COMPLETE CONVEX METRICS FOR

GENERALIZED CONTINUA

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PREFACE

If as Bing [5] suggests, "topology may be regarded as an offshoot of geometry," then the definition of the convex metric by Menger [18] in 1928 must be regarded as the beginning of one of the most perceptive and profitable attempts to link it with its origin. For the property of metric convexity together with completeness provides, as Menger [18] showed, that every two points in the space are endpoints of an arc along which distances are additive. The existence of these arcs, so-called "segments" in a metric sense, along with the notions of lines [8], parallels [8], and angles [21] that are also definable in abstract metric spaces, gives to these complete convex metric spaces an unmistakable euclidean flavor.

In proving the existence of segments in complete convex metric spaces, Menger [18] in effect showed that if a compact space admits a convex metric, then the space must be locally connected, hence a Peano continuum. Then he posed the well-known Konvexierungsproblem: Does every Peano continuum admit a convex metric? This problem, which claimed the attention of several eminent mathematicians over a period of two decades, was finally answered in the affirmative by Bing [7] in 1949. In the pursuit of the Konvexierungsproblem, and in the aftermath of its solution, there grew up a rich body of techniques and results that include partitioning [7], grille decomposition [19], characterization theorems [25], and metric extension theorems [4].

Subsequently, a start has been made in extending these techniques and results to non-compact spaces; notably, in 1955 Tominaga and Tanaka [24] obtained an affirmative answer for the Konvexierungsproblem for more general spaces by means of partitioning locally connected generalized continua. Here it should be noted that, since a convex metric on a compact space is complete, its most natural counterpart in the more general case is not merely a convex metric, but a complete convex metric. Thus, for example, the Konvexierungsproblem for the more general spaces is the question: Does each locally connected generalized continuum admit a complete convex metric?

The present paper can best be considered as a continuation of the process mentioned above as having already begun, that of generalizing to a non-compact setting some of the results obtained originally for convex metrics on Peano continua; hence the title, "Complete Convex Metrics for Generalized Continua." In particular, the core of this dissertation lies in a series of theorems generalizing a result of Bing [4] on the extension of a convex metric to the union of two Peano continua. The rest of the paper is logically related to this core of results, either in providing material to be used in proving it or in furnishing applications of it. Chapter I lays the conceptual foundation by providing definitions, by stating some of the previously obtained results that are of interest to this paper, and by giving a few revealing applications of these results. The next three chapters cover independently three topics that are necessary to accomplish the goal of the paper. Chapter II presents three particular types of complete convex metrics in preparation for some straightforward applications of the extension theorems; this material is placed early in the thesis because of the rich variety of examples of complete convex metric spaces that accompanies it. In Chapter III the segmented convex metric is introduced for its usefulness in the proof of the extension theorems; it is discussed at some length, and its crucial role is shown in a result dealing with closed balls as Peano continua. A certain natural topology for the union of two spaces is discussed in Chapter IV; the choice and properties of this topology become crucial in generalizing the extension theorem of Bing [4] to possibly non-compact settings. The extension theorems themselves are now proved in Chapter V. These theorems are applied in Chapter VI to characterize certain classes of locally connected generalized continua by the variety of complete convex metrics they admit, using those types of convex metrics that were discussed in Chapter II. The results of the paper are summarized and a few suggestions for further research are given in Chapter VII.

All the results of this paper, including theorems, corollaries, and examples, will share a common sequence for numbering; the two numbers, separated by a period, that accompany a result are the number of the chapter where it first appears and its order within that chapter, respectively. Each later reference to this result will give these two identification numbers enclosed in parentheses. Single numerals in parentheses refer to formulas displayed and numbered in the text; this sequence of numbers will be re-initiated at the beginning of each chapter. Numbers enclosed in square brackets refer to the bibliography at the end of the paper. The "proofs" for some of the examples are not so much proofs as they are constructions, with only the non-obvious assertions in them receiving actual proofs. The simplest examples, as well as those results that are found in the literature, are stated without any proof at all. Conversely, where a proof is given, then the result is the author's, although some of the results proved in Chapter I and possibly elsewhere are doubtless well known as part of the "folklore" of the subject.

I would like to express my appreciation, at least in this small way, to those who have helped in the preparation of this dissertation. To Professor John W. Jewett for arranging a graduate assistantship and securing National Science Foundation Traineeship GZ-1694, and to Mrs. Mary Bonner and Mrs. Cynthia Wise for their generous advice in the typing of the manuscript, I would like to render my thanks. I am grateful also for the interest and cooperation shown by my committee, consisting of Professor E. K. McLachlan as chairman, Professors Marvin S. Keener and Donald E. Boyd, and especially I want to express my appreciation to my thesis adviser, Professor John M. Jobe, who has the rare ability to offer positive guidance in a way that offers both freedom and incentive to the student in the pursuit of his research. I would also like to acknowledge the kind attention of Professor A. Lelek of the University of Houston, with whom I have had the honor of communicating both by conversation and by correspondence on the subject of convex metrics. Special thanks are due to my wife Kathie and son Nathan for their continual patience and encouragement. And, if the full story were known, this final acknowledgment would be the most fitting of all: "Blessed be the Lord, because He hath heard the voice of my supplications" (Psalm 28:6).

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CHAPTER I

PRELIMINARY CONCEPTS

This paper will be devoted to the study of properties of complete convex metrics on locally connected generalized continua.

Topological and Metric Spaces

Terminology and notation that is not defined in this paper is assumed to have the meaning assigned by Hall and Spencer [13], Dugundji [11], Whyburn [26], or Moore [20].

A topological space consisting of a set M with a topology $\mathcal{T}$ is denoted by $(M, \mathcal{T})$, or more briefly by M when the topology is clear from context. Similarly a metric space induced by a metric D on a set M may be denoted by (M, D) or, when appropriate, simply by M. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on the same set, then $\mathcal{T}$ is said to be stronger than $\mathcal{T}'$, and $\mathcal{T}'$ weaker than $\mathcal{T}$, if $\mathcal{T}'$ is a subset of $\mathcal{T}$, denoted by $\mathcal{T}' \subset \mathcal{T}$. A sequence is denoted by $\langle x_n \rangle$, whereas the union of all of its points is $\{x_n\colon\ n = 1, 2, \ldots\}$. The following metric conventions will also be observed.

<u>Definition</u> 1.1. Let (M, D) be a metric space.

(i) If $p \in M$ and $\delta > 0$, then $D(p;\delta) = \{x \in M\colon\ D(p,x) < \delta\}$ is called an "open ball," and $\overline{D}(p;\delta) = \{x \in M\colon\ D(p,x) \leq \delta\}$ is called a "closed ball."

(ii) If $A \cap M \neq \emptyset$, then $D(A) = \sup\{D(x,y)\colon\ x, y \in A \cap M\}$.

(iii) If $A \cap M \neq \emptyset$ and $B \cap M \neq \emptyset$, then

$D(A,B) = \inf \{D(x,y): x \in A \cap M, y \in B \cap M\}$.

(iv) If $N \subset M$, then "D restricted to N" is the metric F defined by $F(x,y) = D(x,y)$ for all points $x, y \in N$.

(v) If D' is another metric on the set M, then the statement "D and D' are equivalent" means that $(M,D) = (M,D')$; if $N \subset M$, then the statement "D and D' agree on N" means that $D(x,y) = D'(x,y)$ for all points $x, y \in N$.

Certain concepts that are defined with respect to a particular metric D, such as linearity, segment, ball, and convexity, will often appear with the name of the metric prefixed to them, as D linearity, D segment, D ball, and D convexity. However, when the identity of the metric is clear from the context, the name of the metric may be omitted in a discussion of such concepts. Similarly, $D(x,y)$ will be abbreviated to xy when the identity of the metric D is understood.

Linearity

The notion of linearity in a metric space underlies the definition of convexity.

<u>Definition</u> 1.2. A set in a metric space is <u>linear</u> if it is isometric to a subset of the real line E^1; that is, if (M, D) is the given metric space, $N \subset M$ is linear if there exists a function $L : N \to E^1$ such that $D(x,y) = |L(x) - L(y)|$ for every pair of points $x, y \in N$. A <u>segment</u> is a linear arc, and the notation $\overline{pq}$ denotes some segment whose endpoints are p and q.

The following criterion of linearity is due to Menger [18].

Theorem 1.3. A metric space with more than four points is linear if and only if every three point subset is linear.

The necessity for the space in (1.3) to have more than four points is seen from the following example.

Example 1.4. Let the four point set $\{a, b, c, d\}$ be metrized in such a way that $ab = bc = cd = da = ac/2 = bd/2$. Then every three point subset is linear, but the space is not linear.

The criterion of (1.3) reduces many linearity arguments to the consideration of finite sets of points, and the following two results are then useful.

Theorem 1.5. If a set X in a metric space has $n \geq 2$ points, then X is linear if and only if X may be represented as $X = \{x_1, x_2, \ldots, x_n\}$, where

$$x_1 x_n = \sum_{i=1}^{n-1} x_i x_{i+1}.$$

In case X is linear, the subset $\{x_1, x_n\}$ is uniquely determined.

Proof: To prove necessity, suppose X is linear and has $n \geq 2$ points. Then there exists an isometry $L : X \rightarrow E^1$. Since $L(X)$ is a set of n real numbers, let x_i be the point of X such that $L(x_i)$ is the ith number of $L(X)$ in the usual order for E^1, for $i = 1, \ldots, n$. Then $X = \{x_1, x_2, \ldots, x_n\}$ and also $L(x_1) < L(x_2) < \ldots < L(x_n)$, from which follows the desired formula.

In this case, the uniqueness of the set $\{x_1, x_n\}$ is a result of the fact that $x_1x_n > x_ix_j$ for $\{i, j\} \neq \{1, n\}$.

The proof of sufficiency consists of induction on n with the following induction hypothesis: "If $B_n = \{b_1, b_2, \ldots, b_n\}$ is a set of $n \geq 2$ points such that

$$b_1b_n = \sum_{i=1}^{n-1} b_ib_{i+1}$$

holds, then the function $L : B_n \to E^1$, defined by $L(b_i) = b_1b_i$ for $i = 1, \ldots, n$, is an isometry and moreover

$$b_1b_m = \sum_{i=1}^{m-1} b_ib_{i+1}$$

holds for every m, $2 \leq m \leq n$." This hypothesis is clearly true when $n = 2$.

Suppose the induction hypothesis holds for some value $k \geq 2$. Then let $B_{k+1} = \{x_1, x_2, \ldots, x_{k+1}\}$ be a set of $k+1$ points with

$$x_1x_{k+1} = \sum_{i=1}^{k} x_ix_{i+1}. \tag{1}$$

Since

$$x_1x_{k+1} \leq x_1x_k + x_kx_{k+1} \tag{2}$$

holds by the triangle inequality, it follows that

$$\sum_{i=1}^{k-1} x_i x_{i+1} \leq x_1 x_k \tag{3}$$

can be obtained by substituting (1) into (2) and subtracting $x_k x_{k+1}$ from both sides. But

$$\sum_{i=1}^{k-1} x_i x_{i+1} \geq x_1 x_k \tag{4}$$

follows from the triangle inequality. Thus, (3) and (4) yield

$$\sum_{i=1}^{k-1} x_i x_{i+1} = x_1 x_k .$$

If now the set $B_k = \{x_1, x_2, \ldots, x_k\}$ is defined, the induction hypothesis for $n = k$ yields the fact that $L : B_k \to E^1$, defined by $L(x_i) = x_1 x_i$ for $i = 1, \ldots, k$, is an isometry and moreover

$$x_1 x_m = \sum_{i=1}^{m-1} x_i x_{i+1} \tag{5}$$

holds for every m, $2 \leq m \leq k$. Thus, (5) and (1) together yield

$$x_1 x_m = \sum_{i=1}^{m-1} x_i x_{i+1}, \quad 2 \leq m \leq k - 1.$$

If $L' : B_{k+1} \to E^1$ is defined by $L'(x_i) = x_1 x_i$ for $i = 1, \ldots, k + 1$, then L' extends L. Hence, in order to verify that L' is an isometry on B_{k+1} it suffices to show $x_j x_{k+1} = |L'(x_j) - L'(x_{k+1})|$

for $1 \leq j \leq k$. Since

$$x_1x_j + x_jx_{k+1} \leq \sum_{i=1}^{k} x_ix_{i+1}$$

holds by a double application of the triangle inequality, and since

$$\sum_{i=1}^{k} x_ix_{i+1} = x_1x_{k+1}$$

holds by (1), substitution then yields $x_1x_j + x_jx_{k+1} \leq x_1x_{k+1}$. But this inequality with the triangle inequality results in the equalities $x_1x_j + x_jx_{k+1} = x_1x_{k+1}$ and $x_jx_{k+1} = x_1x_{k+1} - x_1x_j$. From this latter equality and the definition of L', it follows that $x_jx_{k+1} = L'(x_{k+1}) - L'(x_j) = |L'(x_j) - L'(x_{k+1})|$. Thus L' is an isometry, and the induction hypothesis holds for $n = k + 1$. The proof of sufficiency is therefore given by the induction principle. ∎

Theorem 1.6. If

$$x_1x_n = \sum_{i=1}^{n-1} x_ix_{i+1}, \tag{6}$$

then

$$x_jx_k = \sum_{i=j}^{k-1} x_ix_{i+1} \tag{7}$$

and

$$x_1x_n = \sum_{i=1}^{j-1} x_ix_{i+1} + x_jx_k + \sum_{i=k}^{n-1} x_ix_{i+1} \tag{8}$$

hold for any $1 \leq j < k \leq n$.

Proof: Let $1 \leq j < k \leq n$. The generalized triangle inequality is

$$x_jx_k \leq \sum_{i=j}^{k-1} x_ix_{i+1}, \tag{9}$$

as well as

$$x_1x_j \leq \sum_{i=1}^{j-1} x_ix_{i+1} \quad \text{and} \quad x_kx_n \leq \sum_{i=k}^{n-1} x_ix_{i+1}.$$

By adding these last two inequalities member by member and subtracting from the respective members of (6), the inequality

$$x_jx_k \geq \sum_{i=j}^{k-1} x_ix_{i+1}$$

is obtained. This final inequality, along with (9), yields (7). Formula (8) now follows by subtracting (7) from (6), member by member. ∎

Betweenness and Metric Convexity

Closely related to the notion of linearity is that of betweenness.

Definition 1.7. If a, b, and c are three distinct points of a metric space, then b is a between point of a and c, written abc, if $ac = ab + bc$. The statement "abc on $\overline{de}$" means that the three points a, b, and c lie on $\overline{de}$ and that abc holds. A between point b of a and c is a midpoint of a and c if $ab = bc$.

The following two theorems are proved by Blumenthal [8].

Theorem 1.8. In a metric space the simultaneous conditions pqr and prs are equivalent to pqs and qrs; and if pqr holds, then both qpr and qrp are false.

Theorem 1.9. In a metric space, the set $\overline{pq} \cup \overline{qr}$ is a segment $\overline{pr}$ if and only if pqr.

The definition of convex metric employed in this paper is the original definition, first given by Menger [18], and more recently used by Moise [19], Lelek and Nitka [17], and others. The other variety of convex metric, which is called "midpoint convex" in this paper, is employed by Bing [7], Tominaga and Tanaka [24], and others.

Definition 1.10. A metric for a metric space is convex (midpoint convex) if every two points in the space have a between point (midpoint). A subset of a metric space is said to be convex (midpoint convex) if the metric restricted to that subset is convex (midpoint convex).

Example 1.11. The usual metric for n-dimensional euclidean space E^n, when restricted to the set of points with all rational coordinates, is both convex and midpoint convex.

While a midpoint convex metric is necessarily convex, the following example shows that the converse is not true.

Example 1.12. The usual metric of E^1 restricted to $(0, 1) \cup (2, 3)$ is convex, but not midpoint convex.

The addition of completeness to the convexity of a metric space produces strong topological properties, as the following theorem, due originally to Menger [18], shows.

<u>Theorem</u> 1.13. In a complete convex metric space, there is a segment joining any pair of points.

<u>Corollary</u> 1.14. If pqr holds in a complete convex metric space, then for any segment $\overline{qr}$ there is a segment $\overline{pr}$ containing $\overline{qr}$.

Proof: Assuming the hypothesis and $\overline{qr}$ as given, by (1.13) there also exists some segment $\overline{pq}$. By (1.9), the set $\overline{pq} \cup \overline{qr}$ is a segment $\overline{pr}$. **I**

Since a segment between any two points contains a midpoint of them, then by (1.13) a complete convex metric is also midpoint convex; that is, a complete metric is convex if and only if it is midpoint convex. Since this paper is concerned primarily with complete metrics, for most of the results it will not matter that there are two definitions of convexity.

The following example shows that the converse of (1.13) is false.

<u>Example</u> 1.15. Let (a, b) be any proper open interval of E^1, and let D be the euclidean metric restricted to (a, b). Any two points are joined by a D segment, yet D is not complete.

One proof of (1.13), due to Aronszajn [2], is given by Blumenthal [8, p. 41] in a form that may be modified slightly to give the following stronger result.

<u>Theorem</u> 1.16. If p and q are two points of a complete convex metric space and if L is a linear set consisting of p, q, and between points of p and q, then there is a segment $\overline{pq}$ containing L.

Continua

In this dissertation the two kinds of continua about to be defined, will be used extensively.

Definition 1.17. A Peano continuum is a compact, connected, locally connected metric space. A generalized continuum is a locally compact, connected, separable metric space.

Theorem 1.18. The property of being a Peano continuum, and the property of being a (locally connected) generalized continuum, are topological properties.

Proof: Every defining condition in (1.17) is a topological property. ∎

Peano continua, also called Peano spaces, Peano curves, and continuous curves, are very common in the literature. The following characterization, known as the Hahn-Mazurkiewicz theorem, is classical; for one proof, see [13].

Theorem 1.19. A Hausdorff space S is a Peano continuum if and only if there is a continuous mapping of the closed interval $[0, 1]$ of E^1 onto S.

Example 1.20. With the usual metric for E^n, each closed ball is a Peano continuum.

Since Peano continua are separable spaces, it follows that a locally connected generalized continuum may be regarded as a "generalization" of a Peano continuum, obtained by relaxing the condition of compactness to that of local compactness. Characterizations of locally connected generalized continua will be given in the following section and in Chapter III. The remaining examples and theorem in this section are intended to provide illustrations of locally connected generalized continua.

Example 1.21. In E^n with the usual metric, each open ball, as well as E^n itself, is a locally connected generalized continuum.

Example 1.22. If f is a continuous function whose domain is a connected subset of E^1, the graph of f is a locally connected generalized continuum.

Proof: Each connected subset of E^1 is a locally connected generalized continuum, and the graph of a continuous function is homeomorphic to its domain [13].

Theorem 1.23. Let M be a dendrite and E its set of endpoints. The subspace $M \setminus E$ is a locally connected generalized continuum if and only if E is closed in M.

Proof: The definitions here and the elementary properties that follow from them are given by Whyburn [26]. First, suppose that E is closed in M. Then, since $M \setminus E$ is an open subset of the compact metric space M, $M \setminus E$ is a locally compact, locally connected, separable metric space. Since M is arcwise connected and a point of E must be an endpoint of any arc in M on which it lies, then $M \setminus E$ is also arcwise connected. Thus, $M \setminus E$ is a locally connected generalized continuum.

On the other hand, suppose that E is not closed in M. Then there is a sequence $\langle e_n \rangle$ of distinct points of E that converges to a point p in $M \setminus E$. Let U be any open set in M that contains p. Then, since M is locally arcwise connected, there is a connected and arcwise connected open set V such that $p \in V \subset U$. There is some point e_j in V, hence the arc A in M from p to e_j lies in V also. Since the set $A \setminus \{e_j\}$ contains a sequence $\langle a_i \rangle$ of points that converges to e_j, and since $A \setminus \{e_j\} \subset U \setminus E$, then $U \setminus E$ contains the infinite set $\{a_i: i = 1, 2, \ldots\}$ of points which has no accumulation point in $M \setminus E$. Since U is arbitrary, then $M \setminus E$ is not locally compact, hence not a generalized continuum. ∎

Convex Metrics on Continua

One immediate result of (1.13) is that every space that admits a complete convex metric is both connected and locally connected; in fact, it is arcwise connected and uniformly locally arcwise connected [22]. Therefore, a compact space that admits a convex metric must be a Peano continuum. The converse to this statement was an open question until proved in 1949 by Bing [7]. Hence, the following characterization of Peano continua is a result of the work of Menger [18] and Bing [7].

<u>Theorem</u> 1.24. A compact space is a Peano continuum if and only if it admits a convex metric.

Bing's result was generalized in 1955 to locally connected generalized continua by Tominaga and Tanaka [14], as follows.

<u>Theorem</u> 1.25. Every locally connected generalized continuum admits a complete convex metric.

Of fundamental importance to this dissertation is the following generalized Bolzano-Weierstrass theorem of Lelek and Mycielski [16].

Theorem 1.26. Every closed and bounded subset of a locally compact, complete convex metric space is compact.

The following theorem shows that the spaces to which (1.26) applies are precisely the locally connected generalized continua. Hence, the following parallel is established to the characterization (1.24) of Peano continua.

Theorem 1.27. A locally compact space is a locally connected generalized continuum if and only if it admits a complete convex metric.

Proof: Necessity is given by (1.25). For sufficiency, let M be a locally compact space with a complete convex metric D. Then M is connected and locally connected [22]. Since by (1.26) each closed ball is compact and thus separable, and since M is a countable union of such closed balls, then M is separable. Therefore, M is a locally connected generalized continuum. ∎

The requirement of local compactness cannot be omitted in the "sufficiency" part of the proof of (1.27), as the following example shows.

Example 1.28. The space L^p, $1 \leq p < \infty$, or in fact any infinite dimensional Banach space with metric given by the norm, is a complete convex metric space which is not locally compact [27].

The proof of (1.27) provides an apt illustration of the usefulness of (1.26), although, as will be seen in (3.12), this tool is actually not required for the above result. Another application of (1.26) can be seen in the following theorem, which states roughly that every non-compact, locally connected generalized continuum with a complete convex metric contains an isometric copy of the closed ray $[0, \infty)$ of E^1.

<u>Theorem</u> 1.29. Let p be any point of a locally connected generalized continuum M with a complete convex metric D. If M is not compact, then there is a subset R_p of M containing p that is isometric with the closed ray $[0, \infty)$ of E^1; moreover, there is a closed retraction of M onto R_p.

Proof: Since M is not compact, then by (1.26) M cannot be D bounded. Therefore, for each non-negative integer n the set $C_n = \{x: px = n\}$ contains some point r_n. Since C_n is closed and bounded, it is compact by (1.26). By (1.13) there is a segment $\overline{pr_n}$ for each positive n, and for each $0 \leq m < n$ the segment $\overline{pr_n}$ intersects C_m in exactly one point $q_{n,m}$. It is noted that

$$\sum_{i=0}^{n-1} q_{n,i}q_{n,i+1} = pr_n \quad \text{for each } n. \tag{10}$$

Since C_1 is compact, some subsequence $\langle q_{n(i),1}\rangle$ of $\langle q_{n,1}\rangle$ converges to a point p_1 of C_1. Denote p by p_0, and assume for an induction hypothesis that for $1 \leq j \leq k$, a point p_j of C_j has been chosen such that

$$\sum_{i=0}^{k-1} p_i p_{i+1} = pp_k$$

and that p_j is the limit of a subsequence of $\langle q_{n,j}\rangle$. In particular, there is a subsequence $\langle q_{n',k}\rangle$ of $\langle q_{n,k}\rangle$ that converges to p_k, where $n' \geq k+2$ for each i. Since C_{k+1} is compact, the subsequence $\langle q_{n',k+1} \rangle$ has a subsequence $\langle q_{n'',k+1} \rangle$ that converges to a point p_{k+1} of C_{k+1}. Since by (10) and (1.6) the point $q_{n'',k}$ is a between point of p and $q_{n'',k+1}$ for each n'', and since the sequences $\langle q_{n'',k} \rangle$ and $\langle q_{n'',k+1} \rangle$ converge respectively to p_k and p_{k+1}, then

$$pp_{k+1} = pp_k + p_k p_{k+1} = \sum_{i=0}^{k} p_i p_{i+1},$$

and the induction argument is completed. Therefore, there is a sequence $p = p_0, p_1, p_2, \ldots$ of points of M such that

$$\sum_{i=0}^{n-1} p_i p_{i+1} = pp_n = n$$

for each positive integer n.

The set R_p is now constructed from $\langle p_n \rangle$ by induction on n. There is a segment $\overline{pp_1}$ by (1.13). If for $1 \leq k$ it is assumed that there are segments $\overline{pp_1} \subset \overline{pp_2} \subset \cdots \subset \overline{pp_k}$, then since $pp_k p_{k+1}$ holds, by (1.14) it follows that there is a segment $\overline{pp_{k+1}}$ containing $\overline{pp_k}$. By induction there is therefore an infinite sequence of segments, $\overline{pp_1} \subset \overline{pp_2} \subset \ldots \subset \overline{pp_n} \subset \ldots$. Let

$$R_p = \bigcup_{n=1}^{\infty} \overline{pp_n}.$$

If $f : R_p \to E^1$ is defined by $f(q) = pq$, it is seen from the construction of $\langle p_n \rangle$ and R_p that $f(R_p)$ is the closed ray $[0, \infty)$.

Moreover, since any two points r and s of R_p lie together in some segment $\overline{pp_n}$, then either $pr + rs = ps$ or $ps + sr = pr$, and in both cases $|f(r) - f(s)| = |pr - ps| = rs$. Hence, f is an isometry.

Now define $g : M \to R_p$ by $g(q) = f^{-1}(pq)$. Since D and f^{-1} are both continuous functions, so is g. Moreover, since g is just the identity on R_p, then g is a retraction of M onto R_p. Suppose now that H is a closed subset of M such that $g(H)$ has an accumulation point t in R_p. If $\delta = pt + 1$, then $H \cap \overline{D}(p;\delta)$ is compact, and the continuity of g insures that $g(H \cap \overline{D}(p;\delta))$ is also compact. The inclusion $g(H) \cap \overline{D}(t;1) \subset g(H \cap \overline{D}(p;\delta))$ follows from the fact that $D(p,g(u)) = D(p,u)$ holds for each u of M. Thus, since t is an accumulation point of $g(H) \cap \overline{D}(t;1)$, then t is also an accumulation point of the compact set $g(H \cap \overline{D}(p;\delta))$; in particular, t is in $g(H)$. Therefore, g is a closed mapping. ∎

Example 1.30. The retraction constructed in the proof of (1.29) may not be open.

Proof: Let M be the planar set composed of the union of the closed unit disc and the strip $[0, \infty) \times [-1, 1]$ in E^2, and let D be the restricted euclidean metric. If p is the origin, then R_p must be the non-negative axis. If q has a negative abscissa and is one unit from p, then q has no local basis consisting of sets whose images, under the retraction defined above, are open. ∎

The obvious parallelism between the characterizations (1.24) and (1.27) of Peano continua and locally connected generalized continua by the admission of complete convex metrics, suggests that certain other results that have been proven for Peano continua might be generalized to locally connected generalized continua, especially where the complete convex metrics play a part. The main objective of this dissertation is to prove results for locally connected generalized continua that parallel the following metric extension theorem of Bing [4].

Theorem 1.31. If M_1 and M_2 are intersecting Peano continua whose topologies agree on their intersection, and if D_1 is a convex metric for M_1, there is a convex metric D_3 for $M_1 \cup M_2$ that extends D_1.

CHAPTER II

VARIETIES OF COMPLETE CONVEX METRICS

Since by (1.13) every two points of a complete convex metric space are joined by a segment, it might be supposed, out of analogy with euclidean space, that much of the classical theory of convexity would transfer easily to such spaces. However, such is far from the truth. Although it is not the purpose of this dissertation to investigate the possibilities of generalizing the theory of convexity in this way, as has been done in part by Blumenthal [8] and Rinow [21], it should become evident from the material now to be presented that complete convex metric spaces may depart drastically from the familiar euclidean geometry. But to illustrate non-euclidean pathologies that appear in complete convex metric spaces is only a secondary purpose of this chapter. The primary purpose is to lay a foundation for applications of the extension theorems of Chapter V; this will be done by introducing three varieties of complete convex metrics from the literature. A certain number of examples and results of an expository nature will be in order here, since "the literature concerning relationships between [these three] properties of convex metric spaces is not satisfactory," according to Lelek [15]. But again, the major objective of the present chapter is to provide preliminary results to be used in applications of the extension theorems.

Definitions and Characterizations

The following definition is due to Lelek and Nitka [17]; these properties are also discussed by Rolfsen [22].

Definition 2.1. A metric is said to satisfy condition (α), (β), or (γ), respectively, if for any points p, q, r and s of the space it holds that:

(α) If prq and psq, then $\{p, q, r, s\}$ is linear.

(β) If pqr and pqs, then $\{p, q, r, s\}$ is linear.

(γ) If pqr and spq, then $\{p, q, r, s\}$ is linear.

A complete convex metric on a space, as well as the space itself, is said to be:

(i) Strongly convex (SC) if it satisfies condition (α).

(ii) Without ramifications (WR) if it satisfies condition (β).

(iii) Without edges (WE) if it satisfies condition (γ).

Moreover, if a metric is both SC and WR, it is described as being SC-WR, and so for other combinations of these three properties.

A simple but useful result is the following.

Theorem 2.2. Let $\{p, q, r, s\}$ be a linear set in a metric space.

(i) If prq and psq, then either $pq = ps + sr + rq$ or $pq = pr + rs + sq$.

(ii) If pqr and pqs, then either $pr = pq + qs + sr$ or $ps = pq + qr + rs$.

(iii) If pqr and spq, then $sr = sp + pq + qr$.

Proof: Since the set $\{p, q, r, s\}$ is linear, the metric space can be assumed to be E^1 with the usual metric. The conclusions are then apparent. ∎

The conditions defining SC, WR, and WE metrics in (2.1) provide little geometrical insight into these properties. The conditions given in the following three characterization theorems, in addition to aiding the geometrical intuition, prove to be quite useful in the sequel.

<u>Theorem</u> 2.3. In a complete convex metric space (M, D), the following statements are equivalent:

(i) The metric D satisfies condition (α), hence is SC.

(ii) Every pair of points of M has a unique midpoint.

(iii) Between every pair of points of M is a unique segment.

Proof: It is shown that (i) → (ii) → (iii) → (i).

(i) → (ii) If M is SC, then let m and m' be midpoints of the two points p and q. Since pmq and $pm'q$ both hold, then the set $\{p, m, m', q\}$ is linear by condition (α), hence without loss of generality $pq = pm + mm' + m'q$ holds by (2.2.i). Since also $pm = mq = pq/2 = pm' = m'q$ holds, then $mm' = 0$ implies that $m = m'$.

(ii) → (iii) If S and S' are both segments from p to q with $S \neq S'$, then there are points u, v of $S \cap S'$ such that the subarcs of S and S' from u to v are independent arcs. But since each one of these subarcs is a segment from u to v by reason of the restricted isometries, then each contains a midpoint of u and v. Thus, u and v have more than one midpoint, contradicting (ii). Therefore, $S = S'$.

(iii) → (i) Let prq and psq hold for two points p and q.

Then by (1.14), points r and s each lie on a segment from p to q, and by (iii) this segment is uniquely $\overline{pq}$. Thus, it holds that $\{p, r, s, q\} \subset \overline{pq}$, and the linearity of $\{p, r, s, q\}$ follows from that of $\overline{pq}$. **I**

<u>Theorem</u> 2.4. In a complete convex metric space (M, D), the following statements are equivalent:

(i) The metric D satisfies condition (β), hence is WR.

(ii) If pqr, pqr', and $qr = qr'$ hold, then $r = r'$ follows.

(iii) Whenever q is a midpoint of p and r, and also of p and r', then $r = r'$.

(iv) If $\overline{pq} \subset \overline{pr} \cap \overline{ps}$ holds, then $\overline{pr} \cup \overline{ps}$ is a segment.

Proof: The plan of the proof is to show (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (iv) $\rightarrow$ (i).

(i) $\rightarrow$ (ii) If (i) holds, let q satisfy the hypothesis of (ii). Then by condition (β), pqr and pqr' imply that $\{p, q, r, r'\}$ is a linear set, and by (2.2.ii) it follows that $pr' = pq + qr + rr'$ without loss of generality. But $qr = qr'$ implies by the triangle inequality that $pr' = pq + pr' + r'r \geq pr' + r'r \geq pr$. Since $pr \geq pr'$ similarly, it follows that $pr' = pr' + r'r = pr$, hence $r'r = 0$. Therefore, $r = r'$.

(ii) $\rightarrow$ (iii) Points p, q, r, and r' that satisfy the hypothesis of (iii) must also satisfy the hypothesis of (ii), since $qr = pq = qr'$. Therefore, (iii) follows.

(iii) $\rightarrow$ (iv) Suppose that $\overline{pq} \subset \overline{pr} \cap \overline{ps}$ holds. It is first shown that $\overline{pr} \cup \overline{ps}$ cannot contain two independent arcs joining two points x and y of $\overline{pr} \cap \overline{ps}$. For suppose there were two such arcs, where without loss of generality x precedes y on $\overline{pr}$ from p to r. Then, since there is a segment $\overline{pq}$ in $\overline{pr} \cap \overline{ps}$, it must be that $p \neq x$. If $\overline{px}$ is the subsegment of $\overline{pr}$ joining p and x, and if $\overline{xs}$ is the subsegment of $\overline{ps}$ joining x and s, then since pxs holds, it follows from (1.9) that $\overline{px} \cup \overline{xs}$ is a segment from p to s. If $\delta = \min \{px, xy/2\}$ and $C = \{z: xz = \delta\}$, then $\overline{px}$, $\overline{xs}$, and the subsegment $\overline{xy}$ of $\overline{pr}$ intersect C in the points p', s', and r', respectively. Then $p'xr'$ and $p'xs'$, along with the fact that $\delta = p'x = xr' = xs'$, show that x is a midpoint of both p', r' and p', s'. Hence by (iii) it follows that $r' = s'$, a contradiction since $\delta < xy$. Therefore, $\overline{pr} \cup \overline{ps}$ cannot contain two independent arcs joining two of the points of $\overline{pr} \cap \overline{ps}$.

Thus, let q' be the last point of $\overline{pr}$, from p to r, that lies on $\overline{ps}$. Since there is a point q such that $\overline{pq} \subset \overline{pr} \cap \overline{ps}$, then $p \neq q'$. Moreover, by the previous paragraph, the subsegment $\overline{pq'}$ of $\overline{pr}$ is also a subsegment of $\overline{ps}$. In the case that $r \neq q' \neq s$, let $\delta = \min \{pq', q'r, q's\}$ and $C = \{z: q'z = \delta\}$. Then C intersects $\overline{pr} \cup \overline{ps}$ in exactly three points p', r', and s', where p' is on $\overline{pq'}$, r' is on the subsegment $\overline{q'r}$ of $\overline{pr}$, and s' is on the subsegment $\overline{q's}$ of $\overline{ps}$. Therefore q' is a midpoint of both p', r' and p', s', and by (iii) it follows that $r' = s'$. Hence, $\overline{pr} \cup \overline{ps}$ contains two independent arcs joining q' to $r' = s'$, in contradiction to the conclusion of the preceding paragraph.

Therefore, it must happen that either $q' = r$ or $q' = s$, in which case $\overline{pq'}$ is either $\overline{pr}$ or $\overline{ps}$, and $\overline{pr} \cup \overline{ps}$ is either $\overline{ps}$ or $\overline{pr}$.

(iv) → (i) Let pqr and pqs hold in M. By (1.13) there is a segment $\overline{pq}$, and by (1.14) there are segments $\overline{pr}$ and $\overline{ps}$ such that $\overline{pq} \subset \overline{pr} \cap \overline{ps}$. Therefore, since by (iv) the set $\overline{pr} \cup \overline{ps}$ is a segment, the subset {p, q, r, s} is linear. Thus, D satisfies condition (β) and is SC. ∎

<u>Theorem</u> 2.5. In a complete convex metric space (M, D), the following statements are equivalent:

(i) The metric D satisfies condition (γ), hence is WE.

(ii) If wxy and xyz hold, then wz = wx + xy + yz follows.

(iii) If wxy and xyz hold with wx = yz, then it follows that wz = wx + xy + yz.

(iv) If wxy and xyz hold with wx = yz and m is a midpoint of x and y, then m is a midpoint of w and z.

(v) If x is a midpoint of w and m, y is a midpoint of m and z, and m is a midpoint of x and y, then m is a midpoint of w and z.

(vi) If it holds that $\overline{sq} \cap \overline{pr} = \overline{pq}$, then $\overline{sq} \cup \overline{pr}$ is a segment from s to r.

Proof: The theorem will be proved by showing that (i) → (ii) → (iii) → (iv) → (v) → (i) and that (i) → (vi) → (i).

(i) → (ii) If it holds that wxy and xyz, then (i) by condition (γ) implies that {w, x, y, z} is linear. By (2.2.iii), it follows that wz = wx + xy + yz.

(ii) → (iii) Statement (iii) has a stronger hypothesis than (ii), but the same conslusion.

(iii) → (iv) If wxy and xyz hold with wx = yz and m is a midpoint of x and y, then (iii) implies wz = wx + xy + yz. Since m is a midpoint of x and y, it holds that xy = xm + my with xm = my. Hence wz = wx + xm + my + yz = wm + mz holds with wm = wx + xm = my + yz = mz. Therefore, m is a midpoint of w and z.

(iv) → (v) Let x be a midpoint of w and m, y a midpoint of m and z, and m also a midpoint of x and y. Then wxm and xmy hold. By (1.14), there is a segment $\overline{xy}$ containing m. If $\overline{xm}$ is a subsegment of $\overline{xy}$, then since wxm holds, by (1.14) again there is a segment $\overline{wm}$ containing $\overline{xm}$. Let n be the midpoint of x and m in $\overline{xm}$. Since wxm and xmy hold with wx = xm = my, and n is a midpoint of x and m, then from (iv) it follows that n is a midpoint of w and y. Thus, because of wny, wxn on $\overline{wm}$, and nmy on $\overline{xy}$, there follows wy = wn + ny = wx + xn + nm + my = wx + xy; that is, wxy.

An argument similar to the one in the preceding paragraph shows that xyz. Since it holds that wxy, xyz, wx = xm = my = yz, and m is a midpoint of x and y, then by (iv) it follows that m is a midpoint of w and z.

(v) → (i) Let pqr and spq hold, and suppose that the set {p, q, r, s} is not linear. By (1.13), there is some segment $\overline{pq}$, and by (1.14), there are segments $\overline{sq}$ and $\overline{pr}$ containing $\overline{pq}$. The set $X = \{h \in \overline{sq}: hqr \text{ or } h = q\}$ is non-empty, since pqr holds. Moreover, X is closed in $\overline{sq}$, for suppose h is the limit of a sequence of points $\langle h_n \rangle$ in $X \setminus \{q\}$. Then from the continuity of the metric it follows that

$$hr = \lim_{n\to\infty} h_n r = \lim_{n\to\infty} (h_n q + qr) = \lim_{n\to\infty} h_n q + qr = hq + qr,$$

so that $h \varepsilon X$. Therefore, since X is closed, then X intersects $\overline{sq}$ in a first point p_0 from x to q. It follows that $p_0 r = p_0 q + qr$ and $sp = sp_0 + p_0 p$ and also that $p_0 \neq s$; for if $p_0 = s$, then $sr = sp + pq + qr$, that is, $\{p, q, r, s\}$ would be linear, contrary to assumption. In a similar manner, the set $Y = \{k \varepsilon \overline{pr}\colon\ spk \text{ or } k = p\}$ intersects $\overline{pr}$ in a last point q_0 from p to r, with $sq_0 = sp + pq_0$, $qr = qq_0 + q_0 r$, and $q_0 \neq r$. Therefore, if $\overline{sp}$ and $\overline{p_0 q}$ are contained in $\overline{sq}$, and if $\overline{qr}$ and $\overline{pq_0}$ are in $\overline{pr}$, then $\overline{sp} \cup \overline{pq_0}$ and $\overline{p_0 q} \cup \overline{qr}$ are segments $\overline{sq_0}$ and $\overline{p_0 r}$, respectively, by (1.19). Moreover, if $\overline{p_0 q_0}$ is a subsegment of $\overline{sq_0}$, then

$$\begin{aligned}
\overline{p_0 q_0} &= (\overline{p_0 q_0} \cap \overline{sp}) \cup (\overline{p_0 q_0} \cap \overline{pq_0}) \\
&= \{x \varepsilon \overline{sp}\colon\ p_0 q_0 = p_0 x + x q_0\} \cup \{x \varepsilon \overline{pr}\colon\ p_0 q_0 = p_0 x + x q_0\} \\
&= \{x \varepsilon \overline{sq}\colon\ p_0 p = p_0 x + xp\} \cup \{x \varepsilon \overline{pq}\colon\ pq_0 = px + xq_0\} \\
&\quad \cup \{x \varepsilon \overline{qr}\colon\ qq_0 = qx + xq_0\} \\
&= \{x \varepsilon \overline{p_0 q}\colon\ p_0 p = p_0 x + xp\} \cup \{x \varepsilon \overline{p_0 q}\colon\ pq_0 = px + xq_0\} \\
&\quad \cup \{x \varepsilon \overline{qr}\colon\ qq_0 = qx + xq_0\} \\
&= \{x \varepsilon \overline{p_0 q}\colon\ p_0 p = p_0 x + xp \text{ or } pq_0 = px + xq_0\} \\
&\quad \cup \{x \varepsilon \overline{qr}\colon\ p_0 q_0 = p_0 x + xq_0\} \\
&= \{x \varepsilon \overline{p_0 q}\colon\ p_0 q_0 = p_0 x + xq_0\} \cup \{x \varepsilon \overline{qr}\colon\ p_0 q_0 = p_0 x + xq_0\} \\
&= (\overline{p_0 q_0} \cap \overline{p_0 q}) \cup (\overline{p_0 q_0} \cap \overline{qr}) \\
&= \overline{p_0 q_0} \cap \overline{p_0 r}.
\end{aligned}$$

Hence, not only does $\overline{p_0 q_0} \subset \overline{sq_0}$ hold, but also $\overline{p_0 q_0} \subset \overline{p_0 r}$.

Let $\overline{sp_0} \subset \overline{sq_0}$ and $\overline{q_0r} \subset \overline{p_0r}$, and define the set

$$E = \{\varepsilon \geq 0: \text{ there are points } a_\varepsilon \in \overline{sp_0},\ b_\varepsilon \in \overline{q_0r} \text{ with } a_\varepsilon p_0 = q_0 b_\varepsilon = \varepsilon,\ a_\varepsilon p b_\varepsilon\}.$$

The set E is non-empty, since $0 \in E$, and is bounded above by $\min\{sp_0, q_0r\}$. Therefore the number $\beta = \sup E$ exists, and there is a sequence $<\varepsilon_n>$ of E such that $\varepsilon_n \leq \varepsilon_{n+1}$ for every n and

$$\lim_{n\to\infty} \varepsilon_n = \beta.$$

Let a_n and b_n denote a_{ε_n} and b_{ε_n}, respectively, and let a and b denote the points of $\overline{sp_0}$ and $\overline{q_0r}$, respectively, such that $ap_0 = q_0b = \beta$. Then

$$\lim_{n\to\infty} aa_n = \lim_{n\to\infty} (ap_0 - a_np_0) = \lim_{n\to\infty} (\beta - \varepsilon_n) = 0,$$

and similarly

$$\lim_{n\to\infty} bb_n = 0.$$

By the triangle inequality, $a_nb_n \leq a_na + ab + bb_n$. Hence,

$$ab = \lim_{n\to\infty} (a_na + ab + bb_n) \geq \lim_{n\to\infty} a_nb_n = \lim_{n\to\infty} (a_np + pb_n) = ap + pb.$$

Therefore, $ab = ap + pb$ holds. If it were the case that $a = s$, then would follow $sb = sp + pb$ and $\beta > 0$, from which it could be concluded that $b \in Y$ and q_0br on $\overline{pr}$, contradicting the definition of q_0. Hence, $a \neq s$ and similarly $b \neq r$.

Let $2\delta = \min\{sa, ab, br\}$, and let w and z be the points of $\overline{sa} \subset \overline{sp_0}$ and $\overline{br} \subset \overline{q_0r}$, respectively, such that $wa = bz = \delta$.

Let m be the midpoint of p_0 and q_0 that lies on $\overline{p_0q_0}$. Let x be the midpoint of w and m in $\overline{sq_0}$, and let y be the midpoint of m and z that lies in $\overline{p_0r}$. Since it is true that $wm = wa + ap_0 + p_0m = mq_0 + q_0b + bz = mz$, then it must hold that $xm = wm/2 = mz/2 = my$. Moreover, $am = ap_0 + p_0m = mq_0 + q_0b = mb$ holds. The relationship amb is shown from the equation $ab = ap + pb = ap_0 + p_0p + pq_0 + q_0b = ap_0 + p_0q_0 + q_0b = ap_0 + p_0m + mq_0 + q_0b = an + mb$. Therefore, m is a midpoint of a and b, from which it follows that $am = mb = ab/2 \geq \delta$. Thus it must be that $xm = wm/2 = (wa + am)/2 = (\delta + am)/2 \leq (am + am)/2 = am$, hence that $am = ax + xm$ holds for $\{a, x, m\} \subset \overline{sq_0}$. Similarly, $mb = my + yb$ holds. Since amb, then $ab = am + mb = ax + xm + my + yb$, from which it follows by (1.6) that $xy = xm + my$, or xmy. Thus, m is a midpoint of x and y, since $xm = ym$ was shown previously. In summary, x is a midpoint of w and m, y is a midpoint of m and z, and m is a midpoint of x and y. From (v) it follows that m is also a midpoint of w and z. Hence, $wz = wm + mz = wp_0 + p_0m + mq_0 + q_0z = wp_0 + p_0q_0 + q_0z = wp_0 + p_0p + pq_0 + q_0z$ holds, implying wpz. But $wp_0 = wa + ap_0 = \delta + \beta = q_0b + bz = q_0z$ holds, and the inequality $wp_0 = q_0z > \beta$ contradicts the definition of β. In this way, the assumption that $\{p, q, r, s\}$ is not linear is shown to be false, and (i) is proved.

(i) → (vi) Suppose $\overline{sq} \cap \overline{pr} = \overline{pq}$ holds. If $p = s$ then $\overline{pq} = \overline{sq}$ implies that $\overline{sq} \cup \overline{pr} = \overline{pq} \cup \overline{pr} = \overline{pr}$. If $q = r$, then similarly $\overline{sq} \cup \overline{pr} = \overline{sq}$ holds. It may therefore be assumed that $p \neq s$ and $q \neq r$, and it follows from $\overline{sq} \cap \overline{pr} = \overline{pq}$ that spq and pqr. By (i) therefore, the set $\{p, q, r, s\}$ is linear, and from (2.2) it follows that $sr = sp + pq + qr$ holds, and in particular spr from (1.6). If $\overline{sp} \subset \overline{sq}$, then $\overline{sq} \cup \overline{pr} = \overline{sp} \cup \overline{pq} \cup \overline{pr} = \overline{sp} \cup \overline{pr}$ follows. Thus by (1.9), it follows that $\overline{sq} \cup \overline{pr}$ is a segment from s to r.

(vi) → (i) Let spq and pqr hold, and by (1.13) pick some segment $\overline{pq}$. From (1.14), there are segments $\overline{sq}$ and $\overline{pr}$ such that $\overline{pq} \subset \overline{sq} \cap \overline{pr}$ holds. To show actual set equality, suppose that there is a point u of $(\overline{sq} \setminus \overline{pq}) \cap (\overline{pr} \setminus \overline{pq})$. Then upq and pqu both hold, a contradiction to (1.8). Therefore, it must be that $\overline{pq} = \overline{sq} \cap \overline{pr}$. By (vi) it follows that $\overline{sq} \cup \overline{pr}$ is a segment, and therefore the subset $\{p, q, r, s\}$ is linear. Hence, (i) holds. **I**

Examples

The independence of the properties SC, WR, and WE is a natural area for investigation, once they have been defined and characterized. For example, the question could be raised, Is every SC-WR metric also WE? To this question no answer has yet appeared in the literature, although Lelek [15] suspects that the answer is negative. If this were indeed the case, then the negative answer, together with the following examples, would show that these properties are entirely independent of one another in the sense of logical implication. But before these examples are presented, it will be convenient to define a particular metric, following Busemann [10].

<u>Definition</u> 2.6. Let (M, E) be a metric space, each two points of which are joined by at least one arc of finite length with respect to the metric E. For points $x \neq y$, define $D(x,y)$ to be the infimum of the lengths of all arcs joining x and y. Then D is a metric

on the set M, called the geodesic metric obtained from E. When M is a subset of euclidean space, then by "the geodesic metric on M" is meant that one obtained from the usual euclidean metric restricted to M.

Example 2.7. The usual metric for E^n is SC-WR-WE.

Example 2.8. The geodesic metric on the union in E^n of three euclidean segments sharing precisely one common endpoint is SC-WE, but not WR.

Example 2.9. The geodesic metric on the 2-sphere S^2 is WR but is neither SC nor WE.

Example 2.10. If the metric of (2.9) is restricted to the part of S^2 that lies in the non-negative x and y half-spaces, then it is WE-WR, but not SC.

Example 2.11. If a euclidean segment has in common with the space of (2.10) exactly one of its endpoints, the resultant geodesic metric is WE but is neither SC nor WR.

Example 2.12. The geodesic metric on the union in E^2 of the unit circle and the segment $[1, 2] \times \{0\}$ is neither SC, WR, nor WE.

It should be noted that the metrics for the examples of (2.7) through (2.12) are all convex, and since the spaces are compact, the metrics are also complete.

The final example in this section is related to an interesting phenomenon in the literature. Busemann [10] has shown that in a locally connected generalized continuum with a SC-WR metric which satisfies the additional property that for every two points x and y there exists a point z with xyz, then any two points uniquely determine a "straight line," that is, a subspace isometric to E^1. Moreover, in such a space a "perpendicular" can be constructed from a point to a "straight line" if and only if the closed balls are convex. The convexity of the closed balls thus becomes a rather significant point. Glynn [12] poses the question whether in a Peano continuum with an SC metric, the closed balls are necessarily convex. The following example answers Glynn's question in the negative. A further result on the convexity of balls will be given in (6.3).

Example 2.13. A 2-cell admits a complete convex metric that is SC but neither WR nor WE, having closed balls that are not convex.

Proof: Define the 2-cell $C = \{(\rho,\Theta): 0 \leq \rho \leq \cos\Theta, -.73 \leq \Theta \leq .73\}$ by using polar coordinates in E^2, as Figure 1 illustrates on the following page. The set C is composed of "ridge sections" $R(\Theta,r_1,r_2) = \{(\rho,\Theta): r_1 \cos\Theta \leq \rho \leq r_2 \cos\Theta$ or $r_2 \cos\Theta \leq \rho \leq r_1 \cos\Theta\}$, defined for all $-.73 \leq \Theta \leq .73$ and for all $0 \leq r_1, r_2 \leq 1$. Set C is also composed fo "arch sections" of the form $A(r, \Theta_1, \Theta_2) = \{(\rho,\Theta): \rho = r\cos\Theta, \Theta_1 \leq \Theta \leq \Theta_2$ or $\Theta_2 \leq \Theta \leq \Theta_1\}$, defined for all $0 \leq r \leq 1$ and for all $-.73 \leq \Theta_1, \Theta_2 \leq .73$. The arclength L of these sections, with respect to the euclidean metric $\|\circ\|$, is given by the expressions

$$L(R(\Theta,r_1,r_2)) = |r_1 \cos\Theta - r_2 \cos\Theta| = |r_1 - r_2| \cos\Theta, \text{ and}$$

$$L(A(r,\Theta_1,\Theta_2)) = \left| \int_{\Theta_1}^{\Theta_2} \sqrt{r^2 \cos^2\Theta + r^2 \sin^2\Theta}\, d\Theta \right| = r\,|\Theta_1 - \Theta_2|.$$

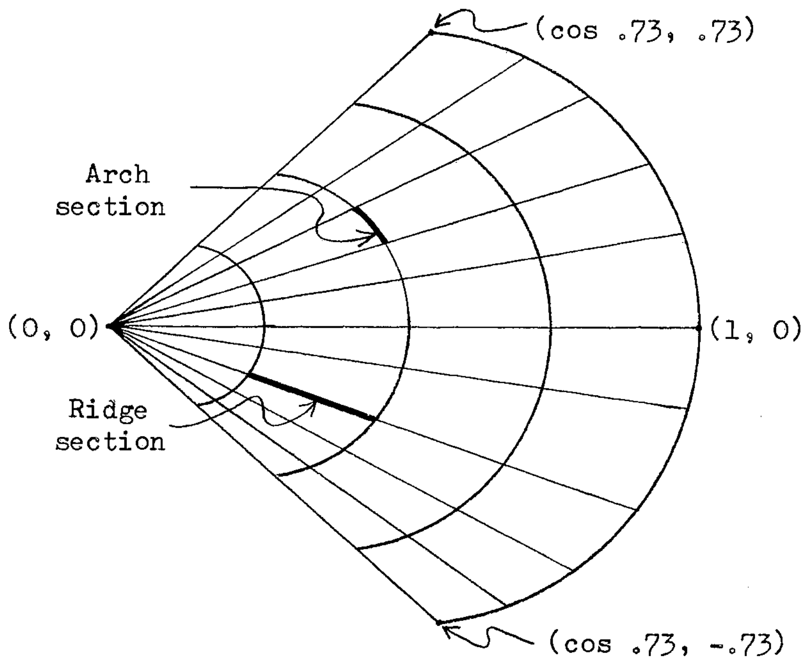

Figure 1. The Space C of (2.13)

The metric D may now be defined on $C \times C$ as follows:

if $x = y$, $D(x,y) = 0$;

if $x \neq y$, $D(x,y) = \inf \{ \sum_{i=1}^{n} L(S_i)$: the S_i are alternately non-degenerate ridge sections or non-degenerate arch sections with endpoints p_i and q_i such that $x = p_1$, $y = q_n$, and $q_i = p_{i+1}$ for all $i = 1, 2, \ldots, n-1$; for all $n = 1, 2, \ldots \}$.

The section S_i may also be designated (S_i, p_i, q_i) when it is desired to specify its endpoints.

The following assertions are now to be proved.

(i) If $x \neq y$ and $\langle S_i \rangle_{i=1}^{n}$ is a finite sequence of ridge sections and arch sections with endpoints p_i and q_i respectively such that $x = p_1$, $y = q_n$, and $q_i = p_{i+1}$ for $i = 1, 2, \ldots, n-1$, then there is a finite sequence $\langle S_i' \rangle_{i=1}^{n'}$ as in the definition of D such that $n' \leq n$ holds, along with

$$\bigcup_{i=1}^{n} S_i' \subset \bigcup_{i=1}^{n} S_i \quad \text{and} \quad \sum_{i=1}^{n'} L(S_i') \leq \sum_{i=1}^{n} L(S_i).$$

(ii) $\|x - y\| \leq D(x,y)$ for all $x, y \in C$.

(iii) D is a metric on the set C.

(iv) If $0 \leq r_1 < r_2 \leq 1$ and $-.73 \leq \theta_1, \theta_2 \leq .73$ with $\theta_1 \neq \theta_2$, then it holds that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) < L(R(\theta_1, r_1, r_2)) + L(A(r_2, \theta_1, \theta_2)).$$

(v) If $0 \leq r_1, r_2 \leq 1$ and $-.73 \leq \theta_1, \theta_2 \leq .73$ with $r_1 \neq r_2$, then it holds that

$$L(A(r_2, \theta_1, \theta_2)) < L(R(\theta_1, r_1, r_2)) + L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)).$$

(vi) If $0 \leq r_1, r_2 \leq 1$ and $-.73 \leq \theta_1, \theta_2 \leq .73$ with $\theta_1 \neq \theta_2$ and $r_1 + r_2 \neq 0$, then it holds that

$$L(R(\theta_1, r_1, r_2)) < L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) + L(A(r_2, \theta_1, \theta_2)).$$

(vii) Each non-degenerate ridge section is the unique D segment joining its endpoints.

(viii) Each non-degenerate arch section is a D segment.

(ix) Each non-degenerate arch section is the unique D segment joining its endpoints.

(x) If $x = (r_1 \cos \Theta_1, \Theta_1)$ and $y = (r_2 \cos \Theta_2, \Theta_2)$ with $r_1 < r_2$ and $\Theta_1 \neq \Theta_2$, then $A(r_1, \Theta_1, \Theta_2)$ $R(\Theta_2, r_1, r_2)$ is a D segment $\overline{xy}$.

(xi) The D segment given in (x) is the only one joining the two points x and y.

(xii) D is equivalent to the usual planar metric restricted to the set C.

(xiii) D is SC.

(xiv) D is not WR.

(xv) D is not WE.

(xvi) The closed ball $\overline{D}((0, 0); \cos .73)$ is not convex.

(i) This assertion is obvious geometrically, and its proof can be formalized by means of an induction argument on n. The sequence $\langle(S_i, p_i, q_i)\rangle_{i=1}^{n}$ is reduced down to $\langle(S_i', p_i', q_i')\rangle_{i=1}^{n'}$ by repeated applications of the following two-step process: (1) omit all degenerate sections; (2) consolidate all adjacent sections of the same type.

(ii) Since the length of any arc from x to y will at least equal the usual distance $\|x - y\|$, and since the union of any finite sequence $\langle S_i \rangle_{i=1}^{n}$ of sections as in the definition of $D(x,y)$ must contain some arc A from x to y with

$$L(A) \leq \sum_{i=1}^{n} L(S_i),$$

then it follows that $\|x - y\| \leq D(x,y)$ holds.

(iii) It is immediate from (ii) and from the definition of D that $D(x,y) = 0$ if and only if $x = y$. Symmetry is observed from the definition of D. The triangle follows in a straightforward manner from (i).

(iv) If $\theta_1 < \theta_2$, then since $\cos x - x$ is a strictly decreasing function for $-.73 \leq x \leq .73$, it follows that the inequalities $\cos \theta_2 - \theta_2 < \cos \theta_1 - \theta_1$, $\cos \theta_2 - \cos \theta_1 < \theta_2 - \theta_1$, and $(r_2 - r_1)(\cos \theta_2 - \cos \theta_1) < (r_2 - r_1)(\theta_2 - \theta_1)$ all hold. Thus,

$$r\,|\theta_1 - \theta_2| + (r_2 - r_1) \cos \theta_2 < (r_2 - r_1) \cos \theta_1 + r_2\,|\theta_1 - \theta_2| \qquad (1)$$

must hold. If $\theta_1 > \theta_2$, then $-\theta_1 < -\theta_2$, so that θ_1 and θ_2 may be replaced in (1) by their negatives, and (1) is again obtained.

(v) If $r_1 < r_2$, then $|\theta_1 - \theta_2| \leq 2(.73) < 2 \cos .73 \leq \cos \theta_1 + \cos \theta_2$, hence $(r_2 - r_1)|\theta_1 - \theta_2| < (r_2 - r_1)(\cos \theta_1 + \cos \theta_2)$, so

$$r_2\,|\theta_1 - \theta_2| < |r_2 - r_1| \cos \theta_1 + r_1\,|\theta_1 - \theta_2| + |r_2 - r_1| \cos \theta_2 \qquad (2)$$

must hold. If $r_2 < r_1$ and $\theta_1 \neq \theta_2$, then (2) is again established by $r_2 |\theta_1 - \theta_2| < r_1 |\theta_1 - \theta_2|$. If $r_2 < r_1$ and $\theta_1 = \theta_2$, then $r_2 |\theta_1 - \theta_2| = 0$ establishes (2).

(vi) Let $w = (r_1 \cos \theta_1, \theta_1)$, $x = (r_1 \cos \theta_2, \theta_2)$, $y = (r_2 \cos \theta_2, \theta_2)$, and $z = (r_2 \cos \theta_1, \theta_1)$. Since $\theta_1 \neq \theta_2$ holds and $A(r_1, \theta_1, \theta_2)$ is not a euclidean segment, then it follows that $\| w - x \| < L(A(r_1, \theta_1, \theta_2))$. Hence,

$$\begin{aligned} L(R(\theta_1, r_1, r_2)) &= \| w - z \| \leq \| w - x \| + \| x - y \| + \| y - z \| \\ &< L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) + L(A(r_2, \theta_1, \theta_2)). \end{aligned}$$

(vii) This assertion follows from (ii), since non-degenerate ridge segments are unique euclidean segments.

(viii) Let $x = (r \cos \theta_1, \theta_1)$ and $y = (r \cos \theta_2, \theta_2)$ with $r > 0$ and $\theta_1 \neq \theta_2$. It will be shown that if the finite sequence $\langle(S_i, p_i, q_i)\rangle_{i=1}^{n}$ is as in the definition of $D(x,y)$ with $n > 1$, then

$$L(A(r, \theta_1, \theta_2)) < \sum_{i=1}^{n} L(S_i). \tag{3}$$

This is established by induction. If $n = 2$, then there is no finite sequence $\langle(S_i, p_i, q_i)\rangle_{i=1}^{2}$ as in the definition of $D(x,y)$, since such a sequence would have to include one non-degenerate ridge section; that is, (3) holds vacuously for $n = 2$. In the case that $n = 3$, if $\langle(S_i, p_i, q_i)\rangle_{i=1}^{3}$ is as in the definition of $D(x,y)$, then as before there must be more than one ridge section; that is, S_1 and S_3 are non-degenerate ridge sections and S_2 is a non-degenerate arch section. Assertion (v) shows that (3) holds for $n = 3$.

Suppose it has been shown that (3) holds for $k - 1 \geq 3$. Let $<(S_i, p_i, q_i)>_{i=1}^{k}$ be as in the definition of $D(x,y)$. Since there are ridge sections in this finite sequence, there are also arch sections $A(s, \alpha, \beta)$ with $s \neq r$, and even $s < r$ without loss of generality. Let $s_0 = \min \{s\colon A(s, \alpha, \beta) = S_i$ for some $i = 1, 2, \ldots, k\}$. Then $A(s_0, \alpha_0, \beta_0) = S_j$ for some $1 < j < k$, $S_{j-1} = R(\alpha_0, s_0, s_1)$ for some $s_0 < s_1$, and $S_{j+1} = R(\beta_0, s_0, s_2)$ for some $s_0 < s_2$; without loss of generality let $s_1 \leq s_2$. The $k - 1$ term sequence $S_1, S_2, \ldots, S_{j-2}, A(s_1, \alpha_0, \beta_0), R(\beta_0, s_1, s_2), S_{j+2}, \ldots, S_k$ satisfies the hypothesis of (i), hence there is a finite sequence $<(T_i, a_i, b_i)>_{i=1}^{k'}$ as in the definition of $D(x,y)$ with $k' \leq k - 1$ and

$$\sum_{i=1}^{k'} L(T_i)$$

$$\leq \sum_{i=1}^{j-2} L(S_i) + L(A(s_1, \alpha_0, \beta_0)) + L(R(\beta_0, s_1, s_2)) + \sum_{i=1}^{k} L(S_i).$$

From (v) it follows that

$L(A(s_1, \alpha_0, \beta_0)) < L(S_{j-1}) + L(S_j) + L(S_{j+1}) - L(R(\beta_0, s_1, s_2))$, thus

$$\sum_{i=1}^{k'} L(T_i) < \sum_{i=1}^{k} L(S_i).$$

But the induction hypothesis yields

$$L(A(r, \theta_1, \theta_2)) \leq \sum_{i=1}^{k'} L(T_i),$$

so that

$$L(A(r, \theta_1, \theta_2)) < \sum_{i=1}^{k} L(S_i)$$

must follow.

The induction is therefore complete, and shows that if the finite sequence $<(S_i, p_i, q_i)>_{i=1}^{n}$ is as in the definition of $D(x,y)$ but distinct from $<(A(r, \theta_1, \theta_2), x, y)>$, then it holds that

$$L(A(r, \theta_1, \theta_2)) < \sum_{i=1}^{k} L(S_i).$$

Hence, $D(x,y) = L(A(r, \theta_1, \theta_2)) = r\ |\theta_1 - \theta_2|$ follows. Moreover, if $z = (r \cos \theta, \theta)$ is any point of $A(r, \theta_1, \theta_2)$, then θ is between θ_1 and θ_2 inclusively, so that the foregoing argument may now be applied to show that

$D(x,z) + D(z,y) = r\ |\theta_1 - \theta| + r\ |\theta - \theta_2| = r\ |\theta_1 - \theta_2| = D(x,y)$;

that is, $A(r, \theta_1, \theta_2)$ is a D segment $\overline{xy}$.

(ix) Let $x = (r \cos \theta_1, \theta_1)$ and $y = (r \cos \theta_2, \theta_2)$ with $\theta_1 \neq \theta_2$. It was shown in (viii) that $A(r, \theta_1, \theta_2)$ is a D segment. Let $z = (\rho \cos \phi, \phi)$ be a point that does not lie on $A(r, \theta_1, \theta_2)$.

If $\rho = r$, then $\emptyset$ does not lie between θ_1 and θ_2, so that one of the distances $D(x,z) = A(r, \theta_1, \emptyset)$ or $D(z,y) = A(r, \emptyset, \theta_2)$ exceeds $D(x,y)$, hence $D(x,z) + D(z,y) > D(x,y)$. If $\rho < r$, then

$$\begin{aligned} D(x,z) &+ D(z,y) \\ &= \rho\,|\theta_1 - \emptyset| + (r - \rho)\cos\theta_1 + \rho\,|\emptyset - \theta_2| + (r - \rho)\cos\theta_2 \\ &\geq (r - \rho)\cos\theta_1 + \rho\,|\theta_1 - \theta_2| + (r - \rho)\cos\theta_2 \\ &> r\,|\theta_1 - \theta_2| = D(x,y) \end{aligned}$$

holds by the real triangle inequality and by (v). If $\rho > r$, then

$$\begin{aligned} D(x,z) &+ D(z,y) \\ &= r\,|\theta_1 - \emptyset| + (\rho - r)\cos\emptyset + r\,|\theta_2 - \emptyset| + (\rho - r)\cos\emptyset \\ &\geq r\,|\theta_1 - \theta_2| + 2\,(\rho - r)\cos\emptyset \\ &> r\,|\theta_1 - \theta_2| = D(x,y) \end{aligned}$$

holds by the real triangle inequality.

Thus in any case, z is not a between point of x and y. Therefore, $A(r, \theta_1, \theta_2)$ is the unique D segment $\overline{xy}$.

(x) The method is induction on n, with the induction hypothesis given as follows: if $\langle(S_i, p_i, q_i)\rangle_{i=1}^{n}$ is a finite sequence as in the definition of $D(x,y)$, then

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) \leq \sum_{i=1}^{n} L(S_i)$$

holds if $n \geq 2$, with strict inequality holding for $n \geq 3$. If $n = 2$ and S_1, S_2 is as in the definition of $D(x,y)$, and if it is not true that $S_1 = A(r_1, \theta_1, \theta_2)$ and $S_2 = R(\theta_2, r_1, r_2)$, then it must hold that $S_1 = A(r_1, \Theta_1, \Theta_2)$ and $S_2 = A(r_2, \Theta_1, \Theta_2)$. Whichever form S_1, S_2 may take, statement (iv) insures that $L(A(r_1, \Theta_1, \Theta_2)) + L(R(\Theta_2, r_1, r_2)) \leq L(S_1) + L(S_2)$ holds, and it should be noted that (iv) does imply strict inequality if S_1, S_2 is not $A(r_1, \Theta_1, \Theta_2), R(\Theta_2, r_1, r_2)$.

Suppose that for $k - 1 \geq 2$ the induction hypothesis holds, and let $\langle(S_i, p_i, q_i)\rangle_{i=1}^{k}$ be as in the definition of $D(x,y)$. It may be assumed that $\Theta_1 < \Theta_2$, since inequalities given in (iv), (v), and (vi), as well as the present line of argument, do not depend essentially on a particular order for Θ_1 and Θ_2. Let $p_i = (\rho_i \cos \varnothing_i, \varnothing_i)$ for $i = 1, 2, \ldots, k$, and let $\rho_{k+1} = r_2$, $\varnothing_{k+1} = \Theta_2$.

Suppose that $\langle S_i\rangle_{i=1}^{k}$ contains a ridge section that goes left; that is, suppose $\rho_i < \rho_{i-1}$ holds for some i. Then either of two cases could hold. In the first case, $r_2 = \rho_{k+1} < \rho_i$ for some i. Let $\rho_M = \max\{\rho_i\colon i = 1, 2, \ldots, k\}$. Then $\rho_M = \rho_j > \rho_{j-1}$ for some index j. Therefore, S_{j-1} is the ridge section $R(\varnothing_j, \rho_{j-1}, \rho_j)$, S_j is the arch section $A(\rho_j, \varnothing_j, \varnothing_{j+1})$, and S_{j+1} is the ridge section $R(\varnothing_{j+1}, \rho_j, \rho_{j+2})$, with $\rho_j = \rho_{j+1} > \rho_{j+2}$. Without loss of generality it may be assumed that $\rho_{j-1} \leq \rho_{j+2}$. Then the $k - 1$ term sequence $S_1, \ldots, S_{j-2}, A(\rho_{j-1}, \varnothing_j, \varnothing_{j+1}), R(\varnothing_{j+1}, \rho_{j-1}, \rho_{j+2}), S_{j+2}, \ldots, S_k$ satisfies the hypothesis of (i). Therefore, there is a finite sequence $\langle S_i'\rangle_{i=1}^{k'}$ as in the definition of $D(x,y)$ satisfying

$$\sum_{i=1}^{k'} L(S_i') \leq \sum_{i=1}^{j-2} L(S_i) + L(A(\rho_{j-1}, \varnothing_j, \varnothing_{j+1})) + L(R(\varnothing_{j+1}, \rho_{j-1}, \rho_{j+2}))$$

$$+ \sum_{i=j+2}^{k} L(S_i)$$

for some $k' \leq k - 1$. Thus by the induction hypothesis, it holds that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) \leq \sum_{i=1}^{k'} L(S_i).$$

But by (v), it must hold that

$$\begin{aligned} &L(A(\rho_{j-1}, \emptyset_j, \emptyset_{j+1})) + L(R(\emptyset_{j+1}, \rho_{j-1}, \rho_{j+2})) \\ &\quad < L(R(\emptyset_j, \rho_{j-1}, \rho_j)) + L(A(\rho_j, \emptyset_j, \emptyset_{j+1})) + L(R(\emptyset_{j+1}, \rho_j, \rho_{j-1})) \\ &\qquad + L(R(\emptyset_{j+1}, \rho_{j-1}, \rho_{j+2})) \\ &\quad = L(S_{j-1}) + L(S_j) + L(S_{j+1}). \end{aligned}$$

Thus, it follows that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) \leq \sum_{i=1}^{k'} L(S_i') < \sum_{i=1}^{k} L(S_i).$$

In the second case, it happens that $\rho_i \leq r_2 = \rho_{k+1}$ holds for each i. Let $\rho_m = \min \{\rho_i: \rho_i < \rho_{i-1}, i = 1, \ldots, k+1\}$. The preceding set is non-empty since $\langle S_i \rangle_{i=1}^{k}$ does contain a ridge section that goes left. Then $\rho_m = \rho_j < \rho_{j-1}$ holds for some $1 < j < k + 1$. In fact, since S_{j-1} is the ridge section $R(\emptyset_j, \rho_j, \rho_{j-1})$, and S_j must be the arch section $A(\rho_j, \emptyset_j, \emptyset_{j+1})$, then it must hold that $j < k$ and S_{j+1} must exist as a ridge section $R(\emptyset_{j+1}, \rho_j, \rho_{j+1})$ with $\rho_j < \rho_{j+1}$. The argument now proceeds as in the first case in showing that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) < \sum_{i=1}^{k} L(S_i).$$

Suppose that $\langle S_i \rangle_{i=1}^{k}$ contains an arch section going down; that is, suppose $\emptyset_i < \emptyset_{i-1}$ holds for some i. Then the proof proceeds by use of (vi) in two cases, exactly as the preceding proof for ridge sections procceded by use of (v) in its two cases, and the result is

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) < \sum_{i=1}^{k} L(S_i).$$

Finally, suppose that all ridge sections go right and all arch sections go up; that is, suppose that $\rho_{i-1} \leq \rho_i$ and $\emptyset_{i-1} \leq \emptyset_i$ hold for all $i = 2, \ldots, k + 1$. Since $k - 1 \geq 2$, there is a first arch section $S_j = A(\rho_j, \emptyset_j, \emptyset_{j+1})$ with $\rho_j > \rho_1 = r_1$, for this will be the section following the first ridge section $S_{j-1} = R(\emptyset_j, \rho_{j-1}, \rho_j)$ with $\rho_{j-1} < \rho_j$. Of course, $j \geq 2$ must hold.

If $j = 2$, define $S_1' = A(\rho_{j-1}, \emptyset_j, \emptyset_{j+1})$; define $S_2' = R(\emptyset_{j+1}, \rho_{j-1}, \rho_j) \cup S_3 = R(\emptyset_{j+1}, \rho_{j-1}, \rho_{j+2})$; and define $S_i' = S_{i+1}$ for $i = 4, \ldots, k - 1$. Then the finite sequence $\langle S_i' \rangle_{i=1}^{k-1}$ is as in the definition of $D(x,y)$, and by the induction hypothesis it follows that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) \leq \sum_{i=1}^{k-1} L(S_i') < \sum_{i=1}^{k} L(S_i)$$

holds, since (iv) implies

$L(S_1') + L(S_2')$

$$= L(A(\rho_{j-1}, \emptyset_j, \emptyset_{j+1})) + L(R(\emptyset_{j+1}, \rho_{j-1}, \rho_j)) + L(R(\emptyset_{j+1}, \rho_j, \rho_{j+2}))$$
$$< L(S_1) + L(S_2) + L(S_3).$$

If $3 \leq j = k$, define $S_i' = S_i$ for $i = 1, \ldots, j - 3$ in the case that $j = 4$; define
$S_{j-2}' = S_{j-2} \cup A(\rho_{j-1}, \emptyset_j, \emptyset_{j+1}) = A(\rho_{j-1}, \emptyset_{j-2}, \emptyset_{j+1})$; and define
$S_{j-1}' = R(\emptyset_{j+1}, \rho_{j-1}, \rho_j)$. Then it happens that the finite sequence $\langle S_i' \rangle_{i=1}^{k-1}$ is as in the definition of $D(x,y)$, and by the induction hypothesis it follows that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_3)) \leq \sum_{i=1}^{k-1} L(S_i') < \sum_{i=1}^{k} L(S_i)$$

must hold, since (iv) implies

$L(S_{j-2}') + L(S_{j-1}')$

$$= L(S_{j-2}) + L(A(\rho_{j-1}, \emptyset_j, \emptyset_{j+1})) + L(R(\emptyset_{j+1}, \rho_{j-1}, \rho_j))$$
$$< L(S_{j-2}) + L(S_{j-1}) + L(S_j).$$

If $3 \leq j < k$, then define $S_i' = S_i$ for $i = 1, \ldots, j - 3$ in the case that $j \geq 4$; define
$S_{j-2}' = S_{j-2} \cup A(\rho_{j-1}, \emptyset_j, \emptyset_{j+1}) = A(\rho_{j-1}, \emptyset_{j-2}, \emptyset_{j+1})$; and define
$S_i' = S_{i+2}$ for $i = j, \ldots, k - 2$ in the case that $k \geq j + 2$.
Then $\langle S_i' \rangle_{i=1}^{k-2}$ is as in the definition of $D(x,y)$, and it follows from the induction hypothesis that

$$L(A(r_1, \theta_1, \theta_2)) + L(R(\theta_2, r_1, r_2)) \leq \sum_{i=1}^{k-2} L(S_i') < \sum_{i=1}^{k} L(S_i)$$

holds, since (iv) implies

$$
\begin{aligned}
&L(S'_{j-2}) + L(S'_{j-1}) \\
&\quad = L(S_{j-2}) + L(A(\rho_{j-1}, \phi_j, \phi_{j+1})) + L(R(\phi_{j+1}, \rho_{j-1}, \rho_j)) + L(S_{j+1}) \\
&\quad < L(S_{j-2}) + L(S_{j-1}) + L(S_j) + L(S_{j+1}).
\end{aligned}
$$

Therefore, the induction is complete, with the result that if $x = (r_1 \cos \Theta_1, \Theta_1)$ and $y = (r_2 \cos \Theta_2, \Theta_2)$ with $r_1 < r_2$ and $\Theta \neq \Theta$, then

$$L(A(r_1, \Theta_1, \Theta_2)) + L(R(\Theta_2, r_1, r_2)) < \sum_{i=1}^{n} L(S_i)$$

holds for any finite sequence $\langle S_i \rangle_{i=1}^{n}$ as in the definition of $D(x,y)$ besides the finite sequence $A(r_1, \Theta_1, \Theta_2)$, $R(\Theta_2, r_1, r_2)$. Hence,

$$
\begin{aligned}
D(x,y) &= L(A(r_1, \Theta_1, \Theta_2)) + L(R(\Theta_2, r_1, r_2)) \\
&= r_1 \, |\Theta_1 - \Theta_2| + (r_2 - r_1) \cos \Theta_2.
\end{aligned}
$$

Moreover, from this fact and from (vii) and (viii) it follows easily that $A(r_1, \Theta_1, \Theta_2) \cup R(\Theta_2, r_1, r_2))$ is a D segment $\overline{xy}$.

(xi) To prove the uniqueness of the above segment, let $x = (r_1 \cos \Theta_1, \Theta_1)$ and $y = (r_2 \cos \Theta_2, \Theta_2)$ with $r_1 < r_2$ and $\Theta_1 \neq \Theta_2$. By (x), it holds that $A(r_1, \Theta_1, \Theta_2) \cup R(\Theta_2, r_1, r_2)$ is a D segment $\overline{xy}$. Let $z = (\rho \cos \phi, \phi)$ be a point not lying on $A(r_1, \Theta_1, \Theta_2) \cup R(\Theta_2, r_1, r_2)$. If $\rho < r_1$, then it holds that

$$
\begin{aligned}
&D(x,z) + D(z,y) \\
&\quad = \rho \, |\phi - \Theta_1| + (r_1 - \rho) \cos \Theta_1 + \rho \, |\phi - \Theta_2| + (r_2 - \rho) \cos \Theta_2
\end{aligned}
$$

$\geq (r_1 - \rho) \cos \theta_1 + \rho\, |\theta_2 - \theta_1| + (r_1 - \rho) \cos \theta_2 + (r_2 - r_1) \cos \theta_2$

$> r_1 |\theta_2 - \theta_1| + (r_2 - r_1) \cos \theta_2 = D(x,y)$

by the real triangle inequality and (v). If $r_1 \leq \rho \leq r_2$, then

$D(x,z) + D(z,y)$

$= r_1\, |\phi - \theta_1| + (\rho - r_1) \cos \phi + \rho\, |\theta_2 - \phi| + (r_2 - \rho) \cos \theta_2$

$> r_1\, |\phi - \theta_1| + r_1\, |\theta_2 - \phi| + (\rho - r_1) \cos \theta_2 + (r_2 - \rho) \cos \theta_2$

$\geq r_1\, |\theta_2 - \theta_1| + (r_2 - r_1) \cos \theta_2 = D(x,y)$

holds by (v) and the real triangle inequality. If $r_2 < \rho$, then

$D(x,z) + D(z,y)$

$= r_1\, |\phi - \theta_1| + (\rho - r_1) \cos \phi + r_2\, |\phi - \theta_2| + (\rho - r_2) \cos \phi$

$= r_1\, |\phi - \theta_1| + (\rho - r_2) \cos \phi + (r_2 - r_1) \cos \phi + r_2\, |\phi - \theta_2|$

$\quad + (\rho - r_2) \cos \phi$

$> r_1\, |\phi - \theta_1| + (\rho - r_2) \cos \phi + r_1\, |\phi - \theta_2| + (r_2 - r_1) \cos \theta_2$

$\quad + (\rho - r_2) \cos \phi$

$\geq r_1\, |\theta_2 - \theta_1| + (r_2 - r_1) \cos \theta_2 + 2\, (\rho - r_2) \cos \phi$

$> D(x,y)$

holds by (v) and the real triangle inequality.

In any case, $D(x,z) + D(z,y) > D(x,y)$ holds, so that z cannot be a between point of x and y. Thus, the D segment $A(r_1, \theta_1, \theta_2) \cup R(\theta_2, r_1, r_2)$ is unique as $\overline{xy}$.

(xii) Let $x = (r_1 \cos \theta_1, \theta_1)$ and $y = (r_2 \cos \theta_2, \theta_2)$ be distinct points of C, with $r_1 \leq r_2$. It was shown by (ii) that $D(x,y) \geq \| x - y \|$. It is now shown that $D(x,y) \leq (k/\sqrt{2})\, \| x - y \|$, where $k = \pi / (2 \cos .73)$. From either (x) or (viii) it follows that

$D(x,y) = r_1\ |\theta_1 - \theta_2| + (r_2 - r_1) \cos \theta_1$. Since $t \leq \pi/2 \sin t$ holds for all $0 \leq t \leq \pi/2$, then $|\theta_1 - \theta_2| \leq 1.46$ implies that $|\theta_1 - \theta_2| \leq\ /2 \sin |\theta_1 - \theta_2|$. Moreover, since $1 \leq \cos \theta_1 / \cos .73$,

$$r_1\ |\theta_1 - \theta_2| \leq k\ r_1 \cos \theta_1 \sin |\theta_1 - \theta_2| \qquad (4)$$

must hold.

Now, in the case that $\cos \theta_2 - \cos \theta_1 \cos (\theta_2 - \theta_1) \geq 0$, then $\cos \theta_2 + \cos \theta_1 \cos (\theta_2 - \theta_1) \leq 2 \cos \theta_2$ holds, hence

$$\begin{aligned} &\cos^2 \theta_2 - \cos^2 \theta_1 \cos^2 (\theta_2 - \theta_1) \\ &\quad \leq 2 \cos \theta_2 [\cos \theta_2 - \cos \theta_1 \cos (\theta_2 - \theta_1)] \end{aligned} \qquad (5)$$

is obtained upon multiplication by $\cos \theta_2 - \cos \theta_1 \cos (\theta_2 - \theta_1)$. If, on the other hand, it happens that $\cos \theta_2 - \cos \theta_1 \cos (\theta_2 - \theta_1) \leq 0$ holds, then $\cos \theta_2 + \cos \theta_1 \cos (\theta_2 - \theta_1) \geq 2 \cos \theta_2$, and again (5) is obtained upon multiplication by $\cos \theta_2 - \cos \theta_1 \cos (\theta_2 - \theta_1)$; that is, (5) holds in either case. When (5) is multiplied through by the corresponding members of $r_1^2 \leq r_1\ r_2$, there results the inequality

$$\begin{aligned} &r_1^2 [\cos^2 \theta_2 - \cos^2 \theta_1 \cos^2 (\theta_2 - \theta_1)] \\ &\quad \leq 2\ r_1\ r_2 \cos \theta_2 [\cos \theta_2 - \cos \theta_1 \cos (\theta_2 - \theta_1)]. \end{aligned}$$

By distributing the multiplications over the differences and by adding $r_1^2 \cos^2 \theta_1 \cos^2 (\theta_2 - \theta_1) - 2\ r_1\ r_2 \cos^2 \theta_2 + r_1^2 \cos^2 \theta_2$ to both members, the inequality

$$\begin{aligned} &r_2^2 \cos^2 \theta_2 - 2\ r_1\ r_2 \cos^2 \theta_2 + r_1^2 \cos^2 \theta_2 \\ &\quad \leq r_2^2 \cos^2 \theta_2 - 2\ r_1\ r_2 \cos \theta_1 \cos \theta_2 \cos (\theta_2 - \theta_1) \\ &\qquad + r_1^2 \cos^2 \theta_1 \cos^2 (\theta_2 - \theta_1) \end{aligned}$$

is obtained, which may be rewritten as

$$(r_2 - r_1)^2 \cos^2 \theta_2 \leq [r_2 \cos \theta_2 - r_1 \cos \theta_1 \cos (\theta_2 - \theta_1)]^2.$$

Therefore, since $k > 1$, this last inequality becomes

$$(r_2 - r_1) \cos \theta_2 \leq k \,|r_2 \cos \theta_2 - r_1 \cos \theta_1 \cos (\theta_2 - \theta_1)|. \qquad (6)$$

Inequalities (4) and (6) add to the inequality

$$D(x,y) \leq k \,[r_1 \cos \theta_1 \sin |\theta_1 - \theta_2| + |r_2 \cos \theta_2 - r_1 \cos \theta_1 \cos (\theta_2 - \theta_1)|]. \qquad (7)$$

But since $(1/2)(s + t)^2 \leq s^2 + t^2$ holds for all real s and t,

$$\begin{aligned}(1/2)\,[r_1 \cos \theta_1 \sin|\theta_1 - \theta_2| &+ |r_2 \cos \theta_2 - r_1 \cos \theta_1 \cos(\theta_2 - \theta_1)|]^2 \\ &\leq r_1^2 \cos^2 \theta_1 \sin^2 (\theta_1 - \theta_2) + r_2^2 \cos^2 \theta_2 \\ &\quad - 2 r_1 r_2 \cos \theta_1 \cos \theta_2 \cos (\theta_1 - \theta_2) + r_1^2 \cos^2 \theta_1 \cos^2(\theta_1 - \theta_2) \\ &= r_1^2 \cos^2 \theta_1 + r_2^2 \cos^2 \theta_2 - 2 r_1 r_2 \cos \theta_1 \cos \theta_2 \cos (\theta_1 - \theta_2) \\ &= (r_1 \cos^2 \theta_1 - r_2 \cos^2 \theta_2)^2 \\ &\quad + (r_1 \cos \theta_1 \sin \theta_1 - r_2 \cos \theta_2 \sin \theta_2)^2 \\ &= \| x - y \|^2\end{aligned}$$

is obtained.

This last inequality, combined with (7), yields the fact that $D(x,y) \leq k/\sqrt{2}\, \| x - y \|$, completing the proof that D is equivalent to the usual planar metric restricted to C.

(xiii) Since C is compact under the usual planar metric, then by (xii) it is shown that (C, D) is compact, and in particular, D is complete. Moreover, since (vii), (ix), and (xi) combine to show that for every two points of C there is a unique D segment joining them, then D is SC according to (2.3).

(xiv) The metric D is not WR, for let $r = 1/1.73$. Since the point (r, 0) is on the segment $A(r, -.73, 0) \cup R(0, r, 1)$, which is $\overline{(r \cos .73, -.73)\ (1, 0)}$, and since $D((r \cos .73, -.73), (r, 0))$ $= L(A(r, -.73, 0)) = .73\, r = .73/1.73 = 1 - r = L(R(0, r\ 1))$ $= D((r, 0), (1, 0))$, then (r, 0) is the midpoint of the points (r cos .73, -.73) and (1, 0). Similarly, (r, 0) is the midpoint of the points (r cos .73, .73) and (1, 0). Hence by (2.4), D is not WR.

(xv) Neither is D a WE metric, for let $p = (\cos .73, .73)$, $q = ((\cos .73)/2, .73)$, $r = ((\cos .73)/2, -.73)$, $s = (\cos .73, -.73)$. Then $\overline{pr} \cap \overline{qs} = A(1/2, -.73, .73) = \overline{qr}$, but $\overline{pr} \cup \overline{qs} \neq A(1, -.73, .73)$ $= \overline{ps}$. Thus, by (2.5), D is not WE.

(xvi) The points (cos .73, -.73) and (cos .73, .73) lie in the closed ball $\overline{D}((0, 0); \cos .73)$, but their midpoint (1, 0) does not. Similarly, it may be shown that any closed ball centered at the origin, unless of course it is the entire set C, cannot be D convex. ∎

SC and WE Metrizability

The following result is due to Borsuk [9].

Theorem 2.14. Every compact space which admits an SC metric is contractible.

Corollary 2.15. No n-sphere S^n in E^{n+1} admits an SC metric.

Proof: According to Brouwer's Theorem, S^n is not contractible [11]. Hence, (2.14) applies. ∎

In regard to (2.14), Krakus and Trybulec [14] have given an example of a non-compact, non-contractible metric space with unique segments; that is, one whose metric satisfies condition (2.3.iii). Moreover, they left as an open question whether or not there exists a non-contractible space with an SC metric. No answer to their question has thus far appeared in the literature. The following theorem provides a partial answer to this question, in the case of locally compact spaces, while generalizing (2.14).

Theorem 2.16. Every locally compact space that admits an SC metric is contractible.

Proof: Let D be an SC metric for the locally compact space M. Fix any point $p \in M$, and define a function $H : M \times [0, 1] \to M$ as follows: for $(y, t) \in M \times [0, 1]$ there exists a unique point $z \in M$ such that $pz = (t)\ py$ and $zy = (1 - t)\ py$; let $H(y, t) = z$. It follows that $H(y, 0) = p$ and $H(y, 1) = y$ for each $y \in M$. To show that H is continuous, let $\langle (y_n, t_n) \rangle$ be a sequence of points in $M \times [0, 1]$ that converges to a point (y, t). Then $\langle y_n \rangle$ and $\langle t_n \rangle$ converge to y and t, respectively. Since the set $\{y_n: n = 1, 2, \ldots\}$ is bounded, there exists a number $\delta > 0$ such that $\{y_n: n = 1, 2, \ldots\} \subset \overline{D}(p;\delta)$. If $z_n = H(y_n, t_n)$ for each n, then $pz_n = (t_n)\ py_n \leq py_n \leq \delta$ holds, so that $\{z_n: n = 1, 2, \ldots\}$ is contained in $\overline{D}(p;\delta)$, which by (1.26) is a compact set. Let

$z = H(y, t)$. If $\langle z_n \rangle$ does not converge to z, then there is some $\varepsilon > 0$ and a subsequence $\langle z_{n_i} \rangle$ of $\langle z_n \rangle$ such that the set $\{z_{n_i} : i = 1, 2, \ldots\}$ is contained in $\overline{D}(p;\delta) \setminus D(z;\varepsilon)$, which is also a compact set. Thus, $\langle z_{n_i} \rangle$ has a convergent subsequence, and for simplicity it may be assumed that $\langle z_{n_i} \rangle$ itself converges to some point z' in $\overline{D}(p;\delta) \setminus D(z;\varepsilon)$; in particular, it must be that $z' \neq z$. But since $\langle (y_{n_i}, t_{n_i}) \rangle$ converges to (y, t) and D is continuous, then

$$pz' = \lim_{i\to\infty} pz_{n_i} = \lim_{i\to\infty} (t_{n_i})\, py_{n_i} = (t)\, py$$

and

$$z'y = \lim_{i\to\infty} z_{n_i} y = \lim_{i\to\infty} (1 - t_{n_i})\, py_{n_i} = (1 - t)\, py.$$

Thus, by the definition of H it must be that $z' = H(y, t) = z$, a contradiction. Hence, $\langle z_n \rangle$ converges to z, and H is continuous and consequently a homotopy from the constant map p to the identity map on M. Therefore, M is contractible. ∎

It is immediate from (2.15) that there is no SC metric for a simple closed curve. In fact, Glynn [12] has shown that a Peano continuum in E^2 admits an SC metric if and only if it does not separate E^2. It is natural to ask whether results analogous to this and to (2.15) hold for WE metrics. The following theorem is of some interest along this line, and is useful in proving some of the results of Chapter VI.

Theorem 2.17. There is no WE metric for a simple closed curve.

Proof: It suffices to show that there is no WE metric for the unit circle S^1 in the complex plane. Let D be a convex metric for S^1. Suppose that there is some point of S^1 that is not a between point of any two points of S^1, and for simplicity assume that this is the point 1. Define $p_n = \exp[i(\pi/2n)]$ and $q_n = \exp[i(2\pi - \pi/2n)]$, for $n = 1, 2, \ldots$. Since $\langle p_n \rangle$ and $\langle q_n \rangle$ both converge to 1, then

$$\lim_{n\to\infty} D(p_n, q_n) = 0.$$

But since no segment $\overline{p_n q_n}$ can contain 1, then the segment $\overline{p_n q_n}$ must be uniquely $\{\exp(i\theta): \pi/2n \leq \theta \leq 2\pi - \pi/2n\}$, so that the points i and $-i$ are in $\overline{p_n q_n}$ for each n. Therefore, the presence of the bound $D(p_n, q_n) \geq D(i, -i) > 0$ contradicts the above limit. Hence, the point 1 and every other point of S^1 is a between point of some other two points of S^1.

Suppose that D is WE. Define

$$\delta = \sup\{0 < \alpha < 2\pi: \{e^{i\theta}: 0 \leq \theta \leq \alpha\} \text{ is a segment from } 1 \text{ to } e^{i\alpha}\}.$$

By the preceding paragraph such α's exist, so that $\delta > 0$ is well defined. Let $\langle \alpha_n \rangle$ be a sequence of increasing positive numbers whose limit is δ. Since

$$D(1, e^{i\delta}) = \lim_{n\to\infty} D(1, e^{i\alpha_n}) \geq D(1, e^{i\alpha_1}) > 0$$

holds, then $\delta < 2\pi$. But since $e^{i\delta}$ is a between point of some two points of S^1, there are values $0 < \delta_1 < \delta < \delta_2 < 2\pi$ such that $\{e^{i\theta}: \delta_1 \leq \theta \leq \delta_2\}$ is a segment from the point $e^{i\delta_1}$ to $e^{i\delta_2}$. There is an integer n such that $\delta_1 < \alpha_n < \delta$, so that the set

$\{e^{i\theta}: 0 \leq \theta \leq \alpha_n\} \cup \{e^{i\theta}: \delta_1 \leq \theta \leq \delta_2\} = \{e^{i\theta}: \delta_1 \leq \theta \leq \alpha_n\}$ is a segment from $e^{i\alpha_n}$ to $e^{i\delta_1}$. Since D is assumed to be WE, then by (2.5) the set $\{e^{i\theta}: 0 \leq \theta \leq \delta_2\}$ is a segment from 1 to $e^{i\delta_2}$, contradicting the definition of δ. Hence, D is not WE. ∎

CHAPTER III

SEGMENTED CONVEX METRICS AND LOCALLY CONNECTED GENERALIZED CONTINUA

In proving that a certain metric D is complete and convex, it is often possible to conclude that every two points lie on a D segment, before it can be proved that D is complete. This is the case in the proof of certain of the extension theorems in Chapter V. Therefore, it becomes quite useful to have at hand a collection of properties of metric spaces that satisfy the condition that every two points are joined by a segment. This present chapter provides a few elementary results on such metric spaces. Additionally, these metrics are found to characterize locally connected generalized continua among the locally compact spaces and to identify certain Peano continua that are contained within the locally connected generalized continua.

Definition and Examples

Definition 3.1. A metric D is said to be segmented convex if every two points in the space are joined by a D segment.

It is observed that the segmented convex metrics occupy an intermediate position between the convex metrics and the complete convex metrics, in that every segmented convex metric is convex, and by (1.13) every complete convex metric is segmented convex. The following two examples sharpen the distinctions between these three metrics.

Example 3.2. The usual metric of E^1 restricted to the space of rationals is a convex metric, and this space does not admit a segmented convex metric.

Example 3.3. Not every space that admits a segmented convex metric must also admit a complete convex metric.

Proof: Every normed linear space can be given a segmented convex metric, namely, the metric obtained from the norm. But there are normed linear spaces that are not topologically complete, for let Q be the space of rationals in E^1. By the Baire category theorem, Q is not topologically complete [11]. Yet, Q can be embedded isometrically as a closed subset of a normed linear space [1]. Therefore, since the property of topological completeness is inherited from a space by each of its closed subsets, the normed linear space N is not topologically complete [11]. ∎

A Condition Sufficient for Segmented Convexity

A natural question is the following: Under what conditions must a given convex metric be also segmented convex? One answer is given below in (3.5).

Lemma 3.4. For any two points x and y of a midpoint convex metric space, there is a midpoint convex, linear set L(x,y) consisting of x, y, and between points of x and y.

Proof: It is assumed for simplicity that $xy = 1$. Define the set $A = \{m\, 2^{-n}:\ m = 0, 1, \ldots, 2^n;\ n = 1, 2, \ldots\}$, which is the set of all dyadic rationals in the interval $[0, 1]$. If M denotes the given space, an isometry $f : A \to M$ will be defined such that $f(0) = x$, $f(1) = y$, and $f(A)$ will have the properties required of $L(x,y)$.

A sequence $\langle f_n \rangle_{n=0}^{\infty}$ of isometries is now defined by induction. Define $f_0 : \{0, 1\} \to M$ by $f_0(0) = x$, $f_0(1) = y$. Suppose that for $n = 0, \ldots, k$ the isometries $f_n : \{0, 2^{-n}, \ldots, i\,2^{-n}, \ldots, 1\} \to M$ have been defined such that f_n extends f_{n-1}, for all positive n. Define $f_{k+1} : \{0, 2^{-k-1}, \ldots, i\,2^{-k-1}, \ldots, 1\} \to M$ by $f_{k+1}(z) = f_k(z)$ if z is in the domain of f_k; and if $z = i\,2^{-k-1}$ is not in the domain of f_k, then both $(i - 1)\,2^{-k-1}$ and $(i + 1)\,2^{-k-1}$ are in the domain of f_k, so that $f_{k+1}(z)$ is defined by choice to be some midpoint of $f_k((i - 1)\,2^{-k-1})$ and $f_k((i + 1)\,2^{-k-1})$. Then f_{k+1} extends f_k. Further, f_{k+1} is an isometry, for let $i\,2^{-k-1}$ and $j\,2^{-k-1}$ be two points in the domain of f_{k+1}, where without loss of generality $0 \leq i < j \leq 2^{k+1}$. There are even integers $0 \leq i' \leq i'' \leq j' \leq j'' \leq 2^{k+1}$ such that $i = (i' + i'')/2$ and $j = (j' + j'')/2$ hold, which may be found by taking $i' = i'' = i$ if i is even and $i' = i - 1$, $i'' = i + 1$ if i is odd, and similarly for j. Since, for example, i' is even, then $i'\,2^{-k-1}$ is in the domain of f_k. Moreover, the definition of f_{k+1} implies that $f_{k+1}(i\,2^{-k-1})$ is a midpoint of $f_{k+1}(i'\,2^{-k-1})$ and $f_{k+1}(i''\,2^{-k-1})$, if these last are actually distinct points, and similarly for j, j', and j''. Therefore,

$$f_{k+1}(i'\ 2^{-k-1})\ f_{k+1}(j''\ 2^{-k-1})$$

$$= f_k(i'\ 2^{-k-1})\ f_k(j''\ 2^{-k-1})$$

$$= f_k(i'\ 2^{-k-1})\ f_k(i''\ 2^{-k-1}) + f_k(i''\ 2^{-k-1})\ f_k(j'\ 2^{-k-1})$$

$$+ f_k(j'\ 2^{-k-1})\ f_k(j''\ 2^{-k-1})$$

$$= f_{k+1}(i'\ 2^{-k-1})\ f_{k+1}(i''\ 2^{-k-1}) + f_{k+1}(i''\ 2^{-k-1})\ f_{k+1}(j'\ 2^{-k-1})$$

$$+ f_{k+1}(j'\ 2^{-k-1})\ f_{k+1}(j''\ 2^{-k-1})$$

$$= f_{k+1}(i'\ 2^{-k-1})\ f_{k+1}(i\ 2^{-k-1}) + f_{k+1}(i\ 2^{-k-1})\ f_{k+1}(i''\ 2^{-k-1})$$

$$+ f_{k+1}(i''\ 2^{-k-1})\ f_{k+1}(j'\ 2^{-k-1}) + f_{k+1}(j'\ 2^{-k-1})\ f_{k+1}(j\ 2^{-k-1})$$

$$+ f_{k+1}(j\ 2^{-k-1})\ f_{k+1}(j''\ 2^{-k-1})$$

holds since f_k is an isometry. Now, (1.6) implies that

$$f_{k+1}(i\ 2^{-k-1})\ f_{k+1}(j\ 2^{-k-1})$$

$$= f_{k+1}(i\ 2^{-k-1})\ f_{k+1}(i''\ 2^{-k-1}) + f_k(i''\ 2^{-k-1})\ f_k(j'\ 2^{-k-1})$$

$$+ f_{k+1}(j'\ 2^{-k-1})\ f_{k+1}(j\ 2^{-k-1})$$

$$= (i'' - i)\ 2^{-k-1} + (j' - i'')\ 2^{-k-1} + (j - j')\ 2^{-k-1}$$

$$= (j - i)\ 2^{-k-1}$$

holds since f_k is an isometry and, for example,

$$f_{k+1}(i\ 2^{-k-1})\ f_{k+1}(i''\ 2^{-k-1}) = [f_{k+1}(i'\ 2^{-k-1})\ f_{k+1}(i''\ 2^{-k-1})]/2$$

$$= [f_k(i'\ 2^{-k-1})\ f_k(i''\ 2^{-k-1})]/2$$

$$= [(i'' - i')\ 2^{-k-1}]/2 = (i'' - i)\ 2^{-k-1}$$

holds by the definition of i' and i''. Thus, f_{k+1} is an isometry.

Hence, the nested sequence $<f_n>$ of isometries is inductively defined.

Let $f = \bigcup \{f_n: n = 0, 1, \ldots\}$. Since f_{n+1} extends f_n for each n, then f is well defined as a function. The domain of f is the union of the domains of the functions f_n, which is the set A. To show that $f : A \to M$ is an isometry, let p and q be two points of A. For some n the points p and q are in the domain of the isometry f_n, so that $f(p)\, f(q) = f_n(p)\, f_n(q) = |p - q|$ holds, and f is thus an isometry. Therefore, the set $L(x,y) = f(A)$ is linear. Since $f(0) = x$, $f(1) = y$, and any point of $A \setminus \{0, 1\}$ is a between point in E^1 of 0 and 1, then $L(x,y)$ consists of x, y, and between points of x and y. Finally, let s and t be two points of $L(x,y)$. The dyadic rationals $f^{-1}(x)$ and $f^{-1}(y)$ have a midpoint u in A, hence $f(u)$ is a midpoint of s and t. Therefore, $L(x,y)$ is midpoint convex. ∎

<u>Theorem</u> 3.5. If in a locally compact metric space every two points have a unique midpoint, then the metric is segmented convex.

Proof: Let p and q be two distinct points of the space (M, D). By (3.4) there is a linear set L consisting of p, q, and between points of p and q, and containing a midpoint of every two of its points. Let $g : L \to E^1$ be an isometry, where it may be assumed without loss of generality that $g(p) < g(q)$, and furthermore $g(p) = 0$. Since $g(q) = pq$ holds with $pq = pz + zq$ for each z in L, then $g(L) \subset [0, 1]$. Moreover, since L is midpoint convex, then $g(L)$ is midpoint convex also, and therefore is dense in $[0, pq]$.

Denote by $\overline{L}$ the closure of L, and define $G : \overline{L} \to E^1$ by $G(z) = pz$ for z in $\overline{L}$. Then G extends g, for is z is any point of L, $G(z) = pz = |g(z) - g(p)| = g(z)$ holds. If x and y are any two points of $\overline{L}$, there are sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of points of L such that

$$x = \lim_{n\to\infty} \langle x_n \rangle, \quad y = \lim_{n\to\infty} \langle y_n \rangle$$

hold. From the continuity of the metric, it follows that

$$|G(y) - G(x)| = |py - px| = |\lim_{n\to\infty} py_n - \lim_{n\to\infty} px_n| = \lim_{n\to\infty} |py_n - px_n|$$

$$= \lim_{n\to\infty} x_n y_n = xy.$$

Therefore, G is an isometry. Since G is continuous and since $G(x_n)$ is a point of $[0, pq]$ for each n, it follows that

$$G(x) = \lim_{n\to\infty} G(x_n)$$

is a point of $[0, pq]$ also. Thus, $G(\overline{L}) \subset [0, pq]$. Note also that $G(L) = g(L)$ is dense in $[0, pq]$.

Suppose the number $\delta = \sup \{\alpha \in [0, pq]\colon [0, \alpha] \subset G(\overline{L})\}$ is less than pq. Then δ cannot be in $G(\overline{L})$, for assume that $G(d) = \delta$ holds for some $d \in \overline{L} \setminus \{q\}$. Since the space is locally compact, there is a number $0 < \varepsilon < dq$ such that $\overline{D}(d;\varepsilon)$ is compact, hence the closed subset $\overline{L} \cap \overline{D}(d;\varepsilon)$ is also compact. Since G is continuous, then $G(\overline{L} \cap \overline{D}(d;\varepsilon))$ is a compact subset of $[0, pd]$. Moreover, $G(\overline{L} \cap \overline{D}(d;\varepsilon))$ is dense in $[0, pd] \cap [\delta - \varepsilon, \delta + \varepsilon] = [\alpha, \delta + \varepsilon]$, where $\alpha = \max \{0, \delta - \varepsilon\}$, since there is a subset A of $G(\overline{L})$ which is dense in $[\alpha, \delta + \varepsilon]$, with $G^{-1}(A) \subset \overline{D}(d;\varepsilon)$. Therefore, $G(\overline{L} \cap \overline{D}(d;\varepsilon)) = [\alpha, \delta + \varepsilon]$ must hold. But then $[0, \delta + \varepsilon] \subset G(\overline{L})$ holds, contrary to the definition of δ. Thus δ is not in $G(\overline{L})$, and in particular, $0 < \delta < pq$ holds since $G(p) \neq \delta$.

It is now shown that since $0 < \delta < pq$ holds, then δ must be in $G(\overline{L})$, contradicting the preceding conclusion. For now, the definition of δ implies that $[0, \delta) \subset G(\overline{L})$, and a subset of $G(\overline{L})$ is dense in $(\delta, pq]$. There is thus a decreasing sequence $<\beta_n>$ of points of $G(\overline{L})$ such that $\delta < \beta_n < 2\delta$ holds for each n, and

$$\delta = \lim_{n\to\infty} \beta_n.$$

Thus, $\alpha_n = 2\delta - \beta_n$ is a point of $[0, \delta)$ for each n. Hence, for each n there are points s_n and t_n of $\overline{L}$ such that $G(s_n) = \alpha_n$ and $G(t_n) = \beta_n$. Let d_n be the unique midpoint of s_n and t_n for each n. Then since $s_n s_{n+1} = \alpha_{n+1} - \alpha_n = \beta_n - \beta_{n+1} = t_{n+1}t_n$, and since $s_n t_n = \beta_n - \alpha_n = (\beta_n - \beta_{n+1}) + (\beta_{n+1} - \alpha_{n+1}) + (\alpha_{n+1} - \alpha_n)$ $= s_n s_{n+1} + s_{n+1}t_{n+1} + t_{n+1}t_n = s_n s_{n+1} + s_{n+1}d_{n+1} + d_{n+1}t_{n+1} + t_{n+1}t_n$ $= s_n d_{n+1} + d_{n+1}t_n$ with $s_n d_{n+1} = s_n s_{n+1} + s_{n+1}d_{n+1} = d_{n+1}t_{n+1} + t_{n+1}t_n$ $= d_{n+1}t_n$ by (1.6), then d_{n+1} is the midpoint d_n of s_n and t_n; that is, $d_n = d_1$ holds for each n. Since

$$\lim_{n\to\infty} s_n d_1 = \lim_{n\to\infty} s_n d_n = (1/2) \lim_{n\to\infty} s_n t_n = (1/2) \lim_{n\to\infty} (\beta_n - \alpha_n) = 0,$$

then

$$d_1 = \lim_{n\to\infty} s_n$$

holds; that is, $d_1 \in \overline{L}$. Moreover, since $s_1 d_1 = d_1 t_1$, then $G(d_1) = (\alpha_1 + \beta_1)/2 = \delta$. Therefore, $\delta \in G(L)$ holds, a contradiction.

The foregoing argument has shown that $\delta = pq$ must hold, so that $[0, pq] = G(\overline{L})$. Therefore, $\overline{L}$ is a segment $\overline{pq}$, and D is segmented convex. ∎

The converse of (3.5) is not true, as (2.9) shows. Moreover, (3.2) shows that local compactness cannot be omitted from the hypothesis, and (1.12) shows that "midpoint" cannot be replaced by "between point." The following example completes the discussion of the above theorem.

Example 3.6. The uniqueness of the midpoint in the hypothesis of (3.5) cannot be omitted.

Proof: This example is constructed in E^2 from the union of a certain collection of right isosceles triangles, each denoted ΔABC, where C is the hypoteneuse and A and B are the equal sides. Moreover, each hypoteneuse C will have slope 0 or -1: if the slope is 0, then A and B will lie above C with A to the right of B; if the slope of C is -1, then A and B will lie to the left of C with A horizontal. This convention will hold for each ΔABC under discussion. Let $|S|$ denote the usual length of any line segment S. For a line segment S with slope 0 or -1, define $N(S) = \{\Delta ABC\colon\ C \subset S,\ \text{midpoint of } C = \text{midpoint of } S,\ |C| = |S|\, 2^{-n}$ for some integer $n \geq 2\}$. Finally, if (x_1, y_1) and (x_2, y_2) are points in E^2 identified by their cartesian coordinates, let $[(x_1, y_1), (x_2, y_2)]$ denote the line segment joining them.

To construct the space of the example, let $A_0 = [(1, 0), (0, 1)]$, $B_0 = [(-1, 0), (0, 1)]$, $C_0 = [(-1, 0), (1, 0)]$. Define collections Q_i of triangles as follows: $Q_0 = \{\Delta A_0B_0C_0\} \cup N(C_0)$, and recursively $Q_n = \bigcup \{N(A)\colon\ \Delta ABC \in Q_{n-1}\}$ for integers $n > 0$. For each $n \geq 0$, let $Q_n^* = \bigcup \{\Delta ABC\colon\ \Delta ABC \in Q_n\}$, and define $P_n = \bigcup \{Q_i^*\colon i = 0, 1, \ldots, n\}$. The set $N = \bigcup \{P_n\colon\ n = 0, 1, \ldots\}$ is illustrated in Figure 2.

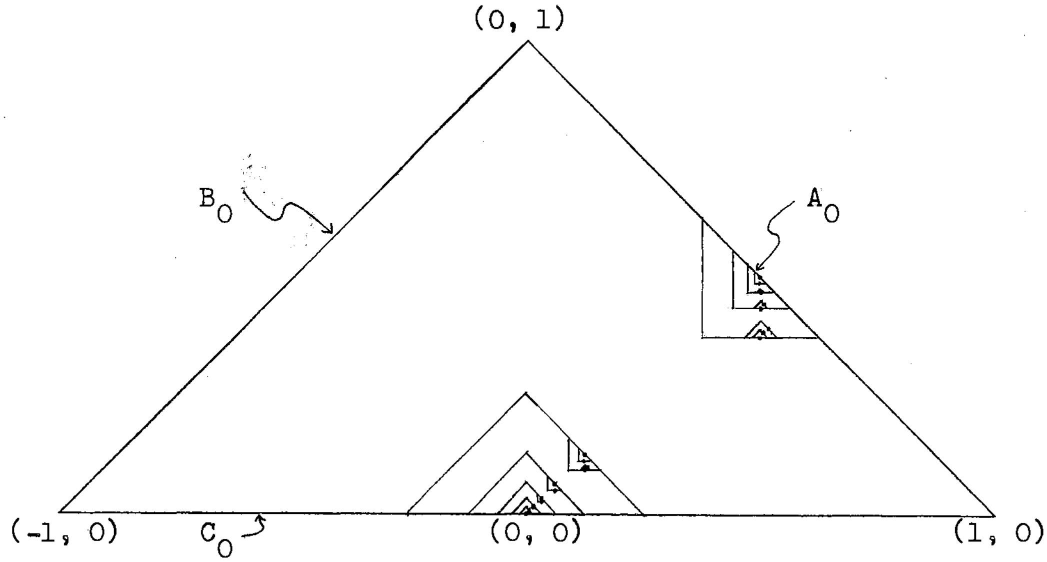

Figure 2. The Set N of (3.6)

A metric E is now defined inductively on N as follows. If x and y are points of C_0, then $E(x,y) = \| x - y \|$, where $\| \circ \|$ is the usual norm for E^2. If points x and y lie together on $A \cup B$ for some $\triangle ABC \varepsilon Q_0$, then $E(x,y) = E(p(x),p(y))$, where p(z) is the perpendicular projection of z to C_0. At this point, E has been defined on each side of each triangle of Q_0; moreover, since every two points of P_0 are joined by a polygonal arc in P_0 whose line segments are subarcs of sides of triangles $\triangle ABC \varepsilon Q_0$, then for such arcs, arclength is well defined by summing lengths of contiguous line segments in the arc, length here being taken with respect to the metric E, to the extent that it has been defined. Thus, for any two points x and y of P_0, define

$E(x,y) = \inf \{E$ length of T: T is a polygonal arc in P_0 from

x to $y\}$. Then the triangle inequality holds for E, and E is a metric on P_0. Moreover, given any two points of P_0 there is actually a shortest polygonal arc T in P_0 joining them, as can be verified from the geometry of P_0.

The definition of E proceeds by induction, under the hypothesis that E has been defined on P_n. For any $\triangle ABC \in Q_{n+1}$, E has already been defined for points of C, since $C \subset P_n$. Define E on $A \cup B$ by $E(x,y) = E(p(x),p(y))$ as above. Then each side of each $\triangle ABC \in Q_{n+1}$ has E defined on it. Since P_{n+1} is arcwise connected by polygonal arcs and, as above, arclength of polygonal arcs is well defined, let $E(x,y) = \inf \{E$ length of T: T is a polygonal arc in P_{n+1} from x to $y\}$ for any two points x and y of P_{n+1}. Again the triangle inequality holds, so that E is a metric on P_{n+1}. Moreoever, it is true that, given any two points of P_{n+1}, there is a shortest polygonal arc T in P_{n+1} joining them. Also, a shortest polygonal arc in (P_n, E) remains a shortest polygonal arc between its endpoints in (P_{n+1}, E), again by appeal to the geometric construction of P_{n+1}; that is, E on P_{n+1} agrees with its previous definition on P_n. Define E on N as the union of E on each P_n, when the induction principle has been applied. The triangle inequality for each P_n insures the triangle inequality for E on N, so that E is a metric on N. Moreover, since polygonal segments are preserved under the induction process, then between any two points x and y of N there is a segment $\overline{xy}$ that is a polygonal arc, so that in particular, E is a segmented convex metric on N.

Define $K = \{p$: p is a midpoint of A or C, $\triangle ABC \in Q_n$ for some $n = 0, 1, \ldots\}$; the points of K are shown as dots in Figure 2.

Let $M = N \setminus K$, and let D be the metric E restricted to M. Then (M, D) is a metric space, the space of interest in the present example. The following assertions are now proved.

(i) If $q \in M \cap Q_m^*$ and $D(q;\varepsilon) \cap Q_{m+1}^* = \emptyset$, then $D(q;\varepsilon) \subset P_m$.

(ii) There exists a basis for (M, D) consisting of sets G_q which are arcs or simple triods, but without their endpoints.

(iii) D is equivalent to the metric given by the norm $\|\circ\|$ restricted to M.

(iv) (M, D) is locally compact.

(v) (M, D) is not connected.

(vi) D is midpoint convex, but not segmented convex.

(i) The proof is given by induction on all $n > m$ to show that $D(q;\varepsilon) \cap Q_n^* = \emptyset$. If $n = m + 1$, this fact is given as hypothesis. If $D(q;\varepsilon) \cap Q_k^* = \emptyset$ for some $k > m$, suppose there is a point $r \in D(q;\varepsilon) \cap Q_{k+1}^*$. Since E is segmented convex, there is an E segment $\overline{qr}$ in N, thus $\overline{qr} \subset E(q;\varepsilon)$. The point r lies in $(A \cup B) \setminus C$ for some $\triangle ABC \in Q_{k+1}$, since $D(q;\varepsilon) \cap Q_k^* = \emptyset$. But since $q \notin A \cup B$, then $(A \cap C) \subset \overline{qr}$ or $(B \cap C) \subset \overline{qr}$, and without loss of generality let $A \cap C = \{s\} \subset Q_k^*$. Since $s \notin K$, then $s \in D(q;\varepsilon)$. But $s \in C \subset Q_k^*$, hence $D(q;\varepsilon) \cap Q_k^* \neq \emptyset$, contrary to the induction hypothesis. Therefore, by induction it has been demonstrated that $D(q;\varepsilon) \cap (M \setminus P_n) = D(q;\varepsilon) \cap (\bigcup\{Q_{i+1}^* \setminus Q_i^*\colon\ i = n, n+1, \ldots\})$ $= \bigcup\{D(q;\varepsilon) \cap (Q_{i+1}^* \setminus Q_i^*)\colon\ i = n, n+1, \ldots\} = \emptyset$, hence $D(q;\varepsilon) \subset P_m$.

(ii) At each point $q \in M$ a local basis of sets G_q is constructed consisting of sets of the required form. Let $q \in M$ and $\varepsilon > 0$.

There is a smallest n such that $q \varepsilon P_n$. Suppose that $n = 0$. Then $q \varepsilon Q_0^* \setminus K$. If $y \varepsilon C_0 \setminus \{(0, 0)\}$, two cases may arise. The first case is that $q \varepsilon A \cup B$ for some $\triangle ABC \varepsilon N(C_0)$. If this is true, let t_0 be the first point of the arc $A \cup B$, ordered from q, such that $t_0 \varepsilon Q_1$. Since $|C| = |C_0| 2^{-n} = 2^{-n+1}$ for some $n \geq 2$, then $D(q;\delta) \cap Q_1^* = \emptyset$, where $\delta = \min \{\varepsilon, qt_0, 2^{-n+1}\}$. Since it holds that $D(q;\delta) \subset P_1 = Q_0^*$, it follows that $G_q = D(q;\delta)$ is a simple triod with ramification point q, but without its endpoints. The second case that may arise is that q is not in $A \cup B$ for any triangle $\triangle ABC \varepsilon N(C_0)$. Then there is a first point t_1 of $[q, (0, 0)]$, ordered from q, such that $t_1 \varepsilon A$ for some $\triangle ABC \varepsilon N(C_0)$. There is also a number $\alpha > 0$ such that $D(q;\alpha) \cap \triangle ABC = \emptyset$ for every triangle $\triangle ABC \varepsilon N(A_0)$. If $\delta = \min \{\varepsilon, qt_1, \alpha\}$, then $D(q;\delta) \cap Q_1^* = \emptyset$, hence $D(q;\delta) \subset Q_0^*$. Actually, $G_q = D(q;\delta)$ is an arc in $\triangle A_0B_0C_0$, lacking its endpoints. If it happens that $q \varepsilon A \setminus (C_0 \cup K)$ for some $\triangle ABC \varepsilon Q_0$, then a demonstration similar to the preceding, but with distances properly scaled, shows that for some $0 < \delta \leq \varepsilon$ the ball $G_q = D(q;\delta)$ is either an arc or a simple triod with ramification point q, but lacking its endpoints, where $D(q;\delta) \subset P_1$. If $q \varepsilon B \setminus (A \cup C)$ for some $\triangle ABC \varepsilon Q_0$, then define the number $\delta = \min \{\varepsilon$, one-half the D length of $B\}$. Then in this case, $G_q = D(q;\delta) \cap [B \setminus (A \cup C)]$ is a D neighborhood of q, being an arc without its endpoints.

If $n \geq 1$, then $q \varepsilon P_n \setminus P_{n+1}$. Hence, $q \varepsilon (A \cup B) \setminus C$ for some $\triangle ABC \varepsilon Q_n$, since $C \subset P_{n-1}$. Thus, there is some number $0 < \beta \leq \varepsilon$ such that $D(q;\beta) \cap C = \emptyset$, hence $D(q;\delta) \cap P_{n-1} = \emptyset$ holds. A demonstration similar to that of the preceding paragraph now shows that there is an open set $G_q \subset D(q;\delta)$ containing q such that $G_q \cap (P_{n+1} \setminus P_{n-1})$ is either an arc or a simple triod with ramification point q, but without its endpoints, and $G_q \cap Q^*_{n+2} = \emptyset$. Thus $G_q \subset P_{n+1} \setminus P_{n-1}$, so that $G_q = G_q \cap (P_{n+1} \setminus P_{n-1})$. Hence, the induction is complete.

(iii) For a set G_q as given in (ii), let $\overline{G}_q$ denote its closure in (N, E), and let $\delta = \min \{D(q,e)\colon\ e$ is an endpoint of $\overline{G}_q\}$. Since $D(q,e) \leq \|q - e\|$ for each such endpoint e, and by the construction of G_q the cartesian ball $G'_q = \{x \in M\colon\ \|q - x\| < \delta\}$ is a subset of G_q, then $\|\cdot\|$ restricted to M is stronger than D. On the other hand, the set G'_q is open in (M, D), as is any of the usual spheres about q with radius less than δ. Therefore, D is stronger than the metric of $\|\cdot\|$ restricted to M, so that the two metrics are equivalent on M.

(iv) Each of the basis sets G_q is locally compact in the usual planar topology, hence by (iii) is locally compact in (M, D).

(v) For each n, each triangle $\triangle ABC \in Q_n$ is a simple closed curve that is separated by the omission of midpoints of C and A into a "left side" and a "right side." In precise terms, the left side of $\triangle ABC$ is the component of $\triangle ABC \setminus K$ which contains B, and the right side of $\triangle ABC$ is the component of $\triangle ABC \setminus K$ which contains $A \cap C$. Then there is a decomposition $M = L \cup R$, where $L = \bigcup\{\text{left side of } \triangle ABC\colon\ \triangle ABC \in Q_n,\ n = 0, 1, \ldots\}$ and $R = \bigcup\{\text{right side of } \triangle ABC\colon\ \triangle ABC \in Q_n,\ n = 0, 1, \ldots\}$. To show that $L \cap R = \emptyset$, suppose that, on the contrary, there is a point x of M which is both in the left side of some $\triangle ABC \in Q_n$ and in the right side of some $\triangle A'B'C' \varepsilon Q_m$, where $m \leq n$ holds without loss of generality. If $m = n$, then $x \varepsilon C \cap C'$, where without loss of generality $C \subset C'$. But by the orientation and naming of the sides of these triangles, x is in the left side of $\triangle ABC$ if and only if x is in the left side of $\triangle A'B'C'$, and a contradiction is reached. If $n = m + 1$, then $\triangle ABC \varepsilon N(A')$ and $x \varepsilon C \subset A'$. Again a contradiction arises from the geometry of M. If $n > m + 1$, then the fact that $x \varepsilon Q_m^* \cap Q_n^*$ is itself a contradiction, for $Q_m^* \cap Q_n^* = \emptyset$. Hence, $L \cap R = \emptyset$. Moreover, the following property of the sets G_q holds from their particular construction: If $q \varepsilon L$, then $G_q \subset L$, and if $q \varepsilon R$, then $G_q \subset R$. This last property shows that L and R are actually separated sets in M, so that M is not connected.

(vi) Since every space that admits a segmented convex metric is connected, then (v) implies that (M, D) cannot admit a segmented convex metric. However, D is now shown to be midpoint convex. As a preliminary case, it is shown that two given points x and z lying together on a line segment in N have a point in M that is a midpoint of them. Since each line segment in N lies on a side of some $\triangle ABC \varepsilon Q_n$ for some n, then it is true that x and z lie on one of the sides of some $\triangle ABC$. It should be noted that on line segments, the E midpoint and the euclidean midpoint coincide. If $x, z \varepsilon B$, then since $B \subset M$, then also the euclidean midpoint of x and z is in M, which is a D midpoint of x and z. If $x, z \varepsilon C$ and if the euclidean midpoint of x and z is not the midpoint of C, then the euclidean midpoint of x and z is not in K, hence is a D midpoint of x and z. If $x, z \varepsilon C$ and the euclidean midpoint of x and z is the midpoint of C, then N(C) is contained in Q_{n+1}, and there is some $\triangle A'B'C' \varepsilon N(C)$ such that $C' \subset [x, z]$. Let $\{y\} = A' \cap B'$. Then the point y lies in M, and $D(x,y) = E(x,y) = E(x,p(y)) = E(x,y') = E(y',z) = E(p(y),z) = E(y,z) = D(y,z)$ and $E(x,y') = E(x,z)/2 = D(x,z)/2$, where y' is the midpoint of C. Hence, y is indeed a D midpoint of x and z.

Now, let x and z be any two points of M. Since $x, z \varepsilon N$, there is a polygonal arc in N which is an E segment $\overline{xz}$. Therefore, the E midpoint y of $\overline{xz}$ is also an E midpoint of x and z. If $y \not\varepsilon K$, then y is also a D midpoint of x and z. If $y \varepsilon K$, then y cannot be at the junction of two non-parallel line segments in $\overline{xz}$. Thus, y is a non-cut point of some non-degenerate line segment $[p, q] \subset \overline{xz}$. There is some $\triangle ABC \varepsilon Q_n$ for some n, such that y is the midpoint of C and $C \subset [p, q]$. Let e and e' denote the endpoints of C, and let $\{y'\} = A \cap B$. As in the preceding paragraph, it may be shown that y' is an E midpoint of e and e'; it is also true that $y' \varepsilon M$ and that the equalities $E(p,y) = E(p,y')$ and $E(y,q) = E(y',q)$ hold. Without loss of generality $E(x,z) = E(x,p) + E(p,q) + E(q,z)$ holds, so that $E(x,z) = E(x,p) + E(p,y) + E(y,q) + E(q,z) = E(x,p) + E(p,y') + E(y',q) + E(q,z)$ holds, and $E(x,z) = E(x,y') + E(y',z)$ holds by (1.6). Also, since $E(x,z)/2 = E(x,y) = E(x,p) + E(p,y) = E(x,p) + E(p,y')$ holds, then the triangle inequality implies $E(x,z) \geq E(x,y')$, and similarly $E(x,z) \geq E(y'z)$ holds. Since $E(x,z) = E(x,y') + E(y',z)$ holds, this implies $E(x,y') = E(x,z)/2 = E(y',z)$, and y' is an E midpoint of x and z. Since $y' \varepsilon M$, then y' is also a D midpoint of x and z. Therefore, D is midpoint convex, and the demonstration is complete. ∎

Property S and Peano Continua

The following definition is given by Whyburn [26].

Definition 3.7. A point set P in a metric space is said to have property S provided that for each $\varepsilon > 0$, P is the union of a finite number of connected sets, each of diameter less than ε.

Whyburn [26] has shown that every locally connected generalized continuum has property S locally; in fact, at each point there is a local basis of connected open sets having property S. It follows that every locally connected generalized continuum has a basis of connected open sets whose closures are Peano continua, since the closure of a set with property S is locally connected [26]. If a locally connected generalized continuum is given a segmented convex metric, it is possible to specify exactly which open balls have property S and which closed balls are Peano continua. This result, and a useful corollary in the case that the segmented convex metric is complete, are the main results of this section.

Theorem 3.8. In a locally connected generalized continuum with a segmented convex metric D, if for a point p and a number $\varepsilon > 0$ the closed ball $\overline{D}(p;\varepsilon)$ is compact, then $D(p;\varepsilon)$ has property S and $\overline{D}(p;\varepsilon)$ is a Peano continuum.

Proof: In order to show that $D(p;\varepsilon)$ has property S, it is necessary, for a given $\alpha > 0$, to show that $D(p;\varepsilon)$ is a finite union of connected sets, each of diameter less than α. If $\varepsilon < \alpha/2$, then $D(p;\varepsilon)$ itself is connected and of diameter less than α. Therefore, it may be assumed that

$$0 < \alpha/2 \leq \varepsilon \tag{1}$$

holds. Let ρ be a number with the property that

$$\varepsilon - \alpha/4 < \rho < \varepsilon \tag{2}$$

holds, and let

$$\beta = \min \{(\varepsilon - \rho)/2,\ \alpha/8\}. \tag{3}$$

For each point x of $\overline{D}(p;\varepsilon)$ there is a number $0 < \beta_x \leq \beta$ such that $\overline{D}(x;\beta_x)$ is compact, since the space is locally compact. The remainder of the proof is suggested by a proof scheme given by Hall and Spencer [13, p. 216]. Since $\{D(x;\beta_x)\colon\ x \in \overline{D}(p;\varepsilon)\}$ is an open cover of the compact set $\overline{D}(p;\varepsilon)$, it contains a finite subcover F of $\overline{D}(p;\varepsilon)$. Let $S_1, S_2, \ldots, S_n$ denote the elements of F that intersect $D(p;\rho)$. For any points $x \in S_i$ and $y \in S_i \cap D(p;\rho)$, it holds that $px \leq py + yx < \rho + 2\beta \leq \rho + \varepsilon - \rho = \varepsilon$. Therefore,

$$D(p;\rho) \subset \bigcup_{i=1}^{n} S_i,\quad D(S_i) \leq 2\beta,\quad \text{and}\quad S_i \subset D(p;\varepsilon)\quad \text{for}\quad i = 1, \ldots, n. \tag{4}$$

For $i = 1, \ldots, n$ define C_i to be the set of all points of $D(p;\varepsilon)$ that lie, along with a point of S_i, in a connected subset of $D(p;\varepsilon)$ whose diameter does not exceed $\alpha/4$. Note that since $D(S_i) \leq \alpha/4$ holds by (3) and (4), and since S_i is connected, then $S_i \subset C_i$. If x and y are any two points of C_i, there are points x' and y' of S_i such $xx' \leq \alpha/4$ and $yy' \leq \alpha/4$. Thus, the inequality $xy \leq xx' + x'y' + y'y \leq \alpha/2 + D(S_i) \leq 3\alpha/4 < \alpha$ holds, and therefore $D(C_i) < \alpha$. Furthermore, the set C_i is connected, since it consists of the connected set S_i and a collection of connected sets each intersecting S_i.

To show that $D(p;\varepsilon)$ is the union of the sets C_i it suffices to show that $D(p;\varepsilon)$ is contained in that union, since $C_i \subset D(p;\varepsilon)$ holds for each i. Let x be any point of $D(p;\varepsilon)$. If $x \varepsilon D(p;\rho)$, then from (4) it follows that

$$x \varepsilon \bigcup_{i=1}^{n} S_i \subset \bigcup_{i=1}^{n} C_i.$$

Therefore, assume $x \not\varepsilon D(p;\rho)$; that is, assume

$$0 < \rho \leq px < \varepsilon. \tag{5}$$

If $\delta = px - \varepsilon + \rho$, it will now be shown that

$$0 < \delta < \rho \leq px \tag{6}$$

holds. If $px \leq \alpha/4$ were true, then combining (2) with (5) would yield $\varepsilon - \alpha/4 < \rho \leq px \leq \alpha/4$, hence $\varepsilon < \alpha/2$, contradicting (1). Thus, it must hold that $\alpha/4 < px$. This inequality, combined with one form of (2), yields $\varepsilon - \rho < \alpha/4 < px$, hence $0 < \delta$. Also, it follows from (5) that $\delta = px - \varepsilon + \rho < \varepsilon - \varepsilon + \rho = \rho \leq px$ holds, establishing inequality (6).

If $\overline{px}$ is a segment from p to x, then $\overline{px} \subset D(p;\varepsilon)$. By (6) there is a point y of $\overline{px}$ such that $py = \delta$. By (6) again, the point y lies in $D(p;\rho)$, hence from (4) there is some ball S_j that contains y. The diameter of the subsegment $\overline{yx}$ of $\overline{px}$ is given by $yx = px - py = px - \delta = \varepsilon - \rho < \alpha/4$ by use of (2). Thus, by virtue of the connected set $\overline{yx}$, the point x belongs to C_j. This completes the proof that $D(p;\varepsilon)$ is the union of the sets C_i.

Therefore, $D(p;\varepsilon)$ has property S, and it follows that $\overline{D}(p;\varepsilon)$ is a Peano continuum. **I**

Corollary 3.9. In a locally connected generalized continuum with a complete convex metric, each open ball has property S and each closed ball is a Peano continuum.

Proof: Since by (1.13) a complete convex metric is segmented convex, and by (1.26) each closed ball is compact, then (3.8) applies to give the desired result. ∎

Corollary 3.10. Every locally connected generalized continuum is the image of the closed ray $[0, \infty)$ of E^1 under a continuous mapping.

Proof: Let M be a locally connected generalized continuum, which by (1.25) admits a complete convex metric D. Pick a point p of M. By (3.9), every closed ball $\overline{D}(p;n)$ is a Peano continuum, hence by (1.17) there is a continuous mapping f_n of the closed interval $[2n, 2n + 1]$ onto $\overline{D}(p;n)$, for $n = 0, 1, \ldots$. Since M is arcwise connected, let g_n be a continuous map of $[2n + 1, 2n + 2]$ into M with $g_n(2n + 1) = f_n(2n + 1)$ and $g_n(2n + 2) = f_{n+1}(2n + 2)$, for $n = 0, 1, \ldots$. Then $f = \bigcup\{f_n \cup g_n : n = 0, 1, \ldots\}$ is a continuous mapping of $[0, \infty)$ onto M. ∎

The following example shows that one converse to (3.8) is not true.

Example 3.11. There is a locally connected generalized continuum M with a segmented convex metric D, a point $p \in M$, and a number $\varepsilon > 0$, such that the open ball $D(p;\varepsilon)$ has property S, yet the closed ball $\overline{D}(p;\varepsilon)$ is not compact.

Proof: In E^2, define $M = [(-\infty, 0) \times E^1] \cup [\{0\} \times (-1, 1)]$ and let D be the geodesic metric on M. Then the point $p = (0, 0)$ and any number $\varepsilon \geq 1$ satisfy the requirements. ∎

Characterizations of Locally Connected Generalized Continua

One characterization of locally connected generalized continua was given in (1.27), by means of complete convex metrics. That result is included in the following theorem.

Theorem 3.12. For a space M, the following statements are equivalent.

(i) M is a locally connected generalized continuum.

(ii) M is a locally compact space that admits a complete convex metric.

(iii) M is a locally compact space that admits a segmented convex metric.

(iv) M is a connected Hausdorff space with a countable basis of connected open sets whose closures are Peano continua.

Proof: The proof of (i) → (ii) is given by (1.25), and (ii) → (iii) follows from (1.13). For (iii) → (iv) it is noted that if M is a locally compact space with a segmented convex metric D, then M is locally separable and connected, hence separable by a result of Sierpiński [23]. Thus, let $\{p_i: i = 1, 2, \ldots\}$ be a countable dense subset of M. For each i there is a local basis $\{D(p_i;\delta): \delta \in U_i\}$ at p_i, where $U_i = \{\delta > 0: \delta$ is rational and $\overline{D}(p_i;\delta)$ is compact$\}$. Further, the set $\{D(p_i;\delta): \delta \in U_i, i = 1, 2, \ldots\}$ is a countable basis for M, and each $D(p_i;\delta)$, for $\delta \in U_i$, is a connected open set whose closure, by (3.8), is a Peano continuum. Therefore, (iv) is established from (iii). For the proof of (iv) → (i), it is simply noted that a space satisfying (iv) is a separable, locally connected, locally compact, and metrizable space since it is regular and second countable [11]. ∎

CHAPTER IV

THE UNION TOPOLOGY

Definition and Elementary Properties

For reasons that are to be made more specific at the beginning of Chapter V, the main results of this paper require a certain topology to be specified for the union of two topological spaces. The most important properties of this union topology are given by the results of the present chapter. It is noted that a topological space consisting of a topology $\mathcal{T}$ on a set M is designated by $(M, \mathcal{T})$, or simply by M when the topology is clear from context.

<u>Theorem</u> 4.1. Let $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ be two topological spaces whose topologies agree on $M_1 \cap M_2$. Let $\mathcal{T}_0 = \{R \subset M_1 \cup M_2: R \cap M_1 \varepsilon \mathcal{T}_1, R \cap M_2 \varepsilon \mathcal{T}_2\}$. Let $\mathcal{T}_3$ be a topology on the set $M_1 \cup M_2$ such that both $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ are subspaces of $(M_1 \cup M_2, \mathcal{T}_3)$. Then $\mathcal{T}_0$ is a topology on the set $M_1 \cup M_2$ that is stronger than $\mathcal{T}_3$, and both $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ are subspaces of $(M_1 \cup M_2, \mathcal{T}_0)$.

Proof: It is first shown that $\mathcal{T}_0$ is indeed a topology on the set $M_1 \cup M_2$. Since $(M_1 \cup M_2) \cap M_i = M_i \varepsilon \mathcal{T}_i$ for $i = 1, 2$, then $M_1 \cup M_2 \varepsilon \mathcal{T}_0$. Since $\emptyset \varepsilon \mathcal{T}_i$ for $i = 1, 2$, then $\emptyset \varepsilon \mathcal{T}_0$. If A is any index set and $R_\alpha \varepsilon \mathcal{T}_0$ for every $\alpha \varepsilon A$, then

$[\cup\{R_\alpha: \alpha \in A\}] \cap M_i = \cup\{R_\alpha \cap M_i: \alpha \in A\} \in \mathcal{T}_i$ since $R_\alpha \cap M_i \in \mathcal{T}_i$ for each $\alpha \in A$, $i = 1, 2$. Thus $\cup\{R_\alpha: \alpha \in A\} \in \mathcal{T}_0$ by definition of $\mathcal{T}_0$. Similarly, if R_1 and R_2 are members of $\mathcal{T}_0$, then $R_1 \cap M_i$ and $R_2 \cap M_i$ are members of $\mathcal{T}_i$, hence $(R_1 \cap R_2) \cap M_i \in \mathcal{T}_i$, for $i = 1, 2$; hence, $R_1 \cap R_2 \in \mathcal{T}_0$. Thus, $\mathcal{T}_0$ is a topology on the set $M_1 \cup M_2$.

It is now shown that $(M_i, \mathcal{T}_i)$ is a subspace of $(M_1 \cup M_2, \mathcal{T}_0)$, where $i = 1$ without loss of generality. Let $\mathcal{T}_{12}$ denote the subspace topology on $M_1 \cap M_2$ induced from $\mathcal{T}_1$, which by hypothesis is the same as that induced from $\mathcal{T}_2$. For any set $R \in \mathcal{T}_1$, there is a set $P \in \mathcal{T}_2$ such that $R \cap M_2 = M_1 \cap P \in \mathcal{T}_{12}$. In particular, since $P \cap M_1 \subset R$, then $(R \cup P) \cap M_1 = R \in \mathcal{T}_1$. But similarly $R \cap M_2 \subset P$, so that $(R \cup P) \cap M_2 = P \in \mathcal{T}_2$. Thus it follows that $R \cup P \in \mathcal{T}_0$, and the equation $(R \cup P) \cap M_1 = R$ shows that R is a member of the subspace topology on M_1 induced from $\mathcal{T}_0$. On the other hand, let Q be any member of $\mathcal{T}_0$ restricted to M_1, that is, $Q = M_1 \cap S$ for some $S \in \mathcal{T}_0$. Since $S \in \mathcal{T}_0$, then Q is a member of $\mathcal{T}_1$. This completes the proof that $(M_1, \mathcal{T}_1)$ is a subspace of $(M_1 \cup M_2, \mathcal{T}_0)$.

To show that $\mathcal{T}_3 \subset \mathcal{T}_0$, let $R \in \mathcal{T}_3$. Since $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ are subspaces of $(M_1 \cup M_2, \mathcal{T}_3)$, then $R \cap M_1 \in \mathcal{T}_1$ and $R \cap M_2 \in \mathcal{T}_2$, so that $R \in \mathcal{T}_0$. Therefore, $\mathcal{T}_0$ is stronger than $\mathcal{T}_3$. ∎

For the remainder of this chapter the definitions and notation introduced in (4.1) will be assumed, although for future reference the reader will be reminded of these in the hypotheses of the theorems. The main fact stated in (4.1) is that $\mathcal{T}_0$ is the strongest topology for $M_1 \cup M_2$ that leaves M_1 and M_2 as subspaces.

Corollary 4.2. If $M_1 \subset M_2$, then $\mathcal{T}_0 = \mathcal{T}_2 = \mathcal{T}_3$.

Proof: Since in this case $(M_2, \mathcal{T}_2)$ is required to be a subspace of both $(M_2, \mathcal{T}_3)$ and $(M_2, \mathcal{T}_0)$, then $\mathcal{T}_0 = \mathcal{T}_2 = \mathcal{T}_3$.

Theorem 4.3. With the notation of (4.1), if $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $(M_1 \cup M_2, \mathcal{T}_3)$, then $\mathcal{T}_3 = \mathcal{T}_0$. As a partial converse, if $\mathcal{T}_0 = \mathcal{T}_3$ and $M_1 \cap M_2$ is closed in both M_1 and M_2, then $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $(M_1 \cup M_2, \mathcal{T}_3)$.

Proof: Assume first that $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $(M_1 \cup M_2, \mathcal{T}_3)$. To show that $\mathcal{T}_3 = \mathcal{T}_0$, it suffices by (4.1) to show that $\mathcal{T}_0 \subset \mathcal{T}_3$. Let $R \in \mathcal{T}_0$ and $p \in R$. By the definition of $\mathcal{T}_0$, $R \cap M_1 \in \mathcal{T}_1$ and $R \cap M_2 \in \mathcal{T}_2$. Since for $i = 1, 2$ the space $(M_i, \mathcal{T}_i)$ is a subspace of $(M_1 \cup M_2, \mathcal{T}_3)$, there is a set $R_i \in \mathcal{T}_3$ such that $R_i \cap M_i = R \cap M_i$. In the case that $p \in M_1 \cap M_2$, then $p \in R \cap M_1 \cap M_2 \subset R_1 \cap R_2 \in \mathcal{T}_3$. Further, since $(R_1 \cap R_2) \cap M_i \subset R_i \cap M_i = R \cap M_i \subset R$, then $R_1 \cap R_2 = [(R_1 \cap R_2) \cap M_1] \cup [(R_1 \cap R_2) \cap M_2] \subset R$. In the case that p lies outside of $M_1 \cap M_2$, then without loss of generality let p lie in $M_1 \setminus M_2$. Since $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $(M_1 \cup M_2, \mathcal{T}_3)$, there is a set $Q \in \mathcal{T}_3$ with $p \in Q$ and $Q \cap (M_2 \setminus M_1) = \emptyset$. Thus $p \in R_1 \cap Q \in \mathcal{T}_3$, and since $Q \subset M_1$, then $R_1 \cap Q = R_1 \cap (Q \cap M_1) = Q \cap (R \cap M_1) \subset R$. Hence, regardless of where the point p lies, there is an element of $\mathcal{T}_3$ that contains p and is contained in R. Therefore $R \in \mathcal{T}_3$, hence $\mathcal{T}_0 = \mathcal{T}_3$.

For the partial converse, assume that $\mathcal{T}_0 = \mathcal{T}_3$ and that $M_1 \cap M_2$ is closed in both $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$. Then $M_1 \setminus M_2 \in \mathcal{T}_0 = \mathcal{T}_3$ and $M_2 \setminus M_1 \in \mathcal{T}_0 = \mathcal{T}_3$ by the definition of $\mathcal{T}_0$. Since $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are disjoint members of $\mathcal{T}_3$, then these sets are separated in $(M_1 \cup M_2, \mathcal{T}_3)$. ∎

<u>Corollary</u> 4.4. If M_1 and M_2 are both open in $\mathcal{T}_3$ or both closed in $\mathcal{T}_3$, then $\mathcal{T}_0 = \mathcal{T}_3$.

Proof: Given either hypothesis, then $M_1 \setminus M_2$ and $M_2 \setminus M_1$ will be disjoint sets that are both closed or are both open in $\mathcal{T}_3$, hence separated sets in $(M_1 \cup M_2, \mathcal{T}_3)$. The conclusion follows by (4.3). ∎

<u>Corollary</u> 4.5. If M_1 and M_2 are compact and $\mathcal{T}_3$ is Hausdorff, then $\mathcal{T}_0 = \mathcal{T}_3$.

Proof: If the hypothesis holds, then M_1 and M_2 will both be closed in $\mathcal{T}_3$, and (4.4) gives the conclusion. ∎

The next two examples show why the converses to the two statements of (4.3) cannot be proved.

<u>Example</u> 4.6. The sets $M_1 \setminus M_2$ and $M_2 \setminus M_1$ need not be separated in $(M_1 \cup M_2, \mathcal{T}_0)$.

Proof: Let M_1 be the set of all rational numbers in the interval $[0, 1) \subset E^1$, and let $M_2 = (0, 1) \subset E^1$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the subspace topologies induced from the usual topology for E^1. Then $\mathcal{T}_0$ is not the usual topology on $M_1 \cup M_2 = [0, 1)$, for the set $\{x_i : i = 1, 2, \dots\}$ is in $\mathcal{T}_0$, where $\langle x_i \rangle$ is any sequence of irrational numbers in M_2 that decreases to 0. However, it is true that $\{0\} = M_1 \setminus M_2$ is a $\mathcal{T}_0$ limit point of $M_2 \setminus M_1$, for let $0 \in Q \in \mathcal{T}_0$. Then $Q \cap M_1 \in \mathcal{T}_1$, hence there is a rational $0 < r < 1$ such that $r \in Q$. Thus $r \in Q \cap M_2 \in \mathcal{T}_2$, so that there must be an irrational p in $Q \cap M_2$ also, showing that $Q \cap (M_2 \setminus M_1)$ is not empty. **I**

<u>Example</u> 4.7. The sets $M_1 \setminus M_2$ and $M_2 \setminus M_1$ may be separated in $(M_1 \cup M_2, \mathcal{T}_3)$ and yet $M_1 \cap M_2$ may not be closed in either M_1 or M_2.

Proof: Let $M_1 = [0, 2)$ and $M_2 = (1, 3]$ in E^1, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the subspace topologies induced from the usual topology. Then $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $[0, 3]$ with the usual topology, but $(1, 2)$ is closed in neither $\mathcal{T}_1$ nor $\mathcal{T}_2$. **I**

If $M_1 \cap M_2$ is a closed subset of both M_1 and M_2, then $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are disjoint members of $\mathcal{T}_0$, hence separated sets in $(M_1 \cup M_2, \mathcal{T}_0)$. In particular, if M_1 and M_2 are disjoint sets, then M_1 and M_2 are separated sets in $(M_1 \cup M_2, \mathcal{T}_0)$. However, as the following example shows, even this situation need not compel $\mathcal{T}_3$ to be identical with $\mathcal{T}_0$.

<u>Example</u> 4.8. It may happen that $\mathcal{T}_3 \neq \mathcal{T}_0$, even when M_1 and M_2 are disjoint.

Proof: Let M_1 be the portion of the unit circle in the cartesian plane consisting of all points with non-negative ordinates, and let M_2 be the remainder of the unit circle. If $\mathcal{T}_3$ is the usual topology of E^2 restricted to $M_1 \cup M_2$, then $\mathcal{T}_0$ is strictly stronger than $\mathcal{T}_3$. This follows from the last statement of (4.3) by contraposition, since $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are not separated sets in $\mathcal{T}_3$. **I**

Conditions on $\mathcal{T}_3$ have been given in (4.2) through (4.5) which ensure that $\mathcal{T}_3$ must be identical with $\mathcal{T}_0$. The following theorem furnishes a condition on M_1 and M_2 which ensures the existence of some topology $\mathcal{T}_3$ that is different from $\mathcal{T}_0$.

<u>Theorem</u> 4.9. With the notation of (4.1), suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are both T_1 topologies. If for $i = 1, 2$ the set M_i contains some point that is not in the $\mathcal{T}_i$ closure of $M_1 \cap M_2$, then there is some topology $\mathcal{T}$ on the set $M_1 \cup M_2$, different from $\mathcal{T}_0$, such that $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ are subspaces of $(M_1 \cup M_2, \mathcal{T})$.

Proof: Let Q_i be the set of all points of M_i that are not in the $\mathcal{T}_i$ closure of $M_1 \cap M_2$, and pick points $p_i \in Q_i$, $i = 1, 2$. Define $\mathcal{T} = \{R \subset M_1 \cup M_2\colon\ R \cap M_1 \in \mathcal{T}_1,\ R \cap M_2 \in \mathcal{T}_2,$ and $p_1 \in R$ if and only if $p_2 \in R\}$. Since the requirement in this definition that involves the simultaneous inclusion or exclusion of the points p_i is preserved under arbitrary unions and intersections, then it can be shown exactly as in the proof of (4.1) that $\mathcal{T}$ is a topology on the set $M_1 \cup M_2$. To see that $\mathcal{T}_0 \neq \mathcal{T}$, it suffices to note that $Q_1 \in \mathcal{T}_1$ and $Q_1 \cap M_2 = \emptyset$, hence $Q_1 \in \mathcal{T}_0$; but since $p_1 \in Q_1$ while $p_2 \notin Q_1$, then $Q_1 \notin \mathcal{T}$.

For the proof that $(M_1, \mathcal{T}_1)$ is a subspace of $(M_1 \cup M_2, \mathcal{T})$, let $R \in \mathcal{T}_1$. As in the proof of (4.1) it can be shown that there is a set $P \in \mathcal{T}_2$ such that $(R \cup P) \cap M_1 = R$ and $(R \cup P) \cap M_2 = P$. If $p_1 \in R$, then since $p_2 \in Q_2 \in \mathcal{T}_2$, it follows that p_1 and p_2

both lie in $R \cup P \cup Q_2$. Moreover, since $(R \cup P \cup Q_2) \cap M_1 = [(R \cup P) \cap M_1] \cup [Q_2 \cap M_1] = R \cup \emptyset = R \in \mathcal{T}_1$ and $(R \cup P \cup Q_2) \cap M_2 = [(R \cup P) \cap M_2] \cup [Q_2 \cap M_2] = P \cup Q_2 \in \mathcal{T}_2$, then $R \cup P \cup Q_2 \in \mathcal{T}$ by the definition of $\mathcal{T}$. If p is not in R, then since $(M_2, \mathcal{T}_2)$ is a T_1 space, the set $P \setminus \{p_2\}$ is $\mathcal{T}_2$ open. Hence, $[R \cup (P \setminus \{p_2\}] \cap M_1 = [R \cup P] \cap M_1 = R \in \mathcal{T}_1$ and $[R \cup (P \setminus \{p_2\})] \cap M_2 = P \setminus \{p_2\} \in \mathcal{T}_2$. But since p_1 is not in $R \cup (P \setminus \{p_2\})$ and p_2 is not in $R \cup (P \setminus \{p_2\})$, then $R \cup (P \setminus \{p_2\}) \in \mathcal{T}$. Thus, regardless of the position of the point p_1 relative to the set R, there is an element of $\mathcal{T}$ whose intersection with M_1 is the set R; that is, the subspace topology on M_1 induced from $\mathcal{T}$ is stronger than $\mathcal{T}_1$. Since, conversely, $R \cap M_1 \in \mathcal{T}_1$ holds for any set $R \in \mathcal{T}$, then $(M_1, \mathcal{T}_1)$ is indeed a subspace of $(M_1 \cup M_2, \mathcal{T})$. The similar statement is true for $(M_2, \mathcal{T}_2)$. **I**

The following simple result is quite useful in the sequel.

<u>Theorem</u> 4.10. With the notation of (4.1), suppose a point x lies in $Q_1 \cap Q_2$, where $Q_1 \in \mathcal{T}_1$ and $Q_2 \in \mathcal{T}_2$. Then x is $\mathcal{T}_3$ interior to $Q_1 \cup Q_2$.

Proof: Assume the hypothesis, and further suppose that x is not $\mathcal{T}_3$ interior to $Q_1 \cup Q_2$. Then there is some net $\langle x_\alpha \rangle$ that is $\mathcal{T}_3$ convergent to x, yet x_α lies outside $Q_1 \cup Q_2$ for each index α. Since this net is in $M_1 \cup M_2$, there is a subnet $\langle x_\beta \rangle$ of $\langle x_\alpha \rangle$ such that, without loss of generality, $x_\beta \in M_1$ for each β. Since x is in M_1 and $(M_1, \mathcal{T}_1)$ is a subspace of $(M_1 \cup M_2, \mathcal{T}_3)$, then $\langle x_\beta \rangle$ is $\mathcal{T}_1$ convergent to x. But since $Q_1 \in \mathcal{T}_1$, then $\langle x_\beta \rangle$ is eventually in Q_1, contrary to the fact that the net $\langle x_\alpha \rangle$ lies in the complement of $Q_1 \cup Q_2$. Hence, x must be $\mathcal{T}_3$ interior to $Q_1 \cup Q_2$. ∎

Results More Closely Related to

Generalized Continua

Up to this point, Chapter IV has defined the union topology and demonstrated some fundamental results concerning it. This much is only to be expected, since in Chapter V the very statements of the extension theorems assume a familiarity with this topology. The three theorems now to be proved, however, are of a different nature than the foregoing, for these in essence will contribute the proofs of "necessity" and of "sufficiency" in (5.4) and (5.6), respectively, the two main theorems of this dissertation. Because of this purpose, the following three theorems relate more closely to generalized continua than did the preceding ones.

Theorem 4.11. With the notation of (4.1), let $M_1 \cap M_2$ be closed in $\mathcal{T}_2$ and let the $\mathcal{T}_2$ boundary of $M_1 \cap M_2$ be closed in $\mathcal{T}_1$. If both $(M_1, \mathcal{T}_1)$ and $(M_2, \mathcal{T}_2)$ are locally compact, and if $\mathcal{T}_0$ is Hausdörff, then $(M_1 \cup M_2, \mathcal{T}_0)$ is locally compact.

Proof: Pick any point x in $M_1 \cup M_2$. For each i with x in M_i, there is some open $\mathcal{T}_i$ neighborhood R_i of x such that the $\mathcal{T}_i$ closure of R_i is $\mathcal{T}_i$ compact, hence also $\mathcal{T}_0$ compact. If x is in $M_2 \setminus M_1$, the set $Q_2 = R_2 \cap (M_2 \setminus M_1)$ is a $\mathcal{T}_2$ (and $\mathcal{T}_0$) neighborhood of x whose $\mathcal{T}_2$ (and $\mathcal{T}_0$) closure is compact. If x is in $M_1 \cap M_2$, then the fact that x is in $R_1 \cap R_2$ implies by (4.10) that some $\mathcal{T}_0$ neighborhood R of x lies in $R_1 \cup R_2$, so that R is conditionally compact. If x is in $M_1 \setminus M_2$, then $M_1 \setminus B$ is a $\mathcal{T}_1$ neighborhood of x, where B is the $\mathcal{T}_2$ boundary of $M_1 \cup M_2$. Also, $S = (M_1 \cap M_2) \setminus B$ is $\mathcal{T}_2$ open and $Q_1 = R_1 \cap (M_1 \setminus B)$ is a $\mathcal{T}_1$ neighborhood of x whose closure is compact. Since S is a subspace of $(M_1, \mathcal{T}_1)$, then $Q_1 \cap M_2 = Q_1 \cap S$ is open in S, hence $Q_1 \cap M_2$ is $\mathcal{T}_2$ open. Therefore Q_1 is in $\mathcal{T}_0$, and $(M_1 \cup M_2, \mathcal{T}_0)$ is locally compact. **I**

The following example relates both to the preceding theorem and to the one which follows it.

Example 4.12. There are locally connected generalized continua M_1 and M_2 in the plane with $M_1 \cap M_2$ closed in M_2, but with $(M_1 \cup M_2, \mathcal{T}_0)$ neither locally compact nor first countable.

Proof: Let $M_1 = E^1 \times (-\infty, 0]$ and $M_2 = (-\infty, 0) \times E^1$ be given their respective subspace topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets of E^2. First it is shown that the space $(M_1 \cup M_2, \mathcal{T}_0)$ is not locally compact at the point $p = (0, 0)$. If U is any $\mathcal{T}_0$ neighborhood of p, then U contains a segment $[-\varepsilon, \varepsilon] \times \{0\}$ for some $\varepsilon > 0$, since $U \cap M_1$ is a member of $\mathcal{T}_1$. Each point $p_n = (-\varepsilon/n, 0)$ is $\mathcal{T}_2$ interior to U, since $p_n \in U \cap M_2 \in \mathcal{T}_2$, where $n = 1, 2, \ldots$. Thus, for each n there is a number $\delta_n > 0$ such that $q_n = (-\varepsilon/n, \delta_n) \in U$. If $Q = \{ q_n : n = 1, 2, \ldots\}$ and $V = U \setminus Q$, then V is a set containing p but no point q_n, and furthermore $V \cap M_1 = U \cap M_1 \in \mathcal{T}_1$ while $V \cap M_2 = (U \cap M_2) \setminus Q \in \mathcal{T}_2$ since Q is $\mathcal{T}_2$ closed. Thus, p is not a $\mathcal{T}_0$ accumulation point of Q, but neither is any other point of $M_1 \cup M_2$. Hence Q is an infinite subset of U with no accumulation point, so that the $\mathcal{T}_0$ closure of U is not compact.

Therefore, $(M_1 \cup M_2, \mathcal{T}_0)$ is not locally compact at the point p.

Similarly, it could be shown that $(M_1 \cup M_2, \mathcal{T}_0)$ is not first countable at p. More simply, however, the following theorem shows that $(M_1 \cup M_2, \mathcal{T}_0)$ is not first countable, since the $\mathcal{T}_2$ boundary $(-\infty, 0) \times \{0\}$ of $M_1 \cap M_2$ is not $\mathcal{T}_1$ closed. ∎

Theorem 4.13. With the notation of (4.1), let B be the $\mathcal{T}_2$ boundary of $M_1 \cap M_2$. If $(M_1 \cup M_2, \mathcal{T}_0)$ is both Hausdorff and first countable, then $B \cap M_1$ is $\mathcal{T}_1$ closed.

Proof: If $(M_1 \cup M_2, \mathcal{T}_0)$ is Hausdorff and first countable and some point p of $M_1 \cap B$ is a $\mathcal{T}_1$ accumulation point of $B \cap M_1$, let $\langle U_n \rangle$ be a countable local $\mathcal{T}_0$ base at p. For each positive integer n there is some point b_n in $B \cap (M_1 \cap U_n)$, hence there is also a point c_n in $U_n \cap (M_2 \setminus M_1)$. The set $C = \{c_n : n = 1, 2, \ldots\}$ is a $\mathcal{T}_2$ closed subset of $M_2 \setminus M_1$, so that $U = M_1 \cap (M_2 \setminus C)$ is a $\mathcal{T}_0$ neighborhood of p. But this is impossible, since U contains no set U_n. ∎

Theorem 4.14. With the notation of (4.1), let the space M_1 have a complete metric D_1. In order that $(M_1 \cup M_2, \mathcal{T}_0)$ be a connected space and admit some metric D_3 that extends D_1, it is necessary that $M_1 \cap M_2$ be a non-empty subspace of both M_1 and M_2 which is closed in M_2, and that the M_2 boundary of $M_1 \cap M_2$ be closed in M_1.

Proof: In order for the space $(M_1 \cup M_2, \mathcal{T}_0)$ to be defined, it is necessary that $M_1 \cap M_2$ be a subspace of both M_1 and M_2. Since $(M_1 \cup M_2, \mathcal{T}_0)$ is connected, then by (4.3) it follows that $M_1 \cap M_2$ is non-empty. If $M_1 \cap M_2$ is not closed in M_2, there is a point p of $M_2 \setminus M_1$ and a sequence $\langle p_n \rangle$ of points in $M_1 \cap M_2$ such that $\langle p_n \rangle$ converges in M_2 to p. Therefore, it holds that

$$\lim_{n \to \infty} D_3(p_n, p) = 0,$$

and $\langle p_n \rangle$ is thus a D_1 Cauchy sequence which converges in M_1 to some point q of M_1. Hence, in topology $\mathcal{T}_0$ the sequence $\langle p_n \rangle$ converges to the two distinct points p and q, contradicting the fact that $\mathcal{T}_0$ is Hausdorff. Since $M_1 \cap M_2$ must therefore be closed in M_2, then (4.13) shows that the M_2 boundary of $M_1 \cap M_2$ must also be closed in M_1. ∎

CHAPTER V

EXTENSION OF COMPLETE CONVEX METRICS

Background

In 1949 Bing [4] proved that *if M_1 and M_2 are intersecting Peano continua whose topologies agree on their intersection, and if D_1 is a convex metric for M_1, there is a convex metric D_3 for $M_1 \cup M_2$ that extends D_1.* In the present chapter the compactness of M_1 and M_2 is deleted, and the question is addressed: *if M_1 and M_2 are intersecting locally connected generalized continua whose topologies agree on their intersection, and if D_1 is a complete convex metric for M_1, under what conditions will there be a complete convex metric D_3 for $M_1 \cup M_2$ that extends D_1?* In (5.4) a necessary and sufficient condition for the existence of D_3 is obtained by specifying two topological properties of the intersection $M_1 \cap M_2$; actually, in this result the space M_1 is not required to be a locally connected generalized continuum, but merely any space with a complete convex metric D_1. In order to establish this result, several others must first be obtained.

Before this program is begun, however, a word of explanation is due on what is meant by references to the space $M_1 \cup M_2$. In the case of Bing's extension theorem (1.31) that was cited at the beginning of this section, a topology for $M_1 \cup M_2$ was not specified for the following reason: according to (4.5) there is only one possible Hausdorff topology on the set $M_1 \cup M_2$ that leaves M_1 and M_2 as subspaces, namely, the topology $\mathcal{T}_0$ discussed in Chapter IV. If, as will be the case in the present chapter, the spaces M_1 and M_2 are not required to be compact, there is in general more than one topology on the set $M_1 \cup M_2$ for which M_1 and M_2 are subspaces, as (4.9) clearly shows. It is for this reason that, for the results that are now to be proved, a topology must be specified on the set $M_1 \cup M_2$. Accordingly, the convention is hereby adopted that whenever "$M_1 \cup M_2$" is written without further explanation, the topological space $(M_1 \cup M_2, \mathcal{T}_0)$ is intended. There will be no need to consider any other topology on the set $M_1 \cup M_2$.

Extension Theorems

The first extension theorem to be proved provides a sufficient condition for a complete convex metric to be extended to the union of two spaces.

<u>Theorem</u> 5.1. Let M_1 be a space with complete convex metric D_1 and let M_2 be a locally connected generalized continuum with complete convex metric D_2, whose intersection with M_1 is a non-empty, compact subspace of both M_1 and M_2. Then for any $\varepsilon > 0$ and for any two non-empty subsets C and H of M_2 with $D_2(C, H \cup (M_1 \cap M_2)) > 0$, there is a complete convex metric D_3 for $M_1 \cup M_2$ that extends D_1, satisfies $D_3(C,H) \geq \varepsilon$, and has the property that if $D_3(x,y) < D_2(x,y)$ for points x, y of $M_2 \setminus M_1$, then x and y have a D_3 between point in M_1.

Proof: Let $\delta = D_2(C, H \cup (M_1 \cap M_2))$. The proof follows the general pattern of the proof of the extension theorem of Bing [4]. There is a real-valued function $F(x)$, $x > 0$, satisfying the following conditions: $F(x) \geq \sup \{D_1(p,q): p, q \in M_1 \cap M_2, D_2(p,q) \leq x\}$ holds for all $x > 0$, $F(x)$ approaches 0 as x approaches 0 from the right, $F'(x)$ is a continuous non-increasing function which exceeds both ε/δ and 1, and the improper integral

$$\int_0^\alpha F'(x)\, dx$$

exists for all $\alpha > 0$. Such a function is obtained exactly as in [4]; in fact, several statements asserted in this proof are restatements of facts used in [4], and thus are left unproven here. For every two points x, y of M_2 let $A(x,y)$ be the set of all D_2 rectifiable arcs C from x to y that lie, except for possibly their endpoints, in $M_2 \setminus M_1$ and for which the (possibly improper) Riemann integral

$$\int_C F'[D_2(p(s), M_1)]\, ds$$

exists. Here, s denotes D_2 length along C from a fixed endpoint and $p(s)$ is the point of C whose D_2 distance along C from the fixed endpoint is s. If x lies in $M_2 \setminus M_1$, y is a point of M_1 such that $D_2(x,y) = D_2(x, M_1)$ holds, and $\overline{xy}$ is a D_2 segment from x to y, then the integral

$$\int_{\overline{xy}} F'[D_2(p(s), M_1)]\, ds$$

exists and has the value $F[D_2(x,M_1)]$. For all points x, y of M_2 with $A(x,y) \neq \emptyset$, let

$$D_0(x,y) = \inf \{\int_C F'[D_2(p(s),M_1)]\, ds: \ C \in A(x,y)\}.$$

If $D_0(x,y)$ exists for two points x and y then $D_0(x,y) \geq D_2(x,y)$ holds, and $D_0(x,y) \geq D_1(x,y)$ holds if x and y are points of M_1. Define D_3 on the set $M_1 \cup M_2$ as follows: if $x, y \in M_1$, then $D_3(x,y) = D_1(x,y)$; if $x \in M_1$ and $y \in M_2 \setminus M_1$, then define $D_3(x,y) = D_3(y,x) = \inf \{D_1(x,a) + D_0(a,y): \ a \in M_1, \ A(a,y) \neq \emptyset\}$; if $x, y \in M_2 \setminus M_1$, then $D_3(x,y)$ is the minimum of $D_0(x,y)$ and $\inf \{D_0(x,a) + D_1(a,b) + D_0(b,y): \ a, b \in M_1, A(x,a) \neq \emptyset \neq A(b,y)\}$; if $x = y$, then $D_3(x,y) = 0$. It follows as in [4] that D_3 is a metric on the set $M_1 \cup M_2$ whose restriction to M_2 is equivalent to D_2. Since also D_1 is the restriction of D_3 to M_1, then both M_1 and M_2 are subspaces of $(M_1 \cup M_2, D_3)$. The proof is now completed by proving assertions (i) through (vii).

(i) $(M_1 \cup M_2, D_3) = (M_1 \cup M_2, \mathcal{T}_0)$.

(ii) If x is in $M_2 \setminus M_1$ and y is a point of M_1 such that $D_2(x,y) = D_2(x,M_1)$, then $D_0(x,y) = \inf \{D_0(x,a): \ a \in M_1, A(x,a) \neq \emptyset\}$.

(iii) For every point x of M_2, $D_3(x,M_1) \geq D_2(x,M_1)$ holds.

(iv) Every closed and D_3 bounded subset of M_2 is compact.

(v) D_3 is complete and convex.

(vi) $D_3(C,H) \geq \varepsilon$.

(vii) If $D_3(x,y) < D_2(x,y)$ holds for points x, y of $M_2 \setminus M_1$, then x and y have a D_3 between point in M_1.

(i) It has been noted that M_1 and M_2 are subspaces of $(M_1 \cup M_2, D_3)$. It is now shown that $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $(M_1 \cup M_2, D_3)$. To this end, let $x \in M_1 \setminus M_2$ and $y \in M_2 \setminus M_1$; then $D_3(x,y) = \inf \{D_1(x,a) + D_0(a,y)\colon\ a \in M_1, A(a,y) \neq \emptyset\}$. For any point $a \in M_1$ with $A(a,y) \neq \emptyset$ it must hold that $a \in M_1 \cap M_2$, hence $D_1(x,a) + D_0(a,y) \geq D_1(x,a) \geq D_1(x,M_1 \cap M_2)$ and $D_1(x,a) + D_0(a,y) \geq D_0(a,y) \geq D_2(a,y) \geq D_2(M_1 \cap M_2,y)$. Upon taking infima, it is seen that $D_3(x,y) \geq D_1(x,M_1 \cap M_2) > 0$ and also $D_3(x,y) \geq D_2(M_1 \cap M_2,y) > 0$ by reason of the compactness of $M_1 \cap M_2$. Therefore, it follows that $D_3(x,M_2 \setminus M_1) \geq D_1(x,M_1 \cap M_2) > 0$ and $D_3(M_1 \setminus M_2,y) \geq D_2(M_1 \cap M_2,y) > 0$ both hold, and the sets $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are indeed separated in $(M_1 \cup M_2, D_3)$. It follows now from (4.3) that $(M_1 \cup M_2, D_3) = (M_1 \cup M_2, \mathcal{T}_0)$.

(ii) With x and y as given in (ii), it suffices to show that for any point a in M_1 with $A(x,a) \neq \emptyset$ and for any $C \in A(x,a)$,

$$\int_{\overline{xy}} F'[D_2(q(s),M_1)]\, ds \leq \int_C F'[D_2(p(s),M_1)]\, ds \tag{1}$$

holds, where $\overline{xy}$ is a D_2 segment from x to y, $q(s)$ is the point u of $\overline{xy}$ with $D_2(y,u) = s$, and $p(s)$ is the point of C whose D_2 distance along C from a is s. In fact, inequality (1) shows that

$$D_0(x,y) = \int_{\overline{xy}} F'[D_2(q(s),M_1)]\, ds$$

holds, being obtained by setting $a = y$.

Let α and β be the D_2 lengths of $\overline{xy}$ and C, respectively. Since $D_2(p(s),M_1) \leq D_2(p(s),a) \leq s$ and F' is monotone non-increasing, then $F'[D_2(q(s),M_1)] = F'(s) \leq F'[D_2(p(s),M_1)]$ holds for $0 < s \leq \beta$. Further, since $0 < \alpha = D_2(x,M_1) \leq \beta$ holds, then

$$\int_0^{\alpha} F'[D_2(q(s),M_1)]\, ds \leq \int_0^{\beta} F'[D_2(p(s),M_1)]\, ds$$

follows by elementary properties of improper integrals, and is the desired inequality (1). Thus, (ii) is proved.

(iii) Pick a point $x \in M_2$, where for (iii) it may be assumed that x is in $M_2 \setminus M_1$. Let y be a point of M_1 such that $D_2(x,y) = D_2(x,M_1)$ and let z be any point of M_1. Then from (ii) and the definition of D_3, it holds that

$$\begin{aligned} D_3(x,z) &= \inf \{D_0(x,a) + D_1(a,z)\colon\ a \in m_1,\ A(x,a) \neq \emptyset\} \\ &\geq \inf \{D_0(x,a)\colon\ a \in M_1,\ A(x,a) \neq \emptyset\} = D_0(x,y) \\ &\geq D_2(x,y) = D_2(x,M_1). \end{aligned}$$

Since z is arbitrary in M_1, it follows that $D_3(x,M_1) \geq D_2(x,M_1)$.

(iv) By (iii), every closed and D_3 bounded subset of M_2 is also D_2 bounded, hence is compact according to (1.26).

(v) The convexity of D_3 is proved by applying the local compactness of M_2 in much the same way that compactness is used in the proof of Bing [4]. To show that D_3 is complete, let $\langle x_i \rangle$ be a D_3 Cauchy sequence in $M_1 \cup M_2$. If some subsequence $\langle y_j \rangle$ of $\langle x_i \rangle$ lies in M_1, then $\langle y_j \rangle$ is D_1 Cauchy since D_3 extends D_1.

Hence, $\langle y_j \rangle$ converges to some point y in M_1, to which $\langle x_i \rangle$ must also converge. If no subsequence of $\langle x_i \rangle$ lies in M_1, then it may be assumed that the entire set $W = \{x_i: \ i = 1, 2, \ldots\}$ is contained in M_2. Since $\langle x_i \rangle$ is a D_3 Cauchy sequence, then W is D_3 bounded. Hence, the M_2 closure $\overline{W}$ of W is a closed and D_3 bounded subset of M_2 which, according to (iv), is compact. Therefore, D_3 restricted to $\overline{W}$ is complete, and $\langle x_i \rangle$ converges to some point in $\overline{W}$. Thus, D_3 is a complete metric.

(vi) Let $x \in H$ and $y \in C$. Then $y \in M_2 \setminus M_1$ since the given inequality $D_2(C, H \cup (M_1 \cap M_2)) > 0$ implies that $C \cap M_1 = \emptyset$. If $x \in M_1 \cap M_2$, then by the definition of D_3, it is given that $D_3(x,y) = \inf \{D_1(x,a) + D_0(a,y): \ a \in M_1, \ A(a,y) \neq \emptyset\}$. For any point $a \in M_1$ with $A(a,y) \neq \emptyset$ the inequality $D_1(x,a) + D_0(a,y) \geq D_0(a,y) \geq (\varepsilon/\delta)\, D_2(a,y) \geq (\varepsilon/\delta)\, D_2(C, H \cup (M_1 \cap M_2)) = \varepsilon$ holds by the definition of δ. Therefore, $D_3(x,y) \geq \varepsilon$ holds in this case.

If, however, $x \in M_2 \setminus M_1$ holds, then either $D_3(x,y) = D_0(x,y)$ or $D_3(x,y) = \inf \{D_0(x,a) + D_1(a,b) + D_0(b,y): \ a, b \in M_1, \ A(x,a) \neq \emptyset \neq A(b,y)\}$. If $D_3(x,y) = D_0(x,y)$, then as above, it holds that $D_3(x,y) = D_0(x,y) \geq (\varepsilon/\delta)\, D_2(x,y) \geq \varepsilon$. If $D_3(x,y)$ equals the above infimum, then for any points $a, b \in M_1$ with $A(x,a) \neq \emptyset \neq A(b,y)$ it follows as before that the inequality

$D_0(x,a) + D_1(a,b) + D_0(b,y) \geq D_0(b,y) \geq (\varepsilon/\delta)\, D_2(b,y) \geq \varepsilon$ holds, and therefore $D_3(x,y) \geq \varepsilon$.

Since in any case $D_3(x,y) \geq \varepsilon$ holds, then $D_3(C,H) \geq \varepsilon$.

(vii) With x and y as given in (vii), then $D_3(x,y)$ cannot equal $D_0(x,y)$ since $D_0(x,y) \geq D_2(x,y)$. Therefore, it must be that

$D_3(x,y) = \inf \{D_0(x,a) + D_1(a,b) + D_0(b,y)\colon\ a, b \in M_1,\ A(x,a) \neq \emptyset \neq A(b,y)\}$. The existence in $M_1 \cap M_2$ of a D_3 between point of x and y now follows, exactly as in the original proof of Bing [4], from the compactness of $M_1 \cap M_2$. ∎

The following example shows that a complete convex metric on a locally connected generalized continuum need not have the property that every compact subset is contained in a compact subset on which the metric is convex. It is shown in (5.3), however, that the space in question can be remetrized with a complete convex metric for which the stated property holds.

Example 5.2. There is a one dimensional locally compact space **X** in E^2 containing three points and having a complete convex metric D that is not convex on any compact subset of X which contains those three points.

Proof: Using cylindrical coordinates in E^3, for each odd m let $p_m^n = (1, (2n+1)\pi/3, 1-2^{-m})$ and for each even m let $p_m^n = (1, 2n\pi/3, 1-2^{-m})$, $n = 0, 1, 2$ and $m = 0, 1, \ldots$. Construct the euclidean segments $[p_m^i p_{m+1}^j]$ for all m and for all $(i, j) = (0, 0), (0, 2), (1, 0), (1, 1), (2, 1)$, and $(2, 2)$. If X is the union of all such segments and if D is defined to be the geodesic metric on X, then D is a complete convex metric. Moreover, the only D convex subsets of X containing the three points p_0^0, p_0^1, and p_0^1 are dense in the space X. The space (X, D), shown in Figure 3 on the following page, is one dimensional and can be embedded in the plane $z = 0$ by projecting along lines through $(0, 0, 1)$. ∎

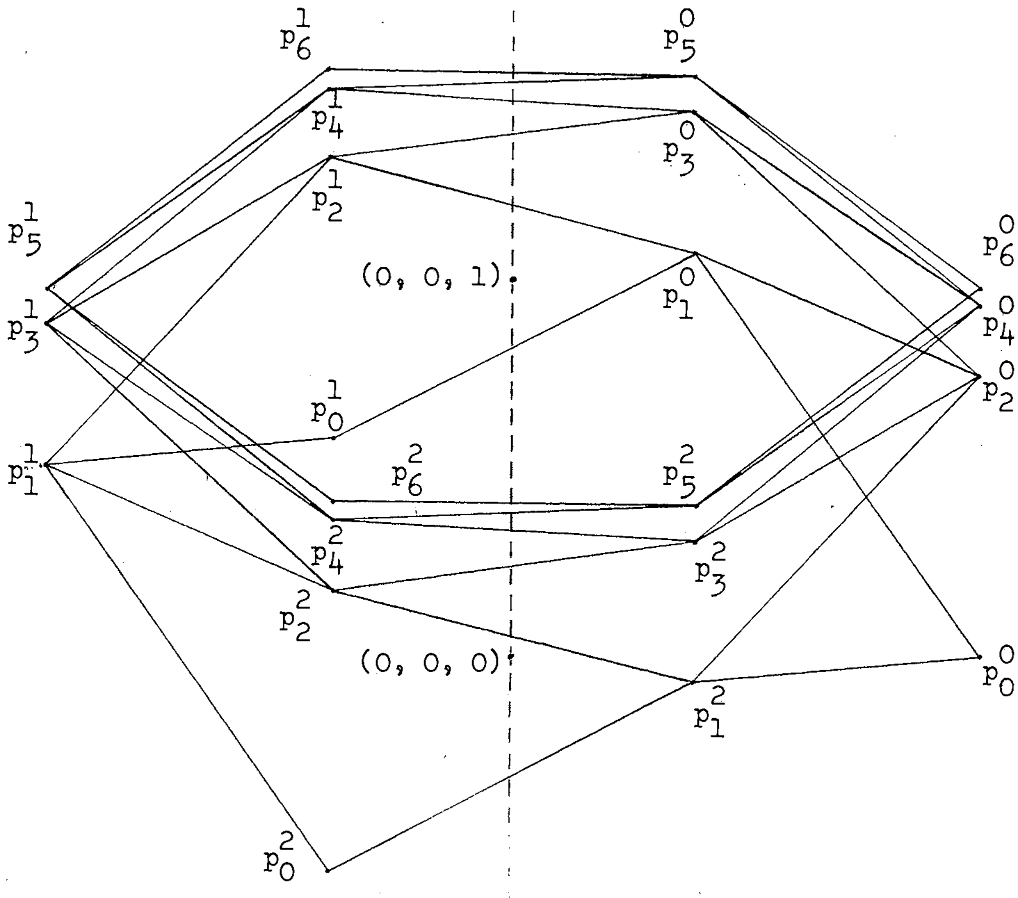

Figure 3. The Set X of (5.2)

Theorem 5.3. Let M be a locally connected generalized continuum with complete convex metric D. Then for any point p of M, there is a complete convex metric E for M that is convex on $\overline{D}(p;n)$ and has the property that if $D(p,x) = n$ then $E(x,\overline{D}(p;n - 1/2)) \geq 1$, for each positive integer n.

Proof: For each non-negative integer n, it follows from (3.9) that the set $P_n = \overline{D}(p;n)$ is a Peano continuum. Moreover, for $n \geq 1$ the two sets $C_n = \{x\colon\ D(p,x) = n\}$ and $H_{n-1} = \{x\colon\ D(p,x) = n - 1/2\}$

are compact and disjoint. By (5.1) there is a convex metric E_1 for P_1 such that $E_1(C_1,H_0) \geq 1$ if $C_1 \neq \emptyset$. By repeated use of (5.1), a sequence $E_1, E_2, \ldots$ of convex metrics respectively for $P_1, P_2, \ldots$ may be defined inductively so that E_{n+1} extends E_n and the inequality $E_n(C_n,H_{n-1}) \geq 1$ holds whenever $C_n \neq \emptyset$. If E is the union of all these metrics E_n, the conclusion of the theorem is given by the following statements (i) through (iv).

(i) E is a segmented convex metric for the space M, E is convex on each P_n, and $E(C_n,\overline{D}(p;n - 1/2)) \geq 1$ holds whenever $n \geq 1$ and $C_n \neq \emptyset$.

(ii) $E(C_n,p) \geq n$ holds whenever $n \geq 1$ and $C_n \neq \emptyset$.

(iii) Every E bounded set is D bounded.

(iv) E is complete.

(i) It is clear that the metric E is segmented convex, that E is convex on each P_n, and that $E(C_n,H_{n-1}) \geq 1$ holds for $n \geq 1$, since E extends each of the complete convex metrics E_n. Now for any point x in C_n let $\overline{xz}$ be an E segment from x to any point z in $\overline{D}(p;n - 1/2)$. Then $\overline{xz}$ meets H_{n-1} in at least one point y, so that $E(x,z) = E(x,y) + E(y,z) \geq E(y,z) \geq E(C_n,H_{n-1}) \geq 1$ demonstrates that $E(C_n,\overline{D}(p;n - 1/2)) \geq 1$.

The metric E induces the same topology as D, for an arbitrary D ball $D(x;\varepsilon)$ is contained in some $D(p;n)$. Since E_n is a metric for the space P_n, then $E(x;\delta) \subset D(x;\varepsilon)$ for some $\delta > 0$. Since E is segmented convex, any point z of $E(x;\delta)$ that lies outside P_n must be joined to x by a segment $\overline{xz}$ that contains a point y of

$C_n \cap E_n(x;\delta)$, in contradiction to the choice of δ. Hence, it holds that $E(x;\delta) = E_n(x;\delta) \subset D(x;\varepsilon)$.

Now let $E(x;\varepsilon)$ be an arbitrary E ball, where x is in P_{n-1} for some n. Since E_n is a metric for the space P_n, there is a number $0 < \delta \leq 1$ such that $D(x;\delta) \cap P_n \subset E_n(x;\varepsilon)$ holds. But the choice of δ insures $D(x;\delta) \subset P_n$, so that $D(x;\delta) \subset E_n(x;\varepsilon) \subset E(x;\varepsilon)$.

(ii) Statement (ii) is implied by (i) in the case $n = 1$ and follows by induction for general n, by use of (i) and E segments having p as one endpoint.

(iii) Let Q be an E bounded set. If a point x in Q lies outside some P_i, then a point y of C_i lies on some E segment $\overline{xp}$, and from (ii) it follows that $E(x,p) \geq E(y,p) \geq i$ holds. Since Q is E bounded, it follows that Q must lie in some P_n.

(iv) Claim (iv) now follows from (iii), since by (1.26) every closed and D bounded set is compact. ▮

The following theorem can be regarded as the main result of this dissertation.

<u>Theorem</u> 5.4. Let M_1 be a space with a complete convex metric D_1 and let M_2 be a locally connected generalized continuum. In order for there to be a complete convex metric for $M_1 \cup M_2$ that extends D_1, it is necessary and sufficient that $M_1 \cap M_2$ be a non-empty subspace of both M_1 and M_2 which is closed in M_2, and that the M_2 boundary of $M_1 \cap M_2$ be closed in M_1.

Proof: Necessity is given by (4.1). For the proof of sufficiency, let

p be any point of $M_1 \cap M_2$ and let D be any complete convex metric for M_2. By (5.3) there is a complete convex metric D_2 for M_2 whose restriction D_2^n to $P_n = \overline{D}(p;n)$ is convex, and which has the property that if $D(p,x) = n$ then $D_2(x,\overline{D}(p;n - 1/2)) \geq 1$ holds, for $n = 1, 2, \ldots$.

Since $M_1 \cap M_2$ is closed in M_2, then $M_1 \cap P_1$ is compact. Hence, (5.1) may be applied by replacing M_1, D_1, M_2, and D_2 by M_1, D_1, P_1, and D_2^1 respectively in the hypothesis; let D_0^1 and D_3^1 be the D_0 and D_3 given respectively by the proof and conclusion. (The sets C and H in (5.1) will not be used here.) Then D_3^1 is a complete convex metric for $M_1 \cup P_1$ that extends D_1 and has the property that whenever $D_3^1(x,y) < D_2^1(x,y)$ holds for two points x, y of $P_1 \setminus M_1$, then x and y have a D_3^1 between point in M_1. It is noted that $D_0^1(u,v) \geq D_2^1(u,v) = D_2(u,v)$ whenever $D_0^1(u,v)$ is defined, and that if x lies in $P_1 \setminus M_1$ and y in M_1, $D_3^1(x,y)$ is defined to be the infimum of sums $D_0^1(x,a) + D_1(a,y)$ for certain points a in the P_1 boundary of $M_1 \cap P_1$.

Proceeding inductively, suppose that D_3^n is a complete convex metric for $M_1 \cup P_n$ which extends D_1. Again apply (5.1) by replacing M_1, D_1, M_2, and D_2 by $M_1 \cup P_n$, D_3^n, P_{n+1}, and D_2^{n+1} respectively, and obtain D_0^{n+1} and D_3^{n+1} in place of D_0 and D_3. The conclusion of (5.1) gives that D_3^{n+1} is a complete convex metric for $M_1 \cup P_{n+1}$ that extends D_3^n, with the property that whenever the inequality $D_3^{n+1}(x,y) < D_2^{n+1}(x,y)$ holds for points x, y of $P_{n+1} \setminus (M_1 \cup P_n)$, then x and y have a D_3^{n+1} between point in $M_1 \cup P_n$. Again, it should be noted that $D_0^{n+1}(u,v) \geq D_2^{n+1}(u,v) = D_2(u,v)$ holds whenever $D_0^{n+1}(u,v)$ is defined. Further, for points x in $P_{n+1} \setminus (M_1 \cup P_n)$

and y in $M_1 \cup P_n$, the value D_3^{n+1} is defined to be the infimum of sums $D_0^{n+1}(x,a) + D_3^n(a,y)$ for certain points a in the P_{n+1} boundary of $(M_1 \cup P_n) \cap P_{n+1}$. The induction principle is now applied.

Define D_3 to be the union of the metrics D_3^n, and for convenience let $P_0 = M_1 \cap P_1$ and $D_3^0 = D_1$. The following assertions (i) through (vii) combine to show that D_3 is a complete convex metric for the space $M_1 \cup M_2$ which extends D_1.

(i) D_3 is a segmented convex metric that extends D_1.

(ii) For points x in P_n, y in M_1, and for $\rho > 0$, there is a point z in $M_1 \cap P_n$ such that $D_3^n(x,y) + \rho > D_2(x,z) + D_1(z,y)$ holds. If x is not in M_1, then z can be chosen in the P_n boundary of $M_1 \cap P_n$, hence in the M_2 boundary of $M_1 \cap M_2$.

(iii) For points x in P_{n+k}, y in $M_1 \cup P_k$ $(k = 0, 1, \ldots;$ $n = 1, 2, \ldots)$, and for $\rho > 0$, there is a point z in the set $(M_1 \cup P_k) \cap P_{n+k}$ such that $D_3^{n+k}(x,y) + \rho > D_2(x,z) + D_3^k(z,y)$ holds.

(iv) D_2 is equivalent to D_3 restricted to M_2.

(v) D_3 is a metric for the space $M_1 \cup M_2$.

(vi) For points t in P_n and v in $M_2 \setminus P_{n+1}$ for some $n > 0$, there is a D_3 between point u of t and v such that $D(p,u) = n + 1/2$ holds and P_n contains no D_3 between point of u and v.

(vii) D_3 is complete.

(i) Claim (i) is immediate from the definition of D_3, since each D_3^n is a complete convex metric.

(ii) Let x be in P_n, y in M_1, and let $\rho > 0$ be given. If x is in M_1, then x itself may be taken for z since it is true that $D_3^n(x,y) = D_1(x,y)$. Therefore, with the assumption that x is not in M_1, the proof of (ii) is given by induction on n. If x is in $P_1 \setminus M_1$, then by the definition of $D_3^1(x,y)$ there is a point z on the P_1 boundary of $M_1 \cap P_1$ such that the inequality $D_3^1(x,y) + \rho > D_0^1(x,z) + D_1(z,y) \geq D_2(x,z) + D_1(z,y)$ holds. Proceeding inductively, assume that (ii) holds for $n = k$ and arbitrary $\rho' > 0$, and let $x \in P_{k+1} \setminus (M_1 \cup P_k)$ with $\rho > 0$. From the definition of $D_3^{k+1}(x,y)$ there is a point z' on the P_{k+1} boundary of $(M_1 \cup P_k) \cap P_{k+1}$ such that the inequality

$$D_3^{k+1}(x,y) + \rho/2 > D_0^{k+1}(x,z') + D_3^k(z',y) \geq D_2(x,z') + D_3^k(z',y) \qquad (2)$$

holds. If z' is in $M_1 \cap P_{k+1}$, then z' is on the P_{k+1} boundary of $M_1 \cap P_{k+1}$, and since $D_3^k(z',y) = D_1(z',y)$ holds, inequality (2) shows that $z = z'$ satisfies (ii). If, however, z' is not in M_1, then z' is in P_k. Thus, by the induction hypothesis for the points z' and y, there is a point z on the P_k boundary of $M_1 \cap P_k$, hence on the P_{k+1} boundary of $M_1 \cap P_{k+1}$, such that the inequality $D_3^k(z',y) + \rho/2 > D_2(z',z) + D_1(z,y)$ holds. By combining this last inequality with (2) and the triangle inequality, the desired inequality in x, y, and z is obtained. Claim (ii) is now established by the induction principle.

(iii) Assertion (iii) can be proved by the technique of double induction on k and n, by using (ii) as the initialization $k = 0$

and an argument similar to the proof of (ii) to complete the induction.

(iv) To prove that D_2 and D_3 induce the same topology on M_2, first let $D_3(x;\varepsilon)$ be given with x a point of M_2. Since by (1.26) the set $\overline{D}_2(x;\varepsilon)$ is compact, it is contained in P_n for some n. But since $D_3^n(x;\varepsilon) \cap M_2$ is D_2^n open, there is some $\varepsilon \geq \delta > 0$ such that $D_2^n(x;\delta) \subset D_3^n(x;\varepsilon) \cap M_2$ holds. Since $D_2(x;\delta) \subset P_n$, then $D_2(x;\delta) = D_2^n(x;\delta) \subset D_3(x;\varepsilon)$ holds.

Now let $D_2(x;\varepsilon)$ be an arbitrary D_2 ball, and first suppose that x is in $M_1 \cap M_2$. Then there is some number $\varepsilon/2 \geq \delta > 0$ such that $D_1(x;\delta) \cap M_2 \subset D_2(x;\varepsilon/2) \cap M_1$ holds. For any point y of $D_3(x;\delta) \cap M_2$, there is by (ii) some point z in $M_1 \cap M_2$ such that $\delta > D_2(y,z) + D_1(z,x)$ holds. Since z is thus in $D_1(x;\delta) \cap M_2$, then $D_2(x,z) < \varepsilon/2$ and the triangle inequality shows that y is in $D_2(x;\varepsilon)$. Hence in this case, $D_3(x;\delta) \cap M_2 \subset D_2(x;\varepsilon)$ holds. If instead x is in $M_2 \setminus M_1$, there is some number $\varepsilon/2 \geq \beta > 0$ such that the compact set $\overline{D}_2(x;\beta)$ is in $M_2 \setminus M_1$ and in some P_n, so that $D_2(x;\beta) = D_2^n(x;\beta)$. Since D_2^n and D_3^n are equivalent on P_n, there is some number $\beta \geq \alpha > 0$ for which $D_3^n(x;\alpha) \cap M_2 \subset D_2(x;\beta)$. Since any point y of $D_3(x;\alpha) \cap M_2$ lies in P_{n+k} for some $k \geq 1$, by (iii) there is a point z of $(M_1 \cup P_n) \cap P_{n+k}$ satisfying $\alpha > D_2(y,z) + D_3^n(z,x)$. Thus, z lies in $D_3^n(x;\alpha) \cap M_2$ and hence in $D_2(x;\beta)$, so that as above the triangle inequality places y in the ball $D_2(x;\varepsilon)$. Therefore $D_3(x;\alpha) \cap M_2 \subset D_2(x;\varepsilon)$ holds, and (iv) is established.

(v) Statement (v) follows from (4.3), since M_1 and by (iv) also M_2 are subspaces of $(M_1 \cup M_2, D_3)$, once it has been shown that $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in $(M_1 \cup M_2, D_3)$. Since $M_1 \cap M_2$ is closed in M_2, for each point x of $M_2 \setminus M_1$ there is some $\varepsilon > 0$ for which $D_2(x;\varepsilon) \subset M_2 \setminus M_1$. If there were some point y in $D_3(x;\varepsilon) \cap M_1$, (ii) would provide a point z in M_1 for which $\varepsilon > D_2(x,z) + D_1(z,y)$, contrary to the choice of ε. Hence, the ball $D_3(x;\varepsilon)$ lies in $M_2 \setminus M_1$, and therefore the set $M_2 \setminus M_1$ is actually D_3 open. Now let y be in $M_1 \setminus M_2$, and denote by B the M_2 boundary of $M_1 \cap M_2$. Since B is closed in M_1, then $D_1(y;\delta) \subset M_1 \setminus B$ holds for some $\delta > 0$. If some point x of $D_3(y;\delta)$ were in $M_2 \setminus M_1$, there would be according to (ii) some point z of B satisfying $\delta > D_2(x,z) + D_1(z,y)$, contrary to the choice of δ. Therefore, the ball $D_3(y;\delta)$ must lie in M_1, and the sets $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are thereby D_3 separated.

(vi) Suppose that points t in P_n and v in $M_2 \setminus P_{n+1}$ have no D_3 between point that satisfies the conditions given in (vi). Since D_3 is segmented convex, for every point t' in P_n there is clearly some D_3 between point u' of t' and v lying in the set $C = \{x \in M_2 \colon D(p,x) = n + 1/2\}$. In particular, points $t_1 = t$ and v have a between point u_1 in C. Since by assumption there must be a between point t_2 of u_1 and v which lies in P_n, it follows that $D_3(t,v) = D_3(t_1,u_1) + D\,(u_1,t_2) + D\,(t_2,v)$ holds. In fact, it is possible to define points $t_1, t_2, \ldots$ of P_n and also $u_1, u_2, \ldots$ of C inductively, satisfying

$$D_3(t,v) = \sum_{i=1}^{k} [D_3(t_i,u_i) + D_3(u_i,t_{i+1})] + D_3(t_{k+1},v)$$

for each k. Therefore, the series

$$\sum_{i=1}^{\infty}[D_3(t_i,u_i) + D_3(u_i,t_{i+1})]$$

converges, implying that $D_3(P_n,C) = 0$. But this is impossible, since P_n and C are disjoint compact sets. Thus, (vi) is established.

(vii) To show that D_3 is complete, let $\langle x_k \rangle$ be a D_3 Cauchy sequence. It may be assumed that $\langle x_k \rangle$ lies entirely in $M_2 \setminus M_1$ and has no subsequence that lies entirely in one of the sets P_n. In fact, if x_k lies in $P_{n_k} \setminus P_{n_k-1}$ for each k, it may be assumed that $n_k + 1 < n_{k+1}$. Suppose that for only a finite set of indices k the points x_k and x_{k+1} have a D_3 between point in M_1. It may be assumed, in fact, that this set of indices is empty. Then for each k, (vi) shows that there is some D_3 between point u of x_k and x_{k+1} such that $D(p,u) = n_k + 1/2$ holds and P_{n_k} contains no D_3 between point of u and x_{k+1}. Since x_{k+1} lies outside P_{n_k+1}, there is a D_3 between point v of u and x_{k+1} satisfying $D(p,v) = n_k + 1$, and moreover P_{n_k} contains no D_3 between point of u and v. Therefore, u and v can have no D_3 between point in the set $M_1 \cup P_{n_k}$. Because of this fact and the particular construction of the metrics $D_3^{n_k+1}$ and D_2, it follows that the inequality $D_3(x_k,x_{k+1}) \geq D_3(u,v) = D_3^{n_k+1}(u,v) \geq D_2^{n_k+1}(u,v) = D_2(u,v) \geq 1$ holds, and $\langle x_k \rangle$ cannot be D_3 Cauchy.

Hence, there must be a subsequence $\langle x_{k_i} \rangle$ of $\langle x_k \rangle$ for which the points x_{k_i} and x_{k_i+1} have a D_3 between point y_i in M_1. Then $\langle y_i \rangle$ is a D_1 Cauchy sequence that converges to some point y

in M_1. Since for each k and i it holds that

$$D_3(x_k,y) \leq D_3(x_k,x_{k_i}) + D_3(x_{k_i},y_i) + D_3(y_i,y)$$
$$< D_3(x_k,x_{k_i}) + D_3(x_{k_i},x_{k_i+1}) + D_1(y_i,y),$$

it follows that $\langle x_k \rangle$ converges to y. Therefore, the metric D_3 is complete. ∎

The justification for stating the next corollary is its likeness to the following classic theorem of Bing [6] on the extension of a general metric: <u>if a closed subspace M_1 of a metric space M_2 has a metric D_1, then D_1 can be extended to a metric for M_2</u>.

<u>Corollary</u> 5.5. If a closed subspace M_1 of a locally connected generalized continuum M_2 has a complete convex metric D_1, then D_1 can be extended to a complete convex metric for M_2.

Proof: Not only is the intersection $M_1 \cap M_2 = M_1$ closed in M_2, but its boundary is also. Thus, (5.4) gives the desired extension of D_1 to a complete convex metric for the space $M_1 \cup M_2$, which by (4.2) is just M_2. ∎

The condition given in (5.4) as being necessary and sufficient for metric extension actually proves to be a sufficient condition to ensure that $M_1 \cup M_2$ is a locally connected generalized continuum whenever M_1 and M_2 are. Thus, (5.4) is included in the following, in the case that M_1 is a locally connected generalized continuum.

<u>Theorem</u> 5.6. Let M_1 and M_2 be locally connected generalized continua. In order for $M_1 \cup M_2$ to be a locally connected generalized

continuum and for a given complete convex metric for M_1 to extend to a complete convex metric for $M_1 \cup M_2$, it is necessary and sufficient that $M_1 \cap M_2$ be a nonempty subspace of both M_1 and M_2 which is closed in M_2 and that the M_2 boundary of $M_1 \cap M_2$ be closed in M_1.

Proof: Necessity is given by (5.4). For sufficiency, assume that $M_1 \cap M_2$ is a non-empty subspace of both M_1 and M_2 which is closed in M_2 and that the M_2 boundary of $M_1 \cap M_2$ is closed in M_1. By (5.4), a given complete convex metric for M_1 does extend to a complete convex metric for $M_1 \cup M_2$. Since by (1.25), it is true that the space M_1 does admit a complete convex metric, then by the previous statement, the space $M_1 \cup M_2$ admits some complete convex metric also. Moreover, since by (4.11) the space $M_1 \cup M_2$ must be locally compact, then by (1.27) it must be a locally connected generalized continuum. ∎

In connection with (5.6), it should be noted that for locally connected generalized continua M_1 and M_2, the fact that $M_1 \cup M_2$ is a locally connected generalized continuum does not, according to (4.7), imply that $M_1 \cap M_2$ is closed in M_2, although by (4.13) this fact does imply that the M_2 boundary of $M_1 \cap M_2$ is closed in M_1.

CHAPTER VI

CHARACTERIZING CLASSES OF LOCALLY CONNECTED GENERALIZED CONTINUA

In 1966 Toranzos [25] used the extension theorem of Bing [4], along with three varieties of convex metrics, to characterize dendrites, arcs, and simple closed curves among the Peano continua. For example, Toranzos [25] proved that *a Peano continuum is a dendrite if and only if each convex metric for it is SC*. It is the purpose of Chapter VI to prove analogous theorems for complete convex metrics on locally connected generalized continua, using the corollary to the main extension theorem (5.4), along with the three varieties of complete convex metrics discussed in Chapter III: SC, WR, and WE. It is noted that, although the three varieties of metrics used in this chapter do not correspond exactly to the three varieties used by Toranzos [25], yet analogues to dendrites, arcs, and simple closed curves are among the classes of locally connected generalized continua identified in the results of this chapter.

The following theorems characterize classes of locally connected generalized continua by using all possible combinations of the properties SC, WR, and WE, beginning with the use of these properties one at a time.

Theorem 6.1. A locally connected generalized continuum contains no simple triod if and only if every complete convex metric for it is WR.

Proof: For a contrapositive proof of necessity, let D be a complete convex metric for a locally connected generalized continuum M, and assume that D is not WR. Then by (2.4), there are four distinct points x, y, y', z in M such that $xz = zy = zy' = (1/2)\, xy = (1/2)\, xy'$. By (1.13), there exist segments $\overline{xz}, \overline{zy}, \overline{zy}'$. Moreover, x is in neither $\overline{zy}$ nor $\overline{zy}'$, since $xy > zy$ and $xy' > zy'$ hold. Thus, there is some number $\varepsilon > 0$ such that $D(x;\varepsilon)$ is disjoint from $\overline{zy} \cup \overline{zy}'$, and if t is a point chosen from $D(x;\varepsilon) \cap \overline{xz}$ that is distinct from x, then the sub-segment $\overline{xt}$ of $\overline{xz}$ lies in $D(x;\varepsilon)$. Since $t \neq z$, then $\overline{xz} \setminus t = A \cup B$, where A and B are separated sets containing x and z, respectively. If $V = \overline{xz} \cup \overline{zy} \cup \overline{zy}'$, then $V \setminus t = A \cup (B \cup \overline{zy} \cup \overline{zy}')$ holds, where again A and $B \cup \overline{zy} \cup \overline{zy}'$ are separated sets, since $A \subset \overline{xt} \subset D(x;\varepsilon)$ holds. Thus the set V, having t as a cut point, cannot be a simple closed curve [20]. But since x, y, and y' are non-cut points of V, then V is not an arc. Thus V, hence also M, must contain a simple triod [20].

The proof of sufficiency is also given by contraposition. Suppose a locally connected generalized continuum M contains a simple triod T. Then there exist four points x, y, y', z and arcs $\widehat{xz}, \widehat{zy}, \widehat{zy}'$ that intersect pairwise only at z, such that $T = \widehat{xz} \cup \widehat{zy} \cup \widehat{zy}'$. The triod T is homeomorphic to a triod T' in E^2 composed of three equal line segments which intersect pairwise only at a common endpoint of each. The geodesic metric on T' is convex, and by the homeomorphism with T induces a convex metric D_T for T such that z is the midpoint of both x, y and x, y'. By (5.5), the metric D_T extends to a complete convex metric D for M, and under D also the point z is a midpoint of both x, y and x, y'. Thus by (2.4), D is not WR. ∎

<u>Theorem</u> 6.2. For a locally connected generalized continuum M, the following statements are equivalent:

(i) M contains no simple closed curve.

(ii) Every complete convex metric for M is SC.

(iii) Every complete convex metric for M is WE.

(iv) Every complete convex metric for M is SC-WE.

(v) Every complete convex metric for M has the property that every closed ball contains every segment between every pair of its points.

Proof: The plan of the proof is to show that (ii), (iii), (iv), and (v) are separately equivalent to (i).

(i) → (ii) Let M satisfy (i), and let D be a complete convex metric for M. Suppose that for some two points p and q of M there are two distinct D segments A_1 and A_2 from p to q. Then $A_1 \cup A_2$ would contain a simple closed curve, contradicting the hypothesis [20]. Hence, between every two points of M there is a unique D segment, and by (2.3) the metric D is SC.

(ii) → (i) Let M be a locally connected generalized continuum containing a simple closed curve C. A homeomorphism from the unit circle in E^2 onto C induces a complete convex metric D_C for C that is not SC, namely, the metric induced from the geodesic metric on the unit circle. By (5.5), D_C can be extended to a complete convex metric D for M, and D is not SC.

(i) → (iii) With M as in (i), let D be a complete convex metric for M. Suppose that $\overline{sq} \cap \overline{pr} = \overline{pq}$ holds in M. If $p = s$ or if $q = r$, then $\overline{sq} \cup \overline{pr}$ is the segment $\overline{pr}$ or $\overline{sq}$, respectively. If $p \neq s$ and $q \neq r$, then $\overline{sq} \cup \overline{pr}$ is at least an arc from s to q. But since in M it holds that any arc joining two points is unique, then $\overline{sq} \cup \overline{pr}$ is the segment $\overline{sr}$ known by (1.13) to exist [20]. By (2.5), it follows that D is WE.

(iii) → (i) Since the geodesic metric for the unit circle in E^2 is not WE, then the above proof of (ii) → (i) suffices in this case also.

(i) → (iv) This implication is just the conjunction of the two assertions, (i) → (ii) and (i) → (iii), proved already.

(iv) → (i) Since (iv) → (iii) and (iii) → (i) both hold, then so does (iv) → (i).

(i) → (v) With M as in (i), suppose that there is some complete convex metric D for M and some closed ball $\overline{D}(p;\varepsilon)$ containing two points x and y for which some segment $\overline{xy}$ does not lie entirely in $\overline{D}(p;\varepsilon)$. Since $\overline{D}(p;\varepsilon)$ is arcwise connected, there is an arc A from x to y that lies entirely in $\overline{D}(p;\varepsilon)$. Hence $A \neq \overline{xy}$, so that $A \cup \overline{xy}$ must contain a simple closed curve [20]. This contradicts (i).

(v) → (i) Assume, for a contrapositive argument, that the locally connected generalized continuum M contains a simple closed curve C. Pick three points x, y, z of C and induce a metric D_C for C via a homeomorphism from the unit circle in E^2 in such a way that x, y, and z are the respective images of the points (0, 1), (1, 0), and (0, -1). By (5.5), extend D_C to a complete convex metric D for M.

The segment $\overline{xz}$ in C that does not contain y does not lie in the set $\overline{D}_C(y;\pi/2)$, although $D(y,x) = D(y,z) = \pi/2$. Thus, (v) cannot hold. ∎

Theorem (6.2) furnishes a simple condition that is sufficient for the convexity of metric balls, which is stated as follows.

Corollary 6.3. Let M be a locally connected generalized continuum which contains no simple closed curve. If D is a complete convex metric for M, then every closed (and open) D ball is convex.

Proof: The conclusion is given by (i) → (v) of (6.2) for closed D balls. But the fact that closed balls are convex implies the same for open balls. ∎

Theorem 6.4. For a non-degenerate, locally connected generalized continuum M, the following statements are equivalent:

(i) M is homeomorphic to an interval of E^1.

(ii) Every complete convex metric for M is SC-WR.

(iii) Every complete convex metric for M is WR-WE.

(iv) Every complete convex metric for M is SC-WR-WE.

Proof: The proof follows (i) → (iv) → (iii) → (ii) → (i).

(i) → (iv) If M is homeomorphic to an interval of E^1, let D be a complete convex metric for M. Since M contains no simple closed curve, then by (6.2) it follows that D is SC-WE, and since M contains no simple triod, then D is WR by (6.1).

(iv) → (iii) This implication is a tautology.

(iii) → (ii) This is established by (6.2).

(ii) → (i) Suppose (ii) holds. Let D be a complete convex metric for M, let x and z be two points of M, and let y denote the midpoint of x and z, which by (2.3) is known to be unique. Suppose y' is a point of M distinct from y such that xy' = xy holds. Choose segments $\overline{xy}$, $\overline{yz}$, and $\overline{yy'}$ by (1.13). If either of the points x or z lay on $\overline{yy'}$, then the set $\overline{xy} \cup \overline{yz} \cup \overline{yy'}$ would contain a simple closed curve. If neither x nor z lay on $\overline{yy'}$, then the set $\overline{xy} \cup \overline{yz} \cup \overline{yy'}$ would contain a simple triod. Since by (6.1) and (6.2) both alternatives are impossible, then y itself is the only point of M satisfying xy = yz. Since M is a connected metric space in which every two points have exactly one point equidistant from them, then according to a theorem of Berard [3], M is homeomorphic to an interval of E^1. ∎

<u>Theorem</u> 6.5. For a locally connected generalized continuum M, the following statements are equivalent:

(i) M contains no simple closed curve, but does contain a simple triod.

(ii) Every complete convex metric for M is SC but is not WR.

(iii) Every complete convex metric for M is WE but is not WR.

(iv) Every complete convex metric for M is SC-WE but is not WR.

Proof: It is shown only that (i) → (ii) holds, since (ii) ↔ (iii) ↔ (iv) is entailed by (6.2), and (ii) → (i) follows from (6.1) and (6.2).

(i) → (ii) With M as in (i), let $\widehat{x_1 z} \cup \widehat{x_2 z} \cup \widehat{x_3 z}$ be a simple triod contained in M, where $\widehat{x_i z}$ is an arc from x_i to z and these arcs intersect pairwise only in the point z. If D is a complete convex metric for M, then D is SC by (6.2). Since there are no simple closed curves contained in M, then there is only one arc joining any two points [20]. From this it follows that $\widehat{x_i z}$ is the unique segment $\overline{x_i z}$, and the arc $\widehat{x_i z} \cup \widehat{x_j z}$ is the unique segment $\overline{x_i x_j}$, for $i \neq j$. Thus $\overline{x_1 z} \subset \overline{x_1 x_2} \cup \overline{x_1 x_3}$ holds, and the set $\overline{x_1 x_2} \cup \overline{x_1 x_3} = \widehat{x_1 z} \cup \widehat{x_2 z} \cup \widehat{x_3 z}$ is not a segment. Therefore, it follows from (2.4) that D is not WR. ∎

Theorem 6.6. For a locally connected generalized continuum M, the following statements are equivalent:

(i) M is a simple closed curve.

(ii) Every complete convex metric for M is WR but is not SC.

(iii) Every complete convex metric for M is WR but is not WE.

(iv) Every complete convex metric for M is WR but is neither SC nor WE.

Proof: Since (iv) is just the conjunction of (ii) and (iii), the proof is completed by showing that (ii) and (iii) are separately equivalent to (i).

(i) → (ii) If M is a simple closed curve and D is a complete convex metric for M, then D is WR by (6.1). But by (2.15), D does not admit an SC metric. Hence, D cannot be SC.

(ii) → (i) If M satisfies (ii), then by (6.2) there is some simple closed curve C contained in M. If there exists some point p in $M \setminus C$, then there would be an arc joining p to C, hence there would be a simple triod in M. Since this is prohibited by (6.1), it must be that $M = C$.

(i) → (iii) The proof of (i) → (ii) suffices here also, if "SC" and "(2.15)" are replaced by "WE" and "(2.17)", respectively.

(iii) → (i) The proof of (ii) → (i) can be used, with "(ii)" replaced by "(iii)". ∎

The space characterized by (6.6) is rather striking in that it is the only one obtained in this chapter that must be compact. The locally connected generalized continua that contain no simple closed curves (6.2) and those that are homeomorphic to an interval of E^1 (6.4) are the possibly non-compact analogues to the dendrites and arcs, respectively, that were characterized by Toranzos [25].

CHAPTER VII

SUMMARY AND PROSPECTS

This paper is an investigation of the properties of complete convex metrics on locally connected generalized continua, and is especially concerned with the question of metric extension. The study of convex metrics on Peano continua was begun in 1928 by Menger [18], who posed the famous question, Does every Peano continuum admit a convex metric? This question was answered affirmatively by Bing [4] in 1949, but the notion of a convex metric continues to provide material for current research.

One of the current areas of research involving convex metrics is in the setting of spaces which, aside from compactness, enjoy the other properties of Peano continua: these are the locally connected generalized continua. Complete convex metrics on locally connected generalized continua seem to have many of the properties possessed by their counterparts in the compact setting, the convex metrics on Peano continua. For example, in 1955 it was proved by Tominaga and Tanaka [24] that every locally connected generalized continuum admits a complete convex metric. In 1967, Lelek and Mycielski [16] showed that whenever a locally connected generalized continuum is given a complete convex metric, then every closed and bounded set is compact. These last two results, which were discussed in Chapter I, have been important tools for the results of this paper.

The primary aim of the dissertation has been to establish results on the extension of complete convex metrics to the union of two spaces; it was the author's intent to generalize a useful theorem of Bing [4] concerning the extension of convex metrics to the union of two Peano continua. In Chapter V, a necessary and sufficient condition for such an extension was found by specifying two simple topological properties of the intersection of the two spaces in question. In proving this main result, it was discovered that a locally connected generalized continuum admits not only a complete convex metric, but also one having the property that every bounded set is contained in a compact, convex set. Two consequences of the main extension theorem were given at the end of Chapter V. One of these, analogous to the classic theorem of Bing [6] on general metric extension, states that a complete convex metric for a closed subspace of a locally connected generalized continuum can be extended to a complete convex metric for the entire space. The second consequence shows that the properties required in the author's main extension theorem on the intersection of two spaces are sufficient to ensure that the union of the spaces is a locally connected generalized continuum whenever both spaces are also.

The entire thesis is closely related to the main body of results of Chapter V, in providing either preparatory material for proving it or applications of it; nevertheless, a few results have emerged that are of some interest in their own right. Chapter III provided three theorems on segmented convex metrics that may be worthy of notice. First, it was found that if a locally compact metric space has a unique midpoint for every two of its points, then the metric is segmented convex. Second, in a locally connected generalized continuum with a segmented convex metric, every compact ball is a Peano continuum; in the case that the convex metric is complete, this result yields the useful corollary that every closed ball is a Peano continuum. And third, the admission of a segmented convex metric by a locally compact space was found to characterize the locally connected generalized continua in a theorem that concluded Chapter III. Chapter IV introduced a particular topology on the union of two spaces, and a few elementary properties were established; these became useful in proving the extension theorems of Chapter V. In Chapter VI it was found that locally connected generalized continua that are either without any simple closed curves, without simple triods, or homeomorphic to an interval of the real line, can be characterized by the admission of complete convex metrics possessing some combination of the three properties SC, WR, and WE; these properties were investigated in Chapter II.

Certain questions have arisen in the course of this research that hopefully will prove to be of interest for further study. The outstanding question of Chapter II is whether an SC-WR metric must also be WE. The question of Krakus and Trybulec [14] remains unanswered, whether every space with an SC metric is contractible. This question may also be restated with "SC" replaced by "WE." Also, it might be of interest to determine which of the plane continua admit WE metrics, much as Glynn [12] has done for SC metrics. In regard to Chapter V, it might be profitable to investigate whether, in the main extension theorem (5.4) or in the subsequent corollary, it is necessary to require that the space M_2 be locally compact, or whether it might suffice that M_2 be simply a space that admits some complete convex metric.

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