NON-STANDARD TOPOLOGY

By

JAMES WESLEY HALL

Bachelor of Science in Education
Eastern Illinois University
Charleston, Illinois
1968

Master of Arts
Eastern Illinois University
Charleston, Illinois
1969

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
July, 1973

NON-STANDARD TOPOLOGY

Thesis Approved:

Forrest D. Whitfield
Thesis Adviser

Vernon Troxel

E. K. McLachlan

Jeanne Agnew

D. D. Durham
Dean of the Graduate College

ACKNOWLEDGMENTS

I wish to thank Dr. Forrest Whitfield for serving as my thesis adviser and for the extra effort he gave so that this paper could be written on "non-standard" material. The other members of my committee, Professor Jeanne Agnew, Professor E.K. McLachlan, and Professor Vernon Troxel have also been most cooperative. I do appreciate your cooperation and assistance. To those who said that the outline of this thesis was interesting, I wish to note the encouragement that this gave me while writing the thesis.

It has been my good fortune to have throughout my education many excellent teachers. To all of you who were more interested in teaching me than in teaching mathematics, I wish to acknowledge the role that you played in helping me reach this goal. In particular, the names Cox, DiPietro, Laible, and Steffensen deserve special recognition.

To my family, I owe the greatest gratitude. My parents have given generous support for many years. This degree would not have been possible without your support and encourgement. Peggy, my wife, not only lightened my burden for the last three years, she also typed this thesis. May she and Manette and Mason fully enjoy the benefits that the completion of this degree may bring.

Lastly, I wish to comment upon the many friends acquired among the other graduate students. Our moments of levity helped make the acquisition of this degree an enjoyable process. I hope our paths cross many times in the future.

TABLE OF CONTENTS

Chapter		Page
	Non-standard Characterization of $\bar{A}$	66
	Near (*Sets to Sets)	68
	Non-standard Characterization of A^{o}	71
	Non-standard Characterization of b(A)	73
	Non-standard Characterization of Dense Sets	75
VI.	CONTINUOUS FUNCTIONS AND HOMEOMORPHISMS	78
	Non-standard Characterization of Continuous Functions	80
	Non-standard Characterization of Convergent Sequences of Reals	84
	Non-standard Characterization of Open Functions . . .	89
	Non-standard Characterization of Homeomorphisms . . .	91
VII.	CONNECTED SPACES	93
	Non-standard Characterization of Separated Sets . . .	94
	Non-standard Characterization of Connected Spaces . .	96
VIII.	THE SEPARATION AXIOMS	102
	Non-standard Characterization of T_0-spaces	103
	Non-standard Characterization of T_1-spaces	104
	Non-standard Characterization of T_2-spaces	107
	Non-standard Characterization of R-spaces	110
	Non-standard Characterization of N-spaces	111
IX.	COMPACT SPACES	116
	Non-standard Characterization of Compact Sets	117
	Non-standard Characterization of Locally Compact Sets	124
X.	PRODUCT SPACES	127
	Product Topology (Tichonov Topology)	128
	Nearness to Points in a Product Space	130
	Tichonov's Theorem	135
XI.	SUMMARY AND CONCLUSIONS	136
BIBLIOGRAPHY	. . . .	138
INDEX	. . . .	140

CHAPTER I

INTRODUCTION

The purpose of this paper is to use the tools of non-standard analysis to develop some of the material found in many introductory topology texts. Every attempt has been made to keep the exposition elementary. The student who is prepared for a normal introductory topology course should also have a sufficient background for this material. All topological and non-standard concepts which are used are defined within the paper. This should minimize any need for an outside reference, and it should also eliminate any questions concerning notation or use of a particular term which might be defined differently in various texts.

Chapter II introduces the concepts from non-standard analysis which are used in the remainder of the paper. The non-standard real numbers, which are introduced in Chapter III, provide the reader with some specific examples of the concepts discussed in the previous chapter. These examples are also referenced when new material is later presented.

Chapter IV introduces topological spaces and initiates a study of these spaces. As a part of this development, the concept of nearness is rigorously defined. Neighborhoods and open sets are then characterized using this definition of nearness.

In Chapter V, a non-standard characterization of closed sets is given. This characterization is the same as that given by Abraham Robinson [14]; however, it is derived somewhat differently. As part of this derivation two new generalized definitions of nearness are made. One of the important aspects of non-standard analysis is its appeal to intuition. Throughout the remainder of the paper, these generalized definitions are used to describe as intuitively as possible many standard concepts. Particular topics that are developed include continuity, connectedness, the separation axioms, compactness, and product spaces.

The term "non-standard analysis" was coined by Abraham Robinson in 1960. His objective on this date was to provide a logically consistent system for calculus using infinitely small and infinitely large numbers. The language of infinitesimals had been used earlier by Leibniz, who stated that the same success could be obtained with infinite numbers as with the finite. Needless to say, Leibniz failed to provide the basis for the success he thought possible. It was not until Robinson decided to use other models for the real numbers that this success was achieved.

This paper avoids Robinson's use of type theory in constructing a non-standard model by using the framework presented by Moshe Machover and Joram Hirschfeld [11]. This simplified framework allows presentation of the material in a manner that conforms closely to the usual usage of set theoretic terminology. As has been indicated earlier, one of the interesting things this allows is a rigorous definition of the intuitive concept of nearness. It is quite natural for one to seek a better intuitive feeling for nearness in an arbitrary topological space. Many mathematicians, either when explaining proofs to others or when seeking proofs for themselves, often use the intuitive idea of a small ball or sphere about a point when examining a neighborhood, regardless of the actual topology involved. What non-standard analysis does in an arbitrary topological space is to allow one to use directly his intuition for nearness by making the term precise, in the same way that the term "infinitesimal" was first made precise in the space of real numbers by Robinson.

A theorem that exemplifies how well this concept of nearness is used in the non-standard characterization of topological concepts is the following:

> A function f from a topological space $(X, \mathfrak{J})$ to a topological space $(X', \mathfrak{J}')$ is continuous at $p \in X$ iff $f(q)$ is near $f(p)$ whenever q is near p.

Although the remainder of this paper will avoid type theory, it is instructive to examine this theorem in terms of types. (Points are called objects of type zero, sets of points are objects of type one, and families of sets of points are objects of type two, etc.) Note that the previous non-standard characterization of continuity involves only objects of type zero, whereas the usual definition involves sets of points, i.e. objects of type 1. As further evidence that non-standard characterizations are often simpler, examine the following non-standard characterization of compactness:

> A set K in a topological space $(X, \mathfrak{J})$ is compact iff every *point of K is near some point of K.

This is also a statement involving only objects of type zero, as contrasted to the usual definition which involves type two objects, i.e. families of sets of points.

Unfortunately, perhaps, the most difficult material will have to be presented first. The concept of a non-standard model and the associated terminology must either be given or referenced. The choice has been to include in Chapter II that which is pertinent to the development of the material in later chapters. The technique used will be that of Machover and Hirschfeld. Although the proofs included here are in more detail than those by Machover and Hirschfeld, the reader is advised that many of their interesting points about non-standard models have not been included, due to the limited purpose of this paper. For the most part it is the terminology and the results of the theorems of Chapter II which will be used later. The reader may well want to skim this chapter and then pursue the remaining chapters in depth before delving into the proofs of this background material. If this is done, the remaining chapters should still be easily comprehensible.

CHAPTER II

NON-STANDARD MODELS

The language of infinitesimals was used even before the time of Leibniz. Due to the lack of a rigorous foundation for such terminology, this language was later abandoned by most mathematicians in favor of Weierstrass's epsilon-delta notation. It may seem surprising that terminology dormant since the nineteenth century has been revived and given a rigorous foundation in 1960. The key to the breakthrough was the newly developing field of mathematical, or symbolic, logic. The notion of a "formal language" enabled Abraham Robinson to make precise the earlier vague claims that had been made concerning infinitesimals.

The concept of a formal language is somewhat difficult to become adjusted to, particularly since the same language will be used to describe different universes of objects. In order to clarify some of this, a couple of analogies will now be made.

The key to working with a non-standard model is to conceive of the system that you are working with as embedded in a somewhat larger system. The role of a non-standard model is analogous to that of an extension field in algebra. To prove results about a system, one might consider the original system as embedded in a larger framework, do most of the work in this framework; and then try to reinterpret these results in the original system.

The primary capability that one must have to use this powerful proof technique is to be able to write precisely a formal sentence expressing the idea under consideration. It will be seen shortly that this sentence is true in the given system if and only if it is true in the enlargement. Anyone who has ever programmed a computer is somewhat aware of the problems encountered when one expresses ideas using a fixed set of symbols according to specified rules. Mathematicians are not normally so limited and use, in addition to their formal language, the even more complex and less formally understood language of society. Thus it is difficult to express precisely and symbolically all ideas which can be stated in an informal language. In fact, not all properties of the real numbers are expressible formally. Rather than being a weakness this is the very feature which makes non-standard models so useful. Any property formally expressible will be formally shared by both the original system and the non-standard model. For example, both the reals and the non-standard reals are fields. However, the original system and the non-standard model may well differ on properties which are not formally expressible. For example, the Archimedean property, which is not formally expressible, holds for the standard real numbers but does not hold for the non-standard real numbers.

The following example may be instructive when one later considers interpreting a formal sentence in more than one system. Consider the statement: $\forall x\ [x \neq 0 \rightarrow \exists y\ (xy = 1)]$. To determine the truth value of this statement, one must assume a particular number system and the operation of multiplication upon that system. Interpreted in the system of integers it is false, while it is true in both the rational and the real number systems. In which number systems is the following statement true: $\forall x\ (x \leq x^2)$?

It is now time to answer some of the questions that the previous discussion has hopefully led you to formulate. First of all, how are ideas expressed precisely and formally?

If you are working with a given system, such as a topological space $(X, \mathcal{J})$, then a universe of discourse will be assumed. Before indicating a construction of this universe, note that the most important feature is that it will be quite large enough to contain as objects all concepts commonly studied in mathematics. In particular, this universe will contain all points of the set X above, and it will be closed under finite unions, finite cartesian products, and the power set operator.

All concepts that are commonly used are considered as sets in this universe. For example, if X is the set of real numbers, then the concept of addition (+) is the object represented by a certain set of ordered pairs whose first term is itself an ordered pair of real numbers (e.g. $((3, 5), 8) \in (+)$). Customary abbreviations will be followed in this paper. For example, $((3, 5), 8) \in (+)$ will be abbreviated as $3 + 5 = 8$.

Since the universe of discourse for the reals is closed under finite cartesian products, it will contain all n-ary relations and in particular all real functions of n-1 real variables as objects.

The universe of discourse U is converted into a mathematical system by specifying one binary relation and two binary operations over U.

The binary relation is $\in$, the membership relation on U. The first binary operation is pr, forming an ordered pair (i.e. a pr b = (a, b)). The second is ap, applying a function to its argument. In this paper f ap x will be customarily abbreviated f(x) [e.g. cos ap π = cos(π) and sin ap{+ap(π pr x)} = sin (π + x)]. So that U will be closed under ap as well as under pr, f ap x will be defined to be $\emptyset$ when f is not a function or when f is a function otherwise undefined for x.

As indicated earlier, the existence of the universe of discourse U is actually more important to this paper than its actual construction; nonetheless, for the sake of completeness the construction is outlined below.

If a topological space is to be considered or, as will be the case later, if a collection of spaces is under consideration, then let V be the set of all points in these spaces.

2.1 Definition: The union set of A, $\cup A$ = {a : a $\in$ b for some b $\in$ A}.

Note that the union set of A may be considered as an extension of the usual set theoretic union. If A is merely a collection of sets, then $\cup A$ is just the usual union of all sets in A. The union set of A is defined even if A is not a set or if A has some members which are not sets. When A is not a set or when it is a set with no non-empty set as a member, then $\cup A = \emptyset$.

2.2 Definition: A set S is transitive if a $\in$ b and b $\in$ S imply a $\in$ S.

In order to extend V to a set U_o which is transitive, let $\underline{V_o} = V$ and $\underline{V_k} = \cup V_{k-1}$ for each natural number k. Then let $\underline{U_o} = \cup\{V_k : k = 0, 1, 2, \ldots\}$.

2.3 Lemma: U_o is a transitive set such that $V \subset U_o$.

Proof: Clearly $V \subset U_o$ by the definition of U_o. Now assume $a \in b$ and $b \in U_o$. Since $b \in U_o$, $b \in V_j$ for some natural number j. Note from the definition of V_{j+1} that $V_{j+1} = \{a : a \in b$ for some $b \in V_j\}$. It then follows that $a \in V_{j+1}$. Since $V_{j+1} \subset U_o$, $a \in U_o$. Hence U_o is transitive. □

The power set of A, $\{a : a \subset A\}$, will be denoted by $P(A)$. If A is not a set, then $P(A) = \emptyset$.

Now let $\underline{U_k} = U_{k-1} \cup P(U_{k-1})$ for each natural number k. Finally let the universe of discourse U be defined by $U = \cup\{U_k : k = 0, 1, 2, \ldots\}$.

2.4 Lemma: Each U_k, $k = 0, 1, 2, \cdots$ is transitive.

Proof: (by induction on k) First recall that U_o has already been shown to be transitive. Secondly, assume that U_k is transitive, that $b \in U_{k+1}$, and that $a \in b$. By definition of U_{k+1}, it follows that either $b \in U_k$ or $b \in P(U_k)$. If $b \in U_k$, then the transitive property of U_k implies that $a \in U_k$. If on the other hand $b \in P(U_k)$, then $b \subset U_k$. In this case, $a \in b$ implies that $a \in U_k$. In either case $a \in U_k$, which is a subset of U_{k+1}. Therefore $a \in U_{k+1}$, which means U_{k+1} is transitive. □

<u>2.5 Theorem:</u> The universe of discourse U is transitive.

Proof: Assume that $a \in b$ and that $b \in U$. From the definition of U, it follows that $b \in U_j$ for some j. Since each U_j is transitive, $a \in b$ and $b \in U_j$ imply that $a \in U_j$. U_j is a subset of U, thus $a \in U$ and U is therefore transitive. □

<u>2.6 Lemma:</u> If S is a transitive set and the set b is an element of S, then b is a subset of S.

Proof: Let the set b be an element of the transitive set S. If $b = \emptyset$, then clearly $b \subset S$. If $b \neq \emptyset$, then let a be an arbitrary element of b. Since $a \in b$, $b \in S$, and S is transitive, it follows that $a \in S$. Hence $b \subset S$. □

<u>2.7 Theorem:</u> The universe of discourse U is closed under $\mathcal{P}$, the power set operator.

Proof: If $a \in U$ then $a \in U_j$ for some j. If a is a set, then the previous lemma and the transitivity of U_j imply that $a \subset U_j$. Hence $\mathcal{P}(a) \subset \mathcal{P}(U_j)$. If a is not a set, then $\mathcal{P}(a) = \emptyset$ and again $\mathcal{P}(a) \subset \mathcal{P}(U_j)$. Since $\mathcal{P}(U_j) \subset U_{j+1}$, $\mathcal{P}(a) \subset \mathcal{P}(U_j)$ implies $\mathcal{P}(a) \subset U_{j+1}$. Thus $\mathcal{P}(a) \in \mathcal{P}(U_{j+1})$. Since $\mathcal{P}(U_{j+1}) \subset U_{j+2}$ and $U_{j+2} \subset U$, it follows that $\mathcal{P}(a) \in U$. Therefore U is closed under $\mathcal{P}$. □

<u>2.8 Lemma:</u> If $b \in U$, the universe of discourse, and $a \subset b$, then $a \in U$.

Proof: If $a \subset b$ then $a \in \mathcal{P}(b)$. Since $b \in U$ and U is closed under $\mathcal{P}$, $\mathcal{P}(b) \in U$. Thus $a \in U$ since $a \in \mathcal{P}(b)$, $\mathcal{P}(b) \in U$, and U is transitive. □

<u>2.9 Theorem</u>: The universe of discourse U is closed under $\cup$, the union set operator.

Proof: Assume that $a \in U$ and that $c \in \cup(a)$. Since $a \in U$, $a \in U_j$ for some j. From the definition of $\cup$, it follows that $c \in \cup(a)$ implies that $c \in b$ for some $b \in a$. Since U_j is transitive, $b \in a$, and $a \in U_j$, it follows that $b \in U_j$. Then $c \in b$, $b \in U_j$, and U_j transitive imply that $c \in U_j$. Hence $\cup(a) \subset U_j$ and therefore $\cup(a) \in \mathcal{P}(U_j)$. Since $\mathcal{P}(U_j) \subset U_{j+1}$ and $U_{j+1} \subset U$, it follows that $\cup(a) \in U$. Therefore U is closed under the union set operator. □

<u>2.10 Lemma</u>: If $a \subset U$ and $a \subset U_j$ for some j, then $a \in U$.

Proof: If $a \subset U_j$, then $a \in \mathcal{P}(U_j)$. Thus $a \in U$ since $\mathcal{P}(U_j) \subset U_{j+1}$ and $U_{j+1} \subset U$. □

<u>2.11 Corollary</u>: $U_k \in U$ for each non-negative integer k.

Proof: $U_k \subset U$, $U_k \subset U_k$, and the previous lemma yield this immediately. □

<u>2.12 Lemma</u>: If a is a finite subset of U, then $a \in U$.

Proof: For $a \neq \emptyset$, let $a = \{c_1, c_2, \cdots, c_n\}$ be a subset of U for some natural number n. Since $a \subset U$ each $c_j \in U$. Thus for each j a k may be selected such that $c_j \in U_k$. Let m be the maximum value of the k's as j varies from 1 to n. Since $U_0 \subset U_1 \subset U_2 \cdots$, each $c_j \in U_m$. For $a = \emptyset$ let $m = 0$. In either case $a \subset U_m$ and so by the preceding lemma it may be concluded that $a \in U$. □

<u>2.13 Theorem</u>: The universe of discourse U is closed under finite unions (i.e. usual set theoretic unions of sets).

Proof: If the sets $a_1, a_2, \cdots, a_n$ are elements of U, where $n > 0$, then $\{a_1, a_2, \cdots, a_n\}$ is a finite subset of U. Therefore the preceding lemma guarantees that $\{a_1, a_2, \cdots, a_n\} \in U$. As previously indicated the union set $\cup\{a_1, a_2, \cdots, a_n\} = (a_1 \cup a_2 \cup \cdots \cup a_n)$. Since U is closed under $\cup$, $(a_1 \cup a_2 \cdots \cup a_n) \in U$. Thus U is closed under finite unions. □

For a, $b \in U$, the ordered pair (a, b) is identified with the set $\{\{a\}, \{a, b\}\}$. Thus for A, $B \in U$, $A \times B = \{\{\{a\}, \{a, b\}\} : a \in A, b \in B\}$. By considering $(a) = a$, the ordered n-tuple for $n \geq 2$ can be defined recursively as $(a_1, a_2, \cdots, a_n) = ((a_1, a_2, \cdots, a_{n-1}), a_n)$.

<u>2.14 Theorem</u>: The universe of discourse U is closed under finite cartesian products.

Proof: (by induction on k) Assume that A, $B \in U$. If either A or B have no elements, then $A \times B = \emptyset$ and trivially $A \times B \in U$. So suppose both A and B contain some elements. Note then that $a \in A$ and $b \in B$ imply that $\{a\}, \{a, b\} \in \mathcal{P}(A \cup B)$. Thus $\{\{a\}, \{a, b\}\} \in \mathcal{P}\,\mathcal{P}(A \cup B)$. Therefore $A \times B \in \mathcal{P}\,\mathcal{P}\,\mathcal{P}(A \cup B)$. Since U is closed under $\mathcal{P}$, $\mathcal{P}\,\mathcal{P}\,\mathcal{P}(A \cup B) \in U$. Hence $A \times B \in \mathcal{P}\,\mathcal{P}\,\mathcal{P}(A \cup B)$, $\mathcal{P}\,\mathcal{P}\,\mathcal{P}(A \cup B) \in U$, and U transitive imply that $A \times B \in U$. For $n \geq 2$, let

$$\prod_{k=1}^{n} A_k = \left(\prod_{k=1}^{n-1} A_k \right) \times A_n.$$

The proof follows without difficulty. □

The construction of the universe is now complete. For emphasis note that if the construction of U is based upon a set X, then U contains all families of subsets of X and so will contain all topologies (which are certain families of subsets of X) on X. The next task is to describe how to express formally a mathematical concept about the objects in the universe of discourse.

The formal language $\mathcal{L}$ will be very similar to the usual set theoretic language with the restriction that it will be more precise and systematic, and it will use some new symbols to replace verbal expressions (e.g. ap).

The language $\mathcal{L}$ is required to have at least one symbol to denote each object in the universe of discourse. These symbols have a fixed meaning and are called constants. The distinction being made between an object in U and the symbol in $\mathcal{L}$ representing that object should not be foreign since this is similar to the distinction made in some texts between the number "two" and the numeral "2" representing that number.

All objects that have well-known symbols representing them, such as "$\emptyset$" for the empty set, "<" for the less than relation, "+" for addition, etc., will have the same symbols used to name them in $\mathcal{L}$. When symbols from $\mathcal{L}$ are used as variables, the range of the variables is all of U. This is the expected thing, but it becomes extremely important to be cognizant of this when more than one universe of discourse is being considered.

In addition to the symbols for the objects in U and other symbols which are used as variables with range U, $\mathcal{L}$ has the symbols "$\in$", "pr", and "ap" to denote the binary relation and binary operations on U. $\mathcal{L}$ also contains "=" to designate equality.

A properly combined expression involving a finite number of constants, variables, and operational symbols "pr" and "ap" is called a <u>term</u> of $\mathcal{L}$. Formulas obtained by properly combining "=" and "$\in$" with two terms of $\mathcal{L}$ are called <u>atomic formulas</u>, i.e. these are the atoms from which the world of mathematics will be described.

$\mathcal{L}$ also contains the <u>logical connectives</u> "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", and "$\leftrightarrow$" which designate respectively "not" or "it is not the case that", "and", "or", "implies", and "if and only if" or "iff". In this paper $\neg$ will often be abbreviated in context, e.g. $\neg \in$ will be abbreviated as $\notin$. It is important that the symbols "$\forall$" and "$\exists$" which represent the <u>quantifiers</u> "for all" and "there exists at least one" be interpreted with respect to some universe of discourse. A variable will always follow each of these quantifiers.

By combining the atomic formulas with connectives and quantifiers, and iterating a finite number of times, all formulas of $\mathcal{L}$ are obtained. Some mathematical concepts are not expressible formally because it would take an infinite iteration to express the concept. For example, the Archimedean property (nonexistence of infinitesimals) of the standard reals can be expressed using an infinite set of sentences. For each $r \in R$ (reals), all but a finite number of the following sentences are true:

$$r > 1,\ r + r > 1,\ r + r + r > 1,\ \cdots.$$

The principle cannot be expressed by a single sentence, however, and this is precisely why the property can hold for the standard reals but does not hold for the non-standard reals. Note that

$$\forall\, r\{r \in R\ \exists n[n \in N \wedge \forall\, m(m \in N \wedge m > n) \rightarrow m \cdot r > 1]\}$$

is not a sentence in $\mathcal{L}$ the language for the universe of discourse for the reals. The reason for this is that $m \cdot r$ need not make sense. For more on this see Lightstone [7]. (Note that $m > n$ is an abbreviation for $(m, n) \in (>)$.)

Terms with variables do not denote objects in the universe until a constant or another term without variables is substituted for each variable. Similarly some formulas do not have a truth value unless the variables are all "bound" either to a specific set of constants or by a formula Φ. To clarify some of the vagueness of this concept of bound variables consider the following examples:

(i) any occurence of the variable x in the form "$\forall\, x\ \Phi$" or "$\exists\, x\ \Phi$" is said (by Machover and Hirschfeld) to be <u>bound</u>.

(ii) $\exists x\ \ \forall y[y \notin x]$ has all variables bound (the formula is true).

(iii) $\forall y[\neg(y \in x)]$ does not have x bound (has no truth value).

(iv) $\forall x\ \ \forall y[y \notin x]$ has all variables bound (formula is false).

Formulas in which all variables are bound, and hence which have a truth value, are called <u>sentences</u>.

So far a universe of discourse U has been constructed and a brief description of the language $\mathcal{L}$, which can be used to express propositions about U, has been given. It is now time to see how the language $\mathcal{L}$ can be used to describe other universes of discourse (such as non-standard ones) and what possible relationships might exist between universes so described.

Suppose some people on a world in another universe use exactly the same symbols that we use to name the objects (perhaps quite different from the objects in our universe) in their universe. Then the same language would be describing two universes. Naturally, objects in the two universes which bear the same name are of special interest. It might be interesting to see what "James W. Hall 322-38-4674" is in another universe. This is analogous to the situation about to be examined.

Let I be the mapping which assigns to each constant of $\mathcal{L}$ the object in U which that constant names. Since there is a constant in $\mathcal{L}$ for each object in U, I is an onto map. The system $\mathfrak{A} = (U; I; \in; pr, ap)$ concisely represents a summary of the discussion so far. The universe of discourse U indicates the range of the quantifiers. It tells how the constants of $\mathcal{L}$ are to be interpreted, while $\in$, pr, and ap specify how the relation and operations on U are to be interpreted. Note that the other connectives $(=, \rightarrow,$ etc.) always have the same meaning. The system $\mathfrak{A}$ is called an <u>interpretation of $\mathcal{L}$</u>.

It will turn out that a non-standard model will also be an interpretation of $\mathcal{L}$ (with certain restrictions of course). Another interpretation of $\mathcal{L}$, $\mathfrak{A}^* = (U^*; I^*; {}^*\!\in; pr^*, ap^*)$, will now be compared to $\mathfrak{A} = (U; I; \in; pr, ap)$. Note the symbols in $\mathfrak{A}^*$ have a meaning corresponding to their respective counterparts in $\mathfrak{A}$. I^* is a mapping from $\mathcal{L}$ into (need not be onto) a universe of discourse U^* (perhaps distinct from U). Although $^*\in$ is a binary relation on U^*, it is arbitrary and need not be the membership relation. Similarly pr^* and ap^* are arbitrary binary operations under which U^* is closed.

2.15 Definition: Two interpretations of $\mathcal{L}$,

$$\mathfrak{A}^* = (U^*; I^*; {}^*\in; pr^*, ap^*) \text{ and}$$
$$\mathfrak{A}' = \{U'; I'; {}'\in; pr', ap'\}$$

are isomorphic if there is a 1-1 mapping Ψ of U^* onto U' satisfying the following conditions:

(i) for every constant "c" of $\mathcal{L}$, $I'(\text{"c"}) = \Psi(I^*(\text{"c"}))$.

(ii) for every a and b in U^*, $a \; {}^*\in \; b$ iff $\Psi(a) \; {}'\in \; \Psi(b)$.

(iii) for every a and b in U^*, $\Psi(a \; pr^* \; b) = \Psi(a) \; pr' \; \Psi(b)$.

(iv) for every a and b in U^*, $\Psi(a \; ap^* \; b) = \Psi(a) \; ap' \; \Psi(b)$.

If Ψ is not onto, but satisfies the other properties, Ψ is then called an isomorphic embedding of $\mathfrak{A}^*$ into $\mathfrak{A}'$.

2.16 Definition: An interpretation $\mathfrak{A}^*$ of $\mathcal{L}$ is a model for S a set of sentences of $\mathcal{L}$ if every sentence of S is true in $\mathfrak{A}^*$. (For emphasis note that the interpretation is in $\mathfrak{A}^*$. Hence the quantifiers relate to objects in U^*, $^*\in$ is the relation on U^*, etc.)

$\mathfrak{K}$ will be used to designate the set of all sentences of $\mathcal{L}$ which are true in the original interpretation $\mathfrak{A}$. $\mathfrak{A}$ is certainly a model for $\mathfrak{K}$

and $\mathfrak{A}$ and all interpretations of $\mathscr{L}$ isomorphic to $\mathfrak{A}$ are called the standard model. If $\mathfrak{A}^*$ is a model of $\mathcal{K}$ but is not isomorphic to $\mathfrak{A}$, then $\mathfrak{A}^*$ is called a non-standard model. Our interest will be narrowed shortly to a particular type of non-standard model which will be used throughout the remainder of the paper. Note that the validity of the definition of a standard model depends upon the next lemma.

2.17 Lemma: If $\mathfrak{A}^*$ is a model for S and $\mathfrak{A}^*$ and $\mathfrak{A}'$ are isomorphic, then $\mathfrak{A}'$ is also a model for S.

Proof: Since $\mathfrak{A}^*$ and $\mathfrak{A}'$ are isomorphic the atomic sentences are, by definition of the isomorphism, true in $\mathfrak{A}^*$ iff they are true in $\mathfrak{A}'$. Since sentences are finite combinations of atomic sentences, connectives, and quantifiers, it follows by induction that a sentence is true in $\mathfrak{A}^*$ iff it is true in $\mathfrak{A}'$. □

The following theorem is an important key to understanding how and why to use a non-standard model. It allows one to go freely from a universe to its non-standard counterpart.

2.18 Theorem: If $\mathfrak{A}^*$ is any model (standard or non-standard) of $\mathcal{K}$ then a sentence of $\mathscr{L}$ is true in $\mathfrak{A}^*$ iff it is true in $\mathfrak{A}$.

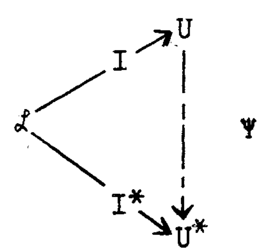
Proof: If a sentence S is true in $\mathfrak{A}$, then $S \in \mathcal{K}$ and is therefore true in $\mathfrak{A}^*$ which is a model for $\mathcal{K}$. Conversely, if S is false in $\mathfrak{A}$ then $\neg S \in \mathcal{K}$ and hence $\neg S$ is true in $\mathfrak{A}^*$, i.e. S is also false in $\mathfrak{A}^*$. □

As observed above, if two interpretations of a language are isomorphic then a given sentence in $\mathscr{L}$ is either true in both interpretations or in neither. On the other hand, if different interpretations of a language are given can one be isomorphically embedded in the other? In the case that an interpretation $\mathfrak{A}^*$ is also a model for K there is a quite natural way of embedding $\mathfrak{A}$ into $\mathfrak{A}^*$. In the following, "c" denotes a symbol in $\mathscr{L}$ while c and c^* denote objects in U and U^* respectively, where $\mathfrak{A} = (U; I; \in; pr, ap)$ and $\mathfrak{A}^* = (U^*; I^*; {}^*\in; pr^*, ap^*)$. It is supposed that

$$I : \mathscr{L} \xrightarrow{\text{onto}} U \text{ is defined by } I(\text{"c"}) = c \text{ and}$$

$$I^* : \mathscr{L} \xrightarrow{\text{into}} U^* \text{ is defined by } I^*(\text{"c"}) = c^*.$$

It then seems quite natural, since the symbol "c" identifies the object c under the interpretation $\mathfrak{A}$ and c^* under $\mathfrak{A}^*$, to identify c and c^*. Following our intuition, let $\psi : U \xrightarrow{\text{into}} U^*$ be defined by $\psi(c) = c^*$.



Notice that ψ maps every object in U to some object in U^*. For if $c \in U$, there exists at least one symbol "c" in $\mathscr{L}$ such that $I(\text{"c"}) = c$. Hence $I^*(\text{"c"}) = c^*$ and therefore c^* is an element in U such that $\psi(c) = c^*$. Secondly, note that ψ is a well-defined correspondence, for if "c" and "d" both denote the same object in U, then

$c = d$. That is, $c = d$ is a sentence in $\mathcal{L}$ which is true in $\mathfrak{A}$ and thus $c^* = d^*$ is true in $\mathfrak{A}^*$ which is also a model of $\mathcal{K}$.

<u>2.19 Theorem</u>: The natural embedding Ψ of $\mathfrak{A}$ into a model $\mathfrak{A}^*$ of $\mathcal{K}$ is an isomorphic embedding. If I^* is a mapping of $\mathcal{L}$ onto U^*, then $\mathfrak{A}^*$ is a standard model.

Proof: First of all it is clear from the definition of Ψ that $I^*("c") = \Psi(I("c"))$. Now if $a \neq b$ is a sentence of $\mathcal{L}$ true in $\mathfrak{A}$ then the interpretation $a^* \neq b^*$ must be true in $\mathfrak{A}^*$. That is, if $a \neq b$ then $\Psi(a) \neq \Psi(b)$. Thus Ψ is a 1-1 mapping. To conclude that Ψ is an isomorphic embedding it must further be shown that:

(i) $a \in b \leftrightarrow \Psi(a) \,{}^*\!\!\in \Psi(b)$ i.e. $a^* \,{}^*\!\!\in b^*$

(ii) $a = b \text{ pr } c \rightarrow \Psi(a) = \Psi(b) \text{ pr}^* \Psi(c)$ i.e. $a^* = b^* \text{ pr}^* c^*$

(iii) $a = b \text{ ap } c \rightarrow \Psi(a) = \Psi(b) \text{ ap}^* \Psi(c)$ i.e. $a^* = b^* \text{ ap}^* c^*$.

Note that each of i, ii, and iii above contains a sentence of $\mathcal{L}$ under the two interpretations, $\mathfrak{A}$ and $\mathfrak{A}^*$ respectively. Thus i, ii, and iii are consequences of the fact that $\mathfrak{A}^*$ is a model of $\mathcal{K}$ and that truth in one interpretation implies truth in the other. If, in addition, I^* is a mapping from $\mathcal{L}$ onto U^*, then Ψ is a mapping from U onto U^*. This follows by first noting that if I^* is onto then for each c^* in U^* there exists a "c" in $\mathcal{L}$ such that $I^*("c") = c^*$. Since I is also onto, there is some c in U such that $I("c") = c$. Then by definition of Ψ, $\Psi(c) = c^*$. Therefore Ψ is onto and so $\mathfrak{A}^*$ is a standard model. □

The previous two theorems make working with non-standard models seem very natural. Rather than considering the non-standard model as completely apart from the standard world, the standard world is considered as part of the larger non-standard world. Thus from now on $\mathfrak{A}^*$ will be considered as merely an extension of $\mathfrak{A}$. Thus any object c in $\mathfrak{A}$ will be identified with the associated c^* in $\mathfrak{A}^*$. Objects of $\mathfrak{A}^*$ that are in $\mathfrak{A}$ are called <u>standard objects</u> while those that are in $\mathfrak{A}^*\setminus\mathfrak{A}$ (i.e. in $\mathfrak{A}^*$ but not in $\mathfrak{A}$) are called <u>non-standard objects</u>. There are non-standard objects in $\mathfrak{A}^*$ iff $\mathfrak{A}^*$ is a non-standard model.

All models used later in connection with topological spaces, other than the original standard model itself, will be assumed to be a specfic type of non-standard model called an enlargement. A description of an enlargement requires a special type of relation which shall now be considered.

A relation R over U is a set of ordered pairs; the left domain of R is the set of all first members of the ordered pairs in the set R. In the following $(a \text{ pr } b) \in R$ will be abbreviated by $a\ R\ b$. A binary relation R is called <u>concurrent</u> if for any natural number n and any n objects $a_1, a_2, \ldots, a_n$ in the left domain of R there exists some b such that $a_1\ R\ b, a_2\ R\ b, \ldots, a_n\ R\ b$. Since U is infinite, "$\neq$" is an example of a concurrent relation over U.

Now every formula Φ of $\mathcal{L}$ with two free (not bound) variables defines a binary relation R_Φ over U where R_Φ is given by $R_\Phi = \{(a, b) : \Phi(a, b) \text{ is true in } \mathfrak{A}\}$.

To verify the existence of enlargements, the compactness theorem from symbolic logic will be assumed [3]. The compactness theorem guarantees that if a language $\mathcal{L}'$ has a set of sentences S such that every finite subset of S has a model, then S also has model.

Since U is infinite, there is at least one formula $\Phi(x, y)$ such that R_Φ is concurrent (e.g. $\Phi(x, y)$ given by "$x \neq y$"). Now for each formula Φ of $\mathcal{L}$ for which R_Φ is concurrent, a new constant "c_Φ" (i.e. a constant not previously in $\mathcal{L}$) is invented. With $\mathfrak{K}$ still designating the set of all sentences of $\mathcal{L}$ true in $\mathfrak{A}$, let $\mathfrak{K}_\Phi$ be the set of all sentences of the form $\Phi(a, c_\Phi)$ where each "a" is a constant of $\mathcal{L}$. These sentences are not in $\mathcal{L}$ but are in an enlarged language $\mathcal{L}'$ obtained from $\mathcal{L}$. Then let $\mathfrak{K}' = \mathfrak{K} \cup [\cup\{\mathfrak{K}_\Phi : R_\Phi$ is concurrent$\}]$. Let S be a finite subset of sentences from $\mathfrak{K}'$. Examining S, note that a sentence of S containing constants only from U is still true in the interpretation $\mathfrak{A}$.

For any fixed Φ there are at most a finite number of sentences of $\mathfrak{K}_\Phi$ in S. Let $\Phi(a_1, c_\Phi), \cdots, \Phi(a_n, c_\Phi)$ designate these sentences. Since R is concurrent there is some $b \in U$ such that $a_1 R_\Phi b, \cdots, a_n R_\Phi b$. Let I" be a mapping from $\mathcal{L}'$ to U such that I" agrees with I for all constants of $\mathcal{L}'$ that are also in $\mathcal{L}$. Then define $I''(c_\Phi) = b$. Thus each of $\Phi(a_1, c_\Phi), \cdots, \Phi(a_n, c_\Phi)$ is true in $\mathfrak{A}'' = (U; I''; \in;$ pr, ap). Hence $\mathfrak{A}''$ is a model for S. By the arbitrariness of S, the compactness theorem guarantees there is a model $\mathfrak{A}' = (U'; I'; '\in;$ pr', ap') for the set $\mathfrak{K}'$ of sentences of $\mathcal{L}'$.

Now let $\mathfrak{A}^* = (U^*; I^*; {}^*\in; pr^*, ap)$ be the same as $\mathfrak{A}'$ with the exception that I^* is the restriction of I' to $\mathcal{L}$. Then $\mathfrak{A}^*$ is an interpretation of $\mathcal{L}$ and a model of K. The interpretation $\mathfrak{A}^*$ is called an <u>enlargement</u> of $\mathfrak{A}$.

In the work which follows an arbitrary but fixed enlargement will always be assumed. The major properties of an enlargement which are used in the remainder of the paper are summarized in the following theorem.

<u>2.20 Theorem:</u> If $\mathfrak{A}^*$ is an enlargement of $\mathfrak{A}$, then

(i) $\mathfrak{A}^*$ is a non-standard model of K

(ii) For every formula $\Phi(x, y)$ of $\mathcal{L}$ for which R is concurrent there is an object c_Φ in U^* such that when $\Phi(a, b)$ is true for some $b \in U$, then $\Phi(a, c_\Phi)$ is true in $\mathfrak{A}^*$.

Proof: As mentioned earlier, the construction of $\mathfrak{A}^*$ forces $\mathfrak{A}^*$ to be a model of K.

The second part also follows from the construction since the compactness theorem guaranteed that $\mathfrak{A}'$ was a model for K'. That is, all sentences of the form $\Phi(a, c_\Phi)$ in K' are true in $\mathfrak{A}'$. Since c_Φ is an object in U', c_Φ is in $U^* = U'$.

That $\mathfrak{A}^*$ is a non-standard model may be verified from examining any formula $\Phi(x, y)$ which yields a concurrent relation R_Φ, such as $\Phi(x, y)$ given by $x \neq y$. Since there is an object in $\mathfrak{A}^*$ corresponding to c_Φ, which is a constant in $\mathcal{L}'$ but not in $\mathcal{L}$, the natural embedding

Ψ which relates $\mathcal{L}$, U, and U^* cannot map any object in U to c_Φ in U^*. Thus Ψ is not onto, c_Ψ is a non-standard object, and $\mathfrak{A}^*$ must be a non-standard model. □

The construction is now complete. A very nice non-standard model, i.e. an enlargement, has been constructed. The natural embedding has provided an intuitive way of treating an enlargement $\mathfrak{A}^*$ of $\mathfrak{A}$ as if U were actually contained in U^*. Further, once any property about $\mathfrak{A}$ is formally stated this property holds when reinterpreted in $\mathfrak{A}^*$. Now a few theorems and remarks will be given so that the powerful proof technique outlined above may be used with dexterity.

2.21 Remark: Since U is closed with respect to pr and ap, an ordered pair of standard objects is standard, as is a standard function applied to a standard object. Also if $a \in A$ where A is a standard set, then a is also a standard element since U is a transitive set. However, if $a \; {}^*\!\in A$ where A is a standard set then a may well be non-standard. In the next chapter it will be seen that this is the case with infinitesimal real numbers.

For each concept which is definable for objects in $\mathfrak{A}$ there is a corresponding concept in $\mathfrak{A}^*$. For example, a *set S is an object in U^* that is either $\emptyset$ or is a collection of objects from U^* such that at least one of these *objects $a \; {}^*\!\in S$. Recall that ${}^*\!\in$ may not be the membership relation on U, but is merely a binary relation on U. Nonetheless, a member a of S for which $a \; {}^*\!\in S$ is called a *member of S. Our interest will be in those members of S which are indeed *members of S. A definition concerning these *objects will be given shortly.

Another *concept is that of a *(ordered pair), i.e. pr^* applied to two objects of U^*. A *relation in U^* is a *set of *(ordered pairs). When considering a *function f it is necessary to "*" many of the words such as saying f *maps the *set A *into the *set B. The reason for this is that the mapping is done by *(ordered pairs) and is performed on *sets.

<u>2.22 Definition</u>: The <u>scope</u> of a *set S, denoted $\hat{S}$, is the set of all *members of S.

$\hat{S}$ is a well-defined collection of *objects, but it need not be a standard set, and in fact it might not even be a *set since it might not be a *object of U^*. Since every standard object is considered as a *object, every standard set will be contained in its scope. That is, every standard element of S is in $\hat{S}$. The next chapter shows that *elements of S need not be elements of S.

<u>2.23 Theorem</u>: If A is a finite standard set then $\hat{A} = A$, i.e. the only *members of A are its standard elements.

Proof: If $A = \emptyset$ the result follows from the definition of the natural embedding, so let $A \in U$ be given by $A = \{a_1, a_2, \cdots, a_n\}$. Then the sentence "$\forall a(a \in A \leftrightarrow a = a_1 \vee a = a_2 \vee \cdots \vee a = a_n)$" is true in $\mathfrak{A}$, and hence in $\mathfrak{A}^*$. The interpretation is

$$\text{"}\forall a\{a \;{}^*\!\!\in A \leftrightarrow a = a_1 \vee a = a_2 \vee \cdots \vee a = a_n\}.\text{"}$$

Therefore $a \;{}^*\!\!\in A$ iff $a = a_1, a_2, \cdots, a_{n-1}$, or a_n. That is $\hat{A} = A$. $\square$

Sentences, recall, have at most a finite number of connectives and quantifiers. Thus the above argument is not applicable to infinite sets since their elements could not be enumerated in a single sentence.

<u>2.24 Theorem</u>: The scope of the union (intersection) of a finite collection of *sets is the union (intersection) of their scopes.

Proof: The following sentence defines the union of a finite collection of sets of U: $\forall x\ \forall A_1\ \forall A_2 \cdots \forall A_n[x \in A_1 \vee x \in A_2 \vee \cdots \vee x \in A_n \leftrightarrow x \in A_1 \cup A_2 \cup \cdots \cup A_n]$. The translation in $^*\mathfrak{A}$ is $\forall x\ \forall A_1\ \forall A_2\ \forall \cdots \forall A_n[x\ {}^*\!\in A_1 \vee x\ {}^*\!\in A_2 \vee \cdots \vee x\ {}^*\!\in A_n \leftrightarrow x\ {}^*\!\in A_1 \cup A_2 \cdots \cup A_n]$. Since x is a member of $\hat{A}_1 \cup \hat{A}_2 \cup \cdots \cup \hat{A}_n$ iff x is a *member of A_1 or of A_2 or of $\cdots$ or of A_n, $\hat{A}_1 \cup \hat{A}_2 \cup \cdots \cup \hat{A}_n = \widehat{A_1 \cup A_2 \cup \cdots \cup A_n}$. By replacing $\cup$ by $\cap$ and "or" by "and" a proof that $\hat{A}_1 \cap \hat{A}_2 \cap \cdots \cap \hat{A}_n = \widehat{A_1 \cap A_2 \cap \cdots \cap A_n}$ will result. □

<u>2.25 Corollary</u>: For *sets X, A, B, if $X = A \cup B$, where $A \cap B = \emptyset$, then $\hat{X} = \hat{A} \cup \hat{B}$ and $\hat{A} \cap \hat{B} = \emptyset$.

<u>2.26 Corollary</u>: $\widehat{A \backslash B} = \hat{A} \backslash \hat{B}$, where A and B are *subsets of the *set X.

<u>2.27 Theorem</u>: For each function f and set A, $\widehat{f(A)} = f(\hat{A})$.

Proof: The following sentence defines $f(A)$ in $\mathfrak{A}$:

$$\forall x[x \in f(A) \leftrightarrow \exists y(y \in A \wedge f(y) = x)].$$

Reinterpreted in $\mathfrak{A}^*$, this sentence says $\widehat{f(A)} = f(\hat{A})$. □

<u>2.28 Theorem</u>: For each function f and set A, $\widehat{f^{-1}(A)} = f^{-1}(\hat{A})$.

Proof: Given f and A this follows from the interpretation of the sentence "$\forall x[x \in f^{-1}(A) \leftrightarrow \exists y(y \in A \wedge f(x) = y)]$" in $\mathfrak{A}^*$. □

<u>2.29 Theorem</u>: The function f maps the set A onto the set B iff f maps $\hat{A}$ onto $\hat{B}$.

Proof: The following sentence defines what it means for the function f to map the set A onto the set B in $\mathfrak{A}$:

$$\forall y[y \in B \rightarrow \exists x(x \in A \wedge f(x) = y)].$$

Reinterpreted in ${}^*\mathfrak{A}$, this sentence says f maps $\hat{A}$ onto $\hat{B}$. □

In the context of the following discussion it may sometimes be said that A is <u>the</u> set whose scope is $\hat{A}$. The following theorem justifies all such statements.

<u>2.30 Theorem</u>: The correspondence which assigns a scope to each *set is a 1-1 correspondence.

Proof: The following sentence defines set equality in $\mathfrak{A}$:

$$\forall A\ \forall B[A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B)].$$

Translated in $\mathfrak{A}^*$ it says *sets are equal iff their scopes are equal. □

The remaining material in this chapter also comes from Machover and Hirschfeld [11]. There are two central concepts, that of "Nuc" and "Fil". The theorems which concern these concepts provide the justification for some later manipulations. Thus one might want to defer this material until the need for such justification arises.

<u>2.31 Definition</u>: A collection of subsets F of X is called a <u>filter</u> on X if:

(i) $A, B \in F \rightarrow A \cap B \in F$

(ii) $A \in F,\ A \subset B \subset X \rightarrow B \in F$.

Note that $\emptyset$ may be an element of F according to this definition. Hence $\mathcal{P}(X)$ is a <u>trivial filter</u> on X. One of the important filters to be considered in Chapter IV will be that of the neighborhood family of a point.

<u>2.32 Lemma</u>: If F_a is a filter on X for each a in some index set A, then $\cap\{F_a : a \in A\}$ is also a filter on X.

<u>2.33 Lemma</u>: If $G \subset \mathcal{P}(X)$ and G' is the family of all finite intersections of members of G, then $G'' = \{A : B \subset A \text{ for some } B \in G'\}$ is a filter on X.

G'' is said to be <u>generated</u> by G. If the intersection of any finite number of members of G has a subset belonging to G, then G is said to be a <u>base</u> for G''.

<u>2.34 Definition</u>: A non-trivial filter F on X is called an <u>ultrafilter</u> if no non-trivial filter on X properly contains F.

For proofs of the next two lemmas concerning ultrafilters see Thron [19].

<u>2.35 Lemma</u>: Every non-trivial filter on X is contained in an ultrafilter.

<u>2.36 Lemma</u>: If F is an ultrafilter on X and $A \subset X$, then exactly one of A and $X \backslash A$ is an element of F.

<u>2.37 Definition</u>: For $G \subset \mathcal{P}(X)$ the <u>nucleus</u> of G is given by <u>NucG</u> $= \cap\{\hat{A} : A \in G\}$. If $N = \text{NucG}$ for some $G \subset \mathcal{P}(X)$, then N is said to be <u>nuclear</u>.

<u>2.38 Theorem</u>: If $G \subset \mathcal{P}(X)$ and F is the filter generated by G, then $\text{NucF} = \text{NucG}$.

Proof: Since $G \subset G' \subset G'' = F$, the definition of Nuc implies $\text{NucF} \subset \text{NucG}$. To establish $\text{NucG} \subset \text{NucF}$, consider an arbitrary set $A \in F$. Since G generates F, there exist a finite number of sets $B_1, B_2, \cdots, B_n$ in G such that $B_1 \cap B_2 \cap \cdots \cap B_n \subset A$. It then follows that $\hat{B}_1 \cap \hat{B}_2 \cap \cdots \cap \hat{B}_n \subset \hat{A}$. Now if $a \in \text{NucG}$, then $a \in \hat{B}_1 \cap \hat{B}_2 \cdots \cap \hat{B}_n \subset \hat{A}$. Since A was arbitrary from F,

$$a \in \cap\{\hat{A} : A \in F\} = \text{NucF}.$$

Therefore $\text{NucG} \subset \text{NucF}$. Hence $\text{NucF} = \text{NucG}$. □

<u>2.39 Definition</u>: If $G \subset \mathcal{P}(X)$ and $A \ {}^*\!\in G$ such that $\hat{A} \subset \text{NucG}$, then A is called an <u>infinitesimal</u> <u>*member</u> of G.

<u>2.40 Theorem</u>: Every filter has an infinitesimal *member.

Proof: Let G be base for the filter F (such a base exists since F is a base for itself) and define a formula $\Phi(x, y)$ by "$y \in G \wedge y \subset x$". Then consider the associated relation R_Φ on U. Since G is a base for F, the left domain of R_Φ is F. Then since the intersection of a finite family $A_1, A_2, \cdots, A_n$ of sets from F is some $A \in F$,

$$A = A_1 \cap A_2 \cap \cdots \cap A_n$$

must contain some set B from the base G. That is, there exists a B such that $A_1 R_\Phi B, \cdots, A_n R_\Phi B$. Hence R_Φ is concurrent, and so in the enlargement $\mathfrak{U}^*$ there is some $C_\Phi {}^*\!\in U^*$ such that $C_\Phi {}^*\!\in G^*$ and for each $A \in F$, it is true that $C_\Phi {}^*\!\subset A$ (i.e. $\widehat{C}_\Phi \subset \widehat{A}$). Thus $\widehat{C}_\Phi \subset [\cap\{\widehat{A} : A \in F\}]$ which equals NucF. Since $G \subset F$, it also follows that $C_\Phi {}^*\!\in F$. Therefore C_Φ is an infinitesimal *member of F. □

<u>2.41 Definition</u>: If $S \subset \widehat{X}$ then <u>FilS</u> = $\{A : A \subset X$ and $S \subset \widehat{A}\}$.

<u>2.42 Theorem</u>: FilS is a filter.

Proof: If A, B ∈ FilS, then $S \subset \widehat{A}$ and $S \subset \widehat{B}$. Hence $S \subset \widehat{A} \cap \widehat{B}$. Since $\widehat{A} \cap \widehat{B} = \widehat{A \cap B}$, $S \subset \widehat{A \cap B}$. Therefore $A \cap B \in$ FilS. Secondly, if $B \subset A$ and B ∈ FilS, then $\widehat{B} \subset \widehat{A}$ and $S \subset \widehat{B}$ imply $S \subset \widehat{A}$. Hence A ∈ FilS. It follows that FilS is a filter. □

<u>2.43 Theorem</u>: If $A \subset B$ and $B \subset \widehat{X}$, then FilB ⊂ FilA. For any filters F and G, Fil (NucF) = F and F ⊂ G iff NucG ⊂ NucF.

Proof: The first statement follows from the definition of Fil. Now assume that $A \in F$. The definition of Nuc implies that NucF $\subset \widehat{A}$. Since Fil (NucF) = $\{A : A \subset X$ and NucF $\subset \widehat{A}\}$, A ∈ Fil (NucF). A was an arbitrary element of F so it follows that F ⊂ Fil (NucF). To establish the other inclusion let A ∈ Fil (NucF). The definition of Fil implies that NucF $\subset \widehat{A}$. Since F is a filter, there exists an infinitesimal *member C of F. Thus $\hat{C} \subset$ NucF which equals $\cap\{\hat{A} : A \in F\}$. Together NucF $\subset \hat{A}$ and $\hat{C} \subset$ NucF imply that $\hat{C} \subset \hat{A}$. Hence the sentence $\exists x[x \in F \wedge x \subset A]$ is true in $\mathfrak{A}^*$. Since F and A are standard, this sentence is also true in $\mathfrak{A}$. Therefore some element B of F is a subset of A. Since F is a filter, $B \in F$, and $B \subset A$, it follows that $A \in F$. Hence the inclusion Fil (NucF) $\subset$ F follows. Therefore Fil (NucF) = F.

Lastly, note that $F \subset G$ implies NucG $\subset$ NucF, by the definition of Nuc. On the other hand, if NucG $\subset$ NucF, then by the first part of the theorem, Fil (NucF) $\subset$ Fil (NucG). Therefore, by the previous result, $F \subset G$. □

<u>2.44 Theorem</u>: For any two filters F and G, Nuc $(F \cap G)$ = NucF $\cup$ NucG.

Proof: $F \cap G$ is a filter since F and G are. Now $F \cap G \subset F$ and $F \cap G \subset G$. Therefore the previous result implies NucF $\subset$ Nuc $(F \cap G)$ and NucG $\subset$ Nuc $(F \cap G)$. Hence NucF $\cup$ NucG $\subset$ Nuc $(F \cap G)$. On the other hand, suppose $x \ {}^*\!\in X$ but $x \notin$ NucF $\cup$ NucG. Then there exists an $A \in F$ and a $B \in G$ such that $x \ {}^*\!\notin A$ and $x \ {}^*\!\notin B$. Therefore $x \notin \hat{A} \cup \hat{B}$ and, since $\hat{A} \cup \hat{B} = \widehat{A \cup B}$, $x \notin \widehat{A \cup B}$. Now note that $(A \cup B) \in (F \cap G)$. So $x \notin \widehat{A \cup B}$ implies $x \notin$ Nuc $(F \cap G)$. Therefore Nuc $(F \cap G) \subset$ NucF $\cup$ NucG. Hence the equality Nuc $(F \cap G)$ = NucF $\cup$ NucG follows. □

Although the union of filters may not be a filter, it is sometimes useful to consider the filter which is generated by a union of filters. The following result is used later in a proof concerning local compactness.

<u>2.45 Theorem</u>: Let $G = \bigcup\{F_i : i \in I\}$ where F_i is a filter for each $i \in I$, an indexing set. Then Nuc $(G'') = \bigcap\{\text{Nuc}F_i : i \in I\}$ where G'' is the filter generated by G.

Proof: Since G'' is generated by G, it follows from Theorem 2.38 that $\text{Nuc}G'' = \text{Nuc}G$. By definition $\text{Nuc}G = \bigcap\{\hat{A} : A \in G\}$. From the definition of G it may be ascertained that

$$\bigcap\{\hat{A} : A \in G\} = \bigcap\{\hat{A} : A \in F_i \text{ for some } i \in I\}.$$

Then note that $\bigcap\{\hat{A} : A \in F_i \text{ for some } i \in I\} = \bigcap\{\bigcap\{\hat{A} : A \in F_i\} : i \in I\}$. Finally, from the definition of Nuc it follows that

$$\bigcap\{\bigcap\{\hat{A} : A \in F_i\} : i \in I\} = \bigcap\{\text{Nuc}F_i : i \in I\}.$$

Thus, from all these equalities, $\text{Nuc}G'' = \bigcap\{\text{Nuc}F_i : i \in I\}$. □

CHAPTER III

THE NON-STANDARD REALS

The previous chapter has outlined a procedure for producing an enlargement for any given set X. There are also other procedures for producing non-standard models. One of these is called the ultrapower technique. This technique has the added feature that in the case of the real numbers it not only guarantees a non-standard model, but it also provides specific examples of elements that are non-standard real numbers. Thus the major purpose of this chapter is to provide the reader with some *points of a set which are not points of that set, that is, to show that the scope of a set may properly contain that set.

Throughout the remainder of this chapter the natural numbers will be denoted by N and the real numbers by R. $\mathfrak{U}$ will designate an ultrafilter containing the cofinite (Frechet) filter on N. That is, $\mathfrak{U}$ is an ultrafilter containing all subsets S of N where $N \setminus S$ is finite.

Now let $R^N = \{f : f \text{ is a function from N into R}\}$. Since R^N may be described equivalently as the set of all real sequences, sequential notation shall be used to represent the elements of R^N. For example, if $\bar{x} \in R^N$ then $\bar{x} = (x_1, x_2, x_3, \ldots)$ where $x_j \in R$ for each $j \in N$. The ultrafilter $\mathfrak{U}$ will now be used to define an equivalence relation on R^N. If $\bar{x}$, $\bar{y} \in R^N$, define "$\equiv$" by $\bar{x} \equiv \bar{y}$ iff $\{n \in N : x_n = y_n\} \in \mathfrak{U}$.

In particular, $\bar{x} \equiv \bar{y}$ if $\bar{x}$ and $\bar{y}$ differ in only a finite number of coordinates.

__3.1 Lemma__: The relation $\equiv$ is an equivalence relation on R^N.

Proof: For each $\bar{x} \in R^N$, $\bar{x} \equiv \bar{x}$ since $\{n \in N : x_n = x_n\} = N \in \mathfrak{J}$. Hence $\equiv$ is reflexive. If $\bar{x} \equiv \bar{y}$, then $\{n \in N : x_n = y_n\} \in \mathfrak{J}$. Note that $\{n \in N : y_n = x_n\} = \{n \in N : x_n = y_n\}$. Hence $y \equiv x$, and so $\equiv$ is symmetric. If $\bar{x} \equiv \bar{y}$ and $\bar{y} \equiv \bar{w}$, then $A = \{n \in N : x_n = y_n\} \in \mathfrak{J}$ and $B = \{n \in N : y_n = w_n\} \in \mathfrak{J}$. Since $\mathfrak{J}$ is a filter, $A \cap B \in \mathfrak{J}$. Note that $A \cap B \subset \{n \in N : x_n = w_n\} = C$. This implies $C \in \mathfrak{J}$ since $\mathfrak{J}$ is a filter. Hence $\bar{x} \equiv \bar{w}$, and so $\equiv$ is transitive. □

Let R^* denote $R^N/\equiv$, the collection of equivalence classes of R^N with respect to the equivalence relation $\equiv$. (This explains the reason for calling this technique the ultrapower technique. R^* is constructed by first forming the cartesian product R^N and then reducing it modulo an ultrafilter.) For simplicity, $\bar{x} = (x_1, x_2, x_3, \cdots)$ will be used to denote the equivalence class to which $\bar{x}$ belongs. R^* will be called the non-standard reals.

There is a natural way of embedding the reals into R^*. Let $\Psi : R \to R^*$ be defined by $\Psi(r)$ equals the equivalence class in R^* to which the constant function $\bar{r}$ belongs. That is,

$$\psi(r) = \bar{r} = (r, r, r, \cdots).$$

__3.2 Definition__: Each $\bar{r} \in R^*$ for which there is some $r \in R$ such that $\bar{r} = \psi(r)$ will be called a standard element of R^* or a __standard__ real number. All other members of R^* are called non-standard real numbers.

For convenience, R will be considered as a subset of R^* and each standard real number $\bar{r} = \Psi(r)$ may be denoted by r. Thus $\bar{0} = (0, 0, 0, \cdots)$ may be identified by 0 and $\bar{1} = (1, 1, 1, \cdots)$ by 1.

3.3 Example: $\bar{x} = (1, 2, 3, \cdots, n, \cdots)$ and $\bar{y} = (1, 1/2, 1/3, \cdots, 1/n, \cdots)$ are non-standard real numbers. To see this, compare $\bar{x}$ to any standard real number r (i.e. $\bar{r} = (r, r, r, \cdots)$). The set $A = \{n \in N : x_n = r\}$ is either $\emptyset$ or contains exactly one natural number as $r \in R \setminus N$ or $r \in N$. (Since $\equiv$ is an equivalence relation it is permissible to let the equivalence class be represented by any of its members, so in particular it was assumed that $\bar{r}$ was represented by $(r, r, r, \cdots)$.) Since A is finite, $N \setminus A \in \mathfrak{U}$. Thus $A \notin \mathfrak{U}$ and so $\bar{x}$ and r are distinct equivalence classes. Thus $\bar{x}$ is a non-standard real number for it is not equal to any standard real number. Similarly $\bar{y}$ may be shown to be non-standard. In fact, any $\bar{x} = (x_1, x_2, x_3, \cdots)$ is non-standard if $\{n \in N : x_n = r\}$ is finite for each $r \in R$. Thus the reader now has an infinite supply of *points which are not points.

The major purpose of the chapter has been achieved provided the reader is willing to accept that R^* is a non-standard model for the reals. Since a rigorous proof of this would entail another discussion similar to that of the preceding chapter, a proof will not be provided. Instead, an indication of how concepts of R can be reinterpreted in R^* will be given. For each concept concerning the real numbers there is a corresponding concept for R^*. Generally speaking, this concept for R^*

is obtained by using the usual definitions associated with operations on functions and perhaps reinterpreting properties of R through the ultrafilter $\mho$. The following definitions and lemmas illustrate these remarks.

3.4 Definition: For $\bar{x} = (x_1, x_2, x_3, \cdots)$ and $\bar{y} = (y_1, y_2, y_3, \cdots)$ in R^*, $\bar{x} < \bar{y}$ iff $\{n \in N : x_n < y_n\} \in \mho$.

3.5 Definition: For $\bar{x} = (x_1, x_2, x_3, \cdots)$ in R^*,

$$|\bar{x}| = (|x_1|, |x_2|, |x_3|, \cdots).$$

3.6 Definition: If $\bar{x} \in R^*$ and $r < |\bar{x}|$ for each standard real number r, then $\bar{x}$ is said to be infinitely large.

Of course no standard real number is infinitely large, but there are many infinitely large non-standard real numbers.

3.7 Example: $\bar{x} = (1, 2, 3, \cdots, n, \cdots)$ is infinitely large. To verify this note that for each $r \in R$ there is some $n \in N$ such that $r < m$ for each $m > n$. Clearly $|\bar{x}| = x$. If $A = \{j \in N : r_j < x_j\}$, then $N \setminus A$ is finite. Hence $A \in \mho$ and so $r < |\bar{x}|$ for each $r \in R$. Therefore $\bar{x}$ is infinitely large. Similarly it may be shown that $\bar{w}$ and $\bar{z}$ are infinitely large where

$$\bar{w} = (0, 1/2, 2, 0, 1/3, 4, 0, 0, \pi, 8, -35, 16, 32, 64, \cdots, 2^n, \cdots)$$

$$\text{and } \bar{z} = (0, -1, 1/2, 1, 0, -2, 1/3, 2, -3, 3, \cdots, -n, n, \cdots).$$

3.8 Definition: If $\bar{x} \in R^*$ and $0 < |\bar{x}| < |r|$ for each non-zero real number r, then $\bar{x}$ is called an infinitesimal.

3.9 Example: $\bar{y} = (1, 1/2, 1/3, \cdots, 1/n, \cdots)$ is an infinitesimal.

Note first of all that $|\bar{y}| = \bar{y}$. It is clear that $0 < \bar{y}$. Now for each non-zero real number r there exists an $n \in N$ such that $m > n$ implies $1/m < |r|$. Thus if $A = \{j \in N : y_j < |r_j|\}$, then $N \setminus A$ is finite. Hence $A \in \mathfrak{U}$ and so $|\bar{y}| < |r|$ for each non-zero $r \in R$. Therefore $\bar{y}$ is an infinitesimal. Similarly it may be shown that $\bar{z}$ and $\bar{w}$ are infinitesimals where

$\bar{z} = (0, 0, 1/2, 0, 0, 1/4, 0, 0, 0, 1/8, 0, 1/16, 1/32, 1/64, \cdots, 1/2^n, \cdots)$ and $\bar{w} = (1/2, 0, 1/3, 0, 1/4, 0, -1/9, 222, 1/8, 1/27, 1/16, -1/81, \cdots, 1/2^n, 1/(-3)^n, \cdots)$.

<u>3.10 Definition</u>: If r is a standard real number and $r - \bar{x}$ is zero or any infinitesimal, then $\bar{x}$ is said to be <u>near</u> r.

<u>3.11 Example</u>:

(i) The set of points near zero is the set containing zero and all infinitesimals.

(ii) If $\bar{x}$ is an infinitesimal then $\bar{y} = \bar{x} + r$ is near r for each standard real number r. Thus the set S of points near r is given by $S = \{\bar{x} + r : \bar{x}$ is zero or an infinitesimal$\}$.

<u>3.12 Definition</u>: For $\bar{x} = (x_1, x_2, x_3, \cdots)$ and $\bar{y} = (y_1, y_2, y_3, \cdots)$ in R^*, $\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots)$ and $\bar{x}\,\bar{y} = (x_1 y_1, x_2 y_2, x_3 y_3, \cdots)$.

<u>3.13 Lemma</u>: 0 is the additive identity for R^* and 1 is the multiplicative identity. The additive inverse of $\bar{x} = (x_1, x_2, x_3, \cdots)$ is given by $-\bar{x} = (-x_1, -x_2, -x_3, \cdots)$. If $\bar{x}^{-1}$ is defined by

$\bar{x}^{-1} = (a_1, a_2, a_3, \cdots)$ where

$$a_n = \begin{cases} x_n^{-1} & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0, \end{cases}$$

then $\bar{x}^{-1}$ is the multiplicative inverse of $\bar{x}$.

Van Osdol [20] proves these results while establishing that R^* is an ordered field. Considered as fields, Ψ is an embedding of R (a complete ordered field) into R^* (an ordered field).

Two other important concepts of R which can be reinterpreted in R^* are now given for future reference.

3.14 Definition: For each subset $S \subset R$, the associated *set S^* in R^* is defined by $S^* = \{\bar{x} \in R^* : \{n \in N : x_n \in S\} \in \mathfrak{U}\}$.

3.15 Example: $\bar{x} = (1, 2, 3, \cdots, n, \cdots)$ is a *element of N^*. For verification note that $\{n \in N : x_n \in N\} = N$ which belongs to $\mathfrak{U}$. Thus $\bar{x}$ is an example of an infinitely large (non-standard) natural number. Another example of an infinitely large natural number is

$$\bar{w} = (0, 1/2, 2, 0, 1/3, 4, 0, 0, \pi, 8, -35, 16, 32, 64, \cdots, 2^n, \cdots).$$

3.16 Definition: If $f : S \to R$ is a function from $S \subset R$, then the associated *function $f^* : S^* \to R^*$ is defined by $f^*(\bar{x}) = \bar{y}$ where

$$y_n = \begin{cases} f(x_n) & \text{if } x_n \in S \\ 0 & \text{if } x_n \notin S. \end{cases}$$

<u>3.17 Example</u>: Let $f : N \to R$ be defined by $f(n) = 1/n$. Then $f^* : N^* \to R^*$ will *map the infinite natural number $\bar{x} = (1, 2, 3, \cdots, n, \cdots)$ onto the infinitesimal $\bar{y} = (1, 1/2, 1/3, \cdots, 1/n, \cdots)$. The function also *maps $\bar{w} = (0, 1/2, 2, 0, 1/3, 4, 0, 0, \pi, 8, -35, 16, 32, 64, \cdots, 2^n, \cdots)$ onto the infinitesimal $\bar{z} = (0, 0, 1/2, 0, 0, 1/4, 0, 0, 0, 1/8, 0, 1/16, 1/32, 1/64, \cdots, 1/2^n, \cdots)$.

CHAPTER IV

TOPOLOGICAL SPACES

The study of topology evolved as a generalization of the notions associated with open subsets on the real line. In the work which follows, it is often helpful to sketch an appropriate "ball" or "sphere" about a fixed point in order to obtain an intuitive idea of what other points are near the fixed point. The actual "picture" of the "ball" may not appear like an open set in the reals, but the purpose for making it is the same — to enlighten its maker. Once a clear conception of an idea has been achieved, non-standard analysis will often allow the idea to be expressed in intuitively worded language. Without further ado, it will be assumed that for each set X under discussion, X is a standard set identified with X^* in some enlargement based upon the universe of discourse constructed from X.

<u>4.1 Definition</u>: Let X be a non-empty set and let $\mathcal{J}$ be a family of subsets of X satisfying:

(i) $X, \emptyset \in \mathcal{J}$.

(ii) The union of any family $\{O_a : a \in A\}$ of members of $\mathcal{J}$ is again a member of $\mathcal{J}$.

(iii) The finite intersection of any family $\{O_k : k = 1, 2, \cdots, n\}$ of members of $\mathcal{J}$ is again a member of $\mathcal{J}$.

$\mathfrak{T}$ is then called a <u>topology</u> for X. The pair $(X, \mathfrak{T})$ is called a <u>topological space</u> (space) and the members of $\mathfrak{T}$ are called <u>open sets</u>.

Note that property (i) is included mainly for emphasis as it follows from properties (ii) and (iii) by using $\emptyset$ as an indexing set.

The following examples may easily be verified to be topological spaces. Spaces which are given names will be referred to later by these names.

<u>4.2 Example</u>:

(i) Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\{a\}, \{b, c\}, X, \emptyset\}$, then $(X, \mathfrak{T})$ is a three-point topological space.

(ii) <u>Discrete topological space</u>

Let $X \neq \emptyset$ be an arbitrary set and $\mathfrak{T} = \mathcal{P}(X)$; $(X, \mathfrak{T})$ is then called a discrete space. ($\mathcal{P}$ is the power set operator.)

(iii) <u>Indiscrete topological space</u>

Let $X \neq \emptyset$ be an arbitrary set and $\mathfrak{T} = \{X, \emptyset\}$; $(X, \mathfrak{T})$ is then called an indiscrete space.

(iv) <u>Cofinite topological space</u>

Let $X \neq \emptyset$ be an arbitrary set and $\mathfrak{T}$ be the family consisting of $\emptyset$ and all subsets of X whose complements in X are finite; $(X, \mathfrak{T})$ is then called a cofinite space.

(v) <u>Co-countable topological space</u>

Let $X \neq \emptyset$ be an arbitrary set and $\mathfrak{T}$ be the family consisting of $\emptyset$ and all subsets of X whose complements in X are countable; $(X, \mathfrak{T})$ is then called a co-countable space.

A given set may have many topologies. For example, if X is the set of real numbers, then X could have the discrete, indiscrete, co-finite, and co-countable topologies (as well as the usual topology and many others). Thus when considering a space, both the set and the topology must be made clear.

The definitions which follow give the terminology which is used to describe the relationship among different topologies on the same set.

<u>4.3 Definition</u>: Let X be a set and let F be the family of all topologies for X. Then F can be partially ordered by set inclusion, i.e. the ordering is given by $\mathfrak{T} \leq \mathfrak{T}'$ for $\mathfrak{T}$, $\mathfrak{T}' \in F$ iff $\mathfrak{T} \subset \mathfrak{T}'$. If $\mathfrak{T} \leq \mathfrak{T}'$, then $\mathfrak{T}$ is said to be <u>weaker</u> or <u>coarser</u> than $\mathfrak{T}'$ while $\mathfrak{T}'$ is said to be <u>stronger</u> or <u>finer</u> than $\mathfrak{T}$. If neither $\mathfrak{T} \leq \mathfrak{T}'$ nor $\mathfrak{T}' \leq \mathfrak{T}$, then $\mathfrak{T}$ and $\mathfrak{T}'$ are said to be <u>non-comparable</u>.

It is easily seen that the discrete topology for X is the finest topology for X while the indiscrete topology is the coarsest topology for X.

One activity that occurs repeatedly in point set topology is the formation of new spaces from old ones. Perhaps the easiest way to do this is given without proof in the following theorem, while a warning is given by the following example.

<u>4.4 Theorem</u>: The intersection of an arbitrary non-empty family of topologies for X is again a topology for X.

<u>4.5 Example</u>: The union of a family of topologies for X need not be a topology for X. Let $X = \{a, b, c\}$, then $\mathfrak{T} = \{\{a\}, \{b, c\}, X, \emptyset\}$ and $\mathfrak{T}' = \{\{b\}, \{a, c\}, X, \emptyset\}$ are topologies for X. Note that $\{b, c\}$ and $\{a, c\}$ are members of $\mathfrak{T} \cup \mathfrak{T}'$. If $\mathfrak{T} \cup \mathfrak{T}'$ is a topology for X, then the intersection of $\{b, c\}$ and $\{a, c\}$ must be a member of $\mathfrak{T} \cup \mathfrak{T}'$. This is not the case.

One more standard definition will be given before enough machinery is present to begin with some non-standard treatment of topology.

<u>4.6 Definition</u>: Let p be a point in the space $(X, \mathfrak{T})$. A subset N of X is called a <u>neighborhood</u> (nbhd) of p iff there exists an open set O such that $p \in O$ and $O \subset N$. The family of all nbhds of p is called the <u>nbhd system</u> at p. This nbhd system will be denoted by <u>Np</u> throughout this paper.

Note that $Np \neq \emptyset$ since X is always a nbhd of p. In fact, even more can be said about the structure of this family.

<u>4.7 Theorem</u>: Np is a filter.

Each open set is a nbhd of each of its points, but a nbhd of a point need not be open. All that is necessary is that it contain some open set about the point. This is sufficient, however, to provide a simple criterion for determining when a subset of a space is open.

<u>4.8 Theorem</u>: Let $(X, \mathfrak{T})$ be a space, and let $O \subset X$. O is open iff O contains a nbhd of each of its points.

This provides sufficient material to introduce the very important non-standard term "monad" and to rigorously define the concept "near". When reading the following definition beware that p must be a standard point and not merely a *point.

<u>4.9 Definition</u>: The <u>monad</u> of the point p, <u>$\mu(p)$</u>, in the space $(X, \mathfrak{J})$ is equal to Nuc Np.

<u>4.10 Lemma</u>: $\mu(p) = \cap\{\hat{N} : N \in Np\}$

The proof of this lemma consists of recalling the definition of Nuc G where $G \subset \mathcal{P}(X)$. The lemma is given for emphasis since this will be the characterization of $\mu(p)$ used most frequently in the work which follows.

<u>4.11 Example</u>:

(i) For the three-point space in Example 4.2.i, $\mu(a) = \widehat{\{a\}} \cap \hat{X}$. Thus $\mu(a) = \{a\}$. (The scope of any finite set is that set.) Similarly, $\mu(b) = \{b, c\}$ and $\mu(c) = \{b, c\}$.

(ii) For each point p in a discrete space $(X, \mathfrak{J})$, $\mu(p) = \{p\}$.

(iii) For each point p is an indiscrete space $(X, \mathfrak{J})$, $\hat{X} = \mu(p)$. For X finite this means $X = \mu(p)$. When X is infinite, $\hat{X}$ may well contain non-standard points and in this case X is a proper *subset of $\mu(p)$.

What the preceding lemma is saying is that $\mu(p)$ consists precisely of the *points which are in every nbhd of p. Thus, intuitively speaking, the points of $\mu(p)$ must be very close to p. This leads to the following definition of the concept of nearness. First note that the topology involved completely determines the monad at each point. No notational change will be made to denote this where the topology on X is considered fixed in any given discussion. It will not be shown now, but it is interesting to point out that a converse to the above statement also holds. Namely, if the monads of every point in X are given then a unique topology $\mathfrak{J}$ is determined for X.

<u>4.12 Definition</u>: In the space $(X, \mathfrak{J})$, a *point q is said to be <u>near</u> the point p if $q \in \mu(p)$. If q is near p for some $p \in X$, then q is called a <u>near-standard</u> *point. Otherwise, q is said to be <u>remote</u>.

<u>4.13 Example</u>:

(i) In the three-point space of Example 4.2.i, a is the only *point near a. The points b and c are both near each other.

(ii) For each point p in a discrete space, the only *point near p is p itself. Thus each non-standard *point of a discrete space is remote.

(iii) In an indiscrete space, every point is near every other point. In addition, when the space is infinite there may be non-standard points near any given point (i.e. there may be non-standard near-standard *points).

(iv) When the usual topology for the reals is given, it can then be noted that all infinitely large real numbers are remote *points while all infinitesimals are non-standard *points which are near-standard. In particular, $\bar{x} = (1, 2, 3, \cdots, n, \cdots)$ is remote while $\bar{y} = (1, 1/2, 1/3, \cdots, 1/n, \cdots)$ is near-standard.

It may now be said rigorously, as well as intuitively, that the points of $\mu(p)$ must be near p. It would certainly be expected that the point p is near itself. Since the point p is in every nbhd of p and thus a *point in every one of its nbhds, it does indeed follow that $p \in \mu(p)$. Thus p is near itself and so each standard point is near-standard. Caution – it does not make sense to say p is near p for non-standard p, and similarly nearness is not a symmetric relationship for *points. In fact, q may be near the point p while $\mu(q)$ is not even defined. This is true when q is a *point but not a point in the space $(X, \mathfrak{T})$. If p and q are standard then ~~it~~ is possible for p and q to be near each other. A characterization of a T_0-space will later involve an examination of when this is possible.

__4.14 Example:__ If a is near b, then b need not be near a. Let $X = \{a, b\}$ and $\mathfrak{T} = \{\emptyset, \{a\}, X\}$. Note that $\mu(a) = \hat{X} \cap \widehat{\{a\}}$ and $\mu(b) = \hat{X}$. Hence $\mu(a) = \{a\}$ and $\mu(b) = X$. Thus a is near b, but b is not near a.

Nearness is a transitive relationship, i.e. if r is near q and q is near p, then r is near p. Before examining the proof of this in the next theorem, note that r near q implies q is standard and q near p implies p is standard while r may be non-standard.

__4.15 Theorem:__ If p and q are points in the space $(X, \mathfrak{T})$ with q near p, then $\mu(q) \subset \mu(p)$.

Proof: If p and q are standard points in the space $(X, \mathfrak{T})$ with q near p, then $q \in \mu(p)$, i.e. q is a *point of every nbhd of p. Since q is a standard point, q is a point of every nbhd of p. Let $N \in Np$ contain the open set O about p. Since O is also a nbhd of p, $q \in O$. Hence N contains an open set O about q and N is thus a nbhd of q. Therefore if $N \in Np$, then $N \in Nq$. It follows that $Np \subset Nq$. Whence $\cap\{\hat{N} : N \in Nq\} \subset \cap\{\hat{N} : N \in Np\}$, i.e. $\mu(q) \subset \mu(p)$. □

Bourbaki [1] gives an intuitive description of open sets. This description says that an open set is a set which is a neighborhood of each of its points; hence an open set contains all points sufficiently close to an arbitrary point in the set. The following development produces a non-standard characterization of open sets which almost uses these same words.

4.16 Lemma: A set N containing the point p in the space $(X, \mathfrak{T})$ is a nbhd of p iff N contains every *point near p (i.e. $\mu(p) \subset \hat{N}$).

Proof: By definition, Fil $\mu(p) = \{N : N \subset X$ and $\mu(p) \subset \hat{N}\}$. Since $\mu(p) = \text{NucNp}$, Fil $\mu(p)$ = Fil (NucNp). Theorem 2.43 guarantees that Fil (NucNp) = Np. From these equalities it follows that $Np = \{N : N \subset X$ and $\mu(p) \subset \hat{N}\}$. That is, $N \in Np$ iff $\mu(p) \subset \hat{N}$. Hence N is a nbhd of p iff N contains every *point near p. □

4.17 Theorem: A subset O of the space $(X, \mathfrak{T})$ is open iff $p \in O$ implies O contains all *points near p (i.e. $\mu(p) \subset \hat{O}$ for each $p \in O$).

Proof: By Theorem 4.8, O is open iff it contains a nbhd of each of its points. Thus by the previous lemma, O is open iff $p \in O$ implies O contains every *point near p. □

If $(X, \mathfrak{J})$ is a topological space where the cardinality of $\mathfrak{J}$ is great, it may be difficult to comprehend which subsets of X are open. In this case, it would be desirable to characterize $\mathfrak{J}$ by some simpler subfamily of $\mathfrak{J}$. As an example, the singletons might be considered to characterize the topology for a discrete space since any open set can be expressed as a union of singletons. As is indicated in the following definition, this subfamily of singletons is in some sense a base upon which the entire topology rests.

<u>4.18 Definition</u>: A subfamily B of a topology $\mathfrak{J}$ of X is a <u>base</u> for $\mathfrak{J}$ iff every member of $\mathfrak{J}$ is the union of members of B. $\mathfrak{J}$ is said to be <u>generated</u> by B.

It now becomes advantageous, although the previous definition considers finding a base for a known topology, to consider what characteristics a family of subsets of X must have to generate some topology.

<u>4.19 Theorem</u>: A family F of subsets of X is a base for a topology $\mathfrak{J}$ for X iff

(i) $X = \cup\{B: B \in F\}$.

(ii) If $p \in A \cap B$ where $A, B \in F$, then there exists a $C \in F$ such that $p \in C$ and $C \subset A \cap B$.

This theorem is a useful tool since not every family of subsets of X is a base for a topology. This fact is obvious from the theorem if X cannot be obtained as a union of members of the family; however, the following example shows that (i) may hold while (ii) fails.

4.20 Example: $B = \{\{a, b\}, \{b, c\}, X, \emptyset\}$ is not a base for a topology for $X = \{a, b, c\}$. For if it were, then both $\{a, b\}$ and $\{b, c\}$ would be in the topology so generated. Thus $\{b\} = \{a, b\} \cap \{b, c\}$ would be in this topology. This cannot be since $\{b\}$ is not expressible as a union of members of B.

When the concept of a base was being introduced, it was mentioned that the base might characterize the open sets in a simpler way. This is exemplified by examining a base for the usual space of real numbers.

4.21 Example: Let R be the set of real numbers. For each $r \in R$ and for every $\nabla > 0$ let $S(r, \nabla) = \{x : x \in R \text{ and } |r - x| < \nabla\}$. Then $B = \{S(r, \nabla): r \in R, \nabla > 0\}$ satisfies the criteria of the previous theorem and is thus a base for a topology for R. This topology E is called the usual topology for R and the space will be denoted by (R, E).

Similar to the way a base simplifies the characterization of open sets a nbhd base might simplify the investigation of the nbhd system about a point in the space. Since the monad of a point is determined by the nbhd system, it might then be advantageous to examine $\mu(p)$ in terms of the nbhd base.

<u>4.22 Definition:</u> A subfamily Mp of Np where p is a point in the space $(X, \mathfrak{J})$ is a <u>nbhd base</u> for Np iff every nbhd of p contains a member of Mp.

<u>4.23 Lemma:</u> If $p \in (X, \mathfrak{J})$, then $\mu(p) = \cap\{\hat{M} : M \in Mp\}$ where Mp a nbhd base for Np.

Proof: Since $Mp \subset Np$, it follows that $\cap\{\hat{N} : N \in Np\} \subset \cap\{\hat{M} : M \in Mp\}$. By Lemma 4.10, $\mu(p) = \cap\{\hat{N} : N \in Np\}$. Hence $\mu(p) \subset \cap\{\hat{M} : M \in Mp\}$. To verify the other set inclusion, assume that $q \notin \mu(p)$. Since $q \notin \cap\{\hat{N} : N \in Np\}$, there must exist some $N \in Np$ such that $q \notin \hat{N}$. Mp is a nbhd base so N contains some $M \in Mp$. It follows that $q \notin \hat{M}$ and thus $q \notin \cap\{\hat{M} : M \in Mp\}$. This verifies that $\cap\{\hat{M} : M \in Mp\} \subset \mu(p)$. Therefore $\mu(p) = \cap\{\hat{M} : M \in Mp\}$. □

<u>4.24 Example:</u> Note that the family of sets $S(r, \nabla)$ where $\nabla > 0$ is a nbhd base at $r \in (R, E)$, the usual space of real numbers.

It is now time to justify the different uses of the term "near" made earlier in this paper. One definition of nearness was given for the non-standard real numbers and another definition was given for an arbitrary topological space. When the topology for the reals is the usual topology E, then these definitions are equivalent. To see this, let us first examine the points near zero in the space (R, E). If q is near 0, then $q \in \mu(0)$. From the preceding example and lemma it follows that $q \ {}^*\!\in S(0, \nabla)$ for each $\nabla > 0$. By the definition of $S(0, \nabla)$, $q \ {}^*\!\in S(0, \nabla)$ implies that $|q| < \nabla$. Since this is true for each $\nabla > 0$, q is either zero or an infinitesimal, i.e. q is near zero in the terminology of Chapter III. Thus if q is near zero in terms of the definition interpreted for the space (R, E), then q is near zero in terms of the definition given earlier in Chapter III specifically for the non-standard Reals. By reversing the last few steps of this argument, one may verify that nearness of q to zero in R^* of Chapter III implies q is near zero in (R, E). Similarly, for each p in (R, E) it follows that $\mu(p) = \{q : p - q \text{ is zero or an infinitesimal}\}$.

The present task will be to continue giving some of the basic definitions in order to form a framework for later discussions.

<u>4.25 Definition</u>: A family S of subsets of X is a <u>subbase</u> for $\mathfrak{J}$ of the space $(X, \mathfrak{J})$ iff the family of all finite intersections of members of S forms a base for $\mathfrak{J}$. $\mathfrak{J}$ is then said to be <u>subgenerated</u> by S.

As was the case when a base was defined, the definition of a subbase is made with regard to an existing topology. It was earlier pointed out that not every family of subsets generates a topology, but it may now be seen that any given family of subsets is contained in some topology. Obviously $S \subset \mathcal{P}(X)$, the discrete topology for X. However, there may exist a coarser topology containing S. In fact, S subgenerates such a topology.

<u>4.26 Theorem</u>: For any family S of subsets of X, S subgenerates a topology $\mathfrak{J}$ which is the coarsest topology containing S.

<u>4.27 Example</u>:

(i) The family of open rays, sets of the form

$\{x : x \in R$ and $x > p\}$ or $\{x : x \in R$ and $x < p\}$ where p is a real number, constitute a subbase for (R, E).

(ii) The family $S = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ subgenerates the topology $\mathfrak{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ where $X = \{a, b, c\}$.

It was mentioned earlier that there are many possible topologies on some sets. Not only was a relationship among the topologies examined, but it was also suggested that new topologies could be formed by intersections of known topologies. Other ways of forming new topologies from known ones will be discussed when the terms "relative topology" and "product topology" are defined. The simplest of these will now be examined while product spaces will be postponed until a later chapter.

4.28 Theorem: If A is subset in the space $(X, \mathfrak{T})$, then the family $\mathfrak{T}_A = \{A \cap O : O \in \mathfrak{T}\}$ is a topology for A.

4.29 Definition: The topology $\mathfrak{T}_A = \{A \cap O : O \in \mathfrak{T}\}$ associated with the set A in the space $(X, \mathfrak{T})$ is called the relative topology of A with respect to $\mathfrak{T}$ and $(A, \mathfrak{T}_A)$ is called a subspace of $(X, \mathfrak{T})$. A property of $(X, \mathfrak{T})$ which is also a property of each subspace of $(X, \mathfrak{T})$ is an hereditary property.

4.30 Example:

(i) Consider $A = [3, 5)$ in the space (R, E). Note that $[3, 4)$, $(3, 4)$, and $(4, 5)$ are open sets in the relative topology of A while $[4, 5)$ is not. Further notice that $[3, 4)$ is open in (A, E_A) but not in (R, E).

(ii) Consider (R, E) and (J, E_J) where J denotes the set of integers. It is easy to verify that E_J is the discrete topology for J.

It was mentioned that the monads of the points of X determine a unique topology for X. This result of Machover and Hirschfeld is interesting and will be examined now, but the reader is advised that no future use will be made of the result in this paper.

<u>4.31 Theorem</u>: Let $\lambda(p)$ be an arbitrary subset of $\hat{X}$ for each $p \in X$ and let $\mathcal{T} = \{O \subset X : \lambda(p) \subset \hat{O}$ for each $p \in O\}$. Then $\mathcal{T}$ is a topology for X.

Proof: First note that $\emptyset$, X are in $\mathcal{T}$. Now suppose that $\{O_a : a \in A\}$ is a family of members of $\mathcal{T}$. If $p \in \cup O_a$, then $p \in O_a$ for some $a \in A$. Since $O_a \in \mathcal{T}$, $\lambda(p) \subset \hat{O}_a$. Thus $\lambda(p) \subset \widehat{\cup O_a}$, and so the union of the members is also in $\mathcal{T}$. (Note that $O_a \subset \cup O_a$ implied $\hat{O}_a \subset \widehat{\cup O_a}$.) Similarly, if $\{O_k : k = 1, 2, \cdots, n\}$ is a finite family of members of $\mathcal{T}$, let $p \in \cap O_k$. Then $p \in O_k$ for each k, and so $\lambda(p) \subset \hat{O}_k$ for each k. Hence $\lambda(p) \subset \cap \hat{O}_k$ which equals $\widehat{\cap O_k}$. Thus $\mathcal{T}$ contains $\cap O_k$. Therefore $\mathcal{T}$ is topology for X. □

<u>4.32 Corollary</u>: The topology $\mathcal{T}$ determined by the family of $\lambda(p)$'s is the strongest topology such that $\lambda(p) \subset \mu(p)$ for each $p \in X$.

Proof: Let $p \in X$ and $N \in Np$ in the space $(X, \mathfrak{J})$ formed by the family of $\lambda(p)$'s. Then there is some $O \in \mathfrak{J}$ such that $p \in O$ and $O \subset N$. Thus $\lambda(p) \subset \hat{O}$ and so $\lambda(p) \subset \hat{N}$. Since this is true for each $N \in Np$, $\lambda(p) \subset \cap\{\hat{N} : N \in Np\}$. Hence $\lambda(p) \subset \mu(p)$ for each p in $(X, \mathfrak{J})$.

Now suppose $\mathfrak{J}'$ is a topology for X strictly stronger than $\mathfrak{J}$, the topology determined by the family of $\lambda(p)$'s. That is, there exists some O in $\mathfrak{J}'$ that is not in $\mathfrak{J}$. Thus for some $p \in O$, $\lambda(p) \not\subset \hat{O}$. However in $(X, \mathfrak{J}')$, $\mu(p) \subset \hat{O}$. Therefore $\lambda(p) \not\subset \mu(p)$. Hence $\mathfrak{J}$ is the strongest topology for X such that $\lambda(p) \subset \mu(p)$. □

No claim of set equality was made in the previous corollary. In general, equality may not hold, as may be inferred from the conditions in the following theorem.

<u>4.33 Theorem</u>: $\lambda(p) = \mu(p)$ iff the following conditions hold:

(i) $\lambda(p)$ is nuclear for each $p \in X$.

(ii) $p \in \lambda(p)$ for each $p \in X$.

(iii) If $A \in \text{Fil}\ \lambda(p)$ then there is some $B \in \text{Fil}\ \lambda(p)$ such that $q \in B$ implies $A \in \text{Fil}\ \lambda(q)$.

Proof: $(\rightarrow)$ Since $\mu(p) = \text{Nuc}\ Np$, $\lambda(p)$ is nuclear whenever $\lambda(p) = \mu(p)$. Further, if $\mu(p) = \lambda(p)$ then $p \in \lambda(p)$ since p is always in $\mu(p)$. Now suppose $A \in \text{Fil}\ \lambda(p)$ which equals

$$\{A \subset X : \lambda(p) \subset \hat{A}\}.$$

Since $\mu(p) = \lambda(p)$, it follows that $\mu(p) \subset \hat{A}$. Hence $A \in Np$ and thus contains some open set B about p. Since B is open, $\mu(r) \subset \hat{B}$ for each $r \in B$. $B \subset A$ and $\mu(r) = \lambda(r)$ imply that $\lambda(r) \subset \hat{A}$ for each $r \in B$. Therefore $A \in \text{Fil}\ \lambda(r)$ for each $r \in B$.

($\leftarrow$) Due to the previous corollary, $\lambda(p)$ will equal $\mu(p)$ if it can be established that $\mu(p) \subset \lambda(p)$. Since $\lambda(p)$ is assumed nuclear, Theorem 2.38 guarantees that $\lambda(p) = \text{Nuc}F$ for some filter $F \subset P(X)$. If it can be established that $\text{Fil}\ \lambda(p) \subset Np$, then it will follow that $\mu(p) \subset \lambda(p)$. To see this note that $\text{Fil}\ \lambda(p) \subset Np$ implies, by Theorem 2.43, that $\text{Nuc}\ Np \subset \text{Nuc}(\text{Fil}\ \lambda(p))$. Thus $\mu(p) \subset \text{Nuc}(\text{Fil}\ \lambda(p))$. Now examine $\text{Nuc}(\text{Fil}\ \lambda(p))$. Recall that $\lambda(p) = \text{Nuc}F$. By Theorem 2.43, $\text{Nuc}F = \text{Nuc}(\text{Fil}\ \text{Nuc}F)$. Substitute twice for NucF to obtain $\lambda(p) = \text{Nuc}(\text{Fil}\ \lambda(p))$. From the preceding inclusion, it follows that $\mu(p) \subset \lambda(p)$.

To establish $\text{Fil}\ \lambda(p) \subset Np$, let $A \in \text{Fil}\ \lambda(p)$ and define $C = \{q : A \in \text{Fil}\ \lambda(q)\}$. By the choice of A and the definition of C, $p \in C$. If $q \in C$, then $\lambda(q) \subset \hat{A}$. Thus by the assumption that $q \in \lambda(q)$, it follows that $q \ {}^*\!\in A$ and so $q \in A$ since q is standard. This shows that $C \subset A$. Now A will be a nbhd of p, i.e. $A \in Np$, if it can be shown that C is an open set about p. By definition of the topology formed by the family of $\lambda(p)$'s, C is open if $\lambda(q) \subset \hat{C}$ for each $q \in C$. So suppose $q \in C$. Then $A \in \text{Fil}\ \lambda(q)$ and by assumption there must exist a $B \in \text{Fil}\ \lambda(q)$ such that $r \in B$ implies $A \in \text{Fil}\ \lambda(r)$. Note that $A \in \text{Fil}\ \lambda(r)$ means that $r \in C$ and thus $B \subset C$.

$B \in \text{Fil } \lambda(q)$ implies $\lambda(q) \subset \hat{B}$ and thus $B \subset C$ implies $\lambda(q) \subset \hat{C}$. Hence C is open and A must be a nbhd of p. Thus $\text{Fil } \lambda(p) \subset Np$ and the theorem follows. □

4.34 Example:

(i) Let $X = \{a, b, c\}$, $\lambda(a) = \{a\}$, $\lambda(b) = \{b, c\}$, and $\lambda(c) = \{b, c\}$. Then the topology $\mathfrak{J} = \{O \subset X : \lambda(p) \subset \hat{O}$ for each $p \in O\}$ is given by $\mathfrak{J} = \{\emptyset, \{a\}, \{b, c\}, X\}$. Note that this is the space given in Example 4.11.i. It may be observed that $\lambda(p) = \mu(p)$ for each $p \in X$. Also note that the conditions of Theorem 4.33 are satisfied.

(ii) Let $X = \{a, b, c\}$, $\lambda(a) = \{b\}$, $\lambda(b) = \{b, c\}$, $\lambda(c) = \{c\}$. Then the topology $\mathfrak{J}$ which is formed is given by $\mathfrak{J} = \{\emptyset, \{c\}, \{b, c\}, X\}$. The monads of the points are : $\mu(a) = \{a, b, c\}$, $\mu(b) = \{b, c\}$, and $\mu(c) = \{c\}$. Clearly $\lambda(p) \subset \mu(p)$ for each $p \in X$; however, $\lambda(a)$ is a proper subset of $\mu(a)$. The condition of Theorem 4.33 that fails is that $a \notin \lambda(a)$.

CHAPTER V

GENERALIZATIONS OF NEARNESS

This chapter will continue the development of some of the basic properties of a general topological space. Recall that q is near p iff $q \in \mu(p)$, that is, the monad of a point p is the set of *points near p. An attempt to generalize this concept of nearness will now be made.

<u>5.1 Definition</u>: In the space $(X, \mathfrak{J})$, the *set A is <u>near</u> the point p iff some *point of A is near p (i.e. if $\hat{A} \cap \mu(p) \neq \emptyset$).

That the concept of nearness actually has been generalized may be verified by examining the following theorem.

<u>5.2 Theorem</u>: If p is a point in the space $(X, \mathfrak{J})$ and q is a *point of X then the *set whose scope is $\{q\}$ is near p iff q is near p. In particular, if p and q are points in X then the set $\{q\}$ is near p iff q is near p.

Proof: By definition the *set A is near p iff some *point of A is near p. Hence if $\hat{A} = \{q\}$, then A is near p iff q is near p. Now if q is a point in X then $\{q\}$ is a finite standard set. Hence $\widehat{\{q\}} = \{q\}$. It therefore follows that the point q is near the point p iff the set $\{q\}$ is near p. □

5.3 Example: If $\{q\}$ is near p, then p need not be near q and $\{p\}$ need not be near q. Recall Example 4.14. The point a was near b so $\{a\}$ is near b. However, b is not near a and so $\{b\}$ is not near a.

This example points out that although nearness is an intuitive concept it must be used carefully with regard to the topology in question. It is also important to emphasize that the terminology must be used carefully. While the term "near" has been defined and must be used prudently, the term "close" has not been (and will not be) given a rigorous definition and will be used more loosely in a descriptive context.

The concept of nearness will now be related to the standard concepts of accumulation points and closed sets. An accumulation point of a set is sometimes loosely described as a point that the set is close to. Accumulation points of a set will now be defined and the accuracy of this description will be examined using non-standard techniques.

5.4 Definition: A point p is called an accumulation point (acc pt) of $A \subset X$ in the space $(X, \mathcal{T})$ if every nbhd N of p has a non-void intersection with $A \setminus \{p\}$. The set of all acc pts of A, accA, will be called the derived set of A.

This definition indicates that A must be very close to its acc pts; nonetheless, it is possible for $p \notin A$ to be an acc pt of A and for $p \in A$ not to be an acc pt of A.

5.5 Example:

(i) Let $A = (0, 1)$ in the space (R, E). Note that $\text{acc}A = [0, 1]$, but neither 0 nor 1 is an element of A.

(ii) Let $A = \{0\}$ in the space (R, E). Note that $\text{acc}A = \emptyset$. Thus $0 \in A$ but is not an acc pt of A.

(iii) Let X be the set of real numbers and let $\mathfrak{J}$ consist of X, $\emptyset$ and all left rays where a left ray is a set of the form $\{x : x < r\}$ for r a real number. From now on this topology will be referred to as the left-ray topology on the Reals. If $A = \{0\}$ then $\text{acc}A = \{x : x > 0\}$.

The following theorem relies heavily on the concepts developed in the chapter on non-standard models. In particular, the proof of this non-standard characterization of acc pts uses properties of an enlargement and certain facts about set theoretic operations with the scopes of sets.

5.6 Theorem: A point p is an acc pt of the set A in the space $(X, \mathfrak{J})$ iff some *point of A other than p is near p (i.e. iff $\mu(p) \cap (\hat{A} \setminus \{p\}) \neq \emptyset$).

Proof: $(\rightarrow)$ Since p is an acc pt of A, each nbhd of p contains a point of A other than p. Now the intersection of a finite number of nbhds of p is again a nbhd of p. Hence the relation defined by the formula $x \in Np \wedge y \in A \wedge y \neq p \wedge y \in x$ is concurrent. Since $\mathfrak{A}^*$ is an enlargement, it follows that there exists some $q \;{}^*\!\!\in A$ such that $q \neq p$ and whenever $N \in Np$, $q \;{}^*\!\!\in N$. Hence $q \in \mu(p) \cap (\hat{A} \setminus \{p\})$

which means $\mu(p) \cap (\hat{A}\setminus\{p\}) \neq \emptyset$. Therefore some *point of A other than p is near p.

($\leftarrow$) Suppose $p \notin \text{acc}A$. Then there must exist a nbhd N of p such that $N \cap (A\setminus\{p\}) = \emptyset$. Since the scope of an intersection of two sets is the intersection their scopes, $\hat{N} \cap \widehat{(A\setminus\{p\}} = \emptyset$. Similarly $\hat{N} \cap (\hat{A}\setminus\widehat{\{p\}}) = \emptyset$ and finally $\hat{N} \cap (\hat{A}\setminus\{p\}) = \emptyset$. Hence $\mu(p) \cap (\hat{A}\setminus\{p\}) = \emptyset$. Therefore if some *point of A other than p is near p, then p is an acc pt of A. □

<u>5.7 Corollary</u>: The standard set A is near the point p in the space $(X, \mathfrak{T})$ iff $p \in A$ or $p \in \text{acc}A$.

Proof: By definition, A is near p iff $\hat{A} \cap \mu(p) \neq \emptyset$. Also, $\hat{A} \cap \mu(p) \neq \emptyset$ iff either $p \in A$ or $\mu(p) \cap (\hat{A}\setminus\{p\}) \neq \emptyset$. This last condition is equivalent to saying $p \in A$ or $p \in \text{acc}A$. □

The following example emphasizes that points must not be used in place of *points in Definition 5.1.

<u>5.8 Example</u>: The set A in the space $(X, \mathfrak{T})$ may be near the point q without any point of A being near q. Let the space be (R, E), $A = (0, 1)$, and $q = 1$. Then $(0, 1)$ is near 1, but no point of $(0, 1)$ is near 1.

That the sufficiency part of Theorem 5.2 may not be generalized to infinite sets is clear from the previous example; nonetheless, the theorem may be generalized to finite sets. One way to do this is to prove the result directly using the fact that a finite standard set contains only standard *points. This result is also immediate from the following lemma.

<u>5.9 Lemma</u>: If $A, B \subset X$, then $A \cup B$ is near the point p in $(X, \mathfrak{T})$ iff either A is near p or B is near p.

Proof: $A \cup B$ is near p iff $\mu(p) \cap \widehat{(A \cup B)} \neq \emptyset$. This is true iff either $\mu(p) \cap \hat{A} \neq \emptyset$ or $\mu(p) \cap \hat{B} \neq \emptyset$, that is, iff A is near p or B is near p. □

<u>5.10 Corollary</u>: If A is a finite set in $(X, \mathfrak{T})$ which is near p then some point of A is near p.

<u>5.11 Example</u>: Lemma 5.9 may not be extended to the union of an infinite family of sets. Let (R, E) be the space with $I_n = (1/n, 2)$ for each natural number n. Note that $(0, 2) = \cup I_n$ and that $(0, 2)$ is near 0. However, I_n is not near 0 for any n.

The proof of the following theorem is so similar to the proof of Lemma 5.9 that only the result will be given here.

<u>5.12 Theorem</u>: The point p is an acc pt of $A \cup B$ in the space $(X, \mathfrak{T})$ iff $p \in accA$ or $p \in accB$.

Machover and Hirschfeld state that results are often easier to invent using non-standard analysis since the language is more intuitive and natural. As but one example of this, note the clarity of the idea presented by the next lemma in this paper.

<u>5.13 Lemma</u>: If, in the space $(X, \mathfrak{T})$, the *set A is near the point p and p is near the point q, then A is near q.

Proof: If A is near p, then $\mu(p) \cap \hat{A} \neq \emptyset$. Since p is near q, $\mu(p) \subset \mu(q)$ by Theorem 4.15. Hence $\mu(q) \cap \hat{A} \neq \emptyset$, and therefore A is near q. □

<u>5.14 Example</u>: The converse of Lemma 5.13 does not hold. In the space (R, E), let $A = (0, 1)$, $p = 0$, and $q = 1$. Note that A is near p and A is near q, but p is not near q. Further, let $(X, \mathfrak{T})$ be the left-ray topology on the reals with $A = (1, 2)$, $p = 0$, and $q = 3$. Then A is near q and p is near q, but A is not near p.

<u>5.15 Example</u>: Lemma 5.13 cannot be extended to a theorem about acc pts. It is possible for $p \in \text{acc}A$ and p to be near q with q being an acc pt of A. Consider the three point space of Example 4.2.i with $p = c$, $A = \{a, b\}$, and $q = b$.

A closed set is a standard concept which is sometimes thought of as a set which contains all points that it is close to. That this thought process is essentially correct will be shown now.

<u>5.16 Definition</u> : A set C in the space $(X, \mathfrak{T})$ is said to be <u>closed</u> if $\text{acc}A \subset A$.

<u>5.17 Example</u>:

(i) The sets $\{0\}$ and $[0, 1]$ are closed in (R, E) while $[0, 1)$ and $(0, 1)$ are not closed. Note that $[0, 1)$ is neither closed nor open in (R, E).

(ii) If A is any set in the discrete space $(X, \mathcal{T})$, then $\mathrm{acc}A = \emptyset$. Thus every set in this space is closed.

(iii) If A is any non-void set in the indiscrete space $(X, \mathcal{T})$, then $\mathrm{acc}A = X$ unless A is a singleton. For singleton sets A, $\mathrm{acc}A = X \setminus A$. Thus the only closed sets in this space are $\emptyset$, X.

<u>5.18 Theorem</u>: A set C in the space $(X, \mathcal{T})$ is closed iff C contains all points which it is near to.

Proof: $(\rightarrow)$ Suppose that C is closed and that C is near the point p. Then $p \in C$ or $p \in \mathrm{acc}C$ by Corollary 5.7. Since C is closed, $\mathrm{acc}C \subset C$. So in either case $p \in C$. Therefore C contains all points that it is near to.

$(\leftarrow)$ Assume that C near p implies $p \in C$. Now suppose C is not closed. This means that there is some $p \in \mathrm{acc}C$ such that $p \notin C$. By Corollary 5.7, $p \in \mathrm{acc}C$ implies C is near p. Thus the assumption indicates $p \in C$. From this contradiction its follows that C must be closed. □

The concept of an acc pt is one of the most useful in topology, not just because it is intimately related to the concept of nearness and thus the the topology of the space, but also because it leads to another characterization of open sets. In fact, many books use the following important result to define closed sets.

<u>5.19 Theorem</u>: A set C in the space $(X, \mathcal{T})$ is open iff $X \setminus C$ is closed.

Proof: ($\leftarrow$) If C is a closed subset of X and $p \notin C$, then it follows from Theorem 5.18 that $\mu(p) \cap \hat{C} = \emptyset$. Thus for each $p \in X \backslash C$, $\mu(p) \subset \hat{X} \backslash \hat{C} = \widehat{X \backslash C}$. Hence, by Theorem 4.17, $X \backslash C$ is open.

($\rightarrow$) Similarly, if $X \backslash C$ is open then $\mu(p) \subset \widehat{X \backslash C}$ for each $p \in X \backslash C$. That is, if $p \notin C$ then $\mu(p) \cap \hat{C} = \emptyset$. Hence C is closed by Theorem 5.18. □

It is worthwhile to note, as shown by Example 5.17, that a set need not be either closed or open. A set can be both closed and open as X and $\emptyset$ always are. Such sets are often called clopen and will be discussed later under the topic of connectedness.

Many textbooks in topology combine Theorem 5.19 and the definition of a topology in order to obtain the following lemma.

5.20 Lemma: In the space $(X, \mathfrak{J})$,

(i) X and $\emptyset$ are closed.

(ii) The intersection of any family $\{C_a : a \in A\}$ of closed subsets of X is again closed.

(iii) The finite union of any family $\{C_k : k = 1, 2, \cdots, n\}$ of closed subsets of X is again closed.

Although the intersection of a finite family of open sets is open, the intersection of an arbitrary family of open sets need not be open. Similarly, the union of an arbitrary family of closed sets need not be closed.

5.21 Example: In (R, E), the set $\{0\}$ is not open but it is the intersection of the family of open sets of the form $(-1/n, 1/n)$ where n is a natural number. In this same space, the set $(0, 2]$ is not closed but it is the union of the family of closed sets of the form $[1/n, 2]$ where n is a natural number.

It is a common mistake to conclude that the derived set of A is closed. That accA need not be closed is shown by the following example.

5.22 Example: Let $(X, \mathfrak{J})$ be the left-ray topology on the reals. If $A = \{0\}$, then $\text{acc}A = \{x : x > 0\}$. Let $B = \text{acc}A$ then $0 \in \text{acc}B$ but $0 \notin B$. Hence $B = \text{acc}A$ is not closed.

By definition, a set is closed if it contains its derived set. The concept that will now be considered is that of forcing a set to contain its acc pts.

5.23 Definition: Let $(X, \mathfrak{J})$ be a space with $A \subset X$. The closure of A is the set $\overline{A} = A \cup \text{acc}A$. To distinguish the closure of A in the spaces $(X, \mathfrak{J})$ and $(X, \mathfrak{J}')$ the names $\mathfrak{J}$-closure $(\overline{A}^{\mathfrak{J}})$ and $\mathfrak{J}'$-closure $(\overline{A}^{\mathfrak{J}'})$ may be incorporated.

5.24 Example: Let $(X, \mathfrak{J})$ be (R, E) while $(X, \mathfrak{J}')$ is the reals with the left-ray topology. If A is chosen to be $\{0\}$, then $\overline{A}^{\mathfrak{J}} = \{0\}$ while $\overline{A}^{\mathfrak{J}'} = \{x : x \geq 0\}$.

This example emphasizes the importance of being cognizant of the topology in question before $\overline{A}$ is formed.

5.25 Theorem: Let $A \subset X$ in the space $(X, \mathfrak{J})$. Then

$$\overline{A} = \{p \in X : A \text{ is near } p\}.$$

Proof: The proof of this theorem follows immediately from the definition of $\overline{A}$ and Corollary 5.7. □

It is obvious that $A \subset \overline{A}$; however, it is almost true that $\overline{A} \subset A$. This is shown by an upcoming lemma. The immediate task is to show that the sets for which $\overline{A} \subset A$ are a very special and familiar type of set.

5.26 Theorem: $A = \overline{A}$ in the space $(X, \mathfrak{J})$ iff A is closed.

Proof: ($\rightarrow$) Assume $A = \overline{A}$. If A is near p, then $p \in \overline{A}$ by the previous theorem. Hence $p \in A$ and so A contains all points that it is near to. A is closed by Theorem 5.18.

($\leftarrow$) Conversely, if A is closed then A contains all points which it is near to. Hence $\overline{A} \subset A$. Since A is always a subset of $\overline{A}$, $A = \overline{A}$. □

Note the similarity between Theorem 5.25 and the next theorem. The following result, which is given without proof, is often a very useful characterization of $\overline{A}$ to use when constructing standard proofs. Perhaps the reason for this is the fact that it expresses the intuitive concept of nearness associated with $\overline{A}$ more clearly than the definition which was given. However, this idea was even more precisely spelled out in the non-standard terminology of Theorem 5.25.

<u>5.27 Theorem</u>: Let A be a set in the space $(X, \mathcal{T})$. Then $p \in \overline{A}$ iff $N \cap A \neq \emptyset$ for each $N \in Np$.

<u>5.28 Lemma</u>: If A is a subset of the closed set C in the space $(X, \mathcal{T})$, then $\overline{A} \subset C$.

Proof: Suppose C is closed and $p \in \overline{A}$. Then A is near p and so $\mu(p) \cap \hat{A} \neq \emptyset$. Now $A \subset C$, so $\mu(p) \cap \hat{C} \neq \emptyset$ and thus C is near p. Hence $p \in C$ since C is closed. Therefore $\overline{A} \subset C$. □

This leads to the following alternative for forming $\overline{A}$. In some texts this criterion is used to define $\overline{A}$.

<u>5.29 Theorem</u>: Let $A \subset X$ in the space $(X, \mathcal{T})$. Then $\overline{A}$ is closed and $\overline{A} = \cap\{C : C \text{ is closed and } A \subset C\}$.

<u>5.30 Theorem</u>: Let $(X, \mathcal{T})$ be a space with A, $B \subset X$. Then the following statements involving closures are true:

(i) $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$.

(ii) $\overline{\overline{A}} = \overline{A}$.

(iii) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(v) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Proof:

(i) X and $\emptyset$ are open so $\emptyset$ and X are closed. Thus by Theorem 5.26 $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$.

(ii) Theorem 5.29 guarantees $\overline{A}$ is closed so $\overline{A}$ equals its closure by Theorem 5.26.

(iii) If $A \subset B$ then $A \subset \overline{B}$ and thus Lemma 5.28 assures $\overline{A} \subset \overline{B}$ since $\overline{B}$ is closed.

(iv) By Theorem 5.25, $p \in \overline{A \cup B}$ iff $A \cup B$ is near p. Now $A \cup B$ is near p iff A or B is near p as shown by Lemma 5.9. This is true iff $p \in \overline{A}$ or $p \in \overline{B}$.

(v) Similarly if $p \in \overline{A \cap B}$, then $A \cap B$ is near p. It is easily shown that both A and B must be near p. Thus p $p \in \overline{A} \cap \overline{B}$. □

5.31 Example: $\overline{A} \cap \overline{B}$ need not be contained in $\overline{A \cap B}$. Let the space be (R, E) and $A = (0, 1)$ while $B = (1, 2)$. Then $\overline{A} \cap \overline{B} = \{1\}$, while $\overline{A \cap B} = \emptyset$.

5.32 Example: Theorem 5.30.iv does not necessarily hold for arbitrary unions. For verification recall Example 5.11.

Using intuition and the previous definitions of nearness as guides, the concept of nearness will once again be generalized.

5.33 Definition: In the space $(X, \mathfrak{J})$, the *set A is near the set B iff A is near some point of B.

To verify that this definition is a generalization of the concept of nearness, examine the following theorem.

5.34 Theorem : The *set A in $(X, \mathfrak{J})$ is near $\{p\}$ where $p \in X$ iff A is near p.

5.35 Theorem: Let A, B, and C be subsets of X in the space $(X, \mathcal{T})$ while p and q are points. Then the following statements are true:

(i) $\{p\}$ is near $\{q\}$ iff p is near q.

(ii) If $A \subset C$ and A is near p, then C is near p.

(iii) If $A \subset C$ and A is near B, then C is near B.

(iv) If $C \subset A$ and A is not near p, then C is not near p.

(v) If $C \subset A$ and A is not near B, then C is not near B.

(vi) If $C \subset A$ and B is not near A, then B is not near C.

(vii) A is near $B \cup C$ iff A is either near B or near C.

(viii) $B \cup C$ is near A iff either B or C is near A.

Proof:

(i) By the previous theorem, $\{p\}$ is near $\{q\}$ iff $\{p\}$ is near q. Now $\{p\}$ is near q iff p is near q by Theorem 5.2.

(ii) If $A \subset C$, then $\hat{A} \subset \hat{C}$. A near p implies $\mu(p) \cap \hat{A} \neq \emptyset$. Hence $\mu(p) \cap \hat{C} \neq \emptyset$, and so C is near p.

(iii) If A is near B, then A is near some point of B. Since $A \subset C$, part (ii) implies C is also near this point. Thus C is near B.

(iv), (v), (vi), (vii), (viii) The proofs of these statements are similar to the proofs of the previous parts. □

To illustrate how well this definition coincides with our intuition about closeness the next result will be presented.

5.36 Theorem: The set A is near the set B in the space $(X, \mathcal{T})$ iff $\overline{A} \cap B \neq \emptyset$.

Proof: A is near B iff A is near p for some point p in B. Now A is near p iff $p \in \overline{A}$. Hence A is near B iff $\overline{A} \cap B \neq \emptyset$. □

If one concentrated upon the words rather than the meaning of Lemma 5.13, it might mistakenly be expected that this result would generalize along with the concept of nearness. It is not difficult to construct examples, such as the following, which show that this is not the case.

<u>5.37 Example</u>: The *set A can be near the set B which is near the point p without A being near p. To see this, let $A = (0, 1)$, $B = (0, 2)$, and $p = 2$ in the space (R, E). Similarly, if $C = (2, 3)$ with A and B as before, then A is near B and B is near C, but A is not near C.

<u>5.38 Example</u>: A may be near B without B being near A. Let $A = \{a\}$ and $B = \{b\}$ in Example 4.14. Since a is near b, A is near B; however, b is not near a, so B is not near A.

The concept that shall now be developed is in some sense a dual concept of that of an acc pt. These interior points, once defined, will determine when a set is open in a manner dual to the way acc pts determine when a set is closed.

<u>5.39 Definition</u>: A point p is an <u>interior point</u> (int pt) of the set A in the space $(X, \mathfrak{J})$ iff $A \in Np$. The set of all int pts of A, $\underline{A}^{o}$, will be called the <u>interior of</u> A.

<u>5.40 Example</u>:

(i) Let $A = [0, 1)$ in the space (R, E). Note that $A^{o} = (0, 1)$.

(ii) Let N be the set of natural numbers in (R, E). Then $N^{\circ} = \emptyset$.

(iii) Let $(X, \mathcal{J})$ be the space of integers with the cofinite topology and let $A = X\setminus\{0\}$. Then $A^{\circ} = A$.

It is obvious from the definition of A° that $A^{\circ} \subset A$ and that for each $p \in A^{\circ}$ there must be an open set containing p and contained in A. This will be recorded for future use in an upcoming theorem, but first examine what this means in non-standard terms. What is suggested is that points close to an int pt of A are also in A. Doing what is suggested often results in a theorem in non-standard analysis; this is the case here.

<u>5.41 Theorem</u>: Let $A \subset X$ in the space $(X, \mathcal{J})$. Then $A^{\circ} = \{p \in X : q$ near p implies $q \mathrel{{}^*\!\!\in} A\}$ (i.e. $A^{\circ} = \{p \in X : \mu(p) \subset \hat{A}\}$).

Proof: ($\rightarrow$) If $p \in A^{\circ}$, then $A \in Np$. Hence $\mu(p) = \bigcap\{\widehat{N} : N \in Np\}$ is a subset of $\hat{A}$.

($\leftarrow$) If $\mu(p) \subset \hat{A}$, then Lemma 4.16 guarantees that $A \in Np$. Hence $p \in A^{\circ}$. $\square$

<u>5.42 Theorem</u>: $A = A^{\circ}$ in the space $(X, \mathcal{J})$ iff A is open.

Proof: ($\rightarrow$) If $A = A^{\circ}$ then $\mu(p) \subset \hat{A}$ for each $p \in A$ and thus A is open by the non-standard characterization of open sets.

($\leftarrow$) If A is open, then $\mu(p) \subset \hat{A}$ for each $p \in A$. Hence $p \in A^{\circ}$ by the previous theorem. Therefore $A \subset A^{\circ}$. Since A° is always a subset of A, $A = A^{\circ}$. $\square$

The previous theorem and Theorem 5.26 illustrate the concept of duality between acc pts and int pts. The following theorems not only serve to illustrate the duality between these concepts but also show that interiors and closures of sets determine each other, much as open and closed sets determine each other.

5.43 Lemma: If O is an open subset of A in the space $(X, \mathfrak{T})$, then $O \subset A^{o}$.

Proof: If $p \in O$, $O \subset A$, and O is open, then $A \in Np$. Hence $p \in A^{o}$ and $O \subset A^{o}$. □

5.44 Theorem: Let $A \subset X$ in the space $(X, \mathfrak{T})$. Then A^{o} is open and $A^{o} = \bigcup\{O : O \text{ is open and } O \subset A\}$.

As a matter of convenience, A' will be used to represent $X\backslash A$ in the next two theorems. For example $(\overline{X\backslash A})^{o}$ will be denoted by A'^{-o}.

5.45 Theorem: Let $(X, \mathfrak{T})$ be a space with $A \subset X$. Then the following statements are true:

(i) $A^{o} = A'^{-\prime}$.

(ii) $A^{o\prime} = A'^{-}$.

(iii) $\overline{A} = A'^{o\prime}$.

This theorem may be used as a tool to establish the next theorem, which is the dual of Theorem 5.30.

5.46 Theorem: Let $(X, \mathfrak{T})$ be a space with $A \subset X$. Then the following statements are true:

(i) $\emptyset^{o} = \emptyset$ and $X^{o} = X$.

(ii) $A^{oo} = A^{o}$.

(iii) If $A \subset B$, then $A^{o} \subset B^{o}$.

(iv) $(A \cap B)^{o} = A^{o} \cap B^{o}$.

(v) $A^{o} \cup B^{o} \subset (A \cup B)^{o}$.

5.47 Example: $(A \cup B)^{o}$ need not be contained in $A^{o} \cup B^{o}$. Let $A = [0, 1)$ and $B = [1, 2]$ in the space (R, E). Then $A^{o} \cup B^{o} = (0, 1) \cup (1, 2)$ while $(A \cup B)^{o} = (0, 2)$.

5.48 Example: Theorem 5.46.iv does not necessarily hold for arbitrary intersections. For each natural number n let $I_n = (-1/n, 1/n)$ in (R, E). Then $I_n^{o} = I_n$ for each n. Hence $\bigcap_{n=1}^{\infty} I_n^{o} = \{0\}$ while $(\bigcap_{n=1}^{\infty} I_n)^{o} = (\{0\})^{o} = \emptyset$.

The next concept that will be examined is that of points which both a set and its complement are close to.

5.49 Definition: A point p is a boundary point (bdry pt) of the set A in the space $(X, \mathfrak{T})$ iff every nbhd of p has a non-void intersection with both A and $X\backslash A$. The boundary of A, $b(A)$, is the set of all bdry pts of A.

5.50 Theorem: The point p is a bdry pt of A in the space $(X, \mathfrak{T})$ iff both A and $X\backslash A$ are near p.

Proof: By definition, $p \in b(A)$ iff $N \cap A \neq \emptyset$ and $N \cap X\backslash A \neq \emptyset$ for each $N \in Np$. This is equivalent to saying that $p \in \overline{A}$ and $p \in \overline{X\backslash A}$. Hence by the non-standard characterization of closures, $p \in b(A)$ iff both A and $X\backslash A$ are near p. □

<u>5.51 Theorem</u>: The point $p \in b(A)$ iff $p \notin A^{o}$ and $p \notin (X\backslash A)^{o}$.

Proof: By the previous theorem it follows that $p \in b(A)$ iff $\mu(p) \cap \hat{A} \neq \emptyset$ and $\mu(p) \cap \widehat{X\backslash A} \neq \emptyset$. Hence $p \in b(A)$ iff $\mu(p) \not\subset \widehat{X\backslash A}$ and $\mu(p) \not\subset \hat{A}$. Therefore $p \in b(A)$ iff $p \notin (X\backslash A)^{o}$ and $p \notin A^{o}$. □

The intimate relationships among the concepts of closure, interior, and boundary are reiterated in a corollary to the previous theorems.

<u>5.52 Corollary</u>: If A is a set in the space $(X, \mathfrak{T})$, then the following statements are true:

(i) $b(A) = \overline{A}\backslash A^{o} = \overline{A} \cap \overline{X\backslash A} = b(X\backslash A)$.

(ii) $X\backslash b(A) = A^{o} \cup (X\backslash A)^{o}$.

(iii) $\overline{A} = A \cup b(A)$.

(iv) $A^{o} = A\backslash b(A)$.

Since the open sets of $(X, \mathfrak{T})$ are the members of the topology and thus determine all topological properties, it is very worthwhile to have various criteria available to determine whether a given set is open or not. Several such criteria have already been given. In particular, a set was shown to be open iff its complement was closed. Two additional criteria shall now be formulated by examining (iii) and (iv) above.

5.53 Corollary: The set A in $(X, \mathcal{J})$ is closed iff $b(A) \subset A$ and A is open iff $A \cap b(A) = \emptyset$. □

Two remaining topics which relate to closures and interiors will now be defined but will not be examined in detail. The first topic is a generalization of a property of the set of rational numbers in the space (R, E). The rational numbers, as Arnold Steffensen [18] has described it, are evenly distributed throughout the set of real numbers in the sense that the rationals are close to every real number. In other words, if Q is the set of rationals in (R, E), then $\overline{Q} = R$.

5.54 Definition: A subset D in the space $(X, \mathcal{J})$ is called dense iff $\overline{D} = X$.

5.55 Example:

(i) As previously mentioned Q is dense in (R, E).

(ii) If A is any non-void set in an indiscrete space, then A is dense.

(iii) If $(X, \mathcal{J})$ is an infinite space with the cofinite topology and A is an infinite subset of X, then A is a dense set.

The following non-standard characterization of dense sets verifies that the intuitive description by Steffensen was indeed accurate.

5.56 Lemma: D is a dense subset of $(X, \mathcal{J})$ iff D is near every point in the space.

Proof: By definition, D is dense iff $\overline{D} = X$. Since $\overline{D}$ contains precisely the points which D is near, D is dense iff D is near each point of X. □

A standard version of this lemma which also indicates this notion of "evenly distributed" is given in the following theorem. This theorem is a particularly useful criterion in real analysis.

<u>5.57 Theorem</u>: A subset D of the space $(X, \mathfrak{J})$ is dense iff each non-void open set contains a point of D.

Proof: $(\rightarrow)$ Suppose O is a non-void open set such that $O \cap D = \emptyset$. If $p \in O$, then $\mu(p) \subset \hat{O}$. Now $\hat{O} \cap \hat{D} = \emptyset$, so $\mu(p) \cap \hat{D} = \emptyset$. Therefore D is not near p and so D cannot be dense.

$(\leftarrow)$ If D is not dense, then $\overline{D} \neq X$. So for some point p, D is not near p. Hence $\mu(p) \subset \widehat{X \backslash D}$. This means that $X\backslash D$ must be a nbhd of D and therefore $X\backslash D$ must contain an open set about p missing D. □

The last topic to be defined is that of a set which may be thought of as not covering much of the space.

<u>5.58 Definition</u>: The set N in the space $(X, \mathfrak{J})$ is <u>nowhere</u> <u>dense</u> iff $\overline{N}^{\circ} = \emptyset$.

<u>5.59 Example</u>: If $(X, \mathfrak{J})$ is an infinite space with the cofinite topology, then all finite subsets of X are nowhere dense.

These last two concepts are related by the following theorem.

<u>5.60 Theorem</u>: N is a nowhere dense subset of $(X, \mathfrak{J})$ iff $X\backslash\overline{N}$ is a dense set.

Proof: N is nowhere dense when $\overline{N}^{\circ} = \emptyset$. By Theorem 5.41, this is true iff $\mu(p) \cap \widehat{(X\setminus\overline{N})} \neq \emptyset$ for each $p \in X$. This is true iff $X\setminus\overline{N}$ is near each $p \in X$. Hence N is nowhere dense iff $X\setminus\overline{N}$ is dense. □

CHAPTER VI

CONTINUOUS FUNCTIONS AND HOMEOMORPHISMS

In this chapter, the concept of continuity which is so important in analysis and the concept of a homeomorphism which is so important in topology will be examined. Functions, which are purely set theoretic, will be examined when there are topologies on both the domain and range spaces. Topologies determine nearness of points and hence, as will be noted shortly, the continuity of functions.

A continuous function often is described loosely as one which preserves closeness of points. That is, if p and q are close then their images, $f(p)$ and $f(q)$, also should be close when f is continuous. Non-standard topology, in fact, characterizes continuity in almost these exact terms. The approach will be to define continuity in the standard fashion, to present the intuitively worded non-standard characterization, and then to develop some theorems with non-standard proofs.

6.1 Definition: A function f from the space $(X, \mathcal{T})$ to the space $(X', \mathcal{T}')$, denoted by $f: X \to X'$ or $f: (X, \mathcal{T}) \to (X', \mathcal{T}')$, is *continuous at the point* p in X iff for each nbhd N of $f(p)$ there exists a nbhd M of p such that $f(M) \subset N$. If $A \subset X$ and f is continuous at each point of A, then f is said to be *continuous on* A. If $A = X$, then f is called a *continuous function* from $(X, \mathcal{T})$ to

$(X', \mathfrak{J}')$. If f is not continuous at p or on A then f is called respectively <u>discontinuous at</u> p or <u>discontinuous on</u> A. A property preserved under a continuous map is called a <u>continuous image property</u>.

<u>6.2 Example</u>:

(i) Every function from a discrete space to an arbitrary space is continuous.

(ii) Every function from an arbitrary space to an indiscrete space is continuous.

(iii) Regardless of the topologies involved, every constant function is continuous.

(iv) The identity function from $(X, \mathfrak{J})$ to $(X, \mathfrak{J})$ is continuous.

Throughout the remainder of this paper neighborhoods of points, monads of points, and nearness of points must be observed more carefully than before. From now on there often will be at least two topological spaces under consideration simultaneously. To avoid unduly complicated notation, the usual notation will be continued without explicit mention of the topology involved since this will be clear from the context.

The next example emphasizes the importance of the topology in determining the continuity of a function. As is shown below, a function may be continuous with respect to some topologies and discontinuous with respect to others.

<u>6.3 Example</u>: Let $X = \{a, b, c\}$, $\mathfrak{J}$ be the indiscrete topology, $\mathfrak{J}'$ be the discrete topology, and f be the identity function on X. Then

$f: (X, \mathfrak{J}) \to (X, \mathfrak{J})$ is continuous while $f: (X, \mathfrak{J}) \to (X, \mathfrak{J}')$ is discontinuous.

The following theorem gives the non-standard characterization of continuity. Its occurence has been heralded in the previous pages; however, a word of warning will be sounded. The *points which are near p may be non-standard as well as standard in this characterization.

<u>6.4 Theorem</u>: A function f from the space $(X, \mathfrak{J})$ into the space $(X', \mathfrak{J}')$ is continuous iff q near p implies $f(q)$ is near $f(p)$.

Proof: ($\to$) Assume that f is a continuous function and that q is near p. Let N be an arbitrary nbhd of f(p). Since f is continuous at each point of X and in particular at p, there exists $M \in Np$ such that $f(M) \subset N$. Now $q \in \hat{M}$ since q is near p, and so $f(q) \in \hat{N}$. N was arbitrary so $f(q) \in \hat{N}$ for each nbhd of $f(p)$. Hence $f(q) \in \mu(f(p))$ and $f(q)$ is therefore near $f(p)$.

($\leftarrow$) Assume that q near p implies $f(q)$ is near $f(p)$, but f is not continuous. Since f is not continuous, there is a nbhd N of $f(p)$ such that $M = f^{-1}(N)$ is not a nbhd of p. Hence by the non-standard characterization of nbhds, $\mu(p) \not\subset \hat{M}$. Let $q \in \mu(p) \setminus \hat{M}$. Since q is near p, $f(q)$ is near $f(p)$ by assumption. Hence $f(q) \in \hat{N}$ and therefore $q \in \hat{M}$. From this contradiction it follows that f must be continuous. □

<u>6.5 Corollary</u>: The function f which maps $(X, \mathfrak{J})$ onto $(X', \mathfrak{J}')$ is continuous iff A is a *subset of X near $p \in X$ implies $f(A)$ is near $f(p)$. Sinilarly, f is continuous iff the *subset A of X near $B \subset X$ implies $f(A)$ is near $f(B)$.

Proof: Assume that f is a continuous function from X onto X' and that A is a *subset of X near $p \in X$. Then some *point q of A is near p. By the previous theorem, f(q) must be near f(p). This implies that f(A) is near f(p). Conversely, assume that if A is any *subset of X near $p \in X$ then f(A) is near f(p). Then, in particular, when $A = \{q\}$, $\{q\}$ near p implies $f(\{q\}) = f(q)$ is near f(p). That is, if q is near p then f(q) is near f(p). Therefore f is continuous by the previous theorem.

Similarly, assume that f is continuous and that the *set A is near B. Then A must be near some point $p \in B$. By the preceding proof, f(A) must be near f(p). Hence f(A) is near f(B). Conversely, assume the *set A near $B \subset X$ implies f(A) is near f(B). Then, in particular, this is true when $B = \{p\}$. It follows from the preceding proof that f is continuous. □

Thus together Theorem 6.4 and Corollary 6.5 say that a function is continuous iff it preserves nearness.

The next theorem is a frequently used result in analysis which is given to indicate the ease with which some non-standard proofs of theorems may be written.

<u>6.6 Theorem</u>: If f is a continuous function from $(X, \mathfrak{T})$ onto $(X', \mathfrak{T}')$ and g is a continuous function from $(X', \mathfrak{T}')$ to $(X'', \mathfrak{T}'')$, then the composition function $g \circ f$ is also a continuous function.

Proof: If f is a continuous function from $(X, \mathfrak{T})$ onto $(X', \mathfrak{T}')$ and q is near p, then f(q) is near f(p). Since g is a continuous function from $(X', \mathfrak{T}')$ to $(X'', \mathfrak{T}'')$ and $f(q)$ is near $f(p)$, it follows that $(g \circ f)(q) = g(f(q))$ is near $g(f(p)) = (g \circ f)(p)$. Thus $g \circ f$ is a continuous function from $(X, \mathfrak{T})$ to $(X'', \mathfrak{T}'')$. □

Earlier several different criteria were given to determine when a set is open. The next theorem similarly gives several standard criteria for determining when a function is continuous.

6.7 Theorem: If f is a function from $(X, \mathfrak{T})$ to $(X', \mathfrak{T}')$, then the following statements are equivalent:

(i) f is a continuous function.

(ii) If $O \in \mathfrak{T}'$ then $f^{-1}(O) \in \mathfrak{T}$.

(iii) If B is a basic open set in $(X', \mathfrak{T}')$, then $f^{-1}(B)$ is open in $(X, \mathfrak{T})$.

(iv) If S is a subbasic open set in $(X', \mathfrak{T}')$, then $f^{-1}(S)$ is open in $(X, \mathfrak{T})$.

(v) If C is closed in $(X', \mathfrak{T}')$, then $f^{-1}(C)$ is closed in $(X, \mathfrak{T})$.

(vi) If $A \subset X$, then $f(\overline{A}) \subset \overline{f(A)}$.

(vii) If $B \subset X'$, then $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

Proof: (i) $\rightarrow$ (ii) Assume that f is continuous and that $O \in \mathfrak{T}'$. Let p be an arbitrary point of $f^{-1}(O)$. If q is near p then $f(q)$ is near $f(p)$ by Theorem 6.4. Since O is open about $f(p)$, the non-standard characterization of open sets guarantees that $f(q) \in \hat{O}$. Since q is a *point whose image is a *point of O, it must be true that $q \; {}^*\!\in f^{-1}(O)$. Hence $f^{-1}(O)$ is open for it contains all *points near the arbitrary point p. By standard techniques, it is not difficult to show that each part of this theorem implies the next part until (vi) implies (vii). It can then be shown that (vii) implies (i) in order to complete the proof. □

6.8 Example: Set equality need not hold in either (vi) or (vii) of the previous theorem. For (vi) let $f: (X, \mathcal{T}) \rightarrow (R, E)$ be defined by $f(x) = 1/x$ where $(X, \mathcal{T})$ is the usual subspace of positive real numbers. Also let $A = \{x : x > 1\}$. Then $\overline{A}^{\mathcal{T}} = \{x : x \geq 1\}$, $f(A) = \{y : 0 < y < 1\}$, $f(\overline{A}^{\mathcal{T}}) = \{y : 0 < y \leq 1\}$, and $\overline{f(A)}^{E} = \{y : 0 \leq y \leq 1\}$. For (vii) let $X = \{a, b, c\}$, $\mathcal{T}$ be the discrete topology and $\mathcal{T}'$ the indiscrete topology for X. Let f be the identity function from $(X, \mathcal{T})$ to $(X', \mathcal{T}')$ and $B = \{a, b\}$. Then $\overline{B}^{\mathcal{T}'} = X$ and so $f^{-1}(\overline{B}^{\mathcal{T}'}) = X$ while $\overline{f^{-1}(B)}^{\mathcal{T}} = B$.

In the very special case of (R, E), the usual space of real numbers, different definitions of continuity are often given. Not all of these are equivalent in an arbitrary space, however. As an example, continuity defined in terms of sequences will now be examined.

6.9 Definition: A sequence $(x_1, x_2, \cdots, x_3, \cdots)$ in $(X, \mathcal{T})$ is said to converge to the point $p \in X$ iff each nbhd of p contains all but a finite number of the terms of the sequence. This will be denoted by $\lim x_n = p$.

In particular, a sequence $(x_1, x_2, \cdots, x_n, \cdots)$ in (R, E) converges to the point p iff each nbhd $S(p, \nabla)$ of p contains all but a finite number of the terms of the sequence. Thus by the definition of $S(p, \nabla)$ the sequence converges iff for each real number $\nabla > 0$

there exists an $m \in N$ such that $n > m$ implies $|x_n - p| < \nabla$. Although we try to impress upon freshman calculus students that it is not permissible to place infinity into an expression, what this definition is intuitively saying is that terms infinitely far out in the sequence are infinitesimally close to the limit.

Before showing that in some sense the student is correct to substitute infinity into the expression, consider the following example. For clarity, the sequences (which are functions from N to R) will be examined using functional notation.

<u>6.10 Example</u>: Using the standard definition of convergence, it is easy to verify that the sequence defined by $f(n) = 1/n$ converges to zero in (R, E). Now proceed as in Example 3.17 and evaluate this function at the infinite natural number given by $\bar{x} = (1, 2, 3, \cdots, n, \cdots)$. $f^*(\bar{x}) = (1, 1/2, 1/3, \cdots, 1/n, \cdots)$ which is an infinitesimal. That is, the sequence f is infinitely close to the limit zero when evaluated at $\bar{x}$.

<u>6.11 Theorem</u>: The sequence f converges to p in (R, E) iff $f(\bar{x})$ is near p for each infinite natural number $\bar{x}$.

Proof: $(\rightarrow)$ Assume that the sequence f converges to p and that $\nabla > 0$. Then there exists an $m_\nabla \in N$ such that for each natural number $n > m_\nabla$, $|f(n) - p| < \nabla$. If $\bar{x}$ is an infinite natural number, then certainly $\bar{x} > m_\nabla$. Hence $|f^*(\bar{x}) - p| < \nabla$. The standard natural number m_∇ depends on the standard real number ∇, but regardless of the m_∇ in question, the infinite $\bar{x}$ will be greater than m_∇. Hence $|f^*(\bar{x}) - p| < \nabla$ for every non-zero real number ∇. Thus $f^*(x)$ is near p.

($\leftarrow$) Conversely, assume that for each infinite natural number $\bar{x}$ that $f^*(\bar{x})$ is near p. That is, $|f^*(\bar{x}) - p| < \nabla$ for each non-zero $\nabla \in R$. Hence given ∇, the following sentence is true for R^*:

$$\exists m\ \forall n[(n \in N \wedge n > m) \rightarrow |f(n) - p| < \nabla].$$

Reinterpreted for R, the sentence says that the sequence f converges to p. □

Although, a sequence (in an arbitrary space) which converges to p may be described loosely as becoming very close to p, it need not be the case that any standard point in the sequence is near p.

<u>6.12 Example</u>: The sequence defined by $f(n) = 1/n$ converges to 0 in (R, E); however, $f(n) \notin \mu(0)$ for any $n \in N$. Hence no point in the sequence is near 0.

<u>6.13 Example</u>: A sequence may converge to more than one point. Let $(X, \mathfrak{T})$ be the real numbers with the cofinite topology, and consider the sequence defined by $f(n) = n$. If $p \in X$ then for any $M \in Np$, $X \setminus M$ is finite. Hence M contains all but a finite number of the terms of the sequence. Therefore this sequence converges to p for each $p \in X$.

<u>6.14 Proposition</u>: If $(X, \mathfrak{T})$ is a space with the co-countable topology, then $\lim x_n = p$ iff for some natural number m, $x_n = p$ for $n > m$.

Proof: Clearly if $x_n = p$ for $n > m$ for some m, then each nbhd of p contains all but a finite number of the terms of the sequence. Conversely, if $\lim x_n = p$, then each nbhd of p contains all but a finite number of the terms of the sequence. Let $N = O \cup \{p\}$ where $O = X \setminus \{x_1, x_2, \cdots, x_n, \cdots\}$. N is an open nbhd of p and the only term of the sequence that N can contain is p. Since N contains all but a finite number of the terms of the sequence, x_n must be p for $n > m$ for some m. □

6.15 Definition: Let $f: (X, \mathfrak{J}) \to (X', \mathfrak{J}')$. The function f is called sequentially continuous at $p \in X$ iff for each sequence $(x_1, x_2, \cdots, x_n, \cdots)$ in X which converges to p, the sequence $(f(x_1), f(x_2), \cdots, f(x_n), \cdots)$ in X' converges to f(p). If f is sequentially continuous at each point of X, then f is called sequentially continuous.

6.16 Example: Let $f: (R, E) \to (R, E)$ be defined by $f(x) = 2x$. Suppose $(s_1, s_2, \cdots, s_n, \cdots)$ converges to s. Let $\overline{m}$ be an arbitrary infinite natural number. Then $|s_{\overline{m}} - s|$ is either zero or some infinitesimal $\overline{y}$. Thus $|2s_{\overline{m}} - 2s| = 2|s_{\overline{m}} - s|$ is zero or some infinitesimal $2\overline{y}$. Hence $(2s_1, 2s_2, \cdots, 2s_n, \cdots) = (f(s_1), f(s_2), \cdots, f(s_n), \cdots)$ converges to f(2s). Therefore f is sequentially continuous. The following theorem and example compare continuity and sequential continuity.

6.17 Theorem: If $f: (X, \mathfrak{J}) \to (X', \mathfrak{J}')$ is continuous then f is sequentially continuous.

<u>6.18 Example</u>: A sequentially continuous function need not be continuous. Let $X = \{x \in R : 1 \leq x \leq 3\}$ be given the co-countable topology $\mathfrak{J}$, and let $X' = \{x \in R : 1 \leq x \leq 2\}$ be given the relative topology $\mathfrak{J}'$ as a subspace of (R, E). Define $f: (X, \mathfrak{J}) \rightarrow (X', \mathfrak{J}')$ by

$$f(x) = \begin{cases} x & \text{if } x \in X' \\ 1 & \text{if } x \notin X'. \end{cases}$$

If $(x_1, x_2, \cdots, x_n, \cdots)$ converges to p in $(X, \mathfrak{J})$, then by Proposition 6.14 $x_n = p$ for $n > m$ for some m. Hence $(f(x_1), f(x_2), \cdots, f(x_n), \cdots)$ has $f(x_n) = f(p)$ for $n > m$. Thus this sequence converges to $f(p)$ in $(X', \mathfrak{J}')$ and so f is sequentially continuous. However, f is not continuous. To see this let $M = (1, 2)$ be a subset of X'. Note that M is open in $(X', \mathfrak{J}')$, but $f^{-1}(M) = M$ is not open in $(X, \mathfrak{J})$ since $X \backslash M$ is not countable.

Two other types of functions which might appear to be closely related to continuous functions will be defined. The definitions are given so that an outside reference will not be needed, but there is actually no general relationship among the three types of functions. Since this would necessitate six examples to verify their independence, the examples will be omitted.

<u>6.19 Definition</u>: A function f from the space $(X, \mathfrak{J})$ to the space $(X', \mathfrak{J}')$ is called an <u>open function</u> if $f(O) \in \mathfrak{J}'$ for each $O \in \mathfrak{J}$. If the image of every closed set in $(X, \mathfrak{J})$ is closed in $(X', \mathfrak{J}')$, then f is called a <u>closed function</u>.

<u>6.20 Theorem</u>: The function $f: (X, \mathfrak{T}) \to (X', \mathfrak{T}')$ is open iff $\mu(p) \subset \hat{N}$ implies $\mu(f(p)) \subset \widehat{f(N)}$ for each $N \subset X$.

Proof: ($\to$) Assume that f is an open function and suppose that $\mu(p) \subset \hat{N}$. By the non-standard characterization of nbhds, N is a nbhd of p. Hence there exists an open set $O \subset N$ about p. Since f is an open function, $f(O)$ is open. Thus $\mu(f(p)) \subset \widehat{f(O)}$. Since $\widehat{f(O)} \subset \widehat{f(N)}$, it follows that $\mu(f(p)) \subset \widehat{f(N)}$.

($\leftarrow$) Assume that $\mu(p) \subset \hat{N}$ implies $\mu(f(p)) \subset \widehat{f(N)}$, and suppose that O is open. If $q \in f(O)$, then $q = f(p)$ for some $p \in O$. Since O is open, $\mu(p) \subset \hat{O}$. Thus by assumption $\mu(f(p)) \subset \widehat{f(O)}$. That is, $\mu(q) \subset \widehat{f(O)}$. Hence $f(O) \in \mathfrak{T}'$, and so f is an open function. □

The following lemma is a result given by Machover and Hirschfeld and is useful in establishing the next theorem.

<u>6.21 Lemma</u>: If $\mu(f(p)) \subset f(\mu(p))$ then $\mu(p) \subset \hat{N}$ implies $\mu(f(p)) \subset \widehat{f(N)}$.

Proof: Assume that $\mu(f(p)) \subset f(\mu(p))$ and suppose $\mu(p) \subset \hat{N}$. If $q \in \mu(f(p))$, then $q \in f(\mu(p))$. Since $\mu(p) \subset \hat{N}$, $f(\mu(p)) \subset f(\hat{N})$ which equals $\widehat{f(N)}$. Thus $q \in \widehat{f(N)}$ and the desired inclusion follows. □

By observing both continuous functions and open functions, it may be noted that an open function takes open sets to open sets while the inverse of a continuous function brings open sets back to open sets. Upon the basis of this comparison, it might be expected that a function would be open if q is near p whenever $f(q)$ is near $f(p)$. This expectation could arise since a continuous function maps points near p to points near $f(p)$. This result follows from the previous theorem and lemma in this paper.

<u>6.22 Theorem</u>: Let f be a function from $(X, \mathfrak{J})$ onto $(X', \mathfrak{J}')$. If q is near p whenever $f(q)$ is near $f(p)$ then f is an open function.

Proof: Assume that q is near p whenever $f(q)$ is near $f(p)$. Since f is onto, if $x \in \mu(f(p))$ then $x = f(q)$ for some *point q of X, and so by assumption $q \in \mu(p)$. If $q \in \mu(p)$ then $f(q) \in f(\mu(p))$, i.e. $x \in f(\mu(p))$. Hence $\mu(f(p)) \subset f(\mu(p))$. Therefore by the previous lemma, $\mu(p) \subset \hat{A}$ implies $\mu(f(p)) \subset \widehat{f(A)}$. Thus the previous theorem guarantees that f is an open function. □

<u>6.23 Example</u>: The converse of the previous theorem does not hold. Let $f: (R, E) \to (X, \mathfrak{J})$ be the characteristic function for the rationals where $X = \{0, 1\}$ and $\mathfrak{J}$ is the indiscrete topology on X. The function f is open; however, $f(2) = 1$ is near $0 = f(\pi)$, while 2 is not near π.

To emphasize the importance of f being onto in the previous theorem, the next example is presented.

<u>6.24 Example</u>: If q is near p whenever $f(q)$ is near $f(p)$ then it does not necessarily follow that f is open. Let $(R, \mathfrak{J})$ be the reals with the indiscrete topology and define $f: (R, \mathfrak{J}) \to (R, E)$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

Since $f(R) = \{0, 1\}$ which is not an element of E, f is not open. However, $f(q)$ near $f(p)$ certainly implies q is near p. For regardless of the point p in question, q is near p in the indiscrete space $(R, \mathfrak{J})$.

The topic that shall now be investigated is that of identifying the similarity of topological properties of a space and its image under a mapping. Thus what needs closer consideration is the identification of the concept of nearness by the function. If $f: (X, \mathfrak{J}) \to (X, \mathfrak{J}')$ is the identity map from a discrete space ~~to~~ an indiscrete space, then f is continuous. Further, if q is near p then $f(q)$ is near $f(p)$; however, $f(q)$ may be near $f(p)$ without q being near p. Thus the continuity of f insures that f preserves nearness of points. To guarantee that f^{-1} also preserves nearness, it would be necessary for f^{-1} to also be continuous.

There is a Greek word "homoiomorph" which means of similar form or structure. From this comes the terminology which is used in the next definition to identify spaces of similar structure.

6.25 Definition: If $f: (X, \mathfrak{J}) \to (X', \mathfrak{J}')$ is a one-to-one continuous function from X onto X' such that f^{-1} is also continuous, then f is called a homeomorphism and the spaces are said to be homeomorphic. A property of a space preserved under a homeomorphism is called a topological property.

Topology is sometimes described as the study of topological properties. To one unfamiliar with the development required to define a topological property, this description must surely sound circular. Nonetheless, within the context of this paper, this description is an accurate one. Topology is the study of the abstract equivalence of spaces due to the nearness of points. The spaces may be different when examined under other structures, for example under an algebraic structure; however, from a topological viewpoint they are indistinguishable.

<u>6.26 Example</u>:

(i) Each space is homeomorphic to itself since the identity map is a homeomorphism from $(X, \mathfrak{T})$ to $(X, \mathfrak{T})$.

(ii) The usual space of real numbers (R, E) is homeomorphic to the subspace (A, E_A) where $A = (0, 1)$. A homeomorphism from this subspace to (R, E) is given by $f(x) = \frac{2x-1}{x(x-1)}$.

As part (ii) of the previous example shows, neither length nor distance is a topological property.

<u>6.27 Theorem</u>: Let $f: (X, \mathfrak{T}) \to (X', \mathfrak{T}')$. Then saying f is a homeomorphism is equivalent to stating that f is a 1-1 function from X onto X' and q is near p iff $f(q)$ is near $f(p)$.

Proof: The function f is continuous iff q near p implies $f(q)$ is near $f(p)$. Likewise the function f^{-1} is continuous iff $f(q)$ near $f(p)$ implies q is near p. The theorem follows. □

<u>6.28 Theorem</u>: A function $f: (X, \mathfrak{T}) \to (X', \mathfrak{T}')$ is a homeomorphism iff f is a continuous, one-to-one, open function from X onto X'.

Proof: ($\rightarrow$) If f is a homeomorphism then f(q) near f(p) implies q is near p. Thus by the non-standard characterization of an open function, f is open. The remaining conditions follow from the definition of a homeomorphism.

($\leftarrow$) If f is open and $O \in \mathfrak{J}$, then the conditions imply that $(f^{-1})^{-1}(O) = f(O) \in \mathfrak{J}'$. Hence f^{-1} is continuous and so f is a homeomorphism. □

One example due to Kuratowski will be given to show that, although spaces may be homeomorphic, care must still be exercised in choosing a function that actually performs the identification between the spaces.

<u>6.29 Example</u>: The function f from the space given below onto itself is one-to-one and continuous, but it is not a homeomorphism. Consider the following subspaces of (R, E), $(X, \mathfrak{J})$ and $(X', \mathfrak{J}')$ where X consists of all intervals of the form (3n, 3n+1) and all points 3n + 2 for n a non-negative integer. $X' = (X\backslash\{2\}) \cup \{1\}$. Define

$$h: (X, \mathfrak{J}) \rightarrow (X', \mathfrak{J}', \quad \text{by} \quad h(x) = \begin{cases} x \text{ if } x \neq 2 \\ 1 \text{ if } x = 2 \end{cases}$$

and

$$g: (X', \mathfrak{J}') \rightarrow (X, \mathfrak{J}) \quad \text{by} \quad g(x) = \begin{cases} x/2 & \text{if } x \leq 1 \\ (x/2) - 1 & \text{if } 3 < x < 4 \\ x - 3 & \text{if } x \geq 5. \end{cases}$$

Note that h, g are one-to-one, continuous, and onto. Hence so is the composition $f = g \circ h$. However f is not a homeomorphism of $(X, \mathfrak{J})$ onto $(X, \mathfrak{J})$ since {2} is open but $f(\{2\}) = \{1/2\}$ is not open.

CHAPTER VII

CONNECTED SPACES

The major theme of this chapter will be the concept of connectedness. Following the usual procedure, the concept opposite connectedness will first be defined. Once separated sets have been defined, both of these concepts will be examined in non-standard terms.

From an intuitive viewpoint, sets are connected if they cannot be severed into distinct pieces. This is, of course, a very loose description since it is not clear what is meant by sever, by distinct, or by a piece. Nonetheless, this description does indicate that the emphasis here seems to be more upon non-closeness rather than upon closeness.

This aside has been given not merely to give a preliminary feel for the definitions which follow, but also to provide a point of reference. When the non-standard characterization for these concepts is given, it then can be noted once again how incisively the non-standard terminology portrays the ideas represented by these names.

7.1 Definition: Let $(X, \mathfrak{T})$ be a space with $A, B \subset X$. Then A and B are separated from each other iff $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. A and B separate X or form a separation of $(X, \mathfrak{T})$ iff A and B are non-void separated sets whose union is X.

<u>7.2 Example</u>:

(i) In the space (R, E), $(0, 1)$ and $(1, 2)$ are separated but do not form a separation of (R, E). Likewise, $(0, 1)$ and $[2, 3)$ are separated but do not form a separation of (R, E). The sets $(0, 1)$ and $[1, 2)$ are not separated in (R, E).

(ii) If $(X, \mathcal{T})$ is a discrete space of more than one point, then A and $X\backslash A$ separate X when $\emptyset \neq A \neq X$.

(iii) Let $(X, \mathcal{T})$ be the three point space in Example 4.2.i. Then $\{a\}$ and $\{b, c\}$ separate X.

<u>7.3 Theorem</u>: A and B are separated iff neither A nor B is near the other.

Proof: A and B are separated iff $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. Thus by Theorem 5.36, A and B are separated iff neither A nor B is near the other. □

<u>7.4 Theorem</u>: Let $(X, \mathcal{T})$ be a space, and let A, B, and C be subsets of X. Then the following statements are true:

(i) $\emptyset$ and A are separated.

(ii) If $C \subset A$ and A and B are separated, then B and C are separated.

(iii) If A and B are separated and A and C are separated, then A and $B \cup C$ are separated.

Proof:

(i) A is not near $\emptyset$ and $\emptyset$ is not near A so A and $\emptyset$ are separated.

(ii) If A and B are separated then B is not near A. Since $C \subset A$, Theorem 5.35 guarantees that B is not near C. Also A is not near B. Hence C cannot be near B. Thus B and C are separated.

(iii) If A is separated from both B and C, then A is not near either set. Thus A cannot be near their union. Likewise neither B nor C is near A so their union cannot be near A. Thus A and $B \cup C$ are separated. □

The next theorem gives sufficient conditions that are sometimes useful in standard proofs to establish when two sets are separated in a space.

<u>7.5 Theorem</u>: Let $(X, \mathfrak{T})$ be a space with A and B subsets of X. If there exist disjoint open sets M and N such that $A \subset M$ and $B \subset N$, then A and B are separated.

Proof: Let a be an arbitrary element in A. Since M is an open set containing A, $\mu(a) \subset \hat{M}$. Now $M \cap N = \emptyset$ implies $\hat{M} \cap \hat{N} = \emptyset$. Hence $\mu(a) \cap \hat{N} = \emptyset$, and so N is not near a. Therefore N is not near A. Thus B is not near A since $B \subset N$. Similarly A is not near B. This means that A and B are separated. □

<u>7.6 Example</u>: The converse of the previous theorem is not valid. Let $X = \{a, b, c\}$ with $\mathfrak{T} = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. If $A = \{a\}$ and $B = \{b\}$, then A and B are separated. However the smallest open sets

containing A and B are respectively $\{a, c\}$ and $\{b, c\}$ which have $\{c\}$ as their intersection.

7.7 Definition: The space $(X, \mathfrak{T})$ is connected iff there does not exist a separation of $(X, \mathfrak{T})$. Stated positively, $(X, \mathfrak{T})$ is connected iff $B = \emptyset$ whenever $X = A \cup B$, $A \neq \emptyset$, and A and B are separated. $(X, \mathfrak{T})$ is called disconnected if it is not connected. A subset Y of X is connected if $(Y, \mathfrak{T}_Y)$ is a connected space.

7.8 Theorem: The space $(X, \mathfrak{T})$ is connected iff $A \neq \emptyset \neq B$ and $A \cup B = X$ implies either A is near B or B is near A.

Proof: ($\rightarrow$) Assume $(X, \mathfrak{T})$ is connected. Then the non-void sets A and B cannot separate X. Since A and B are not separated Theorem 7.3 guarantees either A is near B or B is near A.

($\leftarrow$) If $A \neq \emptyset \neq B$ and $A \cup B = X$ implies either A is near B or B is near A, then A and B are not separated. Hence there does not exist a separation of $(X, \mathfrak{T})$. Therefore $(X, \mathfrak{T})$ is connected. □

7.9 Example:

(i) Discrete spaces of more than one point are disconnected.

(ii) Indiscrete spaces are connected.

7.10 Definition: A subset of the space $(X, \mathfrak{T})$ that is both open and closed is called clopen.

This terminology will be used in the next theorem which gives several equivalent standard criteria for determining connectedness.

<u>7.11 Theorem:</u> The following statements are equivalent in the space $(X, \mathfrak{J})$:

(i) $(X, \mathfrak{J})$ is connected.

(ii) The only clopen subsets of $(X, \mathfrak{J})$ are $\emptyset$ and X.

(iii) If $(X', \mathfrak{J}')$ is a discrete space with $X' = \{1, 0\}$, then there does not exist a continuous function from $(X, \mathfrak{J})$ onto $(X', \mathfrak{J}')$.

(iv) X cannot be represented as the union of two non-void disjoint open sets.

Proof: (i) $\rightarrow$ (ii). Suppose that $A \neq X$ is a non-void clopen subset of the connected space $(X, \mathfrak{J})$. Then $X\backslash A$ is also a clopen subset of X which is not equal to X or $\emptyset$, but it is disjoint from A. Hence by Theorem 7.5, A and $X\backslash A$ separate $(X, \mathfrak{J})$. This would mean $(X, \mathfrak{J})$ is disconnected, hence the conclusion follows.

(ii) $\rightarrow$ (iii). Suppose that $f: (X, \mathfrak{J}) \rightarrow (X', \mathfrak{J}')$ is continuous and onto. Then $f^{-1}(\{0\})$ is unequal to $\emptyset$ or X and it is clopen. Since this contradicts (ii), the conclusion follows.

(iii) $\rightarrow$ (iv). Suppose X could be represented by the union of the non-void open sets A and $X\backslash A$. Then the characteristic function of A, $f: (X, \mathfrak{J}) \rightarrow (X', \mathfrak{J}')$, which is defined by

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in X\backslash A \end{cases}$$

is a continuous function from $(X, \mathfrak{T})$ onto $(X', \mathfrak{T}')$. Since this contradicts (iii), the conclusion follows.

(iv) $\rightarrow$ (i). Suppose that $(X, \mathfrak{T})$ is disconnected, then there exist non-void sets A and B which separate X. Hence neither A nor B is near the other. It is then clear that $A \cap B = \emptyset$. It also follows that A and B must both contain all points they are near to. Thus A and B are closed. Since they must be complements of each other, both A and B are open. This contradicts (iv); hence the conclusion follows. □

The next theorem and example relate connectedness to some properties defined in earlier chapters. This will be the style throughout the remaining portion of this paper. Once a concept has been defined and examined with regard to non-standard criteria, it will then be tested to see what properties it possesses.

7.12 Theorem: Connectedness is both a continuous image property and a topological property.

Proof: Assume that f is a continuous function from the connected space $(X, \mathfrak{T})$ onto the space $(X', \mathfrak{T}')$. Suppose that $(X', \mathfrak{T}')$ is disconnected. Then there exist subsets A and B of X' which form a separation of $(X', \mathfrak{T}')$. Thus neither A nor B is near the other, and so by Corollary 6.5 neither $f^{-1}(A)$ nor $f^{-1}(B)$ is near the other. After examining $f^{-1}(A)$ and $f^{-1}(B)$ carefully, it follows that they are not only separated but also separate X. This means that $(X, \mathfrak{T})$ is disconnected which contradicts the hypothesis. Hence $(X', \mathfrak{T}')$

must be connected. Therefore connectedness is preserved under a continuous map and so certainly under a homeomorphism. □

7.13 Example: Connectedness is not an hereditary property. Consider the subspace of (R, E) given by (A, E_A) where $A = B \cup C$, $B = [0, 1)$, and $C = (2, 3)$. Since A is the union of two non-void disjoint relatively open subsets, A is separated by B and C.

7.14 Lemma: If C is a connected subset of $A \cup B$ in the space $(X, \mathfrak{J})$ where A and B are separated, then either $C \subset A$ or $C \subset B$.

Proof: Assume C is a connected subset of $A \cup B$ where A and B are separated. Since neither A nor B is near the other, it follows that neither $A \cap C$ nor $B \cap C$ is near the other. Thus $A \cap C$ and $B \cap C$ are separated subsets of the connected set C. Hence one of $A \cap C$ and $B \cap C$ is empty, and so either $C \subset A$ or $C \subset B$. □

The following example and theorem show that the union of a collection of connected sets need not be connected, but by restricting the sets in the collection a connected set may always be obtained.

7.15 Example: $A = (0, 1)$ and $B = (2, 3]$ are both connected subsets of (R, E), but $A \cup B$ is disconnected.

7.16 Theorem: Let F be a family of connected subsets of $(X, \mathfrak{J})$ such that no two members of F are disjoint. Then $\cup\{C : C \in F\}$ is also connected.

Proof: Let $S = \cup\{C : C \in F\}$ and suppose that A and B separate S. Now by Lemma 7.14, each $C \in F$ is either a subset of A or of B.

Without loss of generality assume $C \subset A$. Then for any other $C' \in F$, $C \cap C' \neq \emptyset$ implies C' is also a subset of A. Hence $S \subset A \subset S$ and $B = \emptyset$. Therefore A and B cannot have separated S after all, and so S is connected. □

7.17 Theorem: If C is a connected subset of $(X, \mathfrak{T})$ such that $C \subset A \subset \overline{C}$, then A is also connected.

Proof: Suppose that B and D separate A. If $C \subset A \subset \overline{C}$, then $C \subset B \cup D \subset \overline{C}$. Since neither B nor D is empty, they each contain some point of $\overline{C}$. Thus C is near both B and D by Theorem 5.36. However, by Lemma 7.14 $C \subset B$ or $C \subset D$. Thus either C and D or C and B are separated. Hence either C is not near D or C is not near B. From this contradiction it follows that A is connected. □

7.18 Corollary: The closure of a connected subset of $(X, \mathfrak{T})$ is also connected.

Although a space may be disconnected, there will always exist connected subsets since the singletons will be connected. Therefore it is always possible to express the space as a union of connected sets; however, it may well be possible to express the space as the union of connected sets with more elements. The concept that will now be examined is that of maximal connected sets.

7.19 Definition: A component of a space $(X, \mathfrak{T})$ is a connected subset A of X such that $A \subset B \subset X$ with B connected implies $B = A$.

7.20 Theorem: Each component of a space is closed.

Proof: If A is a component of $(X, \mathfrak{T})$, then A is connected. Thus by Corollary 7.18, $\overline{A}$ is connected. By the maximality of A, $A = \overline{A}$. Hence A is closed. □

By considering the union of the family of connected subsets containing $p \in X$, the proof of the next theorem follows in the standard fashion.

7.21 Theorem: The components of $(X, \mathfrak{T})$ partition X.

In some sense, it would seem that the number of components of a space could be used to gauge how connected or disconnected the space is. This is partially verified by the following theorem.

7.22 Theorem: A space $(X, \mathfrak{T})$ is connected iff $(X, \mathfrak{T})$ has only one component.

Proof: $(\rightarrow)$ Assume that $(X, \mathfrak{T})$ is connected and that A is an arbitrary component in the space. Since $A \subset X$ and A is maximal, $A = X$. Thus X is the unique component of $(X, \mathfrak{T})$.

$(\leftarrow)$ Conversely, assume that A is the only component of X. Since the components of the space partition X, X is the union of the components. Hence $X = A$, and so X is connected. □

CHAPTER VIII

THE SEPARATION AXIOMS

As remarked earlier, topology is sometimes defined as the study of topological properties; i.e. those properties preserved by homeomorphisms. Theorem 6.4 and Corollary 6.5 have shown that a continuous function is one which preserves nearness. Thus it would seem that a function would have a better chance of being continuous when the domain space has few points near other points and when the range space has many points near other points. In terms of open sets, a function is more likely to be continuous when the domain space has many open sets and the range space has relatively few open sets. As an example of this, recall that each function from a discrete space to an arbitrary space is continuous and each function from an arbitrary space to an indiscrete space is also continuous.

From the preceding discussion, it may seem that many properties of a space are related to the cardinality of its topology. In this chapter, a collection of concepts known collectively as the separation axioms will be examined. Roughly, the separation axioms gauge the availability of open sets for use in separating points, separating points from closed sets, and separating closed sets. The reader is advised that not all texts use the same names or definitions for these axioms. The terminology here will agree with that of Kelley [4] but will disagree with that of Steen [17].

<u>8.1 Definition</u>: A space $(X, \mathfrak{J})$ is a <u>T_0-space</u> (<u>Kolomogorov space</u>) if whenever $p, q \in X$, $p \neq q$, there exists $V \in \mathfrak{J}$ such that either $p \in V$ and $q \notin V$ or else $q \in V$ and $p \notin V$.

Intuitively speaking, what has been said is that no two distinct points of a T_0-space are close to each other. In non-standard terms, this is stated rigorously in the following theorem.

<u>8.2 Theorem</u>: $(X, \mathfrak{J})$ is T_0 iff no two distinct points of X are near each other.

Proof: $(\rightarrow)$ Assume that $(X, \mathfrak{J})$ is T_0 and that p and q are distinct points in the space. Then suppose without loss of generality that N is a neighborhood of p such that $q \notin N$. Since q is standard, $q \notin \hat{N}$ and hence $q \notin \mu(p)$. That is, q is not near p and hence p and q cannot be near each other.

$(\leftarrow)$ Assume that p and q are arbitrary distinct points in $(X, \mathfrak{J})$ and that p and q are not near each other. Then without loss of generality, q is not near p. Hence $q \notin \mu(p) = \bigcap\{\hat{N} : N$ is a nbhd of $p\}$. Thus there exists a nbhd N of p such that $q \notin \hat{N}$. Since q is standard, $q \notin N$. Thus $(X, \mathfrak{J})$ is T_0. □

Examples related to the separation axioms will be given after all levels of separation have been defined. If these definitions are new to you, it might be advisable to examine the examples immediately after reading these definitions.

<u>8.3 Definition</u>: A space $(X, \mathfrak{J})$ is a <u>T_1-space</u> (<u>Frechet space</u>) if

whenever $p, q \in X$, $p \neq q$, there exist $V, W \in \mathfrak{T}$ such that $p \in V$, $q \notin V$ and $q \in W$, $p \notin W$.

Thus a T_1-space is one in which each of two distinct points in the space has an open nbhd missing the other point. Hence the following theorem could be used to intuitively describe the space.

<u>8.4 Theorem</u>: $(X, \mathfrak{T})$ is T_1 iff no point of X is near another point.

Proof: $(\rightarrow)$ Let p be any point in the T_1-space $(X, \mathfrak{T})$. Then for any other $q \in X$, there exists a neighborhood N about q such that $p \notin N$. Since p is a standard point, $p \notin \hat{N}$. Hence $p \notin \mu(q)$ and p is not near q.

$(\leftarrow)$ Let p, q be arbitrary elements in X. Assuming no point is near another, $p \notin \mu(q)$. Hence there exists an open neighborhood V about q such that $p \,{}^*\!\notin V$. Since p is standard, $p \notin V$. Similarly there exists a neighborhood V about p missing q. Hence $(X, \mathfrak{T})$ is a T_1-space. □

<u>8.5 Theorem</u>: Every T_1-space is a T_0-space.

Proof: Assume that p and q are arbitrary distinct points of the T_1-space $(X, \mathfrak{T})$. Then neither p nor q can be near the other. Hence certainly they are not near each other. Thus the space is T_0. □

The next theorem is sometimes a useful criterion for determining when a space is T_1. Its proof recalls some earlier non-standard ideas.

<u>8.6 Theorem</u>: A space $(X, \mathfrak{T})$ is T_1 iff every singleton subset of X is closed.

Proof: ($\rightarrow$) Assume that $p \in X$ in the T_1-space $(X, \mathfrak{J})$. If q is any other point of X, then p is not near q. Hence $\{p\}$ cannot be near q. Thus $\{p\}$ contains all points of X that it is near to, and so $\{p\}$ is closed.

($\leftarrow$) Conversely, assume that $\{p\}$ is closed for each $p \in X$. Then $\{p\}$ is not near any other point q of X. Hence p is not near q. Thus $(X, \mathfrak{J})$ is T_1. □

Example 5.22 showed that accA did not have to be a closed set. The next theorem claims, however, that all T_1-spaces do have accA closed for each subset A. The standard proof of this theorem will be omitted. The policy of this paper is to omit some proofs when a standard proof seems more direct and concise than a contrived non-standard proof. In the case of the following theorem (as well as Theorem 5.29) another point needs to be made. No non-standard proof of this theorem was discovered. Perhaps a lengthy indirect proof using non-standard techniques does exist, but it seems quite unlikely that a concise non-standard proof does exist. The reason for this lies in the definition of the concept of nearness. Nearness is a very intuitive idea, a very useful concept in non-standard analysis, and a fundamental notion of topology. It is not as fundamental, though, as the notion of a neighborhood upon which its definition relies. One standard proof of the following theorem uses the definition of a T_1-space and certain nbhds to produce a straightforward proof. Several obvious attempts to prove this theorem using non-standard terminology, and hence nearness, failed.

The reason for this seems to be that nearness is not a basic enough notion to get at the heart of this relatively straightforward idea. Since nearness to *points is meaningless, it is impossible using the concept of nearness to relate, through accA, any nearness of A to a *point p which is an acc pt of accA. The reader certainly is invited to attempt a proof of the following theorem in order to clarify some of the nebulous comments made above.

<u>8.7 Theorem</u>: If $A \subset X$ in the T_1-space $(X, \mathfrak{J})$ then accA is closed.

The following theorem also is given to show that, although non-standard analysis is a valuable tool, it is not the answer to all problems. The proof of this theorem relies upon the concept of finiteness and the fundamental concept of a neighborhood; these are standard ideas. Therefore, the role of non-standard analysis should be to supplement, but not to replace, standard analysis.

<u>8.8 Theorem</u>: Given $A \subset X$ in the T_1-space $(X, \mathfrak{J})$, $p \in$ accA iff every open nbhd of p contains an infinite number of points of A.

<u>8.9 Definition</u>: A space $(X, \mathfrak{J})$ is a <u>T_2-space</u> (<u>Hausdorff space</u>) if whenever $p, q \in X$ and $p \neq q$, there exist disjoint open sets V and W such that $p \in V$ and $q \in W$.

Thus a T_2-space is one in which there exist distinct neighborhoods for each pair of distinct points. Therefore the separation is more restrictive than saying that two points cannot be close. The separation will not even allow distinct points to be close to the same point.

In non-standard terms, the monads of different points are disjoint, as is shown in the following theorem.

<u>8.10 Theorem</u>: A space $(X, \mathfrak{T})$ is Hausdorff iff no *point is near two distinct points of X.

Proof: $(\rightarrow)$ If $(X, \mathfrak{T})$ is T_2, then for any $p, q, p \neq q$ in X there exist neighborhoods V and W of p and q respectively such that $V \cap W = \emptyset$. This implies that $\widehat{V \cap W} = \hat{\emptyset}$, and hence $\hat{V} \cap \hat{W} = \emptyset$. Thus $\mu(p) \cap \mu(q) = \emptyset$, i.e. no *point can be near both p and q.

$(\leftarrow)$ If no *point is near two distinct points of X, then $\mu(p) \cap \mu(q) = \emptyset$ for distinct p and q in X. While it would be true that $\hat{V} \cap \hat{W} = \emptyset$ implies $V \cap W = \emptyset$ for standard V and W, it is not permissible to conclude immediately from $\mu(p) \cap \mu(q) = \emptyset$ that neighborhoods V and W exist such that $V \cap W = \emptyset$. Therefore, exercising greater caution, note that since the nbhd systems Np and Nq are filters, there exist infinitesimal *nbhds V and W of p and q respectively. That is, $V \,{}^*\!\in Np$ and $W \,{}^*\!\in Nq$ such that $\hat{V} \subset NucNp$ which equals $\mu(p)$ and $\hat{W} \subset NucNq$ which equals $\mu(q)$. Thus the following sentence is true in $\mathfrak{A}^*$:

$$\exists x\, \exists y[x \in Np \wedge y \in Nq \wedge \nexists z[z \in x \wedge z \in y]].$$

Since Np and Nq are standard objects, the interpretation in $\mathfrak{A}$ must also be true. Therefore p and q have disjoint nbhds. Hence $(X, \mathfrak{T})$ is T_2. □

<u>8.11 Theorem</u>: $(X, \mathfrak{T})$ is Hausdorff iff for each $p \in X$, $\cap\{M : M \in Np$ and M is closed$\} = \{p\}$.

Proof: $(\rightarrow)$ Assume that $(X, \mathfrak{T})$ is T_2 and that $p \in X$. If $q \neq p$ is another point of X, then there exist disjoint open sets V and W such that $p \in V$ and $q \in W$. Then $p \in V$ and $V \subset X \backslash W$ imply that $X \backslash W$ is a closed nbhd of p missing q. Hence $q \notin \cap\{M : M \in Np$ and M is closed$\}$ for $q \neq p$. Therefore this intersection must be $\{p\}$.

$(\leftarrow)$ Assume for each $p \in X$ that $\{p\} = \cap\{M : M \in Np$ and M is closed$\}$. If $p \in X$ and q is distinct from p, then there exists a closed nbhd M of p such that $q \notin M$. Hence M is not near q and so no *point of M is near q. Now if r is an arbitrary *point near p then $r \in \mu(p)$ which is contained in $\hat{M}$. Therefore r is not near q, and so $(X, \mathfrak{T})$ is T_2 by the non-standard characterization of T_2-spaces. □

<u>8.12 Theorem</u>: Every T_2-space is also a T_1-space and a T_0-space.

Proof: Assume that p and q are arbitrary distinct points in the T_2-space $(X, \mathfrak{T})$. Both p and q are points (and so are *points) which are near themselves and hence in the T_2-space neither can be near the other. Thus $(X, \mathfrak{T})$ is T_1 and hence must be T_0. □

As was shown in Example 6.13, sequences may converge to more than one point in an arbitrary space. It is well known that this is not true in (R, E). The property of (R, E) that prevents this is the T_2-space property.

<u>8.13 Theorem</u>: In a Hausdorff space limits of convergent sequences are unique.

T_2-spaces are very nice spaces since they behave similarly to (R, E) in so many respects. Another example of this is given by the following proposition.

8.14 Proposition: If f and g are continuous functions into a Hausdorff space, then $\{x : f(x) \neq g(x)\}$ is open.

Proof: Assume that f and g are continuous functions from $(X, \mathfrak{T})$ into the T_2-space $(X', \mathfrak{T}')$. Let $V = \{x : f(x) \neq g(x)\}$ and suppose q is near $p \in V$. Since f and g are continuous, f(q) and g(q) are near f(p) and g(p) respectively. If $f(q) = g(q)$, then that *point would be near both f(p) and g(p) in a T_2-space. By assumption $f(p) \neq g(p)$, therefore it cannot be that the same *point is near two distinct points. Hence $f(q) \neq g(q)$, and so $q \;{}^*\!\in V$. That is, $\mu(p) \subset \hat{V}$. Therefore V is open. □

8.15 Definition: A space $(X, \mathfrak{T})$ is an R-space (regular space) if whenever $p \in X$ and F is a closed subset of X such that $p \notin F$, then there exist disjoint open sets V and W such that $p \in V$ and $F \subset W$. A space $(X, \mathfrak{T})$ is a T_3-space if it is a regular T_1-space.

Thus a space is regular if every closed set and every point not contained in the closed set can be separated by disjoint open sets.

8.16 Theorem: A space $(X, \mathfrak{T})$ is regular iff for each $p \in X$ and for each open nbhd V of p, there exists an open nbhd W of p such that $p \in W$ and $\overline{W} \subset V$.

8.17 Corollary: The space $(X, \mathfrak{T})$ is regular iff the closed nbhds of p form a nbhd base for each $p \in X$.

<u>8.18 Theorem</u>: A space $(X, \mathcal{T})$ is an R-space iff for each point p and each *point q not near p there exist disjoint open sets V and W such that $p \in V$ and $q \; {}^*\!\in W$.

Proof: $(\rightarrow)$ Assume that p is a point and that q is a *point in the R-space $(X, \mathcal{T})$ such that q is not near p. Let Mp denote the family of closed nbhds of p. Since Mp is a nbhd base for Np, $\mu(p) = \cap\{\hat{M} : M \in Mp\}$ by Lemma 4.23. Since $q \notin \mu(p)$, there must exist a closed nbhd M of p such that $q \notin \hat{M}$. Let $V = M^o$ and let $W = X \backslash M$. Then $q \; {}^*\!\in W$ and $p \in V$ since p is an int pt of M. Further, $V \cap W = \emptyset$.

$(\leftarrow)$ Suppose that the space $(X, \mathcal{T})$ is not regular. Then there exists a point p of X such that the family Mp of closed nbhds of p does not form a nbhd base for Np. Thus there must be some nbhd N of p such that no $M \in Mp$ is a subset of N. Therefore if $M \in Mp$, there is some $q \in M$ such that $q \notin N$. Now the intersection of a finite number of closed nbhds of p is again a closed nbhd of p. Hence the relation defined by the formula $x \in Mp \wedge y \in x \wedge y \notin N$ is concurrent. It follows that there is some *point q such that $q \; {}^*\!\notin N$ but $q \; {}^*\!\in M$ whenever $M \in Mp$. Thus $q \in \cap\{\hat{M} : M \in Mp\}$ which equals Nuc Mp, but $q \notin$ Nuc Np which equals $\mu(p)$. Thus q is a *point which is not near p. Now let V be any open set about p. Since $\overline{V} \in Mp$, $q \; {}^*\!\in \overline{V}$ and cannot be a *point of $(X\backslash V)^o$. Hence there cannot exist an open set W containing q and disjoint from V. The statement follows by contraposition. □

<u>8.19 Theorem</u>: Every T_3-space is a T_2-space.

Proof: Assume that p and q are distinct points in the T_3-space $(X, \mathfrak{T})$. Since $(X, \mathfrak{T})$ is T_3, it is by definition T_1. Hence $\{p\}$ is a closed set. Therefore, by regularity, there exist disjoint open sets containing p and q. Thus $(X, \mathfrak{T})$ is Hausdorff. □

<u>8.20 Definition</u>: A space $(X, \mathfrak{T})$ is an <u>N-space</u> (<u>a normal space</u>) if for disjoint closed sets F and G, there exist disjoint open sets V and W such that $F \subset V$ and $G \subset W$. A space is a <u>T_4-space</u> iff it is a normal T_1-space.

<u>8.21 Theorem</u>: A space $(X, \mathfrak{T})$ is normal iff for each closed set F and open set V containing F, there exists an open set W such that $F \subset W$ and $\overline{W} \subset V$.

<u>8.22 Theorem</u>: A space $(X, \mathfrak{T})$ is normal iff for every two *points p and q such that $p \ {}^*\!\in F$ and $q \ {}^*\!\in G$ for some disjoint closed sets F and G, there exist disjoint open sets V and W such that $p \ {}^*\!\in V$ and $q \ {}^*\!\in W$.

Proof: $(\rightarrow)$ Assume that p and q are *points in the N-space $(X, \mathfrak{T})$ and that there are closed sets F and G such that $p \ {}^*\!\in F$ and $q \ {}^*\!\in G$. Since F and G are disjoint closed sets, there exist by the normality of $(X, \mathfrak{T})$ disjoint open sets V and W such that $F \subset V$ and $G \subset W$. Hence $p \ {}^*\!\in V$ and $q \ {}^*\!\in W$.

$(\leftarrow)$ Assume that $(X, \mathfrak{T})$ is not an N-space. Then there must exist disjoint closed sets F and G that cannot be separated by open sets. Thus if V is an open set containing F, the open set $X \setminus \overline{V}$ cannot contain G. Therefore $\overline{V} \cap G \neq \emptyset$. Now the intersection of a finite number of open sets containing F is also an open set containing F. Thus the closure of the intersection of a finite number of such sets must contain a point of G. Hence the intersection of their closures will contain a point of G. Therefore the relation defined by the following formula is concurrent: $x \in \mathfrak{I} \wedge F \subset x \wedge y \in \overline{x} \wedge y \in G$. It follows that there must be some *point q such that $q \,{}^*\!\in G$ and such that $q \,{}^*\!\in \overline{V}$ for each open set V containing F.

Now let S be the family of all open sets A such that $q \,{}^*\!\in A$. If $A \in S$ then $\overline{A} \cap F \neq \emptyset$. Otherwise $X \setminus \overline{A} = V$ would be an open set containing F while $q \,{}^*\!\notin \overline{V}$. This is a contradiction.

Now the intersection of a finite number of open sets with q as a *element is also an open set with q as a *element. Hence the relation defined by the following formula is concurrent:

$$x \in S \wedge y \in F \wedge y \in \overline{x}.$$

It follows that there must exist some *point p such that $p \,{}^*\!\in F$ and $p \,{}^*\!\in \overline{A}$ whenever $A \in S$. Therefore there cannot exist disjoint open sets V and W such that $p \,{}^*\!\in V$ and $q \,{}^*\!\in W$. Otherwise, if V is an open set with q as a *element, then $\overline{V}$ has p as a *element and so $p \,{}^*\!\notin (X \setminus V)^{\circ}$. □

<u>8.23 Theorem</u>: Every T_4-space is a T_3-space.

Proof: Assume that p is not an element of the closed set F in the T_4-space $(X, \mathfrak{J})$. Since $(X, \mathfrak{J})$ must be a T_1-space, $\{p\}$ is a closed set disjoint from F. Therefore normality assures disjoint open sets V and W such that $p \in V$ and $F \subset W$. Hence $(X, \mathfrak{J})$ is a T_3-space. □

The following examples are included for completeness so that a novice will have readily available examples of spaces which satisfy some of the separation axioms but not others.

<u>8.24 Example</u>:

(i) The indiscrete space $(X, \mathfrak{J})$ where X is not a singleton is not a T_0-space.

(ii) The space $(X, \mathfrak{J})$ where $X = \{a, b, c\}$ and $\mathfrak{J} = \{\emptyset, \{a\}, \{b, c\}, X\}$ is not T_0, but neither is it an indiscrete space.

(iii) The space $(X, \mathfrak{J})$ where $X = \{a, b\}$ and $\mathfrak{J} = \{\emptyset, \{a\}, X\}$ is T_0 but not T_1. The left-ray topology on the Reals also has this property.

(iv) The cofinite space $(X, \mathfrak{J})$ where X is infinite is T_1 but not T_2.

(v) The space $(X, \mathfrak{J})$ where X is the set of real numbers and $\mathfrak{J}$ is the topology subgenerated by the family I of intervals (a, b) and the set Q of rational numbers is T_2 but is not T_3. That $(X, \mathfrak{J})$ is not T_3 may be seen by examining the point zero and the closed set $X \backslash Q$.

(vi) The space $(X, \mathfrak{T})$ of part (ii) is regular and normal but it is not T_0 and hence not T_1, T_2, T_3, or T_4.

(vii) The space $(X, \mathfrak{T})$ of part (iii) is normal but not regular.

(viii) Let X be the upper-half plane together with the x-axis, i.e. $X = \{(x, y) : x \in R, y \in R, \text{ and } y \geq 0\}$. A basis for a topology $\mathfrak{T}$ for X is B where B consists of all open spheres in X (sets of form $\{(x, y) : y > 0, (x - x_0)^2 + (y - y_0)^2 < r\}$ where $x_0 \in R$ and $y_0, r > 0$) together with sets of the form $S \cup \{(x, 0)\}$ where S is an open sphere tangent to the real axis at the point $(x, 0)$. $(X, \mathfrak{T})$ is regular but not normal. To see that $(X, \mathfrak{T})$ is not normal, examine Q the set of rationals and I the set of irrationals on the x-axis, R. It may be seen that every subset of R contains all points that it is near to. Thus both Q and I must be closed. Since there do not exist disjoint open sets about Q and I, the result follows.

(ix) Every discrete space is T_4 and hence satisfies all the separation axioms defined in this paper.

(x) The space (R, E) is not discrete, but it does satisfy all of the separation axioms defined in this paper.

As mentioned at the beginning of this chapter, the separation axioms gauge, in some sense, the availability of open sets in the space. Thus it should not be too surprising to find that each of the separation axioms is a topological property. As the proofs are very similar, only the following theorem will be proven in this paper.

8.25 Theorem: The homeomorphic image of a Hausdorff space is a Hausdorff space.

Proof: Assume that $(X, \mathfrak{J})$ is T_2 and that $f: (X, \mathfrak{J}) \rightarrow (X', \mathfrak{J}')$ is a homeomorphism from X onto X'. Let p' and q' be distinct points of $(X', \mathfrak{J}')$. Suppose r' is a *point near both p' and q'. Since f is a homeomorphism, $r = f^{-1}(r')$ is near both $p = f^{-1}(p')$ and $q = f^{-1}(q')$. This is a contradiction since $(X, \mathfrak{J})$ is T_2. Therefore $(X', \mathfrak{J}')$ is T_2. □

8.26 Example: The identity function from the discrete space $(X, \mathfrak{J})$ to the indiscrete space $(X, \mathfrak{J}')$ is continuous. If X contains more than one point, then $(X, \mathfrak{J})$ satisfies all of the separation axioms while $(X, \mathfrak{J}')$ satisfies none of them. Hence, none of the separation axioms is a continuous image property.

All of the separation axioms defined in this paper, with the exception of normality, are hereditary properties. One proof will be given here to illustrate the proof of this claim.

8.27 Theorem: Every subspace of a T_3-space is also a T_3-space.

Proof: Assume that $(A, \mathfrak{J}_A)$ is a subspace of the T_3-space $(X, \mathfrak{J})$, that $p \in A$, and that q is a *point of A not near p. Since $(X, \mathfrak{J})$ is T_3, there exist disjoint open sets V, W in $(X, \mathfrak{J})$ such that $p \in V$ and $q \,{}^*\!\in W$. Now $p \in V \cap A$ and $q \,{}^*\!\in W \cap A$. Since these sets are disjoint and open in $(A, \mathfrak{J}_A)$, the subspace is also T_3. □

CHAPTER IX

COMPACT SPACES

As mentioned at the beginning of Chapter IV, many of the concepts of topology have evolved as generalizations of concepts associated with the particular topological space (R, E). The main concern of this chapter will be the concept of compactness. Compactness in an arbitrary space is another case where the conclusion of an important theorem (in particular the Heine-Borel Theorem) of real analysis becomes a definition in topology. In (R, E) there are several different equivalent criteria for determining the compactness of a set, not all of which generalize or are equivalent in an arbitrary space.

If a compact set is intuitively thought of as a set in which the points are packed fairly close together, then the following standard definition and non-standard characterization are portrayed justly. Much of the importance of compactness is derived from the well behaved way in which continuous functions act upon compact sets.

9.1 Definition: A family $\mathcal{C}$ of subsets of X is called a covering of $(X, \mathcal{T})$ iff $X = \cup\{A : A \in \mathcal{C}\}$. If $\mathcal{C}'$ is a subfamily of $\mathcal{C}$ which also covers X, then $\mathcal{C}'$ is a subcovering of $(X, \mathcal{T})$. In this case $\mathcal{C}$ is said to be reducible to the subcovering $\mathcal{C}'$. By an abuse of language, a family of open sets which covers X is called an <u>open covering</u> of $(X, \mathfrak{J})$. If the covering $\mathcal{C}$ has only a finite number of members, then $\mathcal{C}$ is called a <u>finite covering</u>.

<u>9.2 Definition</u>: A space $(X, \mathfrak{J})$ is <u>compact</u> iff every open covering of $(X, \mathfrak{J})$ is reducible to a finite subcovering. A is a <u>compact subset</u> of X iff $(A, \mathfrak{J}_A)$ is a compact subspace.

For emphasis note that the quantifier in the preceding definition is "every". If just any open covering reducible to a finite subcovering would make the space compact, then each space would be compact. This may be seen by taking the subfamily $\{X\}$ of the family $\mathfrak{J}$.

Machover and Hirschfeld call the next theorem by Robinson one of the most important and useful theorems of non-standard analysis. Their proof will be given below, but to fully appreciate their description of the theorem one would need to pursue deeper results than this paper will present.

<u>9.3 Theorem</u>: A subset K of the space $(X, \mathfrak{J})$ is compact iff every *point of K is near some standard point of K.

Proof: $(\rightarrow)$ Assume that K is a compact subset of $(X, \mathfrak{J})$ and that $q \;{}^*\!\in K$. Suppose, however, that q is not near any point of K. Then for each $p \in K$ there exists some open nbhd M^p of p such that $q \;{}^*\!\notin M^p$. Now the family $\{M^p : p \in K\}$ clearly is an open cover of the compact set K, and so there exists a finite subcovering $\{M^{p_1}, M^{p_2}, \cdots, M^{p_n}\}$ of K. Thus the following sentence is true in $\mathfrak{A}$ and hence in $\mathfrak{A}^*$: $\forall x[x \in K \rightarrow x \in M^{p_1} \vee x \in M^{p_2} \vee \cdots \vee x \in M^{p_n}]$. Thus the interpretation in $\mathfrak{A}^*$ will not allow q to be a *point of K; for else q would be a *element of M^{p_j} for some $j = 1, 2, \cdots, n$. This is a contradiction since the original assumption was that $q \,{}^*\!\in K$; hence the supposition that q was not near any point of K must be incorrect.

($\leftarrow$) Assume that K is not compact. Then let $\mathcal{C}$ be an open covering of K which is not reducible to a finite subcovering. Thus for each finite subfamily $\{O_1, O_2, \cdots, O_n\}$ of $\mathcal{C}$ there exists a $p \in K$ such that $p \notin O_j$ for $j = 1, 2, \cdots, n$. Hence the following formula defines a concurrent relation: $x \in \mathcal{C} \wedge y \in K \wedge y \notin x$. Therefore there exists a *point q such that $q \,{}^*\!\in K$ but q is not a *element of any $O \in \mathcal{C}$. Now $\mathcal{C}$ covers K, so for each $p \in K$ there is some $O \in \mathcal{C}$ such that $p \in O$. This means that O is an open set about p such that $q \notin \hat{O}$; therefore q is not near p. Thus if K is not compact there is a *point q of K which is not near any point $p \in K$. □

<u>9.4 Example</u>:

(i) The sets $\{0\}$ and $[2, 3]$ are compact subsets of (R, E), but (R, E) is not compact. This follows since the open covering consisting of the intervals $(n, n + 2)$ when n is an integer it is not reducible to a finite subcovering.

(ii) Every indiscrete space is compact.

(iii) Finite discrete spaces are compact while infinite discrete spaces are not compact.

(iv) Every cofinite space is compact.

<u>9.5 Example</u>: Compact sets need not be closed in an arbitrary topological space. Recall the left-ray topology on the reals. In this

space, $\{0\}$ is compact but it is not closed.

It is well known, however, that compact subsets of (R, E) are closed. The next theorem presents the topological property of (R, E) that makes this true.

9.6 Theorem: Every compact subset of a Hausdorff space is closed.

Proof: Assume that K is a compact subset of the T_2-space $(X, \mathfrak{J})$. Suppose that K is near the point p. Then some *point r of K must be near p. Since K is compact, this *point r must be near some point q of K. Now the space is T_2, so r cannot be near two distinct points. Hence $p = q$ which is an element of K. K must therefore be closed since it contains all points that it is near to. □

9.7 Example: The closure of a compact set need not be compact. Consider the set $A = (-\infty, 0]$ in the space of Reals with the left-ray topology. A is compact since any basic open set containing the point 0 also covers A. However, $\bar{A}$ is the whole space which is not compact. This may be seen by deliberating upon the open covering consisting of all rays of the form $(-\infty, r)$ where r is a real number.

9.8 Example: The subset $[0, 1)$ of the compact subset $[0, 1]$ of (R, E) is not compact. Hence compactness is not an hereditary property.

The next corollary gives sufficient restrictions upon subsets of compact sets to force them to also be compact. Thus this property of (R, E) does generalize to arbitrary spaces.

<u>9.9 Theorem</u>: The intersection of a compact set and a closed set is compact.

Proof: Assume that K is compact and that C is closed in $(X, \mathcal{T})$. Let $r \ {}^*\!\in K \cap C$. Since r is a *point in the compact set K, r must be near p for some $p \in K$. Now C is closed, so C must also contain p. It follows that $K \cap C$ is compact since there exists a $p \in K \cap C$ such that r is near p. □

<u>9.10 Corollary</u>: Every closed subset of a compact set is compact.

The next few theorems serve to illustrate the nice behavior of compact sets when acted upon by continuous functions.

<u>9.11 Theorem</u>: Compactness is both a continuous image and a topological property.

Proof: Assume that f is a continuous function from $(X, \mathcal{T})$ to $(X', \mathcal{T}')$ and that K is a compact subset of X with image K' under f. Suppose that r' is a *element of K'. Then $r' = f(r)$ for some $r \ {}^*\!\in K$. Since K is compact, r is near p for some $p \in K$. Thus, in K', $r' = f(r)$ is near $p' = f(p)$ since f is continuous. The existence of the point $p' \in K'$ which r' is near to implies that K' is compact. Thus compactness is preserved by a continuous function and so certainly it is preserved by a homeomorphism. □

<u>9.12 Corollary</u>: The continuous image of a compact set K from $(X, \mathcal{T})$ to the Hausdorff space $(X', \mathcal{T}')$ is closed.

Proof: By the previous theorem, $f(K)$ is compact. Then by Theorem 9.6, $f(K)$ is closed since $(X', \mathfrak{J}')$ is T_2. □

The following theorem can be very useful when one is working with functions into T_2-spaces.

<u>9.13 Theorem</u>: A one-to-one continuous function from a compact space onto a T_2-space is a homeomorphism.

Proof: Assume that $f: (X, \mathfrak{J}) \to (X', \mathfrak{J}')$ is a continuous function from a compact space onto a T_2-space. To establish that f is a homeomorphism, it will suffice to show that f is open, i.e. that $f(r)$ near $f(p)$ implies r is near p. Thus suppose that $f(r)$ is near $f(p)$ in the image space. Since f is onto, there exist $r^* \in X$ and $p \in X$ which are the preimages of $f(r)$ and $f(p)$ respectively. The compactness of $(X, \mathfrak{J})$ guarantees that r is near some point $q \in X$. By the continuity of f, $f(r)$ is near $f(q)$ in the T_2-space $(X', \mathfrak{J}')$. Hence $f(q) = f(p)$, since $f(r)$ cannot be near two distinct points in a T_2-space. Since f is 1-1, $q = p$. Therefore r is near p, and the desired conclusion thus follows. □

As an example of the power of this theorem, consider the following proposition.

<u>9.14 Proposition</u>: Let $(X, \mathfrak{J})$, $(X, \mathfrak{J}')$ and $(X, \mathfrak{J}'')$ be three distinct spaces with $\mathfrak{J}''$ strictly stronger than $\mathfrak{J}'$ which is strictly stronger than $\mathfrak{J}$. If $(X, \mathfrak{J}')$ is a compact T_2-space, then $(X, \mathfrak{J}'')$ is Hausdorff but not compact, while $(X, \mathfrak{J})$ is compact but not Hausdorff.

Proof: Assume that $(X, \mathfrak{J}')$ is a compact T_2-space with $\mathfrak{J}$ strictly weaker than $\mathfrak{J}'$ and $\mathfrak{J}''$ strictly stronger than $\mathfrak{J}'$. It is then clear that $(X, \mathfrak{J}'')$ is Hausdorff, since $\mathfrak{J}''$ is stronger than $\mathfrak{J}'$. Now let f be the identity map from $(X, \mathfrak{J}')$ to $(X, \mathfrak{J})$. Since the inverse of any open set is open, f is continuous. $(X, \mathfrak{J})$ is thus compact, for it is the continuous image of a compact set. To see that $(X, \mathfrak{J})$ is not T_2, again consider the function f. By the previous theorem, if $(X, \mathfrak{J})$ is T_2, then f would be a homeomorphism. This is impossible since $\mathfrak{J}$ is distinct from $\mathfrak{J}'$. Similarly, the identity function g from $(X, \mathfrak{J}'')$ to $(X, \mathfrak{J}')$ is a continuous function, but not a homeomorphism. Therefore $(X, \mathfrak{J}'')$ cannot be compact or else g would be a 1-1 continuous function of a compact space onto a T_2-space. □

There are several other types of compactness which might be investigated; however, the only other type to be examined in this paper will be that of local compactness.

<u>9.15 Definition</u>: A space $(X, \mathfrak{J})$ is <u>locally compact at a point</u> p of X if there exists a compact nbhd of p in $(X, \mathfrak{J})$. If $(X, \mathfrak{J})$ is locally compact at each point of X, then $(X, \mathfrak{J})$ is called a <u>locally compact space</u>. A is a <u>locally compact subset</u> of X iff $(A, \mathfrak{J}_A)$ is a locally compact subspace.

<u>9.16 Example</u>:

(i) All discrete, indiscrete, and cofinite spaces are locally compact.

(ii) (R, E) is locally compact but not compact.

(iii) The family of intervals of the form $[a, b)$ where $a, b \in R$ is a base for a topology on the reals Let Υ be the topology generated by this base. Then it may be shown that (R, Υ) is not locally compact. (R, Υ) will henceforth be called the closed-left-interval topology for R.

Now recall that a near-standard *point of $(X, \mathfrak{T})$ is a *point which is near some point of X. The space $(X, \mathfrak{T})$ was shown to be compact precisely when each *point of X is near some point of X. Thus each *point of a compact space is near-standard. However, in spaces which are not compact there are *points which are not near-standard. The relationship between compactness and near-standard points is examined further in the next theorem which also characterizes locally compact spaces. But first, a lemma will be given so that the proof of the theorem may be expedited.

<u>9.17 Lemma</u>: If $\mathfrak{F}$ is a filter in the space $(X, \mathfrak{T})$ and Np is the nbhd system of $p \in X$, then $\text{Nuc}\ \mathfrak{F} \cap \mu(p) \neq \emptyset$ iff $F \cap N \neq \emptyset$ for each $F \in \mathfrak{F}$ and $N \in \text{Np}$ (i.e. iff p is a contact point of $\mathfrak{F}$).

Proof: Let G denote the filter generated by $\mathfrak{F} \cup \text{Np}$. Using the fact that $\text{Nuc}\ G = \text{Nuc}\ \mathfrak{F} \cap \text{Nuc Np}$ which equals $\text{Nuc}\ \mathfrak{F} \cap \mu(p)$, it follows that $G \neq \mathcal{P}(X)$ precisely when $\text{Nuc}\ \mathfrak{F} \cap \mu(p) \neq \emptyset$. For if $G = \mathcal{P}(X)$ then clearly $\text{Nuc}\ G = \emptyset$, and if G is properly contained in $\mathcal{P}(X)$ then Nuc G properly contains $\text{Nuc}\ \mathcal{P}(X)$ which equals $\emptyset$. Since $G \neq \mathcal{P}(X)$, $\emptyset \notin G$. $\mathfrak{F} \cup \text{Np}$ generates G, hence there cannot exist $F \in \mathfrak{F}$ and $N \in \text{Np}$ such that $F \cap N = \emptyset$. □

<u>9.18 Theorem</u>: A space $(X, \mathfrak{T})$ is locally compact iff every near-standard *point of X is a *point of some compact subset of X.

Proof: $(\rightarrow)$ Assume that $(X, \mathfrak{T})$ is locally compact and that q is a near-standard *point of X. Then by definition, q must be near some point p. Since $(X, \mathfrak{T})$ is locally compact, there exists a compact nbhd K of p. Hence $\mu(p) \subset \hat{K}$, and so certainly $q \; {}^*\!\!\in K$.

$(\leftarrow)$ Assume that $p \in X$ and that every near-standard *point of X is a *element of some compact subset of X. Let $\mathfrak{F}$ be the filter generated by the family $\mathbf{C}$ of sets whose complements are compact. Note that, since the union of two compact sets is compact, the intersection of the complements of two compact sets is again the complement of a compact set. Hence $\mathbf{C}$ constitutes a base for the filter $\mathfrak{F}$.

By definition, $\text{Nuc}\ \mathfrak{F} = \bigcap\{\hat{F} : F \in \mathfrak{F}\}$. If r is a near-standard *point, then by assumption $r \; {}^*\!\!\in X\backslash F$ for some $F \in \mathfrak{F}$. Therefore $r \; {}^*\!\!\notin \text{Nuc}\ \mathfrak{F}$. Hence Nuc $\mathfrak{F}$ can contain only remote *points. On the other hand, if r is remote *point then r is not a *point of any compact set. Thus $\text{Nuc}\ \mathfrak{F} = \{r : r \text{ is a remote *point of } X\}$.

From this it follows that $\mu(p) \cap \text{Nuc}\ \mathfrak{F} = \emptyset$. For else some remote *point would be near p which is a contradiction. Thus by the previous lemma, there must exist $F \in \mathfrak{F}$ and $N \in Np$ such that $F \cap N = \emptyset$. Since $\mathbf{C}$ is a base for $\mathfrak{F}$, F contains some $C \in \mathbf{C}$. Note that $C \cap N = \emptyset$ and $X\backslash C$ is compact. Thus $X\backslash C$ which contains N must be a compact nbhd of p. Therefore X is locally compact. □

It is interesting to compare the non-standard characterization of local compactness with the standard definition. As with much of the study involving non-standard analysis there is a "trade-off" involved. In some respects the non-standard characterization is simpler and in other respects it is more complicated. More difficult in the respect that *points and the associated non-standard terminology is involved, but simpler since it is now necessary to consider only compact sets instead of compact nbhds.

It is clear from the definitions that each compact set is locally compact. The converse of this statement was exhibited to be false by Example 9.16.ii. Nonetheless, there is a close relationship between compactness and local compactness in the sense that theorems involving compactness often have duals involving local compactness. The next theorems are given, in part, to illustrate some of these parallel ideas. The proofs are also intended to exhibit the use of the non-standard characterization of local compactness.

9.19 Theorem: The intersection of a locally compact set and a closed set is locally compact.

Proof: Assume that L is a locally compact subset of $(X, \mathfrak{J})$ and that $C \subset X$ is closed. Let r be a near-standard *point of $L \cap C$. Since r is a near standard *point of the locally compact set L, there exists $K \subset L$ such that $r \,{}^*\!\in K$ and K is compact in $(L, \mathfrak{J}_L)$. It then follows that $L \cap C$ is closed in $(L, \mathfrak{J}_L)$ and so $K \cap C$ is a compact subset of $(L, \mathfrak{J}_L)$ which has r as a *point. Then $K \cap C$ is also a compact subset of $L \cap C$. Hence $L \cap C$ is locally compact by the non-standard characterization of local compactness. □

9.20 Corollary: Every closed subset of a locally compact set is locally compact.

9.21 Example: Let Q be the set of rational numbers. Then (R, E) is a locally compact set with a subspace (Q, Q_E) which is not locally compact. Hence local compactness is not hereditary. To see this examine any compact nbhd K of zero. K cannot be a compact subset of Q since it contains *points which are near the irrationals.

9.22 Example: Local compactness is not a continuous image property. Consider $(R, \mathfrak{T})$ the reals with the discrete topology which is locally compact and (R, Υ) (the left-closed-interval topology) which is not locally compact. The identity function from $(R, \mathfrak{T})$ to (R, Υ) is continuous but does not preserve local compactness.

9.23 Theorem: Local compactness is a topological property.

Proof: Assume that f is a homeomorphism from the locally compact space $(X, \mathfrak{T})$ onto $(X', \mathfrak{T}')$. Let r be a near-standard *point of X' and take q to be the *point of X for which $f(q) = r$. Since r is near-standard, r is near some $f(p) \in X'$. It follows that q is near p in X since f is a homeomorphism. Hence q is a near-standard *point and must therefore be a *point of K for some compact set K in X. It follows that f(K) is a compact subset of X' and that $r \ {}^*\!\in f(K)$. Therefore $(X', \mathfrak{T}')$ is locally compact. □

CHAPTER X

PRODUCT SPACES

Product spaces were mentioned briefly at the end of Chapter IV. There it was suggested that one activity that frequently reoccurs in topology is that of forming new spaces from known spaces. The simplest way suggested was to intersect known topologies on a set in order to form a new topology. The other previously mentioned technique, which has been used repeatedly in this paper, is that of forming subspaces.

This chapter will investigate the formation and properties of product spaces. Not only are these ideas of sufficient merit to justify their inclusion as basic standard material, but they also provide material for some good illustrations of further non-standard proofs of some standard theorems. In particular, some theorems involving product invariant properties will be proven.

10.1 Definition: The cartesian product X of a collection of sets X_a where $a \in A$ (an indexing set) is given by $X = \prod X_a = \{f\colon f$ is a function from A into $\cup X_a$ such that $f(a) \in X_a$ for each $a \in A\}$. X_a will be called the a-th coordinate set of X and f(a) will be called the a-th coordinate of the point f. A function P_a which maps the product set X into the coordinate set X_a such that $P_a(x) = x(a)$ for each $x \in X$ is called a projection.

The material now being covered is still assumed to be in the context of the previous discussions concerning a universe of discourse U and an enlargement $\mathfrak{U}^*$ based upon U. A is assumed to be a subset of V, the set which is used to construct U. For each $a \in A$ (standard elements of A only), X_a is also assumed to be a subset of V. These assumptions guarantee that U contains all points and topologies of the product set X.

If each coordinate set has a topology upon it, then it is customary to form a topology on the product set which is somehow related to each of the individual topologies upon the coordinate sets. To be of any significant value, it is crucial that this topology be influenced by the topologies on each of the coordinates. For example, one could give the product space the discrete, indiscrete, or other well-known topologies, but this would be ignoring the topological properties of the coordinate spaces. The customary way of doing this is such that the projection functions are forced to be continuous. The topology about to be defined is also called, in honor of its discoverer, the Tichonov topology.

<u>10.2 Definition</u>: Let $S = \{U \subset X = \prod X_a : U = P_a^{-1}(O_a)$ for some $a \in A$ and some $O_a \in \mathfrak{J}_a\}$. The topology for X subgenerated by this family S of inverse images of the open sets in the coordinate spaces is called the <u>product topology</u> for X.

<u>10.3 Definition</u>: If a property is possessed by the product space whenever it is possessed by each coordinate space, the property is said to be <u>product invariant</u>.

It should be clear from the definition of the product topology, that this is the weakest topology for which all the projection functions are continuous. Also note that since a basic open set is formed by taking finite intersections of subbasic open sets, that each basic open set is restricted only in a finite number of coordinates. Thus if B is basic open then $B = \prod U_a$ where $U_a \in \mathfrak{J}_a$ and $U_a = X_a$ for all but a finite number of coordinates. Since any open set contains a basic open set, points in any open set must be restricted in at most a finite number of coordinates. This is true even when the indexing set is uncountable, such as $A = (0, 1)$.

Product spaces can become awkward to work with even when the coordinate spaces are relatively simple. The following trivial example is given to illustrate the previous definitions.

<u>10.4 Example</u>: Let $X_1 = \{a, b, c\}$, $X_2 = \{1, 2\}$, $\mathfrak{J}_1 = \{\emptyset, \{a\}, \{b, c\}, X_1\}$ and $\mathfrak{J}_2 = \{\emptyset, \{1\}, X_2\}$. Then a subbase for the product topology $\mathfrak{J}$ for

$$X = X_1 \times X_2 = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

is the set S of inverse images of the projection functions.
$S = \{P_1^{-1}(\emptyset), P_1^{-1}(\{a\}), \cdots, P_2^{-1}(X_2)\} =$
$\{\emptyset, \{(a, 1), (a, 2)\} \{(b, 1), (b, 2), (c, 1), (c, 2)\},$
$\{(a, 1), (b, 1), (c, 1)\}, X\}$. Thus a base B for $\mathfrak{J}$ would consist of all finite intersections of members of S.

$$B = \{\emptyset, \{(a, 1), (a, 2)\}, \{(b, 1), (b, 2), (c, 1), (c, 2)\}, \{(a, 1), (b, 1), (c, 1)\}, \{(a, 1)\}, \{(b, 1), (c, 1)\}, X\}.$$

Then $\mathfrak{J}$ consists of the set of all unions of members of B. The monads of all points in each space will now be given:

$\mu(1) = \{1\}$, $\mu(2) = \{1, 2\}$, $\mu(a) = \{a\}$, $\mu(b) = \{b, c\}$
$\mu(c) = \{b, c\}$, $\mu((a, 1)) = \{(a, 1)\}$, $\mu((a, 2)) = \{(a, 1), (a, 2)\}$,
$\mu((b, 1)) = \{(b, 1), (c, 1)\}$, $\mu((b, 2)) = \{(b, 1), (b, 2), (c, 1),$
$(c, 2)\}$, $\mu((c, 1)) = \{(b, 1), (c, 1)\}$, and $\mu(c, 2) = \{(b, 1), (b, 2),$
$(c, 1), (c, 2)\}$.

What is suggested by the monads in this example is that p is near q in the product space iff p(a) is near q(a) for each coordinate of p and q. The next theorem shows that this is true in general.

<u>10.5 Theorem</u>: Let $(X, \mathfrak{J})$ be the product space formed by the spaces $(X_a, \mathfrak{J}_a)$ where $a \in A$. Then p is near q in $(X, \mathfrak{J})$ iff each coordinate of p is near the respective coordinate of q.

Proof: $(\rightarrow)$ Assume that p is near q in $X = \prod X_a$. The a-th coordinate of p is $P_a(p)$ and of q is $P_a(q)$. Since P_a is continuous and so preserves nearness, it follows that $P_a(p)$ must be near $P_a(q)$.

$(\leftarrow)$ Assume that p(a) is near q(a) for each coordinate $a \in A$. To show that p is near q, it will suffice to show that $p \in \hat{B}$ for each basic open nbhd of q. If B is a basic open nbhd of q, then $B = \cap\{S_k: k = 1, \cdots, n\}$ for some finite collection of subbasic open sets S_k about q. If S_k is any subbasic open set about q, then by definition of the product topology S_k is the inverse image $P_{a_k}^{-1}(O_{a_k})$ for some open set O_{a_k} about some a_k-th coordinate of q. Now by assumption $p(a_k)$ is near $q(a_k)$, thus $p(a_k) \in \hat{O}_{a_k}$. Hence for each $k = 1, 2, \cdots, n$, $p \,{}^*\!\!\in P_{a_k}^{-1}(O_{a_k})$ which equals S_k. Therefore $p \in \hat{B}$ and it follows that p is near q. □

This theorem is certainly not surprising when the product space has a finite number of coordinates. When the indexing set A is the set (0, 1), then nearness is still required at each and every coordinate. In particular, if p is near q then it is not permissible for $p(a)$ not to be near $q(a)$ on a non-empty set of coordinates of Lebesgue measure zero.

<u>10.6 Corollary</u>: Let $(X, \mathfrak{J})$ be the product space formed by the spaces $(X_a, \mathfrak{J}_a)$ where $a \in A$. If B is a *subset of X near x, then $P_a(B)$ is near x_a for each $a \in A$. Similarly, if B is near $C \subset X$ then $P_a(B)$ is near $P_a(C)$ for each $a \in A$.

Proof: Assume that the *subset B of the product space $(X, \mathfrak{J})$ is near $x \in X$. Then there exists $r \,{}^*\!\!\in B$ such that r is near x. By the previous theorem, this means r_a is near x_a for each $a \in A$. Since $r_a \,{}^*\!\!\in P_a(B)$, it follows that $P_a(B)$ is near x_a for each $a \in A$. The remaining part of the proof is also straightforward. □

Although it might seem desirable, the converses of the statements in the previous corollary are not true.

<u>10.7 Example</u>: If B is a *subset of the product space $(X, \mathfrak{J})$ and $x \in X$, then it is possible for $P_a(B)$ to be near x_a for each $a \in A$ without B being near x. Similarly, it is possible for $P_a(B)$ to be

near $P_a(C)$ for each $a \in A$ where $C \subset X$ without B being near C. To see this, let $B = \{(a, 2), (b, 1)\}$ and $x = (a, 1)$ in Example 10.4. For the second case let $C = \{(a, 1)\}$.

<u>10.8 Theorem</u>: Every projection function is open.

Proof: Assume that $(X, \mathfrak{J})$ is the product space whose coordinate spaces are $(X_a, \mathfrak{J}_a)$ where $a \in A$. By Lemma 6.21 it will suffice to show that $\mu(P_c(x)) \subset P_c(\mu(x))$ for each $x \in X$. So suppose r is near x_c in the c-th coordinate space. Then define y by

$$y_i = \begin{cases} x_a & a \neq c \\ r & a = c. \end{cases}$$

The previous theorem guarantees that y is near x. Further, $P_c(y) = r$. Hence $r \in P_c(\mu(x))$ and the desired inclusion follows. Therefore P_c is open. □

Since nearness of points in a product space in completely determined by nearness of the coordinates, it should not be surprising that the continuity of a function from a space into a product space can be determined completely by considering each of the coordinate spaces.

<u>10.9 Theorem</u>: Let $f: (X', \mathfrak{J}') \to (X, \mathfrak{J})$ be a function from an arbitrary space to the product space $X = \prod X_a$, $a \in A$. Then f is continuous iff $P_a \circ f$ is continuous for each $a \in A$.

Proof: The function f is continuous iff x' near y' in $(X', \mathfrak{J}')$ implies $f(x') = x$ is near $y = f(y')$ in $(X, \mathfrak{J})$. Now x is near y iff

$P_a(x)$ is near $P_a(y)$ for each $a \in A$. Hence f is continuous iff x' near y' implies $(P_a \circ f)x'$ is near $(P_a \circ f)y'$ for each $a \in A$, i.e. iff each $P_a \circ f$ is continuous. □

The next theorem gives yet another criterion for determining when a space is Hausdorff. It was chosen because of the importance of T_2-spaces and because the proof brings together some of the non-standard characterizations previously developed.

<u>10.10 Theorem</u>: The space $(X, \mathfrak{T})$ is T_2 iff the diagonal of the product space formed by $X \times X$ is a closed set.

Proof: $(\rightarrow)$ Assume $(X, \mathfrak{T})$ is T_2 and suppose the diagonal D of the product space $(X \times X, \mathfrak{T}')$ is near the point (x, y). Then some *point (p, p) of D must be near (x, y). Hence p is near both x and y, which cannot be in a T_2-space unless $x = y$. Thus $(x, y) \in D$ and D is closed.

$(\leftarrow)$ Assume that the diagonal D of the product space $(X \times X, \mathfrak{T}')$ is closed and suppose p is near both x and y in the space $(X, \mathfrak{T})$. Now (p, p) is near (x, y). Thus D is near (x, y) and so $(x, y) \in D$ since D is closed. Therefore $x = y$ and so $(X, \mathfrak{T})$ is T_2. □

One important feature of product spaces is that every coordinate space is homeomorphic to some subspace of the product space. More importantly, the subspace may be chosen so as to contain an arbitrary point in the product space.

<u>10.11 Theorem</u>: Let x be a point in the product space $(X, \mathfrak{T})$. Then each coordinate space $(X_a, \mathfrak{T}_a)$ is homeomorphic to some subspace of $(X, \mathfrak{T})$ containing x.

Proof: Assume that x is a point in the product space $(X, \mathfrak{T})$. For the c-th coordinate space $(X_c, \mathfrak{T}_c)$, define $f: (X_c, \mathfrak{T}_c) \rightarrow (X, \mathfrak{T})$ by $f(p) = y$ where $y_a = x_a$ for $a \neq c$, while $y_c = p$. Note that f is a one-to-one function from X_c onto $f(X_c) \subset X$ and that $x \in f(X_c)$. Since p is near q in $(X_c, \mathfrak{T}_c)$ iff $f(p)$ is near $f(q)$, it follows that f is a homeomorphism to the subspace formed by $f(X_c)$. □

Some important properties that are product invariant are the T_1 and T_2 separation axioms, connectedness, and compactness. Proofs of some of these invariancies are included below.

<u>10.12 Theorem</u>: A product space is T_2 iff each coordinate space is T_2.

Proof: Assume that $(X, \mathfrak{T})$ is a product space and that r is near both x and y. Now r is near both x and y is equivalent to saying r_a is near both x_a and y_a for each coordinate a. The space $(X, \mathfrak{T})$ is Hausdorff iff $x = y$, i.e. iff $x_a = y_a$ for each a, i.e. iff each coordinate space is T_2. □

Although this paper will not pursue further results which avail themselves of the next theorem, it seems wise to include the result since it is referred to by many authors as the most important result of general topology. Those who are familiar with a standard proof of this theorem should appreciate the brevity and clarity of the non-standard proof.

10.13 Tichonov's Theorem: A product space is compact iff each coordinate space is compact.

Proof: ($\rightarrow$) Assume that the product space $(X, \mathfrak{J})$ is compact. To show that an arbitrary X_c is compact it will suffice to show $r_c \; {}^*\!\in X_c$ implies there is some point in X_c which r_c is near to. Now for an arbitrary $x \in X$, define a *point in X by $y_a = x_a$ for $a \neq c$, while $y_c = r_c$. Since the product space is compact, y is near some $z \in X$. Hence y_c which equals r_c is near $z_c \in X_c$. Therefore X_c is compact.

($\leftarrow$) Assume that each coordinate space is compact and suppose that $r \; {}^*\!\in X$. Then $r_c \; {}^*\!\in X_c$ for each $c \in A$ and thus r_c is near x_c for some $x_c \in X_c$. Thus the point r is near x, the point in X defined all the x_c's. □

10.14 Theorem: If a product space is locally compact then each coordinate space is locally compact.

Proof: Assume that the product space $(X, \mathfrak{J})$ is locally compact and suppose that r_c is a near-standard *point of X_c. Then r_c is near some $x_c \in X_c$. Now let x be a point in X whose c-th coordinate is x_c. If $y_a = x_a$ for $a \neq c$ and $y_c = r_c$, then y is near x. Hence y is a near-standard *point of X. Since the product space is locally compact, $y \; {}^*\!\in K$ for some compact subset K of X. Thus $r_c \; {}^*\!\in P_c(K)$, which is a compact subset of X_c since P_c is a continuous function. This verifies that $(X_c, \mathfrak{J}_c)$ is locally compact. □

CHAPTER XI

SUMMARY AND CONCLUSIONS

The primary purpose of this paper was to present the basic ideas of non-standard analysis by illustrating their usage in developing some of the basic concepts of topology. Since the primary advantage gained by using the non-standard approach is the capability of using more intuitively worded terminology, this paper has concentrated upon the concept of nearness in an arbitrary topological space. Thus Chapter V may be considered as the core of this paper. The previous chapters provided a foundation in non-standard analysis and in topology. The remaining chapters illustrated some uses of the concepts of nearness developed in Chapter V and continued the development of other related concepts.

While many of the major theorems and characterizations given in this paper are equivalent to those found in either Robinson [14] or Machover and Hirschfeld [11], the terminology used in all chapters following Chapter V may differ with these sources. The reason for this is that the definition of nearness of *points to points was generalized first to nearness of *sets to points and then generalized to nearness of *sets to sets. These generalizations allow one to make even more use of his intuition.

The reader may question why these particular generalizations of nearness were given instead of others which might seem equally plausible. The answer is that the generalizations given were those which proved to be most useful in describing other concepts. For example, it seemed plausible to define the *point p as being near $A \subset X$ iff p is near some point $q \in A$. It then would be true that p is near $\{q\}$ iff p is near q. However, this definition of nearness has the unpleasant feature that p can be near A without $p \; {}^*\!\in \bar{A}$. To see this consider $X = \{a, b, c\}$, $\mathfrak{J} = \{\emptyset, \{b\}, \{b, c\}, X\}$, the point b, and the set $\{a, c\}$.

Non-standard analysis often makes characterizations of topological concepts more intuitive and incisive. You may have also observed that the construction of proofs is often easier using non-standard terminology. As evidence of the power of non-standard analysis, note that Abraham Robinson has already used non-standard analysis to solve a previously unsolved problem on compact linear operators. This tool is also applicable to many other branches of mathematics, including algebra.

Historically, mathematicians have been extremely conservative — often greeting innovation with such uncomplimentary terms as irrational, imaginary, radical, negative, or non-standard. It may take several years and perhaps a new generation of mathematicians not entrenched in standard analysis before non-standard analysis is commonly used by mathematicians. Nonetheless, the author expects that the usage of non-standard analysis will become widespread. Hopefully, this paper will contribute to this increased usage.

BIBLIOGRAPHY

(1) Bourbaki, Nicolas. Elements of Mathematics, General Topology, Volume I, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1966.

(2) Davis, Martin and Reuben Hersh. "Nonstandard Analysis", Scientific American, June, 1972, Vol. 226, no. 6, pp. 78-86.

(3) Frayne, T., Morel, A. C., Scott, D. S. "Reduced Direct Products", Fundamenta Mathematicae, Vol. 51, 1962, pp. 195-228.

(4) Kelley, John L. General Topology, D. Van Nostrand Company, Inc., New York, New York, 1955.

(5) Kleene, S. C. Introduction to Metamathematics, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1952.

(6) Lightstone, A. H. "Infinitesimals", The American Mathematical Monthly, Vol. 79, no. 3, March, 1972, pp. 242-251.

(7) Lightstone, A. H. Symbolic Logic and the Real Number System, Harper & Row, New York, 1965.

(8) Luxemburg, Wilhelmus A. J. "A General Theory of Monads", Applications of Model Theory to Algebra, Analysis, and Probability, Holt, Rinehart and Winston, Inc., New York, 1969, pp. 18-86.

(9) Luxemburg, Wilhelmus A. J. "Non-Standard Analysis", Lectures on A. Robinson's Theory of Infinitesimals and Infinitely Large Numbers, (Mimeographed Notes), Mathematics Department, California Institute of Technology, Pasadena, California, 1962.

(10) Luxemburg, Wilhelmus A. J. "Two Applications of the Method of Construction by Ultrapowers to Analysis", Bulletin of the American Mathematical Society, Vol. 68, 1962, pp. 416-419.

(11) Machover, Moshe' and Joram Hirschfeld. "Lectures on Non-Standard Analysis", Lecture Notes in Mathematics, Springer-Verlag, Berlin, Germany, 1969.

(12) Mendelson, Elliot. "On Non-Standard Models for Number Theory", Essays on the Foundations of Mathematics, (Fraenkel anniversary volume), North-Holland Publishing Company, Amsterdam, Netherlands, 1961, pp. 259-268.

(13) Robinson, Abraham. Introduction to Model Theory and to the Metamathematics of Algebra, North-Holland Publishing Company, Amsterdam, Netherlands, 1963.

(14) Robinson, Abraham. Non-Standard Analysis, North-Holland Publishing Company, Amsterdam, Netherlands, 1966.

(15) Robinson, Abraham. "Model Theory and Non-Standard Arithmetic", Infinitistic Methods, Symposium on Foundations of Mathematics, Warsaw 1959, Pergamon Press Inc., New York, 1961, pp. 265-302.

(16) Scott, Dana. "On Constructing Models for Arithmetic", Infinitistic Methods, Symposium on Foundations of Mathematics, Warsaw 1959, 1961, pp. 235-255.

(17) Steen, Lynn A. and J. Arthur Seebach, Jr. Counterexamples in Topology, Holt, Rinehart and Winston, Inc., New York, 1970.

(18) Steffensen, Arnold. "Introduction to Point Set Topology", (unpublished), Mathematics Department, Northern Arizona University, Flagstaff, Arizona, 1970.

(19) Thron, Wolfgang J. Topological Structures, Holt, Rinehart and Winston, New York, 1966.

(20) Van Osdel, Donovan H. "Truth With Respect to an Ultrafilter or How to Make Intuition Rigorous", The American Mathematical Monthly, Vol. 79, no. 4, April, 1972, pp. 355-363.

(21) Wilder, Raymond L. Introduction to The Foundations of Mathematics, John Wiley & Sons, Inc., New York, 1952.

INDEX

VITA

James Wesley Hall

Candidate for the Degree of

Doctor of Education

Thesis: NON-STANDARD TOPOLOGY

Major Field: Higher Education

Biographical:

Personal Data: Born in Effingham, Illinois, November 4, 1946, the son of George C. and Ruth M. Hall.

Education: Graduated from Louisville Grade School, Louisville, Illinois, in 1960; graduated from North Clay High School, Louisville, Illinois, in 1964; received the Bachelor of Science in Education degree from Eastern Illinois University in February, 1968, with a major in mathematics; received the Master of Arts degree from Eastern Illinois University in May, 1969, with a major in mathematics; completed requirements for the Doctor of Education degree at Oklahoma State University in July, 1973.

Professional Experience: Graduate Assistant in Mathematics Department at Eastern Illinois University, Summer, 1968; Graduate Assistant Computer Programmer for Speech and Hearing Clinic at Eastern Illinois University, 1968-1969; Mathematics Instructor at Northern Arizona University, 1969-1970; Graduate Teaching Assistant in Mathematics Department, Oklahoma State University, September, 1970 - September, 1972; Temporary Assistant Professor of Mathematics at Southwest Missouri State University, September, 1972 - January, 1973; Graduate Teaching Assistant in Mathematics Department, Oklahoma State University, January, 1973 - May, 1973.