SPHERICAL-WAVE EXPANSIONS FOR THE FIELDS OF THE LINEAR RADIATORS WITH ASSUMED SINUSOIDAL CURRENT AND SOME OPINIONS ON THE THEORY OF SUPERGAIN RADIATORS

By

TAKERU IRABU

Bachelor of Science
Gifu University
Gifu, Japan
1966

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1970

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1973

SPHERICAL-WAVE EXAPNSIONS FOR THE FIELDS OF THE LINEAR RADIATORS WITH ASSUMED SINUSOIDAL CURRENT AND SOME OPINIONS ON THE THEORY OF SUPERGAIN RADIATORS

Thesis Approved:

Bennett Basore
Thesis Adviser

Paul D. Miller

Robert J. Mulholland

James R. McDougal

N N Durham
Dean of the Graduate College

ACKNOWLEDGEMENTS

I wish to express my sincere thanks to my thesis adviser, Dr. James R. McDougal, for his guidance and assistance throughout this study. I also wish to express my appreciation to Dr. Bennett L. Basore, the committee chairman, for reading the manuscript and making many helpful suggestions. Appreciation is extended to Dr. Robert J. Mulholland and Dr. Paul G. Miller for their generous and helpful assistance.

The debt I owe to my wife, Akiko, is too large to measure; this study would never have been completed without her constant encouragement and support.

TABLE OF CONTENTS

Chapter | Page

LIST OF TABLES

Table | Page

LIST OF FIGURES

Figure | Page

CHAPTER I

INTRODUCTION

Problem Definition

One of the significant works in antenna theory has been published by Chu (2). He managed to show the physical limitations of omni-directional radiators by making use of the spherical-wave functions. Extensions of Chu's work have been carried out on a wide variety of subjects by several authors (4,6,8). However, none of the results have been applied to any actual practical antennas. This is because the mechanical configurations of most practical antennas can not be well described in terms of spherical boundaries; therefore, examples where the spherical-wave functions have been applied to actual practical radiators are extremely rare. This would appear to be an omission on the part of the theory, in comparison with specific actual radiators.

Objectives and Procedures

This study is intended to accomplish two objectives. The first objective is to obtain a spherical-wave expansion for the solution of a practical radiator. The second objective is to explore the consequences of the work of Chu (2) and others when applied to practical radiators by making use of the spherical-wave expansion of the radiator.

For the first objective, the half-wave dipole has been selected as an example of a practical radiator and an attempt was made to expand its field solution in the spherical-wave functions. The half-wave dipole was selected because it is an efficient radiator which is often used and, as a consequence, its physical properties are well known. Also its solution in cylindrical coordinates when a sinusoidal current distribution is assumed is well known.

For the second objective, the directive gain, the radiation impedance, the stored energy, and the quality factor of a half-wave dipole have been evaluated by making use of the spherical-wave expansion and the results are compared with those which are expected from the work of Chu (2) and others (4,6,8).

Findings

As an effective tool for exploring the consequences of the work of Chu (2) and others (4,6,8) when applied to practical radiators, the spherical-wave expansion for the fields of a half-wave dipole has been obtained. In doing this, it is found that it takes an infinite number of terms of the spherical-wave functions to represent the field of the half-wave dipole.

On the basis of the work of L. J. Chu (2) and R. F. Harrington (8), large modes of order $n > ka$ (a is radius of an antenna) in a spherical-wave expansion associate only with the supergain. As it has been observed in Chapter III, however, modes of order $n > ka$ in the spherical-wave expansion are necessary to represent the fields of a half-wave dipole; although the directive gain for a half-wave dipole is only about 1.64 as compared with the normal gain 5.60 for an antenna when cutoff mode $N = ka = k\lambda/4$. When the cutoff mode $N > k\lambda/4$, the normal gain G_N for an antenna of half-length $\lambda/4$ becomes (8):

$$G_N = \sum_{n=1}^{N>k\frac{\lambda}{4}} (2n + 1) > \sum_{n=1}^{N\leq k\frac{\lambda}{4}} (2n + 1) \quad .$$

Therefore, it is believed that the normal gain for an antenna of half-length $\lambda/4$ is larger than what has been expected from equations of Chu (2) and Harrington (8); physical realization of supergain antenna becomes more difficult.

Though it might be less practical in some degree, the spherical-wave expansions for the field of linear antennas have also been obtained.

As an application of the spherical-wave expansion, Collin and Rothschild's (4) and Fante's (6) methods for the quality factor have been studied. In doing this, it has been found that the method of Fante gives more appropriate results for the quality factor for a half-wave dipole.

It is of course well known that the current distribution on an actual half-wave dipole is not exactly sinusoidal. Consequently, the ties between the theoretical and physical measurement become even more difficult to determine. However, the half-wave dipole is realized in practice to a more exact degree than most other structures. Hence, it is believed that the spherical-wave expansion developed in this study is a tool of some value in exploring the physical consequences of the theory associated with the spherical-wave functions.

Organization

The rest of this study is organized in the following manner. Chapter II is mainly devoted to finding a spherical-wave expansion of the half-wave dipole with sinusoidal current distribution. It is shown that the well known classical far-field solution is obtained from the spherical-wave expansion. The convergence of the spherical-wave expansion is also studied.

In Chapter III, the more compact and useful form of the spherical-wave expansion for a half-wave dipole is obtained by making use of the results of Chapter II. The directive gain and the radiation impedance are evaluated by using the spherical-wave expansion to compare the results with the well-known values for a half-wave dipole.

In Chapter IV, the spherical-wave expansion for a half-wave dipole is applied to the evaluation of the quality factor and the result is checked for any direct relationship of the quality factor to the reciprocal bandwidth.

The summary and the conclusion are given in Chapter V.

CHAPTER II

A SPHERICAL-WAVE EXPANSION FOR THE FIELDS OF A HALF-WAVE DIPOLE -- (1)

Introduction

The main purpose of this chapter is to obtain a spherical-wave expansion for the fields of a half-wave dipole. In the first part of this chapter, the summary of the well known classical solution for a half-wave dipole is given and in the rest of the chapter, the spherical-wave expansions for the solution is obtained for the near-field and the far-field of the dipole and its behavior is studied.

Classical Solution for a Half-Wave Dipole Antenna

The geometry for a half-wave dipole antenna and its radiation field are shown in Figure 1. It is assumed that the width of the antenna is infinitesimally thin and the antenna has sinusoidal current distribution such as

$$I = I_0 \sin k(\frac{\lambda}{4} - Z) \qquad ,$$

where

$$-\frac{\lambda}{4} \leq Z \leq \frac{\lambda}{4}$$

where

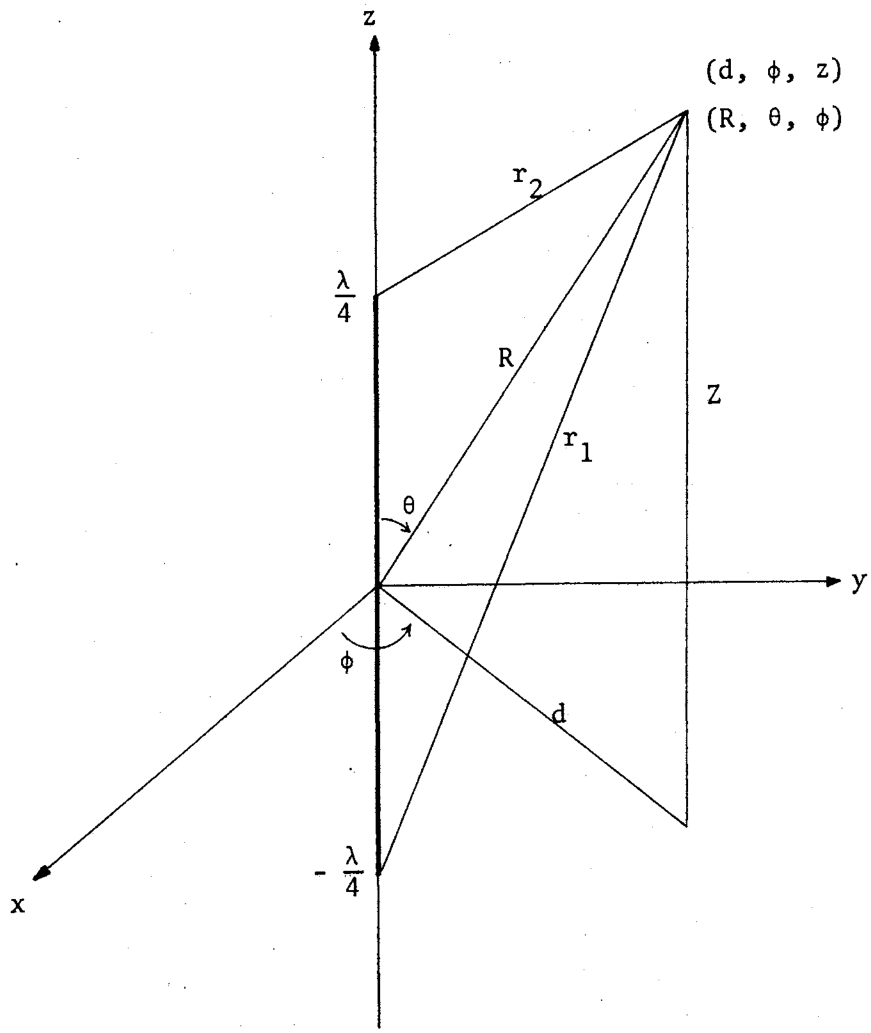

Figure 1. Cylindrical and Spherical Coordinate Geometry for a Half-Wave Dipole

I_0 = maximum value of the current,

$k = \frac{2\pi}{\lambda}$ = wave number, and

λ = wave length.

Then the well-known solution for the half-wave dipole antenna in cylindrical coordinates is given:

$$E_z = -\frac{ikI_0}{4\pi\omega\varepsilon}\left(\frac{e^{-ikr_1}}{r_1} + \frac{e^{-ikr_2}}{r_2}\right) \quad , \qquad (2.1)$$

$$E_d = \frac{ikI_0}{4\pi\omega\varepsilon}\left[(Z + \frac{\lambda}{4})\frac{e^{-ikr_1}}{r_1} + (Z - \frac{\lambda}{4})\frac{e^{-ikr_2}}{r_2}\right] \quad , \qquad (2.2)$$

$$H_\phi = \frac{iI_0}{4\pi d}\left(e^{-ikr_1} + e^{-ikr_2}\right) \quad , \qquad (2.3)$$

where

$i = \sqrt{-1}$,

= imaginary number,

$\omega = k(\varepsilon\mu)^{-1/2}$

= angular frequency,

$\varepsilon = \frac{1}{36\pi} \times 10^{-9}$ farads/meter

= permittivity for free space,

$\mu = 4 \times 10^{-7}$ henry/meter,

= permeability for free space,

$Z = R \cos\theta$, and

$d = R \sin\theta$.

E_z and E_d denote the d- and the z-directional component of the electric field of the half-wave dipole, respectively. H_ϕ denotes the ϕ-directional component of the magnetic-field of the dipole.

Next, the field solution in terms of spherical coordinates is obtained. This is done by combining Equations (2.1) through (2.3) with the following equations:

$$E_\theta = - E_z \sin\theta + E_d \cos\theta \quad , \tag{2.4}$$

$$E_R = E_z \cos\theta + E_d \sin\theta \quad , \tag{2.5}$$

$$H_\phi = H_\phi \quad , \tag{2.6}$$

where E_θ and E_R stand for the θ- and the R-directional component of the electric field of a half-wave dipole, respectively.

Performing the indicated computation, one obtains

$$E_\theta = \frac{ikI_0}{4\pi\omega\varepsilon} \left(\frac{R + \frac{\lambda}{4}\cos\theta}{R\sin\theta}\right) \frac{e^{-ikr_1}}{r_1} + \frac{ikI_0}{4\pi\omega\varepsilon} \left(\frac{R - \frac{\lambda}{4}\cos\theta}{R\sin\theta}\right) \frac{e^{-ikr_2}}{r_2} \quad , \tag{2.7}$$

$$E_R = \frac{ikI_0}{4\pi\omega\varepsilon} \left(\frac{\lambda}{4R}\right) \left(\frac{e^{-ikr_1}}{r_1} - \frac{e^{-ikr_2}}{r_2}\right) \quad , \tag{2.8}$$

$$H_\phi = \frac{iI_0 r_1}{4\pi R\sin\theta} \frac{e^{-ikr_1}}{r_1} + \frac{iI_0 r_2}{4\pi R\sin\theta} \frac{e^{-ikr_2}}{r_2} \quad . \tag{2.9}$$

As it is seen in the above equations, one may expect to obtain a spherical-wave expansion for E_θ, E_R, and H_ϕ, respectively, if it is possible to expand e^{-ikr_1}/r_1 and e^{-ikr_2}/r_2 in terms of the spherical-wave functions.

Transformation Equations

Indeed, it is well known that the following equations are very well satisfied by e^{-ikr_1}/r_1 and e^{-ikr_2}/r_2 in the region of $R > \lambda/4$ in

Figure 1:

$$\frac{e^{-ikr_2}}{r_2} = -ik \sum_{n=0}^{\infty} (2n+1)\, j_n(k\,\frac{\lambda}{4})\, p_n(\cos\theta)\, h_n^{(2)}(kR) \quad , \quad (2.10)$$

$$\frac{e^{-ikr_1}}{r_1} = -ik \sum_{n=0}^{\infty} (2n+1)\, j_n(k\,\frac{\lambda}{4})\, p_n[\cos(\pi-\theta)]\, h_n^{(2)}(kR) \quad . \quad (2.11)$$

Equation (2.11) is rewritten by making use of the following equations:

$$p_n[\cos(\pi-\theta)] = p_n(-\cos\theta)$$

and

$$p_n(-\cos\theta) = (-1)^n\, p_n(\cos\theta) \quad .$$

The result is:

$$\frac{e^{-ikr_1}}{r_1} = -ik \sum_{n=0}^{\infty} (-1)^n (2n+1)\, j_n(k\,\frac{\lambda}{4})\, p_n(\cos\theta)\, h_n^{(2)}(kR) \quad , \quad (2.12)$$

where

$j_n(k\,\frac{\lambda}{4})$ = spherical Bessel function of the first kind,

$p_n(\cos\theta)$ = Legendre function, and

$h_n^{(2)}(kR)$ = spherical Hankel function of the second kind.

If Y(n) is defined as

$$Y(n) = (2n+1)\, j_n(k\,\frac{\lambda}{4})\, p_n(\cos\theta)\, h_n^{(2)}(kR) \quad , \quad (2.13)$$

it becomes

$$\frac{e^{-ikr_2}}{r_2} = -ik \sum_{n=0}^{\infty} Y(n) \quad (2.14)$$

and

$$\frac{e^{-ikr_1}}{r_1} = -ik \sum_{n=0}^{\infty} (-1)^n Y(n) \tag{2.15}$$

A Spherical-Wave Expansion for the Fields of a Half-Wave Dipole in the Region of $R > \frac{\lambda}{4}$

Now it is ready to expand Equations (2.7) through (2.9) in the spherical-wave functions. Substituting Equations (2.14) and (2.15) back into Equations (2.9), (2.7), and (2.8), a spherical-wave expansion is obtained for the field of a half-wave dipole in the region of $R > \lambda/4$. The results are:

$$H_\phi = \frac{kI_0}{4\pi} \frac{r_1}{R \sin\theta} \sum_{n=0}^{\infty} (-1)^n Y(n) + \frac{kI_0}{4\pi} \frac{r_2}{R \sin\theta} \sum_{n=0}^{\infty} Y(n) \quad , \tag{2.16}$$

$$E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{4\pi} \frac{R + \frac{\lambda}{4}\cos\theta}{R \sin\theta} \sum_{n=0}^{\infty} (-1)^n Y(n) + \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{4\pi} \frac{R - \frac{\lambda}{4}\cos\theta}{R \sin\theta} \sum_{n=0}^{\infty} Y(n) , \tag{2.17}$$

$$E_R = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{8} \sum_{n=0}^{\infty} [(-1)^n - 1] \frac{Y(n)}{kR} \quad . \tag{2.18}$$

The above equations for H_ϕ, E_θ, and E_R are rewrittn in terms of the odd and the even modes of the spherical-wave functions. The results are:

$$H_\phi = \frac{I_0 k(r_1 + r_2)}{4\pi} \sum_{n:\text{even}}^{\infty} \frac{Y(n)}{R \sin\theta} + \frac{I_0 k(r_1 - r_2)}{4\pi} \sum_{n:\text{odd}}^{\infty} \frac{Y(n)}{R \sin\theta} , \tag{2.19}$$

$$E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{2\pi} \sum_{n:\text{even}}^{\infty} \frac{Y(n)}{\sin\theta} - \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{I_0}{4} \sum_{n:\text{odd}}^{\infty} \frac{(\cos\theta) Y(n)}{R \sin\theta} , \tag{2.20}$$

$$E_R = -\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{I_0}{4} \sum_{n:\text{odd}}^{\infty} \frac{Y(n)}{R} . \tag{2.21}$$

A Spherical-Wave Expansion for the Fields of a Half-Wave Dipole in the Region of $R < \frac{\lambda}{4}$

It is well known that in the region $R < \lambda/4$ of Figure 1, e^{-ikr_1}/r_1 and e^{-ikr_2}/r_2 are represented by the following equations:

$$\frac{e^{-ikr_1}}{r_1} = -ik \sum_{n=0}^{\infty} (-1)^n A(n) \quad , \tag{2.22}$$

$$\frac{e^{-ikr_2}}{r_2} = -ik \sum_{n=0}^{\infty} A(n) \quad , \tag{2.23}$$

where

$$A(n) = (2n+1) h_n^{(2)}(k \tfrac{\lambda}{4}) p_n(\cos\theta) j_n(kR) \quad . \tag{2.24}$$

For the region of $R < \lambda/4$, a spherical-wave expansion for the field of a half-wave dipole is obtained by substituting Equations (2.22) and (2.23) back into Equations (2.7), (2.8), and (2.9). The results are:

$$H_\phi = \frac{kI_0}{4\pi} \frac{r_1}{R\sin\theta} \sum_{n=0}^{\infty} (-1)^n A(n) + \frac{kI_0}{4\pi} \frac{r_2}{R\sin\theta} \sum_{n=0}^{\infty} A(n) \quad , \tag{2.25}$$

$$E_\theta = (\tfrac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{kI_0}{4\pi} \frac{R + \frac{\lambda}{4}\cos\theta}{R\sin\theta} \sum_{n=0}^{\infty} (-1)^n A(n) + (\tfrac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{kI_0}{4\pi} \frac{R - \frac{\lambda}{4}\cos\theta}{R\sin\theta} \sum_{n=0}^{\infty} A(n) , \tag{2.26}$$

$$E_R = (\tfrac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{kI_0}{8} \sum_{n=0}^{\infty} [(-1)^n - 1] \frac{A(n)}{kR} \quad . \tag{2.27}$$

The above equations for H_ϕ, E_θ, and E_R are rewritten in terms of the odd and the even modes of the spherical-wave functions:

$$H_\phi = \frac{I_0 k(r_1 + r_2)}{4\pi} \sum_{n:\text{even}}^{\infty} \frac{A(n)}{R \sin\theta} + \frac{I_0 k(r_1 - r_2)}{4\pi} \sum_{n:\text{odd}}^{\infty} \frac{A(n)}{R \sin\theta} \quad , (2.28)$$

$$E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{2\pi} \sum_{n:\text{even}}^{\infty} \frac{A(n)}{\sin\theta} - \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{I_0}{4} \sum_{n:\text{odd}}^{\infty} \frac{(\cos\theta)A(n)}{R \sin\theta} \quad , (2.29)$$

$$E_R = -\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{I_0}{4} \sum_{n:\text{odd}}^{\infty} \frac{A(n)}{R} \quad . (2.30)$$

The Continuity of the Spherical-Wave Expansions for $R < \frac{\lambda}{4}$ and $R > \frac{\lambda}{4}$

Setting $R = \lambda/4$ and equating Equation (2.19) to Equation (2.28), Equation (2.20) to Equation (2.29), and Equation (2.21) to Equation (2.30), it is easily seen that the spherical-wave expansions for $R > \lambda/4$ and $R < \lambda/4$, are continuous at $R = \lambda/4$.

The Far-Field Solution for a Half-Wave Dipole

It is well known that the far-field solution for a half-wave dipole is given by the following equations:

$$\lim_{R\to\infty} H_\phi = \frac{iI_0}{2\pi} \frac{e^{-ikR}}{R} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \quad , \quad (2.31)$$

$$\lim_{R\to\infty} E_\theta = iI_0 60 \frac{e^{-ikR}}{R} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \quad , \quad (2.32)$$

$$\lim_{R\to\infty} E_R = 0 \quad . \quad (2.33)$$

In this section it will be shown that the above far-field equations are also obtainable by making use of the spherical-wave expansion for a half-wave dipole. First, the far-field properties of the following equations are studied. From Equations (2.10) and (2.12), it is obtained that

$$\lim_{R\to\infty} \frac{e^{-ikr_2}}{r_2} = -ik \sum_{n=0}^{\infty} T(n) \lim_{R\to\infty} h_n^{(2)}(kR) \tag{2.34}$$

$$\lim_{R\to\infty} \frac{e^{-ikr_1}}{r_1} = -ik \sum_{n=0}^{\infty} (-1)^n T(n) \lim_{R\to\infty} h_n^{(2)}(kR) \tag{2.35}$$

where

$$T(n) = (2n + 1)\, j_n(k \tfrac{\lambda}{4})\, p_n(\cos\theta) \quad . \tag{2.36}$$

In Figure 1, as R goes to infinity, it is easily noticed that r_1 and r_2 are approximated as

$$r_1 \approx R + \frac{\lambda}{4}\cos\theta \quad , \tag{2.37}$$

$$r_2 \approx R - \frac{\lambda}{4}\cos\theta \quad . \tag{2.38}$$

It is also well known that the Hankel function of the second kind is asymptotically given by

$$h_n^{(2)}(kR) \approx (i)^{n+1} \frac{e^{-ikR}}{kR} \quad , \tag{2.39}$$

when R approaches infinity.

Now substituting Equations (2.37), (2.38), and (2.39) back into Equations (2.34) and (2.35), it follows

$$\frac{e^{-ik(R - \frac{\lambda}{4}\cos\theta)}}{R} = \frac{e^{-ikR}}{R} \sum_{n=0}^{\infty} (i)^n\, T(n) \quad , \tag{2.40}$$

and

$$\frac{e^{-ik(R + \frac{\lambda}{4}\cos\theta)}}{R} = \frac{e^{-ikR}}{R} \sum_{n=0}^{\infty} (-i)^n \, T(n) \quad . \tag{2.41}$$

From the above two equations, one obtains

$$e^{ik\frac{\lambda}{4}\cos\theta} = \sum_{n=0}^{\infty} (i)^n \, T(n) \quad , \tag{2.42}$$

$$e^{-ik\frac{\lambda}{4}\cos\theta} = \sum_{n=0}^{\infty} (-i)^n \, T(n) \quad . \tag{2.43}$$

Now it is ready to demonstrate that Equations (2.31), (2.32), and (2.33) are obtained from Equations (2.16), (2.17), and (2.18). Substituting Equations (2.37) through (2.39) back into Equations (2.16), (2.17), and (2.18), one obtains

$$\lim_{R\to\infty} H_\phi = \left(\frac{iI_0}{4\pi}\right)\left(\frac{e^{-ikR}}{R\sin\theta}\right)\left\{\sum_{n=0}^{\infty} (-i)^n \, T(n) + \sum_{n=0}^{\infty} (i)^n \, T(n)\right\} \quad , \tag{2.44}$$

$$\lim_{R\to\infty} E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\left(\frac{iI_0}{4\pi}\right) \frac{e^{-ikR}}{R\sin\theta}\left\{\sum_{n=0}^{\infty} (-i)^n \, T(n) + \sum_{n=0}^{\infty} (i)^n \, T(n)\right\} \quad , \tag{2.45}$$

$$\lim_{R\to\infty} E_R = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\left(\frac{iI_0}{16\pi}\right) \frac{e^{-ikR}}{R^2}\left\{\sum_{n=0}^{\infty} (-i)^n \, T(n) - \sum_{n=0}^{\infty} (i)^n \, T(n)\right\} \quad . \tag{2.46}$$

By making use of the results of Equations (2.42) and (2.43) in Equations (2.44), (2.45), and (2.46), it is expressed that

$$\lim_{R\to\infty} H_\phi = \frac{iI_0}{4\pi}\,\frac{e^{-ikR}}{R\sin\theta}\left[e^{ik\frac{\lambda}{4}\cos\theta} + e^{-ik\frac{\lambda}{4}\cos\theta}\right] = \frac{iI_0}{2\pi}\,\frac{e^{-ikR}}{R}\,\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \tag{2.47}$$

$$\lim_{R\to\infty} E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{iI_0}{4\pi} \frac{e^{-ikR}}{R \sin\theta} \left[e^{ik\frac{\lambda}{4}\cos\theta} + e^{-ik\frac{\lambda}{4}\cos\theta}\right]$$

$$= iI_0 60 \frac{e^{-ikR}}{R} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \quad , \quad (2.48)$$

$$\lim_{R\to\infty} E_R = -\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{iI_0}{16\pi} \frac{e^{-ikR}}{R} \left[e^{ik\frac{\lambda}{4}\cos\theta} - e^{-ik\frac{\lambda}{4}\cos\theta}\right]$$

$$= -15iI_0 \left(\frac{e^{-ikR}}{R^2}\right) \sin\left(\frac{\pi}{2}\cos\theta\right) \quad . \quad (2.49)$$

As R approaches infinity,

$$\frac{e^{-ikR}}{R^2} << \frac{e^{-ikR}}{R} \quad ;$$

therefore, it is concluded that

$$\lim_{R\to\infty} E_R << \lim_{R\to\infty} H_\phi \quad ,$$

and

$$\lim_{R\to\infty} E_R << \lim_{R\to\infty} E_\theta \quad .$$

Therefore, from Equations (2.47), (2.48), and (2.49), one obtains

$$\lim_{R\to\infty} H_\phi = \frac{iI_0}{2} \frac{e^{-ikR}}{R} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \quad , \quad (2.50)$$

$$\lim_{R\to\infty} E_\theta = iI_0 60 \frac{e^{-ikR}}{R} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \quad , \quad (2.51)$$

$$\lim_{R\to\infty} E_R = 0 \quad . \quad (2.52)$$

Equations (2.50), (2.51), and (2.52) are exactly the same as Equations (2.31), (2.32), and (2.33), respectively. Thus it is proved that the spherical-wave expansion which is obtained as Equations (2.16), (2.17), and (2.18) is satisfied with well known classical far-field solutions for a half-wave dipole. Therefore, the proper behavior of the expansion has been demonstrated.

Convergence of the Spherical-Wave Expansion in the Far-Field

It has been found that, in general, a spherical-wave expansion requires an infinite number of terms of the spherical-wave functions to represent the fields of a half-wave dipole. Therefore, it may be of interest to study how fast the spherical-wave expansion converges to the radiation field.

From Equations (2.44) and (2.45), it is found that

$$\lim_{R\to\infty} H_\phi = \frac{iI_0}{2\pi}\frac{e^{-ikR}}{R\sin\theta}\sum_{n:\text{even}}^{\infty}(i)^n\,T(n) = \frac{iI_0}{2\pi}\frac{e^{-ikR}}{R\sin\theta}\sum_{n=0}^{\infty}(-1)^n\,T(2n) \,, \tag{2.53}$$

and

$$\lim_{R\to\infty} E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\frac{iI_0}{2\pi}\frac{e^{-ikR}}{R\sin\theta}\sum_{n:\text{even}}^{\infty}(i)^n\,T(n)$$

$$= (iI_0 60)\,\frac{e^{-ikR}}{R\sin\theta}\sum_{n=0}^{\infty}(-1)^n\,T(2n) \,. \tag{2.54}$$

In the preceding section, it has been proved that Equations (2.44), (2.45), and (2.46) are equal to Equations (2.31), (2.32), and (2.33), respectively. Since Equations (2.53) and (2.54) are rewritten from Equations (2.44) and (2.45),

$$\frac{iI_0}{2}\frac{e^{-ikR}}{R}\frac{1}{\sin\theta}\sum_{n=0}^{\infty}(-1)^n\,T(2n) = \frac{iI_0}{2}\frac{e^{-ikR}}{R}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} ,$$

and

$$(iI_0 60)\,\frac{e^{-ikR}}{R}\frac{1}{\sin\theta}\sum_{n=0}^{\infty}(-1)^n\,T(2n) = (iI_0 60)\,\frac{e^{-ikR}}{R}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} .$$

The above two equations lead to

$$\sum_{n=0}^{\infty}(-1)^n\,(4n+1)\,j_{2n}(\tfrac{\pi}{2})\,\frac{p_{2n}(\cos\theta)}{\sin\theta} = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} . \quad (2.55)$$

At $\theta = \pi/2$, Equation (2.55) becomes

$$\sum_{n=0}^{\infty}(-1)^n\,(4n+1)\,j_{2n}(\tfrac{\pi}{2})\,p_{2n}(0) = 1.0 . \quad (2.56)$$

Therefore, the convergence of the spherical-wave expansion is studied by checking how fast the left-hand side of Equation (2.56) approaches the value 1.0. Keeping this in mind, the first few terms of Equation (2.56) are calculated and examined to see if their total is close enough to 1.0.

It is well known that

$$p_{2n}(0) = \frac{(-1)^n\,(2n-1)!!}{(2n)!!} \quad (2.57)$$

where

$$n!! = \begin{cases} n(n-2)\ldots 3.1 \text{ for } n\text{:odd} \\ n(n-2)\ldots 4.2 \text{ for } n\text{:even,} \end{cases}$$

$0!! = 1$, and

$(-1)!! = 1$.

By making use of Equation (2.57), Equation (2.56) is rewritten in the following form:

$$\sum_{n=0}^{\infty} (-1)^n (4n+1) j_{2n}(\tfrac{\pi}{2}) p_{2n}(0) = \sum_{n=0}^{\infty} (-1)^{2n} (4n+1) j_{2n}(\tfrac{\pi}{2}) \frac{(2n-1)!!}{(2n)!!}$$

$$= \sum_{n=0}^{\infty} (4n+1) \frac{(2n-1)!!}{(2n)!!} j_{2n}(\tfrac{\pi}{2}) = \sum_{n=0}^{\infty} H(n) \quad , \qquad (2.58)$$

where

$$H(n) = (4n+1) \frac{(2n-1)!!}{(2n)!!} J_{2n}(\tfrac{\pi}{2}) \quad . \qquad (2.59)$$

Calculating H(n) for n = 0, 1, 2, 3,:

$$\begin{aligned} H(0) &= j_0(\tfrac{\pi}{2}) = 0.6366197724 \quad , \\ H(1) &= \tfrac{5}{2} j_2(\tfrac{\pi}{2}) = 0.3435426348 \quad , \\ H(2) &= \tfrac{27}{8} j_4(\tfrac{\pi}{2}) = 0.0194170804 \quad , \\ H(3) &= \tfrac{65}{16} j_6(\tfrac{\pi}{2}) = 0.0004149267 \quad . \end{aligned} \qquad (2.60)$$

Substituting Equation (2.60) back into Equation (2.58), the following values for $\sum_{n=0} H(n)$ are found:

$$\begin{aligned} H(0) &= 0.6366197724 \quad , \\ H(0) + H(1) &= 0.9801624072 \quad , \\ H(0) + H(1) + H(2) &= 0.9995794876 \quad , \\ H(0) + H(1) + H(2) + H(3) &= 0.9999944143 \quad . \end{aligned} \qquad (2.61)$$

It is apparent from the above results that the spherical-wave expansion converges very rapidly, and the first three terms of the exapnsion should be quite adequate to represent the field to the degree that could be confirmed by physical measurement.

CHAPTER III

A SPHERICAL-WAVE EXPANSION FOR THE FIELDS OF A HALF-WAVE DIPOLE -- (2)

Introduction

The antenna's radiation field may be divided into two regions. One is the region outside the smallest imaginary sphere which can enclose the antenna. The other is the region inside the smallest imaginary sphere which can enclose the antenna. These two regions are illustrated in Figure 2. The outside region will be called Region A and the inside region will be called Region B.

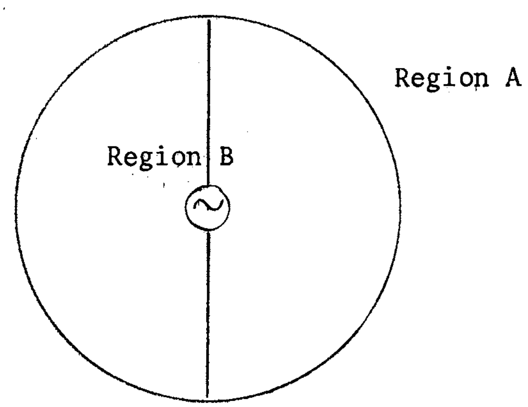

Figure 2. The Smallest Imaginary Sphere Which can Enclose the Antenna

According to L. J. Chu (2), the radiation field of an antenna which has a transverse magnetic field is represented by the following equations in the Region A:

$$H_\phi = \sum_{n=0}^{\infty} A_n P_n^1(\cos\theta) h_n^{(2)}(kR) \quad , \qquad (3.1)$$

$$E_R = -i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n n(n+1) P_n(\cos\theta) \frac{h_n^{(2)}(kR)}{kR} \quad , \qquad (3.2)$$

$$E_\theta = i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n P_n^1(\cos\theta) \frac{1}{kR} \frac{d}{d(kR)} kR h_n^{(2)}(kR) \quad , \qquad (3.3)$$

where

$P_n(\cos\theta)$ = Legendre function,

$P_n^1(\cos\theta) = \sin\theta \dfrac{d}{d(\cos\theta)} P_n(\cos\theta)$

= Associated Legendre function, and

$h_n^{(2)}(kR)$ = Spherical Hankel function of the second kind.

A_n are the coefficients which should be calculated by making use of the boundary conditions or by some other well known results that have been obtained for the antenna.

It is well known that the field of the electric dipole which is shown in Figure 3 is represented by the single term of the spherical-wave function:

$$H_\phi = A_1 P_1^1(\cos\theta) h_n^{(2)}(kR) \quad ,$$

$$E_R = -2i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} A_1 P_1(\cos\theta) \frac{h_1^{(2)}(kR)}{kR} \quad ,$$

$$E_\theta = i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} A_1 P_1^1(\cos\theta) \frac{1}{kR} \frac{d}{d(kR)} kR h_n^{(2)}(kR) \quad ,$$

where

$$A_1 = \frac{k^3 P}{4\pi\sqrt{\mu\varepsilon}},$$

$P = Q_0 L,$

q = electric charge,

$i = \frac{dq}{dt}$ = current, and

Q_0 = effective value of q.

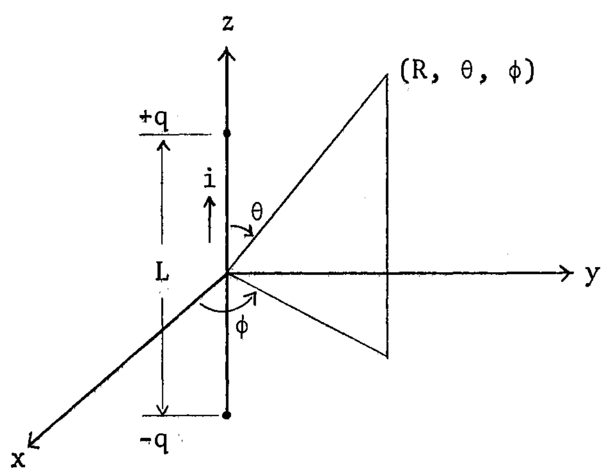

Figure 3. Electric Dipole of Length L

The purposes of this chapter are:

(1) to find the coefficients A_n for a half-wave dipole and for linear radiators with assumed sinusoidal current distribution;

(2) to insure the proper behavior of the obtained spherical wave expansion;

(3) to check the convergence of the wave expansion; and

(4) to discuss the spherical wave expansion's relation to the theory of Chu (2) and Harrington (8).

Finding the Coefficients A_n

In the preceding chapter, it was demonstrated that Equations (2.16), (2.17), and (2.18) behave properly as the electromagnetic field solution for a half-wave dipole antenna. Therefore, it is possible to determine the coefficients A_n by equating Equation (2.18) and Equation (3.2). Then it is found that

$$(\frac{\mu}{\varepsilon})^{\frac{1}{2}}(\frac{I_0}{8}) \sum_{n=0}^{\infty} [(-1)^n - 1] \frac{Y(n)}{R} = - i (\frac{\mu}{\varepsilon})^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n n(n+1) P_n(\cos\theta) \frac{h_n^{(2)}(kR)}{kR} . \tag{3.4}$$

From the above equation, the coefficients A_n are obtained:

$$A_n = i (\frac{kI_0}{8}) \frac{[(-1)^n - 1] (2n + 1) J_n(\frac{\pi}{2})}{n(n + 1)} . \tag{3.5}$$

Equation (3.5) also can be rewritten as

$$A_n = \begin{cases} -\dfrac{ikI_0 (2n + 1) J_n(\frac{\pi}{2})}{4n(n + 1)} & \text{for } n\text{:odd} \\ 0 & \text{for } n\text{:even} \end{cases} . \tag{3.6}$$

Substituting Equation (3.6) back into Equation (3.1) through Equation (3.3), then the field equations for a half-wave dipole antenna become

$$H_\phi = - \frac{ikI_0}{4} \sum_{n:odd}^{\infty} X(n) h_n^{(2)}(kR) , \tag{3.7}$$

$$E_R = -\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{4} \sum_{n:odd}^{\infty} \frac{Y(n)}{kR} \quad , \qquad (3.8)$$

and

$$E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{kI_0}{4} \sum_{n:odd}^{\infty} X(n) \frac{1}{kR} \frac{d}{d(kR)} [kRh_n^{(2)}(kR) \quad , \qquad (3.9)$$

where

$$Y(n) = (2n + 1)\ j_n(\tfrac{\pi}{2})\ P_n(\cos\ \theta)\ h_n^{(2)}(kR) \quad , \qquad (2.13)$$

$$X(n) = \frac{(2n + 1)\ J_n(\frac{\pi}{2})}{n(n + 1)} P_n^1(\cos\ \theta) \quad . \qquad (3.10)$$

Thus using Equation (2.18), new spherical-wave expansion for the field in Region A of a half-wave dipole is obtained.

Convergence of the Expansion

The convergence of the new spherical-wave expansion is checked in the following pages.

From Equation (3.7) through Equation (3.9), it is apparent that the new spherical-wave expansion consists of an infinite number of terms of the spherical functions. How, then, does this expansion converge? To examine the convergence of Equations (3.7) and (3.9), the classical far-field solution for a half-wave dipole is used. It is:

$$\lim_{R\to\infty} H_\phi = i\ \frac{I_0}{2}\ \frac{e^{-ikR}}{R}\ \frac{\cos(\frac{\pi}{2}\cos\ \theta)}{\sin\ \theta} \quad , \qquad (2.31)$$

$$\lim_{R\to\infty} E_\theta = i\ \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{I_0}{2}\ \frac{e^{-ikR}}{R}\ \frac{\cos(\frac{\pi}{2}\cos\ \theta)}{\sin\ \theta} \quad . \qquad (2.32)$$

Since the maximum field intensity occurs at $\theta = \pi/2$, the normalized values of the field intensity become 1.0 at this angle; that is

$$\left| \frac{\lim_{R\to\infty} H_\phi}{i\,\frac{I_0}{2}\,\frac{e^{-ikR}}{R}} \right|_{\theta=\frac{\pi}{2}} = \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \right]_{\theta=\frac{\pi}{2}} = 1.0 \quad , \tag{3.11}$$

and

$$\left| \frac{\lim_{R\to\infty} E_\theta}{i\,(\frac{\mu}{\varepsilon})^{\frac{1}{2}}\,\frac{I_0}{2}\,\frac{e^{-ikR}}{R}} \right|_{\theta=\frac{\pi}{2}} = \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \right]_{\theta=\frac{\pi}{2}} = 1.0 \tag{3.12}$$

Next, Equations (3.7) and (3.9) are used to evaluate

$$\left| \frac{\lim_{R\to\infty} H_\phi}{i\,(\frac{I_0}{2})\,\frac{e^{-ikR}}{R}} \right| , \text{ and } \left| \frac{\lim_{R\to\infty} E_\theta}{i\,(\frac{\mu}{\varepsilon})^{\frac{1}{2}}\,\frac{I_0}{2}\,\frac{e^{-ikR}}{R}} \right| .$$

Recall that as R approaches infinity, the Hankel function $h_n^{(2)}(kR)$ is asymptotically represented as

$$h_n^{(2)}(kR) \approx (i)^{n+1}\,\frac{e^{-ikR}}{kR} \quad , \tag{2.39}$$

and using this result, it follows, in the far-field,

$$\frac{d}{d(kR)}\,[kRh_n^{(2)}(kR)] \approx i(-1)^{\frac{n+3}{2}}\,e^{-ikR} \quad . \tag{3.13}$$

Substituting Equations (2.39) and (3.13) back into Equations (3.7) and (3.9), respectively,

$$\lim_{R\to\infty} H_\phi = i\left(\frac{I_0}{4}\right)\frac{e^{-ikR}}{R}\sum_{n:odd}^{\infty}(-1)^{\frac{n+3}{2}}X(n) \quad , \tag{3.14}$$

$$\lim_{R\to\infty} E_\theta = i\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\left(\frac{I_0}{4}\right)\frac{e^{-ikR}}{R}\sum_{n:odd}^{\infty}(-1)^{\frac{n+3}{2}}X(n) \quad . \tag{3.15}$$

Equations (3.14) and (3.15) are the far-field representation of the spherical-wave expansion for a half-wave dipole. Dividing the both sides of Equation (3.14) by $i\left(\frac{I_0}{2}\right)\frac{e^{-ikR}}{R}$, it is obtained that

$$\left|\frac{\lim_{R\to\infty} H_\phi}{i\left(\frac{I_0}{2}\right)\frac{e^{-ikR}}{R}}\right| = \sum_{n:odd}^{\infty}\frac{\pi}{2}(-1)^{\frac{n+3}{2}}X(n) \quad , \tag{3.16}$$

and dividing Equation (3.15) by $i\left(\frac{\mu}{\varepsilon}\right)^{1/2}\frac{I_0}{2}\frac{e^{-ikR}}{R}$, it is also obtained that

$$\left|\frac{\lim_{R\to\infty} E_\theta}{i\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\frac{I_0}{2}\frac{e^{-ikR}}{R}}\right| = \sum_{n:odd}^{\infty}\frac{\pi}{2}(-1)^{\frac{n+3}{2}}X(n) \quad . \tag{3.17}$$

If Equations (3.16) and (3.17) are compared with Equations (3.12) and (3.13), it is found that

$$\sum_{n:odd}^{\infty}\frac{\pi}{2}(-1)^{\frac{n+3}{2}}X(n) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \quad . \tag{3.18}$$

At $\theta = \pi/2$, the right side of the above equation becomes 1.0 and it follows

$$\sum_{n:odd}^{\infty} \frac{\pi}{2} (-1)^{\frac{n+3}{2}} [X(n)]_{\theta = \frac{\pi}{2}} = 1.0 \quad . \tag{3.19}$$

From Equation (3.19), it will be noted that a measure of the number of terms in the spherical-wave expansion which is required to represent the field might be obtained by determining how rapidly the value of unity is approached. This has been achieved through the following procedure.

First, it is required to know how the associated Legendre function $P_n^1(0)$ is represented. Fortunately, it is well known that

$$P_n^1(0) = \begin{cases} (-1)^{\frac{n-1}{2}} \dfrac{n!}{2^{n-1} \left(\frac{n-1}{2}!\right)^2} & \text{for } n\text{:odd} \\ \\ 0 & \text{for } n\text{:even} \quad . \end{cases} \tag{3.20}$$

Therefore, substituting Equation (3.20) back into Equation (3.19), it becomes

$$\sum_{n:odd}^{\infty} \frac{\pi}{2} (-1)^{n+1} M(n) = \sum_{n:odd}^{\infty} E(n) = 1.0 \tag{3.21}$$

where it is defined that

$$E(n) = \frac{\pi}{2} (-1)^{n+1} M(n) \quad , \tag{3.22}$$

and

$$M(n) = \frac{(2n+1)\ n!\ j_n(\frac{\pi}{2})}{n(n+1)\ 2^{n-1} \left(\frac{n-1}{2}!\right)^2} \quad . \tag{3.23}$$

Performing the indicated computations for the E(n), the following values are obtained:

$$
\begin{aligned}
E(1) &= \frac{3\pi}{4} j_1(\tfrac{\pi}{2}) = 0.9549296588 \\
E(3) &= \frac{7\pi}{16} j_3(\tfrac{\pi}{2}) = 0.0441573110 \\
E(5) &= \frac{11\pi}{32} j_5(\tfrac{\pi}{2}) = 0.0009020484 \\
E(7) &= \frac{75\pi}{256} j_7(\tfrac{\pi}{2}) = 0.0000099328 \\
&\vdots
\end{aligned}
\qquad (3.24)
$$

To observe how fast Equation (3.21) approaches unity, one substitutes Equation (3.24) back into Equation (3.21) and the following partial sums are obtained:

$$
\begin{aligned}
&E(1) = 0.9549296588 \\
&E(1) + E(3) = 0.99908698 \\
&E(1) + E(3) + E(5) = 0.9999899182 \\
&E(1) + E(3) + E(5) + E(7) = 0.9999998510 \quad ,
\end{aligned}
\qquad (3.25)
$$

It is readily apparent from the above results that the spherical-wave expansion converges quite rapidly and that the first two terms should be quite adequate to represent the field that could be confirmed by physical measurement.

It has been seen in Equation (2.61) that the spherical-wave expansion which is found in page 11 needs three terms to represent the field. Therefore, one can say that the spherical-wave expansion given in pages 23 and 24 converges more rapidly.

Directive Gain

In the preceding pages, the coefficients A_n were obtained as Equation (3.6). By substituting Equation (3.6) back into Equation (3.1) through Equation (3.3), a spherical-wave expansion for a half-wave dipole antenna was found as Equations (3.7) through (3.9). Now the general equations for the directive gain and the radiation impedance of an antenna will be obtained by making use of Equation (3.1) through Equation (3.3). Then, substituting A_n into the derived equations, it is demonstrated that the results are the same as those obtained from the classical field solutions for a half-wave dipole.

The definition for the directive gain is given as

$$G(\theta) = \frac{\left|\lim_{R\to\infty} \overline{E}\right|^2}{\frac{1}{4\pi} \oint \left|\lim_{R\to\infty} \overline{E}\right|^2 \sin\theta \, d\theta \, d\phi} \tag{3.26}$$

where

$$\overline{E} = E_R \, \overline{u}_R + E_\theta \, \overline{u}_\theta + E_\phi \, \overline{u}_\phi$$

and $\overline{u}$, $\overline{u}_\theta$, and $\overline{u}_\phi$ are unit vectors, whose direction is toward increasing R, θ, and ϕ, respectively. Recall that when R approaches infinity, $h_n^{(2)}(kR)$ is asymptotically represented as

$$\lim_{R\to\infty} h_n^{(2)}(kR) \approx (i)^{n+1} \frac{e^{-ikR}}{kR} = (-1)^{\frac{n+1}{2}} \frac{e^{-ikR}}{kR} \quad .$$

Therefore,

$$\lim_{R\to\infty} \frac{d}{d(kR)} [kRh_n^{(2)}(kR)] \approx -i(-1)^{\frac{n+1}{2}} e^{-ikR} \quad .$$

Substituting the above two equations back into Equations (3.1) and (3.2), respectively, one obtains

$$\lim_{R\to\infty} E_R = i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^{\frac{n+3}{2}} A_n\, n(n+1)\, P_n(\cos\theta)\, \frac{e^{-ikR}}{(kR)^2} \quad , \qquad (3.27)$$

and

$$\lim_{R\to\infty} E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n\, P_n^1(\cos\theta)\, \frac{e^{-ikR}}{kR} \quad . \qquad (3.28)$$

As R goes to infinity, it is observed that

$$\frac{e^{-ikR}}{(kR)^2} << \frac{e^{-ikR}}{kR} \quad ;$$

therefore, if $\lim_{R\to\infty} E_R$ is compared with $\lim_{R\to\infty} E_\theta$, it is concluded that

$$\left|\lim_{R\to\infty} E_R\right| << \left|\lim_{R\to\infty} E_\theta\right| \quad ;$$

therefore, the magnitude of $\lim_{R\to\infty} E(\theta, \phi)$ becomes.

$$\left|\lim_{R\to\infty} E(\theta, \phi)\right| = \left|\lim_{R\to\infty} E_\theta\right| \quad . \qquad (3.29)$$

By making use of Equation (3.29) in Equation (3.26), the directive gain is expressed in terms of E_θ. The result is:

$$G(\theta) = \frac{\left|\lim_{R\to\infty} E_\theta\right|^2}{\frac{1}{4\pi} \oint \left|\lim_{R\to\infty} E_\theta\right|^2 \sin\theta \;\; d\theta\; d\phi} \quad . \qquad (3.30)$$

Using Equation (3.28), the numerator of Equation (3.30) becomes

$$\left|\lim_{R\to\infty} E_\theta\right|^2 = \left|\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{e^{-ikR}}{kR}\right|^2 \left|\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(\cos\theta)\right|^2 \tag{3.31}$$

$$= \left(\frac{\mu}{\varepsilon}\right) \frac{1}{k^2R^2} \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(\cos\theta)\right] \cdot \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n^* P_n^1(\cos\theta)\right] \tag{3.32}$$

where A_n^* indicates complex conjugate of A_n. Using Equation (3.32), the denominator of Equation (3.30) is

$$\frac{1}{4\pi} \oint \left|\lim_{R\to\infty} E_\theta\right|^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi} \left(\frac{\mu}{\varepsilon}\right) \frac{1}{k^2R^2} \int_0^{2\pi} d\phi \int_0^{\pi} \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(\cos\theta)\right]$$

$$\cdot \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n^* P_n^1(\cos\theta)\right] \sin\theta \, d\theta$$

$$= \left(\frac{\mu}{\varepsilon}\right) \frac{1}{2k^2R^2} \int_0^{\pi} \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(\cos\theta)\right]$$

$$\cdot \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n^* P_n^1(\cos\theta)\right] \sin\theta \, d\theta \quad . \tag{3.33}$$

However, from the orthogonality of the associated Legendre functions, it is well known that

$$\int_0^{\pi} P_n^1(\cos\theta)\, P_m^1(\cos\theta) \sin\theta \, d\theta = \begin{cases} \dfrac{2n(n+1)}{2n+1} & \text{for } n=m \\ 0 & \text{for } n\neq m \end{cases} \quad . \tag{3.34}$$

Therefore, making use of the results of the above,

$$\int_0^{\pi} \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(\cos\theta) \right] \cdot \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n^* P_n^1(\cos\theta) \right] \sin\theta \, d\theta$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \left| A_n \right|^2 \frac{2n(n+1)}{2n+1} \quad . \qquad (3.35)$$

Finally, substituting Equation (3.35) back into Equation (3.33), the denominator of Equation (3.30) becomes

$$\frac{1}{4\pi} \oint \left| \lim_{R\to\infty} E_\theta \right| \sin\theta \, d\theta \, d\phi = \left(\frac{\mu}{\varepsilon}\right) \frac{1}{k^2R^2} \sum_{n=0}^{\infty} (-1)^{n+1} \left| A_n \right|^2 \frac{n(n+1)}{2n+1} \quad . \qquad (3.36)$$

By making use of Equations (3.32) and (3.36) in Equation (3.30), the directive gain for an angle θ is expressed as

$$G(\theta) = \frac{\left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(\cos\theta) \right] \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n^* P_n^1(\cos\theta) \right]}{\sum_{n=0}^{\infty} (-1)^{n+1} \left| A_n \right|^2 \frac{n(n+1)}{2n+1}} \qquad (3.37)$$

Since the directive gain is usually measured at the angle $\theta = \pi/2$, it may be convenient to obtain the expression for $G(\pi/2)$. Setting $\theta = \pi/2$ in Equation (3.37), $G(\pi/2)$ is given by

$$G\left(\frac{\pi}{2}\right) = \frac{\left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n P_n^1(0) \right] \left[\sum_{n=0}^{\infty} (-1)^{\frac{n+1}{2}} A_n^* P_n^1(0) \right]}{\sum_{n=0}^{\infty} (-1)^{n+1} \left| A_n \right|^2 \frac{n(n+1)}{2n+1}} \quad . \qquad (3.38)$$

Substituting Equation (3.19) back into Equation (3.38), $G(\pi/2)$ finally becomes

$$G(\frac{\pi}{2}) = \frac{\left[\sum_{n=0}^{\infty} (-1)^n \frac{n!\, A_n}{2^{n-1} (\frac{n-1}{2}!)^2}\right]\left[\sum_{n=0}^{\infty} (-1)^n \frac{n!\, A_n^*}{2^{n-1} (\frac{n-1}{2}!)^2}\right]}{\sum_{n=0}^{\infty} (-1)^{n+1} |A_n|^2 \frac{n(n+1)}{2n+1}} \qquad (3.39)$$

This is the final form of equation for the directive gain of an antenna at the angle $\theta = \pi/2$. Therefore, if the proper coefficients A_n are available, then $G(\pi/2)$ of any antenna is readily evaluated.

The Directive Gain for a Half-Wave Dipole

By making use of Equation (3.39) with Equation (3.6), it is now possible to evaluate the directive gain for a half-wave dipole.

In Equation (3.6), the coefficients A_n for a half-wave dipole are found as

$$A_n = \begin{cases} -\dfrac{ikI_0\,(2n+1)\,J_n(\frac{\pi}{2})}{4n(n+1)} & \text{for } n\text{:odd} \\ 0 & \text{for } n\text{:even} \end{cases} \qquad (3.6)$$

Then, by making use of the above values, the numerator and the denominator of Equation (3.39) become

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!\, A_n}{2^{n-1} (\frac{n-1}{2}!)^2} = -\left(\frac{ikI_0}{4}\right) \left\{ \sum_{n:odd}^{\infty} (-1)^n M(n) \right\} \quad , \qquad (3.40)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!\, A_n^*}{2^{n-1} (\frac{n-1}{2}!)^2} = \left(\frac{ikI_0}{4}\right) \left\{ \sum_{n:odd}^{\infty} (-1)^n M(n) \right\} \quad , \qquad (3.41)$$

and

$$\sum_{n=0}^{\infty} (-1)^{n+1} |A_n|^2 \frac{n(n+1)}{2n+1} = \left(\frac{kI_0}{4}\right)^2 \left\{ \sum_{n:odd}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} [j_n(\tfrac{\pi}{2})]^2 \right\} . \qquad (3.42)$$

Substituting Equations (3.40), (3.41), and (3.42) back into Equation (3.39), then the directive gain $G_H(\pi/2)$ of a half-wave dipole is expressed by the following equation:

$$G_H(\tfrac{\pi}{2}) = \frac{\left[\sum_{n:odd}^{\infty} (-1)^n M(n) \right]^2}{\sum_{n:odd}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} [J_n(\frac{\pi}{2})]^2} = \frac{\left[\sum_{n:odd}^{\infty} D(n) \right]}{\sum_{n:odd}^{\infty} f(n)} \quad , \qquad (3.43)$$

where it is defined that

$$D(n) = (-1)^n M(n) \quad , \qquad (3.44)$$

and

$$f(n) = (-1)^{n+1} \frac{2n+1}{n(n+1)} [j_n(\tfrac{\pi}{2})]^2 \quad . \qquad (3.45)$$

Performing the indicated computation for D(n) and f(n),

$$D(1) = -\frac{3}{2} j_1(\frac{\pi}{2}) \quad = -0.6079271019$$

$$D(3) = -\frac{7}{8} j_3(\frac{\pi}{2}) \quad = -0.0281114173$$

$$D(5) = -\frac{11}{16} j_5(\frac{\pi}{2}) \quad = -0.0005748348$$

$$D(7) = -\frac{75}{128} j_7(\frac{\pi}{2}) = -0.00000637324 \quad . \qquad (3.46)$$

$$f(1) = \frac{3}{2} [j_1(\frac{\pi}{2})^2 \quad = 0.2463835742$$

$$f(3) = \frac{7}{12} [j_3(\frac{\pi}{2})]^2 = 0.0006020966$$

$$f(5) = \frac{11}{15} [j_5(\frac{\pi}{5})]^2 = 0.0000002563$$

$$\vdots \quad . \qquad (3.47)$$

From Equation (3.46),

$$\sum_{n:odd}^{\infty} D(n) = D(1) + D(3) + D(5) + D(7) + \cdots \approx -0.6366196774 \quad ; \qquad (3.48)$$

therefore, the numerator of Equation (3.43) becomes

$$\left[\sum_{n:odd}^{\infty} D(n)\right]^2 = [D(1) + D(3) + D(5) + D(7) + \cdots]^2 \approx 0.4052846137 \quad . \qquad (3.49)$$

From Equation (3.47), the denominator of Equation (3.43) is given as

$$\sum_{n:odd}^{\infty} f(n) = f(1) + f(3) + f(5) + f(7) + \cdots \approx 0.2469859271 . \qquad (3.50)$$

In Equations (3.48), (3.49), and (3.50), only first few terms of the series are used, because it has been proved that the spherical-wave expansion converges very rapidly, and the first few terms are quite adequate to represent the fields of a half-wave dipole.

Now, Equations (3.49) and (3.50) give us the directive gain of a half-wave dipole as

$$G_H(\frac{\pi}{2}) = \frac{\left[\sum_{n:odd}^{\infty} D(n)\right]^2}{\sum_{n:odd}^{\infty} f(n)} = 1.640921888 \qquad (3.51)$$

It is easily recognizable that the value given in Equation (3.51) is equal to the value of the directive gain which is obtained by making use of a classical field solution for a half-wave dipole.

Radiation Power and Radiation Impedance

Assuming that there is an imaginary sphere enclosing the whole antenna system, the sphere has radius a and its surface is S. Then, the radiation power W out of the surface S will be expressed by the real part of the surface integral of the complex Poynting vector $\overline{P} = (\overline{E} \times \overline{H}^*)$.

$$W = \text{real part of} \int_S (\overline{E} \times \overline{H}^*) \cdot \overline{n}\, dS \qquad , \qquad (3.52)$$

where $\overline{n}$ is a unit vector which is normal to the surface S, and $\overline{H}^*$ is conjugate of $\overline{H}$.

For a half-wave dipole, the electric field consists of the R-directional component and the θ-directional component; and the magnetic field has only the ϕ-directional component; that is,

$$\overline{E} = E_R\, \overline{u}_R + E_\theta\, \overline{u}_\theta \qquad , \qquad (3.53)$$

$$\overline{H} = H_\phi \, \overline{u}_\phi \tag{3.54}$$

where $\overline{u}_R$, $\overline{u}_\theta$, and $\overline{u}_\phi$ are unit vectors, whose direction is toward increasing R, θ, and φ of the spherical coordinates, respectively. It also should be noted that $\overline{n} = \overline{u}_R$. Then, the surface integral of the complex Poynting vector is expressed as

$$\int_S (\overline{E} \times \overline{H}^*) \cdot \overline{n} \, dS = \int_S [(E_R \, \overline{u}_R + E_\theta \, \overline{u}_\theta) \times (H_\phi \, \overline{u}_\phi)] \cdot \overline{u}_R \, dS \quad . \tag{3.55}$$

Applying $\overline{u}_R \times \overline{u}_\phi = - \overline{u}_\theta$, $\overline{u}_\theta \times \overline{u}_\phi = \overline{u}_R$, $\overline{u}_\theta \cdot \overline{u}_R = 0$, and $\overline{u}_R \cdot \overline{u}_R = 1$ to Equation (3.55), it becomes

$$\int_S (\overline{E} \times \overline{H}^*) \cdot \overline{n} \, dS = \int_S E_\theta \, H_\phi^* \, dS = \int_0^{2\pi} \int_0^{\pi} E_\theta \, H_\phi^* \, a^2 \sin\theta \, d\theta \, d\phi . \tag{3.56}$$

By making use of Equations (3.1) and (3.3) in Equation (3.56), the surface integral of the complex Poynting vector is given in terms of the spherical-wave functions and the coefficients A_n. The procedures are given in the following:

$$\int_S (\overline{E} \times \overline{H}^*) \cdot \overline{n} \, dS$$

$$= i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} 2\pi \int_0^{\pi} \left\{ \sum_{n=0}^{\infty} A_n \, P_n^1(\cos\theta) \, \frac{1}{ka} \left[\frac{d}{d(kR)} [kR \, h_n^{(2)}(kR)]\right]_{R=a} \right\}$$

$$\cdot \left\{ \sum_{n=0}^{\infty} A_n^* \, P_n^1(\cos\theta) \, \frac{1}{ka} [kR \, h_n^{(2)}(kR)]_{R=a}^* \right\} a^2 \sin\theta \, d\theta \quad . \tag{3.57}$$

Applying Equation (3.34) to Equation (3.57), it becomes

$$\int_S (\overline{E} \times \overline{H}^*) \cdot \overline{n} \, dS$$

$$= \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{4\pi}{k^2} \sum_{n=0}^{\infty} \left|A_n\right|^2 \frac{n(n+1)}{2n+1} \{ i[\rho \, h_n^{(2)}(\rho)]' [\rho \, h_n^{(2)}(\rho)]^* \}_{\rho=ka} \tag{3.58}$$

where

$$\rho = kR$$

and

$$[\rho\ h_n^{(2)}(\rho)]' = \frac{d}{d\rho}[\rho\ h_n^{(2)}(\rho)] \quad .$$

From the definition of the spherical Hankel function of the second kind, one obtains

$$h_n^{(2)}(\rho) = j_n(\rho) - i\ n_n(\rho) \quad , \tag{3.59}$$

where

$j_n(\rho)$ = spherical Bessel function of the first kind, and

$n_n(\rho)$ = spherical Bessel function of the second kind.

Making use of Equation (3.59), one obtains

$$\{i[\rho\ h_n^{(2)}(\rho)]'[\rho\ h_n^{(2)}(\rho)]^*\}_{\rho=ka} = \{\rho^2[j_n(\rho)\ n_n'(\rho) - j_n'(\rho)\ n_n(\rho)]\}_{\rho=ka}$$

$$+ i\{[\rho\ j_n(\rho)]'[\rho\ j_n(\rho)] + [\rho\ n_n(\rho)]'[\rho\ n_n(\rho)]\}_{\rho=ka} \quad . \tag{3.60}$$

However, from the recursion formula, it is well known that

$$j_n(\rho)\ n_n'(\rho) - j_n'(\rho)\ n_n(\rho) = \frac{1}{\rho^2} \quad . \tag{3.61}$$

Substituting Equation (3.61) back into Equation (3.60), one finds

$$\{i[\rho\ h_n^{(2)}(\rho)]'[\rho\ h_n^{(2)}(\rho)]^*\}_{\rho=ka}$$

$$= 1 + i\{[\rho\ j_n(\rho)]'[\rho\ j_n(\rho)] + [\rho\ n_n(\rho)]'[\rho\ n_n(\rho)]\}_{\rho=ka} \quad . \tag{3.62}$$

Finally, applying Equation (3.62) to Equation (3.58),

$$\int_S (\overline{E} \times \overline{H}^*) \cdot \overline{n}\, dS = (\frac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{4\pi}{k^2} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1} + i(\frac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{4\pi}{k^2} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}$$

$$\cdot \{[\rho\, j_n(\rho)]'[\rho\, j_n(\rho)] + [\rho\, n_n(\rho)]'[\rho\, n_n(\rho)]\}_{\rho=ka} \qquad (3.63)$$

As it is defined in Equation (3.52), the radiation power W is the real part of the surface integral of the complex Poynting vector; therefore, from Equation (3.63),

$$W = (\frac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{4\pi}{k^2} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1} \quad . \qquad (3.64)$$

This is the general equation for the radiation power when the field equations are expressed in terms of the spherical-wave functions.

Radiation Impedance for a Half-Wave Dipole

For a half-wave dipole, it is found that, from Equation (3.6),

$$|A_n|^2 = \begin{cases} (\frac{kI_0}{4})^2 \dfrac{(2n+1)\ j_n(\frac{\pi}{2})^2}{n(n+1)} & \text{for } n\text{:odd} \\ \\ 0 & \text{for } n\text{:even} \end{cases} ;$$

therefore, substituting the above result into Equation (3.64), the power radiated from a half-wave dipole is expressed as

$$W = \frac{\pi}{4} (\frac{\mu}{\varepsilon})^{\frac{1}{2}} (I_0)^2 \{ \sum_{n:odd}^{\infty} \frac{2n+1}{n(n+1)} [j_n(\frac{\pi}{2})]^2 \} \quad . \qquad (3.65)$$

Defining the radiation impedance R_H of a half-wave dipole to be

$$R_H = \frac{W}{(I_0)^2} \quad ,$$

then, from Equation (3.65),

$$R_H = 30\pi^2\{\sum_{n:odd}^{\infty} \frac{2n+1}{n(n+1)} [j_n(\frac{\pi}{2})]^2\} = 30\pi^2\{\sum_{n:odd}^{\infty} f(n)\} \quad .$$

The f(n) is defined in Equation (3.45) and $\sum_{n:odd}^{\infty} f(n)$ is evaluated in Equation (3.50); therefore, by making use of Equation (3.50), one obtains

$$R_H = (30\pi^2)(0.2469859271) = 73.12960179 \quad . \qquad (3.66)$$

It is easily recognizable that the value given in Equation (3.66) is equal to the value of the radiation impedance which is obtained by making use of the classical field solution for a half-wave dipole.

Spherical-Wave Expansion for the Field of Linear Antennas

In the preceding sections, the proper behaviors of the spherical-wave expansion for a half-wave dipole has been insured in many ways.

In this section, it will be shown that the method which is used to obtain the spherical-wave expansion for a half-wave dipole is applicable to finding the spherical wave expansion for the fields of linear antennas.

Let's assume that the antenna has the current distribution which is expressed by the following equation:

$$I = I_0 \sin k \ (m \frac{\lambda}{4} - z) \quad ,$$

where

$$-\frac{\lambda}{4} m < z < \frac{\lambda}{4} m \quad ,$$

I_0 = maximum value of the current,

$k = \frac{2\pi}{\lambda}$, λ = wavelength, and

m = number of the half-wave lengths along the antenna.

Then, the field equations of the antenna are given (27):

$$E_z = \frac{ikI_0}{4\pi\omega\varepsilon} \{(-1)^m \frac{e^{-ikr_2}}{r_2} - \frac{e^{-ikr_1}}{r_1}\} \quad ,$$

$$E_d = \frac{-ikI_0}{4\pi\omega\varepsilon} \{(-1)^m \frac{[z - (\frac{\pi m}{2k})]}{d} \frac{e^{-ikr_2}}{r_2} - \frac{[z + (\frac{\pi m}{2k})}{d} \frac{e^{-ikr_1}}{r_1}\} \quad ,$$

$$H_\phi = - \frac{iI_0}{4\pi d} \{(-1)^m e^{-ikr_1} - e^{-ikr_2}\} \quad , \qquad (3.67)$$

where $d = R \sin \theta$ and $z = R \cos \theta$ (see Figure 1).

From Equation (2.5), the R-directional component of the electric field becomes

$$E_R = E_z \cos \theta + E_d \sin \theta = \frac{ikI_0}{4\pi\omega\varepsilon} (\frac{\frac{\lambda}{4} m}{R}) \{(-1)^m \frac{e^{-ikr_2}}{r_2} + \frac{e^{-ikr_1}}{r_1}\} \ . \qquad (3.68)$$

For the antenna whose half-length is equal to $\frac{\lambda}{4}$ m, e^{-ikr_2}/r_2 and e^{-ikr_1}/r_1 are given by

$$\frac{e^{-ikr_2}}{r_2} = ik \sum_{n=0}^{\infty} Y^{(m)}(n) \qquad (3.69)$$

and

$$\frac{e^{-ikr_1}}{r_1} = - ik \sum_{n=0}^{\infty} (-1)^n Y^{(m)}(n) \quad , \qquad (3.70)$$

where

$$Y^{(m)}(n) = (2n+1)\ j_n(k \frac{\lambda}{4} m)\ p_n(\cos\theta)\ h_n^{(2)}(kR) \quad .$$

By making use of these two equations, E_R is expressed in terms of the spherical-wave functions:

$$E_R = i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{I_0\ km\ [(-1)^m + (-1)^n]}{i\ 8\ kR}\ Y^{(m)}(n) \quad . \tag{3.71}$$

Equating this result with Equation (3.2),

$$A_n = -\frac{I_0\ km\ [(-1)^m + (-1)^n]}{i\ 8}\ \frac{2n+1}{n(n+1)}\ j_n(k \frac{\lambda}{4} m) \quad . \tag{3.72}$$

This result will be rewritten in the following way.

(i) When m is odd:

$$A_n = \begin{cases} \frac{I_0\ km}{4i}\ \frac{2n+1}{n(n+1)}\ j_n(\frac{\pi}{2} m) & \text{for } n\text{:odd} \\ 0 & \end{cases} \quad , \tag{3.73}$$

(ii) When m is even:

$$A_n = \begin{cases} 0 & \text{for } n\text{:odd} \\ -\frac{I_0\ km}{4i}\ \frac{2n+1}{n(n+1)}\ j_n(\frac{\pi}{2} m) & \text{for } n\text{:even} \end{cases} \quad . \tag{3.74}$$

It is easy to recognize that the coefficients A_n for a half-wave dipole antenna are obtained by setting $m = 1$ in Equation (3.73).

Discussion

In Equations (3.51) and (3.66), the directive gain $G_H(\frac{\pi}{2})$ and radiation impedance R_H of a half-wave dipole antenna are obtained as

$$G_H(\tfrac{\pi}{2}) = 1.640921888 \quad ,$$

$$R_H = 73.12960179 \quad .$$

It is readily apparent that these values are equal to those of the well known classical results for a half-wave dipole antenna. Thus, the proper behavior of Equation (3.7) through Equation (3.9) is insured. Therefore, one can say that a spherical-wave expansion for the fields of a half-wave dipole has been derived. Its coefficients are:

$$A_n = \begin{cases} -\dfrac{ikI_0\ (2n+1)}{4n(n+1)}\, j_n(\tfrac{\pi}{2}) & \text{for } n\text{:odd} \\ 0 & \text{for } n\text{:even} \end{cases}$$

and the field equations are:

$$H_\phi = -\frac{ikI_0}{4} \sum_{n:odd}^{\infty} X(n)\ h^{(2)}(kR) \quad ,$$

$$E_\theta = (\tfrac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{kI_0}{4} \sum_{n:odd}^{\infty} X(n)\ \frac{1}{kR} \frac{d}{dR} [Rh_n^{(2)}(kR)] \quad ,$$

$$E_R = -(\tfrac{\mu}{\varepsilon})^{\frac{1}{2}} \frac{kI_0}{4} \sum_{n:odd}^{\infty} T(n)\ \frac{h_n^{(2)}(kR)}{kR} \quad .$$

From the above equations, one may say that it takes an infinite number of terms of a spherical-wave expansion to represent the field of a half-wave dipole. On the other hand, it has been shown that these equations converge very rapidly to the classical results in the far-radiation field. Indeed, the first two terms are quite adequate for representing the field to the degree that could be confirmed by actual measurement.

On the basis of the work of L. J. Chu (2) and R. F. Harrington (8), large modes of order $n > ka$ (a is radius of an antenna) in a spherical-wave expansion associate only with the supergain. As it has been observed in the preceding sections, however, modes of order $n > ka$ in the spherical-wave expansion are necessary to represent the fields of a half-wave dipole; although the directive gain for a half-wave dipole is only about 1.64 as compared with the normal gain 5.60 for an antenna when cutoff mode $N = ka = k\,\lambda/4$. When the cutoff mode $N > k\,\lambda/4$, the normal gain G_N for an antenna of half-length $\lambda/4$ becomes (8):

$$G_N = \sum_{n=1}^{N > k\frac{\lambda}{4}} (2n + 1) > \sum_{n=1}^{N \leq k\frac{\lambda}{4}} (2n + 1) \quad .$$

Therefore, it is believed that the normal gain for an antenna of half-length $\lambda/4$ is larger than what has been expected from equations of Chu (2) and Harrington (8); physical realization of a supergain antenna becomes more difficult.

CHAPTER IV

APPLICATION OF THE SPHERICAL-WAVE EXPANSION TO THE EVALUATION OF QUALITY FACTOR

Introduction

The quality factor Q of an antenna is defined as

$$Q = \frac{2\omega \max[W_m, W_e]}{P_R}$$

where

P_R = the total radiated power,

W_m = time average magnetic stored energy, and

W_e = time average electric stored energy,

and Taylor's supergain ratio γ is defined as

$$\gamma = \frac{\int_{-\infty}^{\infty} |F(u)|^2 \, du}{\int_{-k\ell}^{k\ell} |F(u)|^2 \, du} ,$$

where

$$F(u) = \int_{-1}^{1} g(p) \, e^{ipu} \, dp ,$$

where

$g(p)$ = current distribution along an antenna, and

ℓ = half-length of an antenna.

The quality factor Q and the supergain ratio γ have been used as a measure of the physical limitations of an antenna; that is, high Q and high γ have been associated with an excess amount of stored energy in the near field and with a very limited bandwidth. Collins and Rothschild (4), however, demonstrated that supergain ratio γ does not have a direct relationship with the stored energy of the antenna, and Rhode (23) recently showed that the supergain ratio γ is not equal to Q + 1 in the case of planar aperture antennas. Before the publication of Rhode's study, the supergain ratio γ had been described as equal to Q + 1 without insured proof. Therefore, it is apparent that there exists some ambiguity in the supergain ratio γ as a measure of the physical limitation of an antenna. However, because of its clear physical definition, the quality factor Q is considered as a less ambiguous measure of the physical limitation of an antenna.

The general equations for the quality factor Q have been given in several forms by Chu (2), Harrington (9), Collin and Rothschild (4), and Fante (6). Their equations for the quality factor Q are represented in terms of the spherical-wave functions. But because the mechanical configurations of most practical antennas can not be well described by spherical boundaries, the examples where the fields of actual practical antennas are expanded in terms of the spherical functions are extremely rare. Consequently none of the equations for quality factor Q have been applied to any practical antennas. However since a spherical-wave expansion for the fields of a half-wave dipole has been obtained in Chapters II and III, it is now possible, for the first time, to evaluate the quality factor Q for a practical radiator. In doing this one may also be able to examine the reciprocal relationship between the frequency bandwidth and the quality factor Q of an antenna. The existence or the nonexistence of a reciprocal relationship between the frequency bandwidth and the quality factor Q has been debated, but there are not any accepted answers yet.

In this chapter, the spherical-wave expansion for the field of an infinitesimally thin half-wave dipole is used as a tool to explore the consequences of the work of Collin and Rothschild (4), and Fante (6). For this purpose, their methods are used to evaluate the quality factor Q for an infinitesimally thin half-wave dipole and then the reciprocal relationship between the quality factor Q and the frequency bandwdith will be studied.

General Equation for the Quality Factor Q -- Collin and Rothschild's Method

As it has been mentioned in Chapter III, the radiation field of an antenna, which has a transverse magnetic field, is represented by the following equations in Region A of Figure 2.

$$H_{\phi} = \sum_{n=0}^{\infty} A_n \, P_n^1(\cos\theta) \, h_n^{(2)}(kR) \quad , \tag{3.1}$$

$$E_R = -\, i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n \, n(n+1) \, P_n(\cos\theta) \, \frac{h_n^{(2)}(kR)}{kR} \quad , \tag{3.2}$$

$$E_{\theta} = i \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n \, P_n^1(\cos\theta) \, \frac{1}{kR} \frac{d}{d(kR)} \left[kR \, h_n^{(2)}(kR)\right] \quad . \tag{3.3}$$

However Collin and Rothschild (4) obtained their quality factor Q_n for the n^{th} spherical mode without the coefficients A_n. They used the following equations instead of Equations (3.1), (3.2), and (3.3).

$$H_\phi = \frac{\sin\theta}{R}\frac{d\,P_n(\cos\theta)}{d(\cos\theta)}\,[kR\,h_n^{(2)}(kR)] \quad ,$$

$$E_R = \frac{1}{i\omega\varepsilon}\frac{n(n+1)}{R^2}P_n(\cos\theta)[kR\,h_n^{(2)}(kR)] \quad ,$$

$$E_\theta = -\frac{k\sin\theta}{i\omega\varepsilon R}\frac{d\,P_n(\cos\theta)}{d(\cos\theta)}\frac{d[kR\,h_n^{(2)}(kR)]}{d(kR)} \quad ,$$

and by assuming that there is not any stored energy in Region B of Figure 2, they obtained the quality factor for the n^{th} mode as

$$Q_n = ka - [\frac{(ka)^3}{2} + (n+1)\ ka][j_n^2(ka) + n_n^2(ka)]$$
$$- \frac{(ka)^3}{2}[j_{n+1}^2(ka) + n_{n+1}^2(ka)]$$
$$+ \frac{2n+3}{2}(ka)^2\ [j_n(ka)\ j_{n+1}(ka) + n_n(ka)\ n_{n+1}(ka)] \quad .$$

In the following pages, a quality factor Q will be obtained by following the Collin and Rothschild's (4) method but Equations (3.1), (3.2), and (3.3) are to be used.

When the field solutions for the antenna are represented by Equations (3.1), (3.2), and (3.3), the time-average complex Poynting vector is obtained from the result of Equation (3.63):

$$\frac{1}{2}\oint(\overline{E}\times\overline{H}^*)\cdot\overline{n}\,dS = P_R + 2\omega(W_m - W_e) = (\frac{\mu}{\varepsilon})^{\frac{1}{2}}\frac{2\pi}{k^2}\sum_{n=0}^{\infty}|A_n|^2\frac{n(n+1)}{2n+1}$$
$$+ i\,(\frac{\mu}{\varepsilon})^{\frac{1}{2}}\frac{2\pi}{k^2}\sum_{n=0}^{\infty}|A_n|^2\frac{n(n+1)}{2n+1}\{[\rho\ j_n(\rho)][\frac{d}{d\rho}(\rho\ j_n(\rho)] \qquad (4.1)$$
$$+ [\rho\ n_n(\rho)][\frac{d}{d\rho}(\rho\ n_n(\rho)]\}_{\rho=ka} \quad ,$$

where S is the surface of the smallest imaginary sphere of radius a which can enclose the antenna, and $\overline{n}$ is a unit vector whose direction is normal to the surface of the sphere.

From the real part of Equation (4.1), the total time-average radiated power P_R is obtained:

$$P_R = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{2\pi}{k^2} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1} = \frac{2\mu\pi\omega}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1} \quad . \qquad (4.2)$$

From the imaginary part of Equation (4.1), the difference between the time-average magnetic and the electric energy is given:

$$W_m - W_e = \frac{\mu\pi}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}$$

$$\cdot [\rho\ j_n(\rho)]\left[\frac{d}{d\rho}(\rho\ j_n(\rho))\right] + [\rho\ n_n(\rho)]\left[\frac{d}{d\rho}(\rho\ n_n(\rho))\right]_{\rho=ka} \qquad (4.3)$$

The total time-average reactive energy stored in the region outside of the imaginary sphere is represented by the following equation (4), (6).

$$W_m + W_e = \left\{\int_a^{\infty} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\varepsilon}{4} |E|^2 + \frac{\mu}{4} |H|^2\right) R^2 \sin\theta\ d\theta\ d\phi\ dR\right\} - \left\{\int_a^{\infty} \frac{P_R}{c}\, dR\right\} , \qquad (4.4)$$

where

$$P_R = \text{real part of } \frac{1}{2} \oint_{R\to\infty} (\overline{E} \times \overline{H}^*) \cdot \overline{n}\ dS,$$

$$= \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{2\pi}{k^2} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}, \text{ and}$$

$$c = (\varepsilon\mu)^{-\frac{1}{2}} .$$

Substituting the following equations:

$$E = E_R \overline{u}_R + E_\theta \overline{u}_\theta \quad ,$$

$$H = H_\phi \overline{u}_\phi \quad ,$$

into Equation (4.4), then it becomes

$$W_m + W_e = \{\int_a^\infty \int_0^{2\pi} \int_0^\pi (\frac{\varepsilon}{4} |E_R|^2 + \frac{\varepsilon}{4} |E_\theta|^2 + \frac{\mu}{4} |H_\phi|^2) R^2 \sin\theta \, d\theta \, d\phi \, dR\}$$

$$- \{\int_a^\infty \frac{P_R}{c} dR\} \quad . \tag{4.5}$$

From Equations (3.1), (3.2), (3.3), and (3.34)

$$\int_0^\pi \frac{\mu}{4} |H_\phi|^2 \sin\theta \, d\theta = \frac{\mu}{4} \sum_{n=0}^\infty k^2 |A_n|^2 \frac{2n(n+1)}{2n+1} \cdot \{h_n^{(2)}(\rho)\}\{h_n^{(2)}(\rho)\}^* \quad , \tag{4.6}$$

$$\int_0^\pi \frac{\varepsilon}{4} |E_R|^2 \sin\theta \, d\theta = \frac{\mu}{4} \sum_{n=0}^\infty |A_n|^2 \frac{2[n(n+1)]^2}{2n+1} \cdot \{\frac{[h_n^{(2)}(\rho)][h_n^{(2)}(\rho)]^*}{\rho^2}\} , \tag{4.7}$$

$$\int_0^\pi \frac{\varepsilon}{4} |E_\theta|^2 \sin\theta \, d\theta = \frac{\mu}{4} \sum_{n=0}^\infty \frac{|A_n|^2}{\rho^2} \frac{2n(n+1)}{2n+1} \cdot \{\frac{d}{d\rho} [\rho \, h_n^{(2)}(\rho)]\}\{\frac{d}{d\rho} [h_n^{(2)}(\rho)]^*\} \tag{4.8}$$

where

$\rho = kR.$

Substituting Equations (4.6), (4.7), and (4.8) into Equation (4.5), then integrating with respect to ϕ, one finds

$$W_m + W_e = \frac{\mu\pi}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1} \cdot \{\int_{ka}^{\infty} \{[1 + \frac{n(n+1)}{2}]\rho^2 \, h_n^{(2)}(\rho) \, [h_n^{(2)}(\rho)]^*$$

$$+ [\frac{d}{d\rho} (\rho \, h_n^{(2)}(\rho))][\frac{d}{d\rho} (\rho \, h_n^{(2)}(\rho))^*] - 2\} \, d\rho \quad , \qquad (4.9)$$

For the integral part of the above equation, Collin and Rothschild (4) obtained the following result:

$$\int_{ka}^{\infty} \{[1 + \frac{n(n+1)}{2}]\rho^2 \, h_n^{(2)}(\rho)[h_n^{(2)}(\rho)]^*$$

$$+ [\frac{d}{d\rho} (\rho \, h_n^{(2)}(\rho))][\frac{d}{d\rho} (\rho \, h_n^{(2)}(\rho))^*] - 2\} \, d\rho$$

$$= \{2\rho - (\rho)^3\{[j_n(\rho)]^2 + [n_n(\rho)]^2 - j_{n-1}(\rho) \, j_{n+1}(\rho)$$

$$- n_{n-1}(\rho) \, n_{n+1}(\rho)\} - (\rho)^2\{j_n(\rho) \, j_n'(\rho) + n_n(\rho) \, n_n'(\rho)\}$$

$$- (\rho)\{[j_n(\rho)]^2 + [n_n(\rho)]^2\}\}_{\rho=ka} \quad , \qquad (4.10)$$

where

$$j_n'(\rho) = [\frac{d \, j_n(\rho)}{d\rho}] \quad ,$$

$$n_n'(\rho) = [\frac{d \, n_n(\rho)}{d\rho}] \qquad (4.11)$$

Substituting Equation (4.10) into Equation (4.9), then the total reactive energy stored outside of the sphere is given by the following equation:

$$(W_m + W_e) = \frac{\mu\pi}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}$$

$$\cdot \{2\rho - (\rho)^3 [[j_n(\rho)]^2 + [n_n(\rho)]^2 - j_{n-1}(\rho) \, j_{n+1}(\rho)$$

$$- n_{n-1}(\rho) \, n_{n+1}(\rho)] - (\rho)^2 [j_n(\rho) \, j_n'(\rho) + n_n(\rho) \, n_n'(\rho)]$$

$$- (\rho) [[j_n(\rho)]^2 + [n_n(\rho)]^2]\}_{\rho=ka} \quad . \qquad (4.12)$$

Previously, in Equation (4.3),

$$W_m - W_e = \frac{\mu\pi}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}$$

$$\cdot \{[\rho\, j_n(\rho)][\rho\, j_n(\rho)]' + [\rho\, n_n(\rho)][\rho\, n_n(\rho)]'\}_{\rho=ka} \quad .$$

Substituting the following two relations

$$[\rho\, j_n(\rho)][\rho\, j_n(\rho)]' = \rho\, j_n^2(\rho) + \rho^2\, j_n(\rho)\, j_n'(\rho)$$

and

$$[\rho\, n_n(\rho)][\rho\, n_n(\rho)]' = \rho\, n_n^2(\rho) + \rho^2\, n_n(\rho)\, n_n'(\rho)$$

into Equation (4.3), there results

$$W_m - W_e = \frac{\mu\pi}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}$$

$$\cdot \{ka[j_n^2(ka) + n_n^2(ka)] + (ka)^2\, [j_n(ka)\, j_n'(ka) + n_n(ka)\, n_n'(ka)]\} \quad . \tag{4.13}$$

Subtracting Equation (4.13) from Equation (4.12):

$$2W_e = (W_m + W_e) - (W_m - W_e) = \frac{\mu\pi}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}$$

$$\cdot \{2ka - [(ka)^3 + 2ka][j_n^2(ka) + n_n^2(ka)]$$

$$+ (ka)^3\, [j_{n-1}(ka)\, j_{n+1}(ka) + n_{n-1}(ka)\, n_{n+1}(ka)]$$

$$- 2(ka)^2\, (j_n(ka)\, j_n'(ka) + n_n(ka)\, n_n'(ka)]\} \quad . \tag{4.14}$$

This equation will be further simplified for the numerical calculations by making use of the following recursion formulas:

$$j_{n-1}(ka) = \frac{2n+1}{ka} j_n(ka) - j_{n+1}(ka) \quad ,$$

$$n_{n-1}(ka) = \frac{2n+1}{ka} n_n(ka) - n_{n+1}(ka) \quad ,$$

$$j_n'(ka) = \frac{n}{ka} j_n(ka) - j_{n+1}(ka) \quad ,$$

$$n_n'(ka) = \frac{n}{ka} n_n(ka) - n_{n+1}(ka) \quad .$$

By making use of the above equations, one obtains

$$\begin{aligned}(ka)^3 \, [j_{n-1}(ka) \; j_{n+1}(ka) + n_{n-1}(ka) \; n_{n+1}(ka)] \\ = (ka)^2 \; (2n+1) \; [j_n(ka) \; j_{n+1}(ka) + n_n(ka) \; n_{n+1}(ka)] \\ - (ka)^3 \; [j_{n+1}^2(ka) + n_{n+1}^2(ka)] \quad ,\end{aligned} \tag{4.15}$$

$$\begin{aligned}2(ka)^2 \, [j_n(ka) \; j_n'(ka) + n_n(ka) \; n_n'(ka)] = 2(ka) \; n \; [j_n^2(ka) + n_n^2(ka)] \\ - 2(ka)^2 \; [j_n(ka) \; j_{n+1}(ka) + n_n(ka) \; n_{n+1}(ka)] \quad .\end{aligned} \tag{4.16}$$

Substituting Equations (4.15) and (4.16) back into Equation (4.14), it follows

$$\begin{aligned}2\omega \, W_e = \frac{\mu\pi\omega}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \, \frac{n(n+1)}{2n+1} \cdot \{2ka - [(ka)^3 + 2ka(n+1)][j_n^2(ka) + n_n^2(ka)] \\ - (ka)^3 \; [j_{n+1}^2(ka) + n_{n+1}^2(ka)] \\ + (ka)^2(2n+3) \; [j_n(ka) \; j_{n+1}(ka) + n_n(ka) \; n_{n+1}(ka)]\} \triangleq \sum_{n=0}^{\infty} S(n) \quad .\end{aligned} \tag{4.17}$$

From Equation (4.2), it is defined that

$$P_R = \frac{2\mu\pi\omega}{k^3} \sum_{n=0}^{\infty} |A_n|^2 \, \frac{n(n+1)}{2n+1} \triangleq \sum_{n=0}^{\infty} P_R(n) \quad . \tag{4.18}$$

Therefore, making use of Equations (4.17) and (4.18), quality factor Q may be given by

$$Q = \frac{2\omega W_e}{P_R} = \frac{\sum_{n=0}^{\infty} S(n)}{\sum_{n=0}^{\infty} P_R(n)} \quad . \tag{4.19}$$

Equation (4.19) is rewritten as

$$Q = \frac{\sum_{n=0}^{\infty} P_R(n) \frac{S(n)}{P_R(n)}}{\sum_{n=0}^{\infty} P_R(n)} = \frac{\sum_{n=0}^{\infty} P_R(n)\; Q(n)}{\sum_{n=0}^{\infty} P_R(n)} = \frac{\sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1} Q(n)}{\sum_{n=0}^{\infty} |A_n|^2 \frac{n(n+1)}{2n+1}} \quad , \tag{4.20}$$

where

$$\begin{aligned} Q(n) &= \frac{S(n)}{P_R(n)} \\ &= ka - [\frac{(ka)^3}{2} + ka(n+1)][j_n^2(ka) + n_n^2(ka)] \\ &- (ka)^3[j_{n+1}^2(ka) + n_{n+1}^2(ka)] \\ &+ (ka)^2(\frac{2n+3}{2})[j_n(ka)\; j_{n+1}(ka) + n_n(ka)\; n_{n+1}(ka)] \quad . \end{aligned} \tag{4.21}$$

It should be noted that Q(n) is Q_n of Equation (8) in Collin and Rothschild's paper (4); Equation (4.20) for Q has exactly the same form as the equation for Q of which Chu found (2).

Quality Factor Q for an Infinitesimally Thin Half-Wave Dipole

It has been found that the coefficients A_n for an infinitesimally thin half-wave dipole are given by

$$A_n = \begin{cases} \dfrac{-\,ikI_0\ (2n+1)\ j_n(\frac{\pi}{2})}{4n(n+1)} & \text{for } n\text{:odd} \\ 0 & \text{for } n\text{:even} \end{cases} \qquad (3.6)$$

The quality factor Q for an infinitesimally thin half-wave dipole is found by substituting Equation (3.6) into Equation (4.20) and by setting $ka = \pi/2$. This results in

$$Q = \frac{\sum\limits_{n:odd}^{\infty} \frac{(2n+1)}{n(n+1)} [j_n(\frac{\pi}{2})]^2 [Q(n)]_{ka=\pi/2}}{\sum\limits_{n:odd}^{\infty} \frac{2n+1}{n(n+1)} [j_n(\frac{\pi}{2})]^2} = \frac{\sum\limits_{n:odd}^{\infty} f(n)[Q(n)]_{ka=\pi/2}}{\sum\limits_{n:odd}^{\infty} f(n)} , \qquad (4.22)$$

where f(n) and Q(n) are defined in Equation (3.45) and in Equation (4.21), respectively.

In Equations (3.47) and (3.50), it was calculated that

$$f(1) = 0.2463835742$$

$$f(3) = 0.0006020966$$

$$f(5) = 0.0000002563$$

$$\vdots \qquad (3.47)$$

and

$$\sum_{n:odd}^{\infty} f(n) = f(1) + f(3) + f(5) + \cdots \approx 0.2469859271 \quad . \quad (3.50)$$

After the indicated computation of Equation (4.21), the numerical values for Q(n) were evaluated as

$$\begin{aligned}
[Q(1)]_{ka=\frac{\pi}{2}} &= 0.8946320478 \\
[Q(3)]_{ka=\frac{\pi}{2}} &= 51.96120934 \\
[Q(5)]_{ka=\frac{\pi}{2}} &= 4.124372 \times 10^{4} \\
[Q(7)]_{ka=\frac{\pi}{2}} &= 1.771002 \times 10^{8} \\
&\vdots \qquad .
\end{aligned} \quad (4.23)$$

For the computation of $[Q(n)]_{ka=\pi/2}$, $n \geq 5$, the following equation is used:

$$[Q(n)]_{ka=\frac{\pi}{2}} \approx \frac{\pi}{2} - [(\frac{\pi}{2})^3 + \frac{\pi}{2} n(n+1)][n_n(\frac{\pi}{2})]^2$$

$$- (\frac{\pi}{2})^3 [n_{n+1}(\frac{\pi}{2})]^2 + (\frac{\pi}{2})(\frac{2n+3}{2}) n_n(\frac{\pi}{2}) n_{n+1}(\frac{\pi}{2}) \quad . \quad (4.24)$$

Because, when $n \geq 5$, it is apparent that (see Appendix A)

$$[j_n(\frac{\pi}{2})]^2 << [n_n(\frac{\pi}{2})]^2 \quad ,$$

$$[j_{n+1}(\frac{\pi}{2})]^2 << [n_{n+1}(\frac{\pi}{2})]^2 \quad ,$$

and (4.25)

$$j_n(\frac{\pi}{2})\ j_{n+1}(\frac{\pi}{2}) << n_n(\frac{\pi}{2})\ n_{n+1}(\frac{\pi}{2}) \quad ;$$

therefore, using Equation (4.25) in Equation (4.21), one obtains Equation (4.24).

From Equations (3.47) and (4.23),

$$
\begin{aligned}
f(1)[Q(1)]_{ka=\frac{\pi}{2}} &= 0.220423 \\
f(3)[Q(3)]_{ka=\frac{\pi}{2}} &= 0.031286 \\
f(5)[Q(5)]_{ka=\frac{\pi}{2}} &= 0.010571 \\
f(7)[Q(7)]_{ka=\frac{\pi}{2}} &= 0.005525 \\
&\vdots
\end{aligned}
\qquad (4.26)
$$

Finally, from Equations (3.50) and (4.25), one obtains the quality factor Q for the infinitesimally thin half-wave dipole. This results in

$$
Q = \frac{\sum_{n:odd}^{\infty} f(n)[Q(n)]_{ka=\pi/2}}{\sum_{n:odd}^{\infty} f(n)}
$$

$$
\frac{0.267805}{0.246986} = 1.084 \qquad (4.27)
$$

The values of the spherical Bessel functions, $j_n(\pi/2)$ and $n_n(\pi/2)$, which have been used in the above computation of Q are given in Appendix A.

Relation Between the Reciprocal Bandwidth and the Quality Factor Q

By making use of Collin and Rothschild's (4) method and the spherical-wave function for the infinitesimally thin half-wave dipole, the quality factor Q is evaluated as 1.08. However, these values should not be interpreted as the reciprocal bandwidth of the antenna; because, for the half-wave dipole antenna, it is found (see Appendix C) that the reciprocal bandwidth 1/B is given by

$$\frac{1}{B} = 4.76 \qquad \text{for radius of } \frac{\lambda}{200} \quad ,$$

$$\frac{1}{B} = 6.67 \qquad \text{for radius of } \frac{\lambda}{1000} \quad ,$$

where B is the bandwidth between half-power points. The smaller the radius of the dipole becomes, the larger the quantity 1/B becomes; and the quantity 1/B approaches ∞ for an infinitesimally thin half-wave dipole.

Fante (6) suggested the following equation for the reciprocal bandwidth of an antenna:

$$\frac{1}{B} \approx [Q_{out} + Q_{in} + F(\omega)] \quad , \qquad (4.28)$$

where

$$Q_{out} = \frac{2\omega\, W^{out}}{P_R}$$

$$Q_{in} = \frac{2\omega\, W^{in}}{P_R}$$

$$F(\omega) = \frac{\omega}{\eta\, P_R}\, I_m \left\{ \oint \left(\overline{E}_\infty \cdot \frac{\partial \overline{E}_\infty}{\partial \omega}\, d\Omega \right\} \right.$$

W^{out} = the larger of W_m or W_e which is stored outside of the smallest imaginary sphere which can enclose the antenna,

W^{in} = the larger of W_m or W_e which is stored inside of the smallest imaginary sphere which can enclose the antenna,

P_R = power radiated from the antenna,

$\eta = (\frac{\mu}{\varepsilon})^{1/2} = 120\pi$,

$d\Omega = \sin\theta \, d\theta \, d\phi$, and

$$\overline{E}_\infty = (\lim_{R\to\infty} \overline{E}) \frac{R}{e^{-ikR}} \quad .$$

As it is easily recognized from the above definition, the Q_{out} is equal to the Q of Equation (4.20).

For the infinitesimally thin half-wave dipole, from Equations (3.29) and (3.15), one obtains

$$\overline{E}_\infty = i \left(\frac{k}{\omega\varepsilon}\right)\left(\frac{I_0}{4}\right) \sum_{n:odd}^{\infty} (-1)^{\frac{n+3}{2}} X(n)\, \overline{u}_\theta \quad , \tag{4.29}$$

$$\frac{\partial \overline{E}^*_\infty}{\partial\omega} = + i \left(\frac{k}{\omega^2\varepsilon}\right)\left(\frac{I_0}{4}\right) \sum_{n:odd}^{\infty} (-1)^{\frac{n+3}{2}} X(n)\, \overline{u}_\theta \quad . \tag{4.30}$$

By making use of the orthogonality of the associated Legendre function,

$$\oint \left(\overline{E}_\infty \cdot \frac{\partial \overline{E}_\infty}{\partial\omega}\right) d\Omega = \int_0^{2\pi}\int_0^{\pi} \overline{E}_\infty \cdot \frac{\partial \overline{E}^*_\infty}{\partial\omega} \sin\theta \, d\theta \, d\phi$$

$$= -2\pi \left(\frac{k^2}{\omega^3\varepsilon}\right)\left(\frac{I_0}{4}\right)^2 \sum_{n:odd}^{\infty} (-1)^{n+3} \frac{2(2n+1)}{n(n+1)} \left[j_n\left(\frac{\pi}{2}\right)\right]^2 \quad . \tag{4.31}$$

Since Equation (4.31) is real,

$$I_m \left\{\oint \left(\overline{E}_\infty \cdot \frac{\partial \overline{E}_\infty}{\partial\omega}\right) d\Omega\right\} = 0 \quad ; \tag{4.32}$$

that is, $F(\omega) = 0$. Therefore, from Equation (4.28), for an infinitesimally thin half-wave dipole,

$$\frac{1}{B} \approx [Q_{out} + Q_{in}] \quad . \tag{4.33}$$

From Equation (4.27), $Q_{out} = 1.084$, and it is shown in Equation (B.23) of Appendix B that $Q_{in} \to \infty$; therefore,

$$Q_{out} + Q_{in} \approx \infty \quad . \tag{4.34}$$

From Equation (4.33) and (4.34), one obtains

$$\frac{1}{B} \approx \infty \quad . \tag{4.35}$$

Thus applying the spherical-wave expansion for the field of a half-wave dipole to the equation of Fante (6), it has been demonstrated that there exists the reciprocal relationship between the quality factor and the bandwidth of half-power points.

In conclusion, by making use of the spherical-wave expansion for the field of an infinitesimally thin half-wave dipole as an exploring tool, the method of Collin and Rothschild (4) and the equation of Fante (6) have been applied to find the quality factor Q. It has been found that the direct application of the method of Collin and Rothschild (4) does not give the appropriate value of the quality factor for an infinitesimally thin half-wave dipole. This is because of the method of Collin and Rothschild that does does not take account of the energy in the region inside of the smallest imaginary sphere which can enclose the antenna, and the energy stored in this region is not neglibible for an infinitesimally thin half-wave dipole. On the other hand, it has been shown that the equation of Fante (6) gives proper value for the quality factor. Thus it is found that there exists the reciprocal relationship between the frequency bandwidth and the quality factor.

For the antenna with non-vanishing radius, it is difficult to evaluate the stored energy inside of the smallest imaginary sphere which can enclose the antenna. Therefore, the exact quantity of the quality factor is not obtained.

CHAPTER V

SUMMARY AND CONCLUSION

As an effective tool for exploring the consequences of the theory of supergain radiators when applied to practical radiators, a spherical-wave expansion for the fields of a half-wave dipole has been obtained. It has been presented that the precise values for the directive gain and the radiation impedance of a half-wave dipole have been obtained by using the spherical-wave expansion. By doing this, the proper behavior of the spherical-wave expansion has been assured. It has been found that the spherical-wave expansion takes an infinite number of terms of the spherical-wave functions to represent the fields of a half-wave dipole; however, the spherical-wave expansion for a half-wave dipole converges quite rapidly to the results in the far-field. It is also found that the first three terms of the spherical-wave expansion are quite adequate for representing the field to the degree that can be confirmed by actual measurement.

On the basis of the work of L. J. Chu (2) and R. F. Harrington (8), large modes of order $n > ka$ (a is radius of an antenna) in a spherical-wave expansion associate only with the supergain. As it has been observed in Chapter III, however, modes of order $n > ka$ in the spherical-wave expansion are necessary to represent the fields of a half-wave dipole; although the directive gain for a half-wave dipole is only about 1.64 as compared with the normal gain 5.60 for an antenna when cutoff mode $N = ka = k\ \lambda/4$. When the cutoff mode $N > k\ \lambda/4$, the normal gain G_N for an antenna of half-length $\lambda/4$ becomes (8):

$$G_N = \sum_{n=1}^{N > k\frac{\lambda}{4}} (2n+1) > \sum_{n=1}^{N \leq k\frac{\lambda}{4}} (2n+1) \quad .$$

Therefore, it is believed that the normal gain for an antenna of half-length $\lambda/4$ is larger than what has been expected from equations of Chu (2) and Harrington (8); physical realization of a supergain antenna becomes more difficult.

The technique which was used to obtain the spherical-wave expansion for a half-wave dipole has been applied to obtain the spherical-wave expansions for the fields of the linear radiators with assumed sinusoidal current distribution. Consequently, the spherical-wave expansions for a full-wave and a higher-number-waves dipole have been obtained.

As an example for exploring the consequence of the theory of supergain radiators when it is applied to the practical radiator, the spherical-wave expansion for the fields of the half-wave dipole was used to evaluate the quality factor Q by making use of the equations of Collin and Rothschild (4), and Fante (6). A small quantity of the quality factor Q was obtained when the equation of Collin and Rothschild was used. This is not what was expected, because the quality factor Q for the half-wave dipole with assumed sinusoidal current is to be ∞ (27). On the other hand, the equation of Fante (6) gives ∞ to the quality factor Q. Therefore, one may say that there exists direct relationship between the quality factor Q and the reciprocal bandwidth of the half-wave dipole. Because it is found that the reciprocal bandwidth of the infinitesimally thin half-wave dipole is ∞.

It is, of course, well known that the current distribution on an actual half-wave dipole is not exactly sinusoidal. Consequently, the ties between the theoretical and physical measurement become even more difficult to determine. However, a half-wave dipole is physically realized in practice to a more precise degree than most of the other structures. Hence, it is believed that the spherical-wave expansion which has been developed in this study is an effective tool for exploring the physical consequences of the theory which is associated with the spherical-wave functions.

SELECTED BIBLIOGRAPHY

(1) Borgiotti, G. V. "On the Reactive Energy of an Aperture." IEEE Trans. Antennas and Propagation (Communications), Vol. AP-15 (July, 1967), pp. 565-566.

(2) Chu, L. J. "Physical Limitation of Omni-Directional Antennas." J. Appl. Phys., Vol. 10 (December, 1948), pp. 1163-1175.

(3) Collin, R. E. "Stored Energy Q and Frequency Sensitivity of Planar Aperture Antennas." IEEE Trans. Antennas and Propagation (Communications), Vol. AP-15 (July, 1967), pp. 567-568.

(4) Collin, R. E., and S. Rothschild. "Evaluation of Antenna Q." IEEE Trans. Antennas and Propagation, Vol. AP-12 (January, 1964), pp. 23-27.

(5) Collin, R. E., and S. Rothschild. "Reactive Energy in Aperture Fields and Aperture Q." Canad. J. Phys., Vol. 41 (December, 1963), pp. 1967-1979.

(6) Fante, R. L. "Quality Factor of General Ideal Antennas." IEEE Trans. Antennas and Propagation, Vol. AP-17 (March, 1969), pp. 151-155.

(7) Gradshteyn, I. S., and I. M. Ryzhik. Table of Integrals, Series, and Products. New York: Academic Press, 1972.

(8) Harrington, R. F. Time Harmonic Electromagnetic Fields. New York: McGraw-Hill, 1961.

(9) Harrington, R. F. "Effect of Antenna Size on Gain, Bandwidth, and Efficiency." J. Res. NBS., Vol. 64D (January, 1960), pp. 1-12.

(10) Harrison, C. W. Jr., and R. King. "The Radiation Field of a Symmetrical Center-Driven of Finite Cross Section." Proc. IRE., Vol. 31 (December, 1943), pp. 693-697.

(11) Jones, D. S. The Theory of Electromagnetism. New York: Pergamon Press, 1964.

(12) Jordan, E. C. Electromagnetic Waves and Radiating Systems. Englewood, N. J.: Prentice-Hall, 1950.

(13) Karr, P. R. "Radiation Properties of Spherical Antennas as a Function of the Location of the Driving Force." J. Res. NBS., Vol. 46, No. 5 (May, 1951), pp. 422-436.

(14) King, R. W. P. The Theory of Linear Antennas. Cambridge, Mass.: Harvard University Press, 1956.

(15) King, R., and D. Middleton. "The Cylindrical Antenna: Current and Impedance." Quart. Appl. Math., Vol. 3 (January, 1946), pp. 302-355.

(16) King, R., and C. W. Harrison Jr. "The Distribution of Current Along a Symmetrical Center-Driven Antenna." Proc. IRE., Vol. 31 (October, 1943), pp. 548-567.

(17) Margenau, H., and G. M. Murphy. The Mathematics of Physics and Chemistry. New York: D. Van Nostrand Company, Inc., 1943.

(18) McDougal, J. R., L. C. Surratt, and J. F. Stoops. "Computer Aided Design of Small Superdirective Antennas Using Fourier Integral and Fast Fourier Transform Techniques." 1970 SWIEEECO Record of Technical Papers, April 22-24, 1970, Dallas, Texas, pp. 421-425.

(19) Moriguchi, S., K. Udagawa, and S. Hitotsumatsu. Sugaku Koshiki I. Tokyo: Iwanami Shoten, Inc., 1965.

(20) Mogiguchi, S., K. Udagawa, and S. Hitotsumatsu. Sugaku Koshiki II. Tokyo: Iwanami Shoten, Inc., 1965.

(21) Moriguchi, S., K. Udagawa, and S. Hitotsumatsu. Sugaku Koshiki III. Tokyo: Iwanami Shoten, Inc., 1965.

(22) Ramo, S., and J. R. Whinnery. Fields and Waves in Modern Radio. New York: John Wiley and Sons, Inc., 1946.

(23) Rhodes, D. R. "On the Quality Factor of Strip and Line Source Antennas and Its Relationship to Superdirective Ratio." IEEE Trans. Antennas and Propagation, Vol. AP-20 (May, 1972), pp. 318-325.

(24) Rhodes, D. R. "On a Fundamental Principle in the Theory of Planar Antennas." Proc. IEEE., Vol. 52 (September, 1964), pp. 1013-1021.

(25) Rhodes, D. R. "On the Stored Energy of Apertures." IEEE Trans. Antennas and Propagation, Vol. AP-14 (November, 1966), pp. 676-683.

(26) Riblet, H. J. "Note on the Maximum Directivity of an Antenna." Proc. IRE., Vol. 36 (May, 1948), pp. 620-623.

(27) Schelkunoff, S. A. Advanced Antenna Theory. New York: John Wiley, 1952.

(28) Stratton, J. A. Electromagnetic Theory. New York: McGraw-Hill, 1941.

(29) Taylor, T. T. "Design of Line-Source Antennas for Narrow Beamwidth and Low Side Lobes." _IRE Trans. on Antennas and Propagation_, Vol. AP-3 (January, 1955), pp. 16-28.

(30) Uchida, H., and Y. Mushiake. _Chotanpa Kuchusen (VHF Antenna)_. Tokyo: Koronasha Inc., 1966.

APPENDIX A

NUMERICAL VALUES OF THE SPHERICAL BESSEL FUNCTIONS $j_n(\frac{\pi}{2})$ AND $n_n(\frac{\pi}{2})$

$$j_0\left(\frac{\pi}{2}\right) = 0.6366197724$$

$$j_1\left(\frac{\pi}{2}\right) = 0.4052847346$$

$$j_2\left(\frac{\pi}{2}\right) = 0.1374170539$$

$$j_3\left(\frac{\pi}{2}\right) = 0.0321273340$$

$$j_4\left(\frac{\pi}{2}\right) = 0.0057532090$$

$$j_5\left(\frac{\pi}{2}\right) = 0.0008361234$$

$$j_6\left(\frac{\pi}{2}\right) = 0.0001021358$$

$$j_7\left(\frac{\pi}{2}\right) = 0.0000107920$$

$$n_0\left(\frac{\pi}{2}\right) = 0$$

$$n_1\left(\frac{\pi}{2}\right) = -\ 6.366197724 \times 10^{-1}$$

$$n_2\left(\frac{\pi}{2}\right) = -\ 1.215854204$$

$$n_3\left(\frac{\pi}{2}\right) = -\ 3.233564360$$

$$n_4\left(\frac{\pi}{2}\right) = -\ 1.319400284 \times 10$$

$$n_5\left(\frac{\pi}{2}\right) = -\ 7.236250341 \times 10$$

$$n_6\left(\frac{\pi}{2}\right) = -\ 4.935474020 \times 10^{2}$$

$$n_7\left(\frac{\pi}{2}\right) = -\ 4.012263947 \times 10^{3}$$

$$n_8\left(\frac{\pi}{2}\right) = -\ 3.782075100 \times 10^{4}$$

$$n_9\left(\frac{\pi}{2}\right) = -\ 4.053041802 \times 10^{5}$$

$$n_{10}\left(\frac{\pi}{2}\right) = -\ 4.864647695 \times 10^{6}$$

The numerical values for $j_n(\frac{\pi}{2})$ and $n_n(\frac{\pi}{2})$ have been calculated by making use of the following equation (21):

$$j_n(z) = \frac{1}{z} \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r (n+2r)!}{(2r)!(n-2r)!(2z)^{2r}} \sin\left(z - \frac{n\pi}{2}\right)$$

$$+ \sum_{r=0}^{[\frac{(n-1)}{2}]} \frac{(-1)^r (n+2r+1)!}{(2r+1)!(n-2r-1)!(2z)^{2r+1}} \cos\left(z - \frac{n\pi}{2}\right)$$

and

$$n_n(z) = -\frac{1}{z} \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r (n+2r)!}{(2r)!(n-2r)!(2z)^{2r}} \cos\left(z - \frac{n\pi}{2}\right)$$

$$- \sum_{r=0}^{[\frac{(n-1)}{2}]} \frac{(-1)^r (n+2r+1)!}{(2r+1)!(n-2r-1)!(2z)^{2r+1}} \sin\left(z - \frac{n\pi}{2}\right)$$

where [b] is the maximum integer which does not exceed b.

APPENDIX B

ELECTRIC ENERGY STORED INSIDE OF THE SMALLEST IMAGINARY SPHERE WHICH CAN ENCLOSE AN INFINITESIMALLY THIN HALF-WAVE DIPOLE

In the previous chapters, the field solution for the infinitesimally thin half-wave dipole has been expanded into an infinite series of spherical-wave functions. In this section, the electric energy stored inside of the smallest imaginary sphere which can enclose the infinitesimally thin half-wave dipole will be obtained by making use of the following spherical-wave expansion for the region of $R < \lambda/4$:

$$E_\theta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \left(\frac{kI_0}{2\pi}\right) \sum_{n:\text{even}}^{\infty} \frac{A(n)}{\sin\theta} - \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \left(\frac{I_0}{4}\right) \sum_{n:\text{odd}}^{\infty} \frac{(\cos\theta)A(n)}{\sin\theta} \quad , \qquad (2.29)$$

$$E_R = -\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \left(\frac{I_0}{4}\right) \sum_{n:\text{odd}}^{\infty} \frac{A(n)}{R} \quad , \qquad (2.30)$$

$$E_\phi = 0 \quad ,$$

where

$$A(n) = (2n+1)\, h_n^{(2)}\left(\frac{\pi}{2}\right) P_n(\cos\theta)\, j_n(kR) \quad . \qquad (2.24)$$

Therefore, the electric energy density $\frac{\varepsilon}{4}\,\overline{E} \cdot \overline{E}^*$ for the inside of the imaginary sphere becomes

$$\frac{\varepsilon}{4}\,\overline{E} \cdot \overline{E}^* = \frac{\varepsilon}{4} E_\theta \cdot E_\theta^* + \frac{\varepsilon}{4} E_R \cdot E_R^* \quad . \qquad (B.1)$$

Hence, the total electric energy stored inside the imaginary sphere of radius $\lambda/4$ will be obtained from the following equation:

$$W_e^{in} = \int_0^{\frac{\lambda}{4}} \int_0^{2\pi} \int_0^{\pi} \frac{\varepsilon}{4}\,\overline{E} \quad \overline{E}^* R^2 \sin\theta \, d\theta \, d\phi \, dR$$

$$= \int_0^{\frac{\lambda}{4}} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\varepsilon}{4} E_\theta \cdot E_\theta^* + \frac{\varepsilon}{4} E_R \cdot E_R^*\right) R^2 \sin\theta \, d\theta \, d\phi \, dR \quad . \qquad (B.2)$$

It is well known that the Legendre and the associated Legendre functions give the following results for the definite integrals (7):

$$\int_0^{\pi} P_n(\cos\theta)\, P_k(\cos\theta)\, \sin\theta\, d\theta = \begin{cases} \dfrac{2}{2n+1} & \text{for } n=k \\ 0 & \text{for } n=k \end{cases} \tag{B.3}$$

$$\int_0^{\pi} \frac{P_n^m(\cos\theta)\, P_k^m(\cos\theta)}{1-\cos^2\theta} \sin\theta\, d\theta = \begin{cases} \dfrac{(n+m)!}{(n-m)!m} & \text{for } n=k \\ 0 & \text{for } n=k \end{cases} \tag{B.4}$$

$$\int_0^{\pi} \frac{\cos\theta}{z-\cos^2\theta} P_n(\cos\theta)\, P_k(\cos\theta)\, \sin\theta\, d\theta = 2P_n(z)\, Q_k(z),\ n < k \quad , \tag{B.5}$$

where $Q_k(z)$ is the Legendre function of the second kind.

$$\int_0^{\pi} \frac{\cos\theta}{1-\cos^2\theta} P_n(\cos\theta)\, P_k(\cos\theta)\, \sin\theta\, d\theta$$

$$= \int_0^{\pi} \frac{1}{1-\cos\theta} P_n(\cos\theta)\, P_k(\cos\theta)\, \sin\theta\, d\theta = 2P_n(1)\, Q_k(1) \quad ,$$

$$n < k \text{ and } n \neq k \quad . \tag{B.6}$$

By making use of Equation (2.26), it follows

$$\int_0^\pi \frac{\varepsilon}{4} E_\theta \cdot E_\theta^* \sin\theta\, d\theta = \frac{\varepsilon}{4} \left(\frac{kI_0}{2\pi}\right)^2 \int_0^\pi \left\{ \sum_{n:even}^{\infty} \frac{A(n)}{\sin\theta} \right\} \cdot \left\{ \sum_{n:even}^{\infty} \frac{A_n^*(n)}{\sin\theta} \right\} \sin\theta\, d\theta$$

$$+ \frac{I_0^2}{64} \int_0^\pi \left\{ \sum_{n:odd}^{\infty} \frac{(\cos\theta)A(n)}{R\sin\theta} \right\} \cdot \left\{ \sum_{n:odd}^{\infty} \frac{(\cos\theta)A^*(n)}{R\sin\theta} \right\} \sin\theta\, d\theta$$

$$- \frac{\mu k I_0^2}{32} \int_0^\pi \left\{ \sum_{n:even}^{\infty} \frac{A(n)}{\sin\theta} \right\} \cdot \left\{ \sum_{n:odd}^{\infty} \frac{(\cos\theta)A^*(n)}{\sin\theta} \right\} \sin\theta\, d\theta$$

$$- \frac{\mu k I_0^2}{32} \int_0^\pi \left\{ \sum_{n:even}^{\infty} \frac{A^*(n)}{\sin\theta} \right\} \cdot \left\{ \sum_{n:odd}^{\infty} \frac{(\cos\theta)A(n)}{R\sin\theta} \right\} \sin\theta\, d\theta \qquad (B.7)$$

By making use of Equation (B.4), the first integral of Equation (B.7) becomes

$$\frac{\mu}{4} \left(\frac{kI_0}{2\pi}\right)^2 \int_0^\pi \left\{ \sum_{n:even}^{\infty} \frac{A(n)}{\sin\theta} \right\} \left\{ \sum_{n:even}^{\infty} \frac{A^*(n)}{\sin\theta} \right\} \sin\theta\, d\,\theta$$

$$= \infty \left(\frac{\mu}{4}\right) \left(\frac{kI_0}{2\pi}\right) \sum_{n:even}^{\infty} (2n+1)^2\, h_n^{(2)} \left(\frac{\pi}{2}\right)^2 [j_n(kR)]^2 \quad . \qquad (B.8)$$

Using Equations (B.3) and (B.4), the second integral of Equation (B.7) becomes

$$\frac{\mu I_0^2}{64} \int_0^\pi \left\{ \sum_{n:odd}^{\infty} \left\{ \frac{(\cos\theta)A(n)}{R\sin\theta} \right\} \left\{ \sum_{n:odd}^{\infty} \frac{(\cos\theta)A^*(n)}{R\sin\theta} \right\} \sin\theta\, d\theta$$

$$= \frac{\mu I_0^2}{64} \int_0^\pi \frac{1}{\sin^2\theta} \left\{ \sum_{n:odd}^{\infty} \frac{A(n)}{R} \right\} \left\{ \sum_{n:odd}^{\infty} \frac{A^*(n)}{R} \right\} \sin\theta\, d\,\theta$$

$$- \frac{\mu I_0^2}{64} \int_0^\pi \left\{ \sum_{n:odd}^{\infty} \frac{A(n)}{R} \right\} \left\{ \sum_{n:odd}^{\infty} \frac{A^*(n)}{R} \right\} \sin\theta\, d\theta$$

$$= \infty \left(\frac{\mu I_0^2}{64}\right) \left\{ \sum_{n:odd}^{\infty} (2n+1)^2\, h_n^{(2)} \left(\frac{\pi}{2}\right)^2 \frac{[j_n(kR)]^2}{R^2} \right\}$$

$$- \frac{\mu I_0^2}{32} \left\{ \sum_{n:odd}^{\infty} (2n+1)\, h_n^{(2)} \left(\frac{\pi}{2}\right)^2 \frac{[j_n(kR)]^2}{R^2} \right\} \quad . \qquad (B.9)$$

The third and fourth integrals of Equation (B.7) are combined, and then, by making use of Equation (B.6), one obtains

$$- \frac{\mu k I_0^2}{32\pi} \int_0^\pi \{ \sum_{n:even}^{\infty} \frac{A(n)}{\sin \theta} \} \{ \sum_{n:odd}^{\infty} \frac{(\cos \theta) A^*(n)}{R \sin \theta} \} \sin \theta \, d\theta$$

$$- \frac{\mu k I_0^2}{32\pi} \int_0^\pi \{ \sum_{n:even}^{\infty} \frac{A^*(n)}{\sin \theta} \} \{ \sum_{n:odd}^{\infty} \frac{(\cos \theta) A(n)}{R \sin \theta} \} \sin \theta \, d\theta$$

$$= - \frac{\mu k I_0^2}{32\pi} \int_0^\pi \{ \sum_{n:even}^{\infty} \sum_{t:odd}^{\infty} (2n+1)(2t+1)$$

$$\cdot [h_n^{(2)*}(\tfrac{\pi}{2}) \; h_t^{(2)}(\tfrac{\pi}{2}) + h_n^{(2)}(\tfrac{\pi}{2}) \; h_t^{(2)*}(\tfrac{\pi}{2})]$$

$$\cdot \frac{(\cos \theta) \; P_n(\cos \theta) \; P_t(\cos \theta)}{\sin^2 \theta} \frac{j_n(kR) \; j_t(kR)}{R} \} \sin \theta \, d\theta$$

$$= - \infty [\frac{\mu k I_0^2}{16\pi}] \{ \sum_{n:even}^{\infty} \sum_{t:odd}^{\infty} (2n+1)(2t+1)$$

$$\cdot [j_n(\tfrac{\pi}{2}) \; j_t(\tfrac{\pi}{2}) + n_n(\tfrac{\pi}{2}) \; n_t(\tfrac{\pi}{2})] \frac{j_n(kR) \; j_t(kR)}{R} \} \quad . \qquad (B.10)$$

From Equations (B.8), (B.9), and (B.10), one obtains

$$\int_0^\pi \frac{\varepsilon}{4} E_\theta \cdot E_\theta^* \sin \theta \, d\theta = \infty (\frac{\mu}{4}) \frac{k I_0^2}{2\pi} \{ \sum_{n:even}^{\infty} (2n+1)^2 \; h_n^{(2)}(\tfrac{\pi}{2})^2 \; [j_n(kR)]^2$$

$$+ \infty (\frac{\mu I_0^2}{64}) \{ \sum_{n:odd}^{\infty} (2n+1)^2 \; h_n^{(2)}(\tfrac{\pi}{2})^2 \frac{[j_n(kR)]^2}{R^2} \}$$

$$- \frac{\mu I_0^2}{32} \{ \sum_{n:odd}^{\infty} (2n+1) \; h_n^{(2)}(\tfrac{\pi}{2})^2 \frac{[j_n(kR)]^2}{R^2} \}$$

$$- \infty [\frac{\mu k I_0^2}{16\pi}] \{ \sum_{n:even}^{\infty} \sum_{t:odd}^{\infty} (2n+1)(2t+1)$$

$$\cdot [j_n(\tfrac{\pi}{2}) \; j_t(\tfrac{\pi}{2}) + n_n(\tfrac{\pi}{2}) \; n_t(\tfrac{\pi}{2})] \frac{j_n(kR) \; j_t(kR)}{R} \} \quad . \qquad (B.11)$$

By making use of Equation (B.3), the following result is obtained:

$$\int_0^{\pi} \frac{\varepsilon}{4} E_R \cdot E_R^* \sin\theta \, d\theta = (\frac{\mu I_0^2}{32}) \sum_{n:odd}^{\infty} (2n+1) \, h_n^{(2)} \, \frac{[j_n(kR)]^2}{R^2} \quad . \quad (B.12)$$

From Equations (B.11) and (B.12)

$$\int_0^{\pi} (\frac{\varepsilon}{4} E_\theta \cdot E_\theta^* + \frac{\varepsilon}{4} E_R \cdot E_R^*) \sin\theta \, d\theta$$

$$= \infty(\frac{\mu}{4})(\frac{kI_0^2}{2\pi}) \sum_{n:even}^{\infty} (2n+1)^2 \, [(j_n(\frac{\pi}{2}))^2 + (n_n(\frac{\pi}{2}))^2] \, [j_n(kR)]^2$$

$$+ \infty(\frac{\mu I_0^2}{64}) \sum_{n:odd}^{\infty} (2n+1)^2 \, [(j_n(\frac{\pi}{2}))^2 + (n_n(\frac{\pi}{2}))^2] \, \frac{[j_n(kR)]^2}{R^2}$$

$$- \infty(\frac{\mu k I_0^2}{16\pi}) \sum_{n:even}^{\infty} \sum_{t:odd}^{\infty} (2n+1)(2t+1)$$

$$\cdot \left| j_n(\frac{\pi}{2}) \, j_t(\frac{\pi}{2}) + n_n(\frac{\pi}{2}) \, n_t(\frac{\pi}{2}) \right| \frac{j_n(kR) \, j_t(kR)}{R} \quad . \quad (B.13)$$

Substituting Equation (B.13) into Equation (B.2) and then integrating with respect to ϕ,

$$W_e^{in} = \infty[\frac{\mu I_0^2}{8\pi k}]\{ \sum_{n:even}^{\infty} (2n+1)^2 \cdot [(j_n(\frac{\pi}{2}))^2 + (n_n(\frac{\pi}{2}))^2] \int_0^{\frac{\pi}{2}} [j_n(\rho)]^2 \rho^2 \, d\rho \}$$

$$+ \infty[\frac{\mu I_0^2}{64k}]\{ \sum_{n:odd}^{\infty} (2n+1)^2 \cdot [(j_n(\frac{\pi}{2}))^2 + (n_n(\frac{\pi}{2}))^2] \int_0^{\frac{\pi}{2}} [j_n(\rho)]^2 \, d\rho \}$$

$$- \infty[\frac{\mu I_0^2}{16\ k}]\{ \sum_{n:even}^{\infty} \sum_{t:odd}^{\infty} (2n+1)(2t+1)$$

$$\cdot [j_n(\frac{\pi}{2}) \, j_t(\frac{\pi}{2}) + n_n(\frac{\pi}{2}) \, n_t(\frac{\pi}{2})] \int_0^{\frac{\pi}{2}} [j_n(\rho) \, j_t(\rho)] \, d\rho \} \quad (B.14)$$

where

$$\rho = kR \quad .$$

It is well known that the spherical Bessel functions $j_n(\rho)$ have positive values in the range $0 \le \rho \le \frac{\pi}{2}$. Therefore, it can be said that the following three definite integrals exceed zero. That is:

$$\int_0^{\frac{\pi}{2}} [j_n(\rho)]^2 \rho^2 \, d\rho > 0 \quad , \tag{B.15}$$

$$\int_0^{\frac{\pi}{2}} [j_n(\rho)]^2 \, d\rho > 0 \quad , \tag{B.16}$$

and

$$\int_0^{\frac{\pi}{2}} [j_n(\rho) \; j_t(\rho)] \, \rho \, d\rho > 0 \quad , \tag{B.17}$$

when

$$j_n(\rho) > 0 \text{ for } 0 \le \rho \le \frac{\pi}{2} \quad ,$$

$$j_t(\rho) > 0 \text{ for } 0 \le \rho \le \frac{\pi}{2} \quad .$$

Thus one may say:

$$\sum_{n:\text{even}}^{\infty} (2n+1)^2 \, [(j_n(\tfrac{\pi}{2}))^2 + (n_n(\tfrac{\pi}{2}))^2] \int_0^{\frac{\pi}{2}} [j_n(\rho)]^2 \rho \, d\rho > 0 \quad , \tag{B.18}$$

$$\sum_{n:\text{odd}}^{\infty} (2n+1)^2 \, [(j_n(\tfrac{\pi}{2}))^2 + (n_n(\tfrac{\pi}{2}))^2] \int_0^{2} [j_n(\rho)]^2 \, d\rho > 0 \quad , \tag{B.19}$$

$$\sum_{n:\text{even}}^{\infty} \sum_{t:\text{odd}}^{\infty} (2n+1)(2t+1) \, [j_n(\tfrac{\pi}{2}) \, j_t(\tfrac{\pi}{2}) + n_n(\tfrac{\pi}{2}) \, n_t(\tfrac{\pi}{2})]$$

$$\cdot \int_0^{2} [j_n(\rho) \; j_t(\rho)] \, \rho \, d\rho > 0 \quad . \tag{B.20}$$

Using the above three equations in Equation (B.14), it is concluded that electric energy stored inside of the smallest imaginary sphere which can enclose the infinitesimally thin half-wave dipole approaches ∞; that is,

$$W_e^{in} \to \infty \quad , \tag{B.21}$$

From Equations (4.2) and (3.6), the total power radiated from the half-wave dipole is

$$P_R = \frac{\mu\pi\omega I_0^{\,2}}{8k} \sum_{n:odd}^{\infty} \frac{(2n+1)}{n(n+1)} [j_n(\tfrac{\pi}{2})]^2 \quad , \tag{B.22}$$

and P_R is finite. Therefore, the quality factor Q_{in} for the energy stored inside of the smallest imaginary sphere which can enclose the infinitesimally thin half-wave dipole is obtained as:

$$Q_{in} = \frac{2\omega\ W_e^{in}}{P_R} \to \infty \quad . \tag{B.23}$$

APPENDIX C

NORMALIZED INPUT ADMITTANCE FOR NEAR HALF-WAVE DIPOLES WITH RADIUS OF $\lambda/200$ AND $\lambda/1000$ AND THEIR BANDWIDTH

TABLE I

THE INPUT IMPEDANCE z=R+iX AND THE NORMALIZED ADMITTANCE |Y| FOR NEAR HALF-WAVE DIPOLE WITH RADIUS OF λ/200

$\ell/\frac{\lambda}{4}$	R(Ω)	X(Ω)	$R^2 + X^2$	|Y|	ω_N
0.75	41.52	-117.63	15560.727	0.521	1.240
0.80	47.52	- 85.08	9496.757	0.667	1.163
0.85	54.35	- 52.89	5751.275	0.857	1.094
0.90	61.99	- 20.76	4273.738	0.994	1.030
0.93	65.0	0	4225.0	1.0	1.0
0.95	70.53	11.53	5107.422	0.910	0.979
1.0	80.19	44.40	8401.796	0.709	0.930
1.05	91.24	78.09	14422.786	0.541	0.886
1.10	103.93	112.98	23565.925	0.423	0.845

where

$$|Y| = \left[\frac{4225.0}{R^2 + X^2}\right]^{\frac{1}{2}}$$

$$\omega_N = \left[\frac{0.93}{\ell/\frac{\lambda}{4}}\right]$$

Source: H. Uchida, and Y. Mushiake, Chotanpa Kuchusen (VHF Antenna). Tokyo: Koronasha Inc., 1966.

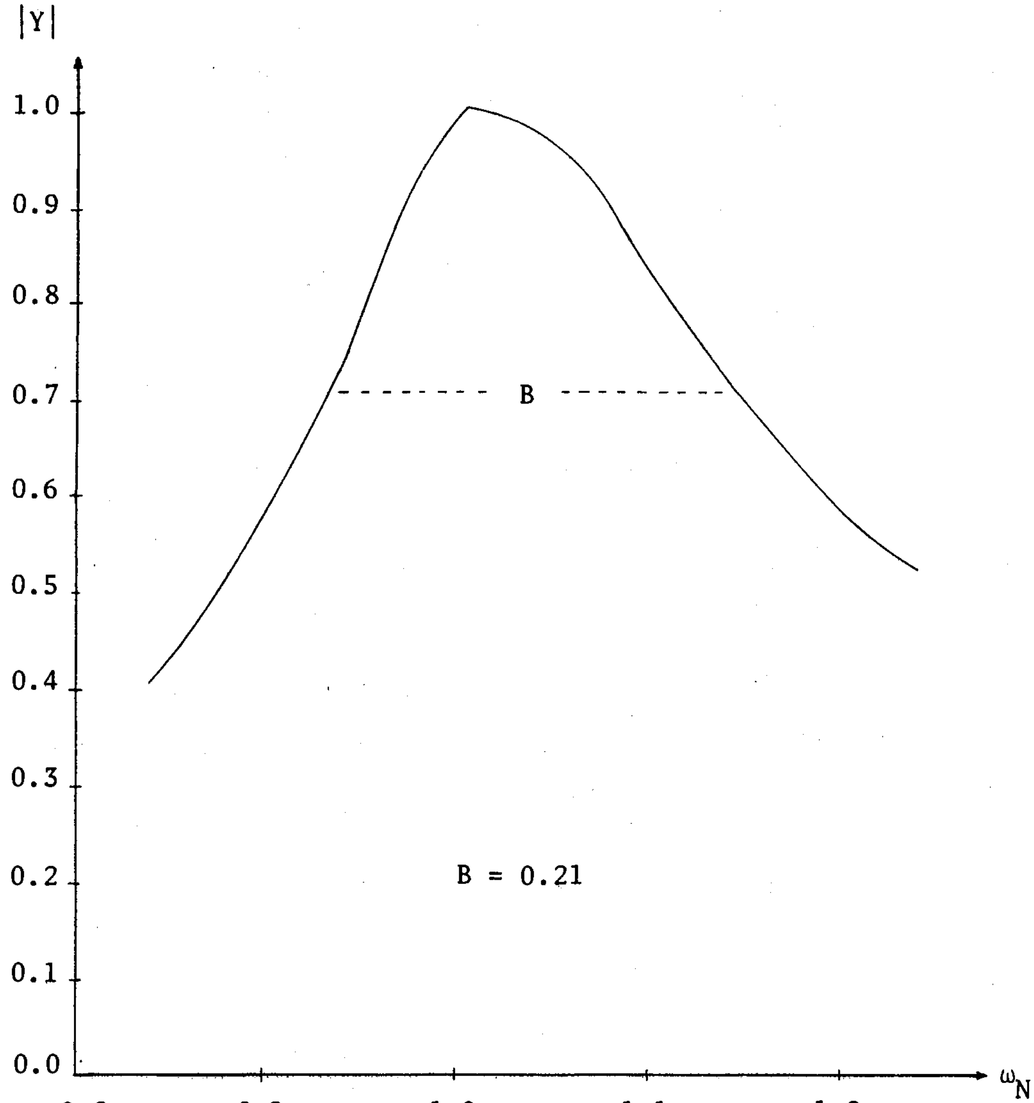

Figure 4. Bandwidth B for the Half-Wave Dipole With Radius $\lambda/200$

TABLE II

THE INPUT IMPEDANCE z=R+iX AND THE NORMALIZED ADMITTANCE |Y| FOR NEAR HALF-WAVE DIPOLE WITH RADIUS OF λ/1000

$\ell/\frac{\lambda}{4}$	R(Ω)	X(Ω)	$R^2 + X^2$	|Y|	ω_N
0.75	39.97	-195.21	39704.545	0.346	1.267
0.80	45.87	-146.34	23519.453	0.450	1.188
0.85	52.66	- 98.48	12471.386	0.617	1.118
0.90	60.28	- 51.11	6245.911	0.872	1.056
0.95	68.84	- 3.78	4753.234	1.0	1.0
1.0	78.57	44.11	8118.937	0.765	0.950
1.05	89.74	93.03	16707.849	0.533	0.905
1.1	102.62	143.55	176.458	0.391	0.864

where

$$|Y| = \left[\frac{4753.234}{R^2 + X^2}\right]^{\frac{1}{2}}$$

$$\omega_N = \left[\frac{0.95}{\ell/\frac{\lambda}{4}}\right]$$

Source: H. Uchida, and Y. Mushiake, Chotanpa Kuchusen (VHF Antenna). Tokyo: Koronasha Inc., 1966.

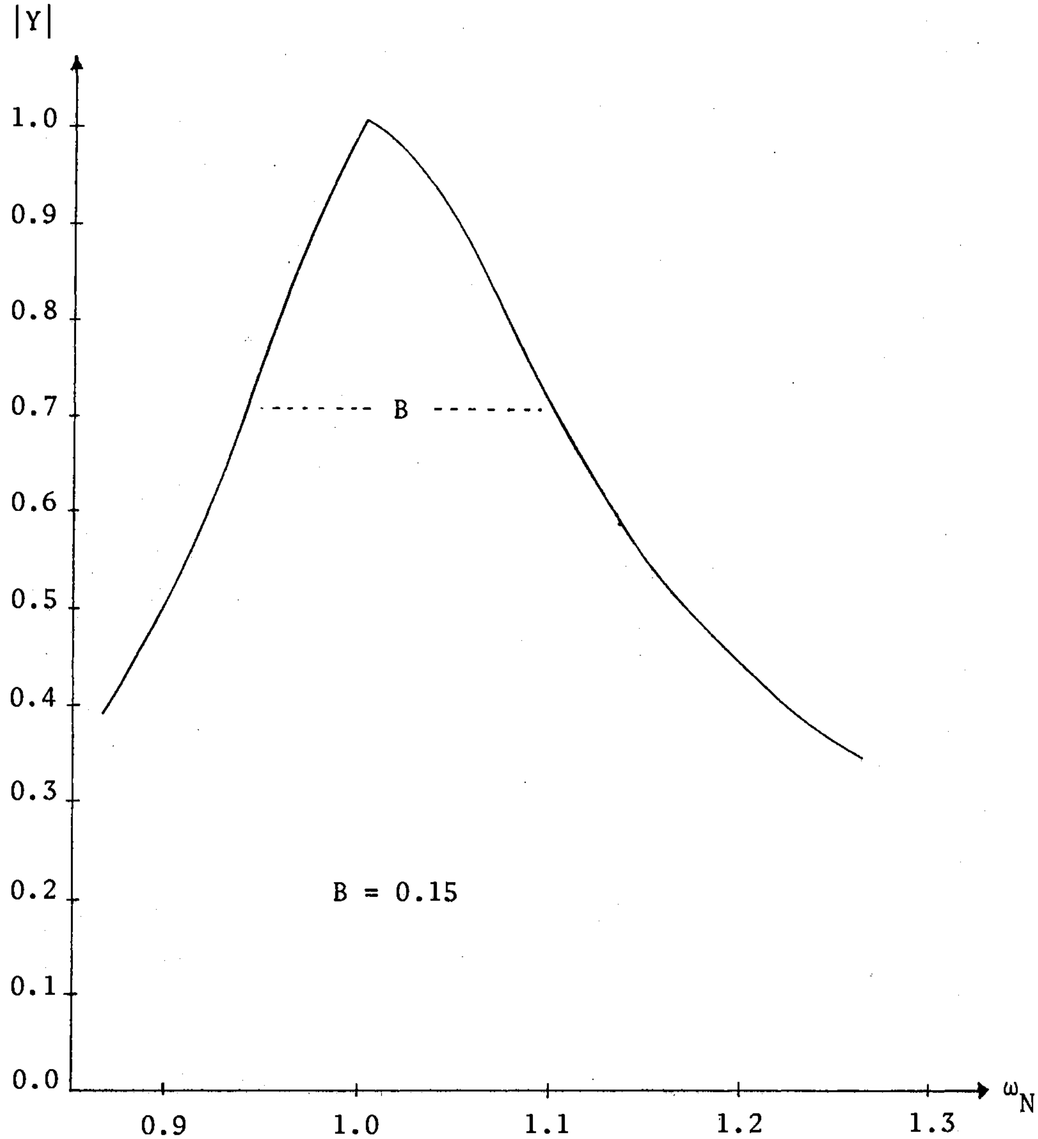

Figure 5. Bandwidth B for the Half-Wave Dipole With Radius $\lambda/1000$

VITA

Takeru Irabu

Candidate for the Degree of

Doctor of Philosophy

Thesis: SPHERICAL-WAVE EXPANSIONS FOR THE FIELDS OF THE LINEAR RADIATORS WITH ASSUMED SINUSOIDAL CURRENT AND SOME OPINIONS ON THE THEORY OF SUPERGAIN RADIATORS

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Ponape, Micronesia, October 21, 1942, the son of Mr. and Mrs. Masayoshi Irabu.

Education: Graduated from Shuri High School, Naha, Okinawa, in March, 1961; received Bachelor of Science degree in Electrical Engineering from Gifu University, Gifu, Japan in 1966; received Master of Science degree in Electrical Engineering from Oklahoma State University in 1970; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1973.

Professional Experience: Research Assistant, Nagoya University, Nagoya, Japan, 1966-1967; Teaching Assistant, School of Electrical Engineering, Oklahoma State University, 1970-1972.

Professional Organizations: Member of Sigma Xi, and the Institute of Electrical and Electronics Engineers.