < c < p$. Then $L(a) \subset U$ and $\overline{U} \cap M(z) = \emptyset$. Let B be the component of $A - U$ which contains b. By (iv'), B must have a zero. But the way U was picked implies that $B \cap L(b) = b$. This implies b is the zero. Hence $B \subset M(b)$ and $B \cap \overline{U}$ is empty. But $\overline{U}$ must contain a limit point of each component of $A - \overline{U}$; otherwise there would exist a separation of A, contradicting the fact that A is a continuum. Since $B \cap \overline{U} = \emptyset$, then $\overline{U}$ does not contain a limit point of B, a contradiction. Therefore, $M(a) \cap L(b) \subset A$.

Now suppose C and D are subcontinua of X, with $C \cap D \neq \emptyset$. If $C \cap D = \{z\}$, $C \cap D$ is connected and is a continuum. Suppose that

$$\{x, y\} \subset C \cap D, \text{ where } x \neq y.$$

Let $p = \sup L(x) \cap L(y)$. Then p does exist, since (iii) implies $L(x) \cap L(y) \neq \emptyset$. From the above

$$P = [M(z) \cap L(x)] \cup [M(z) \cap L(y)] \subset A \cap B$$

and will be connected. Hence every pair of points of $A \cap B$ lies in a connected set which is a subset of $A \cap B$. Therefore, $A \cap B$ is closed and connected. Hence, X is a hereditarily unicoherent continuum.

Let X be a hereditarily unicoherent continuum admitting a partial order which is continuous, order dense, and has a zero. First, we show that condition (iii) holds. To show that $L(x) \cap L(y)$ is a non-empty chain, we need to show that $L(x)$ is a chain for all $x \in X$. Suppose there exist elements a and b of $L(x)$ such that a and b are not comparable. Theorem 2.35 implies that $L(a) \cup [M(a) \cap L(x)]$ and $L(b) \cup [M(b) \cap L(x)]$ are connected and, therefore, are continua. Since X is a hereditarily unicoherent continuum, then

$$[L(a) \cup (M(a) \cap L(x))] \cap [L(b) \cup (M(b) \cap L(x))] = P$$

must be connected. But P can be expressed as follows:

$$\begin{aligned} P = {} & \{[L(a) \cup (M(a) \cap L(x))] \cap [L(b) \cup (M(b) \cap L(x))]\} \cap \{L(a) - a\} \\ & \cup \{[L(a) \cup (M(a) \cap L(x))] \cap [L(b) \cup (M(b) \cap L(x))]\} \\ & \cap \{M(a) - a\}. \end{aligned}$$

This exhibits a separation which contradicts the fact that P is connected. Therefore, $L(x)$ is a chain, for all $x \in X$. For all

$x, y \in X$, $L(x)$ and $L(y)$ contain the zero element of X. Hence, $L(x) \cap L(y) \neq \emptyset$ and is a chain.

Let Y be a closed connected subset of X. Suppose there exist two distinct minimal elements x and y of X. Then, $L(x) \cup L(y)$ is a continuum and

$$[L(x) \cup L(y)] \cap Y = \{x, y\},$$

which is not connected. This is a contradiction to the fact that X is a hereditarily unicoherent continuum. Therefore, there exists only one minimal element of Y. Hence the order on X satisfies the conditions of Definition 4.2 and, therefore, X is a generalized tree. Q.E.D.

Before stating and proving the next theorem regarding generalized trees, we define a new concept and state a lemma that is necessary to establish the theorem.

Definition 4.6 An order is monotone if $L(x)$ is connected for each $x \in X$.

Lemma 4.7 If X is a POTS and $\leq$ is monotone then $\leq$ is order-dense.

Proof. Let x and y be elements of X such that $y < x$. Then $y \in L(x)$, which is a connected chain. By Theorem 2.33, $L(x)$ is order dense and there exist a point z such that $y < z < x$. Hence, $\leq$ is order-dense. Q.E.D.

Theorem 4.8 If X is an hereditarily unicoherent continuum with an order $\leq$ which is a monotone closed partial order with a unique minimal element then X is a generalized tree.

Proof. Lemma 4.7 implies that the order is order-dense. Since the partial order is closed, then, by Theorem 2.33, $\leq$ is continuous. Therefore X is a hereditarily unicoherent continuum with a continuous and order-dense order with a minimal element. Hence, by Theorem 4.5, X is a generalized tree. Q.E.D.

In Chapter III the partial order used to characterize trees was referred to as the cutpoint ordering. In this chapter was introduced a new concept, that of generalized trees. At this point we introduce a new partial order which will be called the weak cutpoint ordering. Before giving the formal definition of the weak cutpoint ordering and the characterization, several definitions and lemmas are needed.

Definition 4.9 If W is an open set, the set $\overline{W} - W$ will be called the boundary or frontier of W and will be denoted by F(W).

Definition 4.10 A space X is said to be an arc if and only if it is homeomorphic with the closed interval [0, 1] of the space R of real numbers (Hall [5]).

Another common definition of an arc is; an arc is a compact non-degenerate continuum that does not have more than two non-cut points (Moore [11]).

Definition 4.11 A set X will be said to be arcwise connected provided every two points of X can be joined by an arc lying in X.

It is possible for a space to be arcwise connected, but not be an arc. The space in Example 4.3 is not an arc, but every two points of X can be joined by an arc lying in X. Hence, X is arcwise connected.

Lemma 4.12 Let X be a compact POTS and let w be an open set in X. If

(i) the graph of $\leq$ is closed and

(ii) for any $x \in W$, each open set about x contains an element y with $y < x$,

then any element x of W belongs to a compact connected chain C with C with $C \cap F(W) \neq \emptyset$ and $x = \sup C$.

Proof. See Koch [9]. Q.E.D.

Corollary 4.13 Let X be a compact POTS with unique minimal element 0. If

(i) the graph of $\leq$ is closed and

(ii) $L(x)$ is connected for each $x \in X$,

then X is arcwise connected.

Proof. See Kock [9]. Q.E.D.

The last theorem of this chapter contains three characterizations of generalized trees. Before stating and proving the theorem, several new concepts will be introduced and the new ordering which was mentioned before, the weak cutpoint ordering, will be defined.

One of the characterizations is stated in terms of nets and below are given the definition of a net and two examples of nets. The reader who would like a more detailed treatment of nets may refer to Kelley [8] and Wilansky [30].

Definition 4.14 A set D is directed if D is non-void and there exist a binary relation $\geq$ such that

(a) if m, n and p are members of D such that $m \geq n$ and $n \geq p$, then $m \geq p$;

(b) if $m \in D$, then $m \geq m$; and

(c) if m and n are members of D, then there is p in D such that $p \geq m$ and $p \geq n$.

A <u>directed set</u> is a pair $(D, \geq)$ such that $\geq$ directs D.

<u>Example 4.15</u> The set of positive integers with the natural order is a directed set. The set of real numbers in $(0,1)$ with the usual order is a directed set.

<u>Definition 4.16</u> A <u>net</u> is a pair $(S, \geq)$ such that S is a function and $\geq$ directs the domain of S.

<u>Example 4.17</u> Let D be the set of positive integers directed by the natural order. Define $S: D \to D$ by $S(n) = 2n$. Then $(S, \geq)$ is a net.

The above net is also a sequence and, in general, if the underlying directed set is isomorphic to the set of positive integers then the notion of a net is equivalent to that of a sequence.

<u>Example 4.18</u> Let D be the set of real numbers in the interval $(0,1)$ with the usual order of the reals. Let f be any real valued function defined on $(0,1)$. Then $(f, \geq)$ is a net.

<u>Definition 4.19</u> A subset D' of a directed set D, is called <u>cofinal</u> if, for any $m \in D$, there exist $m' \in D'$ with $m' \geq m$.

<u>Example 4.20</u> Let D be the directed set in Example 4.18. Let D' be the subset of all rational numbers in D. Then D' is a cofinal subset of D.

<u>Definition 4.21</u> A subset D' of a directed set D, is called <u>residual</u> if there exist $m' \in D'$ such that for all $m \in D$ with $m \geq m'$, $m \in D'$.

<u>Example 4.22</u> Let D be the directed set in Example 4.18 and let D' be the subset $[\frac{1}{2}, 1)$. Then D' is a residual set of D.

<u>Definition 4.23</u> If D is a directed set and if $\{A_\gamma : \gamma \in D\}$ is a family of subsets of X, then we define <u>$\limsup A_\gamma$</u> by: $x \in \limsup A_\gamma$ if for each open set U about x there is a cofinal subset $D(U) \subset D$ with $U \cap A_\gamma \neq \emptyset$, for each $\gamma \in D$.

<u>Definition 4.24</u> If D is a directed set and if $\{A_\gamma : \gamma \in D\}$ is a family of subsets of X, then we define <u>$\liminf A_\gamma$</u> by: $x \in \liminf A_\gamma$ if for each open set U about x there is a residual subset $D(U) \subset D$ with $U \cap A_\gamma \neq \emptyset$, for each $\gamma \in D(U)$.

<u>Definition 4.25</u> We write <u>$\lim A_\gamma = A$</u> or $A_\gamma \to A$ provided $\text{lin inf } A_\gamma = A = \limsup A_\gamma$.

<u>Example 4.26</u> Let $X = \bigcup_{n=0}^{\infty} A_n$, where

$$A_0 = \{(x, y) : x = 0,\ 0 \leq y \leq 1\},$$

$$A_{2k} = \{(\tfrac{1}{2k}, y) : 0 \leq y \leq \tfrac{3}{4}\},\quad k = 1, 2, 3, \ldots, \text{ and}$$

$$A_{2k+1} = \{(\tfrac{1}{2k+1}, y) : \tfrac{1}{4} \leq y \leq 1\},\quad k = 1, 2, 3, \ldots .$$

Let D be the set of positive integers and consider the family of sets $\{A_n : n \in D\}$. Here, $\limsup A_n = A_0$ and $\liminf A_n = \{(x, y) : (x, y) \in A_0, \frac{1}{4} \leq y \leq \frac{3}{4}\}$. Thus, the above is an example of a family of subsets where $\liminf A_n \neq \limsup A_n$.

<u>Example 4.27</u> Let $X = \bigcup_{n=0}^{\infty} A_n$, where

$$A_0 = \{(x, y) : x = 0, \ 0 \leq y \leq 1\} \text{ and}$$

$$A_n = \{(\tfrac{1}{n}, y) : 0 \leq y \leq 1\}, \quad n = 1, 2, 3, \ldots .$$

Let D be the set of positive integers and consider the family of sets $\{A_n : n \in D\}$. In this case $\liminf A_n = \limsup A_n = A_0$.

<u>Lemma 4.28</u> An order $\leq$ is closed if and only if, for any net $\{x_\gamma\}$ in X with $x_\gamma \to x$, it follows that $\limsup L(x_\gamma) \subset L(x)$.

<u>Proof.</u> See Strother [15]. Q.E.D.

<u>Lemma 4.29</u> If X is a hereditarily unicoherent continuum then any two points are contained in a unique minimal continuum.

<u>Proof.</u> Let x and y be distinct points of X, a hereditarily unicoherent continuum. Let $\{A_\alpha\}$ be the collection of all continua containing x and y. $\{A_\alpha\}$ is non-empty since $X \in \{A_\alpha\}$. Define $K = \bigcap A_\alpha$. Since each A_α is closed, K is closed. By the definition of a hereditarily unicoherent continuum, K is connected. Therefore, K is a continuum that contains x and y. We now assert that K is the unique minimal continuum containing x and y. For if not, there exists a continuum K' such that $x, y \in K'$. But $K' = A_\alpha$, for some α. Therefore, $K \subset K'$ and, consequently, K' is not a minimal

continuum containing x and y. Hence, in order to avoid a contradiction, K must be the unique minimal continuum containing x and y, Q.E.D.

Definition 4.30 Let X be a hereditarily unicoherent continuum. Denote by $[x, y]$ the unique continuum irreducible from x to y.

Definition 4.31 Let X be a hereditarily unicoherent continuum. Fix $p \in X$ and define $a \leq_p b$ to mean that any continuum K of X which contains p and b also contains a. This ordering is called the weak cutpoint ordering of X with respect to p.

Example 4.32 Let $X = \bigcup_{n=0}^{\infty} A_n$, where

$$A_0 = \{(x, y) \mid y = 0,\ 0 \leq x \leq 1\} \text{ and}$$

$A_n = \{(x, y) \mid (x, y) \in \ell_n$, where ℓ_n is the closed line segment joining the orgin to the point $(1, \frac{1}{n})\}$, $n = 1, 2, 3, \ldots$.

This space is often called the closed infinite broom. It is easy to see that X is not locally connected by considering any point on A_0 other than $(0, 0)$. Therefore the cutpoint ordering defined in Chapter II does not apply here. However, the space is hereditarily unicoherent and, hence, X can be ordered by the weak cutpoint ordering. The fixed point p may be any point of X. However, if $p \in A_0$, then the corresponding ordering will not be continuous. Theorem 4.5 states that if X admits a continuous order dense order then X is a generalized tree, Consequently, if X is a generalized tree there must exist another order. If $p \in X - A_0$, the weak cutpoint ordering will be continuous and order dense. Therefore X is a generalized tree. This example is to point out that every order dense order of a generalized tree is not continuous. However, a continuous order can be induced by choosing an appropriate point p.

In Theorem 4.8 it was proved that, if X is a hereditarily unicoherent continuum with an order which is a monotone closed partial order with unique minimal element, then X is a generalized tree. In the next theorem there are two statements that are equivalent to requiring that an order be a monotone closed partial order with unique minimal element. From these are obtained two more characterizations of a generalized tree. One of these characterizations is stated in terms of the weak cutpoint ordering.

Theorem 4.33 Let X be a hereditarily unicoherent continuum and let $p \in X$. Then the following statements are equivalent:

(i) $\leq_p$ is a monotone, closed partial order on X,

(ii) there exists a monotone, closed partial order $\leq$ on X with a unique minimal element p,

(iii) X is arcwise connected and, for any net $\{x_\gamma\}$ in X, it is true that $[p, x_\gamma] \to [p, x]$, if $x_\gamma \to x$.

Proof. Proof that (i) implies (ii): From the definition of $\leq_p$, p is a minimal element. Suppose there exists another minimal element p'. Every continuum containing p and p' also contains p. Hence, from the definition of $\leq_p$, $p \leq p'$. Now p' is a minimal element and is related to p. Thus $p' \leq p$. But $p' \leq p$ if and only if every continuum containing p also contains p'. Therefore $\{p\}$ is a continuum containing p but not p'. This implies that $p' \not\leq p$, a contradiction to the fact that p' is a minimal element. Hence, p is the unique minimal element.

Proof that (ii) implies (iii). Since the conditions of Corollary 4.13 are satisfied, then X is arcwise connected. Since $[p, x]$ is an irreducible continuum from p to x, then $[p, x]$ is an arc. We now assert that $[p, x]$ is a chain. From Theorem 4.12, there exists a compact connected chain C from x to p. Since X is compact, then C is closed and is a continuum. We will next show that $[p, x] \subset C$. Suppose $[p, x] \subset C$. Then there exists a point $y \in [p, x]$ such that $y \notin C$. Define x' and x'' as follows:

$$x' = \sup\,[C \cap [p, x]], \text{ and}$$
$$x'' = \min\,[C \cap [y, x]].$$

Let C' denote that part of C between x' and x''. Then C' is connected; for if not, then C is not connected. Since $[x', x''] \subset [p, x]$ and $[x', x''] \cap C' = \{x', x''\}$, then $[x', x'']$ is not a unique minimal continuum between x' and x''. It must be the case, therefore, that $[p, x] \subset C$, which implies that $[p, x]$ is a chain.

Let $\{x_\gamma\}$ be a net in X such that $x_\gamma \to x$. It must be shown that $[p, x_\gamma] \to [p, x]$. To show this, consider the following chain of set inclusions:

$$[p, x] \subset \liminf\,[p, x] \subset \limsup\,[p, x_\gamma]$$
$$\subset \limsup L(x_\gamma) \subset L(x).$$

Suppose there exists a $y \in [p, x] - \liminf\,[p, x_\gamma]$. Then there is an open set V such that $y \in V$ and $V \cap [p, x_\gamma] = \emptyset$, for a cofinal set of γ's. Let $A = \{\gamma \mid V \cap [p, x_\gamma] = \emptyset\}$, and $B = \bigcup \{[p, x_\gamma] \mid \gamma \in A\}$. The closure of B, $\overline{B}$ will be a continuum and, since $x_\gamma \to x$, then $x \in \overline{B}$. Thus, $[p, x] \subset \overline{B}$, $y \in \overline{B}$, and $V \cap [p, x_\gamma] \neq \emptyset$, for some $\gamma \in A$, which is a contradiction. Therefore, we have $[p, x] \subset \liminf [p, x_\gamma]$. From the definitions of lim sup and lim inf, it follows that $\liminf [p, x_\gamma] \subset \limsup [p, x_\gamma]$. Since $[p, x_\gamma]$ is the minimal continuum containing p and x_γ, then $[p, x_\gamma] \subset L(x_\gamma)$. Consequently, $\limsup [p, x_\gamma] \subset \limsup L(x_\gamma)$. This last set inclusion and Theorem 4.28 imply that $\limsup L(x_\gamma) \subset L(x)$.

It is next shown that $\limsup [p, x] \subset [p, x]$. Let $z \in \limsup [p, x_\gamma]$. To show that $z \in [p, x]$, it will first be shown that z compares with each element of $\liminf [p, x_\gamma]$. Suppose $y \in \liminf [p, x_\gamma]$ and that y does not compare with z. By Theorem 2.23, $\leq$ is a continuous order. Hence, there exist open sets U and V such that $z \in U$ and $y \in V$ and such that no element of U compares with an element of V. But there exists an γ such that $U \cap [p, x_\gamma] \neq \emptyset$ and $V \cap [p, x_\gamma] \neq \emptyset$. Since $[p, x_\gamma]$ is a chain, the elements of $U \cap [p, x_\gamma]$ compare with those of $V \cap [p, x_\gamma]$, which contradicts the above. Therefore, z compares with every element of $\liminf [p, x_\gamma]$. Since $\liminf [p, x_\gamma]$ is a chain, then $\{z\} \cup \liminf [p, x_\gamma]$ is a chain in $L(x)$.

We now assert that $z \in [p, x]$. For suppose that $z \notin [p, x]$. Define x' and x'' as follows:

$$x' = \sup \{L(x) \cap [p, x]\} \quad \text{and}$$

$$x'' = \min \{M(z) \cap [p, x]\} .$$

Since $x', x'' \in [p, x]$, then $[x', x''] \subset [p, x]$. Consequently,

$[x', z] \cup [z, x'']$ is a continuum and it follows from the definition of x' and x'' that

$$\{x', x''\} = ([x', z] \cup [z, x'']) \cap [x', x''].$$

Thus $[x', x'']$ is not a unique minimal continuum containing x' and x''. This contradicts the fact that X is a hereditarily unicoherent continuum. Therefore, $z \in [p, x]$ and, by the above,

$$[p, x] \subset \liminf [p, x_\gamma] \subset \limsup [p, x_\gamma] \subset [p, x].$$

Hence

$$[p, x] = \liminf [p, x_\gamma] = \limsup [p, x_\gamma] \quad \text{or} \quad [p, x_\gamma] \to [p, x].$$

Proof that (iii) implies (i): To show that $\leq_p$ is monotone, it will be shown that $L(x) = [p, x]$, for all $x \in X$. Let $z \in [p, x]$. Xince X is unicoherent, each continuum containing p and x also contains z. From the definition of $\leq_p$, it follows that $z \in L(x)$. Hence, $[p, x] \subset L(x)$. Now let $z \in L(x)$. The definition of $\leq_p$ implies that every continuum containing p and x also contains z. Therefore, $[p, x]$ will contain z and $L(x) \subset [p, x]$. Hence, $[p, x] = L(x)$ and, since $[p, x]$ is connected, $L(x)$ is connected and $\leq_p$ is monotone.

Let $x, y \in X$ and suppose that $x \leq_p y$ and $y \leq_p x$. By the above, $[p, y] = L(y)$ and $[p, x] = L(x)$. Now $x \leq_p y$ implies that $L(x) \subset L(y)$ and $y \leq_p x$ implies that $L(y) \subset L(x)$. Therefore $L(x) = L(y)$ or $[p, y] = [p, x]$. Since $[p, y]$ and $[p, x]$ are unique minimal continuum, then $x = y$. Hence, $\leq_p$ is antisymmetric and this implies that $\leq_p$ is a partial order on X.

Now if $\{x_\gamma\}$ is any net in X with $x_\gamma \to x$, then $[p, x_\gamma] \to [p, x]$. But $[p, x] = L(x)$ and $[p, x_\gamma] = L(x_\gamma)$, so $\lim\sup L(x_\gamma) = L(x)$. Consequently, by Theorem 4.28 $\leq_p$ is closed. Therefore, $\leq_p$ is a monotone closed partial order on X. Q.E.D.

After reading the last example and theorem, one might expect that all hereditarily unicoherent continuum admit a monotone, closed partial order with unique minimal element. In other words, it might be expected that every hereditarily unicoherent continuum is a generalized tree. The following example demonstrates that this is not the case.

<u>Example 4.34</u> Let Z be the subset of the plane which consists of the unit segment on the x-axis, the unit segment on the y-axis and the vertical segments of length $\frac{1}{2}$ erected over the points with coordinates $(\frac{1}{n}, 0)$, n a positive integer. Let B be the reflection of A through the line $y = 1$, and let $X = A \cup B$.

Suppose X admits a monotone, closed partial order with unique minimal element. Then, by Theorem 2.23, if the order is closed, the order is continuous. Also, the minimal element p is either in A or in B. If $p \in A$, consider points x and y such that $x = (0, \frac{5}{4})$ and $y = (0, \frac{3}{4})$. Hence, $x < y$ and, if U and V are open sets such that $x \in U$ and $y \in V$, then there exist points of U that are greater than points contained in V. This contradicts the definition of a continuous order. Hence, X does not admit such an order and is not a generalized tree. Therefore, there do exist hereditarily unicoherent continua which are not generalized trees.

CHAPTER V

A FIXED POINT THEOREM FOR GENERALIZED TREES

In this chapter we develop a fixed point theorem for generalized trees by using the order properties of the space. Although the result is not new, the approach is different than that used in most proofs.

The study of the fixed point property was initiated by Brouwer's [2] classical theorem introduced in 1912. Since that time, many mathematicians have spent much time and effort in the study of the fixed point property and from these studies have come a variety of results.

One of the early results in this area was a fixed point theorem for dendrites proved by Scherrer [13] in 1926. Several years later, in 1941, Wallace [20] proved that a tree has the fixed point property and, in 1954, Ward [22] proved the fixed point theorem for trees by using the order-theoretic characterization of trees. When Ward [28] introduced the idea of generalized trees, he also proved the fixed point theorem for generalized trees and his proof depended upon the order properties of these spaces. This is the approach we will use in this chapter.

The reader interested in fixed point properties for a larger variety of spaces may refer to Van Der Walt [12]. This book contains a rather complete history of the development of the fixed point property and, although it does not contain proofs, has a very complete bibliography on the topic.

Before proving the main theorems, we prove several lemmas.

Definition 5.1 A subset A of a QOTS, X, is convex provided $A = E(A)$. X is quasi-locally convex provided, whenever $x \in X$ and $E(x) \subset U$, an open set, there is a convex open set V such that $E(x) \subset V \subset U$. X is locally convex provided, whenever $x \in X$ and $x \in U$, an open set, there is a convex open set V such that $x \in X \subset U$.

Definition 5.2 A net $\{x_\gamma\}$ is monotone increasing (decreasing) if, whenever $\lambda \leq \mu$ in Ω, we have $x_\lambda \leq x_\mu$ $(x_\mu \leq x_\lambda)$.

Definition 5.3 If X is a topological space and $\{x_\gamma\}$ is a net, we say $\{x_\gamma\}$ clusters at the point $z \in X$ provided, whenever $z \in U$, an open set, and $\lambda \in \Omega$, there is $\mu \in \Omega$, $\lambda \leq \mu$, such that $x_\mu \in U$.

Definition 5.4 The net $\{x_\gamma\}$ converges to z provided, whenever $z \in U$, an open set, there is $\lambda \in \Omega$ such that $x_\mu \in U$, for all $\lambda \leq \mu$.

Lemma 5.5 Let X be a compact Hausdorff QOTS with continuous quasi order. Then every monotone net in X clusters and the set of cluster points is contained in $E(z)$, for some $z \in X$.

Proof. Let $\{x_\gamma\}$ be a monotone decreasing net in X. Since X is compact, every net has a cluster point (Kelley [8], p. 136). Let z be a cluster point of $\{x_\gamma\}$ and let U be an open set such that $E(z) \subset U$. Since X is a compact Hausdorff QOTS with continuous quasi order, X is quasi-locally convex (Ward [22], p. 147). Then, from the definition of quasi-locally convex, there exist an open set V such that $V = E(V)$ and $E(z) \subset V \subset U$. Since $\{x_\gamma\}$ clusters at z, there exist a λ such that $x_\lambda \in V$. Let $\mu \geq \lambda$. Then there is $\mu' \geq \mu$ such that $x_{\mu'} \in V$. Since x is monotone decreasing, $x_{\mu'} \leq x_\mu \leq x_\lambda$ and V is convex, so that $x_\mu \in V$. Then, for all $\mu \geq \lambda$, $x_\mu \in V$ and $\{x_\gamma\}$ cannot cluster at a point outside of V. If $y \notin E(z)$, define $U_y = X - y$. Then $E(z) \subset U_y$. From the above argument $\{x_\gamma\}$ can cluster only at points of $E(z)$. The same type of argument holds if $\{x_\gamma\}$ is monotone increasing. Q.E.D.

Corollary 5.6 If X is a compact POTS with continuous order, then every monotone net in X converges.

Proof. By Theorem 2.24, a POTS with a continuous partial order is a Hausdorff space. In every Hausdorff space a net converges to its cluster points. Also, in a Hausdorff space a net converges to one and only one point. We then have the desired result. Q.E.D.

Lemma 5.7 Let X be a topological space, $f : X \to X$ continuous, and $x \in X$ such that the sequence $f^n(x)$, $n = 1, 2, \ldots$, clusters at some $z \in X$. Then $f^n(x)$ clusters at $f(z)$.

Proof. Let $f^{n_i}(x) = y_i$ be a subsequence that converges to z. Since f is continuous, $f(y_i)$ converges to $f(z)$. But $f(y_i) = f[f^{n_i}(x)] = f^{n_i+1}(x)$ and the subsequence $f^{n_i+1}(x)$ converges to $f(z)$ or $f^n(x)$ clusters at $f(z)$. Q.E.D.

Lemma 5.8 Let X be a topological space, $f: X \to X$ continuous and $\{x_n\}$, $n = 1, 2, \ldots$, a sequence in X such that $x_n = f(x_{n+1})$. If $\{x_n\}$ clusters at z, then $\{x_n\}$ clusters at $f(z)$.

Proof. If $\{x_n\}$ clusters at z, then $\{f(x_{n+1})\}$ clusters at z. Since f is continuous, $\{f(x_n)\}$ clusters at $f(z)$. But $f(x_n) = x_{n-1}$, which implies that $\{x_{n-1}\}$ clusters at $f(z)$. Since $\{x_{n-1}\}$ and $\{x_n\}$ are the same sequence, the sequence $\{x_n\}$ clusters at $f(z)$. Q.E.D.

Definition 5.9 If X and Y are quasi ordered sets, a function $f: X \to Y$ is order-preserving provided $f(a) \leq f(b)$ in Y whenever $a \leq b$ in X.

Lemma 5.10 Let X be a Hausdorff QOTS with compact maximal chains, $f: X \to X$ continuous and order preserving. A necessary and sufficient condition that there exist a non-null compact set $K \subset E(z)$, for some $z \in X$, such that $f(K) = K$, is that there exist $x \in X$ such that x and $f(x)$ are comparable.

Proof. Suppose there exist a non-null compact set $K \subset E(z)$, for some $z \in X$, such that $f(K) = K$. Let $x \in K \subset E(z)$. Then $x \in f(K)$ and $f(x) \in K \subset E(z)$. If $x \in f(K)$ and $f(x) \in E(z)$ then $x < z < x$ and $f(x) < z < f(x)$. Hence, $x < f(x)$ and $f(x) < x$. Then, for every $x \in K$, x is comparable to $f(x)$.

Now suppose there exist an $x \in X$ such that x and $f(x)$ are comparable. Then either $x < f(x)$ or $f(x) < x$. Since f is order-preserving, either $x < f(x) < f^2(x)$ or $f^2(x) < f(x) < x$. By induction, the sequence $\{f^n(x) \mid n = 1, 2, \ldots\}$ forms a monotone sequence or chain. Then $\{f^n(x) \mid n = 1, 2, \ldots\}$ is contained in a compact maximal chain. From Theorem 5.5, $\{f^n(x)\}$ clusters at some point z and all cluster points are contained in $E(z)$. Let $x \in E(z)$. Then $z < x < z$ and $f(z) < f(x) < f(z)$. By Theorem 5.7, $f^n(x)$ clusters at $f(z)$, which implies that $f(z) \in E(z)$. Therefore, $f(x) \in E(z)$ which implies that $f(E(z)) \subset E(z)$ and $f^n(E(z)) \subset f^{n-1}(f(E(z)))$. Let $K = \cap \{f^n(E(z)) \mid n = 1,2,\ldots\}$. From the fact that $f^n(E(z)) \subset f^{n-1}(f(E(z)))$, K is non-empty. $E(z)$ is closed and is a subset of the maximal chain, so $E(z)$ is compact. Since f is continuous and $f(E(z)) \subset E(z)$, then $f^n(E(z))$ is compact for each n and K is compact. By the definition of K, $f(K) = K$. Therefore, K is a non-empty compact subset of $E(z)$ and $f(K) = K$. Q.E.D.

Definition 5.11 Let X be a topological space and f a function such that $f(X) \subset X$. A point $x \in X$ is a fixed point for f if $f(x) = x$.

Corollary 5.12 If X is partially ordered, then a necessary and sufficient condition that f have a fixed point is that there exist $x \in X$ such that x and $f(x)$ are comparable.

Proof. In a POTS, $E(x) = x$ for all $x \in X$. Hence, the set K of the theorem will be $K = x$ and $f(x) = x$. Therefore $f(x)$ and x are comparable.

On the other hand, if x and $f(x)$ are comparable then there exists a set $K \subset E(z)$, for some $z \in X$, with $f(K) = K$. But, since X is a POTS, $K = z$ and, therefore, $f(z) = z$. Hence, z is a fixed point. Q.E.D.

We now prove two theorems concerning the fixed point property for generalized trees. The first theorem is rather restrictive and holds for only special types of continuous functions. The second theorem is much more general and the only restriction on the function is that it be continuous.

Theorem 5.13 If X is a generalized tree and $f(X) \subset X$ is continuous and order preserving then $f(x) = x$ for some $x \in X$.

Proof. Since X is a generalized tree, there exist a zero z and $z \leq f(z)$. Then from Corollary 5.12 there exists an $x \in X$ such that $f(x) = x$. Q.E.D.

We are now ready to prove the main theorem of this chapter. As pointed out earlier, this is not a new result, but the approach is not the one commonly used. Since we have shown in Chapter IV that a tree is a generalized tree, the theorem also applies to trees.

Theorem 5.13 If X is a generalized tree and $f(X) \subset X$ is continuous, then $f(x) = x$ for some $x \in X$.

Proof. The set $P = \{x \mid x \leq f(x)\}$ is non-empty since X has a zero. Let C be a maximal chain in P and $z = \sup C$. First we show that $z \in P$. Suppose $z \in X - P$. Then, from the definition of P, either $f(z) < z$ or $f(z)$ is not related to z. $L(z) \cap L(f(z)) \neq \emptyset$ since there exists a zero. Let $y = \sup[L(z) \cap L(f(z))]$. Then $y < z$ and there exists an increasing net $\{x_\alpha\}$ such that $y < x_\alpha < z$ and such that $\lim\{x_\alpha\} = z$. Since $\{x_\alpha\} \subset P$, then $f(x_\alpha) \in M(x_\alpha)$, for each α, and, since f is continuous, $\lim\{f(x_\alpha)\} = f(z)$. Since $f(z) \in M(x_\alpha)$, for all α, then $f(z) \in \bigcap\{M(x_\alpha)\} = K$. Since $\{M(x_\alpha)\}$ is a collection of nested continua, then K is a continuum. Therefore,

K has a zero, k. Since z and $f(z)$ are in K, then k is a predecessor of both z and $f(z)$ and, for some α, $k < x_\alpha$. But then, $k \notin M(x_\alpha)$, which contradicts that $k \in \bigcap \{M(x_\alpha)\}$. Therefore, $z \in P$ or $z \leq f(z)$ and z is maximal with respect to this property.

Suppose $z < f(z)$. Then, there exist y such that $z < y < f(z)$. Since $z = \sup C$, then $f([z, y]) \cap M(y) = f(z)$. Now $L(f(y)) \cap L(y) \neq \emptyset$ and both of these sets are continua. Therefore, $L(f(y)) \cup L(y)$ is a continuum. Now, $f(y) \in f([z, y])$ and $f(y) \in L(f(y)) \cup L(y)$. Therefore, $f([z, y]) \cup L(f(y)) \cup L(y)$ is a continuum. If $x' \in [y, f(z)]$ and $x' \neq y$ or $x' \neq f(z)$, then $x' \notin L(y)$ and $x' \notin L(f(y))$, since $z < x'$ and, by the above, $x' \notin f([z, y])$. If $x' \in f([z, y]) \cup L(f(y)) \cup L(y)$ and $x' \neq y$ or $x' \neq f(z)$, then $x' \notin [y, f(z)]$. Therefore

$$\{[y, f(z)]\} \cap \{f([z, y]) \cup L(f(y)) \cup L(y)\} = \{y, f(z)\},$$

which is not a connected set. This contradicts the hereditary unicoherence of X. Therefore, $z \nless f(z)$, which implies that $z = f(z)$. Q.E.D.

Using the order properties of trees, Smithson [14] has proved a fixed point theorem for lower semicontinuous functions. The following is Smithson's theorem.

<u>Theorem 5.14</u> If X is a tree and if $F : X \to X$ is a lower semicontinuous multifunction such that $F(x)$ is connected for all $x \in X$, then F has a fixed point.

<u>Proof.</u> See Smithson [14]. Q.E.D.

There have been many results developed about the fixed point property for trees and generalized trees, but there still exist unanswered questions. One of the unsolved problems, presented by Isbell [7], is the following: If F is a commutative family of continuous mappings of a tree T into itself, does there exist a point $x \in T$ such that $f(x) = x$, for all $f \in F$?

CHAPTER VI

SUMMARY AND CONCLUSIONS

It was intended that this dissertation be written in such a way that a good undergraduate student who has had a first course in topology might grasp the material presented here and that it could be of some use as a guide for a seminar or independent study course for advanced undergraduate students. It should reinforce many of the basic ideas learned in a beginning course in topology and would introduce the student to the concept of a partially ordered topological space; a notion which is seldom found in elementary topology textbooks or is only briefly discussed there.

Chapter II introduces the basic notions of quasi and partially ordered topological spaces and a glance at the number of papers and books that were referred to here indicates that this chapter could be extended into a study in itself. We have just touched on the material in this area and have included only those results needed to get the desired characterizations of trees.

In Chapter III, we have given several characterizations of trees and have emphasized those characterizations involving partially ordered topological spaces. In Chapter IV, we have discussed the concept of generalized trees, a notion which is gotten by weakening the order conditions in the characterization of trees. In Chapter V, the main emphasis is on the fixed point theorem for generalized trees and on the proof given, using order theoretic methods.

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