CAPITAL BUDGETING: AN EMPIRICAL APPROACH TO A PROBABILISTIC PROBLEM

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PREFACE

Capital budgeting is an exceedingly complex decision-making situation and this is especially true when the associated cash flows are probabilistic. Utility theory has been the backbone of most previous work in the probabilistic case. However, not only is this approach fraught with complexity, but also the validity of utility functions has been questioned. Thus, this research was undertaken to attempt a more practical and simple solution for a particular kind of probabilistic problem without explicitly involving cardinal utility theory.

The dissertation is perhaps the culmination of a student's career and I want to take this opportunity to thank all my teachers, past and present, without whose "developmental" work I would not be writing this today.

Specifically, I would like to acknowledge my gratitude to the late Professor Wilson J. Bentley. His encouragement, guidance and ofttimes most tangible help made it all possible.

Special thanks are due the members of my committee: Dr. James E. Shamblin, the chairman, for his generosity and thoughtfulness throughout the doctoral program; Dr. G. T. Stevens, who has been a most patient and understanding thesis adviser - I am deeply grateful for having had the opportunity of being associated with him; Dr. M. P. Terrell for his consideration in accepting a position on the committee at unavoidably short notice and for his valuable assistance since; Dr. P. L. Claypool for his help with statistical problems in particular and advice in general during my graduate work.

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CHAPTER I

INTRODUCTION

The purpose of this research is to obtain a solution to a probabilistic capital budgeting problem without the explicit use of cardinal utility theory. This basic problem consists of determining the optimum choice from projects (investments) competing for limited resources where project cash flows are probabilistic.

Theoretical solutions for this problem as well as the closely related portfolio selection problem have been obtained through the use of the von Neumann and Morgenstern (15) utility theory. This theory, on the basis of a series of axioms of rational behavior, permits a numerical measure (utility) to be assigned to monetary payoffs which have varying degrees of risk. It accomplishes this by presenting a series of "gambles" to a decision-maker and plotting responses to define his utility function. Through this procedure a preference ordering among alternatives involving risk, for the particular decision-maker, is obtained.

Markowitz (13) uses the von Neumann-Morgenstern utility concept to provide an explanation for the practice of diversification in investment portfolios. Adherence to a simple policy of maximization of expected net present value means the investor will put all his money into what appears to be the "best" security. However, the prudent investor chooses to reduce over-all risk and possibly over-all gain by investing in several securities. Explaining this kind of investor behavior, Markowitz hypothesizes an expectation-variance function of the form:

$$E(U) = \mu - A\sigma^2 \quad (1)$$

where

$E(U)$ = expectation-variance or the expected utility of the return for a particular portfolio.

μ = the mean return for the portfolio.

σ = standard deviation of the return.

A = coefficient of risk aversion.

It can be seen that for a given variance the expected utility is greater for a larger mean. Also, for a given mean, the expected utility decreases as variance increases. Maximization of the expectation-variance function thus leads to an optimum answer for a given coefficient of risk aversion A. Successive repetitions with different coefficients of risk aversion yield a set of such optimums.

A few observations need to be made regarding Markowitz's model. Firstly, in economic terms the model is positive and not normative. In other words, it emphasizes what is, and does not lay any claim to stating what should be. It merely says what a decision-maker does if he has an expectation-variance function of this kind and a particular coefficient of risk aversion A.

Secondly, it explains the logic of diversification of investments. In cases where the expected returns from investments are negatively correlated, the over-all portfolio variance is reduced; and where the expected returns are positively correlated, the over-all variance is increased. Hence, a decision-maker with a large coefficient of risk aversion tends to choose the former, while one with a small coefficient of risk aversion is inclined towards the latter.

Thirdly, the model is designed for portfolio selection and thus a fraction of available funds can be allocated to a security. Therefore, the model as such cannot be directly applied to the attribute (0/1) situation that exists in the capital budgeting problem.[1] However, it is still of considerable consequence since it clearly demonstrates that under conditions of uncertainty maximization of net present value by itself is not sufficient as a criterion for project selection.

Farrar (6) in his doctoral dissertation tests Markowitz's hypothesis using the portfolios of actual mutual funds. He shows that funds can be distinguished in their risk attitudes (different coefficients of risk aversion) on the basis of the variances of the portfolio investments.

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[1]In a capital budgeting problem, a project is either accepted or rejected. Consequently, the decision variables can only have values of zero or one.

He also shows that as long as there is diminishing marginal utility of money, the relationship between the coefficient of risk aversion and the utility function of monetary income is,

$$A = -\frac{U''(\mu)}{2}. \qquad (2)$$

This is, of course, based on an expectation-variance function of the same form as used by Markowitz; namely,

$$E[U(t)] = \mu - A\sigma^2. \qquad (3)$$

However, it needs to be mentioned that Farrar also assumes a utility function of the form

$$U(t) = At - Bt^2 \qquad (4)$$

and proceeds to derive Equation (3) from this by taking expected values. The expectation of Equation (4) does not yield Equation (3). Insead, Equation (3) derives from a utility function of the form

$$U(t) = 1 - e^{-at}. \qquad (5)$$

This error has been noted by footnote in a later edition of the published work. The utility function in Equation (5) is also the basis of Freund's method (8).

Cramer and Smith (3) introduce a further sophistication into the Markowitz and Farrar utility models by including a term for the amount of investment. Their model is of the form:

$$E[U(t)] = \mu - A\sigma^{a}I^{b} \tag{6}$$

where

I = amount of investment in the project.

a = a constant.

b = a constant.

The constants 'a' and 'b' are determined as follows. The utility of money curves are first obtained through direct inquiry. Then appropriate logarithms of the right-hand side terms of Equation (6) are plotted over a range of indifference, that is where $U(t) = 0$. The slopes of these graphs give 'a' and 'b'.

Although these models provide a theoretical solution to the probabilistic capital budgeting problem, they do not give the practitioner a ready answer, largely due to the practical difficulties of establishing a valid utility function. In the first place, it is difficult to persuade decision-makers to participate in such an experiment, and then also to provide them with questions realistic enough to compare with situations they will actually experience. Even when individual utility functions are determined for the few top executives of a firm, there remains the problem of unifying these into a group function representative of company objectives. Swalm (20) raises the question as to how stable these utility functions are over time. These questions concerning cardinal utility theory indicate that it is still in its infancy.

Critics of utility theory also believe that it is normative - that it indicates how decision-makers should behave rather than how they actually behave. In discussing the Savage (17) theory which combines cardinal utility with subjective probability, Raiffa (16, p. 690) says that it is a theory which purports to advise its believers "how he (they) should behave in complicated situations provided he (they) can make choices in a coherent manner in relatively simple, uncomplicated situations." He puts forward the contention that people do not always behave in a manner consistent with maximizing their utility; namely, the theory is not predictive, which is perhaps the most damaging criticism of all from the viewpoint of project selection in capital budgeting.

Other approaches to the problem (without the use of utility theory) are proposed by English (5) and Solomon (19). English presents a varying discount rate model where increasing risk in the more distant future is accounted for by changing the discount rate. Since variable discounting rate functions cannot be readily used, English has developed what he terms an operationally useful one.

$$r(n) = \frac{1}{n} \ln \frac{1}{1 - r_0 n} \tag{7}$$

where

r_0 = the initial rate.

$r(n)$ = the rate at time period n.

The advantage of this model lies in the relatively easy way it compensates for long term risk, though the accuracy of the calculated discount rate as a measure of risk can be debatable. It also fixes a planning horizon N, which is the reciprocal of r_0. Thus $r_0 = 1/N$. The implication is that a long planning horizon yields a low initial discount rate - a result that cannot always be considered reasonable. Solomon's method of varying the discount rate is different, but here again the risk-compensatory rate changes tend to be rather arbitrary.

It can be seen from this review of some of the current literature that neither the utility approach nor the varying discount rate method offer practical solutions to the probabilistic capital budgeting problem. Thus, a solution (or even a good approximation) to this problem without the use of cardinal utility theory would be of great practical value. It is to such an aim that this research is directed.

CHAPTER II

ANALYSIS: THE CHOICE BETWEEN TWO PROJECTS

The probabilistic capital budgeting problem as defined in Chapter I is the optimum choice from projects competing for limited resources where project cash flows follow a probability distribution. More specifically, the problem considered in this dissertation meets the following three conditions:

(1) The net present values for every project are normally distributed.

(2) The net present values for every project are mutually independent.

(3) The budget constraint is based on the expected values of investment.

A solution to this problem can be approached with the hypothesis that given two projects such that,

$$(\text{expected loss})_2 < (\text{expected loss})_1$$

and

$$(\text{expected gain})_2 > (\text{expected gain})_1$$

then a rational decision-maker prefers project 2.

However, before a solution is possible, certain basic concepts must be developed. Thus, the purpose of this chapter is the presentation and explanation of these concepts. One of the fundamental concepts used is the expected loss as proposed by Schlaifer (18) and utilized by Canada (2). A loss, as defined by Canada and also as employed in this dissertation, occurs when a project has a negative net present value.

The Expected Loss (EL)

With the preceding definition of a loss, the expected loss can be mathematically defined as

$$EL = \int_{-\infty}^{0} |NPV| f(NPV)\ d(NPV) \tag{8}$$

for a continuous probability density function of net present value. If the density function, f(NPV), is assumed to be normal, then Equation (8) becomes (see Appendix A):

$$EL = \sigma_{NPV} \cdot G(u). \tag{9}$$

G(u) is the unit normal loss integral defined and evaluated by Schlaifer and

$$u = \mu_{NPV}/\sigma_{NPV} \tag{10}$$

where

μ_{NPV} = the expectation of the net present value distribution,

σ_{NPV} = the standard deviation of the NPV distribution.[1]

As a simple example of the use of Equation (9), consider a project that has a normally distributed net present value with μ = \$2000 and σ = \$1000.[2]

Now, $$u = \frac{\mu}{\sigma} = \frac{2000}{1000} = 2$$

and from the table of unit normal loss integrals (Schlaifer (18), p. 706)

$$G(u) = G(2) = 0.008491.$$

Therefore, by Equation (8), the expected loss is

$$EL = 1000\ (0.008491) = \$8.491.$$

The Expected Gain (EG)

The expected gain is just the converse of expected loss; namely,

$$EG = \int_0^{\infty} |NPV|\ f(NPV)\ d(NPV). \qquad (11)$$

For a normally distributed net present value function, Equation (11) reduces (see Appendix A) to

$$EG = \sigma \cdot G(-u) \qquad (12)$$

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[1]For convenience, the subscript NPV is now dropped so that henceforth $\mu_{NPV} = \mu$ and $\sigma_{NPV} = \sigma$.

[2]Methods of evaluating the mean and variance of the NPV distribution are available in the literature, e. g. Hillier (7).

where

$$G(-u) = u + G(u). \qquad (13)$$

For the example μ = \$2000 and σ = \$1000, the expected gain is calculated as follows:

From Equation (13)

$$G(-u) = 2 + 0.008491$$
$$= 2.008491.$$

Then, applying Equation (12),

$$EG = 1000\ (2.008491)$$
$$= \$2008.491.$$

The Choice Between Two Projects

Given two projects and their net present value distributions, two situations can occur with regard to their expected losses and gains; namely,

A. $EL_2 < EL_1$

$EG_2 > EG_1$

B. $EL_2 < EL_1$

$EG_2 < EG_1$.

In the first situation, the rational decision-maker chooses project 2. In the second situation, the choice is not as obvious and additional criteria are needed before a decision is possible. Which of these two situations occurs can be predicted by considering again the formula for EL and EG.

$$EL = \sigma \cdot G(u)$$

$$EG = \sigma \cdot G(-u)$$

$$= \sigma[u + G(u)]$$

$$= \sigma\left[\frac{\mu}{\sigma} + G(u)\right] \text{ from Equation (10).}$$

Therefore,
$$EG = \mu + EL \qquad (14)^3$$

Thus,
$$EG_1 = \mu_1 + EL_1$$

and

$$EG_2 = \mu_2 + EL_2.$$

Since $EL_2 < EL_1$ (in both situations), it is seen that EG_2 is greater than EG_1 only when the difference of the means is greater than the difference in expected losses; that is, if

$$\mu_2 - \mu_1 > EL_1 - EL_2. \qquad (15)$$

Now, Equation (15) need not always be satisfied as when $\mu_1 = \mu_2$ with the result that situation B occurs. This points out more explicitly the need for additional criteria in order to obtain a solution.

In order to determine these supplementary criteria, the expected loss and expected gain functions are examined

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[3]The mean μ is equal to the difference between the gain and loss expectations and not the sum because EL is always positive; refer to Equation 8 where the absolute value of NPV is used in the calculation of EL.

empirically. A group of projects are constructed by varying μ and σ and the expected loss and expected gain are calculated for each project. In addition, the ratios $\frac{\mu}{\sigma}$, $\frac{EL}{\mu}$, and $\frac{EG}{\mu}$ are also computed. These ratios are designated, respectively, the Worth Ratio, the Loss Ratio, and the Gain Ratio and will be referred to later in this chapter. In Table I, all these results are summarized.

Plots of EL and EG versus the Worth Ratio $\frac{\mu}{\sigma}$ are shown in Figure 1. The curves can be observed to be hyperbolic with the "horizontal" portion extending beyond $\frac{\mu}{\sigma} = 1.9$. This region is called the Low-Risk Zone since EL is very nearly zero throughout the region without any appreciable change. For this reason, it is logical to emphasize EG in any comparison of projects in this region.

From Equation (14), $EG = \mu + EL$.

Since, EL is negligible,

therefore, $EG \approx \mu$.

Thus, in any comparison of projects in the Low-Risk Zone, the emphasis is placed on μ. It is of note that in practice a large group of projects lie in this zone, that is where $\frac{\mu}{\sigma} > 1.9$.

For $\frac{\mu}{\sigma} < 1.9$, the curves rise very steeply and are asymptotic to the vertical EL/EG axis. This region is termed the High-Risk Zone. Since both expected loss and expected gain undergo rapid increases in this area, both are significant and must be considered. Thus, in a

TABLE I

DATA FOR A GROUP OF PROJECTS

Project	μ	σ	$u = \frac{\mu}{\sigma}$	G(u)	EL	EG	$\frac{EL}{\mu}$	$\frac{EG}{\mu}$
1	1000	200	5.00	≈ 0	≈ 0	≈ 1000.00	≈ 0	≈ 1
2	1000	400	2.50	.0020	0.80	1000.80	0.00080	1.00080
3	1000	600	1.67	.0202	12.12	1012.12	0.01212	1.01212
4	1000	800	1.25	.0506	40.48	1040.68	0.04048	1.04048
5	1000	1000	1.00	.0833	83.30	1083.30	0.08330	1.08330
6	1000	2000	0.50	.1978	395.60	1395.60	0.39560	1.39560
7	1000	3000	0.33	.2555	766.50	1766.50	0.76650	1.76650
8	1000	4000	0.25	.2863	1145.20	2145.20	1.14520	2.14520
9	1000	5000	0.20	.3069	1534.50	2534.50	1.53450	2.53450
10	2000	200	10.00	≈ 0	≈ 0	≈ 2000	≈ 0	≈ 1
11	2000	400	5.00	≈ 0	≈ 0	≈ 2000	≈ 0	≈ 1
12	2000	600	3.33	$.0^{3}1135$	0.06	2000.06	.00003	1.00003
13	2000	800	2.50	$.0^{2}2000$	1.60	2001.60	.00080	1.00080
14	2000	1000	2.00	$.0^{2}8500$	8.50	2008.50	.00425	1.00425
15	2000	2000	1.00	.08330	166.60	2166.60	.08330	1.08330
16	2000	3000	0.67	.15300	459.00	2459.00	.22450	1.22950
17	2000	4000	0.50	.19780	791.20	2791.20	.39560	1.39560
18	2000	5000	0.40	.23040	1152.00	3152.00	.57600	1.57600

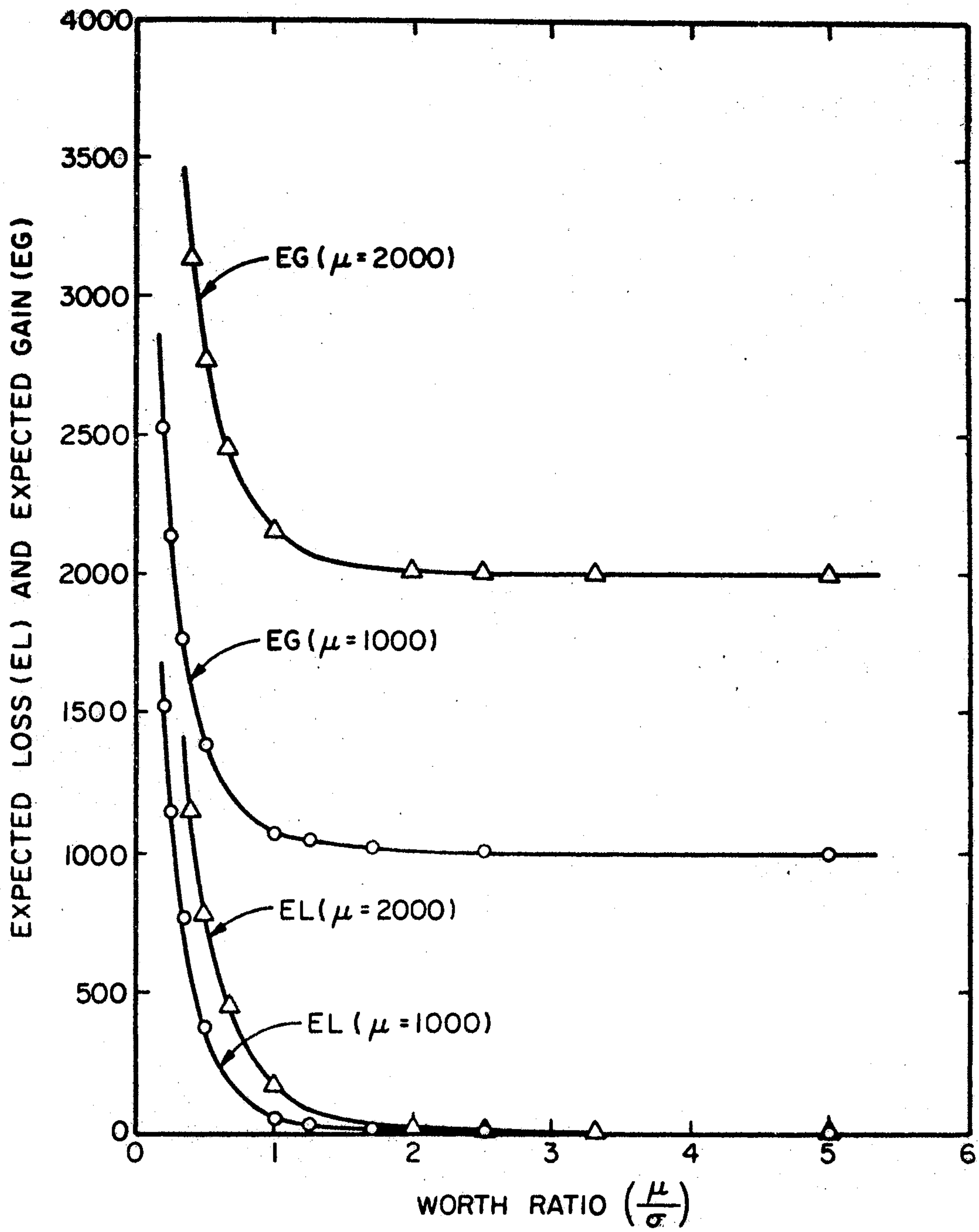

Figure 1. Expected Loss and Expected Gain Curves

comparison of projects, percentage differences between the expected loss of each and between the expected gain of each are evaluated and compared. This can be done according to the following basic premise.

If percentage-wise, the difference between the expected losses is greater, then the project with the lower EL is chosen. Conversely, if the percentage difference between the expected gains is greater, then the project with the larger EG is selected. Expressed as ratios and algebraically, if $EL_1/EL_2 > EG_1/EG_2$, then project 2 is selected. Conversely, if $EG_1/EG_2 > EL_1/EL_2$, then project 1 is preferred. It is to be noted that situation B is being considered; that is, $EL_2 < EL_1$ and $EG_2 < EG_1$. It is implicitly assumed that the above decisions are within the financial capacity of the firm; that is, an adverse project outcome will not result in financial disaster.

Finally, as a result of this premise, in a comparison of projects with one in the high-risk zone and the other in the low-risk zone, the choice must always be the low-risk project. This is because, for comparable projects, the low-risk one must always have a negligible EL. Thus, in any percentage-wise comparisons of the EL and EG differences, the EL percentile must be larger. Hence, the project with the smaller EL, namely the low risk project, is always selected.

A complete selection procedure for comparing two projects can now be summarized as below:

(1) If a situation exists such that $EL_2 < EL_1$ and $EG_2 > EG_1$, then project 2 is selected.

(2) If, however, $EL_2 < EL_1$ and $EG_2 < EG_1$,

then

(i) the projects are examined to see if they lie in the high-risk or low-risk zones.

(ii) If both projects lie in the low-risk zone, then the project with the larger μ is chosen.

(iii) If one project is in the high-risk zone and the other in the low-risk zone, then the low-risk project is selected.

(iv) If both projects lie in the high-risk zone, the percentage-wise changes in EL and EG are examined and an appropriate choice (as explained on the previous page) made.

The comparison of the net present value of projects on a basis of its two parameters μ and σ gives rise to five cases which are shown in Figure 2. All of these are now considered in turn and numerical examples used to illustrate the selection process.

Case I

$$\mu_1 = \mu_2$$

$$\sigma_1 = \sigma_2.$$

This trivial case, included for completeness, yields

$$EL_1 = EL_2$$

and

$$EG_1 = EG_2.$$

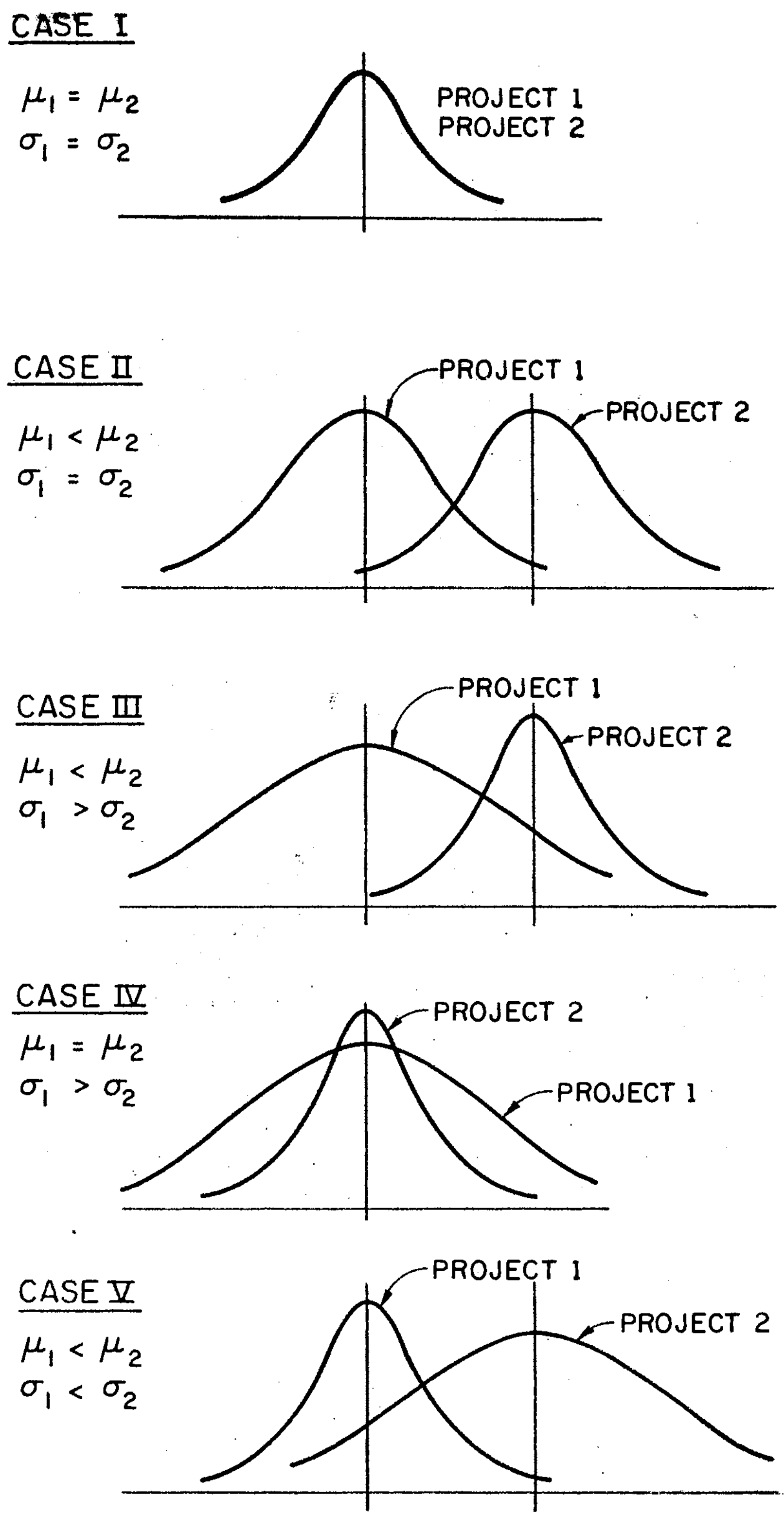

Figure 2. Cases in Project Comparison

The decision-maker is, therefore, indifferent to a choice between the two projects.

Case II

$$\mu_1 < \mu_2$$

$$\sigma_1 = \sigma_2.$$

It is known that $EL = \sigma \cdot G(u)$ (9)

$= \sigma \cdot G(\frac{\mu}{\sigma}).$ (10)

From Appendix A, it can be seen that the function G(u) decreases as u increases. Now, in this particular case, it is always true that,

$$\frac{\mu_1}{\sigma_1} < \frac{\mu_2}{\sigma_2}.$$

Therefore, $G\left(\frac{\mu_1}{\sigma_1}\right) > G\left(\frac{\mu_2}{\sigma_2}\right)$

and, consequently, $\sigma_1 \cdot G\left(\frac{\mu_1}{\sigma_1}\right) > \sigma_2 \cdot G\left(\frac{\mu_2}{\sigma_2}\right)$ (since $\sigma_1 = \sigma_2$).

Thus, it is always true in this case that $EL_2 < EL_1$.

For EG_2 to be greater than EG_1, condition (15) must be met; that is,

$$\mu_2 - \mu_1 > EL_1 - EL_2.$$

In the low-risk region, it is known from Figure 1 that both EL_1 and EL_2 are negligible and, thus, their difference; also, since here $\mu_2 > \mu_1$, the above condition is satisfied. Thus,

$$EL_2 < EL_1$$

and

$$EG_2 > EG_1$$

and project 2 is selected.

If one project is high-risk and the other low-risk, then according to prior discussion the low-risk project (namely 2) is chosen.

If both projects are in the high-risk zone, then percentage-wise changes in EL and EG need to be considered. If percentile increases in EL (EG) are greater than the corresponding percentile changes in EG (EL), then the project with the smaller EL (larger EG) is selected.

Consider the following example:

Project 1	Project 2
$\mu_1 = 1000$	$\mu_2 = 2000$
$\sigma_1 = 800$	$\sigma_2 = 800$
$u_1 = \frac{1000}{800} = 1.25$	$u_2 = \frac{2000}{800} = 2.5.$

From the table of unit normal loss integrals:

$G(u_1) = 0.0509$	$G(u_2) = 0.002004$
$EL_1 = \sigma_1 \cdot G(u_1)$	$EL_2 = \sigma_2 \cdot G(u_2)$
$= 40.472$	$= 1.6032$
$EG_1 = 1040.472$	$EG_2 = 2001.6032.$

Thus,

$$EL_2 < EL_1$$

and

$$EG_2 > EG_1.$$

Hence, a rational decision-maker chooses project 2.

It is noteworthy that in Case II the transition point where the situation $EL_2 > EL_1$ and $EG_2 > EG_1$ occurs is where the worth ratio $(\frac{\mu}{\sigma})$ is of a sufficiently low order to be seldom encountered in practice.

Case III

$$\mu_1 < \mu_2$$

$$\sigma_1 > \sigma_2$$

The analysis in this case is very similar to the previous one. Thus, $\frac{\mu_1}{\sigma_1}$ is always less than $\frac{\mu_2}{\sigma_2}$.

$$\therefore\ G(\frac{\mu_1}{\sigma_1}) \text{ is always greater than } G(\frac{\mu_2}{\sigma_2}).$$

Also, since $\sigma_1 > \sigma_2$

$$EL_1 \text{ is always greater than } EL_2$$

$$\text{or } EL_2 < EL_1.$$

If EG_2 is to be greater than EG_1, then from Equation (15):

$$\mu_2 - \mu_1 > EL_1 - EL_2.$$

In the low-risk region, this condition is met (since $EL_1 \approx EL_2 \approx 0$) and EG_2 is greater than EG_1. Thus,

$$EL_2 < EL_1$$

$$EG_2 > EG_1$$

and project 2 will be selected.

If one project is high-risk and the other is in the low-risk region, then the low-risk project (namely 2) is selected.

If both projects are in the high-risk zone, then the percentage changes in EL and EG need to be considered. If the percentage increase in EG (EL) is greater than the corresponding percentage increase in EL (EG), then the project with the larger EG (smaller EL) is chosen.

Consider the following example:

Project 1	Project 2
$\mu_1 = 1000$	$\mu_2 = 1200$
$\sigma_1 = 1000$	$\sigma_2 = 800$
$u_1 = \frac{1000}{1000} = 1$	$u_2 = \frac{1200}{800} = 1.5$

From the table of unit normal loss integrals:

$G(u_1) = 0.08332$	$G(u_2) = 0.02931$
$EL_1 = 83.32$	$EL_2 = 23.448$
$EG_1 = 1083.32$	$EG_2 = 1223.448$

Thus,

$$EL_2 < EL_1$$

$$EG_2 > EG_1$$

and project 2 is selected.

Again for Case III, it needs to be noted that the transition point, where the situation $EL_2 > EL_1$ and $EG_2 > EG_1$ occurs, is at a very low worth ratio and is seldom encountered in practice.

Case IV

$$\mu_1 = \mu_2$$

$$\sigma_1 > \sigma_2$$

Again, $\frac{\mu_1}{\sigma_1}$ is always less than $\frac{\mu_2}{\sigma_2}$.

$\therefore$ $G(\frac{\mu_1}{\sigma_1})$ is always greater than $G(\frac{\mu_2}{\sigma_2})$

and EL_1 is always greater than EL_2 ($\because$ $\sigma_1 > \sigma_2$).

Thus $$EL_2 < EL_1.$$

Now for EG_2 to be greater than EG,

$$\mu_2 - \mu_1 > EL_1 - EL_2.$$

However, since $\mu_2 = \mu_1$, in this case the above condition can never be met.

Hence EG_1 must always be $> EG_2$.

Thus, the situation that always exists in this case is

$$EL_2 < EL_1$$

$$EG_2 < EG_1.$$

This conflict can be resolved in the following manner.

In the low-risk region, the potential for loss is insignificant since EL_1 and EL_2 are negligible. Also, since $EG = \mu + EL$, the expected gains for the projects are approximately the same. Thus, the rational decision-maker is indifferent as regards choice. However, since the option is available and since there are no obvious advantages in not doing so, it is wise to minimize risk and choose the project with the smaller standard deviation.

In the high-risk region and also when one project is high-risk and other low-risk the loss potential is significant and cannot be ignored. In fact, percentage-wise it is greater than the corresponding potential gain. Thus, the project with the lower EL is chosen. Since $EL_2 < EL_1$, project 2 is preferred.

To illustrate this case, consider the following example where one project is marginally high-risk and the other marginally low-risk:

Project 1	Project 2
$\mu_1 = 1000$	$\mu_2 = 1000$
$\sigma_2 = 800$	$\sigma_2 = 4000$
$u_1 = \frac{1000}{800} = 1.25$	$u_2 = \frac{1000}{400} = 2.5.$

From the table of unit normal loss integrals:

$G(u_1) = 0.05059$	$G(u_2) = 0.002004$
$EL_1 = 40.472$	$EL_2 = 0.8016$
$EG_1 = 1040.472$	$EG_2 = 1000.8016.$

Thus $$EL_2 < EL_1$$

and $$EG_2 < EG_1.$$

However, the loss potential for project 1 is about 50 times that of project 2 while the gain potentials of both are approximately the same. Thus, a rational decision-maker chooses project 2.

Case V

$$\mu_1 < \mu_2$$

$$\sigma_1 < \sigma_2$$

It is not possible in this case to derive generalities as has been done in the previous ones. Consequently, each selection problem has to be dealt with on an individual basis. As in previous cases, there are three types of problems: (i) where both projects are low-risk; (ii) where one project is low-risk and the other high-risk; and (iii) where both projects are high-risk. It is found that the selection algorithm is still applicable and this is illustrated in the following examples of each of the three types of problems.

(i) Low-Risk Projects ($\frac{\mu}{\sigma} > 1.9$):

Project 1	Project 2
$\mu_1 = 1000$	$\mu_2 = 2000$
$\sigma_1 = 500$	$\sigma_2 = 800$
$u_1 = \frac{1000}{500} = 2$	$u_2 = \frac{2000}{800} = 2.5.$

From the table of unit normal loss integrals:

$G(u_1) = 0.008491$	$G(u_2) = 0.002004$
$EL_1 = 4.2455$	$EL_2 = 1.6032$
$EG_1 = 1004.2455$	$EG_2 = 2001.6032.$

Thus, the choice is obviously project 2. It is to be noted that if $\mu_2 = 1600$, then $EL_2 = 6.7928$ and is greater than EL_1. However, the choice must still remain project 2 since EL here is of an order of magnitude that is insignificant.

(ii) Mixed Projects ($\frac{\mu_1}{\sigma_1} < 1.9$; $\frac{\mu_2}{\sigma_2} > 1.9$):

Project 1	Project 2
$\mu_1 = 500$	$\mu_2 = 2000$
$\sigma_1 = 500$	$\sigma_2 = 800$
$u_1 = \frac{500}{500} = 1$	$u_2 = \frac{2000}{800} = 2.5.$

From the table of unit normal loss integrals:

$G(u_1) = 0.08332$	$G(u_2) = 0.002004$
$EL_1 = 41.660$	$EL_2 = 1.6032$
$EG_1 = 541.660$	$EG_2 = 2001.6032.$

Thus, $EL_2 < EL_1$

and $EG_2 > EG_1$.

Project 2 is therefore selected.

(iii) High-Risk Projects ($\frac{\mu}{\sigma} < 1.9$):

Project 1	Project 2
$\mu_1 = 1000$	$\mu_2 = 2000$
$\sigma_1 = 1600$	$\sigma_2 = 2500$
$u_1 = \frac{1000}{1600} = 0.625$	$u_2 = \frac{2000}{2500} = 0.8.$

From the table of unit normal loss integrals:

$G(u_1) = 0.1620$	$G(u_2) = 0.1202$
$EL_1 = 259.2$	$EL_2 = 300.5$
$EG_1 = 1259.2$	$EG_2 = 2300.5$

Thus, $EL_2 > EL_1$

and $EG_2 > EG_1$.

However, an examination of percentage changes in EL and EG shows that the percentage increase in expected gain of doing project 2 outweighs the corresponding percentage increase in expected loss. Thus, project 2 is preferred. At this point, it is desirable to consider the loss and gain ratios mentioned earlier in this chapter.

Loss and Gain Ratios

The loss ratio $(\frac{EL}{\mu})$ and the gain ratio $(\frac{EG}{\mu})$ have been computed for a series of projects and are given in Table I. Each of these ratios are plotted against the worth ratio $(\frac{\mu}{\sigma})$ in Figure 3. The curves produced are hyperbolic in appearance and are nearly horizontal for $\frac{\mu}{\sigma} > 1.9$, that is in the low-risk region; and, the curves are nearly vertical in the high-risk region where $\frac{\mu}{\sigma} < 1.9$.

It is to be noted that these ratios are dimensionless quantities developed from the project parameters. Thus, an inherent property of the loss/gain ratio versus worth ratio curves is that any project must lie on these curves. Consequently, any two projects can be compared. This comparison can be illustrated by using the two preceding examples, V(ii) and V(iii), of mixed and high-risk projects. The data for these projects is repeated with the addition of the loss and gain ratios.

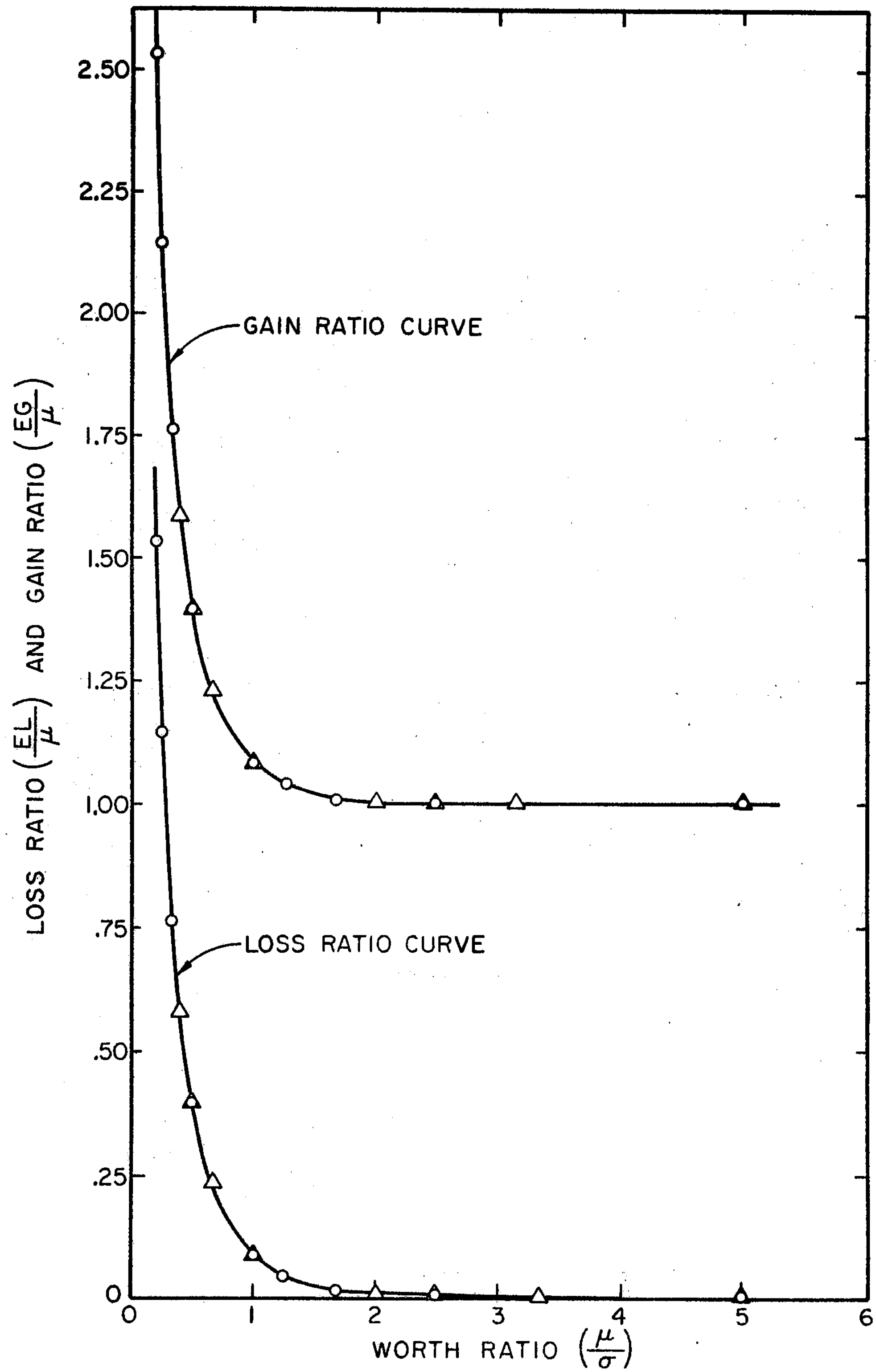

Figure 3. Loss Ratio and Gain Ratio Curves

Mixed Projects

Project 1	Project 2
$\mu_1 = 500$	$\mu_2 = 2000$
$\sigma_1 = 500$	$\sigma_2 = 800$
$\frac{\mu_1}{\sigma_1} = 1$	$\frac{\mu_2}{\sigma_2} = 2.5$
$EL_1 = 41.660$	$EL_2 = 1.6032$
$EG_1 = 541.660$	$EG_2 = 2001.6032$
$\frac{EL_1}{\mu_1} = 0.08332$	$\frac{EL_2}{\mu_2} = 0.008016$
$\frac{EG_1}{\mu_1} = 1.08332$	$\frac{EG_2}{\mu_2} = 1.008016$

High-Risk Projects

Project 1	Project 2
$\mu_1 = 1000$	$\mu_2 = 2000$
$\sigma_1 = 1600$	$\sigma_2 = 2500$
$\frac{\mu_1}{\sigma_1} = 0.625$	$\frac{\mu_1}{\sigma_1} = 0.8$
$EL_1 = 259.2$	$EL_2 = 300.5$
$EG_1 = 1259.2$	$EG_2 = 2300.5$
$\frac{EL_1}{\mu_1} = 0.2592$	$\frac{EL_2}{\mu_2} = 0.15025$
$\frac{EG_1}{\mu_1} = 1.2592$	$\frac{EG_2}{\mu_2} = 1.15025.$

It is noted that in both examples, while EG_2 is greater than EG_1, the gain ratio $\frac{EG_2}{\mu_2}$ is less than $\frac{EG_1}{\mu_1}$.

Figure 3 confirms that as the worth ratio increases the gain ratio decreases. This is also true for the loss ratio. However, the loss ratio decreases at a faster percentile rate (see Figure 3 and the examples). In each of these examples, the loss ratio of project 2 is less than that of project 1 and percentage-wise this reduction is greater than the corresponding reduction in the gain ratio of project 2. Hence, in each case project 2 is selected. This answer is the same as that obtained by examining the percentile changes in the expected gains and losses.

From these results, it is possible to develop a simpler method for comparing two projects. Now, consider that while project 2 is preferred in both examples, in the first example EL_2 is less than EL_1, and in the second example EL_2 is greater than EL_1. The loss ratios, however, show that $(\text{loss ratio})_2$ is always less than $(\text{loss ratio})_1$. From Figure 3, it can be seen that the loss ratio has a base of approximately zero compared to a gain ratio base of about one. Consequently, the percentile change in loss ratio is always greater than the corresponding change in gain ratio.[4] Thus, the project with the lower loss ratio is preferred. Now, a smaller loss ratio corresponds to a larger worth ratio $(\frac{\mu}{\sigma})$. Therefore, for high-risk and mixed projects, a valid means of selection is to pick the larger worth ratio.

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[4]It is not possible to make such a statement for the expectation curves (Figure 1) since, in the high-risk zone, as μ varies the curves are laterally displaced. The preceding high-risk example, V(iii), where the expected gain provided the greater percentile change confirms this.

When considering mixed projects, this policy results in the low-risk project being automatically selected.

In the low-risk region (both projects are low-risk), the change in risk is insignificant (nearly horizontal EL and loss ratio curves) and as has been shown previously the selection criterion is to choose the project with the larger μ.

The techniques developed in this chapter and summarized later provide a basis for the construction of an algorithm for the solution of the multi-project problem. This is the subject of the next chapter.

A Summary of Selection Procedures for the Choice Between Two Projects

Case Classification

Case I: $\mu_1 = \mu_2$

$\sigma_1 = \sigma_2$

Course of action: Indifference between projects.

Case II: $\mu_1 < \mu_2$

$\sigma_1 = \sigma_2$

Course of action: Select project 2 in low-risk region.

Max $\frac{\mu}{\sigma}$ otherwise and, thus, also project 2.

Case III: $\mu_1 < \mu_2$

$\sigma_1 > \sigma_2$

Course of action: Select project 2 in low-risk region.

Max $\frac{\mu}{\sigma}$ otherwise and, thus, also project 2.

Case IV: $\mu_1 = \mu_2$

$\sigma_1 > \sigma_2$

Course of action: Select project 2 in low-risk region.

Max $\frac{\mu}{\sigma}$ otherwise and, thus, also project 2.

Case V:

$\mu_1 < \mu_2$

$\sigma_1 < \sigma_2$

(i) Low-risk projects $(\frac{\mu}{\sigma} > 1.9)$:

Course of action: Select project 2.

(ii) Mixed projects $\left(\frac{\mu_1}{\sigma_1} < 1.9,\ \frac{\mu_2}{\sigma_2} > 1.9\right)$.

Course of action: Select low-risk project - in this instance project 2.

(iii) High-risk projects $(\frac{\mu}{\sigma} < 1.9)$:

Course of action: Select project with the larger $\frac{\mu}{\sigma}$.

Risk Zone Classification

(i) High-Risk Projects ($\frac{\mu}{\sigma} < 1.9$):

Course of action: Select the project with the smaller loss ratio ($\frac{EL}{\mu}$) or equivalently the larger worth ratio ($\frac{\mu}{\sigma}$).

(ii) Mixed Projects $\left(\frac{\mu_1}{\sigma_1} < 1.9, \frac{\mu_2}{\sigma_2} > 1.9\right)$:

Course of action: Select the low-risk project - in this instance project 2.

(iii) Low-Risk Projects ($\frac{\mu}{\sigma} > 1.9$):

Course of action: Select the project with the larger μ.

CHAPTER III

AN ALGORITHM FOR THE MULTI-PROJECT PROBLEM WITH A BUDGETARY CONSTRAINT

The next stage of the probabilistic capital budgeting problem consists of selecting, within a budget, one or more projects from several available. The concepts developed in the last chapter are now used to form an algorithm for the solution of this problem. However, first some comments are necessary on the method of bundling.

Bundling of Projects

The method of bundling is used by several authors, as for example Fleischer (7). It refers to determining all possible combinations of projects that do not violate a constraint. These combinations are evaluated and then compared in pairs to obtain the best bundle. A numerical example of this procedure is included in the next chapter. For n projects, there are $2^n - 1$ combinations. Thus, as n increases, the number of combinations become large enough to make the method impractical. Hence, the need exists for an alternative solution.

An Algorithm for the Multi-Project Problem

From the analysis of the comparison of two projects (Chapter II), certain selection criteria have been determined. These, adapted for consideration of the budgetary constraint, are the following:

(1) In the case of low-risk ($\frac{\mu}{\sigma} > 1.9$) projects, the risk level is negligible and the selection process is based on the maximization of expected net present value (μ) subject to the budgetary constraint.

(2) In the case of high risk ($\frac{\mu}{\sigma} < 1.9$) projects, the procedure consists of maximizing the worth ratio ($\frac{\mu}{\sigma}$) subject to the budgetary constraint.

(3) In a choice between low-risk ($\frac{\mu}{\sigma} > 1.9$) and high-risk ($\frac{\mu}{\sigma} < 1.9$) projects, the low-risk ones are preferred.

The preceding selection criteria provide a basis for an algorithm for the selection of projects with probabilistic parameters within a prescribed budget. In step form, this is as follows:

Step 1: Eliminate all projects with $\frac{\mu}{\sigma} < 1.9$.

Step 2: If the remainder of the projects (i.e., those with $\frac{\mu}{\sigma} > 1.9$) require an investment that is greater than the permissible budget, the choice will be among these. Select by maximizing μ_{NPV} subject to the budgetary constraint and that will be the solution.

Step 3: If this remainder of projects (those with $(\frac{\mu}{\sigma} > 1.9)$ require an investment that is less than the permissible budget, then choose all of them.

Step 4: The budgetary constraint now consists of the budget remaining after Step 3. To utilize this remainder, return to all the projects with $\frac{\mu}{\sigma} < 1.9$ and maximize $\frac{\mu}{\sigma}$ subject to the budgetary constraint.

The precise mathematical statement of the problem resulting from Step 2 (denoted Case A) is the following:

$$\text{Maximize} \sum_{r=1}^{n} \mu_r \; X_r$$

subject to

$$\sum_{r=1}^{n} C_r \; X_r \leq B$$

$$X_r = 0, 1$$

where

μ_r = expected net present value for the r^{th} project

r = project number

C_r = investment for the r^{th} project

B = budget limit

X_r = decision variable for the r^{th} project.

And the precise mathematical statement of the problem resulting from Step 4 (denoted Case B) is as follows:

$$\text{Maximize} \sum_{r=1}^{n} \left(\frac{\mu}{\sigma}\right)_r X_r$$

subject to
$$\sum_{r=1}^{n} C_r X_r \leq b$$

where

σ_r = standard deviation for the r^{th} project

$\left(\frac{\mu}{\sigma}\right)_r$ = worth ratio for the r^{th} project

b = remaining budget limit after the selection of low-risk projects.

In the next chapter, the preceding algorithm is applied to the solution of numerical examples.

CHAPTER IV

APPLICATIONS

The object of this chapter is to apply the concepts developed in Chapters II and III to some numerical examples. In this way, the use of these concepts and their validity are demonstrated.

The Method of Bundling

Bundling, as mentioned previously, refers to determining all possible combinations of projects that do not violate a particular restriction. As an example of this procedure, consider the four projects below:

Project No.	1	2	3	4
μ (\$)	500	500	3000	4000
σ (\$)	500	1500	1000	1000
$\frac{\mu}{\sigma}$	1.0	0.3	3.0	4.0
Investment \$	7000	5000	15000	18000

There is also a budgetary limit of \$41,000.

In the example, there are $2^4 - 1 = 15$ combinations if the budget restriction is not considered. The fifteen combinations and their associated data are shown in Table II. These combinations of projects (Table II) are established without a budget restriction. If a budget constraint is

TABLE II

DATA FOR PROJECT COMBINATIONS IN THE BUNDLING PROBLEM

Project Symbol	Combination	Investment	μ	σ	$u = \frac{\mu}{\sigma}$	G(u)	EL $= \sigma \cdot G(u)$	EG $= \mu + EL$
A	1	7000	500	500	1.000	.08332	41.6600	541.6600
B	2	5000	500	1500	0.667	.15130	227.0000	727.0000
C	3	15000	3000	1000	3.000	$.0^{3}3822$	0.3822	3000.3822
D	4	18000	4000	1000	4.000	$.0^{5}7145$	0.0071	4000.0071
E	1,2	12000	1000	1580	0.633	.15980	252.0000	1252.0000
F	1,3	22000	3500	1120	3.125	$.0^{3}2435$	0.2725	3500.2725
G	1,4	25000	4500	1120	4.020	$.0^{5}6538$	0.0073	4500.0073
H	2,3	20000	3500	1800	1.945	$.0^{2}9827$	17.6700	3517.6700
J	2,4	23000	4500	1800	2.500	$.0^{2}2004$	3.6072	4503.6072
K	3,4	33000	7000	1414	4.950	$.0^{7}6982$	≈ 0	7000
L	1,2,3	27000	4000	1870	2.140	$.0^{2}5788$	10.8800	4010.8800
M	1,2,4	30000	5000	1870	2.675	$.0^{2}1151$	2.1575	5002.1575
N	1,3,4	40000	7500	1500	5.000	$.0^{7}5330$	≈ 0	7500
P	2,3,4	38000	7500	2060	3.640	$.0^{4}3321$	.0685	7500.0685
R	1,2,3,4	45000	8000	2120	3.770	$.0^{4}1933$	.0409	8000.0409

imposed, then those combinations that violate it (or any other restriction) are eliminated. Thus, when the example budget limit of $41,000 is applied, project combination R (projects 1, 2, 3, and 4) is eliminated. Once all the combinations that do not violate any restrictions have been determined, then the concepts developed in Chapters II and III can be used to obtain the best combination of projects. Table III illustrates this selection procedure. The final answer is to do projects 1, 3, and 4.

The Algorithm Approach

For a large number of projects, the bundling method becomes impractical because of the number of combinations that need to be determined. For this reason, the algorithm developed in Chapter III provides a more practical approach. Applying this algorithm to the four-project example results in the initial choice of projects 3 and 4 since they are in the low-risk zone ($\frac{\mu}{\sigma} > 1.9$). The investment required for these is $33000 which leaves 41000 − 33000 = $8000 as the remaining budget. This means that out of the two high-risk projects (1 and 2), just one can be attempted.

The problem has now reduced to

Maximize $\frac{\mu_1}{\sigma_1} X_1 + \frac{\mu_2}{\sigma_2} X_2$; that is, maximize $1\ X_1 + \frac{1}{3} X_2$

subject to $7000\ X_1 + 5000\ X_2 \leq 8000$

$$X_1,\ X_2 = 0,\ 1.$$

TABLE III

SELECTION OF THE OPTIMUM PROJECT COMBINATION IN THE BUNDLING PROBLEM

Comparison	Relationship	Case	Decision
A vs. B	$\mu_B = \mu_A$ $\sigma_B > \sigma_A$	IV	A
A vs. C	$\mu_A < \mu_C$ $\sigma_A < \sigma_C$	V(ii)	C
C vs. D	$\mu_C < \mu_D$ $\sigma_C = \sigma_D$	II	D
D vs. E	$\mu_E < \mu_D$ $\sigma_E > \sigma_D$	III	D
D vs. F	$\mu_F < \mu_D$ $\sigma_F > \sigma_D$	III	D
D vs. G	$\mu_D < \mu_G$ $\sigma_D < \sigma_G$	V(i)	G
G vs. H	$\mu_H < \mu_G$ $\sigma_H > \sigma_G$	III	G
G vs. J	$\mu_J = \mu_G$ $\sigma_J > \sigma_G$	IV	G
G vs. K	$\mu_G < \mu_K$ $\sigma_G > \sigma_K$	III	K
K vs. L	$\mu_L < \mu_K$ $\sigma_L > \sigma_K$	III	K
K vs. M	$\mu_M < \mu_K$ $\sigma_M > \sigma_K$	III	K

TABLE III (Continued)

Comparison	Relationship	Case	Decision
K vs. N	$\mu_K < \mu_N$ $\sigma_K < \sigma_N$	V(i)	N
N vs. P	$\mu_P = \mu_N$ $\sigma_P > \sigma_N$	II	N

Note that combination R cannot be considered since it exceeds the budget limit of $41000. Thus, combination N is the preferred choice and the solution to the problem is to do projects 1, 3, and 4.

By inspection, the solution is $X_1 = 1$, $X_2 = 0$. Thus, the complete solution is that projects 1, 3, and 4 are selected.

The algorithm is next applied to the following, much larger problem:

Project	μ($)	σ($)	$\frac{\mu}{\sigma}$	Investment ($)
A	500	2000	0.25	18000
B	800	1600	0.50	10000
C	1000	1500	0.67	30000
D	900	1200	0.75	26000
E	1200	1000	1.20	25000
F	1500	900	1.67	22000
G	1800	1400	1.28	33000
H	2000	1000	2.00	24000
J	2500	900	2.78	20000
K	3000	800	3.75	16000
L	3500	700	5.00	28000
M	4400	1100	4.00	42000
N	5000	1200	4.17	32000
P	5600	1500	3.73	25000
R	6000	2000	3.00	40000

It can be seen that the method of bundling is practically impossible for this problem because the total number of combinations ($2^{15} - 1$) is very large.

Case A: Where the total investment for all the low-risk projects exceeds the budget limit.

To illustrate this case, a budget of $150,000 is assumed. The low-risk ($\frac{\mu}{\sigma} > 1.9$) projects are H, J, K, L, M, N, P, R with a total required investment of $227,000. For convenience in writing, these are labeled 1 through 8, thus H is 1 and R is 8. Since, in the low-risk region, the object is to maximize μ, the statement of the problem is as follows:

$$\text{Maximize } 2000\,X_1 + 2500\,X_2 + 3000\,X_3 + 3500\,X_4 + 4400\,X_5 + 5000\,X_6 + 5600\,X_7 + 6000\,X_8$$

$$\text{subject to } 24000\,X_1 + 20000\,X_2 + 16000\,X_3 + 28000\,X_4 + 42000\,X_5 + 32000\,X_6 + 25000\,X_7 + 40000\,X_8 \leq 150{,}000$$

and

$$X_1,\ X_2,\ \ldots,\ X_8 = 0,\ 1.$$

This is an integer programming problem. Several algorithms are available for its solution, notably Gomory (10), Glover (9), Land-Doig (12), Dakin (4), Balas (1), and also dynamic programming. It is solved (in Appendix B) by dynamic programming using a method explained by Nemhauser (14). The final result is that projects 3, 4, 6, 7, and 8 (namely, K, L, N, P, and R) are selected for a total capital outlay of \$141,000; hence, \$9000 is left over. The projects yield a total expected net present value of \$23,100.

Case B: Where the total investment for all the low-risk projects is less than the budget.

To illustrate this type of application, consider the same set of projects, but this time with a budget limit of \$300,000. The low-risk ($\frac{\mu}{\sigma} > 1.9$) projects are H, J, K, L, M, N, P, R and they require a total investment of \$227,000. All of these are selected which leaves a remaining budget of \$73,000. To invest this, the high-risk projects are now examined. In the high-risk zone, the criterion for selection is maximization of the worth ratio $\frac{\mu}{\sigma}$. The problem can then be stated as follows: (For convenience, the high-risk projects A through G are numbered 1 through 7, respectively).

$$\text{Maximize } .25\,X_1 + .50\,X_2 + .67\,X_3 + .75\,X_4 + 1.20\,X_5 + 1.67\,X_6 + 1.28\,X_7$$

$$\text{subject to } 18000\,X_1 + 10000\,X_2 + 30000\,X_3 + 26000\,X_4 + 25000\,X_5 + 22000\,X_6 + 33000\,X_7 \leq 73000$$

$$\text{and} \qquad X_1, X_2, \ldots, X_7 = 0, 1.$$

This is solved using dynamic programming in Appendix B. The answer is that projects 4, 5, and 6 (namely, D, E, and F are selected for a total investment of \$73,000). Thus, the complete solution states that projects D, E, F, H, J, K, L, M, N, P, R are selected. All of the budget is utilized for an expected net present value yield of \$35,600.

CHAPTER V

SUMMARY AND CONCLUSIONS

The basic problem considered in this study concerns the optimum choice from projects competing for limited resources where project cash flows follow a probability distribution. Most previous work in this area has centered around the utility theory approach, which involves the determination of the utility of money functions for decision-makers. Because of the difficulties encountered with evaluating such functions, this study provides a solution to this problem without the explicit use of cardinal utility theory.

A primary assumption is made for choosing between two projects, that if

$$(\text{expected loss})_2 < (\text{expected loss})_1$$

and $$(\text{expected gain})_2 > (\text{expected gain})_1$$

then a rational decision-maker selects project 2.

A secondary hypothesis is necessary where the above situation does not occur; that is, when

$$(\text{expected loss})_2 < (\text{expected loss})_1$$

and $$(\text{expected gain})_2 < (\text{expected gain})_1.$$

In this circumstance, percentile changes are examined and if the percentage change in the expected losses is the greater, project 2 is selected; conversely, if the percentage change in the expected gains is the greater, project 1 is chosen.

In addition, the concepts of Worth Ratio $(\frac{\mu}{\sigma})$, Loss Ratio $(\frac{EL}{\mu})$ and Gain Ratio $(\frac{EG}{\mu})$ are introduced. With these basic assumptions and concepts, a selection procedure (summarized at the end of Chapter II) is devised for the choice between two projects.

This selection procedure is next extended to the larger problem of selecting, within a prescribed budget, a number of projects from several available, and an algorithm is developed for this purpose.

While the methods of selection presented in this study do not claim to give the "best" answer to the problem, they do provide a good solution. For example, a particular decision-maker may not always agree with the greater-percentile-change assumption and may make decisions contrary to it. This does not necessarily mean, however, that he is making the best choice, rather that he is being biased by his own personal preferences. From a corporate standpoint and in the long run, decisions based on a comparison of percentage changes are more likely to give consistently better choices.

It is of note that the fundamental assumption, namely, the choice of project 2 if $EL_2 < EL_1$ and $EG_2 > EG_1$, implies <u>only an increasing marginal utility of money</u>. The <u>rate</u> of increase can be constant, decreasing or increasing, that is, the utility function itself can be a straight line, concave downwards or convex, or even a combination of these.

The object of this study has been to obtain a solution that combines conceptual simplicity with ease in application. The worth ratio versus loss/gain ratio plots offer an at-a-glance impression of projects to the "lay" (not mathematically oriented) decision-maker. They exclude the intangibles of utility theory and are based entirely on available project information. Furthermore, the algorithm provides an easily programmed solution to the larger complete selection problem. It is believed that this study is a contribution to the understanding, simplification, and solution of the complex probabilistic capital budgeting problem.

Proposals for Future Investigations

In this work, the mean μ and the standard deviation σ are assumed to be known. As has been mentioned previously, methods are easily available in the literature for the calculation of these parameters of the net present value distribution once the corresponding parameters for the individual annual cash flows are known. However, further work is necessary in improved and more accurate estimation of these parameters for individual cash flows.

Project independence has also been assumed in this study. Even if projects are not independent, the selection procedure for the choice between two projects remains valid.

However, for the larger problem (namely, the selection of a number of projects, within a budget, from several available) the algorithm method becomes limited to the low-risk zone only and cannot be applied when high-risk projects are involved because of the dependence of σ. The bundling method, for when the number of available projects is small, is still applicable, with slight alterations to include co-variance terms in the calculation of the standard deviations of project combinations. Further research is needed to extend the algorithm or develop an alternative to solve the dependent project problem.

Normality of the project net present value distributions is another assumption that has been made in this research. Exactly how essential and necessary this is, is another area for further investigation. Following from this, further work is required for cases where net present value distributions are skewed, that is when moments of higher order than two need to be considered.

Finally, a simulated comparison using the methods developed in this study and those of utility theory, involving both risk-averse and risk-taker behavior, would be worthwhile.

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APPENDIX A

EXPECTED LOSS AND EXPECTED GAIN

Expected Loss (EL):

$$EL = \int_{-\infty}^{0} |NPV| f(NPV)\ d(NPV)$$

$$= \sigma(NPV) \cdot G(u)$$

Schlaifer (12) calls this expected opportunity loss and gives a derivation.

Expected Gain (EG):

$$EG = \int_{0}^{\infty} |NPV| f(NPV)\ d(NPV)$$

For convenience in writing, let x = NPV,

then $$EG = \int_{0}^{\infty} x \cdot f(x)dx.$$

For a normal density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

where σ is the standard deviation and μ is the mean.

$$\therefore EG = \int_{0}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} (x-\mu) e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx +$$

$$+ \mu \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= -\frac{\sigma^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \Bigg|_0^\infty + \mu \cdot P(x > 0)$$

$$= +\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} + \mu \cdot P(x > 0)$$

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{u^2\sigma^2}{2\sigma^2}} + u\,\sigma\,P(z > -u)$$

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} + u\,\sigma(1 - P(z > u))$$

$$= \sigma \cdot g(u) - u\,\sigma \cdot P(z > u) + u\sigma$$

$$= \sigma \cdot [\{g(u) - u \cdot P(z > u)\} + u]$$

$$= \sigma \cdot (G(u) + u)$$ where $G(u)$ is the unit normal loss integral

$$= \sigma \cdot G(-u)$$

The table of the unit normal loss integral is given in Schlaifer (12, pp. 706-707).

APPENDIX B

SOLUTIONS TO NUMERICAL EXAMPLES

In this appendix, the actual solution to problems in Chapter IV are presented.

The first problem (Case A) is

$$\text{Maximize} \quad 2000\, X_1 + 2500\, X_2 + 3000\, X_3 + 3500\, X_4 + 4400\, X_5 + 5000\, X_6 + 5600\, X_7 + 6000\, X_8$$

$$\text{subject to } 24000\, X_1 + 20000\, X_2 + 16000\, X_3 + 28000\, X_4 + 42000\, X_5 + 32000\, X_6 + 25000\, X_7 + 40000\, X_8 \leq 150000$$

and the decision variables $X_1, X_2, \ldots, X_8 = 0, 1.$

The budgetary constraint can be re-written,

$$24\, X_1 + 20\, X_2 + 16\, X_3 + 28\, X_4 + 42\, X_5 + 32\, X_6 + 25\, X_7 + 40\, X_8 \leq 150.$$

The problem is solved by dynamic programming using a method discussed by Nemhauser (9). First, state variables S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7, and S_8 are defined as the feasible values of the budget at the beginning of each stage. Next, these are evaluated.

$S_8 = 150.$

$S_7 = 150 - 40\ X_8 = 150, 110$ (since X_8 can be 0 or 1).

$S_6 = S_7 - 25\ X_7 =$ 150, 110, 125, 85.

$S_5 = S_6 - 32\ X_6 =$ 150, 110, 125, 85,
118, 78, 93, 53.

$S_4 = S_5 - 42\ X_5 =$ 150, 110, 125, 85, 118, 78, 93, 53,
108, 68, 83, 43, 76, 36, 51, 11.

$S_3 = S_4 - 28\ X_4 =$ 150, 110, 125, 85, 118, 78, 93, 53,
108, 68, 83, 43, 76, 36, 51, 11,
122, 82, 97, 57, 90, 50, 65, 25,
80, 40, 55, 15, 48, 8, 23, –.

$S_2 = S_3 - 16\ X_3 =$ 150, 110, 125, 85, 118, 78, 93, 53,
108, 68, 83, 43, 76, 36, 51, 11,
122, 82, 97, 57, 90, 50, 65, 25,
80, 40, 55, 15, 48, 8, 23, –,
134, 94, 109, 69, 102, 62, 77, 37,
92, 52, 67, 27, 60, 20, 35, –,
106, 66, 81, 41, 74, 34, 49, 9,
64, 24, 39, –, 32, –, 7, –.

$S_1 = S_2 - 20\ X_2 =$ 150, 110, 125, 85, 118, 78, 93, 53,
108, 68, 83, 43, 76, 36, 51, 11,
122, 82, 97, 57, 90, 50, 65, 25,
80, 40, 55, 15, 48, 8, 23, –,
134, 94, 109, 69, 102, 62, 77, 37,
92, 52, 67, 27, 60, 20, 35, –,
106, 66, 81, 41, 74, 34, 49, 9,
64, 24, 39, –, 32, –, 7, –,

130, 90, 105, 65, 98, 58, 73, 33,

88, 48, 63, 23, 56, 16, 31, -,

102, 62, 77, 37, 70, 30, 45, 5,

60, 20, 35, -, 28, -, 3, -,

114, 74, 89, 49, 82, 42, 57, 17,

72, 32, 47, 7, 40, 0, 15, -,

86, 46, 61, 21, 54, 14, 29, -,

44, 4, 19, -, 12, -, -, -.

$S_0 = S_1 - 24\ X_1$

Stage 1:

S_1 \ X_1	0	1
0	0	–
3	0	–
4	0	–
5	0	–
7	0	–
8	0	–
9	0*	–
11	0	–
12	0	–
14	0	–
15	0	–
16	0	–
17	0	–
20	0	–
21	0	–
23	0	–
24	0	(2000)
25	0	(2000)
.	.	.
.	.	.
.	.	.
150	0	(2000)

NOTE: Parentheses imply state optimums.

Asterisk implies final solution.

Stage 2:

$S_2 \backslash X_2$	0	1
7	0	–
8	0	–
9	0*	–
11	0	–
15	0	–
20	0	(2500)
23	0	(2500)
24	2000	(2500)
25	2000	(2500)
.	.	.
.	.	.
.	.	.
43	2000	(2500)
48	2000	2000+2500 (4500)
49	2000	(4500)
.	.	.
.	.	.
.	.	.
150	2000	(4500)

Stage 3:

$s_3 \backslash x_3$	0	1
8	0	–
11	0	–
15	0	–
23	2500	(3000)
25	2500	(3000)*
36	2500	2500+3000 (5500)
40	2500	(5500)
43	2500	(5500)
48	4500	(5500)
50	4500	(5500)
51	4500	(5500)
53	4500	(5500)
55	4500	(5500)
57	4500	(5500)
65	4500	4500+3000 (7500)
68	4500	(7500)
.	.	.
.	.	.
.	.	.
150	4500	(7500)

Stage 4:

s_4 \ x_4	0	1
11	0	-
36	(5500)	3500
43	(5500)	3500
51	5500	3000+3500 (6500)
53	5500	(6500)*
68	7500	5500+3500 (9000)
76	7500	(9000)
78	7500	(9000)
83	7500	(9000)
85	7500	(9000)
93	7500	7500+3500 (11000)
108	7500	(11000)
110	7500	(11000)
118	7500	(11000)
125	7500	(11000)
150	7500	(11000)

Stage 5:

s_5 \ x_5	0	1
53	(6500)*	4400
78	9000	5500+4400 (9900)
85	9000	(9900)
93	(11000)	6500+4400 10900
110	11000	9000+4400 (13400)
118	11000	(13400)
125	11000	(13400)
150	11000	(13400)

Stage 6:

s_6 \ x_6	0	1
85	9900	6500+5000 (11500)*
110	13400	9900+5000 (14900)
125	13400	11000+5000 (16000)
150	13400	13400+5000 (18400)

Stage 7:

$s_7 \backslash x_7$	0	1
110	14900	11500+5600 (17100)*
150	18400	16000+5600 (21600)

Stage 8:

$s_8 \backslash x_8$	0	1
150	21600	17100+6000 (23100)*

Thus, the optimum expected net present value return is $23,100.

Tracing back through the tableaus, the projects selected are 3, 4, 6, 7, and 8; namely, K, L, N, P, and R. These require a total investment of $141,000. Hence, $9000 is left over.

The second problem (Case B) from Chapter IV is

$$\begin{aligned} \text{Maximize} \quad & .25\ X_1 + .50\ X_2 + .67\ X_3 + .75\ X_4 + \\ & 1.20\ X_5 + 1.67\ X_6 + 1.28\ X_7 \\ \text{subject to} \quad & 18000\ X_1 + 10000\ X_2 + 30000\ X_3 + \\ & 26000\ X_4 + 25000\ X_5 + 22000\ X_6 + \\ & 33000\ X_7 \leq 73000 \end{aligned}$$

and the decision variables $X_1, X_2, \ldots, X_7 = 0, 1$.
The budgetary constraint can be re-written,

$$\begin{aligned} & 18\ X_1 + 10\ X_2 + 30\ X_3 + 26\ X_4 + 25\ X_5 + 22\ X_6 + \\ & \quad 33\ X_7 \leq 73. \end{aligned}$$

This problem is also solved using dynamic programming. State variables S_0, S_1, S_2, S_3, S_4, S_5, S_6, and S_7 are defined as the feasible values of the budget at the beginning of each stage. They are then determined as follows:

$S_7 = 73.$

$S_6 = S_7 - 33\ X_7 = 73,\ 40.$

$S_5 = S_6 - 22\ X_6 = 73,\ 40,\ 51,\ 18.$

$S_4 = S_5 - 25\ X_5 = 73,\ 40,\ 51,\ 18,\ 48,\ 15,\ 26,\ -.$

$S_3 = S_4 - 26\ X_4 = 73,\ 40,\ 51,\ 18,\ 48,\ 15,\ 26,\ -,$
$47,\ 14,\ 25,\ -,\ 22,\ -,\ 0,\ -.$

$S_2 = S_3 - 30\ X_3 = 73,\ 40,\ 51,\ 18,\ 48,\ 15,\ 26,\ -,$
$47,\ 14,\ 25,\ -,\ 22,\ -,\ 0,\ -,$
$43,\ 10,\ 21,\ -,\ 18,\ -,\ -,\ -,$
$17,\ -,\ -,\ -,\ -,\ -,\ -,\ -,$

$S_1 = S_2 - 10\ X_2 = 73,\ 40,\ 51,\ 18,\ 48,\ 15,\ 26,$
$47,\ 14,\ 25,\ 22,\ 0,$

43, 10, 21, 18, 17,

63, 30, 41, 8, 38, 5, 16 ,

37, 4, 15, 12, - ,

33, 0, 11, 8, 7 .

$S_0 = S_1 - 18\ X_1$.

Stage 1:

$s_1 \setminus x_1$	0	1
0	0*	–
4	0	–
5	0	–
7	0	–
8	0	–
10	0	–
11	0	–
12	0	–
14	0	–
15	0	–
16	0	–
17	0	–
18	0	(.25)
21	0	(.25)
22	0	(.25)
25	0	(.25)
26	0	(.25)
30	0	(.25)
33	0	(.25)
37	0	(.25)
38	0	(.25)
40	0	(.25)
41	0	(.25)
43	0	(.25)
47	0	(.25)
48	0	(.25)
51	0	(.25)
63	0	(.25)
73	0	(.25)

<u>Stage 2</u>:

s_2 \ x_2	0	1
0	0*	-
10	0	(.50)
14	0	(.50)
15	0	(.50)
17	0	(.50)
18	.25	(.50)
21	.25	(.50)
22	.25	(.50)
25	.25	(.50)
26	.25	(.50)
40	.25	.25+.50 (.75)
43	.25	(.75)
47	.25	(.75)
48	.25	(.75)
51	.25	(.75)
73	.25	(.75)

<u>Stage 3</u>:

$S_3 \diagdown X_3$	0	1
0	0*	–
14	(.50)	–
15	(.50)	–
18	(.50)	–
22	(.50)	–
25	(.50)	–
26	(.50)	–
40	.75	.50+.67 (1.17)
47	.75	(1.17)
48	.75	(1.17)
51	.75	(1.17)
73	.75	.75+.67 (1.42)

Stage 4:

$s_4 \backslash x_4$	0	1
15	(.50)	–
18	(.50)	–
26	.50	(.75)*
40	1.17	.50+.75 (1.25)
48	1.17	(1.25)
51	1.17	(1.25)
73	1.42	1.17+.75 (1.92)

Stage 5:

$s_5 \backslash x_5$	0	1
18	(.50)	–
40	1.25	.50+1.20 (1.70)
51	1.25	.75+1.20 (1.95)*
73	1.92	1.25+1.20 (2.45)

Stage 6:

$s_6 \backslash x_6$	0	1
40	1.70	.50+1.67 (2.17)
73	2.45	1.95+1.67 (3.62)*

Stage 7:

$s_7 \backslash x_7$	0	1
73	(3.62)*	2.17+1.28 3.45

Tracing back through the tableaus, the high-risk projects selected are 4, 5, and 6; i.e., D, E, and F. These require a total investment of $73,000 (so that no money is left over) and provide an expected net present value return of $3,600.

Thus, the complete solution to the problem states that projects D, E, F, H, J, K, L, M, N, P, R are selected. The total budget is utilized for an expected net present value return of $35,600.

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