ANALYTIC CHARACTERISTIC FUNCTIONALS AND SOME CHARACTERIZATION PROBLEMS IN PROBABILITY THEORY

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CHAPTER I

PRELIMINARY CONCEPTS

This paper will be devoted to the development of the concept of an analytic characteristic functional and some characterization problems in probability.

Analytic Functions on Banach Spaces

In order to develop the concept of an analytic characteristic functional it will be necessary to outline the development of analytic functions on Banach spaces which appears in Hille [11], Bochnak and Siciak [2] and [3], Bochnak [1] and Ligocka and Siciak [15]. In the following $\mathfrak{X}$ and Z will be Banach spaces over the scalar field F (F is either the real or complex numbers).

Definition 1.1 A function $P: \mathfrak{X} \to Z$ is called a polynomial of degree m if for all $a, h \in \mathfrak{X}$ and all $\alpha \in F$

$$P(a + \alpha h) = \sum_{v=0}^{m} P_v(a, h)\, \alpha^v ,$$

where the P_v are functions which are independent of α. The degree is exactly m if $P_m \neq 0$.

Unless otherwise specified, all polynomials in the following definitions will be assumed to map $\mathfrak{X}$ into Z.

<u>Definition 1.2</u> A polynomial P is said to be homogeneous of degree n if

$$P(\alpha x) = \alpha^n P(x) \qquad \alpha \in F, \; x \in \mathfrak{X}.$$

<u>Definition 1.3</u> The series $\sum_{n=0}^{\infty} P_n$ of homogeneous polynomials is said to converge in the set $E \subset \mathfrak{X}$ provided $\sum_{n=0}^{\infty} P_n(x)$ converges for all $x \in E$.

The preceeding definitions concerning polynomials on a Banach space make possible an approach to generalized analyticity which is both similar to and consistent with analyticity of functions of complex variables.

<u>Definition 1.4</u> The function $f: \mathfrak{X} \to Z$ is said to be $\mathcal{G}$-analytic in the open set $E \subset \mathfrak{X}$ provided that for each $x \in E$ there exists a series of homogeneous polynomials, $\sum_{n=0}^{\infty} P_n$ such that

$$f(x+h) = \sum_{n=0}^{\infty} P_n(h)$$

for all h in a neighborhood of 0. (Here neighborhood of 0 is relative to the norm topology on $\mathfrak{X}$.)

<u>Definition 1.5</u> The function f defined on $\mathfrak{X}$ is said to possess a p^{th} $\mathcal{G}$-differential at a point $x_o \in \mathfrak{X}$, written $f \in \mathcal{G}^p_{x_o}$, if

(1) For each $h \in \mathfrak{X}$ the function g defined on F by $g(\alpha) = f(x_o + \alpha h)$ has a p^{th} derivative at 0; and

(2) The mapping $\delta_{x_o}^n f : \mathfrak{X} \to Z$ defined by

$$\delta_{x_o}^n f(h) = \frac{d^n}{d\alpha^n} f(x_o + \alpha h) \Big|_{\alpha = 0}$$

is a homogeneous polynomial of degree n $(n = 1, 2, \ldots, p)$.

For each $x_o, h \in \mathfrak{X}$ $\delta_{x_o}^n f(h)$ is called the n^{th} $\mathcal{G}$-differential of f at the point x_o with increment h. It should be noted that if the domain of f is the set of real numbers, then $\delta_{x_o}^n f(h) = f^{(n)}(x_o) h^n$. If $f \in \mathcal{G}_x^p$ for all $x \in U \subset \mathfrak{X}$, the notation $f \in \mathcal{G}^p(U)$ is used and if $f \in \mathcal{G}^p(U)$ for all positive integers p, the notation $f \in \mathcal{G}^\infty(U)$ is used.

For a proof of the following theorem see Hille [11].

Theorem 1.1 The function $f : \mathfrak{X} \to Z$ is $\mathcal{G}$-analytic in an open set $E \subset \mathfrak{X}$ if and only if $f \in \mathcal{G}^\infty(E)$ and for every $x \in E$ there is a neighborhood N_x of $0 \in \mathfrak{X}$ such that

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta_x^n f(h), \qquad \forall h \in N_x .$$

For the case $\mathfrak{X} = R$, the series on the right is the usual Taylor series expansion of $f(x+h)$ about the point x.

Definition 1.6 A function $f : \mathfrak{X} \to Z$ is said to be analytic in $E \subset \mathfrak{X}$ if for each $x \in E$ there exists a series, $\sum_{n=0}^{\infty} P_n$, of continuous homogeneous polynomials such that

$$f(x+h) = \sum_{n=0}^{\infty} P_n(h)$$

for all h in some neighborhood of 0.

Definition 1.7 A function $f: \mathfrak{X} \to Z$ is said to be locally bounded in the open set $D \subset \mathfrak{X}$ if for each $a \in D$ there is a sphere S_a containing a and a finite number M_a such that $\|f(x)\| < M_a$ when $x \in S_a$.

Theorem 1.2 A function which is G-analytic on a set D is analytic on D provided it is locally bounded on D.

Theorem 1.3 An analytic function which vanishes in a sphere vanishes identically in its domain of analyticity.

Theorems 1.2 and 1.3 both appear in Hille [11].

In the remainder of this paper the scalar field F is assumed to be the real numbers. Hence $\mathfrak{X}$ is a real Banach space. Bochnak and Siciak [2] showed that for such a space $\mathfrak{X}$ there exists a complex Banach space $\overline{\mathfrak{X}}$ with the following properties.

(1) $\overline{\mathfrak{X}} = \mathfrak{X} \times \mathfrak{X}$ with addition component wise and multiplication by complex scalars defined by

$$(a+ib)(x, y) = (ax - by, bx + ay).$$

(2) $\mathfrak{X}$ can be treated as a subspace of $\overline{\mathfrak{X}}$ by associating $x \in \mathfrak{X}$ with $(x, 0)$.

(3) $\overline{\mathfrak{X}}$ may be treated as the direct topological product of $\mathfrak{X}$ and $i\,\mathfrak{X}$ and every element $(x, y) \in \overline{\mathfrak{X}}$ can be written as $x + iy$.

(4) For any seminorm [norm] q on $\mathfrak{X}$

$$\overline{q}(\overline{x}) = \inf\{\Sigma\, |t_j|\, q(x_j) : \overline{x} = \Sigma\, t_j x_j,\ x_j \in \mathfrak{X},\ t_j \in \mathbb{C}\}$$

for $\overline{x} \in \overline{\mathfrak{X}}$, is a seminorm [norm] on $\overline{\mathfrak{X}}$. In addition $\overline{q}$ has the property that

$$\max\{q(x), q(y)\} \leq \overline{q}(x+iy) \leq q(x) + q(y) .$$

(5) If P is a homogeneous polynomial on $\mathfrak{X}$, then there exists a unique polynomial $\overline{P}$ on $\overline{\mathfrak{X}}$ such that $P = \overline{P}|\mathfrak{X}$. $\overline{P}$ is continuous if and only if P is continuous.

(6) If a series of homogeneous polynomials $\sum_{n=0}^{\infty} P_n$ defined on $\mathfrak{X}$ converges in an open set H, then there exists an open set $\overline{H} \subset \overline{\mathfrak{X}}$ such that $H \subset \overline{H}$ and $\sum_{n=0}^{\infty} \overline{P}_n$ is convergent in $\overline{H}$.

Random Variables and Characteristic Functionals

Let $(\Omega, \mathfrak{F}, \mu)$ be a probability space and let $\mathfrak{X}$ be a real Banach space with dual space $\mathfrak{X}^*$. Let X be a weakly $\mathfrak{F}$ measurable function mapping Ω into $\mathfrak{X}$. Then X is called a random variable (r.v.). An important analytical tool in the study of random variables is the characteristic functional.

<u>Definition 1.8</u> The characteristic functional (c.f.) of the r.v. X is given by

$$f(x^*) = E\, e^{i x^*(X)} = \int_\Omega e^{i x^*(X(\omega))} \mu(d\omega) , \qquad x^* \in \mathfrak{X}^* ,$$

The special case which occurs when $\mathfrak{X}$ is the set of real numbers has been investigated extensively. For this case, the theory of analytic c.f.'s has been based upon the following definition.

Definition 1.9 Let f be the c.f. of a real valued r.v. The c.f. f is said to be an analytic c.f. if there exists a function g of the complex variable z which is regular in the circle $|z| < \rho$ $(\rho > 0)$ and a constant $\delta > 0$ such that $g(t) = f(t)$ for $-\delta < t < \delta$.

That Definition 1.6 and Definition 1.9 are equivalent can be seen by considering the following well known theorem concerning c.f.'s of real r.v.'s.

Theorem 1.4 If f is an analytic c.f. of a real valued r.v., then the function g, of the complex variable z, defined by

$$g(z) = \int_\Omega e^{izX(\omega)} \mu(d\omega) = E(e^{izX})$$

is analytic in a horizontal strip containing the real axis and g coincides with f on the real axis.

A similar result holds for c.f.'s of r.v.'s taking values in an arbitrary Banach space. This result along with other generalizations will be proved later in this chapter. In order to discuss c.f.'s of r.v.'s taking values in the real Banach space $\mathfrak{X}$, it is necessary to make the following observations.

Since the c.f. f of an $\mathfrak{X}$-valued r.v. X is bounded, it is analytic provided it is $\mathcal{G}$-analytic [Theorem 1.2]. For $x^* \in \mathfrak{X}^*$, $x^*(X)$ is a real valued r.v. whose c.f. is given by

$$f_{x^*}(t) = E\, e^{itx^*(X)} = f(tx^*), \qquad t \in R.$$

Definition 1.10 The n^{th} moment functional of the r.v. X is defined by

$$m_n(x^*) = E[x^*(X)]^n, \qquad x^* \in \mathfrak{X}^*, \quad n = 0, 1, \ldots,$$

provided the expectation exists. Clearly, if $m_n(x^*)$ exists, it is equal to the n^{th} moment of the r.v. $x^*(X)$.

The analyticity of f in a neighborhood of $0 \in \mathfrak{X}^*$ determines the analyticity of the c.f.'s of the r.v.'s $x^*(X)$, $x^* \in \mathfrak{X}^*$.

Theorem 1.5 If f is $\mathcal{G}$-analytic in a neighborhood of $0 \in \mathfrak{X}^*$, then for any $x^* \in \mathfrak{X}^*$ the real r.v. $x^*(X)$ has an analytic c.f.

Proof Since f is $\mathcal{G}$-analytic in a neighborhood of 0, there exists $r > 0$ such that $\|y^*\| < r$ implies

$$f(y^*) = \sum_{n=0}^{\infty} \delta_o^n f(y^*) \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\alpha^n} f(\alpha y^*)\Big|_{\alpha=0}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\alpha^n} f_{y^*}(\alpha)\Big|_{\alpha=0} = \sum_{n=0}^{\infty} \frac{i^n}{n!} m_n(y^*).$$

Therefore the series

$$f_{y^*}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} m_n(y^*)$$

converges in $|t| < 1$, and when $\|y^*\| < r$, $y^*(X)$ has an analytic c.f.

Now let $x^* \in \mathfrak{X}^*$. There exists $k > 0$ such that $\|kx^*\| < r$ so $(kx^*)(X)$ has an analytic c.f. But $(kx^*)(X) = kx^*(X)$ having analytic c.f. implies $x^*(X)$ has an analytic c.f. Hence the theorem is proved.

The $\mathcal{G}$-analyticity of f in a neighborhood of $0 \in \mathfrak{X}^*$ can also be used to prove results analogous to well known theorems concerning c.f.'s of real r.v.'s. Examples of such results appear in Theorem 1.6 and the succeeding corollary. The following lemmas will be needed for the proof of Theorem 1.6.

<u>Lemma 1.1</u> If $\{X_n\}$ is a sequence of complex r.v.'s, such that $\sum_{n=0}^{\infty} E|X_n| < \infty$, then

$$E \sum_{n=0}^{\infty} X_n = \sum_{n=0}^{\infty} E X_n .$$

<u>Proof</u> See Halmos [9].

<u>Lemma 1.2</u> If f is $\mathcal{G}$-analytic in $\|y^*\| < r$, then

$$\sum_{n=0}^{\infty} \frac{E|x^*(X)|^n}{n!} < \infty$$

for $\|x^*\| < \frac{r}{2}$.

Proof Suppose f is $\mathfrak{C}$-analytic in $\|y^*\| < r$. Let $x^* \in \mathfrak{X}^*$ such that $\|x^*\| < \frac{r}{2}$. Then $\|2x^*\| < r$ and

$$f(2x^*) = \sum_{n=0}^{\infty} \frac{i^n}{n!} m_n(2x^*) = \sum_{n=0}^{\infty} \frac{(2i)^n}{n!} m_n(x^*) < \infty.$$

Hence the series $\sum_{n=0}^{\infty} \frac{m_n(x^*)}{n!} z^n$ converges for $|z| < 2$, and $x^*(X)$ has a c.f. which is analytic in $|Im(z)| < 2$. Therefore

$$\sum_{n=0}^{\infty} \frac{E|x^*(X)|^n}{n!} < \infty$$

since $E|x^*(X)|^n$ is the n^{th} absolute moment of $x^*(X)$.

Lemma 1.3 Let $y^* \in \mathfrak{X}^*$ and define

$$P_n(x^*;y^*) = E\left[e^{iy^*(X)}[x^*(X)]^n\right] \frac{i^n}{n!}, \qquad x^* \in \mathfrak{X}^*, \; n = 0, 1, \ldots .$$

Then $P_n(x^*;y^*)$ is a homogeneous polynomial of degree n in x^* provided the expectation exists.

Proof Let $a, h \in \mathfrak{X}^*$ and $\alpha \in R$. Then for $n = 0, 1, \ldots$

$$P_n(a + \alpha h; y^*) = \frac{i^n}{n!} E\{e^{iy^*(X)}[(a + \alpha h)(X)]^n\}$$

$$= \frac{i^n}{n!} E\{e^{iy^*(X)}[a(X) + \alpha h(X)]^n\}$$

$$= \frac{i^n}{n!} E \sum_{k=0}^{n} e^{iy^*(X)} \binom{n}{k} \alpha^k [h(X)]^k [a(X)]^{n-k} =$$

$$= \sum_{k=0}^{n} \alpha^k \frac{i^n}{n!} \binom{n}{k} E\{e^{i y^*(X)} [h(X)]^k [a(X)]^{n-k}\}$$

$$= \sum_{k=0}^{n} \alpha^k \tilde{P}_k(a, h)$$

and the $\tilde{P}_n$ are independent of α. Therefore P_n is a polynomial of degree n. To show that P_n is homogeneous consider

$$P_n(\alpha x^*; y^*) = \frac{i^n}{n!} E\{e^{i y^*(X)} [\alpha x^*(X)]^n\}$$

$$= \frac{i^n}{n!} \alpha^n E\{e^{i y^*(X)} [x^*(X)]^n\}$$

$$= \alpha^n P_n(x^*),$$

Hence P_n is a homogeneous polynomial of degree n.

Theorem 1.6 If f is $\mathcal{G}$-analytic in a neighborhood of 0, then it is $\mathcal{G}$-analytic on all of $\mathfrak{X}^*$.

Proof Suppose that f is $\mathcal{G}$-analytic in $\{x^* \in \mathfrak{X}^* : \|x^*\| < \varepsilon\}$. Let $y^*, x^* \in \mathfrak{X}^*$ such that $\|x^*\| < \frac{\varepsilon}{2}$. Then

$$f(y^* + x^*) = E\, e^{i y^*(X) + i x^*(X)} = E\left\{e^{i y^*(X)} \sum_{n=0}^{\infty} \frac{i^n}{n!} [x^*(X)]^n\right\}$$

$$= E \sum_{n=0}^{\infty} e^{i y^*(X)} \frac{i^n}{n!} [x^*(X)]^n$$

$$= \sum_{n=0}^{\infty} E\left\{e^{i y^*(X)} \frac{i^n}{n!} [x^*(X)]^n\right\} \qquad \text{(Lemmas 1.1 and 1.2)}$$

$$= \sum_{n=0}^{\infty} P_n(x^*; y^*)$$

where by Lemma 1.3, the P_n's are known to be homogeneous polynomials of degree n. Therefore f is $\mathcal{G}$-analytic on all of $\mathfrak{X}^*$ since y^* was arbitrary.

Corollary 1.1 If f_1 and f_2 are two c.f.'s which agree on a sphere and are $\mathcal{G}$-analytic in a neighborhood of $0 \in \mathfrak{X}^*$, then $f_1 = f_2$.

Proof Both c.f.'s are $\mathcal{G}$-analytic on all of $\mathfrak{X}^*$ by the previous theorem. Hence both are analytic on $\mathfrak{X}^*$. It is easily seen that $f_1 - f_2$ is analytic on $\mathfrak{X}^*$, and vanishes in a sphere. Therefore $f_1 = f_2$ by Theorem 1.3.

The preceeding result shows that an analytic c.f. is uniquely determined by its values in a neighborhood of zero.

The next theorem is similar to Theorem 1.4. Let $\overline{\mathfrak{X}^*}$ denote the complexification of $\mathfrak{X}^*$. The elements of $\overline{\mathfrak{X}^*}$ may be written in the form $x_1^* + i x_2^*$ where $x_1^*, x_2^* \in \mathfrak{X}^*$. If f is an analytic c.f. on $\mathfrak{X}^*$, the properties (5) and (6) of the complexification $\overline{\mathfrak{X}^*}$ (see pg. 5) show that there exists a function defined on $\overline{\mathfrak{X}^*}$ which is analytic in an open set containing $\mathfrak{X}^*$ and agrees with f on $\mathfrak{X}^*$. The following theorem shows that the integral defining f converges in a "strip" containing $\mathfrak{X}^*$.

Theorem 1.7 If f is an analytic c.f., there exists a $\delta > 0$ such that the function

$$\overline{f}(x_1^* + i x_2^*) = E\, e^{i(x_1^* + i x_2^*)(X)}$$

is defined on the set $U = \{x_1^* + i x_2^* \in \overline{\mathfrak{X}^*} : \|x_2^*\| < \delta\}$.

Proof Since f is analytic on $\mathfrak{X}^*$, there exists $\delta > 0$ such that

$$f(x^*) = \sum_{n=0}^{\infty} \frac{i^n}{n!} m_n(x^*), \quad \text{for} \quad \|x^*\| < 2\delta .$$

Consider

$$|\overline{f}(x_1^* + i x_2^*)| = |E\, e^{i(x_1^* + i x_2^*)(X)}| \quad , \quad \|x_2^*\| < \delta$$

$$\leq E|e^{i x_1^*(X) - x_2^*(X)}|$$

$$= E|e^{-x_2^*(X)}|$$

$$\leq E \sum_{n=0}^{\infty} \frac{1}{n!} [x_2^*(X)]^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E[x_2^*(X)]^n \qquad \text{(Lemmas 1.1 and 1.2)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} m_n(x_2^*) < \infty$$

because

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} m_n(2x_2^*) = \sum_{n=0}^{\infty} \frac{(2i)^n}{n!} m_n(x_2^*)$$

converges. Therefore $\overline{f}$ is defined in $U = \{x_1^* + i x_2^* \in \overline{\mathfrak{X}^*} : \|x_2^*\| < \delta\}$.

Kotlarski [13] and Miller [18] gave theorems which showed that the distributions of the independent r.v.'s X_1, X_2, X_3 can be determined by the distribution of $(X_1 + X_3, X_2 + X_3)$ provided certain assumptions are satisfied. The result in [13] was given for r.v.'s attaining values in stochastic locally convex linear topological spaces. The following theorem obtains the result for r.v.'s taking their values in a reflexive Banach space with the assumption that the c.f.'s of the X_k are never zero replaced by a weaker assumption.

<u>Theorem 1.8</u> Let $\mathfrak{X}$ be a reflexive Banach space and X_k $(k = 1, 2, 3)$ be independent $\mathfrak{X}$ valued r.v.'s whose c.f.'s f_k are analytic. Let $Y_k = X_k + X_3$ $(k = 1, 2)$. The distribution of (Y_1, Y_2) determines the distributions of the X_k up to a change in location.

<u>Proof</u> Let the joint c.f. of (Y_1, Y_2) be denoted by

$$f(x_1^*, x_2^*) = E \exp[i x_1^*(Y_1) + i x_2^*(Y_2)].$$

It can be seen that

$$f(x_1^*, x_2^*) = f_1(x_1^*) f_2(x_2^*) f_3(x_1^* + x_2^*) .$$

If X_k' $(k = 1, 2, 3)$ are other independent $\mathfrak{X}$ valued r.v.'s with analytic c.f.'s f_k', then the joint c.f. of

$$(Y_1', Y_2') = (X_1' + X_3', X_2' + X_3') ,$$

denoted by f', satisfies

$$f'(x_1^*, x_2^*) = f_1'(x_1^*) f_2'(x_2^*) f_3'(x_1^* + x_2^*) .$$

Suppose that

$$f'(x_1^*, x_2^*) = f(x_1^*, x_2^*) . \tag{1.1}$$

This yields the equation

$$f_1'(x_1^*)\, f_2'(x_2^*)\, f_3'(x_1^* + x_2^*) = f_1(x_1^*)\, f_2(x_2^*)\, f_3(x_1^* + x_2^*) .$$

Since f_k and f_k' $(k = 1, 2, 3)$ are all continuous and take on the value one at the point $0 \in \mathfrak{X}^*$, there exists $\delta > 0$ such that for $x \in E = \{x^* \in \mathfrak{X}^* : \|x^*\| < \delta\}$

$$f_k(x^*) \neq 0 \quad \text{and} \quad f_k'(x^*) \neq 0 .$$

For values of $x^* \in E$ let

$$f_k'(x^*) = f_k(x^*)\, g_k(x^*), \qquad k = 1, 2, 3 . \tag{1.2}$$

Therefore when $x_1^*, x_2^*, x_1^* + x_2^* \in E$

$$g_1(x_1^*)\, g_2(x_2^*)\, g_3(x_1^* + x_2^*) = 1 . \tag{1.3}$$

The functionals g_k are continuous, do not vanish in E and satisfy

$$g_k(0) = 1 \quad \text{and} \quad g_k(-x^*) = \overline{g_k(x^*)} .$$

It can then be seen that for $x^* \in E$

$$g_1(x^*) = g_2(x^*) = \frac{1}{g_3(x^*)} , \tag{1.4}$$

or for $x_1^*, x_2^*, x_1^* + x_2^* \in E$

$$g_3(x_1^* + x_2^*) = g_3(x_1^*)\, g_3(x_2^*) . \tag{1.5}$$

Let

$$h(x^*) = h_1(x^*) + i h_2(x^*) = \log[g_3(x^*)], \qquad x^* \epsilon E,$$

where log is the continuous branch of the logarithm which satisfies the conditions

$$h(0) = h_1(0) + i h_2(0) = 1$$

and $h_1(x^*)$, $h_2(x^*)$ are real. Therefore by (1.5)

$$h_k(x_1^* + x_2^*) = h_k(x_1^*) + h_k(x_2^*)$$

for $x_1^*, x_2^*, x_1^* + x_2^* \epsilon E$. Also the h_k are continuous functionals satisfying

$$h_1(-x^*) = h_1(x^*) \quad \text{and} \quad h_2(-x^*) = -h_2(x^*).$$

Therefore $h_1(x^*) \equiv 0$, h_2 is a real valued functional and there exists a real linear functional $q(x^*)$ such that $q(x^*) = h_2(x^*)$ for $\|x^*\| < \frac{\delta}{2}$. Since $\mathfrak{X}$ is reflexive there exists $x_o \epsilon \mathfrak{X}$ such that $q(x^*) = x^*(x_o)$ for all $x^* \epsilon \mathfrak{X}^*$. Then by (1.4)

$$g_1(x^*) = g_2(x^*) = e^{i x^*(x_o)} \quad \text{and} \quad g_3(x^*) = e^{-i x^*(x_o)}$$

for all x^* with $\|x^*\| < \frac{\delta}{2}$. Hence by (1.2)

$$f_k'(x^*) = e^{-i x^*(x_o)} f_k(x^*), \qquad k = 1, 2,$$

and

$$f_3'(x^*) = e^{ix^*(x_o)} f_3(x^*)$$

when $\|x^*\| < \frac{\delta}{2}$. Because

$$f_k'(x^*) \quad \text{and} \quad e^{-ix^*(x_o)} f_k(x^*) \qquad (k = 1, 2)$$

are both analytic c.f.'s which agree on a neighborhood of zero, Corollary 1.1 implies they agree on all of $\mathfrak{X}^*$. Similarly,

$$f_3'(x^*) = e^{ix^*(x_o)} f_3(x^*)$$

on all of $\mathfrak{X}^*$. Hence the theorem is proved.

CHAPTER II

D. VAN DANTZIG'S PROBLEM

Certain classes of analytic c.f.'s of real r.v.'s have been studied little more than analytic c.f.'s of r.v.'s taking their values in arbitrary Banach spaces. Problems concerning one such class were posed by D. van Dantzig and investigated by Lukacs [17] and Ostrovskii [20].

An analytic c.f. g is said to belong to the class D provided $f(t) = \frac{1}{g(it)}$, $t \in R$, also defines a c.f. As late as 1960 only three nontrivial pairs of the class D were known. These were the pairs

$$\left[\cos t, \frac{1}{\cosh t}\right], \quad \left[\frac{\sin t}{t}, \frac{t}{\sinh t}\right] \quad \text{and} \quad \left[e^{-t^2/2}, e^{-t^2/2}\right].$$

The set D is much larger than one might have judged in 1960. Some c.f.'s belonging to D for which the corresponding density functions are readily available, are found in the following example.

<u>Example 2.1</u> Let $p \geq -\frac{1}{2}$ and for $n = 1, 2, \ldots$ let $\lambda_{p,n}$ be the sequence of positive zero points of $J_p(t)$, the Bessel function of order p. Then

$$f_p(t) = \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{\lambda_{p,n}^2}\right) = 2^p \, \Gamma(p+1) \, \frac{J_p(t)}{t^p} \qquad (2.1)$$

is a c.f. belonging to D. If $p > -\frac{1}{2}$, f_p has a corresponding density given by

$$h_p(x) = \frac{(1-x^2)^{p-\frac{1}{2}}}{B(p+\frac{1}{2}, \frac{1}{2})}, \qquad |x| < 1$$
$$= 0 \qquad\qquad , \qquad |x| \geq 1 . \tag{2.2}$$

For each $p \geq -\frac{1}{2}$, $g_p(t) = \frac{1}{f_p(it)}$, $t \in R$, defines a c.f. Two of the original three examples are included. These are

$$f_{-1/2}(t) = \cos t \qquad g_{-1/2}(t) = \frac{1}{\cosh t}$$

and

$$f_{1/2}(t) = \frac{\sin t}{t} \qquad g_{1/2}(t) = \frac{t}{\sinh t} .$$

Lukacs pointed out that c.f.'s belonging to D are both real valued and even functions. He also obtained results indicating that if X_1 and X_2 are independent r.v.'s whose c.f.'s belong to D, , then the c.f. of the r.v. $Y = X_1 + X_2$ is also an element of D. It is interesting to note that the converse of this statement is not true. For example, consider the independent r.v.'s X_1 and X_2, where X_1 has density

$$h_1(x) = \frac{2}{\sqrt{\pi}} \exp(x - e^{2x}), \quad x \in R$$

and X_2 has density

$$h_2(x) = \frac{2}{\sqrt{\pi}} \exp(-x - e^{-2x}), \quad x \in R .$$

Then X_1 and X_2 have c.f.'s given by

$$f_1(t) = \frac{\Gamma\left(\frac{1+it}{2}\right)}{\sqrt{\pi}}, \qquad t \in R$$

and

$$f_2(t) = \frac{\Gamma\left(\frac{1-it}{2}\right)}{\sqrt{\pi}}, \qquad t \in R .$$

Neither f_1 nor f_2 can belong to D since they are not even functions. However, the r.v. $Y = X_1 + X_2$ has c.f. given by

$$f_1(t)\, f_2(t) = \frac{1}{\cosh \frac{\pi t}{2}}$$

which is an element of the class D.

In order to simplify notation, let H denote the set of pairs $[f, g]$ of elements of D satisfying

$$f(t)\, g(it) = 1 , \tag{2.3}$$

Let c_n $(n = 0, 1, \ldots)$ be a sequence of complex numbers such that

$$\sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

is convergent for some $t \neq 0$. Let (X, Y) be a real random vector having analytic c.f. φ, and define a complex r.v. by $Z = X + iY$. Then the following theorem can be formulated.

Theorem 2.1 $E\,Z^n = c_n$ if and only if

$$\varphi(t, it) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

for values of t for which both sides exist.

Proof For values of t for which the expectation exists

$$E\,e^{itZ} = E\,e^{i(tX + itY)} = \varphi(t, it) \tag{2.4}$$

and

$$E\,e^{itZ} = E \sum_{n=0}^{\infty} \frac{Z^n}{n!} (it)^n = \sum_{n=0}^{\infty} \frac{E\,Z^n}{n!} (it)^n \tag{2.5}$$

and the theorem is proved by the uniqueness of the coefficients of a convergent power series. It should be noted that the last step in (2.5) is valid because of the analyticity of φ and the use of the Lebesgue dominated convergence theorem.

If additionally, X and Y are independent with c.f.'s f and g respectively, the following corollaries result.

Corollary 2.1 $E\,Z^n = c_n$ if and only if

$$f(t)\,g(it) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (it)^n$$

where both sides exist.

Corollary 2.2 $E\,Z^n = 0$ $(n = 1, 2, \ldots)$ if and only if

$$f(t)\,g(it) = 1 .$$

Note that Corollary 2.2 could have been stated in the following way. The pair $[f, g]$ is an element of H, if and only if $E Z^n = 0$ for $n \geq 1$.

Example 2.2 Let (X, Y) be a random vector with joint density $h(x, y)$ which depends only on $x^2 + y^2$. It is known that such a random vector has a c.f.

$$f(u, v) = \int_0^\infty J_0\left(r\sqrt{u^2 + v^2}\right) dG(r) ,$$

where G is the distribution function of a nonnegative r.v. Hence

$$f(u, iu) = \int_0^\infty J_0\left(r\sqrt{u^2 - u^2}\right) dG(r) = 1$$

and it follows from Theorem 2.1 that if f is analytic, then $E(X + iY)^n = 0$ for $n = 1, 2, \ldots$.

The previous results are easily generalized to higher dimensions in the following way. Let $c(n_1, \ldots, n_k)$, $(n_j = 0, 1, \ldots; 1 \leq j \leq k)$ be an infinite multiple sequence of complex numbers such that

$$\sum_{\substack{n_j=0 \\ 1 \leq j \leq k}}^{\infty} \frac{c(n_1, \ldots, n_k)}{n_1!, \ldots, n_k!} (it_1)^{n_1} \cdots (it_k)^{n_k}$$

converges for some $(t_1, \ldots, t_k) \neq (0, \ldots, 0)$. Let $(X_1, \ldots, X_k, Y_1, \ldots, Y_k)$ be a real random vector having analytic c.f. φ. Define the complex random vector $Z = (Z_1, \ldots, Z_k)$ by

$Z_j = X_j + iY_j \quad (1 \le j \le k)$. Then the generalization of Theorem 2.1 is the following,

Theorem 2.2 $E\, Z_1^{n_1} \cdots Z_k^{n_k} = c(n_1, \ldots, n_k)$ if and only if

$$\varphi(t_1, \ldots, t_k, it_1, \ldots, it_k) = \sum_{\substack{n_j=0 \\ 1 \le j \le k}}^{\infty} \frac{c(n_1, \ldots, n_k)}{n_1! \cdots n_k!} (it_1)^{n_1} \cdots (it_k)^{n_k}.$$

for values of $(t_1, \ldots, t_k)$ for which both sides exist.

Proof Let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$. Then for values of $t = (t_1, \ldots, t_k)$ for which the expectation exists

$$E\, e^{itZ'} = e^{i(tX' + itY')} = \varphi(t_1, \ldots, t_k, it_1, \ldots, it_k)$$

and

$$E\, e^{itZ'} = E \prod_{j=1}^{k} e^{it_j Z_j}$$

$$= E \prod_{j=1}^{k} \left(\sum_{v=0}^{\infty} \frac{Z_j^v}{v!} (it_j)^v \right)$$

$$= E \sum_{\substack{n_j=0 \\ 1 \le j \le k}}^{\infty} \frac{Z_1^{n_1} \cdots Z_k^{n_k}}{n_1! \cdots n_k!} (it_1)^{n_1} \cdots (it_k)^{n_k}$$

$$= \sum_{\substack{n_j=0 \\ 1 \le j \le k}}^{\infty} \frac{E\, Z_1^{n_1} \cdots Z_k^{n_k}}{n_1! \cdots n_k!} (it_1)^{n_1} \cdots (it_k)^{n_k}$$

and the theorem is proved by the uniqueness of the coefficients of a convergent power series.

If, additionally, X and Y are independent vectors with c.f.'s f and g respectively, then the following corollaries result.

Corollary 2.3 $E Z_1^{n_1} \cdots Z_k^{n_k} = c(n_1, \ldots, n_k)$ if and only if

$$f(t_1, \ldots, t_k)\, g(it_1, \ldots, it_k) = \sum_{\substack{n_j = 0 \\ 1 \le j \le k}}^{\infty} \frac{c(n_1, \ldots, n_k)}{n_1! \cdots n_k!} (it_1)^{n_1} \cdots (it_k)^{n_k}$$

for values of $(t_1, \ldots, t_k)$ for which both sides exist.

Corollary 2.4 $E Z_1^{n_1} \cdots Z_k^{n_k} = 0$ where $n_j = 0, 1, \ldots$ and $\sum_{j=1}^{k} n_j > 0$, if and only if $f(t_1, \ldots, t_k)\, g(it_1, \ldots, it_k) = 1$.

An extension of the preceeding results to include c.f.'s of r.v.'s taking values in an arbitrary real Banach space $\mathfrak{X}$ is the next logical step. Extending the concept of the class D presents no problem. A c.f. g of an $\mathfrak{X}$ valued r.v. is said to be an element of the class D^* provided $f(x^*) = \dfrac{1}{g(ix^*)}$, $x^* \in \mathfrak{X}^*$ also defines a c.f. of an $\mathfrak{X}$ valued r.v. The following examples show that the class D^* has nontrivial members.

Example 2.3 Let $x_o \in \mathfrak{X}$ and suppose the r.v. X assumes the values x_o and $-x_o$ each with probability one half. Then X has c.f. $g(x^*) = \cos[x^*(x_o)]$, $x^* \in \mathfrak{X}^*$. Also $g \in D^*$ since

$$\frac{1}{g(ix^*)} = \frac{1}{\cosh[x^*(x_o)]}, \quad x^* \in \mathfrak{X}^*.$$

Example 2.4 Let X be an $\mathfrak{X}$ valued r.v. and $E\|X\|^2 < \infty$. Define $\psi: \mathfrak{X}^* \times \mathfrak{X}^* \to R$ by $\Psi(x^*, y^*) = E[x^*(X)\, y^*(X)]$, $(x^*, y^*) \in \mathfrak{X}^* \times \mathfrak{X}^*$. The r.v. X is said to be distributed normally with mean 0 and covariance operator Ψ if

$$g(x^*) = E\, e^{i x^*(X)} = e^{-\frac{1}{2} \Psi(x^*, x^*)} .$$

Notice that

$$\frac{1}{g(i x^*)} = \frac{1}{e^{\frac{1}{2} \Psi(x^*, x^*)}} = e^{-\frac{1}{2} \Psi(x^*, x^*)}, \quad x^* \in \mathfrak{X}^* .$$

Therefore g is an element of D if X is distributed normally with mean 0 and covariance operator Ψ.

The following results are analogous to Corollaries 2.1 and 2.2. Let X and Y be independent $\mathfrak{X}$ valued r.v.'s having analytic c.f.'s f and g. Let c_n be a sequence of linear functionals on $\mathfrak{X}^*$ such that $\sum_{n=0}^{\infty} \frac{c_n(x^*)}{n!}$ converges for $\|x^*\| < \delta$, $\delta > 0$. Let $Z = X + iY$.

Theorem 2.3 $E[x^*(Z)]^n = c_n(x^*)$ for each $x^* \in \mathfrak{X}^*$ if and only if

$$f(x^*)\, g(i x^*) = \sum_{n=0}^{\infty} \frac{c_n(x^*)\, i^n}{n!}$$

for each $x^* \in \mathfrak{X}^*$.

Proof Corollary 2.1 implies that

$$E[x^*(X) + i x^*(Y)]^n = c_n(x^*) ,$$

for each $x^* \in \mathfrak{X}^*$ if and only if

$$f_{x^*}(t)\, g_{x^*}(it) = \sum_{n=0}^{\infty} \frac{c_n(x^*)}{n!} (it)^n .$$

where f_{x^*} and g_{x^*} are the c.f.'s of the real r.v.'s $x^*(X)$ and $x^*(Y)$ respectively. Since the series on the right converges for $t = 1$ and

$$f_{x^*}(1) = f(x^*) \quad \text{and} \quad g_{x^*}(i) = g(ix^*)$$

if follows that

$$E[x^*(Z)]^n = c_n(x^*), \qquad \forall x^* \in \mathfrak{X}^*$$

if and only if

$$f(x^*)\, g(ix^*) = \sum_{n=0}^{\infty} \frac{c_n(x^*)}{n!} i^n, \qquad \forall x^* \in \mathfrak{X}^* .$$

Corollary 2.5 $E[x^*(Z)]^n = 0$ for each $x^* \in \mathfrak{X}^*$, $n \geq 1$ if and only if $f(x^*)\, g(ix^*) = 1$.

Hence Corollary 2.5 characterizes the set of corresponding pairs of elements of D^* in the same way that Corollary 2.2 characterizes the set H.

CHAPTER III

CHARACTERIZATIONS USING CONDITIONAL EXPECTATIONS

Shanbhag [23] characterized the exponential and geometric distributions in terms of conditional expectations. This result was generalized by Hamdan [10] to include a characterization of the uniform and Weibull distributions. A more general result was given by Kotlarski in [14], which contained theorems allowing the characterization of several distributions including the Cauchy distribution. In this chapter the concept of conditional expectation is used to characterize probability measures on arbitrary measurable spaces.

Let $(\Omega, \mathfrak{F})$ be a measurable space and P, P_o two probability measures on $(\Omega, \mathfrak{F})$ such that P is absolutely continuous with respect to P_o, that is

$$P_o(A) = 0 \implies P(A) = 0, \quad A \in \mathfrak{F}. \tag{3.1}$$

Let $\mathfrak{F}_o$ be any subcollection of $\mathfrak{F}$ satisfying

(i) $A \in \mathfrak{F}_o \implies A^c \in \mathfrak{F}_o$, where $A^c = \Omega - A$,

(ii) $A \in \mathfrak{F}_o \implies 0 < P(A) < 1$ and $0 < P_o(A) < 1$,

(iii) There exists a sequence $A_n \in \mathfrak{F}_o$ such that

$$P_o(A_n) \geq 1 - \frac{1}{2^n}.$$

Let $h : \Omega \to R$, be $\mathfrak{F}$ measurable, P_o integrable and denote

$$m_o = E_{P_o}[h] = \int_\Omega h\, dP_o .$$

Theorem 3.1 If $P = P_o$, then

$$E_P[h|A] = E_{P_o}[h|A], \qquad \forall A \in \mathfrak{F}_o . \tag{3.2}$$

Proof Obvious.

Theorem 3.2 For each $A \in \mathfrak{F}_o$ such that

$$E_{P_o}[h|A] \neq m_o \tag{3.3}$$

condition (3.2) implies $P(A) = P_o(A)$.

Proof Since for any $A \in \mathfrak{F}_o$

$$E_P[h|A] = \frac{\int_A h\, dP}{P(A)} \quad \text{and} \quad E_{P_o}[h|A] = \frac{\int_A h\, dP_o}{P_o(A)}$$

then (3.2) can be written as

$$\frac{\int_A h\, dP}{P(A)} = \frac{\int_A h\, dP_o}{P_o(A)}, \qquad \forall A \in \mathfrak{F}_o . \tag{3.4}$$

Let $\{A_n\}$ be the sequence in (iii). Then

$$\sum_{n=1}^{\infty} P_o(A_n^c) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

and by the Borel-Cantelli lemma $P_o(\underline{\lim}\, A_n) = 1$. Hence there exists a set $N \in \mathfrak{F}$ such that $\underline{\lim}\, A_n = \Omega - N$ and $P_o(N) = 0$. Likewise, $P(N) = 0$ because of (3.1). Since

$$\frac{\int_{A_n} h\, dP}{P(A_n)} = \frac{\int_{A_n} n\, dP_o}{P_o(A_n)}, \quad \forall n,$$

then

$$\int_{\Omega} h\, dP = \int_{\Omega} h\, dP_o . \tag{3.5}$$

Now suppose that $A \in \mathfrak{F}_o$ such that (3.3) is satisfied. Then $A \cup A^c = \Omega$ and as a result of (3.5)

$$\int_A h\, dP + \int_{A^c} h\, dP = \int_A h\, dP_o + \int_{A^c} h\, dP_o . \tag{3.6}$$

Using (3.4) and (3.6) yields

$$\frac{P(A)}{P_o(A)} \int_A h\, dP_o + \frac{1 - P(A)}{1 - P_o(A)} \int_{A^c} h\, dP_o = \int_A h\, dP_o + \int_{A^c} h\, dP_o$$

or

$$\left[\frac{P(A)}{P_o(A)} - 1\right] \int_A h\, dP_o + \left[\frac{1 - P(A)}{1 - P_o(A)} - 1\right] \int_{A^c} h\, dP_o = 0 .$$

Therefore

$$\left[P(A) - P_o(A)\right] \left[\frac{\int_A h\, dP_o}{P_o(A)} - \frac{\int_{A^c} h\, dP_o}{1 - P_o(A)}\right] = 0$$

or

$$\left[P(A) - P_o(A)\right]\left[\int_A h\, dP_o - m_o P_o(A)\right] = 0 .$$

Hence

$$\left[P(A) - P_o(A)\right]\left[E_{P_o}[h|A] - m_o\right] = 0$$

and because of (3.3) $P(A) = P_o(A)$. This concludes the proof.

Remark: It should be noted that if $\mathfrak{F}_o$ is a collection with the property that the values which a probability measure assigns to those elements of $\mathfrak{F}_o$ satisfying (3.3) uniquely determine the probability measure on all of $\mathfrak{F}$, then $P = P_o$. The following examples display the selection of such an $\mathfrak{F}_o$.

Example 3.1 Let $\Omega = (0, \infty)$ and $\mathfrak{F}$ be the usual σ-field of Borel subsets. Let P_o be the probability measure associated with the exponential distribution whose density function is given by

$$\begin{aligned} f(x) &= 0 , & x \le 0 \\ &= \frac{1}{a} e^{-x/a} , & x > 0 , \end{aligned}$$

where a is a positive constant. Suppose P is a probability measure defined on $\mathfrak{F}$ such that $P << P_o$. Define $h : \Omega \to R$ by $h(x) = x$. For each $x_o > 0$ let $A_{x_o} = \{x : x > x_o\}$. Define $\mathfrak{F}_o$ to be the collection of all such sets and their complements. Since for $x_o > 0$

$$E_{P_o}[h|A_{x_o}] = x_o + a$$

and

$$E_{P_o}\left[h \mid A^c_{x_o}\right] = a - \frac{x_o e^{-x_o/a}}{1 - e^{-x_o/a}} ,$$

for each $A \in \mathfrak{F}_o$ $E_{P_o}[h|A] \neq E_{P_o}[h]$. Hence using Theorem 3.1, Theorem 3.2 and the fact that the values a probability measure assigns to elements of $\mathfrak{F}_o$ uniquely determine the measure on $\mathfrak{F}$, $P = P_o$ if and only if for all $x_o > 0$

$$E_P\left[h \mid A_{x_o}\right] = x_o + a$$

and

$$E_P\left[h \mid A^c_{x_o}\right] = a - \frac{x_o e^{-x_o/a}}{1 - e^{-x_o/a}} .$$

<u>Example 3.2</u> Let $\Omega = (0,1] \times (0,1]$ and $\mathfrak{F}$ be the usual σ-field of Borel subsets. Let $h : \Omega \to R$ be defined by $h(x,y) = x + y$. Let P_o be the probability measure associated with the uniform distribution on Ω and let P be a probability measure on $\mathfrak{F}$ such that $P \ll P_o$. For each $(x_o, y_o) \in \Omega$, $(x_o, y_o) \neq (1,1)$, define the set $A_{x_o, y_o} = \{(x,y) : 0 < x \leq x_o, \ 0 < y \leq y_o\}$. Let $\mathfrak{F}_o$ be the collection of all such sets and their complements. Since for $(x_o, y_o) \in \Omega$

$$E_{P_o}\left[h \mid A_{x_o, y_o}\right] = \frac{x_o + y_o}{2}$$

and

$$E_{P_o}\left[h \mid A^c_{x_o, y_o}\right] = \frac{1 - \frac{1}{2}(x_o y_o^2 + y_o x_o^2)}{1 - x_o y_o} ,$$

for $A \in \mathfrak{F}_o$ $E_{P_o}[h|A] \neq E_{P_o}[h]$. Hence by Theorem 3.1, Theorem 3.2 and the fact that the values a probability measure assigns to elements of $\mathfrak{F}_o$ uniquely determine the measure on all of $\mathfrak{F}$ it follows that

$$P = P_o \quad \text{if and only if for all } (x_o, y_o) \in \Omega$$

$$E_P\left[h|A_{x_o, y_o}\right] = \frac{x_o + y_o}{2}$$

and

$$E_P\left[h|A^c_{x_o, y_o}\right] = \frac{1 - \frac{1}{2}(x_o y_o^2 + y_o x_o^2)}{1 - x_o y_o}$$

Similar results which do not require the integrability of the function h can be obtained by changing the requirements imposed on the set $\mathfrak{F}_o$. Suppose that $(\Omega, \mathfrak{F})$ is a measurable space on which two probability measures P and P_o are defined. Let $h: \Omega \to R$ be $\mathfrak{F}$ measurable. Let $\mathfrak{F}_o$ be any subcollection of $\mathfrak{F}$ which satisfies the condition

$$A \in \mathfrak{F}_o \Rightarrow P(A) > 0, \ P_o(A) > 0, \ \left|\int_A h \, dP\right| < \infty, \ \left|\int_A h \, dP_o\right| < \infty.$$

Let $A_o \in \mathfrak{F}_o$ be a fixed set such that

$$P(A_o) < 1 \quad \text{and} \quad P_o(A_o) < 1$$

and denote

$$K = \frac{P(A_o)}{P_o(A_o)} . \tag{3.7}$$

Theorem 3.3 If $P = P_o$, then

$$E_P[h|A] = E_{P_o}[h|A], \quad \forall A \in \mathfrak{J}.$$

Proof Obvious.

Theorem 3.4 For each $A \in \mathfrak{J}_o$ such that $A \cap A_o = \emptyset$ and

$$E_{P_o}[h|A] \neq E_{P_o}[h|A_o] \tag{3.8}$$

the condition

$$E_P[h|B] = E_{P_o}[h|B] \quad \text{for} \quad B = A,\ A_o,\ A \cup A_o \tag{3.9}$$

implies $P(A) = KP_o(A)$, where K is given by (3.7).

Proof Let $A \in \mathfrak{J}_o$ such that $A \cap A_o = \emptyset$ and (3.8) is satisfied. Condition (3.9) can be written

$$\frac{\int_B h\,dP}{P(B)} = \frac{\int_B h\,dP_o}{P_o(B)}, \quad B = A,\ A_o,\ A \cup A_o.$$

Then

$$\frac{\int_A h\,dP + \int_{A_o} h\,dP}{P(A) + P(A_o)} = \frac{\int_A h\,dP_o + \int_{A_o} h\,dP_o}{P_o(A) + P_o(A_o)}$$

or

$$\frac{\frac{P(A)}{P_o(A)} \int_A h \, dP_o + \frac{P(A_o)}{P_o(A_o)} \int_{A_o} h \, dP_o}{P(A) + P(A_o)} = \frac{\int_A h \, dP_o + \int_{A_o} h \, dP_o}{P_o(A) + P_o(A_o)} .$$

Therefore

$$\frac{P(A) E_{P_o}[h|A] + P(A_o) E_{P_o}[h|A]}{P(A) + P(A_o)} = \frac{P_o(A) E_{P_o}[h|A] + P_o(A_o) E_{P_o}[h|A_o]}{P_o(A) + P_o(A_o)}$$

or

$$\left(E_{P_o}[h|A] - E_{P_o}[h|A_o] \right) \left(\frac{P(A)}{P_o(A)} - \frac{P(A_o)}{P_o(A_o)} \right) = 0 .$$

From condition (3.8) it follows that

$$\frac{P(A)}{P_o(A)} = \frac{P(A_o)}{P_o(A_o)}$$

or using (3.7) yields

$$P(A) = K \, P_o(A)$$

which concludes the proof.

The previous theorems can be used for characterizations provided it can be shown that $K = 1$ and $\mathfrak{F}_o$ has the property that the values of a probability measure on elements of $\mathfrak{F}_o$ uniquely determine the measure on all of $\mathfrak{F}$. The following lemmas can be useful in applying Theorem 3.4 to characterize a probability measure.

Lemma 3.1 If the sequence $\{A_n\}$ is a sequence of sets such that $\lim_{n\to\infty} A_n = \Omega - A_o$, $A_n \cap A_o = \emptyset$ and $P(A_n) = K P_o(A_n)$ for each n, then $K = 1$.

Proof Since $A_n \cap A_o = \emptyset$ for $n \geq 1$ and $P(A_n) = K P_o(A_n)$ for all n, it follows that $P(A_n \cup A_o) = K P_o(A_n \cup A_o)$. Define $B_n = A_n \cup A_o$, $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} B_n = \Omega$. Hence $P(B_n) = K P_o(B_n)$ and taking the limit of both sides as $n \to \infty$ yields $K = 1$.

Lemma 3.2 If $A \in \mathfrak{F}_o$ such that $A \cap A_o = \emptyset$ and $E_{P_o}[h|A] = E_{P_o}[h|A_o]$, then $E_{P_o}[h|A \cup A_o] = E_{P_o}[h|A_o]$.

Proof Suppose $A \in \mathfrak{F}_o$, $A \cap A_o = \emptyset$ and $E_{P_o}[h|A] = E_{P_o}[h|A_o]$. Then

$$\frac{\int_A h\, dP_o}{P_o(A)} = \frac{\int_{A_o} h\, dP_o}{P_o(A_o)}$$

which implies

$$\frac{\int_A h\, dP_o + \int_{A_o} h\, dP_o}{P_o(A) + P_o(A_o)} = \frac{\int_{A_o} h\, dP_o}{P_o(A_o)} .$$

Therefore

$$\frac{\int_{A \cup A_o} h\, dP_o}{P_o(A \cup A_o)} = \frac{\int_{A_o} h\, dP}{P_o(A_o)}$$

which concludes the proof.

The following example shows how Theorem 3.3 and Theorem 3.4 can be used to characterize probability measures.

Example 3.3 Suppose $\Omega = (1, \infty)$ and $\mathfrak{F}$ is the usual σ-field of Borel sets. Let P_o be the probability measure associated with the density function

$$f(x) = 0 \quad , \quad x \leq 1$$
$$= \frac{1}{x^2} \quad , \quad x > 1 \; .$$

Define the collection $\mathfrak{F}_o$ by

$$\mathfrak{F}_o = \{(a, b] : 1 \leq a < b < \infty\} \; .$$

Let P be a probability measure on $\mathfrak{F}$ such that for all $A \in \mathfrak{F}_o$, $P(A) > 0$. Define $h : \Omega \to R$ by $h(x) = x$. Note that for $1 \leq a < b < \infty$ and $A = (a, b]$

$$E_{P_o}[h|A] = \frac{\log \frac{b}{a}}{\frac{1}{a} - \frac{1}{b}} \; .$$

Let $\delta > 1$, $A_o = (1, \delta]$ and $K = \frac{P(A_o)}{P_o(A_o)}$. Define the collection

$$\mathfrak{F}_\delta = \{(\delta, b] : \delta < b < \infty\} \; .$$

Then for $A = (\delta, b] \in \mathfrak{F}_o$

$$E_{P_o}[h|A] \neq E_{P_o}[h|A_o] \; .$$

That (3.10) holds can be seen by observing that if this is not the case Lemma 3.2 implies

$$E_{P_o}[h|A \cup A_o] = E_{P_o}[h|A_o]$$

or

$$\frac{\log b}{1 - \frac{1}{b}} = \frac{\log \delta}{1 - \frac{1}{\delta}} . \qquad (3.11)$$

However, (3.11) cannot hold since the function

$$g(x) = \frac{\log x}{1 - \frac{1}{x}}$$

is strictly increasing on $(1, \infty)$ as is shown by

$$g'(x) = \frac{x - 1 - \log x}{(x - 1)^2} \quad \text{for} \quad 1 < x < \infty .$$

Note that if for all $(a, b] \in \mathfrak{J}_o$

$$E_P[h|(a, b]] = E_{P_o}[h|(a, b]] ,$$

then

$$E_P[h|A_o] = E_{P_o}[h|A_o]$$

$$E_P[h|A] = E_{P_o}[h|A] \qquad , \quad \forall A \in \mathfrak{J}_o$$

and

$$E_P[h|A \cup A_o] = E_{P_o}[h|A \cup A_o] , \quad \forall A \in \mathfrak{J}_o .$$

Then by Theorem 3.4

$$P(A) = K\,P_o(A)\,, \qquad A \in \mathfrak{F}_\delta$$

if for $1 \leq a < b < \infty$

$$E_P[h|(a,b]] = \frac{\log \frac{b}{a}}{\frac{1}{a} - \frac{1}{b}}\,. \tag{3.12}$$

Applying Lemma 3.1 yields $K = 1$ and (3.12) implies

$$P(\delta, b] = P_o(\delta, b], \qquad b > \delta\,.$$

Since δ is arbitrary (3.12) implies

$$P(a, b] = P_o(a, b], \qquad \text{for} \quad 1 < a < b < \infty\,.$$

Therefore $P = P_o$ if and only if (3.12) holds.

The conclusion of Theorem 3.4 remains valid if the requirement $A \cap A_o = \emptyset$ is replaced by some additional assumptions. The following corollaries display the results.

Corollary 3.1 For each $A \in \mathfrak{F}_o$ such that

(i) $A_o \subset A$

(ii) $A - A_o \in \mathfrak{F}_o$ and

(iii) $E_{P_o}[h|A - A_o] \neq E_{P_o}[h|A_o]$

the condition

$$E_P[h|B] = E_{P_o}[h|B], \quad B = A, A_o, A - A_o,$$

implies $P(A) = K P(A)$.

<u>Proof</u> Theorem 3.4 implies that $P(A - A_o) = K P_o(A - A_o)$. Hence

$$\begin{aligned} P(A) &= P(A_o) + P(A - A_o) \\ &= K P_o(A_o) + K P_o(A - A_o) \\ &= K P_o(A), \end{aligned}$$

<u>Corollary 3.2</u> For each $A \in \mathfrak{J}_o$ such that

(i) $A \subset A_o$

(ii) $A_o - A \in \mathfrak{J}_o$ and

(iii) $E_{P_o}[h|A_o - A] \neq E_{P_o}[h|A]$

the condition

$$E_P[h|B] = E_{P_o}[h|B], \quad B = A, A_o, A_o - A,$$

implies $P(A) = K P_o(A)$.

<u>Proof</u> Theorem 3.4 implies that

$$\frac{P(A)}{P_o(A)} = \frac{P(A_o - A)}{P_o(A_o - A)}$$

Hence

$$\frac{P(A)}{P_o(A)} = \frac{P(A) + P(A_o - A)}{P_o(A) + P_o(A_o - A)} = \frac{P(A_o)}{P_o(A_o)}$$

or

$$P(A) = K\, P_o(A) .$$

Corollary 3.3 For each $A \in \mathfrak{F}_o$ such that

(i) $A \cap A_o,\ A - A_o,\ A_o - A \in \mathfrak{F}_o$

(ii) $E_{P_o}[h|A \cap A_o] \neq E_{P_o}[h|A_o - A]$ and

(iii) $E_{P_o}[h|A - A_o] \neq E_{P_o}[h|A_o]$

the condition

$$E_P[h|B] = E_{P_o}[h|B], \quad B = A_o,\ A - A_o,\ A \cap A_o,\ A_o - A ,$$

implies

$$P(A) = K\, P_o(A) .$$

Proof By Corollary 3.2

$$P(A \cap A_o) = K\, P_o(A \cap A_o) .$$

By Theorem 3.4

$$P(A - A_o) = K\, P_o(A - A_o) .$$

Therefore

$$\begin{aligned} P(A) &= P(A \cap A_o) + P(A - A_o) \\ &= K\, P_o(A \cap A) + K\, P_o(A - A_o) \\ &= K\, P_o(A) . \end{aligned}$$

CHAPTER IV

SUMMARY AND CONCLUSIONS

This paper is devoted to a development of the concept of an analytic characteristic functional and some characterization problems in probability. The concept of analytic functionals defined on abstract spaces has as its foundation the work done by Frechet [6] in developing the generalized or abstract polynomial. The work of Bochnak [1], Bochnak and Siciak [2], Ligoca and Siciak [15] and Hille [11] provide the basis for considering the effects of the property of analyticity on characteristic functionals of abstract valued random variables.

In Chapter I it is shown that well known properties of analytic characteristic functions of real valued random variables are possessed by analytic characteristic functionals of abstract random variables. If X is a random variable taking values in the real Banach space $\mathfrak{X}$ and having analytic characteristic functional f, then the following results are given. The values of f in a neighborhood of the origin uniquely determine f. The function f is analytic in an open subset of the complexification of $\mathfrak{X}^*$ which contains all of $\mathfrak{X}^*$ and f has the integral representation

$$f(x_1^* + i x_2^*) = \int_\Omega e^{i(x_1^* + i x_2^*)(X(\omega))} \mu(d\omega)$$

in the "strip" $\|x_2^*\| < \delta$ for some $\delta > 0$. Also included is a characterization problem showing that the distribution of the independent $\mathfrak{X}$ valued random variables X_1, X_2, X_3 having analytic characteristic functionals are determined up to a shift by the distribution of $(X_1 + X_3, X_2 + X_3)$.

In Chapter II the problem of D. van Dantzig concerning the set $D = \{f: f \text{ is an analytic c.f. and } \frac{1}{f(it)} \text{ is a c.f.}\}$ is considered. Many examples of elements of D are generated and the following characterization of D is given. The pair of characteristic functions $[f, g]$ is a corresponding pair of elements of D if and only if for independent random variables X and Y having characteristic functions f and g, $E(X + iY)^n = 0$ for $n = 1, 2, \ldots$. This result is also generalized to include abstract valued random variables.

In Chapter III four theorems are given to allow characterization of probability measures defined on abstract measurable spaces by using the conditional expectations of a real valued function defined on the sample space. Examples given illustrate the use of these theorems to characterize the exponential distribution and the two dimensional uniform distribution. Theorems 3.3 and 3.4 are used to characterize distributions of random variables which do not possess expected values.

The wealth of information about analytic characteristic functionals of real random variables provides many unanswered questions related to Chapter I. It should be possible to extend many of the theorems in Lukacs [16] and Ramachandran [22] to characteristic functionals of abstract valued random variables. A possibility of sharpening the results of Chapter III depends upon constructing proofs of Theorem 3.2 and Theorem 3.4 with less restrictions placed on the set $\mathfrak{J}_o$.

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