CHARACTERIZATIONS OF CHAINABLE CONTINUA

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PREFACE

This thesis is a complete expository presentation of the known characterizations of chainable continua which are derived directly from the definition of a chainable continuum. While this study was originally undertaken as a partial fulfillment of the requirements of the Doctor of Education degree, as the study progressed, the aim shifted from just fulfilling requirements to that of providing a complete and detailed development of the characterizations and to reveal the effective exploitation of several techniques of proof. This objective, with limitation to a direct approach, has been achieved.

The first two chapters contain the only results which are either without proofs or with partial proofs. Those of the first chapter were considered to be well outside the main objective of the thesis and, of course, we must begin somewhere. Those of the second chapter were only indirectly related to the subject at hand and were presented only because I felt their unusually strong conclusions would help reveal the degree of control one has over chainable continua. Except for one reference to the theorems in a later example, no other use was made of the results of the second chapter.

The proofs of all other results are complete. The detail in some cases, may, in fact, obscure the plan and techniques of proof which are the important results for any reader. I have attempted to provide explanation of the plans and to point out the techniques. The level of presentation is aimed primarily at a reader familiar with R. L. Moore's book, Foundations of Point Set Topology. However, the reader who has successfully completed a year of study in general topology and who has ready access to Moore's book as a reference, could read the thesis without too much difficulty. Everyone, regardless of background, is encouraged to read with pencil in hand and to construct chains in every conceivable manner.

The second chapter is primarily a presentation of results concerning the construction of chains on chainable continua. Two important results reveal that all chainable continua are atriodic and hereditarily unicoherent. I have taken the approach that the converse of this would be the ideal result. While the third and fifth chapters present results which successively reduce the necessary additional restrictions, it is also noted in the fifth chapter that this ideal cannot be achieved.

It would be impossible for me to acknowledge, by name, the many individuals, whose instruction and encouragement have played a part in helping me achieve this goal. A special note of gratitude must be given to those friends and colleagues whose support and encouragement have kept me going. Although it is impossible to do so fully, I especially wish to thank Dr. John Jobe for his patient and understanding direction of this thesis, which has perhaps taken longer than necessary because of my interest and desire to take courses beyond those required. His approach to learning in the classroom will remain with me throughout my teaching career.

My thanks also go to Dr. E. K. McLachlan, who has assisted me as an academic advisor, instructor, and commitee chairman, and to committee members and instructors; Dr. Robert Alciatore and Dr. Joe Howard. To the Department of Mathematics at Oklahoma State University for providing a graduate assistantship for these years, thank you.

Finally, to my family I owe a special debt. My wife Doris, son Doug, and daughter Debi have endured too long without husband and father, while I devoted much of my time and energy to this endeavor. May they recieve fully what benefits this completion may bring.

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CHAPTER I

INTRODUCTION

Historical Development

The notion of a simple chain was first defined by R. L. Moore [25] in 1916. Although not always using exactly the same definition, numerous references to or uses of simple chains were made during the next thirty-five years. Notable among these were the writings of B. Naster [27] in 1922, J. H. Roberts [31] in 1930, and E. E. Moise [24], R. H. Bing [2] [3], O. H. Hamilton [15], and Bing and F. B. Jones [7] around 1950. Most of these publications, however, used simple chains or particular sequences of chains to produce results not specifically concerned with the ability to construct chains. Nor did they develop the consequences of a chain on a continuum. All of the publications, except that by Roberts, were concerned with what was eventually called the pseudo-arc and the subject of homogeneous plane continua. Other work was also being done during this period on such related subjects as triodic continua [32] and unicoherent continua [23].

In 1951, the publication by R. H. Bing of "Snake-like Continua", [4] introduced the study of this classification of continua for its own sake. This represents the first published result concerning the characterization of the continua which Bing called, snake-like continua. Bing attributes the actual name to Gustov Choquet [4]. The terms chainable continua or linearly chainable continua have, however, gradually replaced Bing's earlier name. The main result of Bing's work was a characterization of chainable continua in terms of atriodic, hereditarily unicoherent, and hereditarily decomposable continua.

The interest in chainable continua since the early 1950's has centered on characterizing these continua without restriction to hereditarily decomposable continua or with the possibility of removing some of the other restrictions. In 1961, L. K. Barrett [1], utilizing the results of H. C. Miller [23], was able to prove several characterizations equivalent to that by Bing. It was not until 1966 and again in 1969 that significant results appeared, reducing the restrictions in the characterizations of chainable continua. J. B. Fugate [13] and [14] was first able to show that a compact metric continuum is chainable if and only if it is atriodic, hereditarily unicoherent, and is the union of two chainable subcontinua. Among his later results is a characterization of chainable continua in terms of the indecomposable subcontinua of an atriodic and hereditarily unicoherent compact metric continuum. These results represent the existing state of knowledge of the characterizations of chainable continua, approaching the subject directly from the definition.

Objectives

The main objective of this thesis is to present as complete a development of the characterizations of chainable continua as possible. However, one limitation will be placed on the study and that is to limit the characterizations to those obtainable directly from the definition of chainable continua. Thus, the study of chainable continua in terms of algebraic topology and inverse limit spaces will be omitted. This limitation is necessitated by the desire to avoid having to assume too extensive a background by any reader and to keep the thesis manageable in terms of length.

Since it is not assumed that every reader will already be familiar with the terms and basic results needed for this presentation, an aim will be to develop the background necessary to understand and prove the theorems. This will be attempted through examples illustrating the definitions and by discussing the method of a proof both before and during the actual presentation of the proof of a theorem. An additional objective is to present in a single reference the works of several individuals which currently appear only in mathematical journals. Any pedegogical value in this study must ultimately lie in the opportunity which it affords to investigate, in depth, a subject of current interest in mathematical journals and which at present only rarely enters into courses in topology. There is also the aim of developing the ability to analyze research developments in mathematics and interpret them in light of the future of mathematics education.

Thesis Notation and Procedures

In order to help facilitate the reading of this thesis, certain policies regarding notation and procedure should be mentioned. When a reference is made in the form, theorem 3.14 or [3.14], then this reference is to the fourteenth result of Chapter III. Generally, except when the reference is to Chapter I, the complete proof of the indicated result appears in the thesis. Certain theorems which appear in the literature and which are used in this presentation, but which are not as accessible as others, will be stated later, without proof, in this chapter. References to the literature will be made in the form, [5, p. 14], where the first number indicates the source as given in the bibliography of this thesis, and the second number indicates the page number within that source.

Definitions will be numbered consecutively only in the first chapter. In the later chapters, definitions will simply be incorporated into the discussion. The basic examples will first appear in a section of this chapter with complete explanation of their construction. They will be referenced later for illustrating specific definitions and results. To assist the reader in finding definitions and examples, an index of the definitions and examples is included as an appendix.

The completion of a proof will be indicated with the symbol ||. For the reader who does not wish to study each proof in detail, and this is sometimes advisable in order to obtain a good overview of the material, this notation will facilitate their reading of the thesis. Some proofs are quite lengthy and this notation serves to indicate the end of the proof and the beginning of the following discussion.

Basic Definitions and Subject Notation

Certain terms and concepts are considered basic to the complete presentation. Those which relate directly to the subject of chainable continua are presented in this section along with some basic assumptions and notational descriptions. Definitions and theorems which are considered basic to any general development of point set topology are not included. These may be found in any standard reference such as <u>Foundations of General Topology</u> by William J. Pervin [29] or <u>General Topology</u> by John L. Kelley [18]. Additionally, all theorems or definitions which do not appear in this thesis or one of the above references, may be found in Foundations of Point Set Topology by R. L. Moore [26].

The basic topological setting for the definitions and theorems of this thesis is a metric space with the metric being denoted by ρ. This will be assumed, without further mention, in the statement of all results. Examples will generally be constructed in the Cartesian plane although this restriction is certainly unnecessary in most cases.

Since the subject of this thesis deals with continua in a metric space, this is obviously the point at which we should begin the definitions. In order to avoid having to state a condition of compactness in each theorem, we shall simply define a continuum to include this property.

1 Definition A compact connected set is called a continuum.

2 Definition Any finite collection $C = \{d_1, d_2, \ldots, d_m\}$ of open sets is a linear chain, or just chain, if $d_i \cap d_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

It should be noted that the definition of a chain does not require the existence of a continuum; ie, chains exist without reference to a continuum. Also it should be noted that no restrictions are placed on the open sets involved. That is, they may or may not be connected point sets. Theorem 2.8 will be a good example of a result which produces a chain where the open sets may very well not be connected. Chains may also be denoted by $C(d_1, \ldots, d_m)$ or simply by $(d_1, \ldots, d_m)$. Once a particular chain has been defined or denoted in a proof and thus when it is not apt to cause confusion, a chain may be further abbreviated as

$C(1,m)$. A subchain of $C(1,m)$ is the collection $\{d_k, d_{k+1}, \cdots, d_n\}$ $= C(k,n)$ which is a subcollection of $C = \{d_1, d_2, \cdots, d_m\}$. The notation $C^*(1,m)$ will be used to denote the set, $\bigcup_{i=1}^{m} d_i$.

3 <u>Definition</u> If $C = \{d_1, d_2, \cdots, d_m\}$ is a chain then each set d_i is called a <u>link</u> of the chain C. If $|i - j| = 1$ then the links d_i and d_j are called <u>adjacent links</u> of $C(1,m)$. Otherwise, distinct links are called <u>nonadjacent links</u>.

Although definition 2 requires that $d_i \cap d_j = \emptyset$ for $|i - j| > 1$, where d_i and d_j are open sets, it is possible that the links may have some limit points in common. That is, no restrictions have been placed on the possibility that $\overline{d}_i \cap \overline{d}_j \neq \emptyset$ with $|i - j| > 1$. Thus, the following definition describes just this type of chain.

4 <u>Definition</u> A chain $C = \{d_1, d_2, \cdots, d_m\}$ is <u>taut</u> if and only if the intersection of the closures of two nonadjacent links is empty. That is, $C(1,m)$ is taut if and only if $\overline{d}_i \cap \overline{d}_j = \emptyset$ for $|i - j| > 1$. This is equivalent to requiring that $\rho(d_i, d_j) > 0$ for $|i - j| > 1$ [13, p. 460].

It will frequently be necessary to construct a chain which is contained within another chain. That is, if C and F are chains, then F is contained within C if and only if $F^* \subseteq C^*$. This form of containment is usually too general and the following definition places further restrictions on the two chains.

5 <u>Definition</u> Let C and F be two chains. The chain F is said to be a <u>refinement</u> of chain C if and only if each link of the chain F is contained in some link of the chain C. The refinement is

called a <u>closed refinement</u> if and only if the closure of each link of F is contained in some link of C.

Definition 6 is the first definition relating chains to continua and is of course the basis for defining a chainable continuum.

6 <u>Definition</u> Let M be a continuum. If C is a chain such that each link of C contains a point of M and $M \subseteq C^*$, then C is a <u>chain on the continuum</u> M. If C is a chain such that $M \subseteq C^*$, but it is not known whether each link of C contains a point of M or not, then C is said to be a <u>chain covering the continuum</u> M.

This definition does not require a certain type of minimality. That is, if $C = \{d_1, d_2, \cdots, d_m\}$ is a chain on the continuum M, it may also be that C(1,m-1), C(2,m), or even that C(2,m-1) is a chain on M. The definitions of a chain and a chain on the continuum M exclude the possibility of omitting any other links. This property of minimality is sometimes desirous and shall be described in the future simply by the following.

7 <u>Definition</u> If C is a chain on the continuum M then C is a <u>minimal chain</u> on M if and only if no proper subchain of C is also a chain on M.

This property is also a type of irreducibility and thus we may sometimes refer to a chain on the continuum M which is irreducible, meaning that the chain is a minimal chain on the continuum M.

Thus far, none of the definitions have directly required that the topological space under consideration be a metric space. The following two definitions show the necessity of having made this a general requirment when studying chainable continua.

8 Definition If each link of a chain C has a diameter less than ε, then the chain is called an ε-chain.

9 Definition If M is a continuum, then M is a chainable continuum if and only if for every real number $\varepsilon > 0$, there is an ε-chain on the continuum M.

An arc in the plane is the simplest example of a nondegenerate chainable continuum. Because of the ease of constructing such an example in the plane, we shall usually not give special mention to it. Examples of chainable continua which are not arcs will be described in the following section of this chapter.

The remaining definitions relate to specific properties of or requirements for chainable continua.

10 Definition The continuum M is said to be a triod if and only if M is the union of the three proper subcontinua A, B, and C such that $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ and $A \cap B \cap C$ is properly contained in each of A, B, and C. A continuum is said to be atriodic if and only if it contains no triod.

11 Definition A simple triod is a triod such that the common part of the three proper subcontinua which form the triod is degenerate.

One usually thinks of a simple triod as being a simple "Y" or "T" shaped continuum.

12 Definition The continuum M is unicoherent if and only if whenever M is the union of two subcontinua, their intersection is a

continuum. A continuum M is said to be hereditarily unicoherent if and only if every subcontinuum is unicoherent.

13 Definition A continuum is decomposable if and only if it can be expressed as the union of two proper subcontinua. If each nondegenerate subcontinuum of a continuum is decomposable then it is called a hereditarily decomposable continuum.

14 Definition A continuum is indecomposable if and only if it is nondegenerate and not decomposable. It is hereditarily indecomposable if each subcontinuum is indecomposable.

Fundamental Examples

A number of examples are described in this section for reference throughout the remainder of the thesis. They also provide immediate examples of continua which illustrate the definition of a chainable continuum in the preceding section. The definitions of some of the terms used in naming or describing the examples will be given in later chapters and have no significance to the actual description of the examples. Thus, what is meant by an end point or opposite end point is immaterial at present.

A Example Closed Topologist's Sine Curve [33, p. 137]. Let M be the closure of $\{ (x,y) : y = \sin(\frac{2\pi}{x}),\ 0 < x \leq 1 \}$. Because M is the closure of this set, M clearly includes the points $\{ (0,y) : -1 \leq y \leq 1 \}$ (see Figure 1.) For a given $\varepsilon > 0$ the method of constructing an ε-chain on the Closed Topologist's Sine Curve is illustrated in Figure 2.

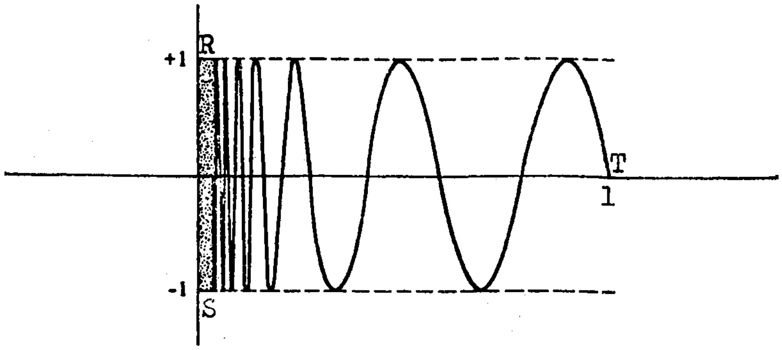

Figure 1. Closed Topologist's Sine Curve

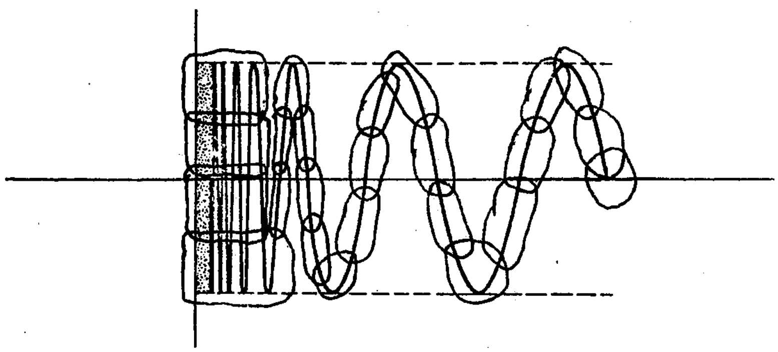

Figure 2. Chain on the Closed Topologist's Sine Curve

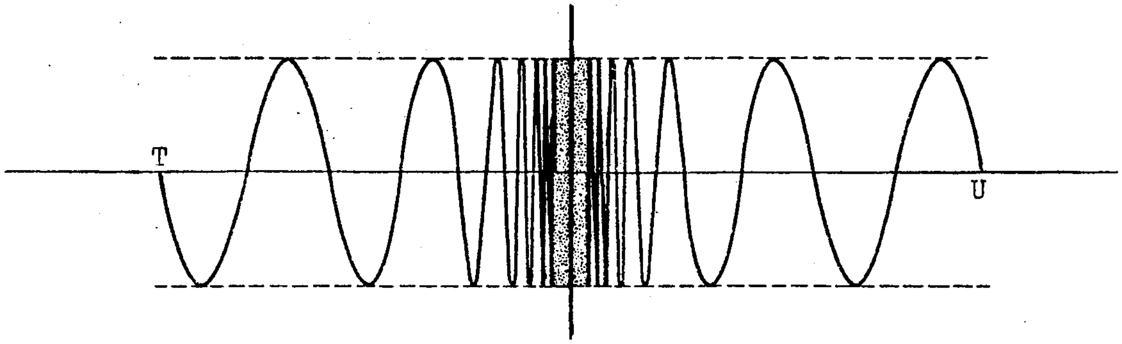

Figure 3. A Chainable Continuum With Two End Points

B <u>Example A Chainable continuum with two end points.</u> This continuum is the union of the continuum of example A and the continuum which is symmetric to it with respect to the origin. Figure 3 illustrates this example which is ε-chainable in a manner similar to example A. The significance of this example is that it is a chainable continuum with the two end points, T and U, but it is not an arc.

C <u>Example A chainable continuum with four end points.</u> This continuum is the union of the continuum of example A and the continuum which is symmetric to it with respect to the point (1,0) or point T of Figure 1. An illustration of this example is given in Figure 4. Again, this example is ε-chainable in a manner similar to example A.

D <u>Example A chainable continuum (indecomposable) with one end point</u> [4, p. 662]. Let $C = \bigcap_{k=1}^{\infty}\left(\bigcup_{n=0}^{\frac{3^k-1}{2}}\left[\frac{2n}{3^k}, \frac{2n+1}{3^k}\right]\right)$, where [a,b] denotes the closed interval from a to b. The point set C is the Cantor set in the plane. Let M_0 be the union of all semicircles in the upper half plane with both end points being elements of C and center at the point $(\frac{1}{2},0)$. Let M_i, i = 1, 2, 3, ..., be the union of all semicircles in the lower half plane with both end points being elements of C and center at the point $(\frac{5}{2\cdot 3^i}, 0)$. Then $M = \bigcup_{i=0}^{\infty} M_i$ is a chainable continuum with only one end point, namely the origin. This continuum is also indecomposable [19, p. 204] and is illustrated in Figure 5.

E <u>Example A chainable continuum (indecomposable) with no end points</u> [4, p. 662]. This continuum is the union of the continuum M of

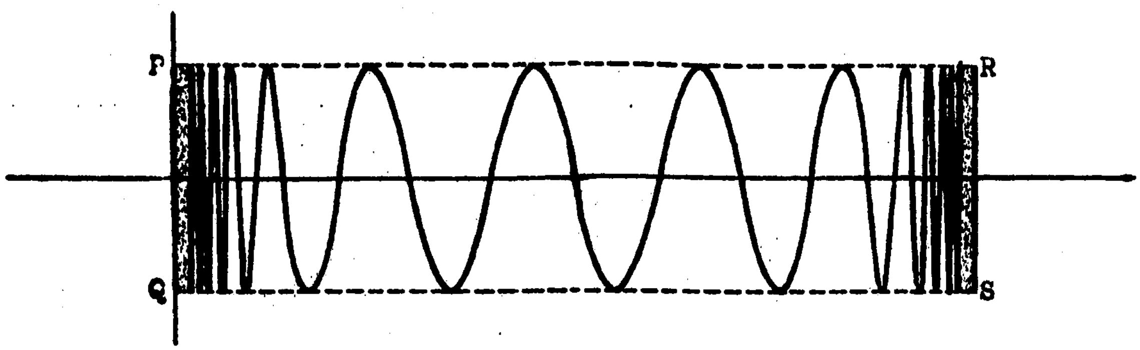

Figure 4. A Chainable Continuum With Four End Points

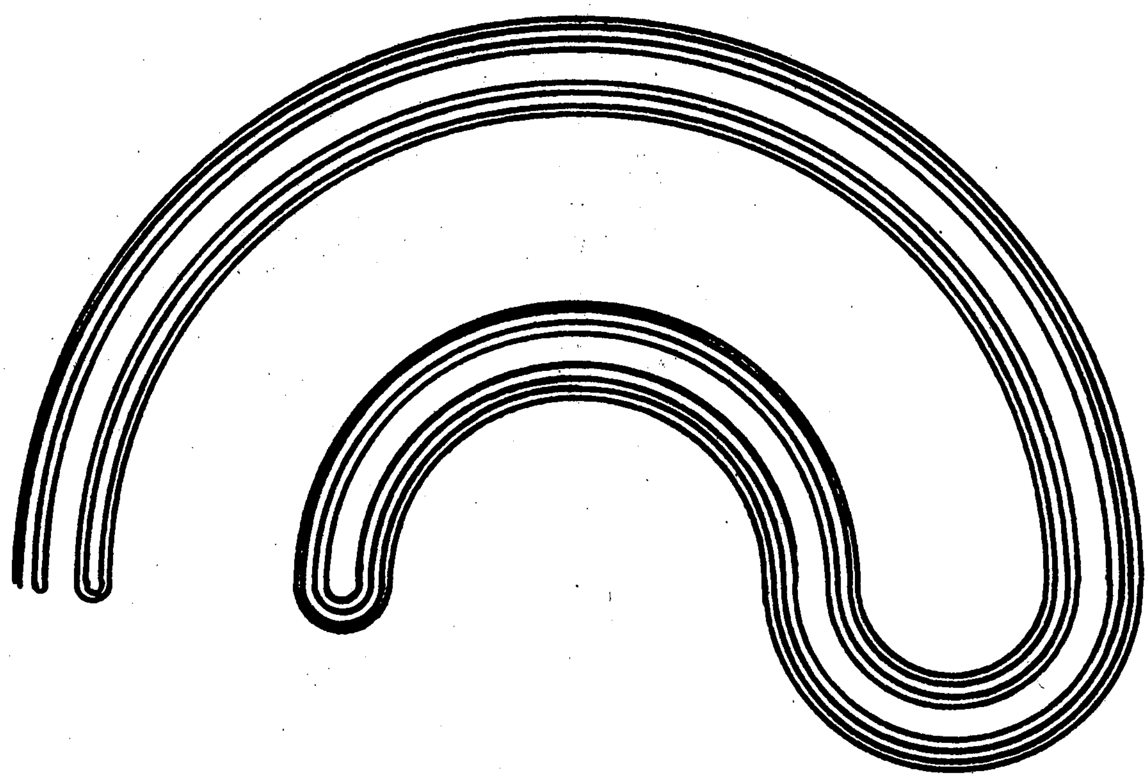

Figure 5. A Chainable Continuum (Indecomposable) With One End Point

example D and the continuum which is symmetric to it with respect to the origin. Figure 6 illustrates this example.

F <u>Example</u> <u>Chainable continuum (indecomposable) with two end points</u>. Let $E = \bigcap_{k=1}^{\infty}\{ \bigcup_{i=0}^{3^{k-1}-1} (\bigcup_{j=0}^{2}[\frac{5\cdot 2^i+2j}{5^k}, \frac{5\cdot 2^i+2j+1}{5^k}]) \}$, where $[a,b]$ denotes the closed interval from a to b. Then E is the set of all numbers of the unit interval which can be written in the base five enumeration system without the digits 1 and 3. Let $E_n = E \cap [\frac{2}{5^{n+1}}, \frac{1}{5^n}]$ for $n = 0, 1, 2, \cdots$ and let $F_n = \{ x : 1-x \in E_n \}$. Let M_n be the union of all semicircles in the lower half plane with centers at the point $(\frac{7}{10\cdot 5^n}, 0)$ and end points elements of E_n, $n = 0, 1, 2, \cdots$. Let N_n be the union of all semicircles in the upper half plane with center at the point $(1-\frac{7}{10\cdot 5^n}, 0)$ and end points elements of F_n, $n = 0, 1, 2, \cdots$. Then $M = \bigcup_{n=0}^{\infty}(M_n \cup N_n)$ is a chainable continuum with two end points, namely the origin and the point $(1,0)$. This continuum is indecomposable [19, p. 205] and is illustrated in Figure 7.

G <u>Example</u> <u>Hereditarily indecomposable chainable continuum</u>. This final example is the pseudo-arc and except for listing it here with the fundamental examples, we shall not further define or discuss it. The pseudo-arc is the subject of a thesis by McKellips [22] and any reader wishing to pursue the subject further is refered to that source or one of the original sources [27] or [24].

Reference Theorems

This chapter will conclude with the statement of several results

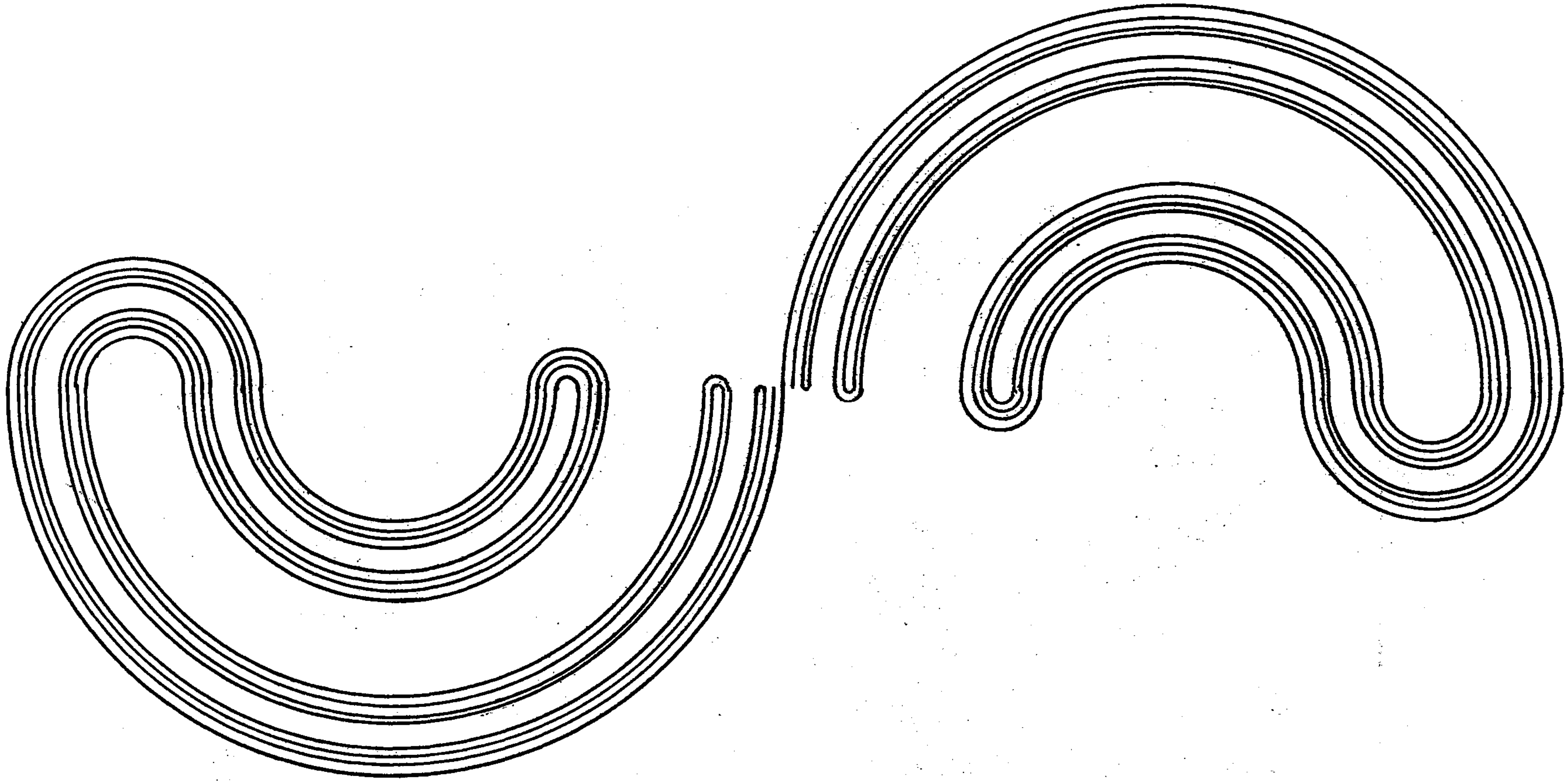

Figure 6. A Chainable Continuum (Indecomposable) With No End Points

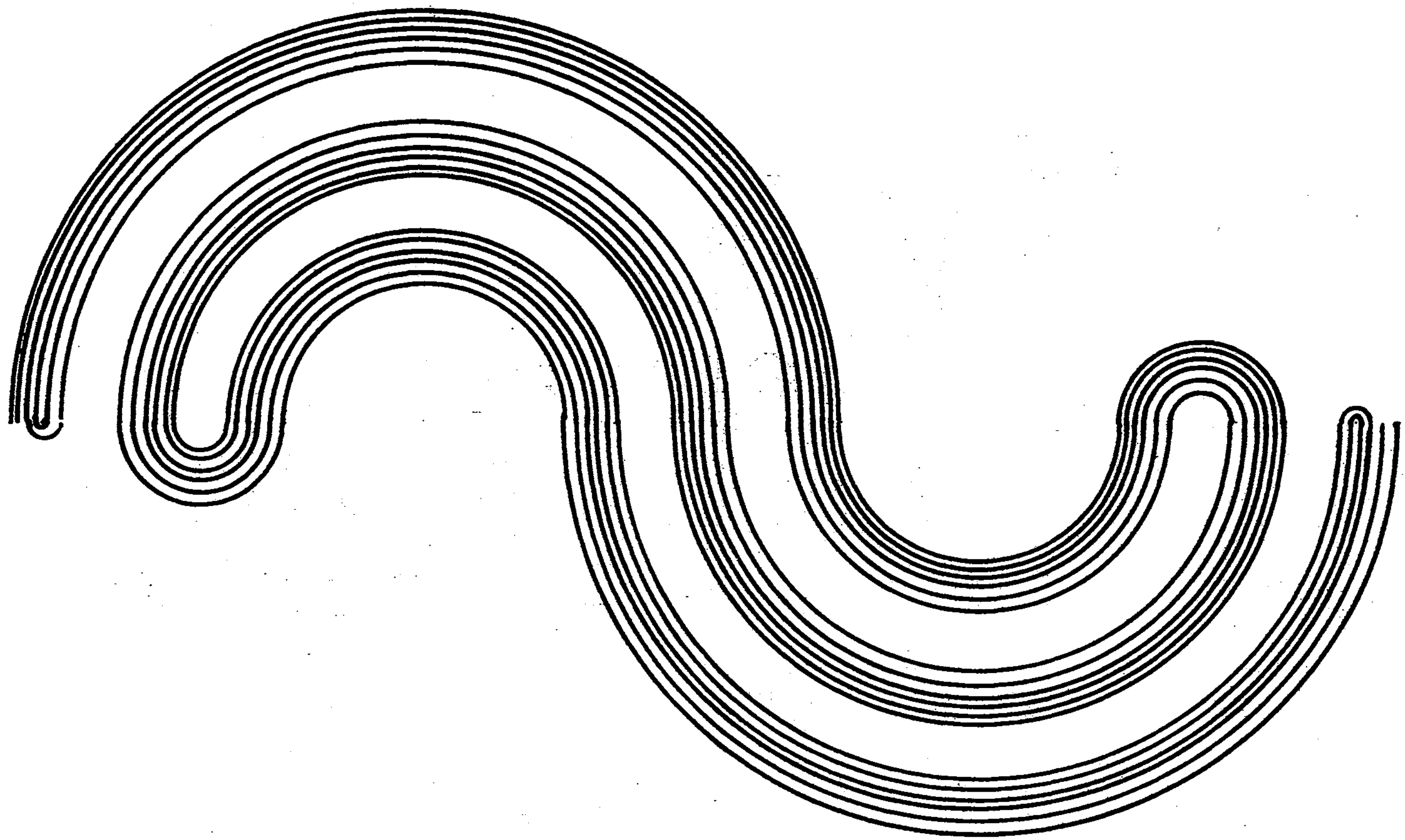

Figure 7. A Chainable Continuum (Indecomposable) With Two End Points

from the literature which will prove to be essential in the later chapters. They are stated without proof since their purpose in being presented is to provide easy access by the reader to their statements. These results do not appear in texts which could be considered generally available to the reader.

1.1 Theorem Every infinite sequence of sets contains a convergent subsequence [34, p. 11].

1.2 Theorem If $\{A_i\}$ is an infinite sequence of sets such that (a) $\bigcup_{i=1}^{\infty} A_i$ is conditionally compact, (b) for each i, any pair of points of A_i can be joined in A_i by an ε_i-chain and $\varepsilon_i \to 0$ as $i \to \infty$, (c) $\lim\inf \{A_i\} \neq \emptyset$, then $\lim\sup \{A_i\}$ is connected [34, p. 14].

1.3 Theorem If $\{A_i\}$ is a convergent sequence satisfying (a) and (b) of [1.2], then $\lim \{A_i\}$ is connected [34, p. 15].

1.4 Theorem Any monotone transformation $f(A) = B$ on a compact space A is equivalent to an upper semi-continuous decomposition of A into continua. Conversely, any upper semi-continuous decomposition of A into continua with hyperspace A' is equivalent to a monotone transformation $f(A) = A'$ [34, p. 127].

1.5 Theorem If T is compact and the union of three continua which have a point in common and such that no one of them is a subset of the union of the other two, then T contains a triod [32, p. 443].

1.6 Theorem Every compact nondegenerate unicoherent continuum which is not a triod is irreducible between some two points [32, p. 456].

CHAPTER II

FUNDAMENTAL THEOREMS OF CHAINABILITY

Some Necessities for Chainability of Continua

In this section some of the consequences of a continuum being chainable and in particular its unicoherence and its atriodicity will be presented.

The first several results illustrate that if C is a chain on a continuum M and no subchain of C is a chain on M then in a sense each part of the chain must meet M. The first result states simply that each link must meet M and in particular that part of each link which is not in any other link meets M. Since there is no requirement that links be connected, it is possible for a link to be the union of several disjoint open sets with some of these not meeting M. Proposition 2.3 carries these ideas even further by showing that the common part of two adjacent links must also contain a point of M.

2.1 Proposition Let M be a connected set and $C = \{d_1, \cdots, d_n\}$ be a chain on M such that $d_1 \cap M \neq \emptyset$ and $d_n \cap M \neq \emptyset$. Then $d_i \cap M \neq \emptyset$ for each $i = 1, 2, \cdots, n$. In fact, $M \cap [d_i \setminus (d_{i-1} \cup d_{i+1})] \neq \emptyset$ for $i = 2, 3, \cdots, n-1$. Also, if C is a minimal chain on M, then $M \cap (d_1 \setminus d_2)$ and $M \cap (d_n \setminus d_{n-1})$ are both nonempty.

Proof: Assume that for some integer j, $1 < j < n$, the set

$M \cap [d_j \setminus (d_{j-1} \cup d_{j+1})]$ is empty. Let $H = M \cap C^*(1,j-1)$ and $K = M \cap C^*(j+1,n)$. Then $M = H \cup K$. Now $x \in H$ implies that $x \in d_k$ for some k, $1 \leq k \leq j-1$. The definition of a chain implies that $d_k \cap d_i$ is empty for each i, $j+1 \leq i \leq n$. Since $K \subseteq C^*(j+1,n)$ this implies that $d_k \cap K = \emptyset$ which implies that $x \notin \overline{K}$. Likewise, $x \in K$ implies that $x \notin \overline{H}$. Hence, H and K form a separation of M which is a contradiction to M being connected. Thus, the integer j does not exist and for each i, $1 < i < n$, $M \cap [d_i \setminus (d_{j+1} \cup d_{j-1})] \neq \emptyset$. Therefore, we also have that for each i, $1 \leq i \leq n$, $M \cap d_i \neq \emptyset$. The completion of the theorem is immediate. ||

This proposition can be extended to a result which is more useful in later theorems and is stated below. Since its proof is similar to that given above, it is omitted.

2.2 <u>Proposition</u> Let M be a connected set and C a chain such that $M \subseteq C^*$. If $d_i \cap M \neq \emptyset$ and $d_j \cap M \neq \emptyset$ for some d_i and d_j in C with $i \leq j$, then each link of the subchain $C_1 = C(i,j)$ contains a point of M.

2.3 <u>Proposition</u> Let M be a connected set and $C = \{d_1, \cdots, d_n\}$ be a chain on M such that $d_1 \cap M \neq \emptyset$ and $d_n \cap M \neq \emptyset$. Then for each i, $1 \leq i \leq n-1$, $M \cap (d_i \cap d_{i+1}) \neq \emptyset$.

Proof: Suppose there is an integer j, $1 \leq j \leq n-1$, such that $M \cap (d_j \cap d_{j+1}) = \emptyset$. Let $H = M \cap C^*(1,j)$ and $K = M \cap C^*(j+1,n)$, both of which are nonempty sets. We must now have that $M = H \cup K$. The point sets H and K do not meet since $M \cap d_j \cap d_{j+1} = \emptyset$ and $H \subseteq C^*(1,j-1) \cup (d_j \setminus d_{j+1}) \cup (d_j \cap d_{j+1})$ which implies that

$H \subseteq C^*(1,j-1) \cup (d_j \setminus d_{j+1})$, This latter quantity being denoted by G_1. Likewise, K is contained in the set $C^*(j+2,n) \cup (d_{j+1} \setminus d_j)$ which we shall denote by G_2. Since $d_j \setminus d_{j+1} \subseteq d_j$ and $d_j \cap C^*(j+2,n) = \emptyset$, this with the fact that $d_j \cap (d_{j+1} \setminus d_j) = \emptyset$ implies that $d_j \cap G_2 = \emptyset$. Likewise, $d_{j+1} \cap G_1 = \emptyset$. Now by the definition of G_1 and G_2 and the definition of a chain, $G_1 \cap G_2 = \emptyset$. Thus, $H \cap K \subseteq G_1 \cap G_2$ which implies that $H \cap K = \emptyset$.

Now suppose there is a point x in $\overline{H} \cap K$. Then $x \in K$ which implies that $x \in G_2$ and thus $x \in d_{j_x}$, $j+1 \leq j_x \leq n$. Since x is a limit point of H, $d_{j_x} \cap H \neq \emptyset$. Let $y \in d_{j_x} \cap H$, $y \neq x$. Then there is an integer j_y, $1 \leq j_y \leq j$, such that $y \in d_{j_y}$ and thus $d_{j_x} \cap d_{j_y} \neq \emptyset$. This implies that $|j_x - j_y| \leq 1$ and thus $j_y = j$, $j_x = j+1$. But now $y \in d_j \cap H \subseteq d_j \cap M$ and $y \in d_{j+1}$ implies that $y \in d_j \cap d_{j+1} \cap M$, a contradiction to the original assumption. Therefore, $d_i \cap d_{i+1} \cap M \neq \emptyset$, for each i, $1 \leq i \leq n-1$. ||

With the three preceding propositions it is now possible to establish several important properties of chainable continua. The final result is stated as theorem 2.7 but it is the following which makes this possible. It should be recalled that by a continuum being chainable we mean that the links can be made arbitrarily small. Thus, many of the following results are established by simply requiring the links of a chain to be too small to allow an assumed property to occur.

2.4 <u>Proposition</u> Each chainable continuum is unicoherent.

Proof: Let M be a chainable continuum and suppose that

$M = M_1 \cup M_2$, with M_1 and M_2 proper subcontinua of M. Let p and q be arbitrary points of $M_1 \cap M_2$ which must be nonempty since to assume to the contrary would produce a separation of M. We may assume that p and q are distinct for if $M_1 \cap M_2 = \{p\}$ then M is unicoherent.

Let $C = \{d_1, d_2, \cdots, d_n\}$ be a chain on M. Then $p \in d_i$ and $q \in d_j$ for some i and j such that without loss of generality, $i \leq j$, and $p \notin d_{i+1}$, $q \notin d_{j-1}$. Proposition 2.2 now implies that each link of $C_1 = C(i,j)$ contains a point of M_1. Let $\{p = p_i, p_{i+1}, \cdots, p_{j-1}, p_j = q\}$ be a set of points such that for each m, $i \leq m \leq j$, $p_m \in M_1 \cap d_m$.

Now if $\{d_1, d_2, \cdots, d_n\}$ is an ε-chain on M, then $\rho(p_m, M_2) < \varepsilon$ since each link must contain a point of M_2. We also have that $\rho(p_m, p_{m+1}) < 2\varepsilon$. Since M is chainable, for each $\varepsilon > 0$, M can be covered by an ε-chain. Thus, there is a sequence of point sets R_1, R_2, $\cdots$ such that for each k, $R_k = \{p = p_{k,1}, p_{k,2}, \cdots, p_{k,n_k} = q\}$ such that for each i, $1 \leq i \leq n_k$, $p_{k,i} \in M_1$, $\rho(p_{k,i}, M_2) < \frac{1}{2k}$, and $\rho(p_{k,i}, p_{k,i+1}) < \frac{1}{k}$.

Since $\{R_1, R_2, \cdots\}$ is an infinite sequence of point sets, some subsequence has a connected sequential limiting set L [1.1] and [1.2]. But L is contained in $M_1 \cap M_2$ and contains p and q. Since p and q were arbitrary members of $M_1 \cap M_2$, this intersection is connected and M is unicoherent. ||

2.5 <u>Proposition</u> A chainable continuum is not a triod.

Proof: Assume that the chainable continuum M is a triod. Let

M_1, M_2, and M_3 denote the three proper subcontinua such that M is their union and $M_i \cap M_j = K$, where i, $j \varepsilon \{1, 2, 3\}$ and $i \neq j$, and $K = M_1 \cap M_2 \cap M_3$. For each $i = 1, 2$, and 3, let $p_i \varepsilon M_i \setminus K \neq \emptyset$. Let $\delta_1 = \rho(p_1, M_2 \cup M_3)$, $\delta_2 = \rho(p_2, M_1 \cup M_3)$, and $\delta_3 = (p_3, M_1 \cup M_2)$. Now let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and let ε be a number such that $0 < \varepsilon < \frac{\delta}{2}$.

Let C be an ε-chain on M. Then no single link of C intersects any two of $\{p_1\}$, $\{p_2\}$, $\{p_3\}$, and K. Now these four distinct point sets must intersect the chain C in some order. Without loss of generality, suppose that they meet the links of the chain in the order named. Suppose that the chain C is denoted by $\{d_1, \cdots, d_{p_1}, \cdots, d_{p_2}, \cdots, d_{p_3}, \cdots, d_K, \cdots, d_n\}$, with $p_i \varepsilon d_{p_i}$, $i = 1, 2$, or 3, and $d_K \cap K \neq \emptyset$. Then $d_{p_2} \cap M_2 \neq \emptyset$ and $d_K \cap M_2 \neq \emptyset$ which implies that $d_{p_3} \cap M_2 \neq \emptyset$ by proposition 2.2 since M_2 is connected. But this contradicts the fact that $\rho(p_3, M_2) > \delta_3 \geq \delta > 2\varepsilon \geq$ diameter of d_{p_3}. Similar arguments hold for any order of these four point sets. Hence, M is not a triod. ||

The following proposition will enable us to conclude that each chainable continuum is in fact hereditarily unicoherent and atriodic. This will then establish the previously mentioned theorem 2.7.

2.6 <u>Proposition</u> Each subcontinuum of a chainable continuum is chainable.

Proof: Let K be a subcontinuum of a chainable continuum M. Let $C = \{d_1, d_2, \cdots, d_n\}$ be an ε-chain on M. Since C contains a finite collection of links and each point of M is in at least one link, there is a first link, d_i, $1 \leq i \leq n$ such that $d_i \cap K \neq \emptyset$ and there is a last link, d_j, $1 \leq j \leq n$ such that $d_j \cap K \neq \emptyset$, with $i \leq j$. Then K is a continuum with $C_1 = C(i,j)$ a covering of K. Proposition 2.2 now implies that for each k, $d_k \cap K \neq \emptyset$, $i \leq k \leq j$. Thus, C_1 is an ε-chain on K and since ε was arbitrary, K is chainable. ||

2.7 Theorem Every chainable continuum is hereditarily unicoherent and atriodic.

Proof: This theorem follows immediately from propositions 2.4, 2.5, and 2.7. ||

Now that several results have been obtained pertaining to chainable continua perhaps it is appropriate to pause from the formal presentation to reflect on the consequences of these results. These informal reflections will obviously be nontechnical with the inclusion of such undefined yet descriptive terms as "thin," "line-like," "circle-like," and "looping."

a) b) c)

Figure 8. Examples of a) A Simple Triod, b) Looping or Circle-like, and c) Non Line-like or Non Thin

The concept of chainability, which requires chains with arbitrarily small links, denotes a sort of thinness or line-likeness to the continuum. Perhaps an idea of local thinness is more appropriate since a chainable continuum can appear to be dense in places. A good example is example A, the Closed Topologist's Sine Curve, which is quite dense along the interval from (0,-1) to (0,1).

Since every chainable continuum is atriodic by theorem 2.7, any triod cannot be chainable. This excludes such continua in the plane as the union of the segments joining (0,-1) to (0,1) and (0,0) to (1,0), the unit disk, and the union of sin x and cos x for $0 \leq x \leq 2\pi$. The first example, which is a simple triod, is excluded because with chains having links of diameter less than one-half, a linear chain cannot cover the continuum. Likewise, the unit disk is excluded, but also because it contains the afore mentioned simple triod. Thus, continua which in a sense are not thin or line-like must be excluded because they contain triods.

The third example is one which might informally be described as containing loops. Again such a continuum is not chainable because locally, about a point of intersection, the continuum contains a triod which creates the same difficulty as in the first example.

The necessity of being unicoherent excludes from consideration such continua as circles since, for example, the unit circle can be written as the union of two proper subcontinua whose intersection is not connected. The requirement of hereditary unicoherence further excludes continua such as that previously mentioned which contain loops since a loop is circle-like.

In Chapter III it will be shown that these requirements of atriodicity and hereditary unicoherence are actually sufficient for hereditarily decomposable continua to be chainable. Such is not the case for other types of continua. There are however numerous other fundamental results which must be obtained before considering these conditions of sufficiency further.

Construction Properties of Chains

The results which follow in this section of Chapter II might best be described as structural theorems since they deal with the ability to construct new chains from given chains. The first two propositions follow rather easily and it should be noted that they are stated without reference to any particular continuum.

Proposition 2.8 will provide the ability to construct, from the chain $C(d_1, d_2, \cdots, d_n)$, a new chain, denoted by $C \cap S$, where S is an open set with a certain property given in the proposition, and the ith link of $C \cap S$ is given by $d_i \cap S$, for $1 \leq i \leq n$. Thus, this proposition defines and establishes the collection $\{d_1 \cap S, d_2 \cap S, \cdots, d_n \cap S\}$ as a chain denoted by $\underline{C \cap S}$.

2.8 __Proposition__ If $C = \{d_1, d_2, \cdots, d_n\}$ is a chain and S is an open set such that $S \cap (d_i \cap d_{i+1}) \neq \emptyset$ for each i, $1 \leq i \leq n-1$, then $C \cap S = \{d_1 \cap S, d_2 \cap S, \cdots, d_n \cap S\}$ is a chain.

Proof: Clearly, for each i, $1 \leq i \leq n$, $d_i \cap S$ is an open set. Now $(d_i \cap S) \cap (d_j \cap S) = (d_i \cap d_j) \cap S$. Thus, $|i - j| \leq 1$ implies that $j = i-1$ or $j = i+1$ and consequently $S \cap (d_i \cap d_j) \neq \emptyset$ by hypothesis. Conversely, $S \cap (d_i \cap d_j) \neq \emptyset$ implies that $d_i \cap d_j \neq \emptyset$ which then implies that $|i - j| \leq 1$. Therefore, $(d_i \cap S) \cap (d_j \cap S)$

$\neq \emptyset$ if and only if $|i - j| \leq 1$, and $C \cap S$ is a chain. ||

While the preceding proposition in a sense provides a means of reducing the sizes of the links of a chain and thus of producing a refinement of the given chain, the following simple proposition provides a means of lengthening chains by joining two chains together. Thus, the following proposition will define and justify the notation for the chain denoted by <u>$C(1,n) \oplus G(1,m)$</u>.

2.9 <u>Proposition</u> If $C = \{d_1, d_2, \cdots, d_n\}$ and $G = \{g_1, g_2, \cdots, g_m\}$ are each chains such that $d_i \cap g_j \neq \emptyset$ if and only if $i = n$ and $j = 1$, then $C(1,n) \oplus G(1,m) = \{d_1, d_2, \cdots, d_n, g_1, g_2, \cdots, g_m\}$ is a chain.

Proof: This result is apparent from the hypothesis and the definition of a chain. ||

The use of the preceding proposition will usually be with the joining of two chains which may not necessarily have their links numbered beginning with one. The validity of this should, however, be apparent from the preceding.

The next several results continue in the same vein as the two preceding propositions. They deal primarily with the ability to construct a particular type of chain or to guarantee that a sequence of chains with a particular structure exists. These results will, however, again be relative to an arbitrary but given chainable continuum. Since the details of the proofs become quite involved, although not difficult, any reader might find it advantageous to initially omit a detailed study of the proofs and to first seek to understand the conclusions of the lemmas and their significance in arriving at the primary result which is stated as theorem 2.13.

2.10 Lemma If M is an ε-chainable continuum, then there is a taut ε-chain on M.

Proof: Let $C = \{d_1, d_2, \cdots, d_n\}$ be any ε-chain on M. Although $d_k \cap d_j \neq \emptyset$ if and only if $|k - j| \leq 1$, it is possible that $\overline{d}_k \cap \overline{d}_j \neq \emptyset$ and $|k - j| \geq 2$. Suppose first that $\overline{d}_k \cap \overline{d}_j \neq \emptyset$ and $|k - j| = 2$. Then there is an integer i such that, without loss of generality, $k = i - 1$ and $j = i + 1$. We note at this point that this intersection is compact. Let $y_1 \varepsilon d_{i-1} \cap d_i$ and $y_2 \varepsilon d_i \cap d_{i+1}$. Then $y_1 \not\varepsilon d_{i+1}$ and $y_2 \not\varepsilon d_{i-1}$.

Now for each $x \varepsilon (\overline{d}_{i-1} \cap \overline{d}_{i+1}) \cap d_i$, there is an open ball $B(x;\varepsilon_x)$ contained in d_i and such that $\varepsilon_x < \rho(x,y_i)$, $i = 1$ and 2. Also for each $x \varepsilon (\overline{d}_{i-1} \cap \overline{d}_{i+1}) \setminus d_i$, there is a $\delta_x > 0$ such that the open ball $B(x;\delta_x)$ does not intersect M and $\rho(x,y_i) > \delta_x$ for $i =$ 1 and 2. The collection $\{ B(x;\frac{\varepsilon_x}{2}) : x \varepsilon \overline{d}_{i-1} \cap \overline{d}_{i+1} \cap d_i \} \cup \{ B(x;\frac{\delta_x}{2}) : x \varepsilon (\overline{d}_{i-1} \cap \overline{d}_{i+1}) \setminus d_i \}$ is an open cover of $\overline{d}_{i-1} \cap \overline{d}_{i+1}$.

Let the collection $\{ B(x_1;\frac{\varepsilon_1}{2}), \cdots, B(x_k;\frac{\varepsilon_k}{2}), B(x_{k+1};\frac{\delta_1}{2}), \cdots,$ $B(x_{k+m};\frac{\delta_m}{2}) \}$ denote a finite subcover of $\overline{d}_{i-1} \cap \overline{d}_{i+1}$.

Let $\delta = \min\{ \frac{\varepsilon_1}{2}, \cdots, \frac{\varepsilon_k}{2}, \frac{\delta_1}{2}, \cdots, \frac{\delta_m}{2} \}$. Then for all x in $\overline{d}_{i-1} \cap \overline{d}_{i+1}$, $\rho(x,y_i) > 2\delta$, for $i = 1$ and 2, and for all $x \varepsilon (\overline{d}_{i-1} \cap \overline{d}_{i+1}) \setminus d_i$, $\rho(x,y) > 2\delta$, for all y in M. Let $O = \bigcup \{ B(x,\delta) : x \varepsilon \overline{d}_{i-1} \cap \overline{d}_{i+1} \}$. Then $\overline{O} \supseteq \overline{d}_{i-1} \cap \overline{d}_{i+1}$ and $y_i \not\varepsilon \overline{O}$, $i = 1$ and 2.

Let $h_{i-1} = d_{i-1} \setminus \overline{O}$ and $h_{i+1} = d_{i+1} \setminus \overline{O}$. Then h_{i-1} and h_{i+1}

are disjoint nonempty open sets such that $h_{i-1} \cap d_i \neq \emptyset$ and $h_{i+1} \cap d_i \neq \emptyset$ since they contain y_1 and y_2 respectively. Because h_{i-1} and h_{i+1} are contained in d_{i-1} and d_{i+1} respectively, their diameters are less than ε. If $y \in M \cap \overline{d}_{i-1} \cap \overline{d}_{i+1}$ then since C is a chain on M, $y \in d_i$. Finally, h_{i-1} meets M since $y \in (d_{i-1} \setminus d_i) \cap M \neq \emptyset$ and $x \in (d_{i-1} \setminus d_i) \cap \overline{O}$ implies that $\overline{B}(x;\delta) \cap M = \emptyset$ and thus $y \notin \overline{O}$. Likewise, $h_{i+1} \cap M \neq \emptyset$. Now $C_1 = \{d_1, \ldots, h_{i-1}, d_i, h_{i+1}, \ldots, d_n\}$ is an ε-chain on M and $\overline{h}_{i-1} \cap \overline{h}_{i+1} = \emptyset$.

Now suppose that $\overline{d}_k \cap \overline{d}_j \neq \emptyset$ and $|k - j| > 2$. Then there does not exist a link d_i of C such that $d_i \cap d_k \neq \emptyset$ and $d_i \cap d_j \neq \emptyset$. Also, $(\overline{d}_k \cap \overline{d}_j) \cap M = \emptyset$ since to assume not implies there is a point x of $\overline{d}_k \setminus d_k$ and $\overline{d}_j \setminus d_j$ which is also in M. But because C covers M, $x \in d_i$ for some link d_i of C and thus $d_k \cap d_i$ and $d_j \cap d_i$ are both nonempty which is contrary to the above. Hence, $\overline{d}_k \cap \overline{d}_j \cap M = \emptyset$.

Since this intersection is empty and M is compact, there exists a number $\delta > 0$ such that for all x in $\overline{d}_k \cap \overline{d}_j$, $\rho(x,M) > 2\delta$. Then $\bigcup\{ B(x;\delta) : x \in \overline{d}_k \cap \overline{d}_j \}$ is an open set O such that $\overline{O} \cap M = \emptyset$. Thus, M is contained in the complement of $\overline{O}$ and if $g_k = d_k \setminus \overline{O}$ and $g_j = d_j \setminus \overline{O}$, then M is covered by the collection $C_2 = \{d_1, \ldots, g_k, \ldots, g_j, \ldots, d_n\}$. Since $g_k \subseteq d_k$ and $g_j \subseteq d_j$, both g_k and g_j have diameters less than ε. To see that g_k and g_j are links of a chain, we note that $d_{k-1} \cap d_k \cap M \neq \emptyset$ and $\overline{O} \setminus M = \emptyset$ and hence $d_{k-1} \cap (d_k \setminus \overline{O}) \cap M \neq \emptyset$. Therefore, $d_{k-1} \cap g_k \neq \emptyset$. Likewise, we obtain $g_k \cap d_{k+1} \neq \emptyset$, $d_{j-1} \cap g_j \neq \emptyset$, and $g_j \cap d_{j+1} \neq \emptyset$. Also, g_k and g_j are themselves nonempty. Thus, C_2 is a chain on M and $\overline{g}_k \cap \overline{g}_j = \emptyset$.

Since either of the above procedures can be repeated finitely many times, the chain C may be made into a taut ε-chain on M. ||

2.11 <u>Lemma</u> Let M be a chainable continuum and C a taut ε-chain on M. Then there is a taut $\frac{1}{2}\varepsilon$-chain C_1 on M such that C_1 is a closed refinement of C.

Proof: Let $C = \{d_1, d_2, \cdots, d_n\}$. Without loss of generality, $n \geq 2$. Since M is compact and the complement of C^*, $\sim C^*$, is closed, there exists a number $\alpha > 0$ such that $\rho(M, \sim C^*) > 2\alpha$. Let $D = \bigcup_{x \varepsilon M} B(x;\alpha)$. Then $\overline{D} \subseteq C^*$.

For each i, $1 \leq i \leq n-1$, let $H_i = \overline{D} \cap \sim C^*(i+1,n)$ and $K_i = \overline{D} \cap \sim C^*(1,i)$. Now for each i, H_i and K_i are closed subsets of $\overline{D} \subseteq C^*$ such that $H_i \subseteq (d_i \setminus d_{i+1}) \cup C^*(1,i-1) = C^*(1,i) \setminus d_{i+1}$ which is contained in $C^*(1,i+1)$ and $K_i \subseteq (d_{i+1} \setminus d_i) \cup C^*(i+2,n) = C^*(i+1,n) \setminus d_{i+1} \subseteq C^*(i,n)$. To see that $H_i \cap K_i = \emptyset$ for each i, let $x \varepsilon H_i$. Then $x \varepsilon C^*(1,i)$ which implies that $x \not\varepsilon d_j$ for $i+2 \leq j \leq n$ and $x \not\varepsilon d_{i+1} \setminus d_i$ and thus $x \not\varepsilon K_i \subseteq (d_{i+1} \setminus d_i) \cup C^*(i+2,n)$. Similarly, $x \varepsilon K_i$ implies that $x \not\varepsilon H_i$ and we have that $H_i \cap K_i = \emptyset$ for $1 \leq i \leq n-1$. Then $\delta_i = \rho(H_i, K_i)$ is positive for $1 \leq i \leq n-1$.

Since C is a taut chain, $\overline{d}_i \cap \overline{d}_k = \emptyset$ for $|i - k| \geq 2$, and thus $\rho(\overline{d}_i, \overline{d}_k) > 0$. Now let $\beta = \frac{1}{2}\min\{\ \rho(\overline{d}_i, \overline{d}_k) : 1 \leq i \leq n,\ 1 \leq k \leq n,$ and $|i - k| \geq 2\ \}$. Then $\beta > 0$.

Let $\delta = \min\{\ \frac{\alpha}{2}, \frac{\varepsilon}{2}, \beta, \frac{\delta_1}{2}, \frac{\delta_2}{2}, \cdots, \frac{\delta_{n-1}}{2}\ \}$. Since M is chainable there is a δ-chain C_1 on M which by lemma 2.10 can be considered taut. With $\delta < \frac{1}{2}\varepsilon$, we have that C_1 is a taut $\frac{1}{2}\varepsilon$-chain on M. Let d be any link of C_1. Since the diameter of d is less than or equal to δ with $\delta < \alpha$, and $d \cap M \neq \emptyset$, we have that $d \subseteq D$ which implies that $\overline{d} \subseteq \overline{D} \subseteq C^*$. Also, $\delta < \beta$ implies that $\overline{d}$ cannot meet two nonadjacent links since to assume that $y \in \overline{d} \cap d_j$ and $z \in \overline{d} \cap d_k$ with $|j - k| \geq 2$ implies that $\rho(y,z) < \delta < \beta$. But by the definition of β, $\rho(y,z) > \beta$ and the result follows.

Since $d \cap M \neq \emptyset$, let $x \in d \cap M$. Then $x \in d_i$ for some $d_i \in C$. If $\overline{d} \subseteq d_i$, then we are done. Thus, suppose that $\overline{d} \not\subseteq d_i$ nor for that matter, any other link of C. Again, by the definition of β, $\overline{d}$ can meet only adjacent links of d_i and then at most one of them. Thus, assume $\overline{d} \cap d_{i+1} \neq \emptyset$. By the supposition we have that there is a point y in $(d_i \setminus d_{i+1}) \cap \overline{d}$ and a point $z \in (d_{i+1} \setminus d_i) \cap \overline{d}$. Because $\overline{d} \subseteq \overline{D}$, $y \in \overline{D}$ and $z \in \overline{D}$. Hence, $y \in H_i$ and $z \in K_i$ which implies that $\rho(y,z) \geq \delta_i > \delta$. But y and z being elements of d implies that $\rho(y,z) < \delta$. Therefore, $\overline{d}$ must be contained in either d_i or d_{i+1} and C_1 is a taut $\frac{1}{2}\varepsilon$-chain on M which is a closed refinement of C. ||

Before stating and proving the concluding theorem of this section, the following corollary is established. Although it is not necessary for theorem 2.13, it is a structural result and further illustrates the degree of control possible over chains on a continuum.

2.12 <u>Corollary</u> Let M be a chainable continuum, k a positive integer, and C a taut ε-chain on M. Then there is a taut $\frac{1}{2}\varepsilon$-chain C_2 on M such that C_2 is a closed refinement of C and any subchain of less than k+1 links of C_2 cannot intersect two nonadjacent links of C.

Proof: Lemma 2.11 implies there is a taut $\frac{1}{2}\varepsilon$-chain $C_1 = \{d_1, \ldots, d_n\}$ on M such that C_1 is a closed refinement of C. Let β be the $\min\{ \rho(\overline{d}_i, \overline{d}_j) : d_i, d_j \in C_1, |i - j| \geq 2 \}$. Since $\overline{d}_i \cap \overline{d}_j = \emptyset$ for $|i - j| \geq 2$, $\beta > 0$. Let $\delta = \min\{ \frac{\varepsilon}{2}, \frac{\beta}{k+1} \}$ and C_2 be a taut δ-chain on M such that C_2 is a closed refinement of C_1. Then C_2 is a taut $\frac{1}{2}\varepsilon$-chain on M which is a closed refinement of C.

Suppose there is a subchain $H = \{h_1, h_2, \cdots, h_k\}$ of k or less links of C_2 such that $h_1 \cap d_j \neq \emptyset$ and $h_k \cap d_m \neq \emptyset$ with $|j - m| \geq 2$; ie, H intersects two nonadjacent links of C. Let $x \in h_1 \cap d_j$, $y \in h_k \cap d_m$, and $x_i \in h_i \cap h_{i+1}$ for $1 \leq i \leq k-1$. Then $\rho(x,x_1) < \delta$, $\rho(y,x_{k-1}) < \delta$, and $\rho(x_i,x_{i+1}) < \delta$ for $1 \leq i \leq k-1$, and $\rho(x,y) \leq \rho(x,x_1) + \rho(x_1,x_2) + \cdots + \rho(x_{k-1},y) < (k+1)\delta < \beta$. Thus, $\rho(\overline{d}_j,\overline{d}_k) < \beta$. However, the definition of β implies that $\rho(\overline{d}_j,\overline{d}_k) > \beta$ and we have that the subchain H cannot exist. Therefore, any subchain of less than $k+1$ links of C_2 cannot meet two nonadjacent links of C. ||

Theorem 2.13 concludes this section of Chapter II. When needed it allows us to work with a particular sequence of chains on a continuum. A sequence $\{C_i\}$ of chains on a continuum M will be called a <u>defining sequence of chains</u> on M if and only if $\{C_i\}$ is a sequence of taut ε_i-chains on M such that C_{i+1} is a closed refinement of C_i and $\varepsilon_i \to 0$ as $i \to \infty$. Theorem 2.13 asserts the existence of a particular defining sequence of chains on any chainable continuum.

2.13 <u>Theorem</u> If M is a chainable continuum then there is a defining sequence $\{C_i\}$ of chains on M with $\varepsilon_i = \frac{1}{2^i}$.

Proof: Lemma 2.10 implies the existence of a taut $\frac{1}{2}$-chain C_1 on M and lemma 2.11 implies the existence of a taut $\frac{1}{4}$-chain C_2 on M which is a closed refinement of C_1. Induction on i with lemma 2.11 implies there is a taut $\frac{1}{2^i}$-chain C_i on M, such that C_{i+1} is a closed refinement of C_i. Thus, the defining sequence $\{C_i\}$ of chains on M with $\varepsilon_i = \frac{1}{2^i}$ exists. ||

Homeomorphisms and Chainable Continua

In this section some consequences of chainability will be presented which are concerned with the existence of homeomorphisms. The first concerns a homeomorphism between any two hereditarily indecomposable chainable continua and the second, a homeomorphism between any chainable continuum and a plane continuum. While these theorems are not specifically needed for the development of later material, they are presented because they are important consequences of chainability, because they might be considered as unexpected consequences of chainability, and because they are certainly not obvious.

The first theorem is just stated since its development and proof appear in the thesis by McKellips [22, p. 99].

2.14 Theorem If M and N are two nondegenerate, hereditarily indecomposable, chainable continua then they are homeomorphic.

Before preceding to the second of the two theorems, the following lemma is stated. The statement is not easily followed and except for the requirement that T is a homeomorphism mapping M_1 onto M_2, the proof may be found in either [2, p. 738] or with more detail in [22, p. 50].

2.15 Lemma Suppose that M_1 and M_2 are chainable continua in the metric spaces (X_1,ρ_1) and (X_2,ρ_2) respectively. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers with $\sum_{i=1}^{\infty}\varepsilon_i$ finite. If $\{C_{1,i}\}$ and $\{C_{2,i}\}$ are sequences of chains on M_1 and M_2 respectively such that

1) $C_{1,i}$ and $C_{2,i}$ are ε_i-chains on M_1 and M_2 respectively

for each i, and

2) if the jth link of $C_{n,i+1}$ intersects the kth link of $C_{n,i}$, $n = 1$ or 2, then the distance between the jth link of $C_{m,i+1}$ and the kth link of $C_{m,i}$, $m = 1$ and 2, is less than ε_i,

then there is a homeomorphism T mapping M_1 onto M_2.

Since the ability to duplicate the pattern of one chain within another is required in the next proof, the following notation is defined. The <u>chain</u> $C(d_1, d_2, \cdots, d_n)$ <u>follows the pattern</u> $(1,k_1)$, $(2,k_2)$, $\cdots$, (n,k_n) in the chain $G(g_1, g_2, \cdots, g_m)$ if and only if the ith link of C lies in the k_ith link of G. Theorem 2.16 is the final result of this chapter and as was noted earlier it may seem to be an unusually strong consequence of chainability. A brief outline of the proof is included.

2.16 <u>Theorem</u> Each chainable continuum is homeomorphic with a plane continuum.

Proof: Theorem 2.13 states that there is a defining sequence $\{C_i\}$ of chains on M with $\varepsilon_i = \frac{1}{2^i}$. These chains are, without loss of generality, minimal in the sense that no proper subchain of C_i contains M. Corollary 2.12 implies that each member of the sequence may be selected such that no subchain of C_{i+1} of less than nine links intersects two nonadjacent links of C_i.

It can be shown that in the plane there is a sequence of chains $D_1, D_2, \cdots$ such that D_i is a $\frac{1}{2^i}$-chain whose elements are the interiors of rectangles, D_{i+1} follows a pattern in D_i that C_{i+1} follows in C_i, and each element of D_i contains the closure of an element of

D_i [4, p. 654]. Lemma 2.15 will apply to this construction and implies that M is homeomorphic with $D_1^* \cap D_2^* \cap \cdots$. ||

CHAPTER III

SOME CHARACTERIZATIONS OF THE CHAINABILITY OF HEREDITARILY DECOMPOSABLE CONTINUA

The objectives of Chapter II were to acquire the ability to construct chains satisfying criteria of refinement, tautness, and chaining, and to show some consequences of a continuum being chainable. We were particularly interested in the atriodicity and unicoherence of a chainable continuum. In the present chapter this latter objective is reversed in that our attention will be directed toward showing atriodic and hereditarily unicoherent continua to be chainable. However, this is accomplished in this chapter, as theorem 3.20, only under the additional hypothesis that the continuum be hereditarily decomposable.

The proof of this theorem will be by contradiction. If a continuum is not chainable, a subcontinuum will exist which is irreducible with respect to not being ε-chainable for some ε. By then splitting this subcontinuum into two proper subcontinua, chaining on each of these, and then fitting the chains together, an ε-chain on the irreducible subcontinuum is produced thus contradicting its existence. The idea for the proof is simply stated but as will be evident, it is not easily achieved.

There are a number of preliminary results necessary before attempting to prove the above assertion. These include first some consequences of continua being atriodic, hereditarily unicoherent, or hereditarily decomposable. These are followed by the development of an upper semicontinuous decomposition of an atriodic, hereditarily unicoherent continuum which contains no indecomposable subcontinua with interior points relative to the continuum. The third section of this chapter, containing an introductory development of terminal subcontinua and end points of continua, is an essential ingredient in the process of fitting chains on subcontinua together to form a chain. When all of this has been developed, then the objective of this chapter will be attained, along with some equivalent conditions.

Preliminary Properties

With hereditarily unicoherent continua the definition implies immediately that the intersection of any finite collection of subcontinua is a continuum. Proposition 3.1 shows that this result extends to any collection of subcontinua.

3.1 <u>Proposition</u> Let M be a nondegenerate continuum which is hereditarily unicoherent. The intersection of any collection of subcontinua of M is a continuum.

Proof: Let $\{M_\alpha\}$, $\alpha \varepsilon I$, be a collection of subcontinua of M indexed by I. If the intersection of all the M_α's is empty then this intersection is a continuum. Thus, suppose that $K = \bigcap \{ M_\alpha : \alpha \varepsilon I \}$ is nonempty. Consider I to be a well ordering of the M_α's with M_1 denoting the first element and M_{1+} denoting its successor (the first element of $\{ M_\alpha : \alpha \varepsilon I \setminus \{1\} \}$.) Since $M_1 \cap M_{1+} \neq \emptyset$, $M_1 \cup M_{1+}$ is a subcontinuum of M. If $M_1 \cup M_{1+}$ is degenerate; ie, $M_1 \cup M_{1+} = \{x\}$ for some $x \varepsilon M$, then $M_1 \cap M_{1+} = \{x\}$ and is a continuum. If $M_1 \cup M_{1+}$

is nondegenerate, M being hereditarily unicoherent implies that $M_1 \cap M_{1+}$ is a continuum.

Let $\beta \in I$ and assume that $\bigcap\{ M_\alpha : \alpha < \beta, \alpha \in I \}$ has been shown to be a nonempty continuum. Since $K \subseteq M_\beta \cap (\bigcap\{ M_\alpha : \alpha < \beta \})$, this latter intersection is nonempty and thus $M_\beta \cup (\bigcap\{ M_\alpha : \alpha < \beta \} = M_0$ is a subcontinuum of M. If M_0 is degenerate, then $M_0 = \{x\} = \bigcap\{ M_\alpha : \alpha < \beta \}$ for some $x \in M$ and the intersection is a continuum. If M_0 is nondegenerate then it is unicoherent and $M_\beta \cap (\bigcap\{ M_\alpha : \alpha < \beta \}) = \bigcap\{ M_\alpha : \alpha \leq \beta \}$ is a continuum. Transfinite induction now implies that K is a continuum. ||

Focusing attention on the property of atriodicity, we have the following intuitively obvious result. If one removes a subcontinuum from the "middle" part of an atriodic continuum, then two connected parts remain. If the subcontinuum removed is at one end then the single part remaining is connected.

3.2 <u>Proposition</u> Let M be a nondegenerate continuum which is atriodic. If H and K are subcontinua of M then $M \setminus H$ does not have more that two components and $M \setminus (H \cup K)$ does not have more than three.

Proof: Assume that $M \setminus H$ has at least three components C_1, C_2, and C_3. Now the C_i's are mutually exclusive although $\overline{C}_i$ may meet $\overline{C}_j$ for some $i \neq j$, but only if $\overline{C}_i \cap \overline{C}_j \subseteq H$. Also, for each i, $\overline{C}_i \cap H \neq \emptyset$ since to assume $\overline{C}_k \cap H = \emptyset$ for some k implies a separation of M by $\overline{C}_k$ and $H \cup (\bigcup\{ C_i : i \neq k \})$. Thus, for each i, $H \cap \overline{C}_i$ is connected. Considering only C_1, C_2, and C_3, let $M_1 = H \cup \overline{C}_1 \cup \overline{C}_2 \cup \overline{C}_3$, a subcontinuum of M which is also the union of

$H \cup \overline{C}_1$, $H \cup \overline{C}_2$, and $H \cup \overline{C}_3$. Since $C_2 \not\subset H \cup \overline{C}_1 \cup \overline{C}_3$ and is nonempty, M_1 is nondegenerate. Since the intersection of any two or all three of $H \cup \overline{C}_1$, $H \cup \overline{C}_2$, and $H \cup \overline{C}_3$ is H, M_1 is a triod. But this is not possible since M is atriodic. Thus, $M \setminus H$ cannot have three or more components.

Now suppose that $M \setminus (H \cup K)$ has at least four components C_1, C_2, C_3, and C_4. If $H \cap K \neq \emptyset$, then $H \cup K$ is a subcontinuum of M and the first part of this proposition implies that $M \setminus (H \cup K)$ has at most two components. Thus, suppose that $H \cap K = \emptyset$. Then H and K are mutually separated. As before, each $\overline{C}_i$ must meet $H \cup K$ or M would have a separation. If we suppose that each $\overline{C}_i$ meets at most one of H and K, then $M = (H \cup \bigcup\{ \overline{C}_i : \overline{C}_i \cap H \neq \emptyset \}) \cup (K \cup \bigcup\{ \overline{C}_i : \overline{C}_i \cap K \neq \emptyset \})$ and this would form a separation of M. Thus, there is a component C of $M \setminus (H \cup K)$ such that $\overline{C}$ meets both H and K. Without loss of generality, $C_1 = C$. Considering only C_1, C_2, C_3, and C_4, if we let $N = H \cup K \cup \overline{C}_1$, then N is a continuum and C_2, C_3, and C_4 are components of $M \setminus N$ which is a contradiction of the first part of the proposition. Thus, $M \setminus (H \cup K)$ has at most three components. ||

Capitalizing on the combined properties of atriodicity and hereditarily unicoherence, the following proposition shows that if a nondegenerate continuum with these properties has three subcontinua, each meeting the other two, then one of them is contained in the union of the other two. This will prove to be an extremely useful result and will normally be applied as a contradiction to a result which has been proven based on some assumption.

3.3 <u>Proposition</u> Let M be a nondegenerate continuum which is atriodic and hereditarily unicoherent. If each pair of the subcontinua H_1, H_2, and H_3 of M intersect then one of them is a subset of the union of the other two.

Proof: Since $H_1 \cap H_2$ and $H_2 \cap H_3$ are nonempty, $M_1 = H_1 \cup H_2 \cup H_3$ is a subcontinuum of M. If M_1 is degenerate, then $H_1 = H_2 = H_3$ and the proof is complete. If M_1 is nondegenerate, M_1 is the union of the subcontinua $H_1 \cup H_2$, and H_3. If $H_1 \cup H_2$ or H_3 is not proper in M_1, then again the proof is complete. Thus, we may assume that M_1 is the union of the proper subcontinua $H_1 \cup H_2$ and H_3. The unicoherence of M_1 implies that $(H_1 \cup H_2) \cap H_3$ is a continuum. Now $H_1 \cap H_2 \cap H_3 = (H_1 \cap H_3) \cap (H_2 \cap H_3)$ is nonempty since to assume so implies that $(H_1 \cup H_2) \cap H_3 = (H_1 \cap H_3) \cup (H_2 \cap H_3)$ is the union of two closed, disjoint, nonempty sets $H_1 \cap H_3$ and $H_2 \cap H_3$ which is a separation. Hence, $H_1 \cap H_2 \cap H_3 \neq \emptyset$. By an argument similar to the above, $(H_1 \cap H_2) \cup (H_2 \cap H_3)$ is a continuum. Thus, $H_0 = (H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_1)$ is a continuum and nonempty.

Proposition 3.2 now implies that $M_1 \setminus H_0$ has at most two components. If $M_1 \setminus H_0 = \emptyset$, then $H_0 = M_1$ and $H_1 \cup H_2 \cup H_3 = (H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_1) \subseteq H_1 \cup H_2$. Thus, $H_3 \subseteq H_1 \cup H_2$ and the proof is complete. Thus, suppose that $M_1 \setminus H_0 \neq \emptyset$ and let C denote one of its components.

If any two of $C \cap (H_1 \setminus H_0)$, $C \cap (H_2 \setminus H_0)$, and $C \cap (H_3 \setminus H_0)$ are nonempty, say the first two, then $C = [C \cap (H_1 \setminus H_0)] \cup [C \cap (H_2 \cup H_3) \setminus H_0]$. But this is a separation of C since $C \cap (H_1 \setminus H_0) \neq \emptyset$, $C \cap (H_2 \cup H_3) \setminus H_0 \neq \emptyset$, $[C \cap (H_1 \setminus H_0)] \cup \{C \cap [(H_2 \cup H_3) \setminus H_0]\} = C \cap [(H_1 \setminus H_0) \cup (H_2 \setminus H_0) \cup (H_3 \setminus H_0)] =$

$C \cap (M_1 \setminus H_0) = C$, and x a limit point of $C \cap (H_1 \setminus H_0)$ implies that either $x \in H_1 \setminus H_0$ and thus $x \notin (H_2 \cup H_3) \setminus H_0$ or $x \in H_0$ and thus $x \notin (H_2 \cup H_3) \setminus H_0$. In either case $x \notin C \cap [(H_2 \cup H_3) \setminus H_0]$. Similarly, x a limit point of $C \cap [(H_2 \cup H_3) \setminus H_0]$ implies either $x \in (H_2 \cup H_3) \setminus H_0$ and thus $x \notin H_1$ or $x \in H_0$ and hence $x \notin C \cap (H_1 \setminus H_0)$. Because C is connected we have that C meets at most one of $H_1 \setminus H_0$, $H_2 \setminus H_0$, and $H_3 \setminus H_0$. It follows that C meets at most one of H_1, H_2, and H_3. If C is the only component of $M_1 \setminus H_0$ and without loss of generality, C meets H_1, then $H_2 \cup H_3 \subseteq H_0$ which implies that $H_3 \subseteq (H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_1) \subseteq H_1 \cup H_2$ and the proof is complete.

Finally, if $M_1 \setminus H_0$ does have a second component B, then B by the preceding argument will meet at most one of H_1, H_2, or H_3. Thus, one of these subcontinua fails to meet $C \cup B$, say H_3, which implies that $H_3 \subseteq H_0$ and thus $H_3 \subseteq H_1 \cup H_2$ and again the proof is complete. ||

Proposition 3.4 illustrates the use of proposition 3.3 to prove results by contradiction. This proposition requires that the three mutually exclusive subcontinua H_1, H_2, and H_3 of M contain interior points relative to M. The condition is clearly necessary when one considers the continuum M to be the Closed Topologist's Sine Curve, example A, with the three subcontinua being $H_1 = \{ (x,y) \in M : x = 0, \frac{1}{2} \leq y \leq 1 \}$, $H_2 = \{ (x,y) \in M : x = 0, -1 \leq y \leq -\frac{1}{2} \}$, and $H_3 = \{ (x,y) \in M : \frac{1}{2} \leq x \leq 1 \}$. Both H_1 and H_2 fail to contain interior points relative to M and no one of the three subcontinua separates the other two from each other in M. That is, $M \setminus H_i$ cannot be written as the union of two separated sets each containing one of the other two subcontinua for any i.

3.4 Proposition Let M be a nondegenerate continuum which is atriodic and hereditarily unicoherent. If each of three mutually exclusive subcontinua H_1, H_2, and H_3 of M contain interior points relative to M, then one of the continua separates the other two from each other in M.

Proof: Assume to the contrary that neither H_1, H_2, nor H_3 separates the other two from each other in M. Since $M \neq H_i$, $i = 1, 2, 3$ and by proposition 3.2, $M \setminus H_i$ has at most two components, then some component C_i of $M \setminus H_i$ must contain H_k, $k \neq i$. Now for each i, $\overline{C}_i \cap H_i \neq \emptyset$ and since $C_i \cap H_i = \emptyset$, $\overline{C}_i$ contains no points of the interior of H_i with respect to M.

Considering specifically C_1 and C_2, we have that $\overline{C}_1 \supseteq H_2 \cup H_3$, $\overline{C}_2 \supseteq H_1 \cup H_3$, and both are subcontinua of M. Hence $\overline{C}_1 \cup \overline{C}_2$ is a nondegenerate subcontinuum of M. Thus, $\overline{C}_1 \cap \overline{C}_2$ is a subcontinuum and it contains H_3, at least one limit point of each of H_1 and H_2, but no interior points of these two subcontinua relative to M. Let $K_3 = \overline{C}_1 \cap \overline{C}_2$. The continua K_1 and K_2 are similarly defined such that K_i contains H_i but no interior points of the other two continua H_k, $k \neq i$. Thus, each pair of the subcontinua K_1, K_2, and K_3 of M intersect and proposition 3.3 implies one of them is contained in the union of the other two. Suppose $K_3 \subseteq K_1 \cup K_2$. Then $H_3 \subseteq K_1 \cup K_2$ and one of K_1 or K_2 must contain the interior points of H_3 relative to M. But this contradicts the definition of K_1 or K_2. Therefore, our assumption is false and one of H_1, H_2, or H_3 separates the other two from each other in M. ||

A partial converse to the above is given in proposition 3.5 which asserts the existence of a third subcontinuum which separates two mutually exclusive subcontinua of a continuum M and which contains relative interior points of M. The additional hypothesis condition is that M contain no indecomposable subcontinua with interior points relative to M. That this additional condition is necessary is illustrated by a continuum M formed with example F, which is indecomposable. Let M be the union of the two segments $L_1 = \{ (x,y) : y = 0, -1 \leq x \leq 0 \}$ and $L_2 = \{ (x,y) : y = 0, 1 \leq x \leq 2 \}$ and the continuum of example F. Then L_1 and L_2 are mutually exclusive subcontinua of M which no subcontinuum of M can separate.

3.5 <u>Proposition</u> Let M be a nondegenerate continuum which is atriodic, hereditarily unicoherent, and which contains no indecomposable subcontinua with interior points relative to M. If each of two mutually exclusive subcontinua H_1 and H_2 contain interior points with respect to M, then there exists a subcontinuum with interior points which separates H_1 from H_2.

Proof: Clearly $M \setminus (H_1 \cup H_2) \neq \emptyset$ since to assume so implies a separation of M. Now assume that there does not exist a component C of $M \setminus (H_1 \cup H_2)$ with limit points in both H_1 and H_2. There is at least one component and by proposition 3.2, at most three. Each of these components has a limit point in one of H_1 or H_2 but not both by the assumption. Now let N be the union of H_1 and all components of $M \setminus (H_1 \cup H_2)$ with limit points in H_1. Let K be similarly defined with respect to H_2. Since $H_1 \subseteq N$ and $H_2 \subseteq K$, these are nonempty disjoint sets whose union is M. Thus N and K form a separation of

M since H_2 is closed and has no points in common with H_1 or the components included in N and the components in K are disjoint from the components in N and H_1 and have no limit points in the components in N or H_1. A similar argument holds for H_1 and the components in N relative to K. Since M is a continuum our assumption is false and we let C denote the component of $M \setminus (H_1 \cup H_2)$ which has limit points in H_1 and H_2.

Now $\overline{C}$ is a subcontinuum of M and C is open relative to M since to assume that every set Q open relative to M and containing a point x of C meets either H_1, H_2, or some other component of $M \setminus (H_1 \cup H_2)$ would imply that x is a limit point of one or more of these sets which is impossible.

Since $\overline{C}$ is then a nondegenerate subcontinuum of M with interior points relative to M, the hypothesis implies that it is decomposable into two proper subcontinua C_1 and C_2. By the unicoherence of $\overline{C}$, $C_1 \cap C_2$ is a continuum. Now C_2 closed in $\overline{C}$ implies that C_2 is closed in M. Also, $C \setminus C_2 \neq \emptyset$ since to assume that $C \setminus C_2 = \emptyset$ implies $C \subseteq C_2$ which implies $\overline{C} \subseteq C_2$ and thus $\overline{C} = C_2$, a contradiction of its being proper. Thus, for $x \in C \setminus C_2$, $x \in C_1$ and there is an open set Q relative to M such that $x \in Q$, $Q \cap C_2 = \emptyset$, and $Q \subseteq C$ which implies that $Q \subseteq C \setminus C_2$. Thus, $C_1 \setminus C_2$ has an interior point relative to M as likewise does $C_2 \setminus C_1$.

To see that $K_1 = \overline{C_1 \setminus C_2} = \overline{C_1 \setminus (C_1 \cap C_2)} = \overline{C} \setminus C_2$, is a continuum, it suffices to show that $C_1 \setminus (C_1 \cap C_2)$ is connected. Hence suppose that K and N form a separation of $C_1 \setminus (C_1 \cap C_2)$. Then $\overline{K} \cap (C_1 \cap C_2) \neq \emptyset$ and $\overline{N} \cap (C_1 \cap C_2) \neq \emptyset$ which implies that $T = \overline{K} \cup (C_1 \cap C_2)$ and $T' = \overline{N} \cup (C_1 \cap C_2)$ are connected and in fact both are subcontinua of M. Now $\overline{C} = T \cup T' \cup C_2$ implies that $\overline{C}$ is a triod which is impossible since M is atriodic and $\overline{C}$ is nondegenerate. Thus, K_1 is a subcontinuum of M and contains an interior point relative to M. Similarly $\overline{C_2 \setminus C_1}$ is a subcontinuum and contains interior points of M.

Arguing in a manner similar to the above except with C_2 instead of $\overline{C}$, we can produce subcontinua K_2 and K_3 such that $C_2 = K_2 \cup K_3$, each of K_2 and K_3 have interior points of C_2 relative to M which are not in each other and necessarily not in K_1, and only one of K_2 or K_3 intersects K_1 to form $C_2 \cap K_1$. Thus, $\overline{C} = K_1 \cup K_2 \cup K_3$ with each containing an interior point relative to M which is not in either of the other two.

Now if $K_i \cap (H_1 \cup H_2) = \emptyset$ for one of the K_i's then H_1, H_2, and this K_i satisfy proposition 3.4 and we have in fact that this K_i separates H_1 from H_2 in M. If, however, each K_i meets at least one of H_1 or H_2, suppose that $K_1 \cap H_1 \neq \emptyset$, $K_2 \cap H_2 \neq \emptyset$, and $K_3 \cap H_2 \neq \emptyset$. Also, since K_1 meets one of K_2 or K_3 suppose $K_1 \cap K_2 \neq \emptyset$. Then $H_1 \cup K_1 \cup K_2$, $H_2 \cup K_2 \cup K_3$, and $\overline{C}$ are three subcontinua which are pairwise intersecting and by proposition 3.3, one is a subset of the union of the other two. However, $H_1 \cup K_1 \cup K_2$ contains a point interior to K_1 relative to M but not in H_2, K_2, or K_3. Also, $H_1 \not\subseteq \overline{C}$. Thus, $H_1 \cup K_1 \cup K_2$ cannot be contained in the union of the other two. Arguing similarly we have that no one of the three is contained in the union of the other two and thus one of the K_i's must fail to meet $H_1 \cup H_2$ and will therefore separate H_1 from H_2 in M. Thus, we have a subcontinuum with interior points relative to M which separates H_1 from H_2 in M. ||

The concluding result of this section is designed to provide a means of constructing a particular nested sequence of continua. The particular property desired is that each subcontinuum contain the succeeding subcontinua interiorly. A set A will be said to contain a set B <u>interiorly</u> if and only if A contains an open set which contains B. Since the following proposition, and all others which apply this result, are stated in terms of a continuum M containing A and B, we shall take this definition to mean that A contains an open set relative to M which contains B.

3.6 <u>Proposition</u> Let M be a nondegenerate continuum which is atriodic, hereditarily unicoherent, and which contains no indecomposable subcontinua with interior points relative to M. If H_1 and H_2 are subcontinua with interior points relative to M such that H_1 contains H_2 interiorly, then there exists a continuum H_3 such that H_1 contains H_3 interiorly and H_3 contains H_2 interiorly.

Proof: Proposition 3.2 implies that $M \setminus H_1$ has at most two components. If it has none, then $M = H_1$ which implies that H_1 is open relative to M and we let $H_3 = H_1$ to satisfy the theorem. However, if $M \setminus H_1 \neq \emptyset$, let C_1 denote a component of $M \setminus H_1$. Since H_1 contains H_2 interiorly and C_1 is in the complement of H_1 relative to M, $C_1 \cap H_2 = \emptyset$ and in fact $\overline{C}_1 \cap H_2 = \emptyset$. Thus, $\overline{C}_1$ and H_2 are two mutually exclusive subcontinua of M, H_2 contains interior points relative to M, and since $C_1 \subseteq \overline{C}_1$ is open relative to M, it also contains relative interior points.

Proposition 3.5 implies the existence of a subcontinuum K of M which separates H_2 from $\overline{C}_1$ in M. Therefore, $M \setminus K = S \cup N$ with

$H_2 \subseteq S$, $\overline{C}_1 \subseteq N$, $\overline{S} \cap \overline{N} = \emptyset$, $K \cap N = \emptyset$, $\overline{S} \cap K \neq \emptyset$, $(\overline{S} \cup K) \cap \overline{C}_1 = \emptyset$, and $\overline{S} \cup K$ is a subcontinuum of M containing H_2 interiorly. If $M \setminus H_1$ has only the one component, let $H_3 = \overline{S} \cup K$. Then H_3 is a subcontinuum of M containing H_2 interiorly and since $M \setminus H_1 = C_1$ and $(\overline{S} \cup K) \cap \overline{C}_1 = \emptyset$, we have $\overline{S} \cup K \subseteq M \setminus \overline{C}_1$. But $M \setminus \overline{C}_1$ is open relative to M and $M \setminus \overline{C}_1 \subseteq H_1$. Thus, H_1 contains H_3 interiorly and we are done.

If $M \setminus H_1$ has a second component C_2, we may repeat the above procedure to produce a subcontinuum T of M containing H_2 interiorly and such that $T \subseteq M \setminus \overline{C}_2$. If we let $H_3 = T \cap (\overline{S} \cup K)$ then H_3 will also contain H_2 interiorly and since $H_3 \subseteq (M \setminus \overline{C}_2) \cap (M \setminus \overline{C}_1) = M \setminus (\overline{C}_1 \cup \overline{C}_2)$, which is open relative to M and is properly contained in $M \setminus (C_1 \cup C_2) = H_1$, H_3 is contained interiorly in H_1. Thus, we have found the desired continuum. ||

An Upper Semi-Continuous Decomposition

This section of Chapter III has a single purpose and that is to show that a particular collection of subcontinua of an atriodic and hereditarily unicoherent continuum is upper semi-continuous and to show certain properties of this collection. These results are vitally important for the proofs of theorems 3.21 and 5.11.

Throughout this section, M will denote an atriodic, hereditarily unicoherent continuum which contains no indecomposable subcontinua with interior points relative to M and except for special emphasis, this will not be repeated in the statements of the results. For each point p in M, let $\underline{G_p}$ denote the intersection of all subcontinua of M which contain interiorly a continuum that contains p interiorly.

Then $\mathfrak{G}$ will denote the collection of all such G_p for $p \in M$. Another approach to defining G_p is to first define, for $p \in M$, the collection $\mathcal{H}_p = \{ H : H$ is a subcontinuum of M containing interiorly a continuum K which contains p interiorly $\}$. Such a collection will be called a defining sequence of continua for G_p since $G_p = \bigcap \{ H : H \in \mathcal{H}_p \}$. The objective of this section is to show that $\mathfrak{G}$ is an upper semi-continuous decomposition of M.

A collection $\mathfrak{G}$ of mutually exclusive closed point sets is said to be upper semi-continuous if and only if whenever G is a member of the collection $\mathfrak{G}$ and $G_1, G_2, \ldots$ is a sequence of point sets in this collection and, for each n, a_n and b_n are points of G_n and the sequence $a_1, a_2, \ldots$ has a sequential limit point lying in G, then every infinite subsequence of $b_1, b_2, \ldots$ has a subsequence having a sequential limit point that lies in G [26, p. 273]. If $\mathfrak{G}$ is an upper semi-continuous collection of compact subsets of a metric space M and every point of M belongs to a set of $\mathfrak{G}$, then $\mathfrak{G}$ is said to be an upper semi-continuous decomposition of M. When this latter property holds, which is equivalent to requiring that $\mathfrak{G}^* = M$, then $\mathfrak{G}$ is said to fill up M. Further information regarding upper semi-continuous collections or decompositions may be found in [26, p. 273], [34, p. 122], and [28].

The following several results establish respectively that the collection $\mathfrak{G}$ is a collection of mutually exclusive continua filling up M, that no element of $\mathfrak{G}$ contains an interior point relative to M, that for any three distinct elements of $\mathfrak{G}$ one of them separates the other two from each other in M, and as theorem 3.12, that $\mathfrak{G}$ is an upper semi-continuous decomposition of M. Another important result will be that $\mathfrak{G}$ is an arc with respect to its elements. That is, considering its elements as points and appropriately defining various terms relative to these elements, $\mathfrak{G}$ is an arc. There will be more on this later.

To illustrate the preceding and hopefully to clarify the following, we might consider several examples of continua and the definition of $\mathfrak{G}$ on them. If M is an arc which is also atriodic and hereditarily unicoherent, then the elements of the collection $\mathfrak{G}$ in this case are simply the points of M. However, if M is the Closed Topologist's Sine Curve, example A, then the collection $\mathfrak{G}$ consists of the segment $\{ (x,y) : x = 0, \ -1 \leq y \leq 1 \}$ as a single element along with each point (x,y) of M where $0 < x \leq 1$.

3.7 Lemma The collection $\mathfrak{G}$ is a collection of mutually exclusive continua filling up any nondegenerate, atriodic, hereditarily unicoherent continuum M which contains no indecomposable subcontinua with interior points relative to M.

Proof: From proposition 3.1 we have that G_p is a continuum for each $G_p \in \mathfrak{G}$. To see that the elements of $\mathfrak{G}$ are mutually exclusive we shall first show that $q \in G_p$ implies that $G_p = G_q$. Assume that G_q does not contain a point p' of G_p. Then there exists a continuum H_1 of $M \setminus \{p'\}$ such that H_1 contains interiorly a continuum H_2 containing q interiorly. Applying lemma 3.6, we have the existence of a continuum H_3 contained interiorly in H_1 and containing interiorly H_2. Repeating the process, we have a continuum H_4 interiorly contained in H_3 and containing interiorly H_2. Thus, if $i_M(H)$ denotes the interior of H relative to M, then from the above we now have that

$q \in i_M(H_2) \subseteq H_2 \subseteq i_M(H_4) \subseteq H_4 \subseteq i_M(H_3) \subseteq H_3 \subseteq i_M(H_1) \subseteq H_1$.

If we now assume that $p \in H_4$ then $p \in i_M(H_3) \subseteq H_1$. By definition then $G_p \subseteq H_1$ and hence $p' \notin G_p$. Since this contradicts our initial assumption we must conclude that $p \notin H_4$ and consequently $p \notin H_2$.

Let K_2 denote the closure of the component of $M \setminus H_2$ which contains p. Likewise, let K_4 denote the closure of the component of $M \setminus H_4$ containing p. Now $H_2 \subseteq i_M(H_4) \subseteq H_4 \subseteq M$ implies that $M \setminus H_4 \subseteq M \setminus H_2$. Also, the component of $M \setminus H_4$ which contains p is open relative to M and is contained in K_2 which in turn is contained interiorly in the component of $M \setminus H_2$ which contains p, a subset of K_2. Hence, $G_p \subseteq K_4$ and because $q \in H_2$, $q \notin G_p$. Since this contradicts the fact that q is an element of G_p, our original assumption is false and $p' \in G_p$. Arguing in a similar manner, the assumption that G_p does not contain $q' \in G_q$ leads to a contradiction and we have that $G_p = G_q$. Thus, if $G_p \cap G_q \neq \emptyset$ for any G_p and G_q in $\mathfrak{G}$, then $G_p = G_q$ and the elements of $\mathfrak{G}$ are mutually exclusive. Since it is clear that $\mathfrak{G}^*$ is M, the proof is complete. ||

It is the following result which directly uses the hypothesis condition that M contain no indecomposable subcontinuum with interior points relative to M.

3.8 <u>Lemma</u> No element of the collection $\mathfrak{G}$ on M contains an interior point relative to M.

Proof: Suppose there is an element G_1 of $\mathfrak{G}$ such that G_1 contains an open set relative to M. If G_1 is degenerate then G_1 is itself open relative to M which implies that G_1 is both open and closed. Hence, G_1 and $M \setminus G_1$ form a separation of M, if $M \setminus G_1$ is nonempty. Since M is connected we must conclude that $M \setminus G_1 = \emptyset$ and thus $M = G_1$. But this implies that M is degenerate which contradicts the hypothesis. Therefore, G_1 is nondegenerate and since M is a continuum containing no indecomposable subcontinua with interior points relative to M, by the construction of lemma 3.5, G_1 is the union of three continua C_1, C_2, and C_3 each of which contains interior points relative to M which are not in either of the other two. It follows from lemma 3.3, that one pair of these three continua must fail to intersect. Without loss of generality, suppose that $C_1 \cap C_3 = \emptyset$.

Lemma 3.5 now implies the existence of a continuum C_4 with interior points relative to M such that C_4 separates C_1 from C_3 in G_1. Hence, $C_4 \subseteq G_1 \setminus (C_1 \cup C_3)$ which is open relative to M and is contained in C_2. Let p be a point of C_4 which is an interior point relative to M. By definition, $G_p \subseteq C_2$ since C_2 contains interiorly the continuum C_4 which contains interiorly p. Also, G_p is a proper subset of G_1 since C_1 and C_3 each have interior points relative to M which are not contained in C_2. Since we have previously shown that any two distinct members of $\mathfrak{G}$ must be disjoint, it is impossible for G_p to be properly contained in G_1. Hence our assumption is false and G_1 contains no interior points relative to M. ||

3.9 <u>Lemma</u> The collection $\mathfrak{G}$ is such that if $G_p \in \mathfrak{G}$ and C is any continuum of M with $C \cap G_p = \emptyset$, then there is a continuum H_0 containing G_p and interior points relative to M such that $H_0 \cap C = \emptyset$.

Proof: Let $\mathcal{H}_p$ denote a defining sequence for G_p. We note that if H_1 and H_2 are elements of $\mathcal{H}_p$, then there exist continua K_1 and K_2 of M such that $p \varepsilon i_M(K_i) \subseteq K_i \subseteq i_M(H_i) \subseteq H_i$ for $i = 1, 2$. Thus, $p \varepsilon i_M(K_1) \cap i_M(K_2) = i_M(K_1 \cap K_2) \subseteq K_1 \cap K_2 \subseteq i_M(H_1 \cap H_2) \subseteq H_1 \cap H_2$. Now if either K_1 or K_2 were degenerate, this would imply that $\{p\}$ is open relative to M and since it is also closed, that $\{p\}$ and its complement relative to M would form a separation of M. Thus, K_1 and K_2 are both nondegenerate and consequently so also are H_1 and H_2. Then $K_1 \cup K_2$ and $H_1 \cup H_2$ are nondegenerate subcontinua of M which implies they are unicoherent. The unicoherence of $K_1 \cup K_2$ will imply that $K_1 \cap K_2$ is a continuum. Likewise, $H_1 \cap H_2$ is a continuum. Finally, we have that $H_1, H_2 \varepsilon \mathcal{H}_p$ implies that $H_1 \cap H_2 \varepsilon \mathcal{H}_p$. Therefore, for any finite collection of elements of $\mathcal{H}_p$, their intersection is in $\mathcal{H}_p$.

If we assume that every member of a defining sequence of continua for G_p meets C, then consider the collection $\mathcal{K} = \{ K : K = H \cap C, H \varepsilon \mathcal{H}_p \}$. By assumption, each K in $\mathcal{K}$ is nonempty. Let $K_1, K_2, \ldots, K_n$ be any finite collection of members of $\mathcal{K}$. Then $\bigcap_{i=1}^{n} K_i = (\bigcap_{i=1}^{n} \{ H_i : K_i = H_i \cap C, H_i \varepsilon \mathcal{H}_p \}) \cap C = H_{-1} \cap C$ for some $H_{-1} \varepsilon \mathcal{H}_p$. Thus, $\bigcap_{i=1}^{n} K_i \neq \emptyset$ and $\mathcal{K}$ has the finite intersection property. Since M is compact, $\bigcap \{ K : K \varepsilon \mathcal{K} \} = (\bigcap \{ H : H \varepsilon \mathcal{H}_p \}) \cap C$ is nonempty which implies that $G_p \cap C \neq \emptyset$. Since this contradicts the hypothesis, there exists a member H_0 of $\mathcal{H}_p$ such that $H_0 \cap C = \emptyset$. Therefore, H_0 contains G_p and contains p interiorly which satisfies the conclusion. ||

The following lemma makes it possible to show that $\mathfrak{G}$ is an arc with respect to its elements.

3.10 <u>Lemma</u> The collection $\mathfrak{G}$ of mutually exclusive continua filling up M is such that for any three distinct elements of $\mathfrak{G}$, one of them separates the other two from each other in M.

Proof: The collection $\mathfrak{G}$ is nondegenerate since to assume otherwise would contradict lemma 3.8. Let G_p, G_q, and G_r denote any three distinct members of $\mathfrak{G}$. Lemma 3.9 implies the existence of a continuum H_{pq} such that $G_p \subseteq H_{pq}$, $p \varepsilon i_M(H_{pq})$, and $H_{pq} \cap G_q = \emptyset$. A continuum H_{pr} is similarly defined. Now if $H_{pq} = H_{pr}$, then define K_p to be H_{pq}. If $H_{pq} \neq H_{pr}$, then as is argued in lemma 3.7, $H_{pq} \cap H_{pr}$ is a member of $\mathfrak{H}_p$, a defining sequence of continua for G_p, and consequently this intersection contains G_p, is a continuum, and contains points of the relative interior of M. In this case we define K_p to be $H_{pq} \cap H_{pr}$ and in either case K_p is a continuum with interior points relative to M which contains G_p and fails to meet $G_p \cup G_r$.

Applying lemma 3.9 to the continua K_p, G_q, and G_r, we can by the above method produce a continuum K_q which contains G_q, contains q interiorly, and fails to meet $K_p \cup G_r$ and consequently $G_p \cup G_r$. Similarly, the continuum K_r is produced containing G_r, containing r interiorly, and failing to meet $K_p \cup K_q$ and $G_p \cup G_q$.

Thus, the continua K_p, K_q, and K_r are mutually exclusive and contain respectively G_p, G_q, and G_r and contain interior points relative to M. Proposition 3.4 now implies that one of K_p, K_q, or K_r separates the other two from each other in M. Without loss of generality, we may assume that S and N form a separation of M by K_r with $K_p \subseteq S$ and $K_q \subseteq N$. It now remains to show that $G_r \subseteq K_r$

actually separates G_p from G_q in M.

If $K_r = G_r$ then we are done. However, if $K_r \neq G_r$ then lemma 3.2 implies that $K_r \setminus G_r$ has at most two components and by the first statement, at least one component. If $K_r \setminus G_r = C$, where C is a component, then $\overline{S} \cap G_r$ and $\overline{N} \cap G_r$ are empty since $G_r \subseteq K_r$, $K_r \in \mathcal{K}_r$, and lemma 3.6 implies the existence of a continuum K_r'' which is contained interiorly in K_r and which contains interiorly K_r' the continuum associated with K_r by the definition of G_r. Thus, $G_r \subseteq K_r'' \subseteq i_M(K_r) \subseteq K_r$ and G_r contains no limit points of S or N. However, since $K_r \cap \overline{S}$ and $K_r \cap \overline{N}$ are nonempty, we must have $\overline{S} \cap \overline{C}$ and $\overline{N} \cap \overline{C}$ nonempty as well as $G_r \cap \overline{C}$ nonempty. Thus, $M = \overline{S} \cup \overline{N} \cup \overline{C} \cup G_r$ is a triod with continua $\overline{S} \cup \overline{C}$, $\overline{N} \cup \overline{C}$, and $G_r \cup \overline{C}$. Since M is not a triod, $K_r \setminus G_r$ must have two components C_1 and C_2.

Again, we must have that $\overline{C}_1 \cap G_r$ and $\overline{C}_2 \cap G_r$ are nonempty. Also, $\overline{S} \cap K_r$ and $\overline{N} \cap K_r$ are nonempty while $\overline{S} \cap G_r$ and $\overline{N} \cap G_r$ are empty. Since $K_r = G_r \cup C_1 \cup C_2$ this implies that $\overline{S}$ meets C_1 or C_2 and $\overline{N}$ meets C_1 or C_2.

To see that $\overline{S}$ cannot meet both C_1 and C_2, we assume that $\overline{S} \cap C_1 \neq \emptyset$ and $\overline{S} \cap C_2 \neq \emptyset$. Since $\overline{C}_1 \cap G_r \neq \emptyset$ and $\overline{C}_2 \cap G_r \neq \emptyset$, we have that $\overline{S} \cup \overline{C}_1 \cup G_r$ and $\overline{S} \cup \overline{C}_2 \cup G_r$ are distinct nondegenerate continua. It also follows that their union $\overline{S} \cup \overline{C}_1 \cup \overline{C}_2 \cup G_r$ is a nondegenerate subcontinuum of M and is hence unicoherent. But $(\overline{S} \cup \overline{C}_1 \cup G_r) \cap (\overline{S} \cup \overline{C}_2 \cup G_r) = \overline{S} \cup G_r$ which is not a continuum. Therefore, $\overline{S}$ cannot meet both C_1 and C_2. Likewise, $\overline{N}$ cannot meet both C_1 and C_2.

Since $\overline{S} \cap (C_1 \cup C_2) \neq \emptyset$ and $\overline{N} \cap (C_1 \cup C_2) \neq \emptyset$, $\overline{S}$ and $\overline{N}$ must each meet one of C_1 or C_2. Now we assume that $\overline{S}\cap C_1 \neq \emptyset$ and $\overline{N}\cap C_1 \neq \emptyset$. Then, $\overline{S}\cup\overline{C}_1$, $\overline{N}\cup\overline{C}_1$, and $G_r\cup\overline{C}_1$ are each distinct continua with a common intersection of $\overline{C}_1$ since $\overline{S}\cap\overline{N}$, if nonempty, is a subset of K_r but not in G_r or C_2 as a result of the preceding, which means that $\overline{S}\cap\overline{N}\subseteq\overline{C}_1$. Hence, the union of these three continua forms a triod in M which is not possible. Thus, $\overline{S}$ and $\overline{N}$ cannot both meet C_1 and finally we have that, without loss of generality, $\overline{S}\cap C_1 \neq \emptyset$, $\overline{S}\cap C_2 = \emptyset$, $\overline{N}\cap C_2 \neq \emptyset$, and $\overline{N}\cap C_1 = \emptyset$.

Now we have that $M\setminus G_r = (\overline{S}\cup C_1)\cup(\overline{N}\cup C_2)$ and this is a separation of $M\setminus G_r$ with $G_p\subseteq\overline{S}\cup C_1$ and $G_q\subseteq\overline{N}\cup C_2$ and the lemma is proven. ||

The statement that the collection $\mathfrak{G}$ has certain properties with respect to its elements should perhaps now be clarified. If $\mathfrak{G}$ is an upper semi-continuous collection of mutually exclusive closed point sets then by a <u>region with respect to</u> $\mathfrak{G}$ is meant a subcollection $\mathfrak{R}$ of $\mathfrak{G}$ such that $\mathfrak{R}$ contains no limit point of $\mathfrak{G}^*\setminus\mathfrak{R}^*$ [26, p. 273]. With this definition, if $\mathfrak{G}$ is an upper semi-continuous collection of mutually exclusive closed point sets, then words and phrases, previously defined in terms of points and regions, can be extended to apply to $\mathfrak{G}$ by replacing the term point by element of $\mathfrak{G}$ and the term region by region with respect to $\mathfrak{G}$. The following few definitions will illustrate this process. The element G of $\mathfrak{G}$ will be said to be a <u>limit element</u> of the subcollection $\mathfrak{H}$ of $\mathfrak{G}$ if and only if every region with respect to $\mathfrak{G}$ which contains G contains at least one element of $\mathfrak{H}$ distinct from G [26, p. 274]. The subcollection $\mathfrak{H}$ of $\mathfrak{G}$ is said to be closed if and only if every limit element of $\mathfrak{H}$ belongs to $\mathfrak{H}$ [26, p. 274]. Finally, of immediate importance to the development of this section, the collection $\mathfrak{G}$ <u>is an arc with respect to its elements</u> if and only if $\mathfrak{G}$ is a compact nondegenerate continuum with respect to its elements which does not have more than two non-cut elements. Further definitions and discussion of this process and particularly of the relationship between M and $\mathfrak{G}$ with respect to its elements may be found in [26, p. 273].

3.11 <u>Theorem</u> If M is a nondegenerate, atriodic, hereditarily unicoherent continuum which contains no indecomposable subcontinua with interior points relative to M, then $\mathfrak{G}$ is an upper semi-continuous decomposition of M.

Proof: By the preceding lemmas we have shown that $\mathfrak{G}$ is a collection of mutually exclusive continua filling up M. Assume that $\mathfrak{G}$ is not upper semi-continuous. Because of this, there are two sequences of points $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ converging to points p and q respectively such that for each n, p_n and q_n are both members of some G_n in $\mathfrak{G}$ but p and q do not belong to the same element of $\mathfrak{G}$. Let G_p and G_q denote these two distinct continua.

Suppose there are only finitely many distinct continua in the sequence $G_1, G_2, \ldots$. By applying lemma 3.9 finitely many times it is possible to produce a continuum H such that H contains interiorly a continuum K containing p interiorly and thus, a set O, open relative to M, containing p and lying in H. This continuum and hence the set O may be selected to be disjoint from each of the finitely many distinct G_i's which are distinct from G_p. But since $p_1, p_2, \ldots$ converges to p, for any set open in M, namely O, there exists an integer N such that for each $n \geq N$, $O \cap G_n \neq \emptyset$. Thus, for all

$n \geq N$, $G_n = G_p$ and $q_1, q_2, \cdots$ must converge to an element q in G_p. This contradicts our assumption and $\mathfrak{G}$ is upper semi-continuous in this case.

If, however, there are infinitely many distinct continua in the sequence $G_1, G_2, \cdots$, then applying lemma 3.9 we may, as in lemma 3.10, produce two disjoint continua H_p and H_q such that each interiorly contains respectively continua K_p and K_q which contain respectively p and q interiorly. Hence, $G_p \subseteq H_p$ and $G_q \subseteq H_q$ but more importantly there are sets O_p and O_q both of which are open relative to M, contain p and q respectively, and lie in H_p and H_q respectively. Since the sequence of p_i's converges to p, there exists an integer N such that for all $n \geq N$, $O_p \cap G_n \neq \emptyset$. Because only $N - 1$ continua of the sequence of G_i's are being omitted there must exist two distinct continua among those remaining, say G_{N_1} and G_{N_2}, with $O_q \cap G_{N_1} \neq \emptyset$ and $O_q \cap G_{N_2} \neq \emptyset$. Now we note that for $i = 1$ or 2, $G_{N_i} \cap O_p \neq \emptyset$ implies that $G_{N_i} \cap H_p \neq \emptyset$, $G_{N_i} \cap O_q \neq \emptyset$ implies $G_{N_i} \cap H_q \neq \emptyset$, and $H_p \neq H_q$ implies $G_{N_i} \subseteq H_p$ and $G_{N_i} \subseteq H_q$. Thus, $H_p \cup G_{N_1} \cup H_q$ and $H_p \cup G_{N_2} \cup H_q$ are distinct continua in M whose union is $H_p \cup G_{N_1} \cup G_{N_2} \cup H_q$, a nondegenerate subcontinuum of M. But this implies that $H_p \cup G_{N_1} \cup G_{N_2} \cup H_q$ is unicoherent and yet $(H_p \cup G_{N_1} \cup H_q) \cap (H_p \cup G_{N_2} \cup H_q) = H_p \cup H_q$ which is not a continuum. Thus, again our assumption is false and therefore, $\mathfrak{G}$ is an upper semi-continuous collection. That it is also an upper semi-continuous decomposition follows immediately. ||

3.12 Corollary The upper semi-continuous collection $\mathfrak{G}$ on M is an arc with respect to its elements.

Proof: Using lemma 3.8 we again note that $\mathfrak{G}$ is nondegenerate. By theorem 3.11, $\mathfrak{G}$ is an upper semi-continuous collection on M. Theorems 4, 6, and 13 [26, p. 275] show that $\mathfrak{G}$ is closed, compact, and connected and is thus a continuum with respect to its elements. Lemma 3.10 implies that $\mathfrak{G}$ has at most two non-cut points; ie, for any three elements of $\mathfrak{G}$, one of them separates the other two from each other in M and thus in $\mathfrak{G}$ [26, p. 275]. Therefore, $\mathfrak{G}$ is an arc with respect to its elements. ||

The final result of this section establishes an additional relationship between the collection $\mathfrak{G}$ and the continuum M upon which it is based. Since corollary 3.12 asserts that $\mathfrak{G}$ is an arc with respect to its elements, it has two end elements. Thus, in the continuum M of example C, the line segments PQ and RS are end elements of the arc $\mathfrak{G}$ with respect to its elements which is the upper semi-continuous decomposition $\mathfrak{G}$ of M.

3.13 Corollary If a and b are points of different end elements of the collection $\mathfrak{G}$ on M, then M is irreducible from a to b.

Proof: Let G_a and G_b denote the elements of $\mathfrak{G}$ containing a and b respectively. By hypothesis, $G_a \neq G_b$ and thus $G_a \cap G_b = \emptyset$. Let N be a subcontinuum of M containing a and b. Assume that N is not M. Let C_1 be a component of $M \setminus N$. If $x \in C_1$, then $G_x \not\subseteq N$.

Suppose that $G_x \cap N = \emptyset$. Then $a \notin G_x$ and $b \notin G_x$ imply that $G_a \cap G_x = \emptyset$ and $G_b \cap G_x = \emptyset$, and since $G_a \cap G_b = \emptyset$, G_a, G_b, and G_x are three distinct members of $\mathfrak{G}$. Lemma 3.4 implies that one of them must separate the other two from each other in M. Since G_a and G_b are end elements of the arc $\mathfrak{G}$ with respect to its elements, we have that H and K form a separation of $M \setminus G_x$ with $G_a \subseteq H$ and $G_b \subseteq K$. Since N is a continuum and $N \cap G_x = \emptyset$, $N \subseteq H$. But N must also be contained in K which is impossible. Thus, $N \cap G_x \neq \emptyset$ and yet $G_x \not\subseteq N$.

If $N \cup G_x = M$ then $C_1 \subseteq G_x$. However, C_1 is open relative to M and G_x, by lemma 3.8, contains no interior points relative to M. Thus, $N \cup G_x \subsetneq M$ and $M \setminus (N \cup G_x)$ has a component C_2. Arguing as before, we can produce an element G_y of $\mathfrak{G}$ such that $y \in C_2$, $G_y \neq G_x$, G_a, or G_b, $N \cap G_y \neq \emptyset$, and $N \cup G_y \subsetneq M$. Finally, a similar argument produces a third element G_z of $\mathfrak{G}$ such that $z \in M \setminus (G_y \cup G_x \cup N)$, $G_z \neq G_y$, G_x, G_a, or G_b, and $N \cap G_z \neq \emptyset$. Then $N \cup G_x \cup G_y \cup G_z$ is a nondegenerate subcontinuum of M which by considering the continua $G_x \cup N$, $G_y \cup N$, and $G_z \cup N$ forms a triod. Since M is atriodic, this is impossible and N must equal M. Therefore, M is irreducible from a to b. ||

An Introduction to Terminal Subcontinua

The notion of a terminal subcontinuum is a generalization by Fugate [13, p. 461] of the term terminal point as defined by Miller [23, p. 90] and later the term end point of a chainable continuum as defined and used by Bing [4, p. 660]. Intuitively, terminal subcontinua and terminal points are subcontinua of a continuum which are at the end of the continuum, for example, an end point of an arc. The definition of a terminal subcontinuum is as follows. A subcontinuum T of a continuum M is said to be a <u>terminal subcontinuum</u> of M if and only if whenever A and B are both subcontinua of M meeting T, then one of A or B is contained in the union of T and the other subcontinuum [13, p. 461]. That is, either $A \subseteq T \cup B$ or $B \subseteq T \cup A$. A <u>terminal point</u> is then a degenerate terminal subcontinuum. Since it will be shown in Chapter IV that the notions of a terminal point, as defined by Fugate and Miller, and the notion of an end point, as defined by Bing, are equivalent, these terms will be used interchangably.

Examples A, B, C, D, E, and F illustrate the terms, terminal subcontinua and terminal point. The Closed Topologist's Sine Curve, example A, has exactly three end points, namely the points R, S, and T. The segment $RS = \{ (x,y) : x = 0, \ -1 \leq y \leq 1 \}$ of example A, is a terminal subcontinuum and any terminal subcontinuum or continuum meeting RS and included in the continuum must contain RS. By appropriately joining two Closed Topologist's Sine Curves, examples can be created of a continuum with two end points which is not an arc, example B, and a continuum with four end points, example C.

Examples D, E, and F provide illustrations of indecomposable continua with one, zero, and two terminal points respectively. Bing shows that every point of a pseudo-arc, example G, is in fact an end point of it [4, p. 602].

The purpose of considering terminal subcontinua, and in this chapter end points in particular, is that they provide a means of chaining on certain subcontinua of a continuum, with some degree of control or assurance that certain links contain the terminal subcontinua. This ability is established, with considerable effort, for terminal subcontinua in proposition 3.18. Corollary 3.19 then follows for terminal points. Theorem 3.20 will then illustrate the use of these results and the previously mentioned process of fitting chains together. Further consideration will be given to terminal subcontinua in Chapter IV where some sufficient conditions will be presented for terminal subcontinua and certain results concerning chains will be established.

Proposition 3.14 and its corollary 3.15 provide a characterization of terminal subcontinua in terms of the irreducibility of the continuum M between a pair of points, one of which must belong to the terminal subcontinuum. Corollary 3.14 will then establish that the definitions of a terminal point by Miller and Fugate, are equivalent.

3.14 <u>Proposition</u> If M is an atriodic, hereditarily unicoherent continuum and A is a subcontinuum of M, then A is a terminal subcontinuum of M if and only if for each subcontinuum B of M such that B meets A and $A \cup B$ is nondegenerate, $A \cup B$ is irreducible between some pair of points, one of which belongs to A.

Proof: Suppose that A is a terminal subcontinuum of M and B is a subcontinuum of M meeting A such that $A \cup B$ is nondegenerate. Since $A \cup B$ is again a subcontinuum of M, $A \cup B$ is atriodic and hereditarily unicoherent. Theorem 1.6 implies that there are two points p and q of $A \cup B$ such that $A \cup B$ is irreducible between them.

Now assume that the two points p and q are always in $B \setminus A$. Let $r \in A$. By the assumption, $A \cup B$ is not irreducible between p and r nor between q and r. Thus, there are proper subcontinua L and K of $A \cup B$ such that $\{p, r\} \subseteq L$ and $\{q, r\} \subseteq K$. Since A is a terminal subcontinuum of M, either $K \subseteq A \cup L$ or $L \subseteq A \cup K$. Without loss of generality, suppose that $L \subseteq A \cup K$. Then $p \in A \cup K$. Since $p \notin A$, this implies that $p \in K$ and thus K is a proper subcontinuum of $A \cup B$ containing p and q which is contrary to the earlier conclusion that $A \cup B$ was irreducible between p and q. Hence, our assumption that $\{p, q\} \subseteq B \setminus A$ is false and one of p or q is contained in A and this part of the lemma is established.

For the converse statement suppose that A is not a terminal subcontinuum of M. This assumption then implies that there are subcontinua D and E of M, each meeting A, such that $A \cup E$ and $A \cup D$ are nondegenerate, $D \not\subseteq A \cup E$, and $E \not\subseteq A \cup D$. Then $A \cup D \cup E$ is a subcontinuum of M. If $p \in A$ and $q \in A \cup D \cup E$, $p \neq q$, then $A \cup D \cup E$ is not irreducible between p and q since A, $A \cup D$, and $A \cup E$ are each proper subcontinua of $A \cup D \cup E$ and one of them contains both p and q. Since p and q were arbitrary with one of them in A, we have established the contrapositive of the statement and the proposition is proven. ||

3.15 <u>Corollary</u> If M is an atriodic, hereditarily unicoherent continuum and $p \in M$, then p is a terminal point of M if and only if for each nondegenerate subcontinuum B of M which contains p, B is irreducible from p to some other point of B.

Proof: The proof follows immediately from proposition 3.14 and the definition of a terminal point. ||

Lemma 3.16 will make it possible to show the existence of two particular terminal subcontinua of a continuum M under the stated hypothesis. This hypothesis is sufficient to allow the use of the upper semi-continuous decomposition $\mathfrak{G}$ of the continuum M which was presented in the preceding section. It was shown there that $\mathfrak{G}$ is an arc with respect to its elements and lemma 3.16 makes extensive use of this fact. The result concerning terminal subcontinua is stated as corollary 3.17.

3.16 <u>Lemma</u> Let M be an atriodic, hereditarily unicoherent continuum which contains no indecomposable subcontinua with interior points relative to M. Each end element of the upper semi-continuous collection $\mathfrak{G}$ on M contains a point p such that each nondegenerate subcontinuum of M containing p is irreducible from p to some other point of M.

Proof: Let G_1 and G_2 denote the two end elements, non-cut points, of $\mathfrak{G}$, considering $\mathfrak{G}$ as an arc with respect to its elements. Let $\mathfrak{H}$ be the collection of all proper subcontinua H of M such that if H is a proper subcontinuum of a subcontinuum N of M then there is a point p of N (and consequently of $N \setminus H$) such that N is irreducible from H to p. The following will show that $\mathfrak{H}$ is non-empty by showing that the end elements G_1 and G_2 of $\mathfrak{G}$ are in fact in $\mathfrak{H}$.

Let N be a subcontinuum of M such that G_1 is properly contained in N. If N = M, then by corollary 3.13, M is irreducible from any point of G_1 to any point of G_2 and consequently is irreducible from G_1 to a point $p \in G_2 \subseteq M$. If N is properly contained in M then by the preceding argument, N must not meet G_2.

Let $\mathfrak{K}$ denote the collection of all members of $\mathfrak{G}$ such that $K \in \mathfrak{K}$ if and only if $K \cap N \neq \emptyset$. Also, let $\mathfrak{C} = \mathfrak{G} \setminus \mathfrak{K}$ which is nonempty since $G_2 \cap N = \emptyset$. Now let $H \in \mathcal{E}$ and $K \in \mathcal{K}$. If $H = G_2$ then clearly H precedes K in the order from G_2 to G_1 in $\mathfrak{G}$. Thus, assume that $H \neq G_2$. Then $\mathfrak{G} \setminus \{H\} = \mathfrak{R} \cup \mathfrak{S}$, where $\mathfrak{R}$ and $\mathfrak{S}$ are each connected and mutually separated with respect to their elements. Since $N \cap H = \emptyset$, N is connected, and without loss of generality, $G_1 \in \mathfrak{R}$, $N \subseteq \mathfrak{R}^*$ which is also connected. Thus, $K \cap \mathfrak{R}^* \neq \emptyset$. Since $\mathfrak{R}^*$ and $\mathfrak{S}^*$ are also mutually separated [26, p. 275] and $K \cap H = \emptyset$, K connected implies $K \subseteq \mathfrak{R}^*$ and consequently $K \in \mathfrak{R}$. Hence, each element of $\mathcal{E}$ precedes each element of $\mathcal{K}$ in the order from G_2 to G_1 in $\mathfrak{G}$.

Since $\mathfrak{G}$ is an arc with respect to its elements and $\mathcal{K} \cup \mathcal{E} = \mathfrak{G}$, $\mathcal{K} \cap \mathcal{E} = \emptyset$, and each element of $\mathcal{E}$ precedes each element of $\mathcal{K}$ in the order from G_2 to G_1, there exists an element G of $\mathfrak{G}$ such that either G is the first member of $\mathcal{K}$ or the last member of $\mathcal{E}$ [26, p. 42]. If G is the first member of $\mathcal{K}$ then $\mathcal{K}$ is an arc from G to G_1 and $\mathcal{K}$ is an upper semi-continuous decomposition of $\mathcal{K}^*$ [26, p. 273]. Corollary 3.13 is again used to imply that $\mathcal{K}^*$ is irreducible from G to G_1. Thus, since N meets both G and G_1 and is a subcontinuum of $\mathcal{K}^*$, $N = \mathcal{K}^*$ and N is irreducible from G to G_1.

If G is the last member of $\mathcal{E}$ then adapting a theorem to $\mathfrak{G}$ with respect to its elements, $\mathcal{K}$ is connected with respect to its elements, $\mathcal{K} \cup \{G\}$ is an arc, and G is a limit element of $\mathcal{K}$ [26, p. 40] and [26, p. 25]. Then for any open set $\mathfrak{O}$ with respect to $\mathfrak{G}$ which contains G, $\mathfrak{O}$ contains an element K of $\mathcal{K}$ and consequently a point of N. Let $\mathfrak{O}_1, \mathfrak{O}_2, \cdots$ be a sequence of open sets with respect to $\mathfrak{G}$ which closes down on G. Then for each n, there is an element K_n of $\mathcal{K}$ such that $K_n \in (\mathfrak{O}_n \cap \mathcal{K})$. Thus, for each n,

$K_n \subseteq \mathfrak{G}_n^*$ and $G \subseteq \mathfrak{G}_n^*$. Now for each n, let $x_n \varepsilon N$ such that $x_n \varepsilon K_n$. Since M is compact, some subsequence x_{n_i} converges to a point x. By the definition of $\{x_n\}$, x is in G. However, N closed and compact implies that $x \varepsilon N$. This is impossible, thus, G is not the last member of $\mathcal{E}$. Therefore, N is irreducible from G_1 to some point p of N. The same argument also shows that G_2 has this property and thus G_1 and G_2 are members of $\mathfrak{H}$.

Let $H_1, H_2, \cdots$ be a sequence of members of $\mathfrak{H}$ such that $H_{n+1} \subseteq H_n$. We wish to show that $H = \bigcap_{i=1}^{\infty} H_i$ is a member of $\mathfrak{H}$ and thus that the property of being in $\mathfrak{H}$ is inductive. Assume that H is not in $\mathfrak{H}$. Then H is a continuum properly contained in a subcontinuum N of M such that for each $x \varepsilon N$, there is a proper subcontinuum N_x of M with $N_x \cap H \neq \emptyset$ for $x \varepsilon N_x$.

If for some i_0, H_{i_0} is properly contained in N, then for all $i \geq i_0$, H_i is properly contained in N. Now since H_{i_0} is a member of $\mathfrak{H}$, there exists a point p_{i_0} of $N \setminus H_{i_0}$ such that N is irreducible from H_{i_0} to p_{i_0}. But $H \subseteq H_{i_0}$ implies that $p_{i_0} \varepsilon N \setminus H$ and hence that there is a subcontinuum N_{i_0} of M such that $N_{i_0} \cap H \neq \emptyset$ and consequently $N_{i_0} \cap H_{i_0} \neq \emptyset$, $p_{i_0} \varepsilon N_{i_0}$, and N_{i_0} is properly contained in N. But this contradicts the irreduciblity of N and therefore, for all i, H_i cannot be properly contained in N.

Since H is properly contained in N there is a point x of $N \setminus H$ such that x is eventually not in the intersection of the H_i's.

That is, there exists an integer i_1 such that for all $i \geq i_1$, $x \notin H_i$. Now $H_{i_1} \cup N$ is a subcontinuum of M properly containing H_{i_1}. Since H_{i_1} is a member of $\mathcal{H}$, there is a point $p_{i_1} \in (H_{i_1} \cup N) \setminus H_{i_1}$ such that $H_{i_1} \cup N$ is irreducible from H_{i_1} to p_{i_1}. But p_{i_1} is then in $N \setminus H_{i_1}$ which implies that $p_{i_1} \notin H$. Thus, there is a proper subcontinuum N_{i_1} of N such that $H \cap N_{i_1} \neq \emptyset$ and hence $H_{i_1} \cap N_{i_1} \neq \emptyset$ and $p_{i_1} \in N_{i_1}$. But again this implies that $H_{i_1} \cup N$ is not irreducible from H_{i_1} to p_{i_1}. Since this contradicts the membership of H_{i_1} in $\mathcal{H}$, $H \in \mathcal{H}$ and membership in $\mathcal{H}$ is inductive.

Now if we consider $\mathcal{H}_1$ to be the collection of all members of $\mathcal{H}$ contained in G_1 and partially order $\mathcal{H}_1$ by set inclusion, then for any linearly ordered subcollection $\mathcal{P}$ of $\mathcal{H}_1$, the common part of the members of $\mathcal{P}$ is a subcontinuum of G_1 and is a member of $\mathcal{H}$ by the preceding. Thus, applying Zorn's lemma, there is a minimal (irreducible) element with respect to being in $\mathcal{H}$ and being contained in G_1. A similar argument produces a minimal element with respect to being in $\mathcal{H}$ and being contained in G_2.

Suppose that G is one of the irreducible elements produced above and assume that G is nondegenerate. Theorem 1.6 implies that G is irreducible between some two points a and b. Let M_1 denote any subcontinuum of M properly containing a. Then either $M_1 \cup G = G$ or $M_1 \cup G$ is a subcontinuum properly containing G. In the first case we have that G, and hence M_1 is irreducible from $\{a\}$ to b. But this implies that $\{a\} \in \mathfrak{H}$ which contradicts the irreducibility of G. In the second case, by the membership of G in $\mathfrak{H}$, there is a point c of $M_1 \setminus G$ such that $M_1 \cup G$ is irreducible from G to c. Since G is irreducible, the sets {a} and {b} are not in $\mathfrak{H}$ and hence there exists a proper subcontinuum M_2 of $M_1 \cup G$ such that $M_2 \cap \{a\} \neq \emptyset$ and $c \in M_2$; ie, a and c are elements of M_2. But now M_2 is a proper subcontinuum of $M_1 \cup G$ such that $M_2 \cap G \neq \emptyset$ and $c \in M_2$ which is contrary to the membership of G in $\mathfrak{H}$. Thus, either case leads to a contradiction and irreducible elements of $\mathfrak{H}$ are degenerate. Therefore, G_1 and G_2 each contain points p_1 and p_2 such that each nondegenerate subcontinuum of M containing p_1 or p_2 is irreducible from p_1 or p_2 to some other point of M. ||

3.17 Corollary The two end elements of the upper semi-continuous decomposition $\mathfrak{G}$ on M are terminal subcontinua of M.

Proof: Let B be a subcontinuum of M which meets G_1. If $B \cup G_1 = G_1$ then $B \cup G_1$ is irreducible between some pair of points of G_1. If $B \cup G_1$ properly contains G_1 then lemma 3.16 implies that $B \cup G_1$ is irreducible between G_1 and some point p of B and thus of $B \setminus G_1$. Therefore, with the two cases, lemma 3.14 implies that G_1 is a terminal subcontinuum of M. ||

Proposition 3.18, which follows, is one of the most involved of the results which will be presented in full detail and will be cited frequently later. Although it is quite involved and while any reader enduring to this point might justifiably skip over the proof, the details are presented here because they use two important methods which will be used repeatedly, and because they show the complications encountered in joining two chains together. In the proof, two of three possible situations are easily dispensed with and it is the third possibility which proves to be difficult.

The continuum M, of the proposition, is shown to contain a proper subcontinuum M_1 which is irreducible with respect to a defined property P. A similar procedure of producing an irreducible subcontinuum will be used in several later results.

A proper subcontinuum N_1 of M_1 is constructed on which a chain will exist satisfying the results of the proposition. Using this chain on N_1 and the given chain which covers M_1, a chain is constructed on M_1 which will contradict the existence of M_1. This brief sketch of the plan of the proof, and descriptive remarks throughout the proof are intended to help clarify what has occurred and where the proof is headed. The primary difficulty is in verifying that the claimed collections do actually form chains.

3.18 <u>Proposition</u> Let M be an atriodic, hereditarily unicoherent continuum with A a terminal subcontinuum of M and $C = \{d_1, \ldots, d_n\}$ a chain on M. Then there is a chain $C_1 = \{f_1, f_2, \ldots, f_{n_1}\}$ on M such that

1) C_1 is a refinement of C.
2) $(f_{n_1} \setminus \overline{f}_{n_1-1}) \cap A \neq \emptyset$.
3) if C is taut then so is C_1.

Proof: If $(d_n \setminus \overline{d}_{n-1}) \cap A \neq \emptyset$ or $(d_1 \setminus \overline{d}_2) \cap A \neq \emptyset$, then the results clearly follow with either $C_1 = C$ or $C_1(1,n_1) = C(n,1)$. If the preceding does not occur and yet $A \cap d_n \neq \emptyset$, then let U be an open set such that $U \cap A \neq \emptyset$ and $\overline{U} \subseteq d_n$. This is possible by the normality of the space. Now for $i \neq n-1$ and $1 \leq i \leq n$, let $f_i = d_i$. Let $f_{n-1} = d_{n-1} \setminus \overline{U}$. Then $f_{n-1} \cap M \neq \emptyset$ since $[d_{n-1} \setminus (d_{n-2} \cap d_n)] \cap M \neq \emptyset$ by proposition 2.1. It is now claimed that $C_1 = \{f_1, f_2, \ldots, f_{n-1}, f_n\}$ is a chain on M satisfying properties 1, 2, and 3 of the lemma.

Clearly, C_1 is a refinement of C. Since $U \cap A \neq \emptyset$, let $x \in U \cap A$. Then $x \in A$ and $\overline{U} \subseteq d_n = f_n$ implies $x \in f_n$. Also, $x \in \overline{U}$ implies that $x \notin d_{n-1} \setminus \overline{U} = f_{n-1}$. Since $x \in U$ and $U \cap f_{n-1} = \emptyset$, $x \notin f_{n-1}$. Thus, $x \in (f_n \setminus \overline{f}_{n-1}) \cap A$ and $(f_n \setminus \overline{f}_{n-1}) \cap A \neq \emptyset$. If we suppose that C is taut, then $\overline{f}_{n-2} \cap \overline{f}_n = \overline{d}_{n-2} \cap \overline{d}_n = \emptyset$, $\overline{f}_{n-3} \cap \overline{f}_{n-1} = \overline{d}_{n-3} \cap \overline{(d_{n-1} \setminus \overline{U})} \subseteq \overline{d}_{n-3} \cap \overline{d}_{n-1} = \emptyset$, and $\overline{f}_i \cap \overline{f}_{i+2} = \overline{d}_i \cap \overline{d}_{i+2} = \emptyset$ for $1 \leq i \leq n-4$. Thus, C_1 is also taut and C_1 satisfies properties 1, 2, and 3 if $A \cap d_n \neq \emptyset$.

If we now suppose that $A \cap d_1 \neq \emptyset$, then C_1 is produced in the same manner as above except that C_1 is based on the chain $C' = \{d_1', d_2', \ldots, d_n'\}$ where $d_i' = d_{n-i+1}$ for $1 \leq i \leq n$.

Hence we now consider the case where $(d_1 \cup d_n) \cap A = \emptyset$. Let us assume that the lemma fails and define a property P as follows. The set B has property P if and only if B is a subcontinuum of M, $A \subseteq B$, and no refinement of C covers B and satisfies properties 1, 2, and 3 of the proposition. By assumption, M has property P and certainly the terminal subcontinuum A does not. The following will show that the property P is inductive.

Assume there is a sequence $K_1, K_2, \ldots$ of subcontinua of M, necessarily compact, with $K_{n+1} \subseteq K_n$, and each member of the sequence having property P, but such that $\bigcap_{i=1}^{\infty} K_i$ does not have property P. Since $A \subseteq \bigcap_{i=1}^{\infty} K_i$, this intersection is a nonempty subcontinuum of M

containing A. By assumption there is a refinement C_2 of C covering $\bigcap_{i=1}^{\infty} K_i$ which satisfies properties 1, 2, and 3.

Now C_2^* is an open set containing $\bigcap_{i=1}^{\infty} K_i$. Because the sequence $\{K_i\}$ is decreasing, there exists an integer j such that $K_j \subseteq C_2^*$ since to assume not would produce a sequence $p_1, p_2, \cdots$ of points in $K_i \setminus C_2$ which would necessarily have a subsequence converging to a point p not in C_2^* and hence not in $\bigcap_{i=1}^{\infty} K_i$. But this is impossible and $K_j \subseteq C_2^*$. This implies that K_j does not have property P which is contrary to its existence. Thus, property P is inductive and the Brouwer Reduction Theorem [18, p. 61] implies the existence of a minimal element M_1 of M with property P. This set M_1 is then irreducible with respect to having property P and clearly A is properly contained in M_1.

Since M_1 is a subcontinuum of M, the chain C covers M_1. Let $G = \{d_j, d_{j+1}, \cdots, d_k\}$, $1 \leq j \leq k \leq n$, denote the minimal subchain of C which covers M_1, minimal in the sense that no proper subchain of G covers M_1. Since M_1 has property P, we must have that $A \cap (d_j \cup d_k) = \emptyset$ and thus, G contains at least three links.

Since M_1 is a subcontinuum of M, M_1 is atriodic and hereditarily unicoherent and A is also a terminal subcontinuum of M_1. Proposition 3.14 implies that M_1 is irreducible from some point p of A to some point q of $M_1 \setminus A$. Now the composant of M_1 determined by p is dense in M_1 [26, p. 58] and thus for each i, $j \leq i \leq k$, there exists a proper subcontinuum of M_1 containing p and meeting d_i. Their union, denoted by N, is a proper subcontinuum of M_1 which meets each link of G and contains p. The subcontinuum N is proper since q cannot be contained in any of the proper subcontinua forming

N. Since $q \notin A$ and $p \varepsilon A \cap N$, $A \cup N$ is a proper subcontinuum of M_1 which meets each link of G.

If we let $N_1 = A \cup N$ then N_1 is a proper subcontinuum of M_1 and hence does not have property P. Thus, there is a chain $H = \{h_1, h_2, \cdots, h_m\}$ covering N_1 such that H is a refinement of C, $A \cap (h_m \setminus \bar{h}_{m-1}) \neq \emptyset$, and C taut implies that H is taut. We may assume that no chain with fewer links than H will also have these properties; ie, H is minimal with respect to being a chain with the three properties. Without loss of generality we may also assume that H is a refinement of G. It should be noted that by its existence, H has at least two links.

To show that either the first link of H or the last link of H meets $(d_j \cup d_k) \cap N_1$, we assume instead that $(h_1 \cup h_m) \cap (d_j \cup d_k) \cap N_1 = \emptyset$. Since N_1 meets each link of G, there must be some link of H which meets $(d_j \cup d_k) \cap N_1$. Let h_s denote the first such link and, to be specific, assume that $h_s \cap d_j \cap N_1 \neq \emptyset$. Now let h_t denote the first link of H such that $h_t \cap d_k \cap N_1 \neq \emptyset$. By their selection under the assumption, $1 < s \leq t < m$. It is important to note that $h_t \subseteq d_{k-1}$ and $h_s \subseteq d_{j+1}$. Because H is a refinement of G there must be some link of G containing h_1. Let d_r denote this link and then we have that $j+1 \leq r \leq k-1$ since $(h_1 \cup h_m) \cap (d_j \cup d_k) \cap N_1 = \emptyset$.

Since H is a refinement of G, $h_s \cap d_j \neq \emptyset$, and $h_t \cap d_k \neq \emptyset$, it follows that some link of the subchain $H(s,t) = \{h_s, h_{s+1}, \cdots, h_t\}$ must be contained in d_r. For, assume that $h_{i_0} \not\subseteq d_r$ for $s \leq i_0 \leq t$. The above implies $j+1 < r \leq k-1$. Then either $h_{i_0} \subseteq \bigcup_{i=j}^{r-1} d_i$ or

$h_{i_0} \subseteq \bigcup_{i=r+1}^{k} d_i$. Since $(\bigcup_{i=j}^{r-1} d_i) \cap (\bigcup_{i=r+1}^{k} d_i) = \emptyset$ and because $h_s \subseteq \bigcup_{i=j}^{r-1} d_i$ and $h_t \subseteq \bigcup_{i=r+1}^{k} d_i$, there must be some integer j_0, $s \leq j_0 \leq t$, such that $h_{j_0} \subseteq \bigcup_{i=j}^{r-1} d_i$ and $h_{j_0+1} \subseteq \bigcup_{i=r+1}^{k} d_i$. But this implies that $h_{j_0} \cap h_{j_0+1} = \emptyset$ which is impossible. Thus, d_r contains some link of $H(s,t)$ and hence some link of $H(1,t)$ distinct from h_1.

Since G could not previously be required to be minimal on N_1, let $G(j_1,k_1) = \{d_{j_1}, \ldots, d_{k_1}\}$ denote such a minimal chain. Then, $j \leq j_1 \leq j+1$ and $k-1 \leq k_1 \leq k$ because N_1 meets each link of G. Also, $h_s \cap d_{j_1} \neq \emptyset$ and we have the relationship $j \leq j_1 \leq j+1 \leq r \leq k-1 \leq k_1 \leq k$.

At this point in the proof the objective is to construct a new chain covering N_1 from the chains $G(j_1,k-1)$ and $H(1,m)$ which will satisfy properties 1, 2, and 3 and also have fewer links that H. Since this is contrary to the existence of H the assumption that $(h_1 \cup h_m) \cap (d_j \cup d_k) \cap N_1 = \emptyset$ will be false. The construction and verification of the afore mentioned new chain however requires the consideration of several very involved cases. Any reader still intent upon continuing may well wish to just scan the descriptions of the cases and the development of one of them and then skip to the point in the proof where the objective is acknowledged.

Case i. Suppose that $h_{t+1} \subseteq d_k$ and that $h_{t+1} \not\subseteq d_{k-1}$. Then we shall show that $[G(j_1,k-1) \cap H^*(t+1,t)] \oplus H(t+1,m)$ is a chain covering N_1 with the required three properties. The collection $G(j_1,k-1) \cap H^*(1,t)$ must first be shown to be a chain for which, by proposition

2.8, it sufficies to show that the open set $H^*(1,t)$ meets the common part of each pair of adjacent links of $G(j_1,k-1)$.

Suppose that for some j_0, $j_1 \leq j_0 \leq k-1$, $(d_{j_0} \cap d_{j_0+1}) \cap H^*(1,t) = \emptyset$. Then $(\bigcup_{i=j_1}^{j_0} d_i) \cap (\bigcup_{i=j_0+1}^{k-1} d_i) \cap H^*(1,t) = \emptyset$. But as argued previously, this contradicts the fact that $H(1,t)$ is a chain. Therefore, $G(j_1,k-1) \cap H^*(1,t)$ is a chain by proposition 2.8.

To show that $[G(j_1,k-1) \cap H^*(1,t)] \oplus H(t+1,m)$ is a chain, it is sufficient by a previous proposition to show that the last link of $G(j_1,k-1) \cap H^*(1,t)$, namely, $d_{k-1} \cap H^*(1,t)$, and the first link of $H(t+1,m)$, h_{t+1}, are the only links of the two chains which meet. Clearly, for $t+1 < i \leq m$, $h_i \cap H^*(1,t) = \emptyset$ which implies that $[d_j \cap H^*(1,t)] \cap h_i = \emptyset$ for $j_1 \leq j \leq k-1$. Since $h_t \subseteq d_{k-1}$, we have $h_t \cap h_{t+1} \subseteq d_{k-1}$, and consequently $d_{k-1} \cap H^*(1,t) \cap h_{t+1} \neq \emptyset$. Finally, $[d_j \cap H^*(1,t)] \cap h_i = \emptyset$ for $j_1 \leq j < k-1$ and $t+1 \leq i \leq m$ since $h_i \cap H^*(1,t) \neq \emptyset$ if and only if $i = t+1$ and $H^*(1,t) \cap h_{t+1} \subseteq d_k$ with $d_k \cap d_j = \emptyset$ for $j_1 \leq j < k-1$. Therefore, proposition 2.9 implies that $[G(j_1,k-1) \cap H^*(1,t)] \oplus H(t+1,m)$ is a chain.

It remains to be shown that this chain satisfies properties 1, 2, and 3 of the proposition and covers N_1. Clearly, this chain is a refinement of G and hence of C. For the second property, two possibilities arise. If $t+1 < m$, then $H(t+1,m)$ includes h_{m-1} and h_m. By their existence, $(h_m \setminus \bar{h}_{m-1}) \cap A \neq \emptyset$ and thus the new chain has this property.

If $t+1 = m$, then the link adjacent to $h_m = h_{t+1}$ is $d_{k+1} \cap H^*(1,t)$. Again by selection, $(h_m \setminus \bar{h}_{m-1}) \cap A \neq \emptyset$ and we let $x \in (h_m \setminus \bar{h}_{m-1}) \cap A$. Then, $x \notin \overline{\bigcup_{i=1}^{m-1} h_i} = \bigcup_{i=1}^{m-1} \bar{h}_i$ and hence $x \in \sim(\overline{\bigcup_{i=1}^{m-1} h_i})$

or $x \in h_m \cap \sim(\overline{\bigcup_{i=1}^{m-1} h_i})$. Thus, $x \in (h_m \setminus \overline{\bigcup_{i=1}^{m-1} h_i}) \cap (h_m \setminus \overline{d}_{k-1}) =$ $h_m \setminus (\overline{\bigcup_{i=1}^{m-1} h_i} \cap \overline{d}_{k-1}) \subseteq h_m \setminus \overline{[d_{k-1} \cap (\bigcup_{i=1}^{m-1} h_i)]}$. Therefore, in either possibility, this chain satisfies the second property.

For the third property, if C is taut we necessarily have that G is taut since G is a subchain of C and H is taut by its selection. Thus, the chain $G(j_1,k-1) \cap H^*(1,t)$ is taut as is the chain $H(t+1,m)$. The only problem which might occur is with the joining of the two chains. Since $\overline{H(1,t)} \cap \overline{h}_i = \emptyset$ for $t+2 \leq i \leq m$, $\overline{d_j \cap H^*(1,t)} \cap \overline{h}_i \subseteq \overline{d}_j \cap \overline{H^*(1,t)} \cap \overline{h}_i = \emptyset$, for $j_1 \leq j \leq k-1$. Thus, the links of $G(j_1,k-1) \cap H^*(1,t)$ are a positive distance from the links of $H(t+1,m)$.

Since $h_{t+1} \subseteq d_k$ and $\overline{d}_k \cap \overline{d}_j = \emptyset$ for $j_1 \leq j \leq k-2$, h_{t+1} is a positive distance from the links of $G(j_1,k-1) \cap H^*(1,t)$ and the constructed chain is taut. It is easily shown that the chain covers N_1.

Since d_r contains at least two links of H and each link d_j, $j_1 \leq j \leq k-1$, contains at least one link of H, the new chain has fewer links than H and this contradicts the existence of H. Therefore, $(h_1 \cup h_m) \cap (d_j \cup d_k) \cap N_1 \neq \emptyset$.

Case ii. Suppose that $h_{t+1} \subseteq d_{k-1}$ and $m \geq t+2$. Then we wish to show that $[G(j_1,k-2) \cap H^*(1,t)] \oplus [(d_{k-1} \cap H^*(1,t)) \cup h_{t+1}] \oplus H(t+2,m)$ is a chain covering N_1 with the required three properties.

The argument to show that $G(j_1,k-2) \cap H^*(1,t)$ is a chain is essentially the same as that given for the corresponding part of case i. Thus, we shall conclude that $G(j_1,k-2) \cap H^*(1,t)$ is a chain.

Since $h_t \cap h_{t+1} \subseteq d_{k-1}$, the set $[(d_{k-1} \cap H^*(1,t)) \cup h_{t+1}] =$ $d_{k-1} \cap H^*(1,t+1)$ is nonempty and also open. To show that

$[G(j_1,k-2)\cap H^*(1,t)] \oplus [d_{k-1}\cap H^*(1,t+1)]$ is a chain, only the last link of $G(j_1,k-2)\cap H^*(1,t)$, namely, $d_{k-2}\cap H^*(1,t)$, must meet $d_{k-1}\cap H^*(1,t+1)$. Since $d_{k-1}\cap d_i = \emptyset$ for $j_1 \leq i < k-2$, $[d_i\cap H^*(1,t)]\cap[d_{k-1}\cap H^*(1,t)] = \emptyset$. Also, $h_{t+1} \subseteq d_{k-1}$ implies that $[d_i\cap H^*(1,t-1)]\cap h_{t+1} = \emptyset$. Thus, for $j_1 \leq i < k-2$, $[d_i\cap H^*(1,t-1)] \cap[d_{k-1}\cap H^*(1,t+1)] = \emptyset$. Finally, $[d_{k-2}\cap H^*(1,t)]\cap[d_{k-1}\cap H^*(1,t+1)] \neq \emptyset$ since again as argued previously, $H^*(1,t)$ meets the common part of d_{k-1} and d_{k-2} or otherwise $H(1,t)$ would fail to be a chain. Thus, $[G(j_1,k-2)\cap H^*(1,t)] \oplus [d_{k-1}\cap H^*(1,t+1)]$ is a chain.

To include the last part in the chain we must argue much the same as in the preceding paragraph. Clearly, each link of $H(t+2,m)$ fails to meet $H^*(1,t)$ and hence none of its links meet a link of $G(j_1,k-2)\cap H^*(1,t)$. For $t+2 < i \leq m$, if any such links exist, $h_i\cap H^*(1,t+1) = \emptyset$. Hence, $h_i\cap[d_{k-1}\cap H^*(1,t+1)] = \emptyset$. Since $h_{t+1}\cap h_{t+2} \neq \emptyset$ and $h_{t+1} \subseteq d_{k-1}$, $[d_{k-1}\cap H^*(1,t+1)]\cap h_{t+2} \neq \emptyset$ and the last link of $[G(j_1,k-2)\cap H^*(1,t)] \oplus [d_{k-1}\cap H^*(1,t+1)]$ meets the first link, h_{t+2}, of $H(t+2,m)$. Again, proposition 2.9 implies that $[G(j_1,k-2)\cap H^*(1,t)] \oplus [d_{k-1}\cap H^*(1,t+1)] \oplus H(t+2,m)$ is a chain.

To complete case ii it remains to show that this chain covers N_1, satisfies properties 1, 2, and 3, and that the chain has fewer links than H. Clearly, N_1 is covered by the chain and the chain is a refinement of G and hence C. Under the supposition of this case, that $m \geq t+2$, $(h_m\setminus\overline{h}_{m-1})\cap A \neq \emptyset$, as a property of H. Hence $(h_m\setminus\overline{g})\cap A \neq \emptyset$ where g denotes the link preceding h_m whether it actually is h_{m-1} or, as might well be the case, it is $d_{k-1}\cap H^*(1,t+1)$.

Since C taut implies G is taut and also that H is taut, the chain $[G(j_1,k-2)\cap H^*(1,t)] \oplus [d_{k-1}\cap H^*(1,t+1)] \oplus H(t+2,m)$ is easily shown to be taut. Thus this chain satisfies properties 1, 2, and 3 of the proposition. Since each link of $G(j_1,k-2)$ contains at least one link of H, d_r contains two links of H, and h_{t+1} is joined with $d_{k-1}\cap H^*(1,t)$ to form a single link, this new chain has fewer links than H which is contrary to the selection of H. Therefore, one of h_1 or h_m meets $(d_j\cup d_k)\cap N_1$ for this case.

Case iii. Suppose $h_{t+1}\subseteq d_{k-1}$ and $m = t+1$. Then an argument similar to that of case ii yields the fact that $[G(j_1,k-2)\cap H^*(1,t)\setminus(\overline{h_{t+1}\cap d_{k-2}})] \oplus [d_{k-1}\cap H^*(1,t+1)]$ is a chain on N_1 with properties 1 and 3. Only the second property requires special mention. The set $(h_{t+1}\setminus\overline{h}_t)\cap A$ is nonempty as a result of the existence of H. Thus, it can be shown that $\{ [d_{k-1}\cap H^*(1,t+1)]\setminus [\overline{d_{k-2}\cap H^*(1,t)}] \}\cap A$ is nonempty and the conclusion of case ii follows.

Case iiii. Suppose $h_{t+1}\subseteq d_{k-2}$ and $h_{t+1}\not\subseteq d_{k-1}$. With methods basically similar to those of the preceding cases it can be shown that $[G(j_1,k-2)\cap(H^*(1,t)\setminus\overline{h}_{t+1})] \oplus [d_{k-1}\cap H^*(1,t)] \oplus H(t+1,m)$ is a chain covering N_1 with properties 1 and 2. To insure that the constructed chain is taut when C is taut we may need to consider a set Q which is open, contains $\overline{h}_{t+1}$, and is such that $\overline{Q}\cap\overline{h}_{t-1} = \emptyset$. This is possible since the space is normal and $\overline{h}_{t-1}\cap\overline{h}_{t+1} = \emptyset$. Then the chain $[G(j_1,k-2)\cap(H^*(1,t)\setminus\overline{Q})] \oplus [d_{k-1}\cap H^*(1,t)] \oplus H(t+1,m)$ will cover N_1, satisfy properties 1, 2, and 3, and have fewer links than H. Since this again contradicts the existence of H, we have that $(h_1\cup h_m)\cap(d_j\cup d_k)\cap N_1 \neq \emptyset$.

The four cases have now all shown that one of h_1 or h_m must meet $(d_j \cup d_k) \cap N_1$. In order to be specific, let us suppose that $h_1 \cap d_j \cap N_1 \neq \emptyset$. Since the original supposition was that $(d_1 \cup d_m) \cap A = \emptyset$ and consequently that $(d_j \cup d_k) \cap A = \emptyset$, we must have that $A \cap (h_1 \cap d_j \cap N_1) = \emptyset$.

Let $x \in h_1 \cap d_j \cap N_1$. Since $x \notin A$ and $h_1 \cap d_j$ is open, there is an open set Q containing x such that $Q \subseteq (h_1 \cap d_j) \setminus A$. Without loss of generality, we may assume that $\overline{Q} \subseteq (h_1 \cap d_j) \setminus A$ by applying the normality of the space. Since $A \subseteq M_1 \setminus Q$, let R denote the component of $M_1 \setminus Q$ containing A. Because $M_1 \setminus Q$ is closed, $\overline{R} \subseteq M_1 \setminus Q$ and hence $R = \overline{R}$. Now each of R and N_1 are subcontinua of M_1 intersecting A. Since A is a terminal subcontinuum of M_1, either $N_1 \subseteq R \cup A$ or $R \subseteq A \cup N_1 = N_1$. Now $Q \cap N_1 \neq \emptyset$, they both contain x, and since $Q \subseteq \sim A$ and $R \subseteq M_1 \setminus Q$, $Q \cap (R \cup A) = \emptyset$. Hence, $N_1 \nsubseteq A \cup R$ and necessarily then $R \subseteq N_1$.

Since R is a component of $M_1 \setminus Q$ and N_1 is covered by H, $R \subseteq N_1$ implies that $R \subseteq H^*$ and thus $R \subseteq (M_1 \setminus Q) \cap H^*$. Then no continuum in $M_1 \setminus Q$ meets both A and $(M_1 \setminus Q) \setminus H^* = M_1 \setminus H^*$ since $Q \subseteq h_1$. Recall that $M_1 \setminus H^*$ cannot be empty since to assume so would have H being a chain on M_1 satisfying properties 1, 2, and 3, contrary to M_1 having property P. Thus, since $M_1 \setminus H^*$ is closed and nonempty, $M_1 \setminus Q$ is the union of two disjoint closed sets, E_1 and E_2, with E_1 containing A and E_2 containing $M_1 \setminus H^*$ [26, p. 15]. By the normality of the space, there must be two open sets S and T such that $A \subseteq E_1 \subseteq S$, $M_1 \setminus H^* \subseteq E_2 \subseteq T$, $\overline{S} \cap \overline{T} = \emptyset$, and $M_1 \setminus Q \subseteq E_1 \cup E_2 \subseteq S \cup T$.

We now claim that $E_2 \cap \overline{Q} \neq \emptyset$. For suppose that $E_2 \cap \overline{Q} = \emptyset$. Since E_1 and E_2 are closed and disjoint, we will have that

$M_1 = E_2 \cup [(M_1 \cap \bar{Q}) \cup E_1]$ which is a separation of M_1 by E_2 and $(M_1 \cap \bar{Q}) \cup E_1$. Hence, $E_2 \cap \bar{Q} \neq \emptyset$ and similarly $E_1 \cap \bar{Q} \neq \emptyset$. With $\bar{Q} \subseteq h_1 \cap d_j$ it follows that $E_2 \cap d_j \neq \emptyset$. Now suppose that $E_2 \cap d_{j+1} \neq \emptyset$ and that $E_2 \cap (d_j \cap d_{j+1}) = \emptyset$. Let $E_3 = E_2 \setminus d_j$. Then E_3 is closed. Also, $E_2 \setminus E_3 \subseteq E_2 \setminus (d_{j+1} \cup \cdots \cup d_k)$ and thus $E_2 \setminus E_3$ is closed. Because $\bar{Q} \subseteq d_j$, $E_3 \cap \bar{Q} = \emptyset$ and because $E_3 \subseteq E_2$, $E_3 \cap E_1 = \emptyset$. Then E_3 and $[(E_2 \setminus E_3) \cup (\bar{Q} \cap M_1) \cup E_1]$ form a separation of M_1. Since this contradicts the existence of M_1, we must have that $E_2 \cap (d_j \cap d_{j+1}) \neq \emptyset$. Similarly, if $E_2 \cap d_i \neq \emptyset$ and $E_2 \cap d_{i+1} \neq \emptyset$, then $E_2 \cap (d_i \cap d_{i+1}) \neq \emptyset$ and also $E_2 \cap d_{i_0} \neq \emptyset$ and $E_2 \cap d_{i_2} \neq \emptyset$ implies that $E_2 \cap d_{i_1} \neq \emptyset$ for $j \leq i_0 < i_1 < i_2 \leq k$.

With the preceding, let $G(j,k_2) = \{d_j, \ldots, d_{k_2}\}$, $j \leq k_2 \leq k$, denote the minimal subchain of $G(j,k)$ containing E_2. Then since $E_2 \subseteq S$ and E_2 meets the common part of each pair of adjacent links of $G(j,k_2)$, proposition 2.8 implies that $G(j,k_2) \cap S$ is a chain covering E_2. Now if $k_2 \geq j+1$, $G(j+1,k_2) \cap S$ is also a chain. Otherwise, consideration of $G(j+1,k_2) \cap S$ in the following may simply be omitted.

Since $(d_j \cap T) \cap (d_i \cap T) \neq \emptyset$ if and only if $|i - j| \leq 1$, we have that this intersection is nonempty if and only if $i = j+1$ (other than for $i = j$.) With $Q \subseteq d_j$, $d_i \cap Q = \emptyset$ for $j+2 \leq i \leq k_2$. Thus, $[(d_j \cap T) \cup Q] \cap (d_i \cap T) \neq \emptyset$ if and only if $i = j+1$ or j. Therefore, $[G(k_2,j+1) \cap T] \oplus [(d_j \cap T) \cup Q]$ is a chain by proposition 2.9.

Since $S \cap T = \emptyset$, $h_1 \cap S$, which is nonempty, has an empty intersection with each link of $G(k_2,j+1) \cap T$. However, $S \cap Q \neq \emptyset$ and

$Q \subseteq h_1$ implies that $[(d_j \cap T) \cup Q] \cap [h_1 \cap S] \neq \emptyset$. Thus, we also have by proposition 2.9 that $[G(k_2,j+1) \cap T] \oplus [(d_j \cap T) \cup Q] \oplus [h_1 \cap S]$ is a chain.

Now consider E_1. As noted earlier, $E_1 \cap \bar{Q} \neq \emptyset$ and hence $E_1 \cap h_1 \neq \emptyset$. Also, if $E_1 \cap h_{i_0} \neq \emptyset$, then $E_1 \cap h_i \neq \emptyset$ for $1 \leq i \leq i_0$ and $E_1 \cap (h_i \cap h_{i+1}) \neq \emptyset$ for $1 \leq i < i_0$. Thus, $E_1 \subseteq S$ implies that S meets the common part of each pair of adjacent links of $H(1,m)$ since $A \subseteq E_1$ and $A \cap (h_m \setminus \bar{h}_{m-1}) \neq \emptyset$.

Since h_2 exists, $[(h_2 \cap S) \setminus \bar{Q}] \cap E_2 \neq \emptyset$. Since $T \cap S = \emptyset$, $[(h_2 \cap S) \setminus \bar{Q}] \cap \{[G(k_2,j+1) \cap T] \oplus [(d_j \cap T) \cup Q]\}^* = \emptyset$. However, $[h_1 \cap S] \cap [(h_2 \cap S) \setminus \bar{Q}] \neq \emptyset$. Thus, $[G(k_2,j+1) \cap T] \oplus [(d_j \cap T) \cup Q] \oplus [h_1 \cap S] \oplus [(h_2 \cap S) \setminus \bar{Q}]$ is a chain.

Finally, if $m \geq 3$, then S meets the common part of each pair of adjacent links of $H(3,m)$ as noted earlier and thus $H(3,m) \cap S$ is a chain. Each link of $H(3,m) \cap S$ fails to meet each link of $[G(k_2,j+1) \cap T] \oplus [(d_j \cap T) \cup Q] \oplus [h_1 \cap S]$ either because $T \cap S = \emptyset$ or because it fails to meet h_1. Since $h_2 \cap h_3 \subseteq \sim\bar{Q}$, and $h_2 \cap h_3 \cap S \neq \emptyset$, $[(h_2 \cap S) \setminus \bar{Q}] \cap [h_3 \cap S] \neq \emptyset$ while $[(h_2 \cap S) \setminus \bar{Q}] \cap (h_i \cap S) = \emptyset$ for $4 \leq i \leq m$. Therefore, $F = [G(k_2,j+1) \cap T] \oplus [(d_j \cap T) \cup Q] \oplus [h_1 \cap S] \oplus [(h_2 \cap S) \setminus \bar{Q}] \oplus [H(3,m) \cap S]$ is a chain.

Let $x \in M_1$. Then $x \in E_1 \cup Q \cup E_2$. If $x \in Q$ then $x \in F^*$. If $x \in E_2$ then $x \in T$ and $x \in d_i$ for some i, $j \leq i \leq k_2$. Hence, $x \in T \cap d_i$ and $x \in F^*$. If $x \in E_1 \setminus \bar{Q}$ then $x \in S$ and $x \in h_i$ for some, $1 \leq i \leq m$. Thus, $x \in S \cap h_i$ and $x \in F^*$. If $x \in E_1 \cap \bar{Q}$ then $x \in S$ and $x \in h_1$. Thus, $x \in F^*$. Therefore, F is a chain covering M_1.

By its construction, F is a refinement of C. Since

$A \cap (h_m \setminus \overline{h}_{m-1}) \neq \emptyset$ and $A \subseteq S$, $A \cap [(h_m \cap S) \setminus (\overline{h_{m-1} \cap S})] \neq \emptyset$ and F satisfies properties 1 and 2.

As noted earlier, C taut implies G and H are taut. Thus, the links of F in $[G(k_2, j+1) \cap T]$ and in $[h_1 \cap S] \oplus [(h_2 \cap S) \setminus \overline{Q}] \oplus [H(3,m) \cap S]$ will be taut. Since $Q \subseteq d_j$, $[\overline{(d_j \cap T) \cup Q}] \cap (\overline{d_i \cap T}) = \emptyset$ for $j+2 \leq i \leq k$. Because $\overline{T} \cap \overline{S} = \emptyset$, $(\overline{d_{j+1} \cap T}) \cap (\overline{h_1 \cap S}) = \emptyset$ as well as any other comparison of links involving only T or S. By definition we have that $[(d_j \cap T) \cup Q] \cap [(h_2 \cap S) \setminus \overline{Q}] = \emptyset$. Thus, by considering individually any pair of nonadjacent links, F can be shown to be taut when C is taut.

Thus, we have finally produced a chain covering M_1 satisfying properties 1, 2, and 3. Since this is contrary to the existence of M_1, we have that there must exist a chain on M satisfying conditions 1, 2, and 3 of the proposition. Therefore, the proposition is proven. ||

The latter part of the preceding proof presents the most important method of this thesis. Based on a theorem by Moore [26, p. 15], Fugate uses this method repeatedly to decompose point sets into two disjoint closed sets. The normality of the space then permits the separation of these two sets by disjoint open sets which can be used to produce disjoint chains on the two closed sets. From these disjoint chains, a chain on the continuum is constructed. This same basic method is applied to several theorems in Chapter V.

Corollary 3.19 applies to a terminal point p and follows very easily after the effort expended on proposition 3.18. The corollary is actually the result needed for proving theorem 3.20.

3.19 <u>Corollary</u> If $C = \{d_1, d_2, \ldots, d_n\}$ is a chain covering a continuum M and p is a point such that each nondegenerate subcontinuum of M containing p is irreducible from p to some other point of M, then there is a chain $G = \{g_1, g_2, \cdots, g_m\}$ covering M such that G is a refinement of C, $p \in g_1 \setminus \overline{g}_2$, and C taut implies that G is taut.

Proof: Corollary 3.15 implies that p is a terminal point of M and hence {p} is a terminal subcontinuum of M. Proposition 3.18 now implies there is a chain $C_1 = \{f_1, f_2, \cdots, f_m\}$ covering M such that C_1 is a refinement of C and $(f_m \setminus \overline{f}_{m-1}) \cap \{p\} \neq \emptyset$, which implies that $p \in f_m \setminus \overline{f}_{m-1}$. Let $G(1,m) = C_1(m,1)$. If C is taut, proposition 3.18 implies that C_1, and consequently, G is taut, and we are done. ||

Some Sufficient Conditions For Chainability

We finally arrive at the first of several characterizations of hereditarily decomposable chainable continua. Theorem 3.20, the first major theorem attempting to characterize chainable continua, was proven by R. H. Bing and appeared in 1951 [4, p. 660]. L. K. Barrett, also working under the restriction of hereditary decomposability, extended the result slightly by using characterizations of atriodic and hereditarily unicoherent, hereditarily decomposable continua [1, p. 517]. Three of these results appear here as theorem 3.26, summarizing the efforts of Chapter III and in particular, this section of the chapter. Aside from these two theorems, this section is devoted to proving each of the characterizations of atriodic, hereditarily unicoherent, and hereditarily decomposable continua used in theorem 3.26. The hereditary decomposability is essential to the proof of these theorems but, as will be seen in theorems 5.1, 5.4, 5.11, and 5.12, this restriction can be considerably relaxed. Some restrictions, other than those of atriodicity and hereditary unicoherence, will, however, remain.

Theorem 3.20 again uses a method noted earlier in proposition 3.18 and is typical of the approach taken in several forthcoming results. That is, showing the existence of a subcontinuum of M which is irreducible with respect to not being ε-chainable. An ε-chain is then constructed covering this subcontinuum contradicting its existence. This theorem is the culmination of the large number of intermediate results which have been presented. The hereditary decomposability implies that M contains no indecomposable subcontinua with interior points relative to M. Thus, the upper semi-continuous decomposition $\mathfrak{G}$ of M may be and is used extensively. In particular, the fact that $\mathfrak{G}$ is an arc with respect to its elements provides the key to proving theorem 3.20.

Before presenting the theorem and its proof, it is necessary to introduce some of the notation and definitions used. Since $\mathfrak{G}$ is an arc with respect to its elements, $\mathfrak{G}$ has two end elements as has been noted earlier. If A and B denote these two end elements then we could equivalently denote $\mathfrak{G}$ by AB, meaning the arc from A to B. Similarly, subarcs may be denoted by their end elements.

For any three distinct elements, P, Q, and R of $\mathfrak{G}$, one must separate the other two from each other in $\mathfrak{G}$ by virtue of its being an arc with respect to its elements. That is, without loss of generality, subsets $\mathfrak{H}$ and $\mathfrak{K}$ of $\mathfrak{G}$ must exist such that they form a separation of $\mathfrak{G} \setminus \{Q\}$, $P \in \mathfrak{H}$, and $R \in \mathfrak{K}$. This fact is denoted by PQR. Similarly, the notation PQRS means, PQR, PRS, PQS, and QRS.

To return to the discussion of the method of proving theorem 3.20, we make the final note that the irreducible subcontinuum is chained by appropriately decomposing it into two chainable proper subcontinua and then joining these two chains together.

3.20 Theorem A hereditarily decomposable continuum M is chainable if and only if it is atriodic and hereditarily unicoherent.

Proof: Since every chainable continuum is atriodic and hereditarily unicoherent by theorem 2.7, the conditions are necessary. To see that they are sufficient, suppose the contrary. Then there exists a positive real number ε such that M is not ε-chainable. Clearly, M is nondegenerate. Let $\mathfrak{N}$ denote the collection of all subcontinua of M which are not ε-chainable. Since $M \in \mathfrak{N}$, $\mathfrak{N}$ is nonempty. The collection $\mathfrak{N}$ can be partially ordered by set inclusion and we let $\mathfrak{S}$ be any linearly ordered subcollection of $\mathfrak{N}$. Let $N = \bigcap\{ L : L \in \mathfrak{S} \}$. If $N \notin \mathfrak{N}$, then since N is a subcontinuum of M, N is ε-chainable. Let $F = \{f_1, f_2, \cdots, f_k\}$ be an ε-chain on N. Then there is an element L_0 of $\mathfrak{S}$ such that F is a chain covering L_0 since to assume not implies that for all $L \in \mathfrak{S}$, F is not a chain covering L. Since F is a chain, it follows that $L \not\subseteq F^*$. Thus, for each $L \in \mathfrak{S}$, let $x_L \in L \setminus F^*$. Since M is compact, this collection $\{ x_L : L \in \mathfrak{S} \}$ has a subcollection converging to a point which is not in F^* and hence not in N. Since this is impossible, there is a member L_0 of $\mathfrak{N}$ such that $L_0 \subseteq F^*$ and F is a chain covering L_0. Since $N \subseteq L_0$, F is a chain on L_0. This contradicts the fact that $L_0 \in \mathfrak{N}$ and hence the assumption is false and $N \in \mathfrak{N}$.

Zorn's lemma now implies that there exists a minimal subcontinuum in $\mathfrak{N}$, minimal in the sense that every proper subcontinuum is ε-chainable. Let N denote such a minimal element. By selection, N is is irreducible with respect to not being coverable by any ε-chain. Again, N is necessarily nondegenerate.

Since M is hereditarily decomposable, N contains no indecomposable subcontinua. Thus, let $\mathfrak{G}$ denote the upper semi-continuous decomposition of N described by theorem 3.11. Corollary 3.12 shows that $\mathfrak{G}$ is an arc with respect to its elements. Let A and B denote the two distinct end elements of $\mathfrak{G}$. Let $C \in \mathfrak{G} \setminus \{A, B\}$. Let $p \in C$ such that every nondegenerate subcontinuum of C containing p is irreducible from p to some other point of C. With C nondegenerate, lemma 3.16 assures the existence of such a point p by considering C to be the continuum of the lemma.

Let $\mathfrak{J} = \{C\} \cup \{ G \in \mathfrak{G} : C$ separates G from B in $\mathfrak{G} \}$, and similarly, let $\mathfrak{K} = \{C\} \cup \{ G \in \mathfrak{G} : C$ separates G from A in $\mathfrak{G} \}$. This is the same as to say that $\mathfrak{J}$ and $\mathfrak{K}$ are the subarcs of $\mathfrak{G}$ from A to C and from B to C respectively. Now, $\mathfrak{J}^* \cup \mathfrak{K}^* = N$ since to assume not implies that $\mathfrak{J}^* \cup \mathfrak{K}^*$ is properly contained in N and thus there is an element G_0 of $\mathfrak{G}$ such that $G_0 \cap (\mathfrak{J}^* \cup \mathfrak{K}^*) = \emptyset$. Thus, C does not separate G_0 from A or B in $\mathfrak{G}$. Since $A \subseteq \mathfrak{J}^*$ and $B \subseteq \mathfrak{K}^*$, $G_0 \neq A$ or B and thus G_0 separates A from B in $\mathfrak{G}$. Because C separates A from B in $\mathfrak{G}$, either AG_0CB or ACG_0B [26, p. 32]. In the first case $G_0 \subseteq \mathfrak{J}^*$ and in the second case $G_0 \subseteq \mathfrak{K}^*$. Thus, G_0 cannot exist and $\mathfrak{J}^* \cup \mathfrak{K}^* = N$. Also, $\mathfrak{J}^* \cap \mathfrak{K}^* = C$ since $\mathfrak{J}$ and $\mathfrak{K}$ are subarcs of $\mathfrak{G}$ with end elements C.

It is the present objective of the proof to produce proper subcontinua of N, namely, $\mathfrak{J}^*$ and $\mathfrak{K}^*$, which would then be ε-chainable and which could be used to produce an ε-chain on N. We must first show that $\mathfrak{J}^*$ and $\mathfrak{K}^*$ are connected. Since $\mathfrak{J}$ and $\mathfrak{K}$ are subarcs of $\mathfrak{G}$, and hence are connected with respect to their elements, $\mathfrak{F}^*$ and $\mathfrak{K}^*$ are connected [26, p. 275]. It also follows that since $\mathfrak{F}$ and $\mathfrak{K}$ are closed with respect to their elements, so also are $\mathfrak{F}^*$ and $\mathfrak{K}^*$ [26, p. 275] and hence $\mathfrak{F}^*$ and $\mathfrak{K}^*$ are proper subcontinua of N. By the selection of N, $\mathfrak{F}^*$ and $\mathfrak{K}^*$ are both ε-chainable. Let $C_1 = \{f_1, f_2, \ldots, f_n\}$ and $D_1 = \{k_1, k_2, \ldots, k_m\}$ be the chains on $\mathfrak{F}^*$ and $\mathfrak{K}^*$ respectively which by the following may be assumed to have the property that $p \in f_n \setminus \overline{f}_{n-1}$ and $p \in k_1 \setminus \overline{k}_2$.

In order to justify the assertion of the preceding paragraph, we shall show that corollary 3.19 applies. Suppose there is a nondegenerate subcontinuum L of $\mathfrak{F}^*$ such that $p \in L$ and L is reducible from p to any point of L. If $L \subseteq \mathfrak{K}^*$ also, then $L \subseteq C$ which is contrary to the selection of p in C. Thus, L is not contained in $\mathfrak{K}^*$ but $L \cap \mathfrak{K}^* \neq \emptyset$ since both contain p. Now, $L \cup \mathfrak{K}^*$ is a nondegenerate subcontinuum of N such that $L \setminus \mathfrak{K}^* \neq \emptyset$ and $A \subseteq \mathfrak{K}^*$ which implies, by lemma 3.16, that there exists an element r of A and an element q of $L \cup \mathfrak{K}^*$ such that $L \cup \mathfrak{K}^*$ is irreducible from r to q. If $q \in \mathfrak{K}^*$ then $\mathfrak{K}^*$ is a proper subcontinuum of $L \cup \mathfrak{K}^*$ containing both q and r which contradicts the irreducibility of $L \cup K^*$ from r to q. Therefore, $q \notin \mathfrak{K}^*$, $q \in L \setminus \mathfrak{K}^*$. Since L is reducible from p to q, there exists a proper subcontinuum L_1 of L such that $\{p, q\} \subseteq L_1$. Then $L_1 \cup \mathfrak{K}^*$ is properly contained in $L \cup \mathfrak{K}^*$ and $L \cup \mathfrak{K}^*$ is not irreducible. Again this is a contradiction and therefore every nondegenerate subcontinuum of $\mathfrak{F}^*$ containing p is irreducible from p to some other point of $\mathfrak{F}^*$. Likewise, the subcontinuum $\mathfrak{K}^*$ has this property. Corollary 3.19 now asserts that the chains C_1 and D_1 on $\mathfrak{F}^*$ and $\mathfrak{K}^*$ respectively, may be selected such that

$p \in f_n \setminus \overline{f}_{n-1}$ and $p \in k_1 \setminus \overline{k}_2$.

We begin the process of fitting these two chains together to form a chain on N which of course is contradictory to the selection of N. Let U be a neighborhood of p such that $U \subseteq (f_n \setminus \overline{f}_{n-1}) \cap (k_1 \setminus \overline{k}_2)$. Furthermore, U exists such that $U \cap (A \cup B) = \emptyset$. Since $U \cap N \neq \emptyset$, $U \cap N$ is open relative to N, and consequently, $U \cap N \not\subseteq C$ by lemma 3.8. Also, $N \setminus U = N \cap \sim U$ is a nonempty closed proper subset of N. If $N \setminus U$ is connected it is a proper subcontinuum of N containing both A and B. This is impossible since N would be reducible from $a \in A$ to $b \in B$ contrary to corollary 3.13. Thus, let H_1 and K_1 form a separation of $N \setminus U$ such that $K_1 \subseteq K^*$ and $H_1 \subseteq G^*$.

Since H_1 and K_1 are both closed and compact, let δ denote a positive real number less than half the distance between H_1 and K_1. Let $O_H = \bigcup \{ B(x;\delta) : x \in H_1 \}$ and $O_K = \bigcup \{ B(x;\delta) : x \in K_1 \}$. Then $H_1 \subseteq O_H$, $K_1 \subseteq O_K$, $O_H \cap O_K = \emptyset$, and O_H and O_K are both open sets. Now C_1 and D_1 are still chains covering H_1 and K_1 respectively. Since $f_i \cap f_{i+1} \cap O_H \neq \emptyset$, $1 \leq i \leq n-1$, and $k_j \cap k_{j+1} \cap O_K \neq \emptyset$, $1 \leq j \leq m-1$, proposition 2.8 asserts that $C_1 \cap O_H$ and $D_1 \cap O_K$ are ε-chains covering H_1 and K_1 respectively. Furthermore, the links of $C_1 \cap O_H$ are respectively disjoint from the links of $D_1 \cap O_K$. Now since U meets C_1 only in the link f_n, $(f_n \cap O_H) \cup U$ is an open set of diameter less than ε which contains p and hence meets N. Likewise, so does $(k_1 \cap O_K) \cup U$. Then $C_2 = \{f_1 \cap O_H, f_2 \cap O_H, \ldots, f_{n-1} \cap O_H, (f_n \cap O_H) \cup U\}$ is a chain covering H_1 as is $D_2 = \{(k_1 \cap O_K) \cup U, k_2 \cap O_K, \ldots, k_m \cap O_K\}$ a chain covering K_1.

Finally, $[(f_n \cap O_H) \cup U] \cap [(k_1 \cap O_K) \cup U] = U$ and $f_i \cap O_H$ fails to meet any link of D_2 for $1 \leq i \leq n-1$ and $k_j \cap O_K$ fails to meet any link of C_2 for $2 \leq j \leq m$. Thus, $C_2 \oplus D_2 = \{f_1 \cap O_H, \ldots, f_{n-1} \cap O_H, (f_n \cap O_H) \cup U, (k_1 \cap O_K) \cup U, k_2 \cap O_K, \ldots, k_m \cap O_K\}$ is an ε-chain covering N.

To see that $C_2 \oplus D_2$ is an ε-chain on N, let $x \in N$. Then $x \in H_1 \cup K_1 \cup U$. If $x \in H_1$, $x \in [C_2 \oplus D_2]^*$ since $x \in O_H$ and $x \in f_r$ for some r, $1 \leq r \leq n$. Similarly, $x \in K_1$ implies $x \in [C_2 \oplus D_2]^*$. Also, $x \in U$ implies that $x \in [C_2 \oplus D_2]^*$ since $U \subseteq [C_2 \oplus D_2]^*$. Finally, since $H_1 \subseteq N$, $K_1 \subseteq N$, $f_i \cap H_1 \neq \emptyset$ for $1 \leq i \leq n-1$, $k_j \cap K_1 \neq \emptyset$ for $2 \leq j \leq m$, $H_1 \subseteq O_H$, and $K_1 \subseteq O_K$, we have that $(f_i \cap O_H) \cap N \neq \emptyset$ and $(k_j \cap O_K) \cap N \neq \emptyset$. Also, $p \in U$ implies that both $(f_n \cap O_H) \cup U$ and $(k_1 \cap O_K) \cup U$ meet N and therefore $C_2 \oplus D_2$ is an ε-chain on N.

Since this contradicts the selection of N as a member of $\mathcal{N}$, the subcontinua of M which were not ε-chainable, M must be ε-chainable. Therefore, M is chainable and the theorem is proven. ||

The next several results are presented to yield characterizations of atriodic and hereditarily unicoherent, hereditarily decomposable continua, which can then be used to provide variations on the preceding theorem. If a continuum is irreducible between some two of its points, then the continuum will be called an <u>irreducible continuum</u>. If a and b denote these two points, then the notation ab is intended to emphasize this irreducibility of M between a and b. A continuum is <u>hereditarily irreducible</u> if and only if each subcontinuum is irreducible between some two of its points [23, p. 180]. Propositions 3.21, 3.22, and 3.23, and corollary 3.24 deal with the relationship between the hereditary irreducibility and the hereditary unicoherence and atriodicity of a hereditarily decomposable continuum.

3.21 <u>Proposition</u> The continuum M is hereditarily unicoherent if and only if for a, $b \in M$, there exists only one irreducible subcontinuum ab of M.

Proof: Let M be a hereditarily unicoherent continuum and suppose there are two points a and b of M such that A and B are distinct subcontinua of M each containing both a and b. Then $A \cup B$ is a subcontinuum of M which by the hereditary unicoherence of M implies that $A \cap B$ is a subcontinuum properly contained in A and B and containing a and b. Thus, neither A nor B is irreducible.

Now suppose that for any two points a and b of M, there exists only one irreducible subcontinuum ab of M. To assume that M is not hereditarily unicoherent implies there exists a subcontinuum N of M such that $N = A \cup B$ with $A \cap B$ not a continuum. Since $A \cap B$ is closed, let R and T denote the separation of $A \cap B$. If $r \in R$ and $t \in T$, then by hypothesis, the irreducible subcontinuum rt of M exists. Because r, $t \in A$ implies that $rt \subseteq A$ and r, $t \in B$ implies that $rt \subseteq B$, $rt \subseteq A \cap B$. But this implies that $rt \subseteq R \cup T$, the separation of $A \cap B$.

Since $r \in R$ implies $rt \subseteq R$ and $t \in T$ implies $rt \subseteq T$, clearly r and t cannot exist and hence neither can N. Therefore, M is hereditarily unicoherent if and only if for any two points of M there is only one irreducible subcontinuum of M containing these two points. ||

3.22 <u>Proposition</u> The point set M is an atriodic, hereditarily unicoherent continuum if and only if

1) if a and b are any two points of M, then there does not

exist more than one irreducible subcontinuum ab of M, and 2) every nondegenerate subcontinuum of M is irreducible between some two of its points.

Proof: Suppose that M is atriodic and hereditarily unicoherent. Proposition 3.21 implies that there does not exist more than one irreducible subcontinuum ab of M. Theorem 1.6 implies that every nondegenerate subcontinuum of M is irreducible between some two of its points.

Suppose that conditions 1 and 2 are true. Again, proposition 3.21 implies that M is hereditarily unicoherent. Now assume that there is a subcontinuum T of M such that T is a triod. Then $T = U \cup V \cup W$ with subcontinua $U \cap V = U \cap W = V \cap W = U \cap V \cap W$, no one of U, V, or W is contained in the union of the other two. Since T is irreducible between some two points, let a and b denote these points. If $a \varepsilon U \cap V \cap W$ and $b \varepsilon U$, then U is a proper subcontinuum of T containing a and b. Since this contradicts the fact that T was irreducible between a and b, neither a nor b is in $U \cap V \cap W$. Also, a and b cannot both be contained in either U, V, or W. Thus, suppose that $a \varepsilon U \setminus (V \cup W)$ and $b \varepsilon V \setminus (U \cup W)$. Then a and b are elements of $U \cup V$, a proper subcontinuum of T. Again, this contradicts the existence of a and b. Hence, M does not contain a triod and M is atriodic and hereditarily unicoherent. ||

3.23 <u>Proposition</u> If a hereditarily irreducible continuum M is the union of two continua H and K whose common part is the union of two mutually separated sets U and V, then H and K contain indecomposable continua whose common part is the union of two mutually separated sets lying in U and V respectively.

Proof: By hypothesis we have that $M = H \cup K$ and that U and V form a separation of $H \cap K$. Since $U \subseteq H$ and $V \subseteq H$, there is a subcontinuum H' of H where H' is irreducible from U to V. Likewise, since $H' \cap U \subseteq K$ and $H' \cap V \subseteq K$, there is a subcontinuum K' of K such that K' is irreducible from $H' \cap U$ to $H' \cap V$. Now $H' \cap K' \neq \emptyset$, $H' \cup K' \subseteq U \cup V$, $H' \cap K' \cap U \neq \emptyset$, and $H' \cap K' \cap V \neq \emptyset$, imply that there are sets U' and V' which form a separation of $H' \cap K'$ with $U' \subseteq U$ and $V' \subseteq V$. The point set H' is clearly irreducible from U' to V'. So also is K' irreducible from U' to V'.

Suppose that $H' = W \cup Z$ where W and Z are each proper subcontinua of H'. Then without loss of generality, W contains no point of V' for suppose $W \cap U'$ and $W \cap V'$ are nonempty. Then H' is not irreducible from U' to V' which is contrary to the above. Thus, W contains no point of V' and Z contains no point of U'.

Now $W \cap K' \subseteq H' \cap K' = U' \cup V'$ and $W \cap V' = \emptyset$ implies that $W \cap K' \subseteq U'$. Also, $Z \cap K' \subseteq V'$. Hence, $W \cap Z \cap K' \subseteq U' \cap V' = \emptyset$. Since $W \cap K'$, $Z \cap K'$, and $W \cap Z$ are mutually exclusive closed sets, no one of the continua W, Z, or K' is a subset of the union of the other two. For suppose that $W \subseteq Z \cup K'$. Then $W \subseteq (Z \cap W) \cup (K' \cap W)$ which implies that either $W \subseteq Z \cap W$ or $W \subseteq K' \cap W$ since these two sets are mutually separated and W is connected. Now $W \subseteq Z \cap W$ implies that $W \subseteq Z$. But then $W \cap K' \subseteq Z \cap K'$ which contradicts their being mutually exclusive. Thus, $W \not\subseteq Z \cap W$. Similarly, $W \not\subseteq Z \cap K'$. Therefore, no one of W, Z, or K' is a subset of the union of the other two.

Since $H' \cup K'$ is irreducible between some two points by hypothesis, let a and b denote two such points. Without loss of generality, $a \in H' \setminus K'$ and $b \in K' \setminus H'$. Then $a \in W \cup V$. But $a \in W$ implies that $a \in W \cup K'$ which is a continuum properly contained in $K' \cup H'$. Similarly, $a \in V$ implies that $a \in V \cup K'$, a proper subcontinuum of $H' \cup K'$. This contradicts the irreducibility of $H' \cup K'$ and therefore, H' is not decomposable. Therefore, H' and K' are indecomposable. ||

Proposition 3.23 now makes it possible to prove the first characterization of atriodic, hereditarily unicoherent, hereditarily decomposable continua.

3.24 <u>Corollary</u> A hereditarily decomposable continuum M is atriodic and hereditarily unicoherent if and only if M is hereditarily irreducible.

Proof: That the result is necessary follows from proposition 3.22. Now assuming M to be hereditarily irreducible it follows from the proof of proposition 3.22 that M is atriodic. Suppose that M contains a subcontinuum N which is not unicoherent. Then $N = H \cup K$ with H and K proper subcontinua of N and the sets U and V form a separation of $H \cap K$. Proposition 3.23 implies that each of H and K must contain indecomposable subcontinua. Since M is hereditarily decomposable, this is impossible and M is hereditarily unicoherent. ||

The second characterization is presented as proposition 3.25 and is stated in terms of the property of a point weakly separating two other points. If a, b, and c are three points of a continuum M, then b <u>weakly separates</u> a from c in M if and only if every subcontinuum of M which contains a and c, also contains b [23, p. 180]. Consider for example the three end points of a "Y" shaped simple triod. No one of the three weakly separates the other two. Proposition 3.25, of course, shows that every triod must contain three points such that each of them fails to weakly separate the other two from each other in M. Actually, the proof as presented here, is restricted to hereditarily decomposable continua. However, it is claimed that this restriction can be removed [23, p. 180].

3.25 <u>Proposition</u> The hereditarily decomposable continuum M is atriodic and hereditarily unicoherent if and only if for any three points of M, there is one which weakly separates the other two from each other in M.

Proof: Let a, b, and c be three distinct points in M and assume that no one of a, b, and c weakly separates the other two from each other in M. Then there exists a subcontinuum M_{ab} of M such that $c \notin M_{ab}$, M_{ac} of M such that $b \notin M_{ac}$, and M_{bc} of M such that $a \notin M_{bc}$. Since $a \in M_{ab} \cap M_{ac}$, $M_{ab} \cup M_{ac}$ is a continuum containing a and $T_a = M_{ab} \cap M_{ac}$ is a continuum containing a.

Let $L_{bc} = M_{bc} \cap (M_{ab} \cup M_{ac})$. Since b and c are members of $M_{ab} \cup M_{ac}$, L_{bc} is a continuum containing b and c. Now consider $L_{bc} \cap M_{ab}$. We have that $L_{bc} \cap M_{ab} = M_{bc} \cap (M_{ab} \cup M_{ac}) \cap M_{ab} =$ $[M_{bc} \cap M_{ab} \cap M_{ab}] \cup [M_{bc} \cap M_{ac} \cap M_{ab}] = (M_{bc} \cap M_{ab}) \cup [M_{bc} \cap M_{ac} \cap M_{ab}]$. Since $[M_{bc} \cap M_{ac} \cap M_{ab}] \subseteq M_{bc} \cap M_{ab}$, we have that $L_{bc} \cap M_{ab} =$ $M_{bc} \cap M_{ab}$ which we denote by T_b. Then T_b is a continuum containing b. Similarly, $L_{bc} \cap M_{ac} = T_c$, a continuum containing c.

Now $T_b \cup T_c = (L_{bc} \cap M_{ab}) \cup (L_{bc} \cap M_{ac}) = L_{bc} \cap [M_{ab} \cup M_{ac}] =$

$M_{bc} \cap (M_{ab} \cup M_{ac}) \cap (M_{ab} \cup M_{ac}) = L_{bc}$. Therefore, $T_b \cup T_c$ is a continuum which implies that $T_b \cap T_c$ is a nonempty continuum. By definition $T_a \cap T_b = (M_{ab} \cap M_{ac}) \cap (L_{bc} \cap M_{ab}) = M_{ab} \cap (L_{bc} \cap M_{ac}) =$
$M_{ab} \cap [(M_{bc} \cap (M_{ab} \cup M_{ac})) \cap M_{ac}] =$
$(M_{ab} \cap M_{bc}) \cap [(M_{bc} \cap (M_{ab} \cup M_{ac}) \cap M_{ac}] = T_b \cap T_c$. Thus, by similar arguments we have that $T_a \cap T_b = T_a \cap T_c = T_b \cap T_c$.

Since $a \notin M_{bc}$, $a \notin M_{bc} \cap M_{ac} = T_c$. Also, $a \notin T_b$, which implies that $a \notin T_a \cap T_b$. Likewise, $b \notin T_a \cap T_b$ and $c \notin T_a \cap T_b$. Thus, neither a, b, nor c is an element of $T_a \cap T_b = T_a \cap T_c = T_b \cap T_c$. Since $T_a \cap T_b$ is contained in each of T_a, T_b, and T_c, $T_a \cup T_b \cup T_c$ is a continuum. Now, $T_a \cap T_b \cap T_c =$
$(M_{ab} \cap M_{ac}) \cap (M_{ab} \cap M_{bc}) \cap (M_{ac} \cap M_{bc}) = M_{ab} \cap M_{ac} \cap M_{bc} =$
$[M_{ab} \cap M_{ac}] \cap [M_{ac} \cap M_{bc}] = T_a \cap T_c$. Thus, $T_a \cap T_b \cap T_c = T_a \cap T_b =$
$T_a \cap T_c = T_b \cap T_c$ and $T_a \cup T_b \cup T_c$ is a triod in M. Since M is atriodic, this is impossible and therefore for any three points of M one weakly separates the other two from each other in M.

For the converse, suppose that for any three points of M there exists one which weakly separates the other two from each other in M. If M contains a triod N, then $N = A \cup B \cup C$ with A, B, and C continua, $A \cap B \cap C = A \cap B = A \cap C = B \cap C$, and each of A, B, and C containing a point not in the other two. Assume that a, b, and c denote these points respectively. Then $A \cup B$ is a subcontinuum of M containing a and b but not c. Also, $A \cup C$ contains a and c but not b and $B \cup C$ contains b and c but not a. Thus, no one of a, b, and c weakly separates the other two from each other in M.

Assume that M is not hereditarily unicoherent. Proposition 3.21

then implies that there exist two points a and b in M for which there exist two distinct irreducible subcontinua, A and B, in M from a to b. Let $x \in A \setminus B$ and $y \in B \setminus A$. Since A is a subcontinuum of M containing a, b, and x, clearly, y does not weakly separate any pair of these elements from each other in M. Likewise, x does not weakly separate any pair of a, b, and y from each other in M. Thus, every subcontinuum containing x and y must also contain a and b.

Let $H_1 = \{ z \in A : A$ is irreducible from a to z $\}$. Then H_1 is a proper subset of A and by [26, p. 61] H_1 is a subcontinuum of A. Now, H_1 is nonempty since $b \in H_1$. Let $H_2 = \{ z \in A : A$ is irreducible from b to z $\}$. As before, H_2 is a nonempty proper subcontinuum of A containing a. To see that $H_1 \cap H_2 = \emptyset$, assume to the contrary that $c \in H_1 \cap H_2$. Then by definition, A is irreducible from a to c and also from b to c. Therefore, A is indecomposable [26, p. 59]. But since M is hereditarily decomposable, we must have that $H_1 \cap H_2 = \emptyset$.

Now let $K_1 = \{ z \in B : B$ is irreducible from a to z $\}$ and let $K_2 = \{ z \in B : B$ is irreducible from b to z $\}$. From the preceding paragraph we have that K_1 and K_2 are disjoint nonempty subcontinua of B containing respectively, b and a. The following cases will show that the assumption that M was not hereditarily unicoherent has led to a contradiction.

Case i. Suppose that $x \in H_1$ and $y \in K_1$ (equivalently, $x \in H_2$ and $y \in K_2$.) Then $H_1 \cup K_1$ is a subcontinuum of M containing a, x, and y but not b. This contradicts the earlier conclusion that every subcontinuum containing x and y must also contain b.

Case ii. Suppose that $x \varepsilon H_1$ and $y \notin K_1$ (equivalently, $x \varepsilon H_2$ and $y \notin K_2$, or $y \varepsilon K_1$ and $x \notin H_1$, or $y \varepsilon K_2$ and $x \notin H_2$.) Since $y \notin K_1$, there must be a proper subcontinuum K_0 of B such that $\{a, y\} \subseteq K_0$. Because B is irreducible from a to b this implies that $b \notin K_0$. Thus, $H_1 \cup K_0$ is a proper subcontinuum of M containing a, x, and y but not b. Again this is a contradiction of an earlier conclusion.

Case iii. Suppose that $x \notin (H_1 \cup H_2)$ and that $y \notin (K_1 \cup K_2)$. Then $x \notin H_1$ implies that there is a proper subcontinuum H of A containing a and x but consequently not b. Likewise, there is a proper subcontinuum K of B containing a and y but not b. Then $H \cup K$ is a subcontinuum of M containing a, x, and y but not b which is a contradiction.

Since these cases have exhausted the possible combinations of locations of x and y in A and B respectively, and each has led to a contradiction, the assumption that M was not hereditarily unicoherent is false and the proposition is proven. ||

Theorem 3.26, which follows, concludes Chapter III. Summarizing the results of theorem 3.20 and the characterizations of atriodic and hereditarily unicoherent, hereditarily decomposable continua just presented, this theorem presents three characterizations of hereditarily decomposable chainable continua.

3.26 Theorem If M is a hereditarily decomposable continuum then the following are equivalent:

1) M is chainable.

2) M is atriodic and hereditarily unicoherent.

3) M is hereditarily irreducible.

4) M is such that for any three points of M there is one which weakly separates the other two from each other in M.

Proof: 1) is equivalent to 2) by theorem 3.20, 2) is equivalent to 3) by corollary 3.24, and 3) is equivalent to 4) by proposition 3.25. Therefore, the theorem is proven. ||

CHAPTER IV

TERMINAL SUBCONTINUA

Chapter III included an introduction to the subject of terminal subcontinua. That material, and in particular the results on terminal points, was sufficient for the purposes of Chapter III. However, the remaining attempts to characterize chainable continua are very dependent on the ability to chain on terminal subcontinua and in a very restricted manner. For this purpose, Chapter IV continues the development of terminal subcontinua and in particular the definition and development of a property for chaining on terminal subcontinua called exact containment. In addition, the intuitively obviously concept of opposite end points and opposite terminal subcontinua will be defined and developed for later use.

Proposition 4.1 asserts that it is always possible to chain on a continuum M with terminal subcontinuum A, such that the last link of the chain must meet A. This illustrates, as is its eventual purpose, an additional degree of control over an ε-chain on a chainable continuum M. The second result extends this property to a characterization of terminal points. Since this is the definition of an end point of a chainable continuum as used by Bing [4, p. 660], proposition 4.2 establishes the equivalence of the three definitions of terminal point or end point by Miller, Bing, and Fugate.

4.1 <u>Proposition</u> Suppose that M is a chainable continuum and A is a subcontinuum of M with the property that for every $\varepsilon > 0$, there is an ε-chain $C = \{d_1, d_2, \cdots, d_m\}$ covering M such that $A \cap d_m \neq \emptyset$. Then A is a terminal subcontinuum of M.

Proof: Assume that A is not a terminal subcontinuum of M. Then by definition there exist two subcontinua H and K of M, each meeting A, but such that neither is contained in the union of A and the other. Let $p \in H \setminus (K \cup A)$, $q \in K \setminus (H \cup A)$, and ε be a positive number such that $\varepsilon = \frac{1}{3}\min\{ \rho(p,K\cup A), \rho(q,H\cup A) \}$.

Since M is chainable, let $C = \{d_1, d_2, \cdots, d_m\}$ denote an ε-chain on M. By hypothesis we may have that $d_m \cap A \neq \emptyset$. Let d_j denote the first link of C such that $d_j \cap (H \cup K) \neq \emptyset$. Let d_s and d_t denote links of C such that $p \in d_s$ and $q \in d_t$. By their selection, $j \leq s \leq m$, $j \leq t \leq m$, $s \neq t$, and in fact $d_s \cap (K \cup A) = \emptyset$ and $d_t \cap (H \cup A) = \emptyset$.

Without loss of generality, we may assume that $d_j \cap H \neq \emptyset$. Since $H \cup A$ is connected with $d_j \cap (H \cup A) \neq \emptyset$ and $d_m \cap (H \cup A) \neq \emptyset$, proposition 2.2 implies that $d_i \cap (H \cup A) \neq \emptyset$ for $j \leq i \leq m$ and hence $H \cup A$ meets each link of $C(j,m)$. Thus, $d_t \cap (H \cup A) \neq \emptyset$. But this is contrary to the existence of d_t and therefore A is a terminal subcontinuum of M. ||

4.2 <u>Proposition</u> If M is a chainable continuum then $p \in M$ is an end point of M if and only if p is a terminal point of M.

Proof: Proposition 4.1 clearly implies that if p is an end point of M, then p is a terminal point of M. For the converse, if p is a terminal point of M, corollary 3.15 implies that every nondegenerate subcontinuum B of M which contains p is irreducible from p to some other point of B. But now corollary 3.19 implies that for every ε-chain on M, there is a refinement of this chain on M, which will of course still be an ε-chain, such that $p \in d_1 \setminus \overline{d_2}$. Therefore, p is an end point of M and the proposition is established. ||

Construction Properties of Chains on Terminal Subcontinua

It has been previously noted that the purpose of studying terminal subcontinua and the chains on such continua, is to provide a means of isolating this part of a continuum in a particular part of a chain. The concept involved is that of exact containment. If M is a continuum, A is a subcontinuum of M, $C = \{d_1, d_2, \cdots, d_m\}$ is a chain on M, and $1 \leq j \leq k \leq m$, then A is said to be contained exactly in the subchain $C(j,k)$, denoted $A \subseteq^e C^*(j,k)$, if and only if

1) $A \subseteq C^*(j,k)$,

2) A is not contained in any proper subchain of $C(j,k)$, and

3) $A \cap [\overline{C^*(1,j-1)} \cup \overline{C^*(k+1,m)}] = \emptyset$ [13, p. 463].

Thus, not only can the link d_{j-1} not meet A but neither may its closure. Since our concern is primarily with terminal subcontinua and particularly with chains having an end link meeting the terminal subcontinuum, this definition of exact containment will usually involve one end of the chain. That is, j may be one and hence $A \subseteq^e C^*(1,k)$ or conversely, k may be m.

Proposition 4.3 shows that if M is an atriodic and hereditarily unicoherent continuum with terminal subcontinuum A and chain C on M, then there will exist a refinement of C exactly containing A in an end subchain. Also the property of tautness is inherited by the refinement. It is of course noted that the subcontinuum A may be M itself, in which case if C is the chain on M, then C and the described subchain are considered to be the same. This result is achieved rather easily compared to previous chain construction type theorems, but it is lemma 3.18 which eases the burden of this result. There is ample opportunity later in this and the following chapter for the reader to exercise his ability to construct illustrations for the proofs encountered.

4.3 Proposition Suppose M is an atriodic and hereditarily unicoherent continuum, A is a terminal subcontinuum of M, and $C = \{d_1, d_2, \cdots, d_m\}$ is a chain on M. Then there is a chain $G = \{g_1, g_2, \cdots, g_n\}$ on M and an integer s, $1 \leq s \leq n$, such that

1) G is a refinement of C.
2) $A \subseteq^e G^*(s,n)$.
3) If C is taut then G is taut.

Proof: By lemma 3.18, there exists a chain $F = \{f_1, f_2, \cdots, f_n\}$ on M such that F is a refinement of C, $(f_n \setminus \overline{f}_{n-1}) \cap A \neq \emptyset$, and C taut implies F is taut. Then there exists an integer s such that $A \subseteq F^*(s,n)$ but is not contained in any proper subchain of $F(s,n)$. Because $F(s,n)$ is a chain, $F^*(1,s-2) \cap F^*(s,n) = \emptyset$ (particularly if $F(1,s-2)$ contains no links.) Thus, $\overline{F^*(1,s-2)} \cap F^*(s,n) = \emptyset$ and hence, $\overline{F^*(1,s-2)} \cap A = \emptyset$. If $\overline{f}_{s-1} \cap A = \emptyset$, then let $G = F$ and the proposition is proven.

Suppose that $\overline{f}_{s-1} \cap A \neq \emptyset$. Since $A \subseteq F^*(s,n)$, A and $M \setminus F^*(s,n)$ are disjoint closed subsets of a normal space. Thus, there exist disjoint open sets U and V such that $A \subseteq U$ and $M \setminus F^*(s,n) \subseteq V$. For each i, $1 \leq i \leq s-1$, let $g_i = f_i \cap V$ and for each i, $s \leq i \leq n$, let $g_i = f_i$. Then $G = \{g_1, g_2, \cdots, g_n\}$ is clearly a chain which is a refinement of F and hence C.

By the definition of G, $A \subseteq G^*(s,n)$ and A is not contained in any proper subchain of $G(s,n) = F(s,n)$. Also, $A \cap \overline{G^*(1,s-1)} \subseteq A \cap \overline{V} = \emptyset$. Thus, $A \subseteq^e G^*(s,n)$. Finally, since nonadjacent links of G are contained in nonadjacent links of F, C taut implies F is taut which implies that G is taut. ||

Lemma 4.4 is another of the highly specialized results required to produce the proper type of chain on a continuum. It should be noted that the hypothesis only requires that the continuum A be atriodic, hereditarily unicoherent, and ε-chainable. Also, the hypothesis requires that $A \cap B$ be a terminal subcontinuum of both A and B. Proposition 4.5 will show that if M is the union of proper subcontinua A and B, then this will in fact always be the case. Again, the objective here is to increasingly isolate terminal subcontinua in a chain. If M is the union of proper subcontinua A and B, perhaps each being chainable, then by chaining on A and B with chains exactly containing $A \cap B$ and satisfying the other consequences of the following lemma, then a chain may be constructed on M. There will be more on this later.

Lemma 4.4 uses another technique which will be exploited in later results and that is the existence of a Lebesque number for an open cover of a compact subset of a metric space. Any reader unfamiliar with this term is refered to the reference by Kelley [18, p. 154] for an unusually nice proof of this result. The use of a Lebesque number guarantees the existence of an ε-chain with links sufficiently small in diameter to accomplish desired results. Although the complete proof is somewhat involved, this technique is developed in the early part of the proof and should be understood because of its later use.

4.4 Lemma Suppose that A is an atriodic, hereditarily unicoherent ε-chainable continuum, B is a chainable continuum, $A \cap B \neq \emptyset$, and $A \cap B$ is a terminal subcontinuum of A and B. Then, there exist ε-chains $C = \{d_1, d_2, \cdots, d_m\}$ and $F = \{f_1, f_2, \cdots, f_n\}$ such that

1) C and F are taut ε-chains on A and B respectively.

2) $A \cap B \subseteq^e C^*(j,m)$ and $A \cap B \subseteq^e F^*(k,n)$.

3) $F(k,n)$ is a closed refinement of $C(j,m)$.

4) $\overline{C^*(1,j-1)} \cap F^* = \emptyset$.

5) $\overline{F^*(1,k-2)} \cap C^* = \emptyset$.

6) $\overline{f}_{k-1} \cap A = \emptyset$ and there is a positive integer t, $j \leq t \leq m$, such that $\overline{f}_{k-1} \cap C^* \subseteq d_t$ and $\overline{f}_{k-1}$ meets no other link of C.

Proof: Since A is ε-chainable, let C_0 denote an ε-chain on A. Using lemma 2.10 we may, without loss of generality, assume that C_0 is taut. Proposition 4.3 implies the existence of a refinement G of C_0, $G = \{g_1, g_2, \cdots, g_m\}$, such that G is a taut ε-chain on A and $A \cap B \subseteq^e G^*(j,m)$. Let α denote the minimum distance between the non-adjacent links of G.

Since the collection $\{[g_j \setminus \overline{G^*(1,j-1)}], g_{j+1}, \cdots, g_m\}$ is now a finite open cover of $A \cap B$, recall that the definition of exact containment requires that $(A \cap B) \cap \overline{G^*(1,j-1)} = \emptyset$, let $\beta > 0$ be a Lebesque number for this cover of $A \cap B$ [18, p. 154]. Let δ denote half the minimum of α, β, ε, and $\rho(A \cap B, \overline{G^*(1,j-1)})$ (if $\overline{G^*(1,j-1)}$

is empty, which is possible, then this latter value is infinite.)

By considering theorem 2.13, we have the existence of a taut δ-chain H_1 on B. Since $\delta \leq \frac{1}{2}\beta$, the closure of each link of H_1 which meets $A \cap B$ is contained in some link of $G(j,m)$. Also, since $\delta \leq \frac{1}{2}\rho(A \cap B, \overline{G^*(1,j-1)})$, the closure of each link of H_1, which meets $A \cap B$, fails to meet $\overline{G^*(1,j-1)}$. Applying proposition 4.3 to B relative to the chain H_1, there exists a taut δ-chain H on B which is a refinement of H_1 and such that $A \cap B \subseteq^e H^*(k,n)$. Thus, the closure of each link of $H(k,n)$ is a subset of some link of $G(j,m)$ and does not meet $\overline{G^*(1,j-1)}$.

Let t = greatest lower bound of $\{ i : \overline{h}_k \subseteq g_i,\ j \leq i \leq m \}$; ie, g_t denotes the first link of $G(j,m)$ properly containing $\overline{h}_k$. At this point we note that should it be the case that $A \cap B = B$; ie, $B \subseteq A$, then the desired chains are $C = G$ and $F = H$. Since this case is now easily shown to satisfy properties 1 thru 6, sometimes vacuously, we continue the proof considering $A \setminus B$ and $B \setminus A$ to be nonempty. Since $B \setminus A$ and $A \setminus B$ are nonempty separated point sets, and a metric space is completely normal, there exist two disjoint open sets S and T such that $A \setminus B \subseteq S$ and $B \setminus A \subseteq T$.

Let $f_i = h_i \cap T$, if $1 \leq i \leq k-1$, and $f_i = h_i$, if $k \leq i \leq n$. By applying proposition 2.8, $F = \{f_1, f_2, \dots, f_n\}$ is a chain since T is open and $(h_i \cap h_{i+1}) \cap B$ is nonempty by proposition 2.1 for $1 \leq i \leq k-2$, and is contained in T. We also claim that $\{ d_i = g_i \cap S : 1 \leq i \leq j-1 \} \cup \{ d_i = g_i \setminus \overline{F^*(1,k-1)} : j \leq i \leq m,\ i \neq t \} \cup \{ d_t = g_t \setminus \overline{F^*(1,k-2)} \}$ is a chain. Since $A \cap (g_i \cap g_{i+1}) \neq \emptyset$ for $1 \leq i \leq m$ and because $A \cap B \subseteq G^*(j,m)$, $A \setminus B$ meets each pair of adjacent links of $G(1,j-1)$, and the collection

$\{ d_i = g_i \cap S : 1 \leq i \leq j-1 \}$ is a chain by proposition 2.8.

Similarly because $A \cap B \subseteq^e H^*(k,n)$, $A \subseteq \sim\overline{H^*(1,k-1)}$ and since A meets the common part of each pair of adjacent links of $G(j,m)$, proposition 2.8 again implies that $G(j,m) \cap [\sim\overline{F^*(1,k-1)}]$ is a chain. Since $\overline{F^*(1,k-2)} \subseteq \overline{F^*(1,k-1)}$, $\sim\overline{F^*(1,k-1)} \subseteq \sim\overline{F^*(1,k-2)}$. Thus, the effect of making $d_t = g_t \setminus \overline{F^*(1,k-1)}$ is simply to include $f_{k-1} \setminus \overline{f}_{k-2}$, an open set, in the t-th link. Since $f_{k-1} \setminus \overline{f}_{k-2} \subseteq g_t$, $\{d_j, \cdots, d_t, \cdots, d_m\}$ will still form a chain.

Finally, since $\emptyset \neq (g_{j-1} \cap g_j) \cap A \subseteq (g_{j-1} \cap g_j) \cap (A \setminus B) \subseteq S$ and $g_{j-1} \cap A \subseteq \sim\overline{F^*(1,k-1)}$, $d_{j-1} \cap d_j \neq \emptyset$. However, $d_{i_1} \cap d_{i_2} = \emptyset$ for $1 \leq i_1 \leq j-2$ and $j \leq i_2 \leq m$ or for $1 \leq i_1 \leq j-1$ and $j+1 \leq i_2 \leq m$, because d_{i_1} and d_{i_2} are contained respectively in g_{i_1} and g_{i_2}. Therefore, the collection $\{d_1, d_2, \cdots, d_m\}$ is a chain.

For the chains C and F, properties 1 and 2 are immediate from the chains G and H. Let $f_{k_1} \varepsilon F(k,n)$. Since $F(k,n) = H(k,n)$, $F(k,n)$ is a closed refinement of $G(j,m)$ and hence for some j_1, $j \leq j_1 \leq m$, $\overline{f}_{k_1} \subseteq g_{j_1}$. If $k_1 = k$, then $\overline{f}_{k_1} \subseteq \sim\overline{F^*(1,k-2)}$ because F is taut and hence $\overline{f}_{k_1} \subseteq g_t \setminus \overline{F^*(1,k-2)} = d_t$. If $k_1 \neq k$, $k < k_1 \leq n$, $\overline{f}_{k_1} \subseteq \sim\overline{F^*(1,k-1)}$ and thus $\overline{f}_{k_1} \subseteq g_{j_1} \setminus \overline{F^*(1,k-1)} = d_{j_1}$. Therefore, $F(k,n)$ is a closed refinement of $C(j,m)$ and property 3 holds.

By the definition of C, $\overline{C^*(1,j-1)} \subseteq \overline{S}$ and by the definition of F, $F^*(1,k-1) \subseteq T$. Thus, $\overline{C^*(1,j-1)} \cap F^*(1,k-1) = \emptyset$. Also, by definition and as noted earlier, $F^*(k,n) \cap \overline{C^*(1,j-1)} = \emptyset$. Therefore, these results imply that $\overline{C^*(1,j-1)} \cap F^* = \emptyset$, and property 4 is satisfied.

Property 5 follows directly from the definition of C.

To see the last property, recall that $A \cap B \subseteq^e F^*(k,n)$, which implies that $(A \cap B) \cap \overline{f}_{k-1}$. Because $f_{k-1} \subseteq T$ and $A \setminus B \subseteq S$, $(A \setminus B) \cap \overline{f}_{k-1} = \emptyset$ and we have that $A \cap \overline{f}_{k-1} = \emptyset$. The definition of d_t is that $d_t = g_t \setminus \overline{F^*(1,k-2)}$ with g_t being the first link of G to contain $\overline{h}_k = \overline{f}_k$. Since $C^* \cap \overline{F^*(1,k-2)} = \emptyset$ by the definition of C, and $d_i \cap \overline{F^*(1,k-1)} = \emptyset$ for $i \neq t$, the previously defined integer t satisfies the remainder of property 6. This completes the proof of the lemma. ||

Some Sufficiencies for Terminal Subcontinua

Several of the results of this section will be intuitively clear and will not rely on the complicated theorems or chain constructions which have been presented. Their significance will, however, be apparent in light of previous discussions of the use of terminal subcontinua in chaining on a continuum. Several more results concerning special chain constructions on terminal subcontinua will also be presented here and in the next chapter.

Proposition 4.5 shows that if M is the union of two proper subcontinua, they are terminal subcontinua of M and their intersection is a terminal subcontinuum of each of them. That it is necessary to have proper subcontinua should be clear by considering an arc and subcontinua of it.

4.5 <u>Proposition</u> If the atriodic, hereditarily unicoherent continuum M is the union of two proper subcontinua A and B, then each of A and B is a terminal subcontinuum of M. Moreover, $A \cap B$ is a terminal subcontinuum of each of A and B.

Proof: Suppose that $M = A \cup B$ with A and B proper subcontinua of M. To assume that A is not a terminal subcontinuum of M implies, by proposition 3.14, the existence of a subcontinuum H of M such that $A \cup H$ is either degenerate (which is impossible) or $A \cup H$ is irreducible between some pair of points p and q with neither in A. Then $\{p, q\} \subseteq H \setminus A \subseteq B$. Since p and q are contained in the continuum B, the irreducible subcontinuum pq, which is unique by proposition 3.21, is contained in B. Thus, $A \cup H \subseteq B$ and hence $A \subseteq B$, which is contrary to B being a proper subcontinuum of M. Thus, A is a terminal subcontinuum of M and likewise B is a terminal subcontinuum of M.

For the remaining part of the proposition, assume that $A \cap B$ is not a terminal subcontinuum of A. Then by the definition, there are subcontinua H and K of A such that $H \cap A \cap B \neq \emptyset$, $K \cap A \cap B \neq \emptyset$, $H \not\subseteq K \cup (A \cap B)$, and $K \not\subseteq H \cup (A \cap B)$. Also note that $B \setminus (H \cup K)$ which contains $B \setminus A$ is nonempty. Thus, $K \cup (A \cap B)$, $H \cup (A \cap B)$, and B are three continua each pair of which intersect and no one of which is contained in the union of the other two. But since M is atriodic and hereditarily unicoherent, this is contrary to proposition 3.3. Thus, $A \cap B$ is a terminal subcontinuum of A and similarly B. ||

If a terminal subcontinuum K of an atriodic and hereditarily unicoherent continuum M is the union of two proper subcontinua, then by the preceding, these subcontinua are terminal subcontinua of K. The following proposition shows that at least one of these subcontinua must also be a terminal subcontinuum of M. The proof uses proposition 3.14 which is a characterization of a terminal subcontinuum T in terms of the irreducibility of $T \cup A$ between some pair of points, one of which must belong to T, where A is a subcontinuum of M and $T \cup A$ is nondegenerate.

4.6 <u>Proposition</u> Let M be an atriodic, hereditarily unicoherent continuum, K a terminal subcontinuum of M, and A and B proper subcontinua of K such that $K = A \cup B$. Then at least one of A and B is a terminal subcontinuum of M.

Proof: Assume the proposition fails. Since A is not a terminal subcontinuum of M, by proposition 3.14, there exists a subcontinuum N of M such that $A \cap N \neq \emptyset$ and $N \cup A$, being atriodic and hereditarily unicoherent, is irreducible between some pair of points, neither of which belongs to A. This property of $N \cup A$ not being irreducible between any pair of points with one of them belonging to A is henceforth refered to as property P.

By an argument similar to that presented for proposition 4.5, we have that $A \subseteq N$. If $N \subseteq K$, proposition 4.5 implies that A is a terminal subcontinuum of K and hence $N \cup K$ is irreducible between some pair of points with one belonging to A. Since this is contrary to the existence of N, we must have that $N \setminus K \neq \emptyset$. Since N meets the terminal subcontinuum K and by hypothesis, K is nondegenerate, applying proposition 3.14 again, we have the existence of the points p and q such that $N \cup K$ is irreducible from p to q and without loss of generality, $q \in K$. Since $N \setminus K \neq \emptyset$, $p \in N \setminus K$. We also have that $q \in B \setminus A$ since to assume that $q \in A$ implies, since $A \subseteq N$, that N is a subcontinuum of $N \cup K$ containing both p and q. Hence, $N \cup K = N$ and $N = N \cup A$ is irreducible from p to q with $q \in A$.

But this is contrary to property P. Thus, $N \cup K$ is irreducible from p to q with $p \in N \setminus K$ and $q \in B \setminus A \subseteq K$.

Arguing similarly to the preceding paragraph, there is a subcontinuum S of M such that $S \cap B \neq \emptyset$ and $S \cup B$ is not irreducible between any pair of points, one of which is in B. Then $S \cup B = S$ and there are points r and s such that $S \cup K$ is irreducible from r to s, with $s \in S \setminus K$, and $r \in A \setminus B \subseteq K$.

Since N and S are both subcontinua of M intersecting K, the definition of terminal subcontinua implies that either $S \subseteq K \cup N$ or $N \subseteq K \cup S$. Suppose $S \subseteq K \cup N$. Since $s \in S \setminus K$, $s \in N \setminus K$. From property P we have that $N = N \cup A$ is reducible from p to $r \in A$, and thus there is a proper subcontinuum L of N such that $\{p, r\} \subseteq L$. Thus, $q \notin L$. Now $r \in L \cap K$ since $r \in L$ and $r \in A \subseteq K$, and hence $L \cup K$ is a subcontinuum of $K \cup N$ containing p and q. Therefore, $K \cup N = K \cup L$ and $s \in L$ since $s \in S \setminus K \subseteq K \cup N$ by supposition. But since $q \in B \setminus [L \cap (S \cup K)]$ which is contained in S, $L \cap (S \cup K)$ is a proper subcontinuum of $S \cup K$ containing s and r which means $S \cup K$ is reducible from s to r, contrary to the existence of s and r. Thus, at least one of A or B is a terminal subcontinuum of M. ||

Proposition 4.7 and lemma 4.8 show how to develop, and then show a consequence of, a terminal subcontinuum which is indecomposable. This will prove helpful later when trying to chain on a continuum, by providing a particular terminal subcontinuum upon which to base a chain construction. It should be noted that the lemma does not require that the continuum be atriodic and hereditarily unicoherent as has generally been the case.

4.7 Proposition Let M be an atriodic, hereditarily unicoherent continuum with a terminal subcontinuum K. Then there is a subcontinuum L of K such that

1) L is a terminal subcontinuum of M.

2) L is irreducible as a terminal subcontinuum of M.

3) L is indecomposable or is a terminal point of M.

Proof: If $B \subseteq M$, then B is said to have property P if and only if B is a terminal subcontinuum of M and $B \subseteq K$. Clearly, K has property P. We shall show that this property is inductive. Suppose that the collection $\{N_1, N_2, \cdots\}$ is a sequence of continua of M such that for each i, N_i has property P and $N_{i+1} \subseteq N_i$. Clearly, $N = \bigcap_{i=1}^{\infty} N_i$ is a subcontinuum of M contained in K. If N does not have property P, then N must fail to be a terminal subcontinuum of M. Thus, there are subcontinua R and S of M, each intersecting N, such that $R \not\subseteq S \cup N$ and $S \not\subseteq R \cup N$.

Let $r \in R \setminus (S \cup N)$ and $s \in S \setminus (R \cup N)$. Since $M \setminus \{r, s\}$ is open in M and contains N, there exists an integer j such that $N_j \subseteq M \setminus \{r, s\}$. Thus, each of R and S intersect N_j and neither is contained in the union of N_j and the other subcontinuum. Hence, N_j is not a terminal subcontinuum of M contrary to its selection. Therefore, property P is inductive. Applying the Brouwer Reduction theorem [18, p. 61] to the terminal subcontinuum K of M, there exists a subcontinuum L of K which is a terminal subcontinuum of M and is irreducible with respect to this property. If L in nondegenerate, the third result of the proposition follows from proposition 4.6 since L decomposable implies the existence of a proper subcontinuum H of L which would be a terminal subcontinuum of M. This contradicts the irreducibility of L and the three results are established. If L is degenerate, then L is a terminal point and the proposition is proven. ||

4.8 <u>Lemma</u> Suppose that M is a continuum and K is an indecomposable terminal subcontinuum of M. Further, suppose that there is a subcontinuum A of M such that $A \cap K \neq \emptyset$, $K \not\subseteq A$, and $A \not\subseteq K$. Let D denote the composant of K containing $A \cap K$. If B is a subcontinuum of M intersecting K, such that $B \not\subseteq K$ and $K \not\subseteq B$, then $B \cap K \subseteq D$.

Proof: Assume there is a subcontinuum B of M such that $B \cap K \neq \emptyset$, $B \not\subseteq K$, $K \not\subseteq B$, and $B \cap K \not\subseteq D$. Then $B \cap K$ is a proper subcontinuum of K not contained in D. Assume $x \in (B \cap K) \cap D$. Then there is a proper subcontinuum L of K such that $x \in L$ and $L \cap (A \cap K) \neq \emptyset$. Then $(B \cap K) \cup L$ is a subcontinuum of K. The point set $(B \cap K) \cup L$ is a proper subset of K since K, by hypothesis, is indecomposable. But now $y \in (B \cap K) \setminus D$ is contained in $L \cup (B \cap D)$ which implies $y \in D$. But this is impossible. Thus, x fails to exist and $(B \cap K) \cap D = \emptyset$.

Thus, certainly, $A \cap K \cap B = \emptyset$. Since K is a terminal subcontinuum of M it follows that either $A \subseteq B \cup K$ or $B \subseteq A \cup K$. Suppose $A \subseteq B \cup K$. Since $A \not\subseteq K$, $A \cap B \neq \emptyset$. Then $(A \cup B) \cap K$ is a subcontinuum of K intersecting disjoint composants of K, namely D and the composant of K containing $B \cap K$. Since $(A \cup B) \cap K$ cannot be a proper subcontinuum of K and meet the two composants, we must have that $(A \cup B) \cap K = K$. But this implies that K is the union of the two proper subcontinua $A \cap K$ and $B \cap K$ contrary to the indecomposability of K. Therefore, the continuum B as described cannot exist and if B is a subcontinuum of M such that $B \cap K \neq \emptyset$, $B \not\subseteq K$, and $K \not\subseteq B$, then $B \cap K \subseteq D$. ||

Opposite end points and more generally opposite terminal subcontinua, are terms which are intuitively easy when applied to continua such as arcs but which have unusual results when applied to other continua. The definition of opposite end points is given in terms of a chain while the general definition of opposite terminal subcontinua does not require a chainable continuum. Proposition 4.9 and lemma 5.9 will establish the equivalence of these definitions for terminal subcontinua of chainable continua. The points p and q of the chainable continuum M are called <u>opposite end points</u> (or <u>opposite terminal points</u>) of M if and only if for every $\varepsilon > 0$, there is an ε-chain $\{d_1, d_2, \ldots, d_n\}$ on M such that $p \in d_1 \setminus d_2$ and $q \in d_n \setminus d_{n-1}$. More generally, if M is a continuum and each of H and K is a terminal subcontinuum of M then H and K are <u>opposite terminal subcontinua</u> of M if and only if there are points h of H and k of K such that M is irreducible from h to k [14, p. 386].

Examples A, B, C, and F are designed to illustrate various possibilities. The Closed Topologist's Sine Curve (example A) has two pairs of opposite end points, namely the pairs (R,T) and (S,T). Example B has one pair while example C has four pairs, namely, (P,R), (P,S), (Q,R), and (Q,S). Finally, example F has one pair of opposite end points and is indecomposable. In example C, the segments PQ and RS are two nondegenerate opposite terminal subcontinua. Other possibilities exist through unions of these examples.

Proposition 4.9 relates the existence of opposite end points to the irreducibility of a chainable continuum between two of its end points. The chainability of the continuum is necessary for the proof since it employs lemma 2.11. The method of proof is similar to the approach taken in proposition 2.4. That is, a sequence of point sets are constructed such that their limit superior is a proper subcontinuum between the two end points, contrary to the hypothesis.

4.9 Proposition If the chainable continuum M is irreducible between two of its end points, then these end points are opposite end points of M.

Proof: Suppose that M is irreducible between two of its end points p and q. Let $\varepsilon > 0$ be arbitrarily given less than $\rho(p,q)$. Lemma 2.10 and corollary 3.19 combine to imply the existence of a taut ε-chain $C = \{d_1, d_2, \cdots, d_m\}$ $(m \geq 2)$ on M such that $q \varepsilon d_m \setminus \overline{d}_{m-1}$. If $p \not\varepsilon d_1 \setminus d_2$ and yet $p \varepsilon d_1$, then there is an open set O containing p such that $\overline{O} \subseteq d_1 \cap d_2$. Then the collection $\{d_1, d_2 \setminus \overline{O}, d_3, \cdots, d_m\}$ is an ε-chain on M and $p \varepsilon d_1 \setminus \overline{(d_2 \setminus \overline{O})}$. Thus, p and q are opposite end points.

Suppose $p \not\varepsilon d_1$. Since C is a chain on M, $M \cap (d_1 \setminus d_2) \neq \emptyset$. Let $x_0 \varepsilon M \cap (d_1 \setminus d_2)$. Since the space is normal, let Q be an open set containing x_0 such that $\overline{Q} \subseteq d_1$. Then p and q are not members of $\overline{Q}$ and the collection $C_0 = \{d_1, d_2 \setminus \overline{Q}, d_3, \cdots, d_m\}$ is an ε-chain on M with $[d_1 \setminus \overline{(d_2 \setminus \overline{Q})}] \cap M \neq \emptyset$. We note that the point x_0 is a member of $d_1 \setminus \overline{(d_2 \setminus \overline{Q})}$ and that $x_0 \not\varepsilon \overline{C_0^*(2,m)}$. Assume that for every positive $\delta < \varepsilon$, if $B = \{b_1, b_2, \cdots, b_v\}$ is a δ-chain on M with $p \varepsilon b_1 \setminus \overline{b}_2$ and $q \varepsilon b_a$, for some a, $1 \leq a \leq v$, then $b_i \not\subseteq d_1$, $1 \leq i \leq a$. Lemma 2.11 and corollary 3.19 combine to imply the existence of a taut $\frac{\varepsilon}{2}$-chain $H_1 = \{h_1, h_2, \cdots, h_t\}$ on M such that H_1 is a refinement of C_0 and $p \in h_1 \setminus \overline{h}_2$. Let h_r denote the last link of H_1 containing q. By assumption, $h_i \not\subseteq d_1$, for $1 \leq i \leq r$. Thus, $h_i \subseteq C_0^*(2,m)$, $1 \leq i \leq r$, and consequently, $x_0 \notin \overline{h}_i \subseteq \overline{C_0^*(2,m)}$. Let $x_1 = p \in h_1 \cap M$, $x_2 \in h_2 \cap M$, $\cdots$, $x_{r-1} \in h_{r-1} \cap M$, and $x_r = q \in h_r \cap M$ and let $R_1 = \{p, x_2, x_3, \cdots, x_{r-1}, q\}$.

Let $\delta_i = \frac{1}{2^i}\varepsilon$. Assume that for $1 \leq i \leq n$, a taut δ_i-chain H_i has been defined on M such that H_i is a refinement of H_{i-1}, p is contained in the first link, $h_{i,1}$, and not in the closure of the second link, $\overline{h}_{i,2}$, of H_i, and h_{i,r_i} is the last link of H_i containing q. By the assumption, of course, no link of H_i between and including $h_{i,1}$ and h_{i,r_i} is contained in d_1. Thus, $x_0 \notin \overline{h}_{i,j} \subseteq \overline{C_0^*(2,m)}$. Assume also that the point set $R_i = \{p, x_{i,2}, x_{i,3}, \cdots, x_{i,r_i-1}, q\}$ has been defined with $x_{i,j} \in h_{i,j} \cap M$.

Lemma 2.11 and corollary 3.19 combine to guarantee the existence of a taut δ_{n+1}-chain H_{n+1} on M such that H_{n+1} is a refinement of H_n, and thus of C_0, with $p \in h_{n+1,1} \setminus \overline{h}_{n+1,2}$. If $h_{n+1,r_{n+1}}$ denotes the last link of H_{n+1} containing q then the assumption implies that $h_{n+1,j} \not\subseteq d_1$ for $1 \leq j \leq r_{n+1}$, and hence $x_0 \notin \overline{h}_{n+1,j} \subseteq \overline{C_0^*(2,m)}$. For each j, $1 < j < r_{n+1}$, let $x_j \in h_{n+1,j} \cap M$ and let $R_{n+1} = \{p, x_{n+1,2}, x_{n+1,3}, \cdots, x_{n+1,r_{n+1}-1}, q\}$. By induction, we have defined an infinite sequence of point sets $R_1, R_2, \cdots$ in M with $R_i \subseteq H_i^*(1,m_i) \subsetneq \overline{C_0^*(2,m)}$, with $x_0 \notin \overline{C_0^*(2,m)}$.

Since M is compact and the limit inferior of $\{R_i\}$ contains

$\{p, q\}$, and is thus nonempty, the method of defining the point sets R_i yields the fact that the limit superior L is connected [1.2]. Since L contains $\{p, q\}$ and is closed, L is a subcontinuum of M containing p and q. Since $R_i \subseteq \overline{C_0^*(2,m)}$ for each i, $L \subseteq \overline{C_0^*(2,m)}$ and $x_0 \notin L$. Thus, L is a proper subcontinuum of M containing p and q contrary to M being irreducible from p to q. Hence, there is a $\delta < \varepsilon$, δ positive, such that $F = \{f_1, f_2, \cdots, f_n\}$ is a δ-chain on M, $p \in f_1 \setminus \overline{f_2}$, and if $q \in f_k$ and $q \notin f_{k+1}$, then for some $f_j \in F$, $1 \leq j \leq k$, $f_j \subseteq d_1$.

Let $N = \bigcup_{i=j}^{n} f_i$. Then $q \in N$ and N is an open set. Let $G = \{f_1, f_2, \cdots, f_{j-1}, N \cap d_1, N \cap d_2, \cdots, N \cap d_m\}$. Then G is a chain on M since $F(1,j-1)$ is a chain, $C(1,m) \cap N$ is a chain by proposition 2.8, and $f_{j-1} \cap (N \cap d_1) \neq \emptyset$ while all other pairs of links from the two chains will have empty intersection which with proposition 2.9 implies that G is a chain. The chain G is clearly on M and is a δ-chain with $p \in f_1 \setminus f_2$ and $q \in (N \cap d_m) \setminus (N \cap d_{m-1})$. Therefore, p and q are opposite end points of M. ||

Proposition 4.10 will provide momentary relief for any detail weary reader. Except for the use of a terminal subcontinuum, which, if not understood may confuse the result, the proof could have been left as an easy exercise for the reader.

4.10 Proposition If M is an atriodic, hereditarily unicoherent continuum and the proper subcontinuum K is a terminal subcontinuum of M, then $\overline{M \setminus K}$ is a subcontinuum of M.

Proof: Proposition 3.2 implies that $M \setminus K$ has at most two components and the fact that K is a proper subset of M implies there is at least one component. Suppose A and B are distinct components of $M \setminus K$. Then $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. Also, $\overline{A}$ and $\overline{B}$ are distinct continua each meeting K. Since K is terminal then either $\overline{A} \subseteq K \cup \overline{B}$ or $\overline{B} \subseteq K \cup \overline{A}$ which is contrary to A and B being distinct. Thus, $M \setminus K$ has one component and $\overline{M \setminus K}$ is a subcontinuum of M. ||

The final result of this chapter is designed for use only in theorem 5.11 and is included here because it deals primarily with terminal subcontinua. This lemma differs from all preceding results in that the hypothesis requires a function f on a continuum which, among other things, is a monotone map. A continuous transformation is called <u>monotone</u> if and only if the inverse image of each connected set is connected. As we shall see later, the upper semi-continuous decomposition described and used extensively in Chapter III gives rise to a monotone map between the continuum and the unit interval. This is therefore the basis for the restrictions in the hypothesis of lemma 4.11. The third conclusion of the lemma is a continuation of proposition 4.7 and is again designed to show the existence of a particular terminal subcontinuum of two proper subcontinua whose union is the continuum. This can be the basis for constructing chains on each of the two proper subcontinua from which a chain on the continuum may be constructed.

4.11 <u>Lemma</u> Let M be an atriodic, hereditarily unicoherent continuum. If f is a continuous, single-valued, monotone mapping of M onto $[0,1]$, $A = f^{-1}[0,\frac{1}{2}]$, and $B = f^{-1}[\frac{1}{2},1]$ then

1) $A \cap (\overline{B \setminus A})$ is a terminal subcontinuum of both A and $\overline{B \setminus A}$.

2) If Q is a subcontinuum of both A and $\overline{B \setminus A}$ such that

$Q \cap A \cap (\overline{B \setminus A}) \neq \emptyset$ and $Q \not\subseteq A \cap (\overline{B \setminus A})$, then $A \cap (\overline{B \setminus A}) \subseteq Q$.

3) There is a subcontinuum K of $A \cap (\overline{B \setminus A})$ such that K is either a terminal point of $\overline{B \setminus A}$ and A or a nondegenerate indecomposable terminal subcontinuum of both $\overline{B \setminus A}$ and A.

Proof: Since A and B are proper subcontinua of M, proposition 4.5 implies that A is a terminal subcontinuum of M. Proposition 4.10 implies that $\overline{B \setminus A} = \overline{M \setminus A}$ is a subcontinuum of M which must be proper since $\overline{B \setminus A} \subseteq B$. Applying proposition 4.5 again, $A \cap (\overline{B \setminus A})$ is a terminal subcontinuum of both A and $\overline{B \setminus A}$. Thus, the first conclusion is proven.

Suppose that Q is a subcontinuum of $\overline{B \setminus A}$ such that $Q \cap A \cap (\overline{B \setminus A}) \neq \emptyset$, $Q \not\subseteq A \cap (\overline{B \setminus A})$, and $A \cap (\overline{B \setminus A}) \not\subseteq Q$. Let $p \in [A \cap (\overline{B \setminus A})] \setminus Q$. Let U denote an open set containing p such that $U \cap Q = \emptyset$. Since $Q \not\subseteq A \cap (\overline{B \setminus A})$ and $Q \subseteq \overline{B \setminus A}$, $Q \not\subseteq A$ and hence there is a point q in $Q \setminus A$. Thus, $f(q)$ is greater that $\frac{1}{2}$. Since $p \in A \cap (\overline{B \setminus A})$, $p \in A$ and $p \in B$. This implies that $f(p) = \frac{1}{2}$ and hence $p \in f^{-1}(0,f(q))$. Thus, $U \cap f^{-1}(0,f(q))$ is an open set containing p which necessarily meets $B \setminus A$. Let $r \in U \cap [f^{-1}(0,f(q))] \cap (B \setminus A)$. Since $r \notin A$ and $r \in f^{-1}(0,f(q))$, $\frac{1}{2} < f(r) < f(q)$. Also, since $U \cap Q = \emptyset$, $r \notin Q$.

Now the fact that f is monotone implies that $f^{-1}[\frac{1}{2},f(r)]$ is a subcontinuum of M. Since $(f^{-1}[\frac{1}{2},f(r)]) \cap (\overline{B \setminus A}) \neq \emptyset$, their intersection is a subcontinuum and each of $(f^{-1}[\frac{1}{2},f(r)]) \cap (\overline{B \setminus A})$ and Q is a subcontinuum of $\overline{B \setminus A}$ intersecting $A \cap (\overline{B \setminus A})$. Since $A \cap (\overline{B \setminus A})$ is a terminal subcontinuum of $\overline{B \setminus A}$, either

$$Q \subseteq [A \cap (\overline{B \setminus A})] \cup \{(f^{-1}[\tfrac{1}{2},f(r)]) \cap (\overline{B \setminus A})\} = A \cap (\overline{B \setminus A}) \cap (f^{-1}[\tfrac{1}{2},f(r)])$$

or $(f^{-1}[\frac{1}{2},f(r)])\cap(\overline{B\setminus A}) \subseteq [A\cap(\overline{B\setminus A})]\cup Q$. However, both are impossible since $q \in Q$ but $q \notin [A\cap(\overline{B\setminus A})]\cup f^{-1}[\frac{1}{2},f(r)]$ for the first case, and in the second case, $r \in (\overline{B\setminus A})\cap f^{-1}[\frac{1}{2},f(r)]$ but $r \notin A$ and $r \notin Q$ which implies that $r \notin [A\cap(\overline{B\setminus A})]\cup Q$. This contradiction establishes the second conclusion.

For the final result, proposition 4.7 proves the existence of a subcontinuum K of $A\cap(\overline{B\setminus A})$ such that K is irreducible with respect to being a terminal subcontinuum of A. Also, K is either a terminal point of A or a nondegenerate indecomposable subcontinuum of A. Assume that K is not a terminal subcontinuum of $\overline{B\setminus A}$. Then there are subcontinua L and R of $\overline{B\setminus A}$ such that $L\cap K \neq \emptyset$, $R\cap K \neq \emptyset$, $L \not\subseteq K\cup R$, and $R \not\subseteq K\cup L$. Since K is a terminal subcontinuum of A, $L\cup R \not\subseteq A$. Suppose that $L \not\subseteq A$. Then $L \not\subseteq A\cap(\overline{B\setminus A})$ and the second conclusion implies that $A\cap(\overline{B\setminus A}) \subseteq L$. Since $R \not\subseteq L$, it follows that $R \not\subseteq A\cap(\overline{B\setminus A})$ and thus $A\cap(\overline{B\setminus A}) \subseteq R$. Thus, L, R, and A are three continua which intersect, no one of which is contained in the union of the other two. Since M is atriodic and hereditarily unicoherent, this is contrary to proposition 3.3. Therefore, K is a terminal subcontinuum of $\overline{B\setminus A}$ and the third conclusion follows. ||

CHAPTER V

SOME CHARACTERIZATIONS OF CHAINABILITY ON CERTAIN CONTINUA

Most of the effort expended in the preceding chapters has been to achieve the results of this chapter. Chapter III included some characterizations of hereditarily decomposable chainable continua. The purpose of this chapter is to present primarily the work of Fugate in extending these characterizations to continua which are not hereditarily decomposable. These results can be divided very nicely into two categories. Characterizations concerning finite unions of chainable continua and a characterization requiring that all indecomposable subcontinua be chainable. This latter characterization also produces a result concerning countable unions of chainable continua. The methods of chaining which have been described and used previously will again be utilized to construct chains with certain properties.

Finite Unions of Chainable Continua

The first six results of this chapter deal primarily with finite unions of chainable or ε-chainable continua. Theorem 5.1 requires that the atriodic and hereditarily unicoherent continuum M be the union of an ε-chainable continuum A and a chainable continuum B. The result is that M is only ε-chainable however. This is immediately extended by corollary 5.2 to M being chainable if it is the union of two chainable continua. This could obviously be extended immediately to finite unions. However, we first present lemma 5.3 in order to reduce the hypothesis to require that M be unicoherent instead of hereditarily unicoherent. This is possible as long as M is the union of two hereditarily unicoherent subcontinua such as two chainable subcontinua. This result, in terms of chainability, is presented in theorem 5.4 and extended by the two corollaries which follow theorem 5.4.

Of the results of this section, theorem 5.1 is certainly the most involved. Under the condition that A and B are proper subcontinua whose union is M, the terminal subcontinuum $A \frown B$ of A and B is used to produce a chain on M. Most of the effort is devoted to showing that a defined collection actually is a chain. It should also be noted that the technique of using an open set G to form a closed set $M \setminus G$ which can be separated into two disjoint closed sets is effectively employed to produce a chain on M.

5.1 <u>Theorem</u> Let M be an atriodic, hereditarily unicoherent continuum and $\varepsilon > 0$. If M is the union of two subcontinua A and B with A ε-chainable and B chainable, then M is ε-chainable.

Proof: Clearly, if either of A or B is not proper, then the result is proven. Thus, we may assume that A and B are both proper subcontinua of M. Proposition 4.5 implies that $A \frown B$ is a nonempty terminal subcontinuum of M. Lemma 4.4 implies the existence of two ε-chains, $C = \{d_1, d_2, \dots, d_m\}$ and $F = \{f_1, f_2, \dots, f_n\}$, on A and B respectively, which satisfy the six results of the lemma. Again, $A \setminus F^*$ and $B \setminus C^*$ are nonempty since otherwise the theorem is proven.

Case i. Suppose that $\overline{f}_{k-1} \cap C^* \subseteq d_m$. That is, in terms of result 6 of lemma 4.4, $t = m$. Since $\overline{f}_{k-1} \cap C^*(1,m-1) = \emptyset$, also by result 6, we claim that $C(1,m) \oplus F(k-1,1)$ is an ε-chain on M. Certainly, $C(1,m)$ and $F(k-1,1)$ are chains which together cover M. Result 5 of lemma 4.4 states that no link of $F(k-2,1)$ meets a link of $C(1,m)$ and the above discussion shows that no link of $C(1,m-1)$ meets a link of $F(k-1,1)$. Because $f_{k-1} \cap d_m \neq \emptyset$, proposition 2.9 asserts that $C(1,m) \oplus F(k-1,1)$ is a chain on M. Therefore, the theorem is proven when case i holds.

Now suppose that $\overline{f}_{k-1} \cap C^* \not\subseteq d_m$. Then result 6 of the lemma concludes that $\overline{f}_{k-1} \cap d_m = \emptyset$. An additional consequence of result 6 is the existence of an integer t, $j \leq t < m$, such that $\overline{f}_{k-1} \cap C^* \subseteq d_t$ and $\overline{f}_{k-1} \cap [C^*(1,t-1) \cup C^*(t+1,m)] = \emptyset$. Consider the set, $\{ i : \overline{f}_i \not\subseteq C^*(j,m-1),\ k \leq i \leq n \}$. Since $A \cap B \subseteq^e C^*(j,m)$, by result 2, there exists a point y of $A \cap B$ such that $y \notin C^*(j,m-1)$. But $y \in \overline{f}_{i_0}$, for some i_0, $k \leq i_0 \leq n$, since $A \cap B \subseteq^e F^*(k,n)$. Thus, the above set is nonempty and being bounded, has a greatest lower bound, r. Since $F(k,n)$ is a closed refinement of $C(j,m)$ and $\overline{f}_r \not\subseteq C^*(j,m-1)$, it follows that $\overline{f}_r \subseteq d_m$. Since $\overline{f}_{k-1} \cap C^*(j,m) \subseteq d_t$ with $t < m$, $\overline{f}_k \subseteq C^*(j,m-1)$ and thus $r \geq k+1$ and $r-2 \geq k-1$.

If $r-2 = k-1$, let $d'_m = d_m$ and if $r-2 \geq k$, let $d'_m = d_m \setminus \overline{F^*(k,r-2)}$. Let $C_1 = \{d_1, d_2, \cdots, d_{m-1}, d'_m\}$. If $r-2 = k-1$, then C_1 is the chain C and consequently is a chain on A for which the six results of lemma 4.4 are valid. Also, $\overline{f}_{r-2} = \overline{f}_{k-1}$ and thus, $d_m \cap \overline{f}_{r-2} = d_m \cap \overline{F^*(k-1,r-2)} = \emptyset$ and $F^*(k-1,k) = F^*(k-1,r-1)$.

If $r-2 \geq k$, then $\overline{f}_i \subseteq C^*(j,m-1)$ for $k \leq i \leq r-1$, by result 3 of the lemma and the definition of r. Thus, $A \cap \overline{F^*(k,r-2)} \subseteq C^*(j,m-1)$ and $d_m \cap \overline{F^*(k,r-2)} \subseteq C^*(j,m-1)$ which implies that C_1 covers A. Since $f_{r-1} \cap f_r \neq \emptyset$, $f_{r-1} \subseteq d_{m-1}$, and $f_r \subseteq d_m$, we have that $d_{m-1} \cap d_m$ contains $f_{r-1} \cap f_r$ and is nonempty. Because f_r does not meet $\overline{F^*(k,r-2)}$, $f_{r-1} \cap f_r \subseteq d_m \setminus \overline{F^*(k,r-2)} = d'_m$, and thus $d_{m-1} \cap d'_m \neq \emptyset$.

Finally, by its definition, d'_m fails to meet any link of $C(1,m-2)$ and C_1 is a chain on A. Since C is taut, the defined chain C_1 is taut and the remaining results of lemma 4.4 similary follow from the definition of C_1. For notational convenience, we shall assume that $C = C_1$. Then C and F are chains on A and B respectively, having the six properties of lemma 4.4, and in addition, $d_m \cap \overline{F^*(k-1,r-2)} = \emptyset$ and $F^*(k-1,k) \subseteq F^*(k-1,r-1)$.

Let $D = M \setminus d_m$. Since $t < m$, $d_t \setminus d_m \neq \emptyset$ and hence $M \cap (d_t \setminus d_m) \neq \emptyset$. Thus, D is a nonempty point set.

Case ii. Suppose that $D \subseteq F^*(1,r-1)$. The definition of r then implies that $f_{r-1} \cap d_m \neq \emptyset$ and $F^*(1,r-2) \cap d_m = \emptyset$. Then the collection $\{f_1, f_2, \cdots, f_{r-2}, d_m\}$ is clearly an ε-chain on M and the theorem holds for case ii.

Case iii. Suppose that $D \not\subseteq F^*(1,r-1)$. Since $\overline{F^*(r,n)} \subseteq C^*(j,m)$, by the definition of r and result 3 of lemma 4.4, and $B \subseteq F^*$, $D \setminus F^*(1,r-1) \subseteq C^*$. Of course the definition of D implies that $D \setminus F^*(1,r-1) \subseteq C^*(1,m-1) \setminus d_m$. Let $x \in D \setminus F^*(1,r-1)$. Then $x \in d_i$, for some i, $1 \leq i \leq m-1$. Hence, the set $\{ i : d_i \cap M \not\subseteq F^*(1,r-1) \cup d_m, 1 \leq i \leq m-1 \}$, is nonempty. Again, the greatest lower bound of this set exists and is denoted by s. Then the definition of s implies that $D \setminus F^*(1,r-1) \subseteq C^*(s,m-1)$ and that this latter set is nonempty.

We note that $M \subseteq C^*(s,m) \cup F^*(1,r-1)$.

Since $\overline{f}_{k-1} \cap d_m$ is empty and $f_{k-1} \cap M$ is nonempty, $\overline{f}_{k-1} \cap D$ is certainly nonempty. Thus, the two sets $\overline{f}_{k-1} \cap D$ and $D \setminus F^*(1,r-1)$ are nonempty and we shall show that they are also disjoint. Suppose there is a point p in both sets. Then $p \varepsilon \overline{f}_{k-1} \cap D$ implies that $p \varepsilon B$ since $\overline{f}_{k-1} \cap A = \emptyset$. Since F is a taut chain on B, this implies that $p \varepsilon F^*(k-2,k)$. The definition of r and the conditions of this case resulted in $F^*(k-2,k)$ being contained in $F^*(1,r-1)$. Therefore, $p \not\varepsilon M \setminus F^*(1,r-1)$ and the two sets are disjoint.

To see that no continuum in D intersects both $\overline{f}_{k-1} \cap D$ and $D \setminus F^*(1,r-1)$, assume to the contrary that the continuum K meets both sets. Then $K \cap (\overline{f}_{k-1} \cap D) \neq \emptyset$ and $\overline{f}_{k-1} \cap A = \emptyset$ imply that K meets $B \setminus A$. Now if $K \cap (A \setminus B) \neq \emptyset$, then K meets both of $A \setminus B$ and $B \setminus A$ which form a separation of $M \setminus (A \cap B)$. Thus, since K is a continuum, by the hereditary unicoherence of M, $K \cap (A \cap B) \neq \emptyset$. Therefore, $A \cap K$, $B \cap K$, and $A \cap B$ are three subcontinua of M each pair of which intersect.

Now $(A \cap B) \cap d_m \neq \emptyset$ which implies that $(A \cap B) \setminus D$ and thus $(A \cap B) \setminus K$ is nonempty. Since by supposition, $K \cap (A \setminus B) \neq \emptyset$ and $K \cap (B \setminus A) \neq \emptyset$ by previous proof, no one of $A \cap K$, $B \cap K$, or $A \cap B$ is contained in the union of the other two. However, this is contrary to proposition 3.3. Therefore, $K \cap (A \setminus B)$ must be empty and hence $K \subseteq B$.

Now $K \subseteq B$ and $K \cap [D \setminus F^*(1,r-1)] \neq \emptyset$ imply that there exists a link f_{i_0} of F, $r \leq i_0 \leq n$, such that $K \cap f_{i_0} \neq \emptyset$. Since $K \cap \overline{f}_{k-1} \neq \emptyset$ and thus $K \cap F^*(k-2,k) \neq \emptyset$, as noted earlier, proposition 2.2 implies that K meets each link of the chain $F(k,i_0)$. Since $i_0 \geq r$, $f_r \cap K \neq \emptyset$ and thus $K \cap d_m \neq \emptyset$ contrary to K being a continuum in D. Therefore, the original assumption that K existed if false and no subcontinuum of D meets both of $\overline{f}_{k-1} \cap D$ and $D \setminus F^*(1,r-1)$.

Since D is a closed and compact point set containing the disjoint closed point sets $D \cap \overline{f}_{k-1}$ and $D \setminus F^*(1,r-1)$ and no subcontinuum of D meets both sets, D itself is the union of two disjoint closed sets one containing $D \cap \overline{f}_{k-1}$ and the other containing $D \setminus F^*(1,r-1)$ [26, p. 15]. The normality of the space thus guarantees the existence of two open sets S and T such that $\overline{S} \cap \overline{T} = \emptyset$, $D \cap \overline{f}_{k-1} \subseteq T$, $D \setminus F^*(1,r-1) \subseteq S$, and $D \subseteq S \cup T$.

With these sets S and T the objective is of course to construct a chain on M using C and F. First, however, we shall verify several necessary facts.

Since $D \setminus F^*(1,r-1)$ is, by the supposition of this case, nonempty, and by its definition, $d_s \cap [D \setminus F^*(1,r-1)] \neq \emptyset$ while $d_i \cap [D \setminus F^*(1,r-1)] = \emptyset$, for $1 \leq i \leq s-1$, let $x \in d_s \cap [D \setminus F^*(1,r-1)]$. Let N_x denote the component of $M \setminus (M \cap \overline{d}_m) = M \setminus \overline{d}_m$. Since $M \cap \overline{d}_m$ is a closed proper subset of the continuum M and $\overline{M \setminus \overline{d}_m}$ is compact, $M \cap \overline{d}_m$ contains a limit point p of N_x. Thus, $\overline{N}_x \cap (M \cap \overline{d}_m) \neq \emptyset$. Since $\overline{N}_x \subseteq M$ this implies that $\overline{N}_x \cap \overline{d}_m \neq \emptyset$. Since C is a taut chain $p \in d_{m-1}$. Also, since $N_x \subseteq M \setminus \overline{d}_m$ which is contained in $M \setminus d_m$, $\overline{N}_x \subseteq M \setminus d_m$ and since $\overline{N}_x$ is connected and $\overline{N}_x \cap [D \setminus F^*(1,r-1)] \neq \emptyset$, $\overline{N}_x \subseteq S$. Proposition 2.3 now implies that $\overline{N}_x$ meets the common part of each pair of adjacent links of $C(s,m-1)$. Then S similarly meets the common part of each pair of adjacent links of $C(s,m-1)$ and proposition 2.8 implies that $C(s,m-1) \cap S$ is a chain.

In a manner similar to the preceding, let $y \in \overline{f}_{k-1} \cap D \subseteq F^*(k-2,k)$. If N_y denotes the component of $M \setminus \overline{d}_m$ containing y, then $\overline{N}_y$ is a connected subset of M meeting the common part of each pair of adjacent links of $F(k,r-1)$. Since $y \in T$, $\overline{N}_y \subseteq T$ and thus, T meets the common part of each pair of adjacent links of $F(k,r-1)$. In fact, since $y \in T$, $T \cap f_{k-1} \neq \emptyset$. Therefore, $F(k,r-1) \cap T$ is a chain.

We shall now show that $G = [C(s,m-1) \cap S] \oplus [d_m] \oplus [F(r-1,k) \cap T] \oplus [(f_{k-1} \cap T) \cup (f_{k-1} \setminus \overline{C^*})] \oplus [F(k-2,1)]$ is an ε-chain on M. Clearly, each member of G is an open set of diameter less than ε. From the preceding, $C(s,m-1) \cap S$ and $F(r-1,k) \cap T$ are chains. Also, d_m, $(f_{k-1} \cap T) \cup (f_{k-1} \setminus \overline{C^*})$, and $F(k-2,1)$ are each chains. Thus, it remains to satisfy the remaining conditions of proposition 2.9.

Since $(d_{m-1} \cap d_m) \cap \overline{N}_x \neq \emptyset$ and $\overline{N}_x \subseteq S$, $(d_{m-1} \cap S) \cap d_m \neq \emptyset$. Then $d_i \cap d_m = \emptyset$ for $1 \leq i \leq m-1$ implies that $[C(s,m-1) \cap S] \oplus [d_m]$ is a chain. Similarly, $\overline{N}_y \cap (f_{r-1} \cap d_m) \neq \emptyset$ and $\overline{N}_y \subseteq T$ implies that $(f_{r-1} \cap T) \cap d_m \neq \emptyset$ and since $F^*(1,r-2) \cap d_m = \emptyset$ and $S \cap T = \emptyset$, $[C(s,m-1) \cap S] \oplus [d_m] \oplus [F(r-1,k) \cap T]$ is a chain.

Now $\emptyset \neq (f_{k-1} \cap f_k) \cap M \subseteq \overline{f}_{k-1} \cap D \subseteq T$. Thus, $(f_k \cap T) \cap (f_{k-1} \cap T) \neq \emptyset$ while the conditions $S \cap T = \emptyset$, $f_{k-1} \cap d_m = \emptyset$, and $f_{k-1} \cap f_i = \emptyset$ for $i > k$ all combine to imply that $f_{k-1} \cap T$ meets no link of $[C(s,m-1) \cap S] \oplus [d_m] \oplus [F(r-1,k) \cap T]$ other than $f_k \cap T$. Since $f_{k-1} \setminus \overline{C^*}$ does not meet $C^*(s,m-1)$, d_m, or $F^*(r-1,k)$, $f_{k-1} \setminus \overline{C^*}$ meets no link of $[C(s,m-1) \cap S] \oplus [d_m] \oplus [F(r-1,k) \cap T]$. Therefore,

$[C(s,m-1)\cap S] \oplus [d_m] \oplus [F(r-1,k)\cap T] \oplus [(f_{k-1}\cap T)\cup(f_{k-1}\setminus\overline{C^*})]$ is a chain.

Finally, since $\overline{F^*(k-2,1)}\cap C^* = \emptyset$ by result 5 of lemma 4.4, no link of $F(k-2,1)$ meets a link of $[C(s,m-1)\cap S] \oplus [d_m] \oplus [F(r-1,k)\cap T]$. Yet, since $(f_{k-1}\setminus\overline{C^*})\cap f_{k-2} \neq \emptyset$, $F(k-2,1)$ does meet $(f_{k-1}\cap T)\cup(f_{k-1}\setminus\overline{C^*})$. Since F is a chain, no link of $F(k-2,1)$ meets f_{k-1} except f_{k-2} and therefore, G is a chain.

To see that G covers M, let $x \in D\setminus F^*(1,r-1)$. Then $x \in C^*(s,m-1)$ by definition of D and s. Also, $x \in S$ and is thus contained in $C^*(s,m-1)\cap S$. For $x \in M\cap d_m$, x is still contained in d_m. If $x \in D\cap F^*(1,r-1)$ then either $x \in F^*(r-1,k)$, $x \in f_{k-1} \subseteq \overline{f}_{k-1}$, or $x \in F^*(k-2,1)$. Now $x \in F^*(r-1,k)\cap D$ implies also that $x \in T$ and is thus in $F^*(r-1,k)\cap T$. For $x \in f_{k-1}$, $x \in T$ by the definition of T and hence $x \in f_{k-1}\cap T$. Clearly, $x \in M\setminus C^*$ implies that $x \in F^*(k-1,1)$ and thus G covers M. Therefore, G is an ε-chain covering M and M is ε-chainable. ||

5.2 <u>Corollary</u> If M is an atriodic and hereditarily unicoherent continuum which is the union of two chainable continua A and B, then M is chainable.

Proof: The proof is immediate from the preceding theorem since A is ε-chainable for all $\varepsilon > 0$. ||

As was noted earlier, the following lemma is presented in order to remove the requirement that M be hereditarily unicoherent. This is possible but at the expense of having to require that both A and B be chainable. It is however the first attempt to prove chainability without first requiring that a continuum be atriodic and hereditarily unicoherent. Corollary 5.6 will make a similar effort concerning the atriodicity of M but will necessarily require that M be unicoherent and decomposable with each proper subcontinuum being chainable.

5.3 <u>Lemma</u> If the atriodic, unicoherent continuum M is the union of two proper subcontinua A and B, then A and B are unicoherent, and if N is a non-unicoherent subcontinuum of M intersecting A then $N \subseteq A$ [23, p. 180].

Proof: Suppose to the contrary that B is the union of two subcontinua H and K such that the point sets U and V form a separation of $H \cap K$. Then U and V are mutually exclusive closed sets and clearly H and K must be proper subcontinua. Since M is by hypothesis nondegenerate, not a triod, and unicoherent, M is irreducible between some two points a and b [1.6]. Clearly, these two points must not both belong to either A or B. Thus, without loss of generality, $a \in A \setminus B$ and $b \in B \setminus A$.

Since $B = H \cup K$, we may assume that $b \in H$. If $A \cap H = \emptyset$, then $K \cap A \neq \emptyset$ since $A \cap B \neq \emptyset$. Now $M = (A \cup K) \cup H$ with U and V forming a separation of $(A \cup K) \cap H = K \cap H$, contrary to M being unicoherent. Thus, the supposition $A \cap H = \emptyset$ is false and A and H, having points in common, form the continuum $A \cup H$ of M containing a and b. But M being irreducible from a to b implies that $M = A \cup H$. Since $K \not\subseteq H$, $K \cap A \neq \emptyset$.

Suppose that $b \notin K$. Then $A \cup K$ is a proper subcontinuum of M and hence $(A \cup K) \cap H$ is a continuum by the unicoherence of M. Since U and V are disjoint closed subsets of a normal space, there exist two open sets G_U and G_V containing U and V respectively, such that $\overline{G}_U \cap \overline{G}_V = \emptyset$. Let p and q denote points of U and V respectively and let C_p and C_q denote the components of p and q in $K \cap G_U$ and $K \cap G_V$ respectively. Now $K \cap G_U$ is open relative to K and is a proper subset of K since $q \in K \setminus G_U$. Since $\overline{K \cap C_p}$ is compact, the boundary, with respect to K, of $K \cap G_U$ contains a limit point x of C_p [26, p. 18]. Then $x \in M$ and in fact $x \in K$. If $x \in H$ then $x \in U$ or $x \in V$ since $x \in H \cap K$. However, $\overline{G}_U \cap \overline{G}_V = \emptyset$ and thus $x \notin V$. But $x \in U$ and $x \in K$ implies that $x \in K \cap U \subseteq K \cap G_U$ which thus implies that $x \notin$ boundary of $K \cap G_U$, relative to K, contrary to the preceding. Therefore, $x \notin H$ and $\overline{C}_p \subseteq K \subseteq M$ and contains at least one point not in H. Similarly, $\overline{C}_q \subseteq K \subseteq M$ and $\overline{C}_q \setminus H \neq \emptyset$. Because $\overline{C}_p \subseteq \overline{G}_U$, $\overline{C}_p \subseteq \overline{G}_V$, and $\overline{G}_U \cap \overline{G}_V = \emptyset$, $\overline{C}_p \cap \overline{C}_q = \emptyset$.

It was shown previously that $(A \cup K) \cap H$ was a subcontinuum of M and clearly it contains $H \cap K$. Thus, $(A \cup K) \cap H$ contains points of both $\overline{C}_p$ and $\overline{C}_q$. Consequently, $S = [(A \cup K) \cap H] \cup \overline{C}_p$ and $T = [(A \cup K) \cap H] \cup \overline{C}_q$ are two subcontinua of M which are proper and are not contained in H since $\overline{C}_p$ and $\overline{C}_q$ each contain distinct points of $M \setminus H$. Since $b \notin K$ by supposition, $\overline{C}_p \subseteq K$, and $\overline{C}_q \subseteq K$, b is not in either of $\overline{C}_p$ or $\overline{C}_q$. With $b \in H$ this implies that H, S, and T are three distinct subcontinua of M no one of which is contained in the union of the other two. Also, $S \cap H$ is the subcontinuum $(A \cup K) \cap H$ since $\overline{C}_p \cap H \subseteq H \cap K \subseteq (A \cup K) \cap H$. Similarly, each of $T \cap H$, $S \cap T$, and $S \cap T \cap H$ is the subcontinuum $(A \cup K) \cap H$. But this implies that $S \cup T \cup H$ is a triod contrary to M being atriodic. Therefore, the supposition that $b \notin K$ is false and $b \in H \cap K$.

Since A contains a and M is irreducible from a to b,

$\overline{M \setminus A}$ is a proper subcontinuum of M [26, p. 60]. Also, $b \in K$ implies $A \cup K = M$ since $A \cap K \neq \emptyset$ implies $A \cup K$ is a continuum, $\{a, b\} \subseteq A \cup K$, and M is irreducible from a to b. Thus, $M \setminus A = (A \cup H) \setminus A = H \setminus A$ and $M \setminus A = (A \cup K) \setminus A = K \setminus A$. Therefore, $M \setminus A \subseteq (H \setminus A) \cap (K \setminus A) = (H \cap K) \setminus A \subseteq H \cap K$, a closed set and consequently, $\overline{M \setminus A} \subseteq H \cap K$. Since U and V form a separation of $H \cap K$, $\overline{M \setminus A}$ is contained in one of U or V. Hence, neither of H or K is a subset of $\overline{M \setminus A}$.

Since $M = A \cup H$, $M = A \cup K$, and M is unicoherent, $A \cap H$ and $A \cap K$ are continua. Because M is a continuum, properly containing A, $(\overline{M \setminus A}) \cap A$ is nonempty. With $\overline{M \setminus A} \subseteq H \cap K$ from the above, $(\overline{M \setminus A}) \cap A \subseteq H \cap K \cap A$ and the latter set is nonempty. Thus, the three continua $A \cap H$, $A \cap K$, and $\overline{M \setminus A}$ all have at least one point in common. Now, $b \in \overline{M \setminus A}$ and b not contained in either of $A \cap H$ or $A \cap K$ implies that $\overline{M \setminus A}$ is not contained in either of $A \cap H$ or $A \cap K$. Also, since $H \setminus A = K \setminus A = M \setminus A$ and $H \neq K$, neither of $A \cap H$ nor $A \cap K$ contains the other. Finally, since $\overline{M \setminus A}$ contains neither of H nor K, $\overline{M \setminus A}$ does not contain either of $A \cap H$ and $A \cap K$. Therefore, the three continua $A \cap H$, $A \cap K$, and $\overline{M \setminus A}$ have a common point and no one is a subset of the union of the other two. Theorem 1.5 implies that M contains a triod, contrary to its being atriodic. Therefore, the continuum B, and similarly A, is unicoherent.

Suppose that N is a non-unicoherent subcontinuum of M intersecting A and that N is not contained in A. If A is not contained in N then the contrapositive of the first part of this lemma implies that $A \cup N$ is not unicoherent. If A is contained in N then $A \cup N = N$ and $A \cup N$ is not unicoherent in either case. Since

M is unicoherent, $A \cup N$ is a proper subset of M. Thus, $M = (A \cup N) \cup K$, where each of $A \cup N$ and K is a proper subcontinuum of M and $A \cup N$ is not unicoherent. But this is contrary to the first part of this lemma. Therefore, $N \subseteq A$. ||

Theorem 5.4 and its first corollary utilize the preceding lemma in combination with corollary 5.2 to produce characterizations of atriodic and unicoherent continua.

5.4 Theorem Suppose that each of A and B is a chainable continuum and $A \cap B \neq \emptyset$. Then $A \cup B$ is chainable if and only if $A \cup B$ is atriodic and unicoherent.

Proof: Theorem 5.1 clearly implies that $A \cup B$ is atriodic and unicoherent when chainable. Thus, suppose that $A \cup B$ is atriodic and unicoherent. If there exists a subcontinuum N of $A \cup B$ such that N is not unicoherent, then without loss of generality, $A \cap N \neq \emptyset$. Lemma 5.3 implies that $N \subseteq A$. But A being chainable implies it is hereditarily unicoherent. Therefore, N does not exist and $A \cup B$ is hereditarily unicoherent. Theorem 5.1 now implies that $A \cup B$ is chainable. ||

5.5 Corollary Let $A_1, A_2, \ldots, A_n$ be a finite collection of chainable continua such that $A_i \cap A_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$. Then $M = \bigcup_{i=1}^{n} A_i$ is chainable if and only if M is atriodic and unicoherent.

Proof: The proof follows immediately from theorem 5.4. ||

Corollary 5.6, the last result of this section, presents a sufficient condition for a decomposable unicoherent continuum to be chainable. The proof need only show that M is not a triod since each proper subcontinuum is chainable and hence atriodic.

5.6 Corollary Suppose that M is a decomposable unicoherent continuum and each proper subcontinuum of M is chainable. Then M is chainable.

Proof: Because of theorem 5.4, it sufficies to show that M is atriodic. Suppose to the contrary. The hypothesis forces M to be a triod. Then there are three proper subcontinua A, B, and C of M such that $M = A \cup B \cup C$ and $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ is a proper subcontinuum of each of A, B, and C. Then $A \cup B$ is a proper subcontinuum of M. Let $p \in A \cap B$ and $q \in C \setminus (A \cup B)$. Then $p \in C$ and there exists a subcontinuum D of C such that D is irreducible from p to q. Since $q \notin A \cup B$, a closed set, let G denote an open set containing q such that $G \cap (A \cup B) = \emptyset$.

Since the composant of D determined by p is dense in D and $q \in G \cap D$, there exists a proper subcontinuum E of D such that $E \cap G \neq \emptyset$ and consequently, $E \cap [C \setminus (A \cup B)] \neq \emptyset$. Then the three subcontinua $E \cup (A \cap B)$, A, and B form the triod $E \cup A \cup B$ which is properly contained in M. But since $E \cup A \cup B$ is chainable by hypothesis, this is impossible and M is not a triod. Because every proper subcontinuum of M is chainable, M is in fact atriodic and theorem 5.4 implies that M is chainable. ||

Continua With Chainable Indecomposable Subcontinua

This section includes the final proofs to be presented in this thesis. The eventual result is a characterization of chainability in terms of the indecomposable subcontinua of an atriodic and hereditarily unicoherent continuum M. This result is stated as theorem 5.11 and the proof is accomplished only after showing four rather involved lemmas. The reader might find it advantageous to turn first to the theorem, studying the intervening results only when utilized. Their purpose is to further make it possible to isolate a point in the continuum M upon which to base the construction of a chain. Most of the techniques developed earlier will be used to ease the burden of construction. An effort has been made to reference these methods when possible and thus avoid complicating an already involved proof. It will be necessary to first present a definition of what are called inaccessible points of a terminal subcontinuum. This will be followed by several lemmas.

Let M be an atriodic, hereditarily unicoherent continuum and K an indecomposable terminal subcontinuum of M. If there exists a subcontinuum A of M such that $A \cap K \neq \emptyset$, $K \not\subseteq A$, $A \not\subseteq K$, and D is the composant of K containing $K \cap A$, then D is called the accessible composant of K. All other composants are inaccessible. If no such subcontinuum A exists, then all composants are inaccessible. In either case, a point of an inaccessible composant is an inaccessible point of K [14, p. 386].

5.7 Lemma If M is an atriodic, hereditarily unicoherent continuum, K is an indecomposable terminal subcontinuum of M, w is an inaccessible point of K, and N is a subcontinuum of M containing w, then either $N \subseteq K$ or $K \subseteq N$.

Proof: Since w is an inaccessible point of K, w is contained in an inaccessible composant B of K. Now, $N \cap B \neq \emptyset$, they both contain w, as does $N \cap K$. If there is an accessible composant D of K, then by definition, there exists a subcontinuum A of M such that $A \cap K \neq \emptyset$, $A \not\subseteq K$, $K \not\subseteq A$, and D is the composant of K containing $A \cap K$. Clearly, $D \cap B = \emptyset$ [26, p. 57]. If $N \not\subseteq K$ and $K \not\subseteq N$ then lemma 4.8 implies that $N \cap K \subseteq D$. But this implies that $w \in D$ which is contrary to their being disjoint. Thus, either $K \subseteq N$ or $N \subseteq K$.

If all composants are inaccessible, then the definition implies immediately that either $N \subseteq K$ or $K \subseteq N$. ||

Lemma 5.8 continues in the vein of earlier results. The effort here is to show the existence of a chain on M which contains an inaccessible point w of the terminal subcontinuum K in its last link. The proof is accomplished, as have several others, by first showing the existence of an irreducible subcontinuum of M which does not have a defined property P. This proof is very similar to the approach taken for proposition 3.18 and parts of theorem 5.1.

5.8 <u>Lemma</u> Suppose that M is an atriodic, hereditarily unicoherent continuum, K is a nondegenerate terminal subcontinuum of M, $C = \{d_1, d_2, \cdots, d_m\}$ is a chain on M, $F = \{f_1, f_2, \cdots, f_n\}$ is a chain on K which is a refinement of C, and w is an inaccessible point of K contained in $f_n \cap K$. Then there is a chain $H = \{h_1, h_2, \cdots, h_r\}$ on M such that H is a refinement of C and $w \in h_r$.

Proof: Assume the lemma fails. If $B \subseteq M$, then we shall say that B has property P if and only if B is a subcontinuum of M containing K and no chain on B is a refinement of C and contains w in an end link. The continuum M itself has property P by assumption. Suppose there is a sequence $N_1, N_2, \cdots$ such that for each i,

N_i has property P and $N_{i+1} \subseteq N_i$. If $N_0 = \bigcap_{i=1}^{\infty} N_i$ does not have property P then there is a chain $G = \{g_1, g_2, \cdots, g_t\}$ which is a chain on N_0, is a refinement of C, and $w \in g_t$. Since G^* is an open set containing N_0, the compactness of M implies there is an integer j such that $N_j \subseteq G^*$. But now G is a chain on N_j, G is a refinement of C, and $w \in g_t$. Since this is contrary to the existence of N_j, N_0 has property P and property P is inductive. Since M has property P while K does not, the Brouwer Reduction theorem implies the existence of a subcontinuum M_1 of M such that M_1 is irreducible with respect to having property P. Let $C(i_1, m_1)$ denote the minimal subchain of C on M_1.

Since K is a terminal subcontinuum of M, K will also be a terminal subcontinuum of M_1. Since M_1 properly contains K, $M_1 \cup K$ is nondegenerate and hence proposition 3.15 implies the existence of two points p and q such that $M_1 \cup K = M_1$ is irreducible from p to q and $p \in K$. It necessarily follows that $q \in M_1 \setminus K$. As was argued in proposition 3.18, the fact that the composant of M_1 determined by p is dense in M_1, implies the existence of a proper subcontinuum N of M_1 such that $K \subseteq N$ and N meets each link of $C(i_1, m_1)$. Since N is properly contained in M_1 and M_1 is irreducible with respect to having property P, N does not have property P. Thus, there is a chain $G = \{g_1, g_2, \cdots, g_r\}$ on N such that G is a refinement of C and $w \in g_r$. We may assume that no chain with fewer links than G will cover N and have these properties.

We shall now show that $(g_1 \cup g_r) \cap N \cap (d_{i_1} \cup d_{m_1}) \neq \emptyset$. If $N = K$, this is certainly true and the following may be omitted.

Suppose that K is properly contained in N and that $(g_1 \cup g_r) \cap N \cap (d_{i_1} \cup d_{m_1}) \neq \emptyset$. Then there is a link d_j of $C(i_1,m_1)$ with $i_1+1 \leq j \leq m_1-1$ such that $g_1 \cap N \subseteq d_j$. Since $G^* \cap N$ meets both d_{i_1} and d_{m_1}, let g_s be the first link of G which meets $N \cap (d_{i_1} \cup d_{m_1})$. Clearly, $1 < s$. In order to be specific, let us suppose that $g_s \cap N \cap d_{i_1} \neq \emptyset$. Let g_t denote the first link of G which meets $N \cap d_{m_1}$. Then clearly, $t < r$. Since $G(s,t)$ is a refinement of $C(i_1,m_1)$ which meets d_{i_1} and d_{m_1}, some link of $G(s,t)$ is contained in d_j. Thus, d_j contains a link of $G(1,t)$ distinct from g_1.

Since g_t is the first link of G which meets $d_{m_1} \cap N$, $g_t \subseteq d_{m_1-1}$. Now there is a link d_{i_2} of $C(i_1,m_1)$ such that $G^*(1,t) \subseteq C^*(i_2,m_1-1)$ and $G^*(1,t)$ is not contained in any proper subchain of $C(i_2,m_1-1)$. Because $G^*(1,t)$ meets both d_{i_1} and d_{m_1}, we must have that $i_1 \leq i_2 \leq i_1+1 \leq j \leq m_1-1$. The objective is now to show that $[G^*(1,t-1) \cap C(i_2,m_1-1)] \oplus [G^*(1,t) \cap d_{m_1-1}] \oplus G(t+1,r)$ is a chain covering N, is a refinement of C, and has w in its last link. Since G is a refinement of C, if this collection is a chain, it is clearly a refinement of C. Since $t < r$, $G(t+1,r)$ is a non-empty collection containing g_r. Since $w \in g_r$, we again easily have that w is contained in the last link, if the collection is a chain.

Thus, it remains to show that $[G^*(1,t-1) \cap C(i_2,m_1-1)] \oplus [G^*(1,t) \cap d_{m_1-1}] \oplus G(t+1,r)$ is a chain covering N. The definition of the integer i_2 was such that $G^*(1,t) \subseteq C^*(i_2,m_1-1)$. Since g_t is in fact contained in d_{m_1-1} and i_2 was defined such that no proper subchain of $C(i_2,m_1-1)$ contained $G^*(1,t)$, $G^*(1,t-1) \cap d_{i_2} \neq \emptyset$ and hence, $G^*(1,t-1) \cap d_{m_1-1} \neq \emptyset$. Suppose that for some i, $i_2 < i \leq m_1-2$, $G^*(1,t-1) \cap (d_i \cap d_{i+1}) = \emptyset$. Because G is a refinement of C, there is a link, g_k, of G such that $g_k \subseteq C^*(i_2,i)$ and $g_{k+1} \subseteq C^*(i+1,m_1-1)$. The supposition implies that $g_k \subseteq C^*(i_2,i) \setminus d_{i+1}$ and $g_{k+1} \subseteq C^*(i+1,m_1-1) \setminus d_i$ and consequently that $g_k \cap g_{k+1} = \emptyset$ contrary to G being a chain. Thus, $G^*(1,t-1)$ meets the common part of each pair of adjacent links of $C(i_2,m_1-2)$ and proposition 2.8 implies that $G^*(1,t-1) \cap C(i_2,m_1-2)$ is a chain.

The definition of t implies that $g_{t-1} \cap g_t \subseteq d_{m_1-1}$. Hence, $G^*(1,t) \cap d_{m_1-1}$ is a nonempty open set and from the preceding paragraph, $G^*(1,t) \cap (d_{m_1-2} \cap d_{m_1-1}) \neq \emptyset$. Thus, $[G^*(1,t-1) \cap d_{m_1-2}] \cap [G^*(1,t) \cap d_{m_1-1}] \neq \emptyset$. However, because $d_i \cap d_{m_1-1} = \emptyset$ for $i \leq m_1-3$, $[G^*(1,t-1) \cap d_i] \cap [G^*(1,t) \cap d_{m_1-1}] = \emptyset$ and proposition 2.9 implies that $[G^*(1,t-1) \cap C(i_2,m_1-2)] \oplus [G^*(1,t) \cap d_{m_1-1}]$ is a chain.

Now because $g_t \subseteq d_{m_1-1}$, $g_t \cap g_{t+1} \subseteq d_{m_1-1}$ and $[G^*(1,t) \cap d_{m_1-1}] \cap g_{t+1} \neq \emptyset$. But since $g_i \cap G^*(1,t-1) = \emptyset$ for

$i \geq t+1$, and because $g_i \cap G^*(1,t) = \emptyset$ for $i \geq t+2$, the nonadjacent links of $[G^*(1,t-1) \cap C(i_2,m_1-2)] \oplus [G^*(1,t) \cap d_{m_{11}-1}]$ and $g(t+2,r)$ do not intersect. Therefore, proposition 2.9 again implies that $[G^*(1,t-1) \cap C(i_2,m_1-2)] \oplus [G^*(1,t) \cap d_{m_1-1}] \oplus G(t+1,r)$ is a chain. Since this chain clearly covers N, it is a chain covering N, it is a refinement of C, and $w \in g_r$, the last link. However, since each link of $C(i_2,m_1-1)$ contains at least one link of G and $d_r \in C(i_2,m_1-1)$ contains at least two links of G, the constructed chain has fewer links than G, contrary to its selection. Therefore, $(g_1 \cup g_r) \cap N \cap (d_{i_1} \cup d_{m_{11}}) \neq \emptyset$. We shall in fact suppose that $g_1 \cap d_{i_1} \cap N \neq \emptyset$.

Since $w \in g_r$, $w \notin g_1 \cap d_{i_{11}}$. Thus, $(g_1 \cap d_{i_{11}}) \setminus \{w\}$ is an open set which by the preceding contains a point of N. The normality of the space implies the existence of an open set U such that $U \cap N \neq \emptyset$ and $\overline{U} \subseteq [(g_1 \cap d_{i_1}) \setminus \{w\}]$. Let R denote the component of $M \setminus U$, necessarily closed, which contains w. Since each of N and R is a subcontinuum of M_1 intersecting the terminal subcontinuum K, either $R \subseteq K \cup N = N$ or $N \subseteq K \cup R$. We shall show that the latter alternative is impossible.

If $U \cap N \subseteq K$, then since w is an inaccessible point of K, lemma 5.7 implies that $R \subseteq K$ because $U \cap N \subseteq K$ and $R \subseteq \sim U$. Now, $N \subseteq K \cup R$ implies that $N \subseteq K$ which is contrary to the fact that N properly contains K. Thus, $N \not\subseteq K \cup R$. If $U \cap N \not\subseteq K$ then $U \cap N \not\subseteq K \cup R$ since $U \cap N \not\subseteq R$. Hence, again, $N \not\subseteq K \cup R$. Therefore,

$N \subseteq K \cup R$ is impossible and we have that $R \subseteq K \cup N = N$.

Since G is a chain on $N \subseteq M_1$ and is a refinement of C with w contained in the last link, and M_1 has property P, $M_1 \setminus G^* \neq \emptyset$. Thus, $M_1 \setminus G^*$ and $R \subseteq N \subseteq G^*$ are disjoint closed subsets of $M_1 \setminus U$. Since $\overline{U} \subseteq G^*$, $(M_1 \setminus U) \setminus G^* = M_1 \setminus G^*$. Suppose there is a continuum H in $M_1 \setminus U$ which meets both $M_1 \setminus G^*$ and R. Then $H \cup R$ contains a connected subset of M_1 containing w, properly containing R, and contained in $M_1 \setminus U$. But R, being a component of $M_1 \setminus U$, is a maximal connected subset of $M_1 \setminus U$ containing w. Thus, H cannot exist and no continuum in $M_1 \setminus U$ meets both $M_1 \setminus G^*$ and R. Therefore, $M_1 \setminus U$ is the union of two disjoint closed sets, one containing $M_1 \setminus G^*$, and the other containing R [26, p. 15]. Using the normality of the space, there exist two disjoint open sets S and T such that $M_1 \setminus U \subseteq S \cup T$, $M_1 \setminus G^* \subseteq T$, $R \subseteq S$, and $\overline{S} \cap \overline{T} = \emptyset$.

The following argument is essentially the same as that given in case iii of theorem 5.1. Since $M_1 \setminus G^* \neq \emptyset$, and $M_1 \subseteq C^*(i_1, m_1)$, let m_2 denote the least integral value, $i_1 \leq m_2 \leq m_1$, such that $M_1 \setminus G^* \subseteq C^*(i_1, m_2)$. Let $x \in M_1 \setminus G^* \subseteq M_1 \setminus U$ such that $x \in d_{m_2}$. Let C_x denote the component of $M_1 \setminus \overline{U}$ containing x. Since $M_1 \cap \overline{U}$ is a closed proper subset of the continuum M_1, $\overline{M_1 \setminus (M_1 \cap \overline{U})} = \overline{M_1 \setminus \overline{U}}$ is compact, and C_x is a component of $M \setminus \overline{U}$, $M \cap \overline{U}$ contains a limit point of C_x [26, p. 18]. Since $C_x \cap M_1 \setminus G^* \neq \emptyset$ and $M_1 \setminus U \subseteq S \cup T$ with S and T separated and $x \in T$, $C_x \subseteq T$. Then C_x and consequently T meets the common part of each pair of adjacent links of $C(m_2, i_1)$ by proposition 2.3. Therefore, $C(m_2, i_1) \cap T$ is a chain by proposition 2.8.

From the preceding we also have that $(d_{i_1+2} \cap d_{i_1+1}) \cap T \neq \emptyset$ and since $\bar{U} \subseteq d_{i_1}$ and $d_{i_1} \cap d_{i_1+2} = \emptyset$, $(d_{i_1+2} \cap T) \cap [(d_{i_1+1} \cap T) \setminus \bar{U}] \neq \emptyset$, whereas $(d_i \cap T) \cap [(d_{i_1+1} \cap T) \setminus \bar{U}] = \emptyset$ for $i_1+2 < i \leq m_2$ since $d_i \cap d_{i_1+1} = \emptyset$ for $i_1+2 < i \leq m_2$. Thus, $[C(m_2, i_1+2) \cap T] \oplus [(d_{i_1+1} \cap T) \setminus \bar{U}]$ is a chain by proposition 2.9.

Since $\bar{U} \subseteq d_{i_1}$ and by its existence, $T \cap \bar{U} \neq \emptyset$, $d_{i_1} \cap T \neq \emptyset$. Furthermore, the argument of the above implies that $(d_{i_1+1} \cap d_{i_1}) \cap T \neq \emptyset$ and hence $[(d_{i_1+1} \cap T \setminus \bar{U}] \cap (d_{i_1} \cap T) \neq \emptyset$. Again, because $d_{i_1} \cap d_i = \emptyset$ for $i_1+2 \leq i \leq m_2$, proposition 2.9 implies that $[C(m_2, i_1+2) \cap T] \oplus [d_{i_1+1} \cap T) \setminus \bar{U}] \oplus [d_{i_1} \cap T]$ is a chain.

Because $S \cap T = \emptyset$, any links determined by S and T are necessarily disjoint. Thus, $(g_1 \cap S)$ meets no link of $[C(m_2, i_1+2) \cap T] \oplus [(d_{i_1+1} \cap T) \setminus \bar{U}] \oplus [d_{i_1} \cap T]$. However, since $U \subseteq d_{i_1}$, $U \cap [(d_{i_1+1} \cap T) \setminus \bar{U}] = \emptyset$, and $U \cap d_i = \emptyset$ for $i_1+1 < i \leq m_2$, $(g_1 \cap S) \cup U$ meets only the last link, namely, $d_{i_1} \cap T$, of the above chain. Thus proposition 2.9 again implies that $[C(m_2, i_1+2) \cap T] \oplus [(d_{i_1+1} \cap T) \setminus \bar{U}] \oplus [d_{i_1} \cap T] \oplus [(g_1 \cap S) \cup U]$ is a chain.

Finally, since $U \subseteq g_1$ and $g_i \cap g_1 = \emptyset$ for $3 \leq i \leq r$, $[(g_1 \cap S) \cup U] \cap [g_i \cap s] = \emptyset$ for $3 \leq i \leq r$. In a manner similar to a preceding argument, it can be shown that S meets the common part of each pair of adjacent links of $G(1,r)$ by considering the component C_y of $M \setminus \overline{U}$, where $y \in g_r \cap N$. Thus, $G(2,r) \cap S$ is a chain by proposition 2.8 and $(g_2 \cap S) \cap [(g_1 \cap S) \cup U] \neq \emptyset$ while no other link of $G(2,r) \cap S$ meets any link of the previously determined chain. Therefore, $[C(m_2, i_1+2) \cap T] \oplus [(d_{i_1+1} \cap T) \setminus \overline{U}] \oplus [d_{i_1} \cap T] \oplus$

$[(g_1 \cap S) \cup U] \oplus [G(2,r) \cap S]$ is a chain. Since this chain covers M_1, is a refinement of C, and contains w in its last link, $g_r \cap S$, we have constructed a chain on M_1 with the above properties contrary to its having property P. Therefore, the assumption is false and there is a chain H on M such that H is a refinement of C and w is an element of the last link of H. ||

The following lemma shows the existence of a particular chain on a continuum with disjoint opposite terminal subcontinua. Its results and use will be very similar to those of lemmas 4.3 and 4.4. The purpose is to permit the isolation of opposite terminal subcontinua in different parts of a chain. No effort however, is made here to prevent the two terminal subcontinua from meeting the same links of the chain. Each is required to meet an end link of the chain. The proof is obviously one of involved chain construction and effectively uses the normality of the space to separate disjoint closed sets with open sets which, as in the past, will help produce a chain on the continuum.

5.9 <u>Lemma</u> Suppose that M is an atriodic, hereditarily unicoherent continuum, K and L are disjoint opposite terminal subcontinua of M, and $C_0 = \{d_1', d_2', \cdots, d_{m'}'\}$ is an ε-chain on M. Then there is a chain $F = \{f_1, f_2, \cdots, f_n\}$ on M such that

1) F is a refinement of C_0.

2) $f_1 \cap K \neq \emptyset$ and $f_n \cap L \neq \emptyset$.

3) there are positive integers i and k with $K \overset{e}{\subseteq} F^*(1,i)$ and $L \overset{e}{\subseteq} F^*(k,n)$.

Proof: Lemma 4.3 implies the existence of an ε-chain $C = \{d_1, d_2, \cdots, d_m\}$ on M such that C is a refinement of C_0 and $K \overset{e}{\subseteq} C^*(j_1,m)$. Without loss of generality, we may assume that C is a minimal chain on M. Applying this lemma a second time, we obtain an ε-chain $G = \{g_1, g_2, \cdots, g_t\}$ on M such that G is a refinement of C, and hence of C_0, and $L \overset{e}{\subseteq} G^*(j_2,t)$. We may assume that no chain with fewer links than G will cover M, be a refinement of C_0, and have a subchain exactly containing L.

Case i. If $d_1 \cap L \neq \emptyset$ or $g_1 \cap K \neq \emptyset$ then let $F_0 = C(m,1)$ or $F_0 = G$ respectively. Then $F_0 = \{f_1', f_2', \cdots, f_n'\}$ will be a chain on M satisfying results one and two of the lemma. It remains to construct a refinement of F_0 which will also satisfy the third result.

Since K and L are disjoint closed subsets of M and the space is normal, there exist two disjoint open sets Q and R such that $K \subseteq Q$, $L \subseteq R$, and $\overline{Q} \cap \overline{R} = \emptyset$. Let $F_0(1,i)$ denote the minimal subchain of F_0 covering K and $F_0(k,n)$ the minimal subchain of F_0 covering L. Since $F_0^*(1,i)$ and $F_0^*(k,n)$ are open sets containing K and L respectively, we may assume that $\overline{Q} \subseteq F_0^*(1,i)$ and $\overline{R} \subseteq F_0^*(k,n)$. Let $f_j = f_j'$ for $j \neq i+1$ and $j \neq k+1$. If $k-1 = i+1$, then let $f_{i+1} = f_{i+1}' \setminus (\overline{Q} \cup \overline{R})$. Otherwise, let $f_{i+1} = f_{i+1}' \setminus \overline{Q}$ and $f_{k-1} = f_{k-1}' \setminus \overline{R}$. In either case let $F = \{f_1, f_2, \cdots, f_n\}$. We must show that F is a chain.

If $k-1 = i+1$, then $F(1,i)$ and $F(k,n)$, as defined, are each chains, none of whose links meet. Now, $f_i' \cap f_{i+1}' \neq \emptyset$ and $f_i' \cap \overline{R} = \emptyset$ since $\overline{R} \subseteq F_0^*(k,n) = F_0^*(i+2,n)$, and consequently $f_i' \cap [f_{i+1}' \setminus \overline{R}] \neq \emptyset$. If $f_i' \cap [f_{i+1}' \setminus \overline{Q}] \neq \emptyset$, then $f_i \cap f_{i+1} = f_i' \cap [f_{i+1}' \setminus (\overline{R} \cup \overline{Q})] \neq \emptyset$ and $F(1,i) \oplus [f_{i+1}]$ is a chain. Thus, it remains to show that $f_i' \cap [f_{i+1}' \setminus \overline{Q}] \neq \emptyset$. If the contrary is true then $f_i' \cap f_{i+1}' \subseteq \overline{Q}$ and necessarily $M \cap (f_i' \cap f_{i+1}') \subseteq \overline{Q}$. Let $H = M \cap F_0^*(1,i)$ and $K = M \setminus F_0^*(1,i)$. Then K is clearly a closed point set.

Let x be a limit point of H. Then $x \in M$ and suppose that $x \in K$. By definition, $x \notin F_0^*(1,i)$ and hence $x \in F_0^*(i+1,n)$. Also, $x \notin F_0^*(1,i)$ implies that $x \notin \overline{Q}$ and thus $x \in F_0^*(i+1,n) \setminus \overline{Q}$. Since $F_0^*(i+1,n) \setminus \overline{Q}$ is an open set containing x it necessarily meets H. Thus, $[F_0^*(i+1,n) \setminus \overline{Q}] \cap F_0^*(1,i) \neq \emptyset$ contrary to the fact that $F_0^*(i+1,n) \cap F_0^*(1,i) \subseteq \overline{Q}$. Thus H and K form a separation of M contrary to hypothesis and we have that $f_i' \cap (f_{i+1}' \setminus \overline{Q}) \neq \emptyset$. Therefore, $F(1,i) \oplus [f_{i+1}]$ is a chain. Similarly, $[f_{i+1}] \oplus F(i+2,n)$ is a chain and the collection F, as defined when $k-1 = i+1$, is a chain.

If $k-1 \neq i+1$, then without loss of generality, $i+1 < k-1$ and by an argument similar to the preceding we have that $f_{i+1} \cap f_i \neq \emptyset$, $f_{i+1} \cap f_{i+2} \neq \emptyset$, $f_{k-1} \cap f_k \neq \emptyset$, and $f_{k-1} \cap f_{k-2} \neq \emptyset$. Thus, $F(1,i) \oplus [f_{i+1}] \oplus F(i+2,k-2) \oplus [f_{k-1}] \oplus F(k,n)$ is a chain.

In either of the preceding situations, F is a chain, $K \subseteq F^*(1,i)$, K is not contained in any proper subchain of $F(1,i)$, and $\overline{F^*(i+1,n)} \cap K = (\overline{f}_{i+1} \cap K) \cup [(\bigcup_{j=i+2}^{n} \overline{f}_j) \cap K]$. Since $K \subseteq F^*(1,i)$, $\overline{f}_j \cap K = \emptyset$ for $i+2 \leq j \leq n$ and hence $(\bigcup_{j=i+2}^{n} \overline{f}_j) \cap K = \emptyset$. Also, $K \subseteq \overline{Q}$ implies that $\overline{f}_{i+1} \cap K = \overline{(f_{i+1}' \setminus \overline{Q})} \cap K = \emptyset$. Thus, $K \cap \overline{F^*(i+1,n)} = \emptyset$ and $K \subseteq^e F^*(1,i)$. Similarly, $L \subseteq^e F^*(k,n)$. Therefore, $F = \{f_1, f_2, \ldots, f_n\}$ is a chain on M satisfying the three results of the lemma when the conditions of this case are met.

Case ii. The objective of this case and thus the remainder of the proof is to construct a chain $F_0 = \{f_1', f_2', \ldots, f_n'\}$ from the chains $C(1,m)$ and $G(1,t)$, where $d_1 \cap L \neq \emptyset$ and $g_1 \cap K \neq \emptyset$, such that $f_1' \cap L \neq \emptyset$. The first case of this proof will then apply to F_0 to yield the desired chain F.

Since C is a minimal chain on M and G is a refinement of C, there is a link g_r of G contained in d_1 and a link g_s of G contained in d_m. We wish to show in fact that either g_1 or g_t is contained in $d_1 \cup d_m$ and eventually that $g_1 \subseteq d_1$. Suppose that neither g_1 nor g_t is contained in $d_1 \cup d_m$. Then there is a link d_a of C such that $g_1 \subseteq d_a$ with $2 \leq a \leq m-1$. As was argued in lemma 5.8, since $G(r,s)$ is a refinement of C with $g_r \subseteq d_1$ and $g_s \subseteq d_m$, there is a link g_u of $G(r,s)$ such that $g_u \subseteq d_a$. Thus, g_u is a link of $G(1,s)$ distinct from g_1 and both are contained in d_a. Again, the argument of lemma 5.8 produces the chain $G_0 = [G^*(1,s-1) \cap C(1,m-2)] \oplus [G^*(1,s) \cap d_{m-1}] \oplus G(s+1,t)$ on M which is a refinement of C.

The following will show the existence of a refinement G_1 of G_0 which has the same number of links as G_0, is a refinement of C, covers M, and which has a subchain exactly containing L. However, by its definition, G_0 has fewer links than G and thus the existence of G_1 is contrary to the selection of G. Thus, if G_1 can be constructed, as claimed, its existence will produce a contradiction and the assumption that neither g_1 nor g_t is contained in $d_1 \cup d_m$ is false.

Subcase iia. If $j_2 \geq s+1$ then $L \subseteq^e G^*(j_2,t)$ and $G(j_2,t)$ is a subchain of G_0. Thus, the claim is easily established with $G_1 = G_0$.

Subcase iib. If $j_2 \leq s$, then $L \cap g_{j_2} \neq \emptyset$ and hence if $g_{j_2} \subseteq d_{j_0}$, $1 \leq j_0 \leq m$, then $L \cap d_{j_0} \neq \emptyset$. Since $g_s \subseteq d_m$ and proposition 2.2 implies $g_s \cap L \neq \emptyset$, we have that $d_m \cap L \neq \emptyset$ and by the same proposition, $d_j \cap L \neq \emptyset$ for $j_0 \leq j \leq m$. Let $C(b,m)$ denote the minimal subchain of $C(1,m)$ on L. Since $L \cap d_1 = \emptyset$, $2 \leq b \leq m$. It follows that the collection $\{d_b \cap G^*(1,s-1), \cdots, d_{m-2} \cap G^*(1,s-1), d_{m-1} \cap G^*(1,s), g_{s+1}, \cdots, g_t\}$ is a minimal subchain of G_0 containing L. Because $L \subseteq^e G^*(j_2,t)$, the normality of the space implies there is an open set Q such that $L \subseteq Q \subseteq \overline{Q} \subseteq G^*(j_2,t) \cap C^*(b,m)$. Then $\overline{Q} \cap d_j = \emptyset$, and hence $Q \cap \overline{d}_j = \emptyset$, for $1 \leq j \leq b-2$. This implies that $L \cap \overline{d}_j = \emptyset$ for $1 \leq j \leq b-2$. If we redefine the link d_{b-1} to be $d_{b-1} \setminus \overline{Q}$, then the closure of this link will also fail to meet L. Thus, if $1 \leq b-1 \leq m-2$, then $(d_{b-1} \setminus \overline{Q}) \cap G^*(1,s-1)$ replaces $d_{b-1} \cap G^*(1,s-1)$ and if $b-1 = m-1$ then $(d_{b-1} \setminus \overline{Q}) \cap G^*(1,s)$ replaces $d_{b-1} \cap G^*(1,s)$. The result is the desired chain G_1 which is a refinement of G_0, has the same number of links as G_0, and exactly contains L in the subchain $\{d_b \cap G^*(1,s-1), \cdots, d_{m-2} \cap G^*(1,s-1), d_{m-1} \cap G^*(1,s), g_{s+1}, \cdots, g_t\}$. Therefore, with either of the subcases, the chain G_1 exists contrary to the existence of G and we have that either g_1 or g_t is contained in $d_1 \cup d_m$.

As was noted earlier, we shall now show in fact that $g_1 \subseteq d_1$. For suppose not. Then $g_1 \subseteq d_m$ by the preceding. One of the conditions of this case is that $g_1 \cap K = \emptyset$. Since $K \subseteq^e C^*(j_1,m)$, $(d_m \setminus d_{m-1}) \cap K \neq \emptyset$ and thus there is a link g_r of G, $2 \leq r \leq t$, such that $g_r \subseteq d_m$.

Thus, g_1 and g_r are distinct links of G contained in d_m. Let $C(v,m)$ denote the minimal subchain of C such that $G^*(1,r) \subseteq C^*(v,m)$. Arguing in a manner similar to that in the proof of lemma 5.8, $[C(v,m-1) \cap G^*(2,r-1)] \oplus [d_m \cap G^*(1,r)] \oplus G(r+1,t)$ is a chain which covers M, is a refinement of C, and because each link of C contains at least one link of G and d_m contains at least two links of G, this constructed chain has fewer links than G. It remains to be shown that some subchain of this chain exactly contains L. Since this is argued in exactly the same manner as was presented in subcases iia and iib of this proof, we shall omit repeating the argument and note that the constructed chain now contradicts the selection of the original chain G. Thus, the supposition that $g_1 \not\subseteq d_1$ is false and this claim is established.

Since K and L are opposite terminal subcontinua of M there are points p and q with $p \in K$ and $q \in L$ such that M is irreducible between p and q. Now the composant of M determined by q is dense in M [26, p. 58]. Hence, there is a proper subcontinuum N of M such that $q \in N$ and $N \cap g_1 \neq \emptyset$. Since $g_1 \subseteq d_1$ it also follows that $N \cap d_1 \neq \emptyset$. Because N is a proper subcontinuum of M, $p \notin N$ and since $p \notin L$, $N \cup L$ is a proper subcontinuum of M. Let $N_0 = N \cup L$.

Let V be an open set such that $V \cap N_0 \neq \emptyset$ and $\bar{V} \subseteq g_1 \subseteq d_1$. Since $d_1 \cap L = \emptyset$ and $d_1 \cap K = \emptyset$, $V \cap (K \cup L) = \emptyset$. If H is a subcontinuum of $M \setminus V$ such that H meets both K and L then $H \cup K \cup L$ is a proper subcontinuum of M containing p and q contrary to the existence of p and q. Thus, K and L are closed disjoint subsets of $M \setminus V$ and no subcontinuum of $M \setminus V$ meets both K and L.

Therefore, $M \setminus V$ is the union of two closed disjoint sets one containing K and the other containing L [26, p. 15]. The normality of the space now guarantees the existence of two open sets S and T such that $M \setminus V \subseteq S \cup T$, $K \subseteq S$, $L \subseteq T$, and $\overline{S} \cap \overline{T} = \emptyset$.

Let $F_0 = [C(m,3) \cap S] \oplus [(d_2 \cap S) \setminus \overline{V}] \oplus [d_1 \cap S] \oplus [(g_1 \cap T) \cup V] \oplus [G(2,t) \cap T]$. The argument that F_0 is a chain on M is virtually the same as that given in the conclusion of the proof of lemma 5.8 and is thus omitted. Since F_0 is a refinement of C, F_0 refines C_0 as required in result one of the lemma. Also, $f_1' = d_m \cap S$ meets K and $f_n' = g_t \cap T$ meets L. Now the conditions of case i are met and the existence of the chain F is assured. This case completes the proof of this lemma. ||

One last lemma is needed before we attempt the proof of theorem 5.11. The complexity of just the statement of the lemma indicates that it is intended for specific use in theorem 5.11 to help ease the effort there. Perhaps an illustration will help to understand the objectives of the lemma. The continuum M is the union of subcontinua A, B, and N with $A \cap N$ and $B \cap N$ being opposite terminal subcontinua of

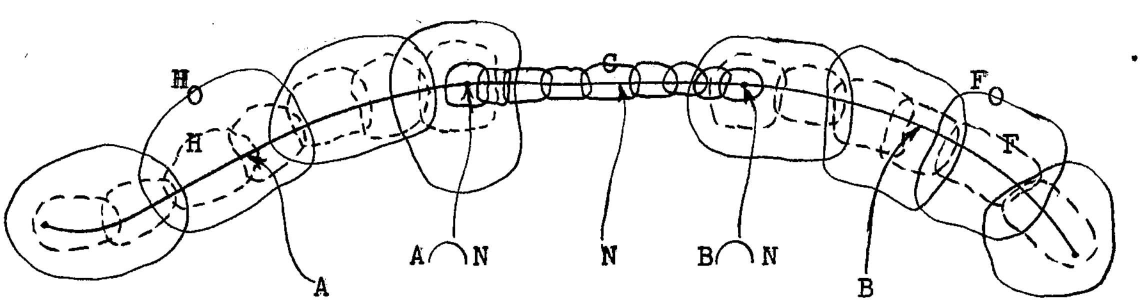

Figure 9. Illustration for Lemma 5.11

N. These terminal subcontinua are in turn exactly contained in subchains of chains on A and B. We wish to produce chains on N, A, and B which will appropriately contain $A \cap N$ and $B \cap N$, which will be refinements of the chains on A and B, and which will also not have too many links in common. This is obviously an over simplification of the problem involved.

5.10 Lemma Let M be a continuum such that M is the union of three distinct proper subcontinua A, B, and N with $\overline{M \setminus N} = A \cup B$, $A \cap B = \emptyset$, $A \setminus N \neq \emptyset$, $B \setminus N \neq \emptyset$, and $A \cap N$ and $B \cap N$ are opposite terminal subcontinua of N. Suppose that N is chainable while $H_0 = \{h_1', h_2', \cdots, h_{r_0}'\}$ and $F_0 = \{f_1', f_2', \cdots, f_{s_0}'\}$ are taut ε-chains on A and B respectively with $A \cap N \subseteq^e H_0^*(t_0, r_0)$ and $B \cap N \subseteq^e F_0^*(u_0, s_0)$. Then, there are taut ε-chains $H = \{h_1, h_2, \cdots, h_r\}$ and $F = \{f_1, f_2, \cdots, f_s\}$ on A and B respectively with $A \cap N \subseteq^e H^*(t, r)$ and $B \cap N \subseteq^e F^*(u, s)$, a taut ε-chain $C = \{d_1, d_2, \cdots, d_m\}$ on N, and positive integers i and k such that $1 \leq i < k-2 < k \leq m$, $A \cap N \subseteq^e C^*(1, i)$, $B \cap N \subseteq^e C^*(k, m)$, $C(1, i)$ and $C(k, m)$ are closed refinements of $H(t, r)$ and $F(u, s)$ respectively, and $\overline{C^*} \cap [\overline{H^*(1, t-1)} \cup \overline{F^*(1, u-1)}] = \emptyset$.

Proof: Since $A \cap N$ and $B \cap N$ must be nonempty or a separation of M would exist, proposition 4.5 implies that $A \cap N$ and $B \cap N$ are also terminal subcontinua of A and B respectively. Since $A \setminus (A \cap N) = A \setminus N$ and $N \setminus (A \cap N) = N \setminus A$ are two separates sets and because the space is completely normal, there are disjoint open sets O_{A_1} and O_{N_1} containing $A \setminus N$ and $N \setminus A$ respectively. Similarly, there are disjoint open sets O_{B_1} and O_{N_2} containing $B \setminus N$ and $N \setminus B$ respectively. Finally, since A and B are disjoint closed sets the normality of the space implies the existence of two open sets O_{A_2} and O_{B_2} containing A and B respectively such that $\overline{O}_{A_2} \cap \overline{O}_{B_2} = \emptyset$. Then $O_A = O_{A_1} \cap O_{A_2}$, $O_B = O_{B_1} \cap O_{B_2}$, and $O_N = O_{N_1} \cap O_{N_2}$ are pair-wise disjoint open sets containing $A \setminus (N \cup B)$, $B \setminus (N \cup A)$, and $N \setminus (A \cup B)$ respectively. Also, $\overline{O}_A \cap \overline{O}_B = \emptyset$.

Since H_0 is a chain on A with $A \cap N \subseteq^e H_0^*(t_0, r_0)$, $(A \setminus N) \cap (h_i' \cap h_{i+1}') \neq \emptyset$ for $1 \leq i \leq t_0 - 1$. Hence, $A \setminus N \subseteq O_A$ implies that O_A meets the common part of each pair of adjacent links of $H_0(1, t_0)$ and by proposition 2.8, $H_0(1, t_0) \cap O_A$ is a chain. Now, $(h_{t_0-1}' \cap h_{t_0}') \cap A \neq \emptyset$ and this intersection must be contained in $A \setminus N$ by the definition of exact containment. Thus, $(h_{t_0-1}' \cap O_A) \cap h_{t_0}' \neq \emptyset$ while d_{t_0} meets no other link of $H_0(1,t) \cap O_A$ and $h_{t_0-1}' \cap O_A$ meets no other link of $H(t_0, r_0)$. Therefore, $H_1 = [H_0(1, t_0) \cap O_A] \oplus H_0(t_0, r_0)$ is a chain on A by proposition 2.9. Since $H_0(t_0, r_0)$ is unaltered in H_1, $A \cap N \subseteq^e H_1^*(t_0, r_0)$. Similarly, there is a chain $F_1 = [F_0(1, u_0) \cap O_B] \oplus F_0(u_0, s_0)$ on B such that $N \cap B \subseteq^e F_1^*(u_0, s_0)$. Since H_1 and F_1 are derived from the taut ε-chains H_0 and F_0, H_1 and F_1 are also taut ε-chains.

Lemma 2.11 and proposition 4.3 combine to guarantee the existence of a closed refinement $H_2(1, r)$ of H_1 such that for some positive integer t, $A \cap N \subseteq^e H_2^*(t, r)$. Then as before, $H' = [H_2(1, t-1) \cap O_A] \oplus$

$H_2(t,r)$ is a taut ε-chain on A which will also be a closed refinement of H_1 and $A \cap N \subseteq^e H^*(t,r)$. In a similar manner, a taut ε-chain F can be produced on B which is a closed refinement of F_1 and such that $B \cap N \subseteq^e F^*(u,s)$, for some s. As a result of their construction $\overline{H^*(1,t-1)} \subseteq O_A$ and $\overline{F^*(1,u-1)} \subseteq O_B$ which implies that $\overline{H^*(1,t-1)} \cap \overline{F^*(1,u-1)} = \emptyset$ and neither $\overline{H^*(1,t-1)}$ nor $\overline{F^*(1,u-1)}$ meets O_N.

Since $N \setminus (A \cup B)$ is a nonempty subset of M which is open relative to M, there is an open set Q_0 such that $Q_0 \cap M = Q_0 \cap N = N \setminus (A \cup B)$. Let $Q = Q_0 \cap O_N$ which implies that Q is an open set such that $Q \cap O_A = \emptyset$, $Q \cap O_B = \emptyset$, $Q \cap M = Q \cap N = N \setminus (A \cup B)$. Since $Q \cap H^*(t,r) = \emptyset$ or $Q \cap F^*(u,s) = \emptyset$ would imply a separation of N, these sets must have nonempty intersections. Thus, $U = \{h_t, h_{t+1}, \cdots, h_r, Q, f_u, f_{u+1}, \cdots, f_s\}$ is a finite open cover of N. Let δ be a Lebesque number for this cover. Let ε_0 be a positive number less than the minimum of ε, δ, and $\frac{1}{4}\rho(A,B)$. Since N is chainable, let $C_0 = \{d'_1, d'_2, \cdots, d'_{m'}\}$ denote an ε_0-chain on N.

If d'_i is a link of C_0 such that $d'_i \cap (A \cap N) \neq \emptyset$, then d'_i is contained in some link of $H(t,r)$ since $\varepsilon_0 < \delta$ and neither Q nor any link of F meets A. Similarly, if $d'_j \cap (B \cap N) \neq \emptyset$, then d'_j is contained in some link of $F(u,s)$. Also, every link d'_i of C_0 is contained in some element of U. Now lemmas 2.11 and 5.9 combined again imply the existence of a taut chain $C = \{d_1, d_2, \cdots, d_m\}$ on N such that C is a closed refinement of C_0, $d_1 \cap (A \cap N) \neq \emptyset$, $d_m \cap (B \cap N) \neq \emptyset$, and for some integers i and k, $A \cap N \subseteq^e C^*(1,i)$ and $B \cap N \subseteq^e C^*(k,m)$. Since C is a refinement of C_0, the definition of ε_0 implies that C is an ε-chain and that

$1 \leq i < k-2 < k \leq m$. Also, $C(1,i)$ is a closed refinement of $H(t,r)$ and $C(k,m)$ is a closed refinement of $F(u,s)$. Because $\overline{C^*(1,m)}$ is contained in $C_0^*(1,m_0) \subseteq U^*$, $\overline{C^*(1,m)} \cap [\overline{H^*(1,t-1)} \cup \overline{F^*(1,u-1)}] = \emptyset$. This completes the proof of the lemma. ||

The final characterization of chainable continua is presented as theorem 5.11. Although theorem 5.12 follows, it does so easily after the effort expended to produce the following. Many previous techniques are again effectively employed and much detail is omitted by referencing similar efforts in other results. There are two basic cases which are further classified for ease of presentation. The two main cases are derived by considering the existence of an indecomposable subcontinuum whose interior relative to M is either nonempty or empty.

5.11 <u>Theorem</u> A continuum M_0 is chainable if and only if M_0 is atriodic, hereditarily unicoherent, and each indecomposable subcontinuum of M_0 is chainable.

Proof: Propositions 2.6 and 2.7 readily imply that the three conditions are necessary for M_0 to be chainable. Thus, suppose that M_0 is an atriodic, hereditarily unicoherent continuum, each indecomposable subcontinuum of which is chainable.

If M_0 fails to be chainable, then there exists an $\varepsilon > 0$ such that no ε-chain covers M_0. The proof of theorem 3.21 implies the existence of a nondegenerate subcontinuum M of M_0 such that M is not ε-chainable but every proper subcontinuum of M is ε-chainable. That is, M is irreducible with respect to the property of not being ε-chainable. Since each indecomposable subcontinuum of M_0 is chainable, M is decomposable.

Case i. Suppose there is an indecomposable subcontinuum N of M such that the interior of N, relative to M, is nonempty. In terms of previous notation, $i_M(N) \neq \emptyset$.

Subcase ia. Suppose N is a terminal subcontinuum of M. Proposition 4.10 implies that $\overline{M \setminus N}$ is a subcontinuum of M and because $i_M(N) \neq \emptyset$, $\overline{M \setminus N}$ is a proper subcontinuum of M. Thus, $M = N \cup (\overline{M \setminus N})$ where N is an indecomposable subcontinuum of M and $\overline{M \setminus N}$ is a proper subcontinuum of M. By hypothesis, N is chainable. Since M is irreducible with respect to the property of not being ε-chainable, $\overline{M \setminus N}$ is ε-chainable. Therefore, theorem 5.1 implies that M is ε-chainable and this contradiction establishes the theorem for subcase ia.

Subcase ib. Suppose N is not a terminal subcontinuum of M. Then $\overline{M \setminus N}$ is not connected since to assume that $\overline{M \setminus N}$ is connected implies that M is the union of two proper subcontinua and proposition 4.5 would imply that N is a terminal subcontinuum. Since this is contrary to the situation in subcase ib, $\overline{M \setminus N}$ is not connected.

Thus, $M \setminus N = A \cup B$ with A and B being the two components of $M \setminus N$ [3.2]. Now, $\overline{M \setminus N} = \overline{A \cup B} = \overline{A} \cup \overline{B}$ with $\overline{A}$ and $\overline{B}$ being disjoint proper subcontinua of M. We also note that the sets $\overline{A} \setminus N$ and $\overline{B} \setminus N$ are nonempty. The existence of M thus implies that $\overline{A}$ and $\overline{B}$ are ε-chainable. Applying proposition 4.5 again, $\overline{A} \cap N$, which must be nonempty or otherwise $\overline{A}$ and $N \cup \overline{B}$ would form a separation of M, is a terminal subcontinuum of $\overline{A}$ and N. Likewise, $\overline{B} \cap N$ is a terminal subcontinuum of $\overline{B}$ and N.

Since $i_M(N) \neq \emptyset$, $i_M(N)$ is not contained in either $\overline{A}$ or $\overline{B}$ and consequently $N \cap \overline{A}$ and $N \cap \overline{B}$ are proper terminal subcontinua of N. Suppose that N_0 is a proper subcontinuum of N which meets both $\overline{A}$ and $\overline{B}$. If $N \setminus N_0 \subseteq \overline{A} \cup \overline{B}$ and, without loss of generality, $(N \setminus N_0) \cap \overline{A} \neq \emptyset$, then $N = (\overline{A} \cap N) \cup [N_0 \cup (\overline{B} \cap N)]$. Since $\overline{A} \cap N$ and $N_0 \cup (\overline{B} \cap N)$ are continua, this contradicts the indecomposability of N and N_0 cannot exist. If $N \setminus (N_0 \cup \overline{A} \cup \overline{B}) \neq \emptyset$, let $K = (\overline{A} \cap N) \cup N_0 \cup (\overline{B} \cap N)$, a proper subcontinuum of M. Then $\overline{A} \cup K$, $\overline{B} \cup K$, and N are distinct subcontinua of M with $(\overline{A} \cup N_0) \cap (\overline{B} \cup N_0) = (\overline{A} \cup N_0) \cap N = (\overline{B} \cup N_0) \cap N = (\overline{A} \cup N_0) \cap (\overline{B} \cup N_0) \cap N = N_0$. Thus, the union of these three continua form a triod in M contrary to its atriodicity and again N_0 cannot exist. Since no proper subcontinuum of N meets both $\overline{A}$ and $\overline{B}$, N is irreducible from $\overline{A} \cap N$ to $\overline{B} \cap N$. Therefore, $\overline{A} \cap N$ and $\overline{B} \cap N$ are opposite terminal subcontinua of N.

Since $\overline{A}$ and $\overline{B}$ are ε-chainable, let $H_0 = \{h_1', h_2', \cdots, h_{r_0}'\}$ and $F_0 = \{f_1', f_2', \cdots, f_{s_0}'\}$ denote taut ε-chains on $\overline{A}$ and $\overline{B}$ respectively. With proposition 4.3 we may assume that $\overline{A} \cap N \subseteq^e H_0^*(t_0, r_0)$ and $\overline{B} \cap N \subseteq^e F_0^*(u_0, s_0)$ for $1 \leq t_0 \leq r_0$ and $1 \leq u_0 \leq s_0$. Lemma 5.10 now implies the existence of taut ε-chains $H = \{h_1, h_2, \cdots, h_r\}$ and $F = \{f_1, f_2, \cdots, f_s\}$ on $\overline{A}$ and $\overline{B}$ respectively such that $\overline{A} \cap N \subseteq^e H^*(t, r)$ and $\overline{B} \cap N \subseteq^e F^*(u, s)$, for $1 \leq t \leq r$ and $1 \leq u \leq s$. Also, there is a taut ε-chain $C = \{d_1, d_2, \cdots, d_m\}$ on N and positive integers i and k such that $1 \leq i < k-2 < k \leq m$, $\overline{A} \cap N \subseteq^e C^*(1, i)$, $\overline{B} \cap N \subseteq^e C^*(k, m)$, $C(1, i)$ and $C(k, m)$ are closed refinements of $H(1, r)$ and $F(1, s)$ respectively, and $\overline{C^*} \cap [\overline{H^*(1, t-1)} \cup \overline{F^*(1, u-1)}] = \emptyset$. The chains $H(1, r)$ and $C(1, m)$ on $\overline{A}$ and N respectively, now satisfy the first four of the six results of lemma 4.4. The proof of lemma 4.4 however shows that H and

C can be modified slightly to also satisfy the last two results. Thus, without loss of generality, we may assume that H and C satisfy all six results of the lemma.

The construction of theorem 5.1 can now be carried out to produce an ε-chain G_0 on $\bar{A} \cup N$ which is a refinement of H and C such that the links of G_0 covering $\bar{B} \cap N$ are precisely those of $C(k,m)$. The chains G_0 and F on $\bar{A} \cup N$ and $\bar{B}$ respectively, with the common terminal subcontinuum $\bar{B} \cap N$, can be joined in exactly the same manner as H and C by the construction of theorem 5.1, to produce an ε-chain G on M contrary to its existence. This contradiction concludes the proof of subcase ib.

Case ii. Suppose that each indecomposable subcontinuum of M has an empty interior relative to M. With this additional hypothesis, theorem 3.11 and corollary 3.12 show that there is an upper semi-continuous decomposition $\mathfrak{G}$ of M which is an arc with respect to its elements. Let $\pi : M \to [0,1]$ denote the composition of the projection map of M onto the decomposition space of $\mathfrak{G}$ and the homeomorphism of $\mathfrak{G}$ onto $[0,1]$. Let $A = \pi^{-1}[0,\frac{1}{2}]$ and $B = \pi^{-1}[\frac{1}{2},1]$. Since π is monotone [1.4], each of A and B is a proper subcontinuum of M. Thus, A and B are both ε-chainable by the existence of M. Proposition 4.7 and lemma 4.11 now imply that $A \cap (\overline{B \setminus A})$ is a terminal subcontinuum of both A and $\overline{B \setminus A}$ and establish the existence of a subcontinuum K of $A \cap (\overline{B \setminus A})$ such that K is either a terminal point or a nondegenerate indecomposable terminal subcontinuum of both A and $\overline{B \setminus A}$.

Let $C_0 = \{d'_1, d'_2, \ldots, d'_{m'}\}$ and $F_0 = \{f'_1, f'_2, \ldots, f'_{n'}\}$ be ε-chains on A and $\overline{B \setminus A}$ respectively. The objective now is to construct refinements C and F of C_0 and F_0 respectively such that the last link of C meets the last link of F in a point of K. If K is a terminal point of A, as may be the case, then K is also a terminal point of $\overline{B\setminus A}$. In this situation, proposition 4.3 immediately implies the existence of the chains $C(1,m)$ and $F(1,n)$ such that C and F are refinements of C_0 and F_0 respectively, with $K \overset{e}{\subseteq} d_m$ and $K \overset{e}{\subseteq} f_n$. Thus, we may assume that K is a nondegenerate indecomposable subcontinuum and is therefore chainable.

Since K is contained in both A and $\overline{B\setminus A}$, the chains C_0 and F_0 both cover K. Let δ_0 and δ_1 denote Lebesque numbers for the covers C_0 and F_0 respectively. If δ is the minimum of ε, δ_0, and δ_1, then as was shown in lemma 5.10, a δ-chain $G(1,k)$ on K exists which is a refinement of both C_0 and F_0. Since K is a nondegenerate indecomposable continuum and $g_k \cap K \neq \emptyset$, every composant of K contains a point of $g_k \cap K$ [26, p. 58]. Since there are uncountably many composants of K [26, p. 59], and at most two of them are accessible from either A or $\overline{B\setminus A}$, there is a point α in $g_k \cap K$ which is inaccessible from either A or $\overline{B\setminus A}$. Lemma 5.8 implies the existence of a chain $C(1,m)$ on A which is a refinement of C_0 and which contains α in $d_m \cap K$. Similarly there is a chain $F(1,n)$ on $\overline{B\setminus A}$ which refines F_0 with $\alpha \in f_n \cap K$.

Let U denote an open set containing α such that $\overline{U} \subseteq d_m \cap f_n$. We shall show that no continuum in $M\setminus U$ meets both $A\setminus U$ and $(M\setminus U)\setminus C^* = M\setminus C^*$. Suppose that N is such a continuum. Then $N \cap (B\setminus A) \neq \emptyset$ since $N \not\subseteq C^*$. Now assume that $N \cap [A \cap (\overline{B\setminus A})] = \emptyset$. Since $M\setminus[A \cap (\overline{B\setminus A})]$ is the union of the two separated sets $A\setminus(\overline{B\setminus A})$ and $B\setminus A$, and N is a connected set, if N fails to intersect $A \cap (\overline{B \setminus A})$, then it is contained in either $B \setminus A$ or $A \setminus (\overline{B \setminus A})$ [26, p. 11]. But this is impossible since $N \cap (B \setminus A) \neq \emptyset$ and $N \cap A \neq \emptyset$ which implies that $N \cap [A \setminus (\overline{B \setminus A})] \neq \emptyset$. Now, $N \cap (\overline{B \setminus A})$ is a subcontinuum of $\overline{B \setminus A}$ which intersects both $A \cap (\overline{B \setminus A})$ and $B \setminus A$. Thus, $N \cap (\overline{B \setminus A}) \not\subseteq A \cap (\overline{B \setminus A})$ and the second conclusion of lemma 4.11 shows that $A \cap (\overline{B \setminus A})$ must be contained in $N \cap (\overline{B \setminus A})$. However, $\alpha \in K \subseteq A \cap (\overline{B \setminus A})$ and $\alpha \notin N$. This contradiction then implies that no subcontinuum of $M \setminus U$ meets both $A \setminus U$ and $M \setminus C^*$.

It follows that $M \setminus U$ is the union of two disjoint closed sets, one of them containing $A \setminus U$ and the other one containing $M \setminus C^*$ [26, p. 15]. Since the space is normal, there must exist two open sets S and T such that $M \setminus U \subseteq S \cup T$, $A \setminus U \subseteq S$, $M \setminus C^* \subseteq T$, and $\overline{S} \cap \overline{T} = \emptyset$. Arguing in exactly the same manner as that given in lemma 5.8, it can be shown that $[C(1,m-2) \cap S] \oplus [(d_{m-1} \cap S) \setminus \overline{U}] \oplus [d_m \cap S] \oplus [(f_n \cap T) \cup U] \oplus [F(n-1,1) \cap T]$ is an ε-chain on M contrary to its existence. Therefore, the theorem is proven. ||

Theorem 5.11 has approached characterizing a chainable continuum in terms of its atriodicity and hereditary unicoherence. However, characterizations of chainable continua will apparently have to include some restrictions concerning the chainability of indecomposable subcontinua similar to those included in theorem 5.11. The dyadic solenoid, S, is an indecomposable, atriodic, and hereditarily unicoherent continuum. In fact, each proper subcontinuum of S is an arc. However, S cannot be embedded in the plane and thus cannot be chainable by theorem 2.16 [14, p. 383].

The concluding theorem of this thesis significantly extends earlier results which had been restricted to finite unions. It does however, follow only after the considerable effort expended to produce theorem 5.11. The ease and brevity of its proof is extremely pleasant relative to those proofs which have preceded it.

5.12 <u>Theorem</u> Suppose that M is an atriodic, hereditarily unicoherent continuum which is the union of countably many chainable continua. Then M is chainable.

Proof: Suppose that $M_1, M_2, \cdots$ is a sequence of chainable subcontinua of M such that $M = \bigcup_{i=1}^{\infty} M_i$. If N is an indecomposable subcontinuum of M, then $N = \bigcup_{i=1}^{\infty}(N \cap M_i)$. Since, for each i, $N \cap M_i$ is a continuum and no indecomposable subcontinuum is the union of countably many proper subcontinua [26, p. 58 and 59], there must exist an integer k such that $N = N \cap M_k$. Thus, N is a subcontinuum of M_k and is hence chainable. Since each indecomposable subcontinuum of M is therefore chainable, theorem 5.11 implies that M is chainable. ||

CHAPTER VI

SUMMARY

The main objective of this thesis has been to present a complete exposition, by a direct approach, of the characterizations of chainable continua. This approach, of working directly from the definition of chainable continua, does not require the additional background required when developing the subject in a more indirect manner, such as the inverse limit approach, to be mentioned later. After a presentation in Chapter I of the historical background of the subject and an introduction which included the basic definitions, the thesis has presented the complete proofs of all results directly involving sufficient conditions for a continuum to be chainable. While some proofs were, perhaps, far more detailed than some readers might have desired, it is hoped that the several methods of approaching problems of chainability are sufficiently detailed so that they might be added to the readers arsenal of research techniques.

Chapter II presented many of the basic consequences of a continuum being chainable or just ε-chainable, including several of the fundamental techniques of constructing chains on a continuum from given chains. These results were repeatedly applied throughout the following chapters. The fact that all chainable continua are atriodic and hereditarily unicoherent was first presented and proven in this chapter. The significance of this lies in the effort expended to show that, while these conditions alone are not sufficient to guarantee chainability, the early additional restrictions could be considerable reduced.

The first notable efforts to show sufficient conditions for chainability included a restriction to hereditarily decomposable continua. This result, along with several equivalent conditions, occupied our attention in Chapter III. A large number of intermediate results and techniques were required, however, before this objective was finally realized. Included in this development was the general idea of splitting a continuum into two disjoint pieces, chaining on each separately, and then joining the chains together to form a chain on the entire continuum. This process necessitated the introduction and eventual development of the subject of terminal subcontinua. This became the objective of Chapter IV along with the further development of chaining abilities on terminal subcontinua.

Chapter V began with the first of several results attempting to weaken the restriction of hereditarily decomposable continua. Additionally, it was possible to show that the conditions of atriodicity or hereditary unicoherence of the continuum could be partially reduced, if other restrictions were imposed on the continuum or its proper subcontinua. This chapter concluded with the proof that any atriodic and hereditarily unicoherent continuum is chainable if each indecomposable subcontinuum of it is chainable. An example was also stated to show that some restrictions along this line are essential to insure chainability.

Closely allied or continuing to build on the material presented in this thesis are several topics which might prove interesting for future study, other Ed. D. theses, and future research. Among these would be the study of chainable continua as the limit of an inverse limit sequence or system. Some results which are laboriously proven directly are apparently simpler when this approach is taken.

Related to this is the study of the continuous images of chainable continua in general and in particular, the continuous images of the pseudo-arc and the pseudo-circle. Parallel to the development of linearly chainable continua, herein called simply chainable continua, and having many of the same results, is the subject of circularly chainable continua or circle-like continua. Each of these subjects would provide a challenge equivalent to or exceeding that encountered in developing the material to the extent attempted in this thesis.

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APPENDIX

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