EXTREME POINTS IN BANACH SPACES

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CHAPTER I

INTRODUCTION

The large volume of research dealing with extreme points makes it apparent that this area is an important segment of functional analysis. Much of this work is scattered in the literature and has not been set forth in a unified way. It is our intent to present an exposition of certain major portions of this area in this paper. We will begin at a basic level and arrive at some of the more interesting results in the literature. The paper is not intended to be comprehensive since such an undertaking would require several volumes.

Extreme points have been studied since the early part of this century. At that time the main concern was with the finite dimensional case. In recent years mathematicians have dealt with extreme points in infinite dimensional spaces. These studies have led to useful theorems concerning the isometric and isomorphic [see Definitions 3.14 and 3.15] properties of Banach spaces. The study of such properties is one of the most active areas in functional analysis.

We hope to present the material in such a manner so that a second year graduate student in mathematics would have little difficulty in understanding the paper. The prerequisite for reading this work is a basic first year course in functional analysis and measure theory (see [40] and [44] for example). For the reader's convenience we list a few of the main results in functional and real analysis that are used frequently and give references: the Hahn-Banach theorem [[44] p. 65]; if E is a normed space then the unit ball of E^* is w*-compact [[44] p. 239]; if X is a Banach space then in X^*, w*-bounded implies norm bounded [[44] p. 245]; in X, w-bounded implies normed bounded [[44] p. 223]; the Baire category theorem [[40] p. 139]; and the Tietze extension theorem [[40] p. 148].

Chapter II will present the basic material needed in the rest of the paper. Some simple examples of extreme points in the plane will be included along with some basic lemmas that will be needed later.

Chapter III will be devoted to three basic theorems concerning extreme points. These theorems will be presented in chronological order so that we may see to a certain extent how the ideas have developed from the first part of the century to the present time. Also included in Chapter III are some applications of extreme points. These should help to explain the interest in extreme points in recent years.

A major portion of the paper will be assigned to Chapters IV and V. The characterizations of the extreme points of the unit balls of five well-known Banach spaces will be covered in Chapter IV. In Chapter V we endeavor to do the same for the unit balls of the duals of these five spaces. Also at the end of each section in Chapter IV we present the results concerning the question of whether the unit ball is the closed convex hull of its extreme points. As previously mentioned, this information is scattered throughout the literature and it is our hope to gather these results together and present them in a readable form.

Chapter VI will be devoted to other distinguished points which are related to extreme points. These ideas give the interested reader an opportunity to investigate some extensions of theorems in Chapters II, III, IV, and V.

CHAPTER II

BASIC CONCEPTS OF EXTREME POINTS

Definitions, Basic Lemmas, and Notation

We begin with the basic definitions and notation to be used throughout the paper.

X will denote a Banach space with the norm being designated by $\|\cdot\|$. (Remark: we will agree that X is not the trivial space consisting of only the zero element.) The dual of a Banach space X will be denoted by X^*; i.e., X^* is the space of continuous linear functionals f defined on X with $\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\}$. The unit ball of a space X is $U(X) = \{x \in X : \|x\| \leq 1\}$. For any subset K of a vector space V, the set of extreme points of K will be denoted by $\operatorname{ext} K$.

Definition 2.1 A subset K of a vector space V is said to be convex if whenever $x, y \in K$, then $\alpha x + (1-\alpha)y \in K$, $0 \leq \alpha \leq 1$.

Definition 2.2 Let K be a convex set and $x \in K$. x is said to be an extreme point of K if whenever $y, z \in K$ with $x = \alpha y + (1-\alpha)z$, $0 < \alpha < 1$, then $x = y = z$.

Intuitively speaking this definition says that $x \in \operatorname{ext} K$ if and only if x does not belong to the interior of a line segment contained in K. We note that in the special case of $\alpha = \frac{1}{2}$, Definition 2.2 could be interpreted as $x \in \text{ext}\, K$ if and only if x is not the midpoint of two distinct elements in K.

We will find the following lemmas helpful in discussing some of the examples. We will therefore establish them first and present the examples later.

If x is not an extreme point of a convex set, it is sometimes to our advantage to be able to write x as the midpoint of two distinct points of the convex set. Therefore the following lemma will be helpful in proving later results.

Lemma 2.3 Let K be a convex subset of a vector space and $x \in K$. $x \notin \text{ext}\, K$ if and only if there exist $v, w \in K$ with $x = (\frac{1}{2})(v + w)$ and $x \neq v$.

Proof: If x is not an extreme point of K, Definition 2.2 implies there exist $y, z \in K$ with $x = \alpha y + (1 - \alpha)z$, where $0 < \alpha < 1$, $x \neq y$, and $x \neq z$. If $\alpha = \frac{1}{2}$ there is nothing to prove. For $0 < \alpha < \frac{1}{2}$ let $v = x - \alpha(y - z)$ and $w = x + \alpha(y - z)$. Then $x = (\frac{1}{2})(v + w)$, $v = \alpha y + (1 - \alpha)z - \alpha y + \alpha z = z$ and $w = \alpha y + (1 - \alpha)z + \alpha y - \alpha z = 2\alpha y + (1 - 2\alpha)z$. Thus w is an element of the line segment joining y and z, since $0 < \alpha < \frac{1}{2}$. Since $y \neq z$ it follows that $v \neq x$. For $\frac{1}{2} < \alpha < 1$ apply the above argument to $1 - \alpha$ instead of α.

The implication in the other direction follows trivially from Definition 2.2. Q. E. D.

Our main concern is with the set of extreme points of $U(X)$. With the help of Lemma 2.3 we prove the following proposition for the convex set $U(X)$.

Remark: when we say $\|x \pm y\| \leq 1$ (or $\|x \pm y\| = 1$), we mean both $\|x + y\| \leq 1$ ($\|x + y\| = 1$) and $\|x - y\| \leq 1$ ($\|x - y\| = 1$).

<u>Proposition 2.4</u> $x \in \operatorname{ext} U(X)$ if and only if whenever $y \in X$ and $\|x \pm y\| \leq 1$ then $y = 0$.

Proof: Assume $x \in \operatorname{ext} U(X)$, $y \in X$, and $\|x \pm y\| \leq 1$. Then $x + y \in U(X)$, $x - y \quad U(X)$ and $x = (\frac{1}{2})[(x+y) + (x-y)]$. Thus by Definition 2.2, $x + y = x - y$ which implies that $y = 0$.

Suppose $x \in U(X)$ and $x \notin \operatorname{ext} U(X)$. Them by Lemma 2.3 there are elements y and z of the unit ball with $x = (\frac{1}{2})(y+z)$ and $x \neq y$. Then

$$\begin{aligned} x - (\tfrac{1}{2})(y-z) &= x - (\tfrac{1}{2})y + x - (\tfrac{1}{2})y \\ &= 2x - y \\ &= z \end{aligned}$$

and

$$\begin{aligned} x + (\tfrac{1}{2})(y-z) &= x - (\tfrac{1}{2})z + x - (\tfrac{1}{2})z \\ &= 2x - z \\ &= y . \end{aligned}$$

Hence $\|x \pm (\frac{1}{2})(y-z)\| \leq 1$ and $y - z \neq 0$. This is contrary to the hypothesis and it follows that $x \in \operatorname{ext} U(X)$. Q. E. D.

The next lemma gives a necessary condition for an element x to be an extreme point of $U(X)$. This lemma will be used in later proofs with greater frequency.

<u>Lemma 2.5</u> If $x \in \operatorname{ext} U(X)$ then $\|x\| = 1$.

Proof: Clearly $x = 0$ is not an extreme point of $U(X)$. (Recall that we assume $X \neq \{0\}$). Suppose $0 < \|x\| < 1$. Let $y = \frac{x}{\|x\|}$ and $z = 2x - y$. Then $\|y\| = 1$ and $\|z\| \leq 1$. The last inequality follows from the fact that

$$0 < 2\|x\| < 2 \Rightarrow -1 < 2\|x\| - 1 < 1 \Rightarrow \|x\| \left| 2 - \frac{x}{\|x\|} \right| < 1 \Rightarrow \|z\|$$

$$= \left\| 2x - \frac{x}{\|x\|} \right\|$$

$$= \|x\| \left| 2 - \frac{x}{\|x\|} \right| < 1 .$$

Hence $y, z \in U(X)$, $x = (\frac{1}{2})(y + z)$ and $x \neq y$. Therefore by Lemma 2.3 $x \notin \text{ext}\, U(X)$. Q.E.D.

Examples in the Plane

We now give some examples of extreme points of convex subsets of the plane.

Example 2.1 Let A be the line segment in the plane between the two distinct points P and Q. Then $\text{ext}\, A = \{P, Q\}$. Notice that any other point of the line segment can be written as the midpoint of two distinct points of the segment.

Example 2.2 Let A be a convex polygonal region in the plane. Then the set $\text{ext}\, A$ is the set of vertices of the polygon. Two examples are given in Figure 2.1.

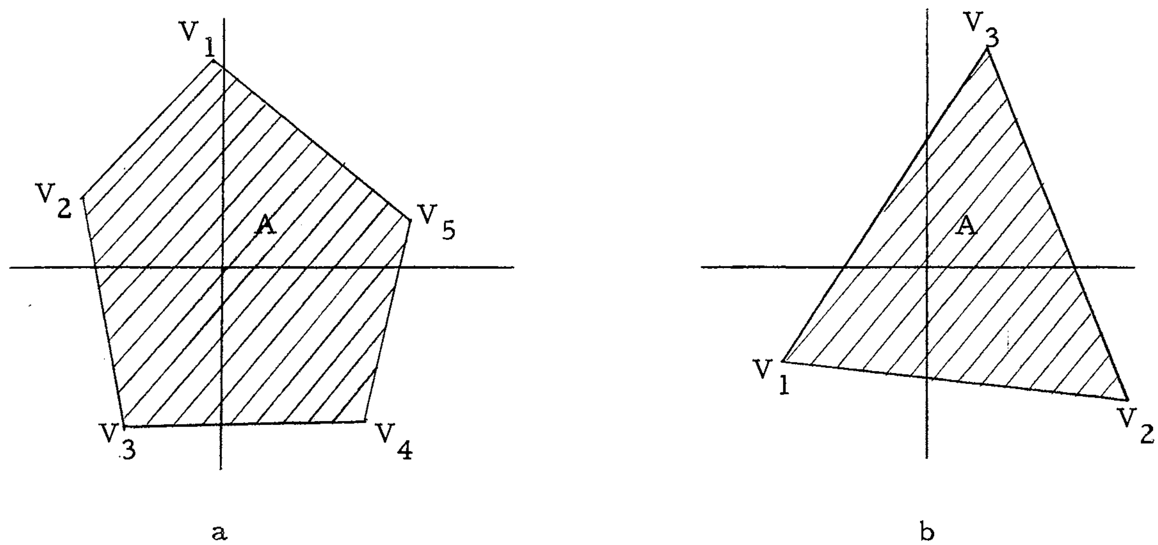

Figure 2. 1. Convex Polygonal Regions

<u>Example 2.3</u> Let $A = \{(x, y) : x^2 + y^2 \leq 1\}$. Then $\operatorname{ext} A = \{(x, y) \in A : x^2 + y^2 = 1\}$. (That the former set is contained in the latter is a consequence of Lemma 2.5. See Lemma 2.6 for a proof of the other containment.) Note that intuitively no extreme point can be positioned on a line segment contained in A. It is also easily seen that the unit ball in the plane has uncountably many extreme points. See Figure 2.2a.

<u>Example 2.4</u> Let the norm in the plane be defined by $\|(x, y)\| = |x| + |y|$. If $A = \{(x, y) : \|(x, y)\| \leq 1\}$ then the set of extreme points of A is precisely the set $\{(0, 1), (1, 0), (-1, 0), (0, -1)\}$ (See Example 2.2). Thus the norm of the space determines the "rotundity" of the unit ball. Note also that the two norms in Examples 2.3 and 2.4 are equivalent but have very different sets of extreme

points. See Figure 2.2b.

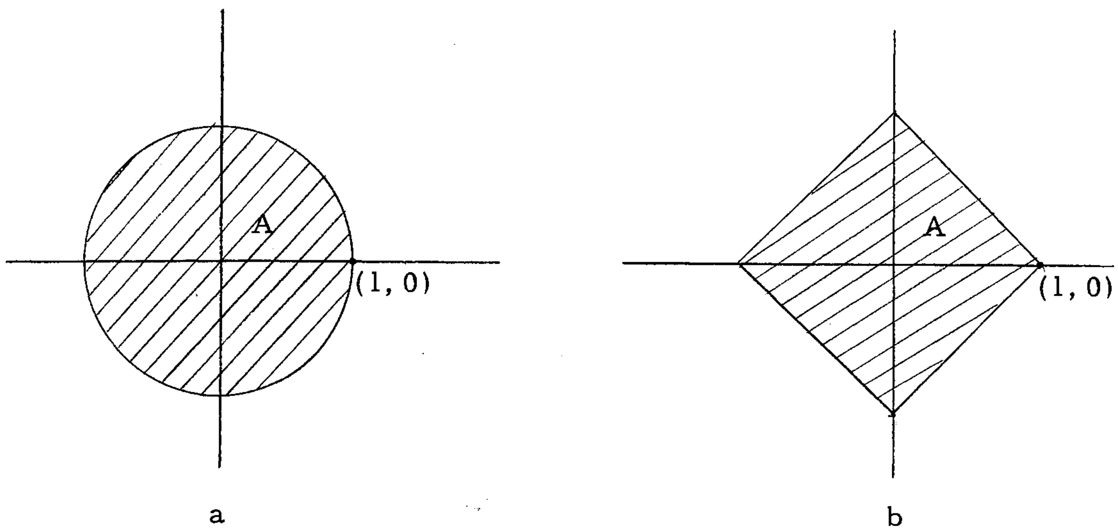

Figure 2.2. Equivalent Unit Balls

Notice that the set A in the above examples is a convex compact subset of the plane and that A always has extreme points. A much stronger result will be proved in Chapter III. Let us now consider convex subsets of the plane which are not compact. Recall that a set in the plane is compact if and only if it is closed and bounded. Example 2.5 is a set which is not closed and Examples 2.6 and 2.7 are unbounded sets. We will see that in these examples, there may or may not be extreme points in the set.

<u>Example 2.5</u> Let $A = \{(x, y) : x^2 + y^2 < 1\}$. Then A has no extreme points since every point of A is on the interior of some line

segment which is contained in A. (See the proof of Lemma 2.6.)

Example 2.6 Let $A = \{(x, y) : y \geq |x|\}$. The origin is the only extreme point of A since it is clear that every other point of the cone A is the midpoint of two distinct points of the set. See Figure 2.3a.

Example 2.7 Let $A = \{(x, y) : -1 \leq y \leq 1\}$. For this example the set ext A is empty. See Figure 2.3b.

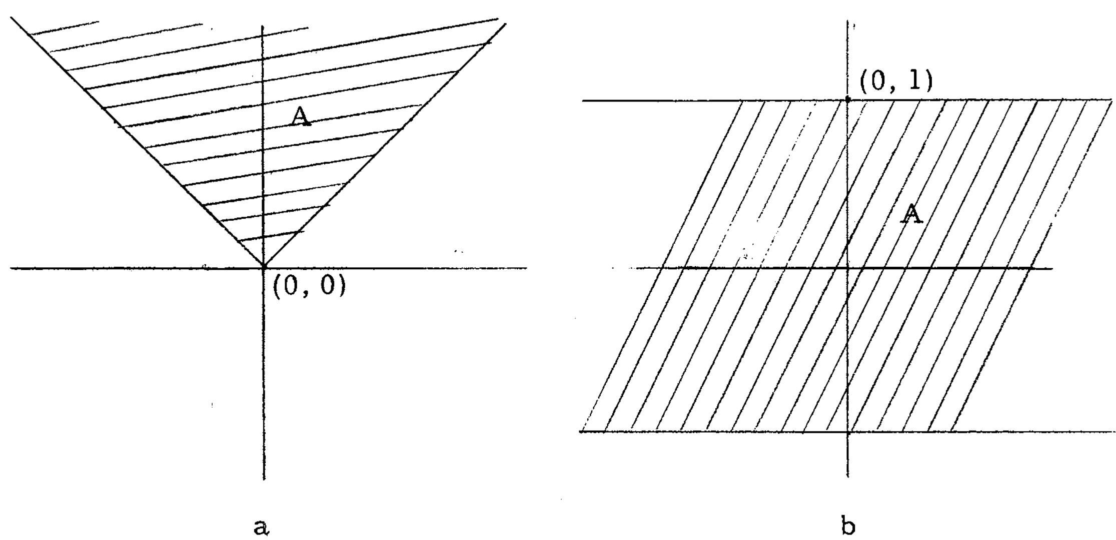

Figure 2.3. Unbounded Regions

The characterization of the extreme points of the unit disk in the complex plane is exactly the same as that of Example 2.3. Since the functions in our Banach spaces are complex-valued, the following lemma will be useful in the proofs of later results.

Lemma 2.6 A complex number λ is an extreme point of the closed unit disk in the complex plane if and only if $|\lambda| = 1$.

Proof: The condition that $|\lambda| = 1$ is necessary by Lemma 2.5.

Suppose $\lambda = a + bi$ is of modulus 1 and $\alpha = c + di$ is any complex number with $|\lambda \pm \alpha| \leq 1$. Then

$$a^2 \pm 2ac + c^2 + b^2 \pm 2bd + d^2 \leq 1.$$

Since $a^2 + b^2 = 1$ we have $\pm 2(ac + bd) + c^2 + d^2 \leq 0$. This inequality is valid only if $c = d = 0$ since either $2(ac + bd) \geq 0$ or $-2(ac + bd) \geq 0$. Thus $\alpha = 0$ and by Proposition 2.4 λ is an extreme point. Therefore $|\lambda| = 1$ is a sufficient condition. Q.E.D.

We will usually be considering Banach spaces X over the complex number field. The following lemma gives us the fact that in such a space, if there is one element in $\operatorname{ext} U(X)$ then there are uncountably many extreme points of the unit ball.

Lemma 2.7 Let λ be a complex number with $|\lambda| = 1$ and X a Banach space over the complex number field. Then $x \in \operatorname{ext} U(X) \Rightarrow \lambda x \in \operatorname{ext} U(X)$.

Proof: Let $x \in \operatorname{ext} U(X)$ and $y \in X$ with $\|\lambda x \pm y\| \leq 1$. Then $\|x + \frac{1}{\lambda} y\| \leq 1$. But since $x \in \operatorname{ext} U(X)$, $\frac{1}{\lambda} y = 0$ so $y = 0$. Thus $\lambda x \in \operatorname{ext} U(X)$. Q.E.D.

Although the concept of "extreme point" is a fairly simple one, it will be seen in Chapters IV and V that the characterization of $\operatorname{ext} U(X)$ and $\operatorname{ext} U(X^*)$ is sometimes a difficult task. In proving the results in the next chapter some fairly deep mathematical tools are sometimes needed.

CHAPTER III

THEOREMS ON EXTREME POINTS AND APPLICATIONS

Three Major Theorems

A few ideas and facts related to convexity and extreme points had been considered earlier, although it was mainly due to the pioneering work of Minkowski that the notions of convexity and extreme points became well known subjects of research. The concept of "extreme point" constitutes an important part of his book published in 1911 (see [30]). We will need the following definitions.

Definition 3.1 Let H be a subset of a vector space. Then the convex hull, denoted $\operatorname{con} H$, is the intersection of all convex sets that contain H. The closed convex hull of H, denoted $\operatorname{clcon} H$, is the intersection of all closed convex sets that contain H.

Definition 3.2 A hyperplane in a vector space V is a set of the form $\{x \in V : f(x) = t\}$ for some linear functional f (not identically zero) defined on V and some scalar t.

For a finite dimensional linear space E, the linear functionals f are of the form $f(x) = \langle x, y\rangle$ for some $y \in E$ where $\langle x, y\rangle$ is the scalar product of x and y. Therefore a hyperplane in E is of the form $\{x \in E : \langle x, y\rangle = t\}$ for some $y \neq 0$ and some scalar t.

Definition 3.3 A hyperplane $H = \{x \in V : f(x) = t\}$ is said to support a convex subset A of a vector space V if there exists an $x_0 \in A$ such that $f(x_0) = t$ and either $\mathrm{Re} f(x_0) \leq \mathrm{Re} f(x)$ for all $x \in A$ or $\mathrm{Re} f(x_0) \geq \mathrm{Re} f(x)$ for all $x \in A$. (Here $\mathrm{Re}\,\lambda$ is the real part of the complex number λ.)

Definition 3.4 Let H be a supporting hyperplane of a convex subset A of a vector space V. The set $F = H \cap A$ is called a face of A. (This is not the usual definition of face, but is convenient for our purpose.)

The next theorem is a classical result which appeared in Minkowski's book. It is a forerunner of the Krein-Milman type theorems which were to appear later.

Theorem 3.5 (Minkowski) Let K be a nonempty compact convex subset of an n-dimensional linear space E. Then $K = \mathrm{con\,ext}\,K$.

Proof: [[16], p. 18] We may assume without loss of generality that the dimension of K in n. Since $\mathrm{ext}\,K \subseteq K$ and K is convex, it follows that $\mathrm{con\,ext}\,K \subseteq \mathrm{con}\,K = K$.

To prove that $K \subseteq \mathrm{con\,ext}\,K$, we use induction on the dimension of K. If the dimension of K is 0 or 1 then it is clear that $K \subseteq \mathrm{con\,ext}\,K$. Let the dimension of K be greater than 1 and $x \in K$. If $x \notin \mathrm{ext}\,K$ then there is a line L such that x is an element of the relative interior of $K \cap L$. Since K is compact and convex, L intersects the boundary of K in exactly two points y and z. Thus there exist faces F_y and F_z of K such that $y \in F_y$ and $z \in F_z$. The dimension of F_y and F_z is less than n and therefore by the induction hypothesis $F_y = \text{con ext } F_y$ and $F_z = \text{con ext } F_z$. To complete the proof we must show that $\text{ext } F_y \subseteq \text{ext } K$. Let $w \in F_y$ and suppose $w \notin \text{ext } K$. Then there are elements u and v of K such that $u \neq w$ and $w = \alpha u + (1 - \alpha)v$ for some $0 < \alpha < 1$. Let f be the linear functional which defines the face F_y with $\sup\{\text{Re} f(t) : t \in K\} = M$. Since $w \in F_y$ it follows that $f(w) = M$. Thus $M = f(w) = \alpha f(u) + (1 - \alpha) f(v) \leq M$ which implies $f(u) = f(v) = M$. Hence $u, v \in F_y$ and $w \notin \text{ext } F_y$. Note that for any two sets A and B, $\text{con } A \cup \text{con } B \subseteq \text{con } (A \cup B)$. We then have that

$$\begin{aligned} x \in \text{con}\{y, z\} &\subseteq \text{con}(F_y \cup F_z) \subseteq \text{con}\{\text{con ext } F_y \cup \text{con ext } F_z\} \\ &\subseteq \text{con}\{\text{ext } F_y \cup \text{ext } F_z\} \subseteq \text{con ext } K . \end{aligned}$$

Hence $K \subseteq \text{con ext } K$. Q.E.D.

The conclusion of this theorem is that every element of a compact convex subset K of E can be written as a convex combination of extreme points of K. The following theorem is a sharper form of Minkowski's theorem.

Theorem 3.6 (Carathéodory) Let K be a nonempty compact convex subset of an n-dimensional linear space E. Then every $x \in K$ can be written as a convex combination of $n + 1$ (or fewer) extreme points of K.

Proof: Assume without loss of generality that the dimension of K is n and as in the proof of Theorem 3.5 we will use induction on the dimension of K. The theorem is trivial if the dimension of K is zero. Let the dimension of K be $n > 0$ and $x \in K$. If x is a boundary point of K then there exists a supporting hyperplane H such that x is an element of the face $F = H \cap K$. The dimension of F is at most $n-1$ and therefore by the induction hypothesis, x can be written as a convex combination of n extreme points of F. It was shown in the proof of Theorem 3.5 that $\operatorname{ext} F \subseteq \operatorname{ext} K$. Thus x is a convex combination of n extreme points of K.

If x is an interior point of K, choose $y \in \operatorname{ext} K$. The line through x and y intersects the boundary of K at some point z. x is a convex combination of y and z and by the first part of the proof, z is a convex combination of n extreme points of K. Thus x is a convex combination of $n+1$ extreme points of K. Q.E.D.

For $n = 2$ we can use Example 2.2, Figure 2.1(b) of Chapter II as an illustration of this theorem. Recall that the extreme points of a triangular region are the three vertices. Thus any non-extreme point on the boundary is a convex combination of the two extreme points on that side of the triangle. For an interior point we must use all three vertices to represent the point as a convex combination of the extreme points.

Probably the most famous theorem concerning extreme points is the Krein-Milman Theorem. There has been a considerable amount of research done attempting to generalize and extend the results of this theorem and for good reason. Its applications in analysis are numerous and important. It first appeared in 1940 and the proof may be found in several standard texts (see [11] and [40]).

<u>Theorem 3.7</u> (Krein-Milman) Let K be a nonempty compact convex subset of a locally convex topological vector space E. Then

$K = \mathrm{cl\,con\,ext}\,K$.

Proof: [[40], p. 207] Let $\mathfrak{H} = \{H : H$ is a supporting hyperplane of $K\}$ and $\mathfrak{F} = \{F : F = H \cap K,\ H \in \mathfrak{H}\}$. $\mathfrak{F}$ is the set of faces of K (see Definition 3.4). Let $\mathfrak{G}$ be the set of all nonempty finite intersections of elements of $\mathfrak{F}$. Let $G \in \mathfrak{G}$ and partially order $\mathfrak{G}$ by inclusion. Then by the Huasdorff maximal principle there is a maximal linearly ordered family $\mathfrak{S}$ in $\mathfrak{G}$ with $G \in \mathfrak{S}$. Since K is compact $S = \bigcap \{S' : S' \in \mathfrak{S}\}$ is nonempty. Furthermore S is minimal in the sense that if S properly contains an element of $\mathfrak{G}$ then the family $\mathfrak{S}$ would not be maximal. Thus any $G \in \mathfrak{G}$ contains a minimal nonempty element Q. We claim that Q can contain only one point. For if Q contains distinct points x_0 and y_0 then since E is locally convex there is a continuous linear functional f with $\mathrm{Re} f(x_0) > \mathrm{Re} f(y_0)$. Thus $H = \{x : \mathrm{Re} f(x) = \sup\{\mathrm{Re} f(x) : x \in Q\}\}$ is a supporting hyperplane of Q (hence of K) that does not contain y_0. Therefore Q properly contains $H \cap Q \neq \emptyset$ which contradicts Q being minimal. Since Q contains only one point, it must be extreme. Hence every $G \in \mathfrak{G}$ contains an extreme point.

If $f \in E^*$ then $\mathrm{Re} f$ assumes its maximum on K since K is compact. Thus $H = \{x : \mathrm{Re} f(x) = \sup\{\mathrm{Re} f(y) : y \in K\}\}$ is an element of $\mathfrak{H}$ and $H \cap K = F \in \mathfrak{G}$. Since F contains an extreme point, we can conclude that the maximum of a continuous linear functional defined on K is equal to its maximum on the extreme points of K. If $x \notin \mathrm{cl\,con\,ext}\,K$ then there is a continuous linear functional g such that $\sup\{\mathrm{Re} g(y) : y \in K\} \leq \sup\{\mathrm{Re} g(y) : y \in \mathrm{cl\,con\,ext}\,K\} < g(x)$. Thus $x \notin K$ and we have $K \subseteq \mathrm{cl\,con\,ext}\,K$. Clearly $\mathrm{cl\,con\,ext}\,K \subseteq K$. Hence $K = \mathrm{cl\,con\,ext}\,K$. Q. E. D.

We note that the converse of this theorem is not true since there are examples of non-compact convex sets which are the closed convex hulls of their extreme points. For example the unit ball of ℓ_1, the space of sequences of complex numbers which are absolutely summable, is the closed convex hull of its extreme points (see Theorem 4.5). The unit ball of ℓ_1 is convex but not compact since ℓ_1 is infinite dimensional.

The Krein-Milman Theorem in its original form stated that every nonempty convex bounded weak*-closed subset of a conjugate Banach space is the weak*-closed convex hull of its extreme points. In general weak*-closed cannot be replaced by norm-closed. For example the unit ball of c_0 (see p. 30) is a norm-closed bounded and convex subset of $\ell_\infty = \ell_1^*$ and has no extreme points (see Theorem 4.1). An interesting result that is closely related to the Krein-Milman Theorem is the following.

<u>Theorem 3.8</u> In a Banach space X, the following two statements are equivalent:

(i) Every closed bounded convex subset K of X has an extreme point.

(ii) Every closed bounded convex subset K of X is the closed convex hull of its extreme points.

Proof: [25] Clearly (ii) $\Rightarrow$ (i). Assume that (i) holds and suppose there is a y in $K \sim \operatorname{cl}\operatorname{con}\operatorname{ext} K$. Then there is an $f \in X^*$ such that $\sup\{\operatorname{Re} f(x) : x \in \operatorname{cl}\operatorname{con}\operatorname{ext} K\} < \operatorname{Re} f(y)$. Let $H = \{x \in X : \operatorname{Re} f(x) = \operatorname{Re} f(y)\}$. $H \cap K$ is disjoint from $\operatorname{cl}\operatorname{con}\operatorname{ext} K$, and is a nonempty bounded closed and convex set. Therefore by hypothesis it has an extreme point. Now

$\emptyset \neq \text{ext}(H \cap K) \subseteq \text{ext}\,K$ (as in the proof of Theorem 3.5). But this contradicts $H \cap K$ being disjoint from cl con ext$K$. Hence K = cl con ext$K$. Q.E.D.

For any Banach space X, the unit ball of X^* with the weak* topology is compact and hence by Krein-Milman Theorem it is the w*-closed convex hull of its extreme points. If the condition of separability is added to X^* then we get the stronger result known as the Bessaga-Pelczyński Theorem. The original proof by Bessaga and Pelczyński [3], which appeared in 1966, is quite involved and uses some deep mathematical tools. We will present a proof due to Namioka [31] which uses only standard techniques in functional analysis. The proof has been distilled from his to coincide with the purposes of this paper. We proceed with four lemmas which lead to the result. (K, T) will denote a subset K of X^* with the topology T. <u>All topological terms in these lemmas refer to the weak*-topology unless otherwise stated.</u>

<u>Lemma 3.9</u> Let K be a compact subset of X^* and $\{C_i : i = 1, 2, \ldots\}$ be a sequence of closed subsets of K such that $K = \bigcup \{C_i : i = 1, 2, \ldots\}$. Then $\bigcup \{\text{int}\,C_i : i = 1, 2, \ldots\}$ is dense in K where $\text{int}\,C_i$ is the interior of C_i in K.

Proof: [31] Assume $K \neq \emptyset$ since otherwise the assertion is trivial. Let V be a nonempty open subset of K. Since K is compact, V is of second category in itself (see [42], p. 8). We have $V = \bigcup \{V \cap C_i : i = 1, 2, \ldots\}$ where $V \cap C_i$ is closed in V, and therefore for at least one i, $V \cap C_i$ has nonempty interior relative

to V and hence relative to K. Thus $V \cap [\bigcup \{\text{int } C : i = 1, 2, \ldots\}] \neq \emptyset$ and since V is arbitrary $\bigcup \{\text{int } C : i = 1, 2, \ldots\}$ is dense in K.
Q. E. D.

Lemma 3.10 Let K be a w*-compact subset of a separable conjugate space X^* and Z be the set of all points of continuity of the identity map: $(K, w^*) \to (K, \|\cdot\|)$. Then Z is a dense subset of (K, w^*).

Proof: [31] For $\varepsilon > 0$ let A_ε be the union of all open subsets of (K, w^*) with norm-diameter $\leq \varepsilon$. Clearly A_ε is open. Let $S = \{x : \|x\| \leq \frac{1}{2}\varepsilon\}$ and let $\{x_i\}$ be a norm-dense sequence in X^*. Then $K = \bigcup \{K \cap (x_i + S) : i = 1, 2, \ldots\}$. Since S is w*-closed, $x_i + S$ is w*-closed for $i = 1, 2, \ldots$. Since K is w*-closed, $K \cap (x_i + S)$ is also. Hence Lemma 3.9 implies that $\bigcup \{\text{int}[K \cap (x_i + S)] : i = 1, 2, \ldots\}$ is dense in (K, w^*) and this union is contained in A_ε since each $x_i + S$ has diameter $\leq \varepsilon$. Thus A_ε is a dense open subset of (K, w^*). We claim that $Z = \bigcap \{A_{1/n} : n = 1, 2, \ldots\}$. If $x \in Z$ then $\{y : \|x - y\| < \frac{1}{2n}\}$ is a w*-neighborhood of x for every $n = 1, 2, \ldots$ and has norm diameter $\leq \frac{1}{n}$. Hence $x \in A_{1/n}$ for every $n = 1, 2, \ldots$. Let $x \in \bigcap \{A_{1/n} : n = 1, 2, \ldots\}$ and let $B(x, \varepsilon) = \{y : \|x - y\| < \varepsilon\}$. Choose n_0 large enough so that $\frac{1}{n_0} < \varepsilon$. Since $x \in A_{1/n_0}$ there is a w*-neighborhood N of x of diameter $\leq \frac{1}{2n_0}$. Since $x \in N$ and the diameter of N is $\leq \frac{1}{2n_0} < \frac{\varepsilon}{2}$, $N \subseteq B(x, \varepsilon)$. Thus $x \in Z$. Since K is of second category in itself, the intersection of a countable family of dense open sets is dense. Therefore Z is a dense subset of (K, w^*). Q. E. D.

Lemma 3.11 Let K be a w*-compact convex subset of a separable conjugate space X^*. Then $Z \cap \text{ext}\, K$ is a dense subset of $(\text{ext}\, K, w^*)$. (Z is defined in Lemma 3.10)

Proof: [31] Assume K contains more than one point since otherwise the assertion is trivial. Given $\varepsilon > 0$, let B_ε be the subset of $\text{ext}\, K$ such that $u \in B_\varepsilon$ if and only if there is a neighborhood of u in (K, w^*) of diameter $\leq \varepsilon$. Clearly B_ε is an open subset of $(\text{ext}\, K, w^*)$. We now show B_ε is dense in $(\text{ext}\, K, w^*)$.

Let W be a w*-open subset of X^* such that $W \cap \text{ext}\, K \neq \emptyset$. We need to show $B_\varepsilon \cap W \neq \emptyset$. Let D be the w*-closure of $\text{ext}\, K$. Then D is w*-compact and $W \cap D \neq \emptyset$. By Lemma 3.10 the set of points of continuity of the identity map: $(D, w^*) \to (D, \|\cdot\|)$ is dense in (D, w^*). Thus there is a w*-open subset V of X^* such that $\emptyset \neq V \cap D \subseteq W \cap D$ and the diameter of $V \cap D \leq \frac{1}{2}\varepsilon$. Let K_1 be the w*-closed convex hull of the w*-compact set $D \sim V$, and K_2 be the w*-closed convex hull of $D \cap V$. K_1 and K_2 are w*-closed subsets of K and therefore are w*-compact. Note that $\text{ext}\, K \subseteq K_1 \cup K_2$. Thus by the Krein-Milman Theorem, $K = \text{con}\,(K_1 \cup K_2)$. Note that $K_1 \neq K$ since $\text{ext}\, K_1 \subseteq D \sim V$ and $D \cap V \neq \emptyset$. Let $r \in (0, 1]$ and let C_r be the image of the map $f_r : K_1 \times K_2 \times [r, 1] \to K$ defined by $f_r(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$. f_r is continuous for the w*-topology on K, so C_r is w*-compact. In addition, $C_r \neq K$ since $\text{ext}\, K \cap C_r \subseteq K_1$ (if $q \in C_r$ then $q = \lambda x_1 + (1 - \lambda)x_2$ and if $q \in \text{ext}\, K$ then $\lambda = 1$) and $K_1 \neq K$. Let $y \in K \sim C_r$. Then y is of the form $y = \lambda x_1 + (1 - \lambda)x_2$, $x_i \in K_i$, $\lambda \in [0, r)$. Hence $\|y - x_2\| = \lambda \|x_1 - x_2\| \leq rd$ where $d = \text{diameter}\, K$.

Since K is w*-compact, K is w*-bounded which implies by the uniform boundedness principle that K is norm bounded. It follows that $d < \infty$. For $y_1, y_2 \in K \sim C_r$ and $x_2 \in K_2$ we have $\|y_1 - y_2\| \leq \|y_1 - x_2\| + \|x_2 - y_2\| \leq 2rd$. Thus diameter $(K \sim C_r) \leq 2rd$. Let $C = C_r$ with $r = \frac{\varepsilon}{2d}$ then diameter $(K \sim C) \leq \varepsilon$. Since $C \neq K$, there is an element u such that $u \in (K \sim C) \cap \operatorname{ext} K$ and $K \sim C$ is a neighborhood of u in (K, w^*) of diameter $\leq \varepsilon$. Hence $u \in B_\varepsilon$. Since $D \sim V \subseteq K_1 \subseteq C$ we have $u \in D \cap V \subseteq W$. Therefore $u \in B_\varepsilon \cap W$ and consequently $B_\varepsilon \cap W \neq \emptyset$. Thus B is dense in $(\operatorname{ext} K, w^*)$.

Finally we see that $Z \cap \operatorname{ext} K = \cap \{B_{1/n}: n = 1, 2, \ldots\}$ and since $\operatorname{ext} K$ is of second category in itself (see [10]) it follows that $Z \cap \operatorname{ext} K$ is a dense subset of $(\operatorname{ext} K, w^*)$. Q. E. D.

<u>Lemma 3.12</u> Let K be a norm-closed, bounded, and convex subset of a separable conjugate space X^* and let K_1 be the w*-closure of K. Then $K \cap \operatorname{ext} K_1$ is a w*-dense subset of $\operatorname{ext} K_1$.

Proof: [31] Since K is bounded, K_1 is bounded and hence w*-compact. Let Z be the set of all points of continuity of the identity map: $(K_1, w^*) \to (K_1, \|\cdot\|)$ and let $z \in Z$. Since K is w*-dense in K_1, there is a net $\{x_\alpha\}$ in K converging to z relative to the w*-topology. Since $z \in Z$, $x_\alpha \to z$ in the norm topology and therefore $z \in K$. Hence $Z \subseteq K$ and $Z \cap \operatorname{ext} K_1 \subseteq K \cap \operatorname{ext} K_1$. By Lemma 3.11 $Z \cap \operatorname{ext} K_1$ is w*-dense in $\operatorname{ext} K_1$, and hence $K \cap \operatorname{ext} K_1$ is w*-dense in $\operatorname{ext} K_1$. Note that since $K \subseteq K_1$, $K \cap \operatorname{ext} K_1 \subseteq \operatorname{ext} K$.
Q. E. D.

We are now ready to state the Bessaga-Pelczyński Theorem which will follow easily from the above lemmas.

Theorem 3.13 Let X be a Banach space such that X^* is separable. Then each norm-closed, bounded, convex subset K of X^* is the norm-closed convex hull of its extreme points.

Proof: [31] Assume $K \neq \emptyset$. Then according to Theorem 3.8 it is sufficient to prove that $\operatorname{ext} K \neq \emptyset$. Let K_1 be the w*-closure of K. Then K_1 is w*-compact and therefore by the Krein-Milman Theorem, $\operatorname{ext} K_1 \neq \emptyset$. Hence it follows from Lemma 3.12 that $\emptyset \neq K \cap \operatorname{ext} K_1 \subseteq \operatorname{ext} K$. Q.E.D.

Applications

In this section we discuss some applications of extreme points in Banach spaces. Some of the material in Chapter IV will be used in this section but it seems appropriate to present the applications first to motivate the study of Chapter IV. The following are standard definitions to be found in almost every functional analysis text book.

Definition 3.14 An isomorphism between two normed linear spaces Y and W is a linear homeomorphism of Y onto W.

Definition 3.15 An isometric isomorphism between two normed linear spaces Y and W is an isomorphism $T : Y \to W$ with $\|T(y)\| = \|y\|$, $y \in Y$. Two such spaces are said to be isometrically isomorphic.

An important problem is classifying Banach spaces as to isomorphic and isometric "types". If two spaces are of the same "type" then the spaces will have many properties in common. In Chapter V we will use the fact that $L_p^* = L_q$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that if spaces Y and W are isometrically isomorphic then the usual notation is $Y = W$. Recall that for any Banach space X, the mapping $Q : X \to X^{**}$ defined by $Q(x) = \hat{x}$ is an isometric isomorphism of X onto its range. $\hat{x}$ is the linear functional defined on X^* by $\hat{x}(f) = f(x)$ for $f \in X^*$. If the mapping Q is onto X^{**} then X is said to be reflexive.

We may apply Theorem 3.13 to determine whether or not a Banach space is isometrically embeddable in any separable conjugate space. Thus we have the following corollary.

<u>Corollary 3.16</u> The space $L_1[0, 1]$ is not isometrically isomorphic to any subspace of a separable conjugate Banach space.

Proof: This follows from the fact that $U(L_1[0, 1])$ has no extreme points (see Corollary 4.7). Q.E.D.

The notion of extreme point also arises in the study of integral representation theory. Phelps' book [34] is suggested to the reader for study in this area. The following definition is needed for the next theorem.

<u>Definition 3.17</u> Let K be a compact convex subset of a locally convex space E. The class of <u>Baire sets</u> of K is defined to be the σ-algebra of subsets of K generated by the sets $\{x \in K : f(x) \geq \alpha\}$ where f is a real-valued continuous function on K.

We say that μ represents x if $f(x) = \int_K f d\mu$ for every $f \in E^*$. Since the proof of Theorem 3.18 is given in [34, p. 30] in a clear readable style and is quite long, we omit it here.

Theorem 3.18 (Choquet-Bishop-de Leeuw) Let K be a compact convex subset of a locally convex space E and denote by $\mathcal{S}$ the σ-algebra of subsets of K which is generated by $\operatorname{ext} K$ and the Baire sets. Then for each point $x \in K$ there exists a nonnegative probability measure μ on $\mathcal{S}$ such that μ represents x and $\mu(\operatorname{ext} K) = 1$.

Let S be a compact Hausdorff space. $C(S)$ will denote the Banach space of continuous complex-valued functions on S with sup-norm (see [11] for properties). Suppose f, f_n, $n = 1, 2, \ldots$ are functions in $C(S)$. A well known theorem states that $\{f_n\}$ converges weakly to f if and only if the sequence $\{f_n\}$ is uniformly bounded and $\lim_n f_n(y) = f(y)$ for each $y \in S$ (see [11] p. 265). Using Theorem 3.18 we are able to prove the following result of which the above mentioned theorem is a special case. It will be shown (see Theorem 5.15) that the extreme points of $U(C(S)^*)$ are the linear functions φ where $\varphi(f) = \lambda f(y)$ for some scalar λ, $|\lambda| = 1$, and some $y \in S$.

Theorem 3.19 (Rainwater) Let E be a normed linear space and suppose x, x_n, $n = 1, 2, \ldots$ are elements of E. Then the sequence $\{x_n\}$ converges weakly to x if and only if $\{x_n\}$ is bounded and $\lim_n f(x_n) = f(x)$ for each $f \in \operatorname{ext} U(E^*)$.

Proof: [34, p. 33] Let Q denote the natural isometry of E into E^{**}. If $\{x_n\}$ converges weakly to x, then $(Qx_n)(f)$ if bounded for each $f \in E^*$ and therefore the uniform boundedness theorem implies that $\{Qx_n\}$, hence $\{x_n\}$, is bounded in norm. That $\lim_n f(x_n) = f(x)$

for each $f \in \text{ext}\, U(E^*)$ follows from the definition of weak convergence.

Suppose $\{Qx_n\}$ is bounded and $f(x_n) = (Qx_n)(f) \to (Qx)(f) = f(x)$ for each $f \in \text{ext}\, U(E^*)$. Let g be an arbitrary element of $U(E^*)$. We need to show that $g(x_n) \to g(x)$ which is equivalent to showing $(Qx_n)(g) \to (Qx)(g)$. In the w*-topology, $U(E^*)$ is compact (and convex) so by Theorem 3.18 there is a σ-algebra $\mathcal{S}$ of subsets of $U(E^*)$ such that $\text{ext}\, U(E^*) \in \mathcal{S}$ and a probability measure μ on $\mathcal{S}$ supported by the extreme points of $U(E^*)$ and such that $F(g) = \int F d\mu$ for each w*-continuous linear functional F on $U(E^*)$. In particular $(Qx_n)(g) = \int (Qx_n) d\mu$ and $(Qx)(g) = \int (Qx) d\mu$. Furthermore $\{Qx_n\}$ converges to Qx on $U(E^*)$ a.e. with respect to μ, so by the Lebesgue bounded convergence theorem $\int (Qx_n) d\mu \to \int (Qx) d\mu$. Hence $(Qx_n)(g) \to (Qx)(g)$. Q. E. D.

We shall see in Theorem 3.21 that it is sometimes useful to know the cardnality of $\text{ext}\, U(E^*)$. We first of all need the following theorem which is of interest in its own right.

<u>Theorem 3.20</u> If X is an infinite dimensional reflexive Banach space, then the set of extreme points of $U(X)$ is uncountable.

Proof: [28] Suppose that $\text{ext}\, U(X) = \{x_n\}$, $n = 1, 2, \ldots$ and for each n let $F_n = \{f \in X^* : \|f\| \leq 1$ and $|f(x_n)| = \|f\|\}$. To show that F_n is weakly closed for every n, let $\{f_i\}$ be a net in F_n such that $\{f_i\}$ converges weakly to f. Since the norm in X^* is w*-lower semicontinuous (see [44], p. 212) we have $\|f\| \leq \liminf \|f_i\| \leq 1$. Weak convergence implies for $x \in X$,

$$\lim_i \hat{x}_n(f_i) = \hat{x}_n(f) \Rightarrow \lim_i f_i(x_n) = f(x_n) \Rightarrow \lim_i |f_i(x_n)|$$
$$= |f(x_n)| .$$

Since $f_i \in F_n$ we have $\lim_i \|f_i\| = |f(x_n)|$. Therefore

$$\|f\| \leq \lim_i \inf \|f_i\| \leq |f(x_n)| \leq \|f\| \ \|x_n\| = \|f\|$$

which implies $|f(x_n)| = \|f\|$. Thus $f \in F_n$.

Next we claim that $U(X^*) = \bigcup \{F_n : n = 1, 2, \ldots\}$. Clearly the latter is a subset of the former. Let $f \in U(X^*)$. $U(X)$ is weakly compact and therefore by the proof of the Krein-Milman Theorem f assumes its maximum on the extreme points of $U(X)$. Thus $\|f\| \leq 1$ and $|f(x_n)| = \|f\|$ for some n which gives us the other set containment.

By the Baire Category Theorem, at least one of the sets F_n (say F_1) has nonempty weak interior relative to $U(X^*)$. Let f_0 be a relative weak interior point. Then there is a ball B centered at f_0 such that $U(X^*) \cap B \subseteq \inf F_1$. Thus we may assume $\|f_0\| < 1$. Since F_1 is a weak neighborhood of f_0 and X is reflexive, it follows that there exist points $y_1, y_2, \ldots, y_n$ in X such that $f \in F_1$ whenever

$$(*) \qquad \|f\| \leq 1 \quad \text{and} \quad |(f - f_0)(y_i)| < 1, \quad i = 1, 2, \ldots, n .$$

Let

$$N = \{f \in X^* : f(y_i) = f_0(y_i), \ i = 1, 2, \ldots, n \ \text{ and } \ f(x_1) = f_0(x_1)\} .$$

Since X is infinite dimensional and N is of finite codimension, N contains a line through f_0 which intersects the boundary of the unit ball in a point g, $\|g\| = 1$. Thus $g \in N$ and (*) implies $g \in F_1$, so $1 = \|g\| = |g(x_1)| = |f_0(x_1)| = \|f_0\|$ which is a contradiction. Q.E.D.

Recall that Lemma 2.7 states if $x \in \text{ext}\, U(E^*)$ then $\lambda x \in \text{ext}\, U(E^*)$ where $|\lambda| = 1$. If, however, we define two extreme points x and y to be equivalent provided $x = \lambda y$ for some $|\lambda| = 1$, then it makes sense to ask whether $U(X)$ can have countably many equivalence classes of extreme points. The proof of Theorem 3.20 applies without change to show that if X is reflexive and infinite dimensional, then the answer is negative.

<u>Theorem 3.21</u> Suppose that E is a normed linear space and that $\text{ext}\, U(E^*)$ is countable. Then

(i) E^* is separable and

(ii) E contains no infinite dimensional reflexive subspace.

Proof: [28] (i) Since $U(E^*)$ is w*-compact and convex, Theorem 3.18 implies that for each $f \in U(E^*)$ there is a probability measure μ on $\text{ext}\, U(E^*)$ such that $f(x) = \int_{\text{ext}\, U(E^*)} g(x)\, d\mu(g)$ for each $x \in E$. Let $\mu_n = \mu(f_n)$ where $\{f_n\} = \text{ext}\, U(E^*)$. Then $\mu_n \geq 0$, $\Sigma\, \mu_n = 1$ and $f(x) = \Sigma\, \mu_n f_n(x)$ for each $x \in E$. Let S be the set of all sequences $\{\mu_n\}$ with $\mu_n \geq 0$ and $\Sigma\, \mu_n = 1$. Then $S \subseteq \ell_1$ and for any $\{\lambda_n\}$ in S, $g = \Sigma\, \lambda_n f_n$ defines a member of $U(E^*)$. Thus there is a map from the norm-separable space S onto $U(E^*)$. Since $\|f - g\| = \sup\{|f(x) - g(x)| : \|x\| \leq 1\} \leq \Sigma\, |\mu_n - \lambda_n|$, the map is norm-to-norm continuous and hence $U(E^*)$ is norm separable which implies that E^* is also.

(ii) If F is an infinite dimensional reflexive subspace of E

then F^* is also reflexive and hence by Theorem 3.20, $\text{ext}\, U(F^*)$ has uncountably many points. But each $f \in \text{ext}\, U(F^*)$ can be extended to an extreme point of $U(E^*)$ which implies $\text{ext}\, U(E^*)$ is uncountable. Q.E.D.

The result of Theorem 3.20 has been improved recently by an application of the following theorem due to [21].

Theorem 3.22 If X is a Banach space and X^{**} is separable then both X and X^* have infinite dimensional reflexive subspaces.

Proof: (see [21]).

We now have the following corollary.

Corollary 3.23 If X is infinite dimensional then $\text{ext}\, U(X^{**})$ is uncountable.

Proof: Suppose that $\text{ext}\, U(X^{**})$ is countable. Then by Theorem 3.21 (i) X^{**} is separable; therefore X^* has an infinite dimensional reflexive subspace by Theorem 3.22. This contradicts Theorem 3.21 (ii). Q.E.D.

We have now presented three important results in the development of the theory of extreme points: the Minkowski Theorem, the Krein-Milman Theorem, and the Bessaga-Pelczyński Theorem. To exemplify the importance of extreme points in the study of functional analysis, we have presented several applications. We are now ready to embark upon the task of characterizing the set $\text{ext}\, U(X)$ where X is one of five well known Banach spaces.

CHAPTER IV

CHARACTERIZATIONS OF ext U(X)

The purpose of this chapter is to characterize the extreme points of the unit ball in some well known Banach spaces. Section one will deal with three of the sequence spaces; section two, the L_p spaces; section three, the Hardy spaces; section four, the Lipschitz space Lip$[0, 1]$; and section five, the space of continuous functions on a compact Hausdorff space S, $C(S)$. At the end of each section we will answer the question whether the unit ball in these spaces is the closed convex hull of its extreme points. This will determine if there are "enough" extreme points in the boundary of the unit ball to, in a sense, span the unit ball.

Sequence Spaces

The sequence spaces to be considered in this section are c_0, ℓ_1, and ℓ_∞. c_0 is the space of complex-valued sequences $x = \{x_n\}$ such that $\lim x_n = 0$ with $\|x\| = \sup |x_n|$. ℓ_1 is the space of complex-valued sequences $x = \{x_n\}$ such that $\sum_n |x_n| < \infty$ with $\|x\| = \sum_n |x_n|$. ℓ_∞ is the space of bounded complex-valued sequences $x = \{x_n\}$ with $\|x\| = \sup |x_n|$. We will not distinguish norms notationally unless it is not clear which one is meant.

<u>Theorem 4.1</u> $\operatorname{ext} U(c_0) = \emptyset$.

Proof: Let $x \in U(c_0)$. Since $\lim x_n = 0$ there exists a positive integer n_0 such that $|x_{n_0}| \leq \frac{1}{2}$. Let

$$y_n = \begin{cases} x_n & , \ n \neq n_0 \\ x_n + \frac{1}{4}, & n = n_0 \end{cases}, \qquad z_n = \begin{cases} x_n & , \ n \neq n_0 \\ x_n - \frac{1}{4}, & n = n_0 \end{cases}$$

and define $y = \{y_n\}$, $z = \{z_n\}$. Clearly $\|y\| \leq 1$, $\|z\| \leq 1$, $x = \frac{1}{2}(y+z)$, and $x \neq y$. Hence x is not an extreme point of $U(c_0)$. Q. E. D.

In the next theorem we will need the following notation: $\delta_j = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ is the element in ℓ_1 which has a 1 in the j-th position and zeros elsewhere. Note that $\|\delta_j\| = 1$.

<u>Theorem 4.2</u> $x \in \operatorname{ext} U(\ell_1)$ if and only if $x = \lambda\delta_j$ for some $j = 1, 2, \ldots$ and complex number λ with $|\lambda| = 1$.

Proof: Let $x = \lambda\delta_j$, $|\lambda| = 1$, and suppose $x = \frac{1}{2}(y+z)$ where $y, z \in U(\ell_1)$. Then $x_k = \frac{1}{2}(y_k + z_k)$, $k = 1, 2, \ldots$. For $k = j, x_j = \frac{1}{2}(y_j + z_j)$ but $x_j = \lambda$ is an extreme point of the unit disk in the complex plane; therefore, $x_j = y_j = z_j = \lambda$. For $k \neq j$, $y_k = z_k = 0$ since $\|y\| = \sum_n |y_n| \leq 1$ and $\|z\| = \sum_n |z_n| \leq 1$. Hence $x = y = z$ which implies that $x \in \operatorname{ext} U(\ell_1)$.

Now suppose $\|x\| = 1$ and $x \neq \lambda\delta_j$ for any $j = 1, 2, \ldots$ and $|\lambda| = 1$. Let n_0 be the first integer such that $x_{n_0} \neq 0$ and $x_{n_0} = re^{i\theta}$. Since $x \neq \lambda\delta_j$ we have $0 < r < 1$; therefore, define $y = \{y_n\}$, $z = \{z_n\}$ where

$$y_n = \begin{cases} e^{i\theta}, & n = n_0 \\ 0, & n \neq n_0 \end{cases}, \qquad z_n = \begin{cases} 0 & , \; n = n_0 \\ \frac{1}{1-r}\, x_n, & n \neq n_0 \end{cases}$$

Note $\|y\| = 1$ and $\|z\| = \sum_n |z_n| = \frac{1}{1-r}(\sum_n |x_n| - |r\,e^{i\theta}|) = \frac{1}{1-r}(1-r) = 1$; thus $y, z \in U(\ell_1)$ and $x = ry + (1-r)z$, $0 < r < 1$, with $x \neq y$. Hence x is not an extreme point of $U(\ell_1)$. Q. E. D.

We now want to consider a more general space than the sequence space ℓ_∞. The characterization of the set $\operatorname{ext} U(\ell_\infty)$ will then be a corollary to the next theorem. Let S be any set, then $\ell_\infty(S)$ is the space of bounded complex-valued functions defined on S. $\ell_\infty(S)$ is a Banach space with $\|f\| = \sup\{|f(x)| : x \in S\}$.

<u>Theorem 4.3</u> $f \in \operatorname{ext} U(\ell_\infty(S))$ if and only if $|f(x)| = 1$ for all $x \in S$.

Proof: Let $f \in \ell_\infty(S)$ with $\|f\| \leq 1$ and suppose for some $x_0 \in S$, $f(x_0) = r\,e^{i\theta}$ where $0 < r < 1$. Define

$$g(x) = \begin{cases} e^{i\theta}, & x = x_0 \\ f(x), & x \neq x_0 \end{cases}, \qquad h(x) = \begin{cases} 0 & x = x_0 \\ f(x), & x \neq x_0 \end{cases}$$

then $f(x) = rg(x) + (1-r)h(x)$. Since $\|f\| \leq 1$ we have that $\|g\| \leq 1$ and $\|h\| \leq 1$; also $f \neq g$, which implies f is not an extreme point of $U(\ell_\infty(S))$.

Let $f \in \ell_\infty(S)$ with $|f(x)| = 1$ for all $x \in S$ and suppose $f = \frac{1}{2}(g+h)$ where $g, h \in U(\ell_\infty(S))$. Since $f(x)$ is an extreme point of the unit disk in the complex plane for all $x \in S$, $f(x) = g(x) = h(x)$ for all $x \in S$. Hence $f = h = g$ and we have that $f \in \operatorname{ext} U(\ell_\infty(S))$. Q. E. D.

Corollary 4.4 $x \in \text{ext}\, U(\ell_\infty)$ if and only if $|x_n| = 1$ for all $n = 1, 2, \ldots$.

It is clear that $U(c_0) \neq \text{cl con ext}\, U(c_0)$ since $U(c_0)$ contains no extreme points. For the space ℓ_1 we have the following result.

Theorem 4.5 $U(\ell_1) = \text{cl con ext}\, U(\ell_1)$.

Proof: ℓ_1 is the conjugate space of c_0 and ℓ_1 is separable (the set of elements of the form $\sum_{k=1}^{n} \lambda_k \delta_{j_k}$ where $\lambda_k = a_k + b_k i$, a_k, b_k rational, is a countable dense subset). Thus the conclusion follows from the Bessage-Pelczyński Theorem (3.13). Q.E.D.

We conclude this section with the result for the space ℓ_∞. We shall delay the proof until the next section.

Theorem 4.6 $U(\ell_\infty) = \text{cl con ext}\, U(\ell_\infty)$.

Proof: (see Theorem 4.15).

L_p Spaces

In this section we will consider the set of extreme points in the space $L_p(S, M, \mu)$ where (S, M, μ) is a measure space (see [40], p. 217). For $1 \leq p < \infty$, $L_p(S, M, \mu)$ is the space of complex-valued, μ-measurable functions f defined on S with $\|f\| = [\int |f|^p d\mu]^{1/p} < \infty$. $L_\infty(S, M, \mu)$ is the space of complex-valued, μ-measurable functions defined on S with $\|f\| = \text{ess sup}\, \{ |f(x)| : x \in S\} < \infty$. The three cases $p = 1$, $1 < p < \infty$, and $p = \infty$ will be treated separately. The next theorem will show that the existence of extreme points in L_1 depends upon the measure μ being atomic.

Definition 4.7 $F \in M$ with $\mu(F) > 0$ is called an atom (with respect to μ) if and only if for all $F' \in M$ such that $F' \subset F$ either $\mu(F') = 0$ or $\mu(F') = \mu(F)$.

Throughout this section χ_F will denote the characteristic function of the set F.

Theorem 4.8 A necessary and sufficient condition that $f \in \operatorname{ext} U(L_1)$ is that $f = \lambda \frac{1}{\mu(F)} \chi_F$ for some atom F and complex constant λ where $|\lambda| = 1$.

Proof: Suppose the condition holds and $g \in L_1$ such that $\|f \pm g\| = 1$. Let $G = \{x \in S : g(x) \neq 0\}$. Then

$$1 = \int |f \pm g| d\mu = \int_{G \cup F} |f \pm g| d\mu$$

$$= \int_{G \cap F} |f \pm g| d\mu + \int_{G \sim F} |g| d\mu + \int_{F \sim G} |f| d\mu .$$

Since F is an atom, either $\mu(G \cap F) = 0$ or $\mu(G \cap F) = \mu F$. Let us first assume $\mu(G \cap F) = \mu F$ (which implies that $G \cap F$ is an atom). $g(x) \neq 0$ for $x \in G \cap F$; therefore, either $|f| < |f+g|$ or $|f| < |f-g|$ on a subset of $G \cap F$ of positive measure (for if not then $g = 0$). But since $G \cap F$ is an atom we have either $|f| < |f+g|$ or $|f| < |f-g|$ a.e. on $G \cap F$. Assume $|f| < |f+g|$ a.e. on $G \cap F$. Then

$$1+\int_{G\sim F}|g|d\mu = \int_{G\cap F}|f|d\mu + \int_{F\sim G}|f|d\mu + \int_{G\sim F}|g|d\mu$$

$$< \int_{G\cap F}|f+g|d\mu + \int_{F\sim G}|f|d\mu + \int_{G\sim F}|g|d\mu$$

$$= 1 .$$

This contradiction implies $\mu(G \cap F) = 0$ which implies $\mu(F \sim G) = \mu(F)$. Thus we have

$$1+\int_{G\sim F}|g|d\mu = \int_{G\cap F}|f \pm g|d\mu + \int_{F\sim G}|f|d\mu + \int_{G\sim F}|g|d\mu = 1$$

and therefore $\int_{G\sim F}|g|d\mu = 0$. $g(x) \neq 0$ for $x \in G \sim F$, hence $\mu(G \sim F) = 0$. This together with $\mu(G \cap F) = 0$ implies $\mu(G) = 0$. Thus $g(x) = 0$ a.e. and $f \in \operatorname{ext} U(L_1)$.

Now let $f \in \operatorname{ext} U(L_1)$. If there exists $A \in M$ such that $0 < \int_A |f|d\mu = \alpha < 1$ then define $g = \frac{1}{\alpha} f \chi_A$ and $h = \frac{1}{1-\alpha} f \chi_{\tilde{A}}$. Clearly $\|g\| = \|h\| = 1$ and $f = \alpha g + (1-\alpha)h$ where $0 < \alpha < 1$. Since $g \neq f$, $f \notin \operatorname{ext} U(L_1)$. This argument shows that $\int_A |f|d\mu = 1$ or 0 for all $A \in M$ such that $\mu(A) > 0$. Let $P = \{x \in S : |f(x)| > 0\}$. Note that $\mu(P) > 0$ since $\int_{S\sim P}|f|d\mu + \int_P |f|d\mu = \int_P |f|d\mu = 1$. Now take $A \subseteq P$ with $\mu(A) > 0$. We then have $\int_A |f|d\mu > 0$ and hence $\int_A |f|d\mu = 1$. For all $A \subseteq P$ with $\mu(A) > 0$, we have

$$1 = \int_{P\sim A}|f|d\mu + \int_A |f|d\mu = \int_{P\sim A}|f|d\mu + 1 .$$

This gives $\mu(P \sim A) = 0$ which implies $\mu(A) = \mu(P)$. Thus P is an atom. For all $A \subseteq P$ with $\mu(A) > 0$ it follows that

$\sup_{x \in A} |f(x)|\, \mu(A) \geq \int_A |f|d\mu = 1$: therefore, $\sup_{x \in A} |f(x)| \geq \frac{1}{\mu(A)} \geq \frac{1}{\mu(P)}$.

For each n such that $\frac{1}{\mu(P)} - \frac{1}{n} > 0$, let $M_n = \{x \in S : 0 < |f(x)| < \frac{1}{\mu(P)} - \frac{1}{n}\}$. $\sup_{x \in M_n} |f(x)| < \frac{1}{\mu(P)}$. Thus $\mu(M_n) = 0$, for if $\mu(M_n) > 0$, then by the previous argument $\sup_{x \in M_n} |f(x)| \geq \frac{1}{\mu(P)}$. Hence $|f(x)| \geq \frac{1}{\mu(P)}$ a.e. on P. It follows that

$$\int_S (|f| - \frac{1}{\mu(P)} \chi_P) d\mu = \int_P (|f| - \frac{1}{\mu(P)} \chi_P) d\mu = 0$$

which implies $|f(x)| = \frac{1}{\mu(P)} \chi_P(x)$ a.e. Therefore there exists a measurable complex-valued function $\varphi(x)$ with $|\varphi(x)| = 1$ for $x \in P$ and $f(x) = \varphi(x) \frac{1}{\mu(P)} \chi_P(x)$. If $\varphi(x)$ is not constant on P then either $\operatorname{Re} \varphi(x)$ or $\operatorname{Im} \varphi(x)$ is not constant. Assume without loss of generality that $\operatorname{Re} \varphi(x)$ is not constant on P. Then there exists a real number c and $B \subseteq P$ with $\mu(B) > 0$ and $\mu(P \sim B) > 0$ where $\operatorname{Re} \varphi(x) > c$ for $x \in B$ and $\operatorname{Re} \varphi(x) \leq c$ for $x \in P \sim B$. But this is impossible since P is an atom. Hence $\varphi(x) = \lambda$ for $x \in P$ where $|\lambda| = 1$. Thus $f(x) = \lambda \frac{1}{\mu(P)} \chi_P(x)$ where $|\lambda| = 1$ and P is an atom. Q.E.D.

As noted in Theorem 4.2 $x \in \operatorname{ext} U(\ell_1)$ if and only if $x = \lambda\delta_j$ for some $|\lambda| = 1$. This is also a consequence of Theorem 4.8 if we consider ℓ_1 as the space $L_1(S, M, \mu)$ where S is the set of positive integers, M the σ-algebra of all subsets of S, and μ the counting measure on M. The result follows by noting that the atoms in this space are the singleton sets.

$L_1[0,1]$ is the space $L_1(S, M, \mu)$ where $S = [0,1]$, M is the family of Lebesgue measurable subsets of S, and μ is Lebesgue measure. Since M contains no atoms, we have the following corollary to Theorem 4.8.

<u>Corollary 4.9</u> $\operatorname{ext} U(L_1[0,1]) = \emptyset$.

Next we shall consider the space $L_p(S, M, \mu)$, $1 < p < \infty$ where $\|f\| = [\int_S |f|^p \, d\mu]^{1/p}$. Note that f and g are in the same equivalence class if and only if $f = g$ a.e. with respect to the measure μ. The following lemma will be used in the proof of the theorem which will characterize the set of extreme points of the unit ball of $L_p(S, M, \mu)$.

<u>Lemma 4.10</u> Let $f, g \in L_p$, $1 < p < \infty$, with $\|f\| = \|g\| = 1$. Then $\|f+g\| = \|f\| + \|g\|$ if and only if $f = g$ a.e.

Proof: If $f = g$ a.e. then $\|f\| = \|g\|$ and $f + g = 2f$ a.e. Thus $\|f+g\| = 2\|f\| = \|f\| + \|g\|$.

Assume $\|f+g\| = \|f\| + \|g\| = 2$. Then

$$1 = \frac{\|f+g\|^p}{2^p} = \int_S \frac{|f+g|^p}{2^p} \, d\mu \leq \int_S \frac{(|f| + |g|)^p}{2^p} \, d\mu \leq \int_S \frac{|f|^p + |g|^p}{2} \, d\mu$$

$$= \frac{\|f\|^p + \|g\|^p}{2} = 1 .$$

The last inequality follows from the fact that the map $t \to t^p$ is a strictly convex function of a real variable for $t > 0$ and $1 < p < \infty$. Therefore equality must hold throughout this expression and since

$$\int_S \frac{|f+g|^p}{2^p} \, d\mu = \int_S \frac{(|f| + |g|)^p}{2^p} \, d\mu ,$$

we have $|f+g| = |f| + |g|$ a.e. This implies there exists a nonnegative real-valued function φ, such that $f(x) = \varphi(x)\, g(x)$ a.e. Also since

$$\int_S \frac{(|f| + |g|)^p}{2^p}\, d\mu = \int_S \frac{|f|^p + |g|^p}{2}\, d\mu$$

we have $\frac{1}{2^p}(|f| + |g|)^p = \frac{1}{2}(|f|^p + |g|^p)$ a.e. Thus $|f| = |g|$ a.e. Hence $|g(x)| = |f(x)| = |\varphi(x)|\ |g(x)|$ a.e. Thus $\varphi(x) \geq 0$ implies $\varphi(x) = 1$ a.e. and we therefore have that $f = g$ a.e. Q.E.D.

Theorem 4.11 $f \in \operatorname{ext} U(L_p)$, $1 < p < \infty$, if and only if $\|f\| = 1$.

Proof: Let $f \in \operatorname{ext} U(L_p)$. Then it follows from Lemma 2.5 that $\|f\| = 1$.

Now let $\|f\| = 1$ and suppose $f = \frac{1}{2}(g+h)$ where $g, h \in U(L_p)$. Then $2 = 2\|f\| = \|g+h\| \leq \|g\| + \|h\| \leq 2$ which implies $\|h\| = \|g\| = 1$ and $\|g+h\| = \|g\| + \|h\|$. It follows from Lemma 4.10 that $h = g$ a.e. Thus $f = g = h$ a.e. and $f \in \operatorname{ext} U(L_p)$. Q.E.D.

The final space to be considered in this section is $L_\infty(S, M, \mu)$, the space of complex-valued measurable functions on S with $\|f\| = \operatorname{ess\,sup} |f| < \infty$.

Theorem 4.12 $f \in \operatorname{ext} U(L_\infty)$ if and only if $|f| = 1$ a.e.

Proof: Suppose $|f| < 1$ on P where $\mu(P) > 0$. Define $g = f + (1 - |f|)$ and $h = f - (1 - |f|)$. Then $\|g\| = \operatorname{ess\,sup} |f + (1 - |f|)| \leq \operatorname{ess\,sup}(|f| + 1 - |f|) = 1$ since $1 - |f| \geq 0$. Similarly we have $\|h\| \leq 1$. Therefore $g, h \in U(L_\infty)$, $f = \frac{1}{2}(g+h)$ and $f \neq g$. Thus $f \notin \operatorname{ext} U(L_\infty)$.

Let $|f| = 1$ a.e. and suppose $f = \frac{1}{2}(g+h)$ a.e. where $g, h \in U(L_\infty)$. Then for almost all $x \in S$, $f(x) = \frac{1}{2}(g(x) + h(x))$, $|f(x)| = 1$, $|g(x)| \leq 1$ and $|h(x)| \leq 1$. Thus Lemma 2.6 implies $f = g = h$ a.e. and hence $f \in \text{ext}\, U(L_\infty)$. Q.E.D.

We observe again that ℓ_∞ is a special case of $L_\infty(S, M, \mu)$ where S is the set of positive integers, M the σ-algebra of all subsets of S, and μ the counting measure on M. Hence Corollary 4.4 follows from Theorem 4.12.

Considering Theorem 4.8, the next statement is not too surprising.

Theorem 4.13 $U(L_1(S, M, \mu)) = \text{cl}\,\text{con}\,\text{ext}\, U(L_1(S, M, \mu))$ if and only if μ is purely atomic, that is, every element of M of positive finite measure can be written as a countable union of disjoint atoms.

Proof: Suppose μ is purely atomic. We first observe that if x and y are elements of a Banach space with $\|x\| = 1$, $y \neq 0$, and $\|x - y\| < \varepsilon$ it follows that $|1 - \|y\|| = |\|x\| - \|y\|| < \varepsilon$. Thus

$$\left\|x - \frac{y}{\|y\|}\right\| \leq \|x - y\| + \left\|y - \frac{y}{\|y\|}\right\| = \|x - y\| + \|y\| \left|1 - \frac{1}{\|y\|}\right|$$

$$= \|x - y\| + |\|y\| - 1| < 2\varepsilon .$$

Returning to the proof of the theorem, we note that it will suffice to show that the set of convex combinations of extreme points is dense in the boundary of $U(L_1(S, M, \mu))$. For suppose $g \in U(L_1(S, M, \mu))$, $g \neq 0$, and $\varepsilon > 0$. Then $h = \frac{g}{\|g\|}$ is of norm one. Therefore there is a convex combination $\sum_{i=1}^{n} a_i f_i$ of extreme points f_i such that

$\|h - \sum_{i=1}^{n} a_i f_i\| < \varepsilon$. Hence

$$\|g\| \, \|h - \sum_{i=1}^{n} a_i f_i\| = \|g - (\sum_{i=1}^{n} \|g\| a_i f_i + \frac{1}{2}(1 - \|g\|) f_1 + \frac{1}{2}(1 - \|g\|)(-f_1))\|$$
$$< \|g\| \varepsilon \leq \varepsilon$$

and g is approximated by a convex combination of extreme points. We proceed to show the density property stated above. Let $f \in U(L_1(S, M, \mu))$ with $\|f\| = 1$. Then there is a simple function Ψ such that $\|f - \Psi\| < \frac{\varepsilon}{4}$. By the definition of purely atomic Ψ may be represented as $\Psi = \sum_{j=1}^{\infty} \alpha_j \chi_{A_j}$ where the A_j's are disjoint atoms and $\alpha_j \neq 0$, $j = 1, 2, \ldots$. $\|\Psi\| = \sum_{j=1}^{\infty} |\alpha_j| \mu A_j = \alpha \neq 0$. Since this series converges there is a number N such that $\sum_{j=N+1}^{\infty} |\alpha_j| \mu A_j < \frac{\varepsilon}{4}$. Let $\varphi = \sum_{j=1}^{N} \alpha_j \chi_{A_j}$. Thus $\|f - \varphi\| \leq \|f - \Psi\| + \|\Psi - \varphi\| \leq \frac{\varepsilon}{2}$. Let $\beta_j = \frac{\alpha_j}{|\alpha_j|}$. Then $\varphi = \sum_{j=1}^{N} |\alpha_j| \mu A_j \frac{\beta_j}{\mu A_j} \chi_{A_j}$ and by Theorem 4.8 $\frac{\beta_j}{\mu A_j} \chi_{A_j}$ is an extreme point of $U(L_1(S, M, \mu))$. Hence

$$\frac{\varphi}{\|\varphi\|} = \sum_{j=1}^{N} \frac{|\alpha_j|}{\|\varphi\|} \mu A_j \frac{\beta_j}{\mu A_j} \chi_{A_j}$$

is a convex combination of extreme points of $U(L_1(S, M, \mu))$ and by the observation at the beginning of the proof $\|f - \frac{\varphi}{\|\varphi\|}\| < \varepsilon$.

Now let A be a subset of S with positive finite measure. Define $\varphi = \frac{1}{\mu A} \chi_A$. Then $\varphi \in U(L_1(S, M, \mu))$ and hence by hypothesis there is a convex combination of extreme points, $\Psi_n = \sum_{j=1}^{k_n} a_j \frac{\lambda_j}{\mu A_j} \chi_{A_j}$, such that $|\lambda_j| = 1$, the A_j's are disjoint atoms, $\sum_{j=1}^{k_n} a_i = 1$ and $\|\varphi - \Psi_n\| < \frac{1}{n}$. Let $B_j = A \cap A_j$. Then

$$\|\varphi - \Psi_n\| = \int_{\cup B_j} |\varphi - \Psi_n| d\mu + \int_{A \sim \cup B_j} |\varphi - \Psi_n| d\mu + \int_{\cup A_j \sim A} |\varphi - \Psi_n| d\mu < \frac{1}{n}.$$

Thus as n increases we see that $\mu(A \sim \bigcup_{j=1}^{k_n} B_j) \to 0$ since $\varphi - \Psi_n = \frac{1}{\mu A}$ on $A \sim \bigcup_{j=1}^{k_n} B_j$. Note that if $\mu B_j > 0$ then B_j is an atom. Hence A is a countable union of atoms together with a set of measure zero. But if P is an atom and Q is a set of measure zero then $P \cup Q$ is an atom. Therefore A is a countable union of atoms and it follows that μ is purely atomic. Q. E. D.

Every element of the boundary of $U(L_p)$, $1 < p < \infty$, is an extreme point (Theorem 4.11). Thus the following result follows readily.

<u>Theorem 4.14</u> $U(L_p) = \mathrm{cl\,con\,ext}\, U(L_p)$, $1 < p < \infty$.

Proof: Let $f \in U(L_p)$, $f \neq 0$. Then $\frac{f}{\|f\|}$ is an extreme point of $U(L_p)$. $\frac{-f}{\|f\|}$ is also an extreme point and

$$f = \frac{1 + \|f\|}{2} \frac{f}{\|f\|} + \frac{1 - \|f\|}{2} \left(\frac{-f}{\|f\|} \right).$$

Thus f is a convex combination of extreme points of $U(L_p)$ and hence $f \in \mathrm{cl\,con\,ext}\, U(L_p)$. Clearly $\mathrm{cl\,con\,ext}\, U(L_p) \subseteq U(L_p)$ (see also Theorem 6.14). Q. E. D.

We now show that $U(L_\infty(S, M, \mu))$ is also the closed convex hull of its extreme points as is the case for L_p, $1 < p < \infty$ and L_1 provided μ is purely atomic.

<u>Theorem 4.15</u> $U(L_\infty) = \mathrm{cl\,con\,ext}\, U(L_\infty)$.

Proof: If is shown in [11, p. 445] that there is a compact Hausdorff space S such that L_∞ and $C(S)$ are isometrically isomorphic.

Thus the result follows from Theorem 4.30. Q.E.D.

Hardy Spaces

The next Banach spaces to be considered are the Hardy spaces. For $1 \le p < \infty$, H_p is the space of all analytic functions in the open unit disk of the complex plane, $|z| < 1$, with

$$\|f\|_p = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})|^p d\theta \right\}^{1/p} < \infty .$$

H_∞ is the space of all bounded analytic functions in $|z| < 1$ with $\|f\| = \sup_{|z|<1} |f(z)|$. Much of the material in this section can be found in Hoffman's book [17], however, we feel that it should be included for completeness.

The characterization of $\operatorname{ext} U(H_p)$ for $1 < p < \infty$, presents no problem since H_p can be considered as a closed subspace of L_p. Hence the extreme points of $U(H_p)$, $1 < p < \infty$, are precisely those functions of norm 1 by Theorem 4.11. Therefore in characterizing the set $\operatorname{ext} U(H_p)$, the spaces H_1 and H_∞ are the only Hardy spaces presenting any difficulty. It might be conjectured that $U(H_1)$ has no extreme points since this is the case in $U(L_1)$, but Theorem 4.20 will show this conjecture to be false. The following will describe the set $\operatorname{ext} U(H_\infty)$.

Theorem 4.16 $f \in \operatorname{ext} U(H_\infty)$ if and only if $|f(z)| \le 1$ for $|z| < 1$ and $\int_0^{2\pi} \log[1 - |f(e^{i\theta})|] d\theta = -\infty$.

Proof: [17, p. 138] Let $|f(z)| \le 1$, $\int_0^{2\pi} \log[1 - |f(e^{i\theta})|] d\theta = -\infty$ and $g \in H_\infty$ with $\|f \pm g\| \le 1$. If $f = u + iv$ and $g = r + it$ then

$|f+g| \leq 1$ implies $(u+r)^2 + (v+t)^2 \leq 1$ and $|f-g| \leq 1$ implies $(u-r)^2 + (v-t)^2 \leq 1$. Adding these two inequalities, we have $u^2 + r^2 + v^2 + t^2 \leq 1$. Thus $|f|^2 + |g|^2 \leq 1$ and $|g(e^{i\theta})|^2 \leq 1 - |f(e^{i\theta})|^2$. It follows that

$$2\int_0^{2\pi} \log|g(e^{i\theta})|\,d\theta \leq \int_0^{2\pi} \log(1+|f(e^{i\theta})|)\,d\theta + \int_0^{2\pi} \log(1-|f(e^{i\theta})|)\,d\theta$$

$$\leq 2\pi\log 2 + \int_0^{2\pi} \log(1-|f(e^{i\theta})|)\,d\theta = -\infty .$$

Since g is analytic in the unit disk it has a Maclaurin's expansion

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = z^m \sum_{n=m}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-m} \equiv z^m h(z)$$

where m is the smallest nonnegative integer such that $g^{(m)}(0) \neq 0$. We have $h \in H_\infty$, $h(0) \neq 0$, and by a simple extension of Jensen's inequality [17, p. 52] $\frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\,d\theta \geq \log|h(0)|$, $0 < r < 1$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\,d\theta = \frac{m}{2\pi}\int_0^{2\pi} \log|re^{i\theta}|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\,d\theta.$$

Taking the limit as r approaches 1 we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|h(e^{i\theta})|\,d\theta \geq \log|h(0)|$$

$$= \log\left|\frac{g^{(m)}(0)}{m!}\right| .$$

The left side of this expression being $-\infty$ implies $g^{(m)}(0) = 0$ for

$m = 0, 1, 2, \ldots$. Hence we have that $g(z) \equiv 0$ for $|z| < 1$ and since $\lim_{r \to 1} g(re^{i\theta})$ exists a.e., $g(e^{i\theta}) = 0$ a.e. Therefore $g = 0$ and $f \in \text{ext}\, U(H_\infty)$.

Now let $f \in \text{ext}\, U(H_\infty)$ and suppose that $\log(1 - |f(e^{i\theta})|)$ is integrable. Define

$$g(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(1 - |f(e^{i\theta})|)\, d\theta\right] .$$

Clearly $g \neq 0$ and $g(z)$ is analytic for $|z| < 1$ by Cauchy's Theorem. It follows from the Poisson integral formula that

$$\begin{aligned} \log|g(e^{it})| &= \lim_{r \to 1} \operatorname{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + re^{it}}{e^{i\theta} - re^{it}} \log(1 - |f(e^{i\theta})|)\, d\theta\right] \\ &= \log(1 - |f(e^{it})|) \end{aligned}$$

(see [12], p. 34). Hence $|g(e^{i\theta})| = 1 - |f(e^{i\theta})|$ which implies g is bounded and therefore $g \in H_\infty$. From the fact that $|g(e^{i\theta})| + |f(e^{i\theta})| = 1$, we have $|g(z)| + |f(z)| \leq 1$ for all z in the unit disk $|z| < 1$. Therefore $\|f \pm g\|_\infty \leq 1$ and since $g \neq 0$, $f \notin \text{ext}\, U(H_\infty)$. This contradiction gives the result, Q.E.D.

It should be mentioned that the result holds for the subspace A of H_∞ of continuous function on the closed unit disk which are analytic in the interior. The first half of the proof is the same as above since $f \in H_\infty$ whenever $f \in A$. In the second half of the proof, the function g must have continuous boundary values. This can be accomplished by setting

$$g(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u\, d\theta\right]$$

where u is continuous on $|z| = 1$, $\log u$ is integrable, $0 \leq u \leq 1 - |f|$, and u is continuously differentiable on each open arc of the set where $|f| \neq 1$. It can then be shown that $|g(e^{i\theta})| + |f(e^{i\theta})| \leq 1$ and hence f is not extreme.

We now turn to H_1, but first we will need some definitions.

Definition 4.17 A function $f \in H_p$ is an outer function if f can be represented as

$$f(z) = e^{i\theta} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})|\,dt\right]$$

where $\int_0^{2\pi} \log|f(e^{it})|\,dt < \infty$.

Definition 4.18 A function $f \in H_p$ is an inner function if $|f(z)| \leq 1$ and $|f(e^{i\theta})| = 1$ a.e.

The following is a known fact from complex variable theory and will be stated without proof.

Lemma 4.19 [17, p. 63] A nonzero function $f \in H_1$ has a unique factorization (up to a constant of modulus 1) of the form $f = M_f Q_f$ where M_f is an inner function, Q_f is an outer function, and $\|Q_f\|_1 = \|f\|_1$.

The next theorem was originally proved by Rudin and de Leeuw in [41].

Theorem 4.20 $f \in \operatorname{ext} U(H_1)$ if and only if $\|f\|_1 = 1$ and f is an outer function.

Proof: [17, p. 139] Assume $\|f\|_1 = 1$ and f is an outer function. Pick $g \in H_1$ such that $\|f \pm g\|_1 = 1$. Define $h = \frac{g}{f}$. (Note that $f(z) \neq 0$ for $|z| < 1$ since f is an outer function. Thus h is analytic in the unit disk.) Define the positive measure $d\mu = |f| d\theta$. Then h is integrable with respect to μ since $\int |h| d\mu = \int \frac{|g|}{|f|} |f| d\theta = \|g\| < \infty$. By our assumption,

$$\int_0^{2\pi} (|f(e^{i\theta}) + g(e^{i\theta})| + |f(e^{i\theta}) - g(e^{i\theta})|) d\theta = 2\int_0^{2\pi} |f(e^{i\theta})| d\theta .$$

By rearranging this expression and introducing our new measure, we have

$$\int_0^{2\pi} (|1 + h(e^{i\theta})| + |1 - h(e^{i\theta})| - 2) d\mu = 0 .$$

Since $f(e^{i\theta})$ does not vanish on any set of positive measure, it follows that $|1 + h(e^{i\theta})| + |1 - h(e^{i\theta})| = 2$ a.e. with respect to μ. Thus $h(e^{i\theta})$ is real and $-1 \leq h(e^{i\theta}) \leq 1$ a.e. In view of the Poisson representation for H_1 functions [12, p. 34], $h(z)$ is real for $|z| < 1$ and hence is constant. Thus $(1 + h) \|f\| = (1 - h) \|f\|$ which implies $h = 0$. Hence $g = 0$ and $f \in \operatorname{ext} U(H_1)$.

Let $f \in \operatorname{ext} U(H_1)$ and suppose f is not an outer function, i.e., $f = M_f Q_f$ where the inner function M_f is not constant. Let $\varphi(\alpha) = \int_0^{2\pi} |f(e^{i\theta})| \operatorname{Re}[e^{i\alpha} M_f(e^{i\theta})] d\theta$. $\varphi(\alpha)$ is a real continuous function and since $\varphi(0) = -\varphi(\pi)$ there exists some β with $0 \leq \beta \leq \pi$ such that $\varphi(\beta) = 0$. Let $u(z) = e^{i\beta} M_f(z)$ and $g(z) = \frac{1}{2} e^{i\beta} Q_f(z)(1 + u^2(z))$. Then $g(z)$ is analytic for $|z| < 1$ and $\|g\| = \frac{1}{2} \|Q_f\| \; \|1 + u^2\| \leq \frac{1}{2}(1 + \|u\|^2) \leq 1$. Therefore $g \in H_1$ and

$g \neq 0$ since M_f is not constant. (Note that $e^{-i\beta}Q_f = \frac{f}{u}$.) Since M_f is an inner function $|u(e^{i\theta})| = 1$ a.e. Whenever $|u(e^{i\theta})| = 1$ we have $2\,\mathrm{Re}\,(u) = u + \bar{u} = u + \frac{1}{u} = \frac{1+u^2}{u^2}$. Therefore $g(e^{i\theta}) = \frac{1}{2} e^{-i\beta} Q_f(e^{i\theta})(1 + u^2(e^{i\theta})) = \frac{1}{2} 2f(e^{i\theta})\,\mathrm{Re}\,[u(e^{i\theta})]$ a.e. on $|z| = 1$. Hence

$$|f(e^{i\theta}) \pm g(e^{i\theta})| = |f(e^{i\theta})|(1 \pm \mathrm{Re}\,[u(e^{i\theta})]) \quad \text{a.e.}$$

and by our choice of β, it follows that $\|f \pm g\| = \|f\| = 1$. Thus $f \notin \mathrm{ext}\, U(H_1)$. This contradiction implies that f is an outer function. Q. E. D.

For the space H_1 we can answer affirmatively the question of whether $U(H_1)$ is the closed convex hull of its extreme points. In fact we have the following stronger result.

<u>Theorem 4.21</u> Let $f \in U(H_1)$.

(i) If $\|f\|_1 = 1$ and f is not an extreme point of $U(H_1)$ then $f = \frac{1}{2}(f_1 + f_2)$ where f_1 and f_2 are distinct extreme points of $U(H_1)$.

(ii) If $\|f\|_1 < 1$ then f is a convex combination of two extreme points of $U(H_1)$.

Proof: We shall present only a sketch of the proof. For details see [17, p. 141-142]. For (i) we construct g as in Theorem 4.20 and define $f_1 = f + g$, $f_2 = f - g$. Then $\|f_1\|_1 = \|f_2\|_1 = 1$ and $f = \frac{1}{2}(f_1 + f_2)$. Once f_1 and f_2 are shown to be outer functions then the proof is complete.

For part (ii) suppose $0 < \|f\| < 1$ ($f = 0$ is trivially the mid-point of two extreme points). If f is outer then f is a convex combination of the two extreme points $\frac{f}{\|f\|}$ and $\frac{-f}{\|f\|}$. If f is not outer, construct g as in Theorem 4.20 and then choose $t_1 > 1$ and $t_2 > 1$ such that $\|f+t_1 g\|_1 = \|f-t_2 g\|_1 = 1$. Once $f+t_1 g$ and $f-t_2 g$ are shown to be outer functions, the proof of (ii) is complete since f lies on the segment joining these two functions. Q.E.D.

That $U(H_p) = \mathrm{cl\,con\,ext}\, U(H_1)$, $1 < p < \infty$, is a consequence of the fact that every element of the boundary of $U(H_p)$ is an extreme point (see proof of Theorem 4.14). The proof of the result for $U(H_\infty)$ is quite long and hence we again only sketch the proof. For the details, see [33]. The following definitions will be needed for the next theorem. A subset A of $C(S)$ is called a <u>function algebra</u> of $C(S)$ if A is a linear subspace and multiplication of functions is closed (multiplication is pointwise). We say A is a <u>logmodular algebra</u> if $\{\log|f| : f \in A \text{ and } \frac{1}{f} \in A\}$ is dense in $C_R(S)$. Denote by $M(A)$ the maximal ideal space of A, i.e., the set of all multiplicative functionals on A (see Theorem 5.6). A <u>part</u> of $M(A)$ is an equivalence class defined by the equivalence relation $\sim$, $\mu_1 \sim \mu_2$ if $\|\mu_1 - \mu_2\| < 2$ (the norm is the one for A^*). Recall that a subset P of A^* is <u>total</u> over A if for $f \in A$, $f \neq 0$, there if $F \in A^*$ such that $F(f) \neq 0$. We now state the theorem.

<u>Theorem 4.22</u> Let A be a logmodular algebra in $C(S)$ with maximal ideal space $M(A)$ and suppose that there is a part P of $M(A)$ which is total over A. Then $U(A)$ is the closed convex hull of its exposed points (see Definition 6.1).

Proof: [33] Recall that every element $\varphi \in P$ may be represented by a measure. Let $\varphi \in P$ with representing measure μ and suppose

$f \in U(A)$ such that $Q = \{x : |f(x)| = 1\}$ has positive μ measure. Then f is an exposed point of $U(A)$. Define $F \in A^*$ by $F(g) = \frac{1}{\mu(Q)} \int_Q g\bar{f}\, d\mu$, $g \in A$. Then $F(f) = 1 = \|F\|$. It can be shown that if $g \in A$ and $F(g) = 1 = \|g\|$ then $g = f$ and hence f is an exposed point.

Next it can be shown that the set of all linear functions $F \in A^*$ such that $1 = \|F\| = F(f)$, where f is an exposed point of $U(A)$, is norm-dense in the boundary of $U(A^*)$. Thus intuitively speaking, we can see that $U(A)$ is the intersection of all closed half-spaces which support $U(A)$ at an exposed point. Thus $U(A)$ is the closed convex hull of its exposed points. Q. E. D.

Corollary 4.23 $U(H_\infty) = \text{cl con ext}\, U(H_\infty)$.

Proof: This follows from the preceding theorem since it can be shown that H_∞ is isometrically isomorphic to a logmodular algebra. The maximal ideal space of H_∞ is complicated but it is known that the open unit disk is a total part of $M(A)$. Q. E. D.

Lipschitz Spaces

We now consider the space $\text{Lip}(S, d^\alpha)$ of complex-valued functions on a compact metric space S with metric d which satisfy a Lipschitz condition.

Definition 4.24 $f \in \text{Lip}(S, d^\alpha)$ if there exists a constant $K > 0$ such that $|f(x) - f(y)| \leq K d^\alpha(x, y)$ for all $x, y \in S$, $0 < \alpha \leq 1$.

It should be noted that if $f \in \text{Lip}(S, d^\alpha)$ then f is continuous and hence bounded. Also it can be shown that $\text{Lip}(S, d^\alpha)$ is a vector space. Define $\|f\|_{d^\alpha} = \sup\{|f(x) - f(y)| d^{-\alpha}(x,y) : x \neq y\}$. Then $\|f\| = \max(\|f\|_\infty, \|f\|_{d^\alpha})$ is a norm on the space. (Recall that $\|f\|_\infty$ is the sup norm on S.) $\mathrm{lip}(S, d^\alpha)$ is the closed linear subspace of $\mathrm{Lip}(S, d^\alpha)$ containing those functions f such that $|f(x) - f(y)| d^{-\alpha}(x,y) \to 0$ as $d^\alpha(x,y) \to 0$.

In general not much is known about the extreme points in $\mathrm{Lip}(S, d^\alpha)$ for arbitrary S and d. Throughout this section we use the term "a.e." to mean almost everywhere with respect to Lebesgue measure. The following is a characterization of $\mathrm{ext}\, U(\mathrm{Lip}[0,1])$ where $\mathrm{Lip}[0,1]$ is $\mathrm{Lip}(S, d^\alpha)$ with $S = [0,1]$, d the usual metric, and $\alpha = 1$. It is due to A. K. Roy [39].

<u>Theorem 4.25</u> If f is not of modulus one everywhere on $[0,1]$ then $f \in \mathrm{ext}\, U(\mathrm{Lip}[0,1])$ if and only if $|f'| = 1$ a.e. on $[0,1] \sim M_f$, where $M_f = \{x \in [0,1] : |f(x)| = \|f\|_\infty\}$.

Proof: If $|f| = 1$ everywhere then it follows that $f \in \mathrm{ext}\, U(\mathrm{Lip}[0,1])$ (see Theorem 4.27). Note that if $f \in \mathrm{ext}\, U(\mathrm{Lip}[0,1])$ then it is necessary that $\|f\|_\infty = 1$; for suppose $\|f\|_\infty = \alpha < 1$. Then $g = f + \frac{1}{2}(1-\alpha)$ and $h = f - \frac{1}{2}(1-\alpha)$ are such that $\|g\|_\infty \leq \|f\|_\infty + (1-\alpha) \leq 1$ and similarly $\|h\|_\infty \leq 1$. Thus $g, h \in U(\mathrm{Lip}[0,1])$ since $\|g\|_{d^\alpha} = \|h\|_{d^\alpha} = \|f\|_{d^\alpha} \leq 1$. We then have $f = \frac{1}{2}(g+h)$, $f \neq g$, and hence $f \notin \mathrm{ext}\, U(\mathrm{Lip}[0,1])$.

Suppose $|f'| < 1$ on a set F of positive measure where $F \subseteq [0,1] \sim M_f$. $[0,1] \sim M_f$ is open; therefore we may assume

(i) F to be compact since otherwise there exists F_1 closed, $F_1 \subseteq F$, whose measure is arbitrarily close to the measure of F;

(ii) $F \subseteq I$ for some closed interval $I \subseteq [0, 1] \sim M_f$;

(iii) $\operatorname{ess\,sup}_{x \in F} |f'(x)| = \alpha < 1$, since if we define $A_n = \{x \in [0, 1] : |f'(x)| < 1 - \frac{1}{n}\}$ then the measure of some A_n is greater than 0. (For if not, the measure of F is 0.); and

(iv) for some $\varepsilon > 0$, $|f(x)| \leq 1 - \varepsilon$ for all $x \in I$ since f is continuous on the closed set I.

Again we let χ_F denote the characteristic function of F. Then the function $g(x) = \int_0^x \chi_F(t)\,dt$ is a continuous function on $[0, 1]$ and by the intermediate value theorem there exists $x_0 \in (0, 1)$ such that $\int_0^{x_0} \chi_F(t)\,dt = \frac{1}{2}\int_0^1 \chi_F(t)\,dt = \frac{1}{2} m(F)$. Define

$$f_0(x) = [\chi_F(x)\, \chi_{[0, x_0]}(x)] - [\chi_F(x)\, \chi_{(x_0, 1]}(x)]$$

and $g_0(x) = \int_0^x f_0(t)\,dt$. Figure 4.1 will help illustrate the functions g, f_0, and g_0 (for real-valued functions).

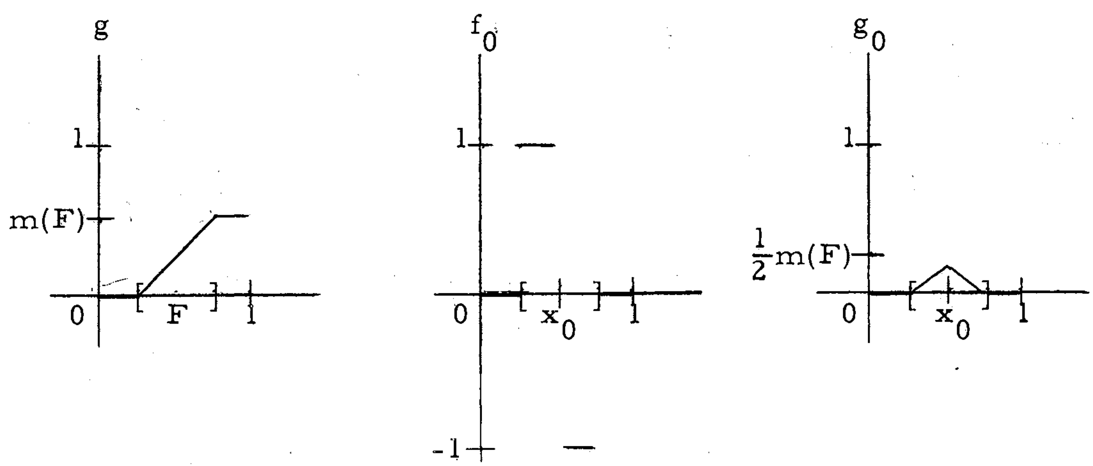

Figure 4.1. Functions in Proof of Theorem 4.25

Since g_0 is absolutely continuous and $|g_0'|$ is bounded we have $g_0 \in \mathrm{Lip}[0, 1]$ (see [, p. 108]). For $\delta > 0$ small enough it follows that $\|f \pm \delta g_0\|_\infty \leq \|f\|_\infty + \delta \|g_0\|_\infty \leq 1$ and $\|f \pm \delta g_0\|_{d^\alpha} \leq \|f\|_{d^\alpha} + \delta \|g_0\|_{d^\alpha} \leq 1$. Hence $\|f \pm \delta g_0\| \leq 1$, $f = \frac{1}{2}(f + \delta g_0) + \frac{1}{2}(f - \delta g_0)$, and $f \neq f + \delta g_0$. Thus $f \notin \mathrm{ext}\, U(\mathrm{Lip}[0, 1])$ which proves the condition is necessary.

Now let $|f'| = 1$ a.e. on $[0, 1] \sim M_f$ and suppose $f = \frac{1}{2}(g + h)$ for some $g, h \in U(\mathrm{Lip}[0, 1])$. $\|f\|_\infty = 1$ implies that $g = h = f$ on M_f. Also $f' = \frac{1}{2}(g' + h')$ a.e. Hence $g' = h' = f'$ a.e. on $[0, 1] \sim M_f$ since $|f'| = 1$, $|g'| \leq 1$ and $|h'| \leq 1$ a.e. If $x \notin M_f$ let y be the closest point of M_f to x. Assume $y < x$. Then $g' = h'$ a.e. on (y, x) and therefore $\int_y^x g'\,dm = \int_y^x h'\,dm$. Thus $g(x) - g(y) = h(x) - h(y)$, and since $y \in M_f$ it follows that $g(x) = h(x)$ for all $x \in [0, 1] \sim M_f$. Hence $g = h = f$ from which we conclude that $f \in \mathrm{ext}\, U(\mathrm{Lip}[0, 1])$. Q.E.D.

Figure 4.2 illustrates the graphs of some extreme functions in the subspace $\mathrm{Lip}_R[0, 1]$ of real-valued functions. Note that $\sup_x |f(x)|$ must be one.

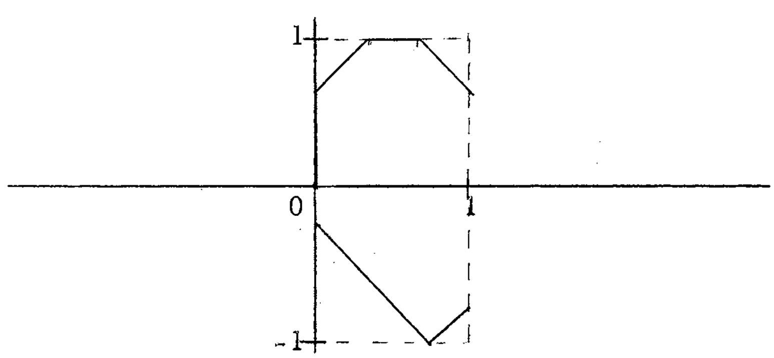

Figure 4.2. Extreme Lipschitz Functions

The proof of the result concerning $\mathrm{cl\,con\,ext}U(\mathrm{Lip}[0,1])$ is quite long and involved. We shall again give only a rough sketch of the proof.

Theorem 4.26 The unit ball of $\mathrm{Lip}[0,1]$ is the closed convex hull of its extreme points.

Proof: [39] Consider $C[0,1] \oplus L_\infty[0,1]$ with norm $\|(f,g)\| = \max(\|f\|_\infty, \|g\|_\infty)$. Define the map: $\mathrm{Lip}[0,1] \to C[0,1] \oplus L_\infty[0,1]$ given by $f \to (f, f')$. This is a linear and isometric map and the image of $\mathrm{Lip}[0,1]$, say A, is a closed subspace of $C[0,1] \oplus L_\infty[0,1]$. Essentially what needs to be shown now is that a dense subset of the boundary of $U(A^*)$ attain a norm of 1 at some extreme point of $U(A)$. This, in fact, is the difficult part of the proof. Once this is shown it will follow that $U(A) = \mathrm{cl\,con\,ext}\,U(A)$ (see [33]). Thus we have the conclusion of the theorem. Q. E. D.

The Spaces C(S)

Let S be a compact Hausdorff space. Then $C(S)$ denotes the space of continuous complex-valued functions on S with $\|f\| = \sup\{|f(x)| : x \in S\}$. The following characterization of $\mathrm{ext}\,U(C(S))$ is not surprising if we consider the graphs of the real-valued functions as was done in Figure 4.2 in the previous section.

Theorem 4.27 $f \in \mathrm{ext}\,U(C(S))$ if and only if $|f(x)| = 1$ for all $x \in S$.

Proof: Let $f \in U(C(S))$ and suppose that $|f(x)| < 1$ on some nonempty subset of S. Define $g = f + \frac{1}{2}(1 - |f|)$ and $h = f - \frac{1}{2}(1 - |f|)$. $g, h \in C(S)$ and $|g(x)| \leq |f(x)| + \frac{1}{2}(1 - |f(x)|) = \frac{1}{2}(1 + |f(x)|) \leq 1$

for all $x \in S$. Similarly $|h(x)| \leq 1$ for all $x \in S$. Therefore $g, h \in U(C(S))$, $f = \frac{1}{2}(g+h)$, and $f \neq g$. This implies that $f \notin \operatorname{ext} U(C(S))$ which proves that the condition is necessary.

Suppose $|f(x)| = 1$ for all $x \in S$ and let $g \in C(S)$ be such that $|f(x) \pm g(x)| \leq 1$ for all $x \in S$. Lemma 2.6 implies that $g = 0$. Hence $f \in \operatorname{ext} U(C(S))$. Q.E.D.

Let $C_R(S)$ denote the subspace of $C(S)$ of real-valued functions on S. We see from the above theorem that the set $\operatorname{ext} U(C_R[0,1])$ contains only two points.

<u>Corollary 4.28</u> $\operatorname{ext} U(C_R[0,1])$ consists of the constant functions 1 and -1.

<u>Corollary 4.29</u> $C_R[0,1]$ is not isometrically isomorphic to a dual space.

Proof: The closed convex hull of the extreme points 1 and -1 is the set of constant functions f such that $|f| \leq 1$. Thus $\operatorname{cl\,con\,ext} U(C_R[0,1]) \neq U(C_R[0,1])$. If $C_R[0,1]$ were a dual space, then its unit ball would be w*-compact. By the Krein-Milman Theorem $U(C_R[0,1])$ would be the closed convex hull of its extreme points. Q.E.D.

We conclude this section with the results concerning the question of whether $U(X) = \operatorname{cl\,con\,ext} U(X)$ for the spaces $C(S)$ and $C_R(S)$. Let μ be a nonnegative Baire measure on S. Then the <u>support</u> of μ is the complement of the union of all open sets G such that $\mu(G) = 0$. Note the support of μ is closed. We can now state the following result.

Theorem 4.30 The unit ball of $C(S)$ is the closed convex hull of its extreme points.

Proof: [33] Once again we will not give all the details to the proof. Suppose $U(C(S))$ is not the closed convex hull of $\operatorname{ext} U(C(S))$. Then there is a nonnegative Baire measure μ on S with $\mu(S) = 1$ and a function $f \in U(C(S))$ with $|f| = 1$ on the support Q of μ such that

$$\sup \operatorname{Re} \int g \bar{f} d\mu < 1, \quad g \in \operatorname{ext} U(C(S)).$$

Thus to prove the theorem we need only to show that for each $\varepsilon > 0$ there exists $g \in \operatorname{ext} U(C(S))$ such that $\operatorname{Re} \int g \bar{f} d\mu > 1 - \varepsilon$.

Partition the unit circle of the complex plane into N equal half-open arcs A_k, where $N \varepsilon > 2$. Since

$$1 = \mu(Q) = \sum_{k=1}^{N} \mu\, [f^{-1}(A_k) \cap Q],$$

we have for at least one of these arcs, say A_1, the subset $f^{-1}(A_1) \cap Q$ of Q must have measure less than $\frac{\varepsilon}{2}$. The same is true for $\operatorname{int} A_1$ (relative to the circle), so let
$Q_1 = \{x : x \in Q \text{ and } f(x) \notin \operatorname{int} A_1\}$. It follows that $\mu Q_1 > 1 - \frac{\varepsilon}{2}$ and that Q_1 is a compact subset of Q. $f = e^{i\varphi}$ where φ is a real-valued continuous function on Q_1. We can extend φ to a real-valued continuous function θ on S. Let $g = e^{i\theta}$. Then by Theorem 4.27, $g \in \operatorname{ext} U(C(S))$ and

$$\operatorname{Re} \int g \bar{f} d\mu = \operatorname{Re} \int_{Q \sim Q_1} e^{i\theta} \bar{f} d\mu + \operatorname{Re} \int_{Q_1} e^{i\theta} \bar{f} d\mu$$

$$\geq -\mu(Q \sim Q_1) + \mu(Q_1) > -\frac{\varepsilon}{2} + 1 - \frac{\varepsilon}{2} = 1 - \varepsilon.$$

Thus g is the function that is needed to complete the proof. Q. E. D.

We have noted (Corollary 4.29) that $U(C_R[0,1]) \neq \mathrm{cl\,con\,ext}\, U(C_R[0,1])$. Thus for real-valued functions on S we must place some restrictions on the compact Hausdorff space S.

<u>Theorem 4.31</u> $U(C_R(S)) = \mathrm{cl\,con\,ext}\, U(C_R(S))$ if and only if S is totally disconnected, i.e., S has a base consisting of sets which are simultaneously open and closed.

Proof: (see [2]; also [14]).

CHAPTER V

CHARACTERIZATIONS OF $\operatorname{ext} U(X^*)$

We now want to characterize the sets of extreme points of the unit balls of the duals of the Banach spaces considered in Chapter IV. Recall that the dual X^* of a Banach space X is the set of continuous linear functionals on X with $\|f\| = \sup\{|f(x)| : x \in X,\ \|x\| \leq 1\}$.

It is very useful if we are able to represent the dual of a Banach as some other known space; i.e., find an isometric isomorphism between the two spaces. For example $C(S)^*$ is represented by the space of regular countably additive measures on the σ-ring of Borel sets in S (see [11], p. 265); i.e., for each $F \in C(S)^*$ there corresponds a measure μ such that

$$F(f) = \int_S f(t)\, d\mu, \quad f \in C(S).$$

Furthermore $\|F\|$ is equal to the total variation of μ. Another example is that l_p^*, $1 < p < \infty$, can be represented by l_q, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus for each $f \in l_p^*$ there is an isometric isomorphism which identifies an element $t = (t_1, t_2, \ldots) \in l_q$ with f such that

$$f(a) = \sum_{i=1}^{\infty} t_i a_i \quad \text{for} \quad a = (a_1, a_2, \ldots) \in l_p.$$

Duals of the Sequences Spaces

We begin with the duals of the sequence spaces. Since $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$ we have the following two theorems immediately.

Theorem 5.1 $\text{ext}\,U(c_0^*) = \{\lambda\delta_j : j = 1, 2, \ldots \text{ with } |\lambda| = 1\}$.

Proof: (see Theorem 4.2)

Theorem 5.2 $\text{ext}\,U(\ell_1^*) = \{x \in \ell_\infty : |x_n| = 1,\ n = 1, 2, \ldots\}$.

Proof: (see Theorem 4.3).

We shall consider ℓ_∞^* as a special case of $L_\infty^*(S, M, \mu)$ and therefore will delay the discussion of $\text{ext}\,U(\ell_\infty^*)$.

Duals of the L_p Spaces

If (S, M, μ) is a positive σ-finite measure space, then there is an isometric isomorphism between $L_1^*(S, M, \mu)$ and $L_\infty(S, M, \mu)$. The isomorphism is $F \to g$ where $F(f) = \int_S g(s) f(s)\, d\mu$ for every $f \in L_1(S, M, \mu)$ (see [11], p. 289). We have previously characterized $\text{ext}\,U(L_\infty)$ but state the result for the sake of completeness.

Theorem 5.3 Let (S, M, μ) be a positive σ-finite measure space. Then $F \in \text{ext}\,U(L_1^*(S, M, \mu))$ if and only if $F(f) = \int g f\, d\mu$ for all $f \in L_1$ where $g \in L_\infty$ with $|g| = 1$ a.e.

Proof: (see Theorem 4.12).

For $1 < p < \infty$, $L_p^*(S, M, \mu) = L_q(S, M, \mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Thus the following theorem is a consequence of an earlier result.

Theorem 5.4 For $1 < p < \infty$, $F \in \text{ext}\, U(L_p^*(S, M, \mu))$ if and only if $\|F\| = 1$.

Proof: (see Theorem 4.11).

We now want to give a necessary condition for an element of $L_\infty^*(S, M, \mu)$ to be an extreme point of the closed unit ball. The problem is not as trivial as in the previous spaces since L_∞^* is an L_1 space for some measure space (S, M, μ). In general little is known about this measure space, but Theorem 5.6 gives a partial description of the extreme points. We need some preliminary remarks before stating the result.

Let (S, M, μ) be a positive σ-finite measure space. Let M_1 be the completion of M; i.e., M_1 contains all sets B such that $B \subseteq A$ for some $A \in M$ with $\mu(A) = 0$. Let μ_1 be the extension of μ to M_1 (see [40], p. 221). Then (S, M_1, μ_1) is a complete, positive, σ-finite measure space. Denote by $ba(S, M_1, \mu_1)$ the space of bounded additive functions on M_1 which vanish on sets of μ_1-measure zero. The norm of an element λ in $ba(S, M_1, \mu_1)$ is its total variation ($\|\lambda\| = \sup\{|\lambda(A)| : A \in M_1\}$). $\nu(\lambda, A)$ denotes the total variation of λ on A; $\nu(\lambda, A) = \sup\{|\lambda(B)| : B \in M_1,\ B \subseteq A\}$.

The following result is needed but the proof is not within the scope of this paper and hence will not be given.

Lemma 5.5 There is an isometric isomorphism between $L_\infty^*(S, M, \mu)$ and $ba(S, M_1, \mu_1)$ determined by the identity

$$F(f) = \int_S f(s)\, d\lambda, \quad f \in L_\infty(S, M, \mu).$$

Proof: (see [11], p. 296).

<u>Theorem 5.6</u> Let (S, M, μ) be a positive σ-finite measure space. $F \in \operatorname{ext} U(L_\infty^*(S, M, \mu))$ only if $F = \alpha G$, where $|\alpha| = 1$ and G is nonzero and multiplicative; i.e., $G(fh) = G(f)\,G(g)$ for all $f, g \in L_\infty(S, M, \mu)$.

Proof: [11, p. 443] By Lemma 5.5 there is a λ in $ba(S, M_1, \mu_1)$ with $\|\lambda\| = 1$ and

$$F(f) = \int_S f(y)\,d\lambda\,, \quad f \in L_\infty(S, M, \mu)\,.$$

We want to show that λ vanishes on at least one of every pair of disjoint sets in M_1 so that we may define a characteristic function later in the proof. Suppose there are disjoint sets P_1 and P_2 in M_1 with $\lambda(P_1) \neq 0$ and $\lambda(P_2) \neq 0$. Let $\lambda_1(P) = \lambda(P \cap P_1)$ and $\lambda_2(P) = \lambda(P \cap (S \sim P_1))$ for $P \in M_1$. It follows that $\lambda_1, \lambda_2 \in ba(S, M_1, \mu_1)$, $\nu(\lambda_1, P) = \nu(\lambda, P \cap P_1)$, and $\nu(\lambda_2, P) = \nu(\lambda, P \cap (S \sim P_1))$ for $P \in M_1$. Since total variation is additive we have $1 = \|\lambda\| = \|\lambda_1\| + \|\lambda_2\|$. Since $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ we may define $\nu_1 = \dfrac{\lambda_1}{\|\lambda_1\|}$ and $\nu_2 = \dfrac{\lambda_2}{\|\lambda_2\|}$. Thus $\nu_1, \nu_2 \in U(L_\infty^*)$ by the isometry between $L_\infty^*(S, M, \mu)$ and $ba(S, M_1, \mu_1)$, and also $\lambda = \|\lambda_1\|\nu_1 + (1 - \|\lambda_1\|)\nu_2$. Since λ is an extreme point, we have $\lambda = \nu_1 = \nu_2$ and thus $0 \neq \lambda(P_1) = \nu_2(P_2) = 0$. This contradiction gives us the desired result.

For $P \in M_1$ we have $\lambda(P)(\lambda(S) - \lambda(P)) = 0$. Hence the function $m = \dfrac{\lambda}{\lambda(S)}$ assumes only the values 0 and 1. Thus

(1) $\qquad m(A \cap B) = m(A)\,m(B) \quad$ for all $A, B \in M$.

For if either $m(A) = 0$ or $m(B) = 0$ then $m(A \cap B) = 0$. If $m(A) = m(B) = 1$ then either $m(A \sim (A \cap B))$ or $m(B \sim (A \cap B))$ is zero (since $A \sim (A \cap B)$ and $B \sim (A \cap B)$ are disjoint). Hence $m(A \cap B) = 1$.

Let G be defined by $G(f) = \int_S f(y)\,dm$ for $f \in L_\infty$. Then $\|G\| = \|m\| = 1$ and $G = \alpha F$ where $|\alpha| = \frac{1}{|\lambda(S)|} = 1$. Let $f = \chi_A$ and $g = \chi_B$ for $A, B \in M_1$. Then

$$G(fg) = \int_S f(y)\,g(y)dm = \int_{A \cap B} f(y)\,g(y)dm = \int_A f(y)dm \int_B g(y)dm .$$

The last equality follows from (1). Thus $G(fg) = G(f)\,G(g)$ where f and g are characteristic functions on sets in M_1. For $g \in L_\infty$ define

$$T_g = \{f \in L_\infty : G(tg) = G(f)\,G(g)\} .$$

It is clear that T_g is a closed linear subspace of L_∞. By the preceding remarks it follows that $T_g = L_\infty$ if g is a characteristic function. (It is known that the characteristic functions form a fundamental set in L_∞.) Thus if f is an arbitrary function in L_∞ then the linear subspace T_f contains all characteristic functions and hence $T_f = L_\infty$. Thus $G(fg) = G(f)\,G(g)$ for every $f, g \in L_\infty$. Q. E. D.

Note that the function λ has an "atomic type" property mentioned in Definition 4.5. This is to be expected since as we have pointed out L_∞^* is an L_1 space for some measure space and the extreme points of $U(L_1)$ were of the form $f = \lambda \frac{1}{\mu(A)} \chi_A$ where A was an atom and $|\lambda| = 1$.

We now have the following corollary.

Corollary 5.7 $F \in \operatorname{ext} U(\ell_\infty^*)$ only if $F = \alpha G$ where $|\alpha| = 1$ and G is nonzero and multiplicative.

Proof: This follows from the previous theorem since the positive integers with the counting measure on all subsets is a σ-finite measure space. Q. E. D.

Duals of the H_p Spaces

It appears that the characterization of $\operatorname{ext} U(H_p^*)$ is still an open question. It is known that for a Banach space X and any closed subspace Y that Y^* is isometrically isomorphic to $\frac{X^*}{Y^\perp}$ where $Y^\perp$ denotes the set of all elements $F \in X^*$ such that $F(y) = 0$ for every $y \in Y$. $Y^\perp$ is called the annihilator of Y. Thus, since H_p is a closed subspace of L_p, $1 \le p < \infty$, the dual of H_p can be described if the annihilator of H_p in L_p^* can be determined. It has been shown that the annihilator of H_p is isometrically isomorphic to H_q where $\frac{1}{p} + \frac{1}{q} = 1$ (see [12, p. 113]). Therefore $H_p^* = \frac{L_q}{H_q}$. Since there is no concrete description of H_p^*, any attempt to identify the extreme points of its unit ball would be rather artificial, hence we will not endeavor to do so.

Much has been written about the maximal ideal space M of H_∞; i. e., the multiplicative linear functionals on H_∞. A famous conjecture which was unsolved until recently is the following: Is the unit disk (when embedded in M) dense in M? Carleson [5] has shown that the answer is affirmative. For more details concerning the "corona theorem" see [12].

It is known that H_∞^* is a proper subset of H_1. No concrete description of H_∞^* has been found and hence the identification of the extreme points of $U(H_\infty^*)$ seems to be an open question.

Duals of the Lipschitz Spaces

We now want to identify the extreme points of the unit ball of the dual of the Lipschitz spaces. This has been done in a more general setting than for the spaces themselves. Some preliminary remarks are needed before we state the main result.

Definition 5.8 Let F be a closed linear subspace of $C(S)$. An evaluation functional on F, denoted by φ_x, is defined by $\varphi_x(f) = f(x)$ where $x \in S$ and $f \in F$.

We will use the following notation: S is a compact metric space with metric d.

$$W = \{(x, y) : x \neq y \quad \text{and} \quad x, y \in S\} .$$

βW is the Stone-Cech compactification of W (see [11, p. 276]).

$$\phi_S = \{\lambda\varphi_x : x \in S, \ |\lambda| = 1\}$$

$$\phi_W = \{\lambda \varphi_w : w = (x, y) \in W, \ 0 < d^\alpha(x, y) < 2, \ |\lambda| = 1\}$$

$$D = \{\lambda \varphi_w : w \in \beta W \sim W, \ |\lambda| = 1\}$$

For $f \in \mathrm{Lip}(S, d^\alpha)$ define $\tilde{f}$ on $S \cup W$ by

$$\tilde{f}(x) = f(x) \qquad , \quad x \in S$$

$$\tilde{f}(w) = \tilde{f}(x, y) = \frac{f(x) - f(y)}{d^\alpha(x, y)} \quad , \quad w = (x, y) \in W$$

Since $\tilde{f}$ is bounded and continuous on W, it has a unique extension $\hat{f} \in C(\beta W)$ with $\|\tilde{f}\| = \|\hat{f}\|$. If we define $\hat{f}(x) = f(x)$, $x \in S$, then $\hat{f}$ is a continuous extension defined on $S \cup \beta W$. For every $f \in \mathrm{Lip}(S, d^{\alpha})$, let $jf = \hat{f}$. j is clearly linear and is also an isometric map since for every $f \in \mathrm{Lip}(S, d^{\alpha})$, we have

$$\begin{aligned}
\|jf\|_{S \cup \beta W} &= \max\left(\sup_{x \in S} |\hat{f}(x)|,\ \sup_{w \in \beta W} |\hat{f}(w)|\right) \\
&= \max\left(\sup_{x \in S} |\hat{f}(x)|,\ \sup_{w \in W} |\hat{f}(w)|\right) \\
&= \max\left(\|f\|, \|f\|_{d^{\alpha}}\right) \\
&= \|f\| .
\end{aligned}$$

<u>Definition 5.9</u> Let F be a closed linear subspace of $C(S)$ and $x \in S$. We say a function f in F <u>peaks</u> at x relative to F if $f(x) = 1 \geq |f(y)|$ with equality holding only for those y in S that satisfy either

$$\begin{aligned}
g(y) &= g(x) \qquad \text{for all} \quad g \in F \quad \text{or} \\
g(y) &= -g(x) \qquad \text{for all} \quad g \in F .
\end{aligned}$$

The next lemma, due to de Leeuw, helps us to identify the extreme points of $U(F^*)$ in terms of the peaking functions relative to F where F is again a closed linear subspace of $C(S)$.

<u>Lemma 5.10</u> Let $x \in S$. If F contains a function f which peaks at x relative to F, then $\varphi_x \in \mathrm{ext}\, U(F^*)$.

Proof: [9] $\varphi_x \in U(F^*)$ since

$$\|\varphi_x\| = \sup\{|\varphi_x(f)| : \|f\| \leq 1\} = \sup\{|f(x)| : \|f\| \leq 1\} \leq 1 .$$

Suppose $\varphi_x = \frac{1}{2}(\gamma_1 + \gamma_2)$ where $\gamma_1, \gamma_2 \in U(F^*)$. γ_1, γ_2 are bounded linear functionals on $C(S)$ and therefore by the Riesz representation theorem (see [40], p. 310) there are unique finite signed Baire measures μ_1 and μ_2 such that

$$\gamma_1(g) = \int_S g\, d\mu_1 \quad \text{and}$$

$$\gamma_2(g) = \int_S g\, d\mu_2 \quad \text{for all } g \in F.$$

Also the total variation of μ_i is equal $\|\gamma_i\| \leq 1$, $i = 1, 2$.

Let f be a function in F which peaks at x relative to F. Then

$$\left|\int_S f\, d\mu_1\right| \leq \sup\{|f(y)| : y \in S\} \leq 1.$$

Similarly $\left|\int_S f\, d\mu_2\right| \leq 1$. Thus

$$1 = f(x) = \varphi_x(f) = \frac{1}{2}(\gamma_1(f) + \gamma_2(f)) = \frac{1}{2}\left[\int_S f\, d\mu_1 + \int_S f\, d\mu_2\right].$$

Thus it follows that $\int_S f\, d\mu_1 = \int_S f\, d\mu_2 = 1$. Define

$$Y_+ = \{y : f(y) = 1\} = \{y : g(y) = g(x) \text{ for every } g \in F\}$$

$$Y_- = \{y : f(y) = -1\} = \{y : g(y) = -g(x) \text{ for every } g \in F\}$$

$$Y_0 = \{y : |f(y)| < 1\} = \{y : y \notin Y_+ \text{ and } y \notin Y_-\}.$$

Hence $1 = \int_S f\, d\mu_i = \mu_i(Y_+) - \mu_i(Y_-) + \int_{Y_0} f\, d\mu_u$. The function

$$f_1(x) = \begin{cases} f(x) , & x \in Y_+ \cup Y_- \\ \frac{1}{2} f(x) , & x \in Y_0 \end{cases} \quad \text{also peaks at } x .$$

Since $\int_{Y_0} \frac{1}{2} f\, d\mu_i \neq \int_{Y_0} f\, d\mu_i$ unless $\mu_i(Y_0) = 0$, it follows that $1 = \mu_i(Y_+) - \mu_i(Y_-)$. Therefore

$$\begin{aligned} \gamma_i(g) &= \int_S g\, d\mu_i = \int_{Y_+} g\, d\mu_i + \int_{Y_-} g\, d\mu_i + \int_{Y_0} g\, d\mu_i \\ &= g(x)\,[\mu_i(Y_+) - \mu_i(Y_-)] \\ &= g(x) = \varphi_x(g) \quad \text{for all } g \in F . \end{aligned}$$

Hence $\varphi_x = \gamma_1 = \gamma_2$ which implies $\varphi_x \in \operatorname{ext} U(F^*)$. Q. E. D.

The next two lemmas will identify the peaking functions in $j(\mathrm{Lip}\,(S, d^\alpha))$ which, as we recall, is the image under j of $\mathrm{Lip}\,(S, d^\alpha)$ in $C(S \cup \beta W)$. We then will use Lemma 5.10 to identify the extreme points of $U(\mathrm{Lip}\,(S, d^\alpha))$ and $U(\mathrm{lip}\,(S, d^\alpha))$.

<u>Lemma 5.11</u> For each point $x_0 \in S$ there is a function $f \in \mathrm{lip}\,(S, d^\alpha)(0 < \alpha < 1)$ such that f peaks at x_0 relative to $j(\mathrm{Lip}\,(S, d^\alpha))$ and hence relative to $j(\mathrm{lip}\,(S, d^\alpha))$.

Proof: [18] Let x_0 be a fixed point of S and define $g(x) = Kd(x, x_0)$ where $K > 0$. For $x \neq y$ we have

$$\frac{K\,|d(x, x_0) - d(y, x_0)|}{d^\alpha(x, y)} = \frac{K\,|d(x, x_0) - d(y, x_0)|}{d(x, y)}\, d^{1-\alpha}(x, y) \leq Kd^{1-\alpha}(x, y) .$$

Clearly g is continuous and since S is compact $d^{1-\alpha}(x, y)$ is bounded. Thus $g \in \mathrm{Lip}\,(S, d^\alpha)$. Letting $d(x, y) \to 0$ we see that

$g \in \text{lip}(S, d^{\alpha})$. By choosing K small enough, $\|g\|_{\infty} \leq 1$ and $\|g\|_{d^{\alpha}} < 1$. Also g is a nonnegative real-valued function vanishing only at x_0. Let $f = 1 - g$. Then

(i) $0 \leq f \leq 1$,

(ii) $f(x) = 1$ if and only if $x = x_0$, and

(iii) $$\|f\|_{d^{\alpha}} = \sup \left\{\frac{|1 - g(x) - 1 + g(y)|}{d(x, y)} : x \neq y\right\}$$
$$\leq \sup\{K d^{1-\alpha}(x, y) : x \neq y\} < 1 .$$

Hence $\hat{f}$ peaks at x_0 relative to $j(\text{Lip}(S, d^{\alpha}))$. Since $f \in \text{lip}(S, d^{\alpha})$ then f also peaks at x_0 relative to $j(\text{lip}(S, d^{\alpha}))$. Q.E.D.

Lemma 5.12 Let $w_0 = (s, t) \in W$ with $d^{\alpha}(s, t) < 2$. Then there is a function $f \in \text{lip}(S, d^{\alpha})$ such that f peaks at w_0 relative to $j(\text{Lip}(S, d^{\alpha}))$ and relative to $j(\text{lip}(S, d^{\alpha}))$.

Proof: [18] Let $0 < \alpha < 1$ and $w_0 = (s, t) \in W$. Suppose $K = d^{\alpha}(s, t) < 2$. Define an auxilliary metric ρ by

$$\rho(x, y) = \min(d^{\alpha-1}(s, t)\, d(x, y), K), \qquad x, y \in S .$$

ρ is a metric and $\rho(x, y) \leq d^{\alpha}(x, y)$ with equality holding if and only if either $d(x, y) = d(s, t)$ or $d(x, y) = 0$. Let g and h be defined by $g(x) = \rho(x, t) - \frac{1}{2}K$ and $h(x) = \frac{1}{2}K - \rho(x, s)$, $x \in S$. It is clear that

(i) $\|g\|_{\infty} = \frac{1}{2}K$ and

(ii) $g(x) = -\frac{1}{2}K$ if and only if $x = t$.

Since $|g(x) - g(y)| = |\rho(x, t) - \rho(y, t)| \leq \rho(x, y) \leq d^{\alpha}(x, y)$ for

$(x, y) \in W$, it follows that

(iii) $\|g\|_{d^\alpha} \leq 1$ and

(iv) $|\hat{g}(x, y)| = \dfrac{|g(x) - g(y)|}{d^\alpha(x, y)} = 1$ implies that either $x = t$ or $y = t$. For if $x \neq y$ then

$$\rho(x, y) \leq d^\alpha(x, y) = |\rho(x, t) - \rho(y, t)| \leq \rho(x, y)$$

which implies that $d^\alpha(x, y) = \rho(x, y)$. Hence $d(x, y) = d(s, t)$ and $|g(x) - g(y)| = d^\alpha(s, t) = K$. (i) implies either $g(x) = -\frac{1}{2}K$ or $g(y) = -\frac{1}{2}K$. Thus it follows from (ii) that either $x = t$ or $y = t$.

Since $|g(x) - g(y)| \leq \rho(x, y)$ then $\|g\|_\rho \leq 1$. Thus for $(x, y) \in W$,

$$\frac{|g(x) - g(y)|}{d^\alpha(x, y)} = \frac{|g(x) - g(y)|}{\rho(x, y)} \cdot \frac{\rho(x, y)}{d^\alpha(x, y)} \leq \frac{\rho(x, y)}{d^\alpha(x, y)} .$$

But from the definition of ρ, $\dfrac{\rho(x, y)}{d^\alpha(x, y)} \to 0$ as $d^\alpha(x, y) \to 0$. Hence Hence $g \in \text{lip}(S, d^\alpha)$. It has been shown [18] that if $g \in \text{lip}(S, d^\alpha)$ then $\hat{g}$ vanishes on $\beta W \sim W$. Thus

(v) $\hat{g}(w) = 0$ if $w \in \beta W \sim W$.

The function h satisfies conditions (i)—(v) with t replaced by s and the minus sign removed in (ii). Let $f = \frac{1}{2}(g + h)$. Since j is linear we have $\hat{f} = \frac{1}{2}(\hat{g} + \hat{h})$. For the fixed point w_0,

$$\hat{f}(w_0) = \frac{f(s) - f(t)}{d^\alpha(s, t)} = \frac{\rho(s, t) + \rho(s, t)}{2d^\alpha(s, t)} = 1 .$$

If $w = (x, y) \in W$ then $|\hat{f}(w)| \leq \|f\|_{d^\alpha} \leq \frac{1}{2}(\|g\|_{d^\alpha} + \|h\|_{d^\alpha}) \leq 1$.

Suppose $|\hat{f}(w)| = 1$ then (iii) implies $|\hat{g}(w)| = |\hat{h}(w)| = 1$. By (iv), $\{x, y\} = \{s, t\}$. Hence $w = (s, t)$ or $w = (t, s)$. Since $\hat{f}$ vanishes on $\beta W \sim W$ and $\|\hat{f}\|_S = \|f\|_\infty \leq \frac{1}{2} K < 1$, the function $|\hat{f}(\cdot)|$ attains the value 1 only at (s, t) and (t, s). By the definition of jg we have $\hat{g}(s, t) = -\hat{g}(t, s)$ for every $g \in \mathrm{Lip}(S, d^\alpha)$ and thus $\hat{f}$ peaks at $w_0 = (s, t)$ relative to $j(\mathrm{Lip}(S, d^\alpha))$ and relative to $j(\mathrm{lip}(S, d^\alpha))$. Q. E. D.

We consolidate the information from these lemmas into the next theorem.

<u>Theorem 5.13</u> For $0 < \alpha < 1$

$$\mathrm{ext}\, U(\mathrm{Lip}(S, d^\alpha)^*) = \phi_S \cup \phi_W \cup D_0$$

where D_0 is some subset of D (see p. 63). For $\mathrm{lip}(S, d^\alpha)$,

$$\mathrm{ext}\, U(\mathrm{lip}(S, d^\alpha)^*) = \phi_S \cup \phi_W .$$

Proof: Identifying the linear functionals on $\mathrm{Lip}(S, d^\alpha)$ with those on the isometric image of $j : \mathrm{Lip}(S, d^\alpha) \to C(S \cup \beta W)$, we see from Theorem 5.15 that every element of $\mathrm{ext}\, U(\mathrm{Lip}(S, d^\alpha)^*)$ has the form $\varepsilon \varphi_v$ where $|\varepsilon| = 1$ and $v \in S \cup \beta W$. If $v = (s, t) \in W$ with $d^\alpha(s, t) \geq 2$ then φ_v can be represented as the convex combination of two elements of $U(\mathrm{Lip}(S, d^\alpha)^*)$ by writing

$$\varphi_v(\hat{f}) = \hat{f}(v) = \frac{f(s) - f(t)}{d^\alpha(s, t)} = \frac{\varphi_s(\hat{f}) - \varphi_t(\hat{f})}{d^\alpha(s, t)}$$

$$= \frac{1}{2}\left(\frac{2}{d^\alpha(s, t)}\, \varphi_s(f) + \frac{-2}{d^\alpha(s, t)}\, \varphi_t(\hat{f})\right) \text{ for all } \hat{f} \in j(\mathrm{Lip}(S, d^\alpha)).$$

Note that $\varphi_s, \varphi_t \in U(\mathrm{Lip}(S, d^\alpha)^*)$ since

$$\|\varphi_s\| = \sup\{|\varphi_s(\hat{f})| : \|\hat{f}\| \le 1,\ \hat{f} \in j(\mathrm{Lip}(S, d^\alpha))\}$$

and $|\varphi_s(\hat{f})| = |f(s)| \le 1$. Thus $\varphi_v \notin \mathrm{ext}\, U(\mathrm{Lip}(S, d^\alpha)^*)$ if $d^\alpha(s,t) \ge 2$ where $v = (s,t)$. Hence

$$\mathrm{ext}\, U(\mathrm{Lip}(S, d^\alpha)^*) \subseteq \phi_S \cup \phi_W \cup D.$$

If $v \in S$ or $v = (s,t) \in W$ with $d^\alpha(s,t) < 2$ then by Lemmas 5.11 and 5.12 there is an $f \in \mathrm{lip}(S, d^\alpha)$ such that $\hat{f}$ peaks at v. Hence by Lemma 5.10, $\phi_S \cup \phi_W \subseteq \mathrm{ext}\, U(\mathrm{Lip}(S, d^\alpha)^*)$. It therefore follows that $\mathrm{ext}\, U(\mathrm{Lip}(S, d^\alpha)^*) = \phi_S \cup \phi_W \cup \phi_{D_0}$ for some subset D_0 of D.

The above argument holds for the unit ball of $\mathrm{lip}(S, d^\alpha)^*$. The elements of D all vanish on $\mathrm{lip}(S, d^\alpha)$ ([18]) and hence cannot be extreme points of $U(\mathrm{lip}(S, d^\alpha)^*)$. Hence $\mathrm{ext}\, U(\mathrm{lip}(S, d^\alpha)^*) = \phi_S \cup \phi_W$.
Q. E. D.

Until recently it was not known whether D_0 was empty. A result of Johnson [19] is that $D_0 \ne \emptyset$. Furthermore it has been shown that if S is countable then D_0 is uncountable (see [20]). An open question is the following: if S is uncountable, is D_0 uncountable? A complete description of D_0 appears to be quite difficult.

Dual of the C(S) Spaces

The final dual to be considered is the dual of $C(S)$. As we shall see in the next theorem, the extreme points of $U(C(S)^*)$ are the point evaluation functionals on $C(S)$. We first of all need the following lemma.

Lemma 5.14 Let K be a compact subset of a locally convex linear topological space E whose closed convex hull is compact. Then the only extreme points of $\mathrm{cl\,con}\,K$ are points of K.

Proof: [11, p. 440] Let $p \in (\mathrm{ext\,cl\,con}\,K) \sim K$. Since K is closed there is a neighborhood V_0 of the origin such that $(p+V_0) \cap K = \emptyset$. Since E is a locally convex linear space there is a convex neighborhood V of the origin such that $V - V \subseteq V_0$. Thus $(p+V) \cap (K+V) = \emptyset$ and hence $p \notin \mathrm{cl}\,(K+V)$. The family $\{k+V : k \in K\}$ is an open covering of K and hence has a finite subcovering $\{k_i + V : i = 1, 2, \ldots, n\}$. Let $K_i = \mathrm{cl\,con}\,[\mathrm{cl}(k_i + V) \cap K] \subseteq \mathrm{cl}\,(k_i + V)$. K_i is a closed and hence compact subset of $\mathrm{cl\,con}\,K$. Thus

$$\mathrm{cl\,con}\,K = \mathrm{cl\,con}\,(K_1 \cup \cdots \cup K_n) = \mathrm{con}\,(K_1 \cup \cdots \cup K_n)$$

(see [11, p. 415]). It follows that p has the form $p = \sum_{i=1}^{n} \lambda_i k_i$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $k_i \in K_i$. Since p is an extreme point we have that $p = k_i$ if $\lambda_i > 0$. Therefore

$$p \in \bigcup_{i=1}^{n} K_i \subseteq \bigcup_{i=1}^{n} \mathrm{cl}\,(k_i + V) \subseteq \mathrm{cl}\,(K + V).$$

This contradiction implies $\mathrm{ext\,cl\,con}\,K \subseteq K$. Q.E.D.

We are now ready to state the main result of this section.

Theorem 5.15 Let F be a closed linear subspace of $C(S)$. Then every extreme point of $U(F^*)$ is of the form $\alpha\varphi_x$ where $|\alpha| = 1$ and $x \in S$. If $F = C(S)$ the converse is also true, i.e., every element of the form $\alpha\varphi_x$, $|\alpha| = 1$, $x \in S$ is an extreme point of $U(C(S)^*)$.

Proof: [11, p. 441] Let $A = \{\alpha\varphi_x : |\alpha| = 1, x \in S\}$. We first need to show that $A \subseteq U(F^*)$. For $f \in F$ we have $\|f\| = \sup\{|f(x)| : x \in S\}$. Thus if $\|f\| \leq 1$ it follows that $|\varphi_x(f)| = |f(x)| \leq 1$ and hence

$$\|\alpha\varphi_x\| = \|\varphi_x\| = \sup\{|\varphi_x(f)| : \|f\| \leq 1\} \leq 1.$$

This implies $\alpha\varphi_x \in U(F^*)$ and $A \subseteq U(F^*)$.

Let F^* have the w^*-topology. Since $U(F^*)$ is w^*-compact, it is w^*-closed and also convex. Thus $w^*\text{-cl con} A \subseteq U(F^*)$. Suppose $\mu \in F^* \sim \text{cl con} A$. Then since F^* has the w^*-topology there is an $f \in F$ and real constants c and ε, with $\varepsilon > 0$; $\operatorname{Re}\hat{f}(\mu) = \operatorname{Re}\mu(f) \geq c$; and $\operatorname{Re}\hat{f}(\nu) = \operatorname{Re}\nu(f) \leq c - \varepsilon$, for $\nu \in \text{cl con} A$. This is a result of a version of the Hahn-Banach Theorem. In particular $\operatorname{Re}\alpha f(x) = \operatorname{Re}\alpha\varphi_x(f) \leq c - \varepsilon$ for $x \in S$ and $|\alpha| = 1$. Hence $\|f\| \leq c - \varepsilon$; for suppose $\|f\| > c - \varepsilon$. Then there is an $x_0 \in S$ such that $|f(x_0)| > c - \varepsilon$. If

$$\alpha = \frac{\overline{f(x_0)}}{|f(x_0)|} \quad \text{then} \quad \alpha f(x_0) = \frac{|f(x_0)|^2}{|f(x_0)|} = |f(x_0)| > c - \varepsilon .$$

But this contradicts $\operatorname{Re}\alpha f(x) \leq c - \varepsilon$ for every $x \in S$ and $|\alpha| = 1$. Therefore

$$\|\mu\| = \sup\left\{\frac{|\mu(g)|}{\|g\|} : g \in F, \ g \neq 0\right\} \geq \frac{\mu(f)}{\|f\|} \geq \frac{c}{c - \varepsilon} > 1$$

Thus $\mu \notin U(F^*)$ and we have $U(F^*) = \text{cl con} A$.

If we show A is w^*-closed then the first part of the theorem will follow from Lemma 5.14. Let C be the unit circle of the complex plane. The map $(\alpha, x) \to \alpha\varphi_x$ is w^*-continuous from $C \times S$ into $C(S)^*$. Since $C \times S$ is compact, the image under the map, namely A, is also.

To prove the converse let $F = C(S)$ and $x \in S$ such that $\varphi_x = \lambda\mu + (1-\lambda)\nu$ where $0 < \lambda < 1$ and $\mu, \nu \in U(C(S)^*)$. We need to show that $\varphi_x = \mu = \nu$. Let $f_0 \in C(S)$, $\|f_0\| \leq 1$ and $f_0(y) = 0$ for y in some neighborhood N of x. By the Tietze extension theorem there is an $h \in C(S)$ such that $\|h\| \leq 1$, $h(x) = 1$ and $h(y) = 0$ for $y \notin N$. Then

$$\lambda\mu(h) + (1-\lambda)\nu(h) = \varphi_x(h) = 1, \quad |\mu(h)| \leq 1, \quad \text{and} \quad |\nu(h)| \leq 1.$$

Thus $\mu(h) = \nu(h) = 1$. Similarly we have $\mu(f_0 + h) = \nu(f_0 + h) = 1$. Hence $\mu(f_0) = \nu(f_0) = 0$. Now let $f_1 \in C(S)$, $\|f_1\| \leq 1$ and $f_1(x) = 0$. Then since f_1 is continuous on S, for each positive integer n there is a neighborhood N_n of x such that $|f_1(y)| < \frac{1}{n}$ for $y \in N_n$. Let M_n be a neighborhood of x such that $\mathrm{cl}\, M_n \subseteq N_n$. Again by the Tietze extension theorem there is an $h_n \in C(S)$ such that $\|h_n\| \leq \frac{1}{n}$, $h_n(y) = 0$, $y \notin N_n$ and $h_n(y) = f_1(y)$ for $y \in M_n$. Then $f_1 - h_n \to f_1$, $\|f_1 - h_n\| \leq 1$ for $n > 1$, and $f_1 - h_n$ vanishes on M_n. Thus $f_1 - h_n$ satisfies the conditions on the function f_0 and hence by what was done above $\mu(f_1 - h_n) = \nu(f_1 - h_n) = 0$. Since $f_1 - h_n \to f_1$ and μ and ν are continuous, we have $\mu(f_1) = \nu(f_1) = 0$. If $h' \in C(S)$ such that $h'(x) = 0$, then for sufficiently large n, $\|\frac{h'}{n}\| \leq 1$ so that $\frac{h'}{n}$ satisfies the conditions on f_1 and thus $\mu(h') = \nu(h') = 0$. If $\varphi_x(h') = 0$ then $\mu(h') = \nu(h') = 0$ and it follows that there are scalars α and γ such that $\mu = \alpha\varphi_x$ and $\nu = \gamma\varphi_x$ (see [11, p. 421]). Since $\mu, \nu \in U(C(S)^*)$, we have $|\alpha| \leq 1$, and $|\gamma| \leq 1$. Since $\varphi_x = (\lambda\alpha + (1-\lambda)\gamma)\varphi_x$ we have $\lambda\alpha + (1-\lambda)\gamma = 1$. This implies $\alpha = \gamma = 1$ since 1 is an extreme point of the unit disk in the complex plane. Hence $\varphi_x \in \text{ext}\, U(C(S)^*)$. That $\alpha\varphi_x \in \text{ext}\, U(C(S)^*)$ is a consequence of Lemma 2.7. Q.E.D.

We have now characterized the extreme points of the unit balls of five Banach spaces in Chapter IV and their duals in the present chapter. It may be noted that is some cases these characterizations were very intuitive and what we might expect them to be. On the other hand some of the cases proved to be difficult and in fact some of the results are not known as was the situation in the duals of the Hardy spaces. In the next chapter we want to look at some extensions of the notion of extreme points.

CHAPTER VI

OTHER DISTINGUISHED POINTS IN BANACH SPACES

In the present chapter we will discuss some other points in Banach spaces which are somewhat related to extreme points. Some of these are generalizations of extreme points while others are special cases of extreme points. Most of the results will be stated without proof. Our intent is to supply some ideas and problems so that the interested reader might pursue the study of extreme and other related points in Banach spaces.

The first topic to be considered is the notion of an exposed point of a convex set K.

Definition 6.1 Let K be a convex subset of a locally convex linear space E. A point x in K is said to be an exposed point of K if there exists $f \in E^*$ such that $\operatorname{Re} f(x) > \operatorname{Re} f(y)$ whenever $y \in K$, $x \neq y$. (The set of exposed points of K will be denoted $\exp K$.)

Intuitively speaking a point x is an exposed point of K provided a closed hyperplane exists which intersects K only at the point x. The following theorem shows that an exposed point is a special case of an extreme point.

Theorem 6.2 Let K be a convex subset of a normed linear space E. Then $\exp K \subseteq \operatorname{ext} K$.

Proof: Let $x \in \exp K$. Thus we have $f \in E^*$ such that $\operatorname{Re} f(x) > \operatorname{Re} f(y)$ for $y \in K$, $x \neq y$. Suppose $x \notin \operatorname{ext} K$. Then there are $w, z \in K$ such that $x = \frac{1}{2}(w+z)$ with $w \neq x$ and $z \neq x$. Therefore $\operatorname{Re} f(x) = \frac{1}{2}\operatorname{Re} f(w) + \frac{1}{2}\operatorname{Re} f(z) < \operatorname{Re} f(x)$. This contradiction implies $x \in \operatorname{ext} K$. Q. E. D.

A simple example in the plane shows that not every extreme point is exposed.

<u>Example 6.1</u> Let K be the convex hull of two disjoint circles in the plane. The four boundary points A, B, C, D where the common tangents intersect the circles are extreme points but not exposed. (See Figure 6.1) A is not exposed since the supporting hyperplane at A is the tangent line through A and B which intersects K at points other than A. On the other hand A is extreme since it is not on an open line segment contained in K.

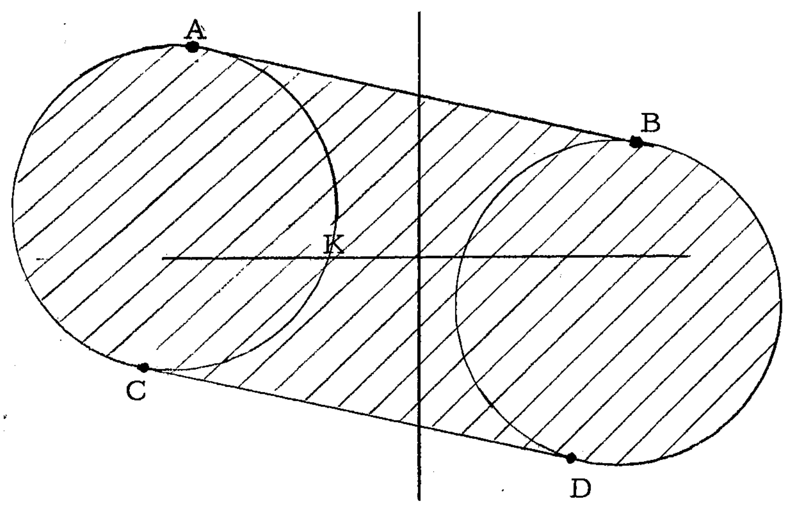

Figure 6.1. Extreme Non-exposed Points

If we restrict our attention to normed linear spaces, then the following is a result similar to the Krein-Milman Theorem.

Theorem 6.3 If K is a compact convex subset of a normed linear space E, then $\operatorname{ext} K \subsetneq \operatorname{cl} \exp K$ and $K = \operatorname{cl} \operatorname{con} \exp K$.

Proof: (see [22]).

The above theorem by Klee appeared in 1958. In the same paper a similar result is proved concerning weakly compact subsets of a separable Banach space. This has been improved [1] and states that every weakly compact subset of a Banach space is the closed convex hull of its exposed points.

A result which is closely associated with the Bessaga-Pelczyński Theorem is the following.

Theorem 6.4 Let X be a reflexive Banach space and K a closed bounded convex subset of X. Then $K = \operatorname{cl} \operatorname{con} \exp K$.

Proof: (see [27]).

By Theorem 3.20 we know that if X is an infinite dimensional reflexive Banach space, then $\operatorname{ext} U(X)$ is uncountable. According to [28] Branko Grunbaum has shown that there is a three dimensional space E such that $\operatorname{ext} U(E)$ is uncountable but $\exp U(E)$ is countable. This leads to the following open question: can the unit ball of an infinite dimensional reflexive Banach space have countably many exposed points?

A refinement of the definition of an exposed point is the notion of a strongly exposed point.

Definition 6.5 Let K be a convex subset of a normed linear space E. A point $x \in K$ is called a strongly exposed point of K if there is an $f \in E^*$ such that

(i) $f(y) < f(x)$ for $y \in K$, $y \neq x$ and

(ii) $f(x_n) \to f(x)$ and $\{x_n\} \subseteq K$ imply $\|x_n - x\| \to 0$.

The following example due to [26] shows that there are separable reflexive Banach spaces whose unit balls have exposed points which are not strongly exposed.

Example 6.2 Let ℓ_2 be the space of real sequences $x = \{x_n\}$ with norm $\|x\| = \{\Sigma |x_n|^2\}^{1/2} < \infty$. Let $e_n = \{1 - \frac{1}{n}, 0, \ldots, 0, \frac{1}{2}, 0, \ldots\}$ for $n = 2, 3, \ldots$ where the number $\frac{1}{2}$ is in the n-th place. e_n is clearly an element of $U(\ell_2)$ for $n = 2, 3, \ldots$. An element $f \in \ell_2^*$ is of the form $f(x) = \Sigma t_n x_n$ where $\{t_n\}$ and $\{x_n\} = x$ are elements of ℓ_2. In particular $g(x) = x_1$ is an element of ℓ_2^*. The element $p = \{1, 0, 0, \ldots\}$ in ℓ_2 is such that $g(p) = 1$ and $g(x) < 1$ for $x \in U(\ell_2)$, $x \neq p$. Thus p is an exposed point, but it is not strongly exposed. To see this let $h \in \ell_2^*$ with $h(p) > h(y)$, $y \in U(\ell_2)$, $y \neq p$ and $h(e_n) \to h(p)$. But we have $\|e_n - p\| \geq \frac{1}{2}$ for $n = 2, 3, \ldots$ and therefore $\|e_n - p\|$ does not converge to zero.

The following refinement of the Krein-Milman Theorem involves strongly exposed points instead of extreme points.

Theorem 6.6 Every weakly compact convex set in a separable Banach space is the closed convex hull of its strongly exposed points.

Proof: (see [26]).

A generalization of Theorem 3.20 is the following: every closed bounded convex set K with nonempty interior of an infinite dimensional reflexive Banach space X, $\operatorname{ext} K$ is uncountable ([28]). It is also shown in the same paper that if X is separable there is a symmetric convex subset with nonempty interior which has countable many strongly exposed points. However, it is an open question whether there are countably many exposed points.

Definition 6.7 Let K be a convex subset of a normed linear space E. An exposed ray of K is a closed half-line L contained in K such that $L = K \cap H$ for some supporting hyperplane H of K. (The union of all exposed rays of K will be denoted by $\operatorname{r exp} K$.)

Theorem 6.8 Suppose K is a locally compact closed convex subset of a normed linear space and K contains no line. Then $\operatorname{ext} K \subseteq \operatorname{cl} \exp K$ and $K = \operatorname{cl} \operatorname{con} (\exp K \cup \operatorname{r exp} K)$.

Proof: (see [22]).

To see the above theorem more clearly, consider the following example in the plane.

Example 6.3 Recall that the only extreme point of the cone $y \geq |x|$ in the plane is the origin (see Figure 2.3a). It is clear that the origin is also an exposed point. We note also that every nonextreme point of the cone is a convex combination of two boundary points. Thus the cone C is the convex hull of $\exp C \cup \operatorname{r exp} C$.

We next want to consider smooth points which are in a sense dual to the notion of an exposed point.

Definition 6.9 Let K be a convex subset of a normed linear space. An element x in K is called a smooth point of K is there exists a unique hyperplane which supports K at x. (The set of smooth points of K will be denoted $\operatorname{sm} K$.)

It is clear that every point on the boundary of the unit disk in the plane is a smooth point (see Figure 2, 2a). The smooth points of a convex polygonal region in the plane are all the points on the boundary which are not vertices. (see Figure 2.1) Recall that the extreme points of such a set were the vertices. Therefore we see that an extreme point may or may not be a smooth point and vice-versa. A problem posed by Klee [22] is the following: if K is a bounded closed convex subset of a reflexive Banach space with $\operatorname{int} K \neq \emptyset$, must K have an exposed point or a smooth point? Theorem 6.4 answers the question for an exposed point. There is however the next result concerning the smooth points of $U(X)$.

Theorem 6.10 Let X be a separable Banach space. Then the smooth points of $U(X)$ form a dense G_δ subset of the boundary of $U(X)$.

Proof: (see [34]).

The dual notion of strongly exposed is strongly smooth. We will use this notion only for $U(X)$ so we define it only in this case.

Definition 6.11 Let E be a normed linear space. $x \in U(E)$ with $\|x\| = 1$ is a strongly smooth point of $U(E)$ if (i) there exists only one $f \in E^*$ satisfying $f(x) = \|f\| = 1$ and (ii) $f_n(x) \to 1$ and

$\{f_n\}_{n=1}^{\infty} \subseteq U(E^*)$ imply that $\|f_n - f\| \to 0$.

Note that by duality and Example 6.2 there are separable reflexive Banach spaces whose unit balls have smooth boundary points which are not strongly smooth. In fact in the next example due to [26] we shall see that there is a separable Banach space whose unit ball has no strongly smooth boundary points.

Example 6.4 Let $x = \{x_n\} \in \ell_1$ with $\|x\| = 1$ (restrict ℓ_1 to real sequences). Suppose x is a strongly smooth point of $U(\ell_1)$. Let f be the unique element of ℓ_1^* such that $f(x) = \|f\| = 1$. For $y \in \ell_1$, $f(y) = \Sigma t_n y_n$ where $t = \{t_n\} \in \ell_1^* = \ell_\infty$. Define f_n by

$$\{\frac{x_1}{|x_1|}, \dots, \frac{x_n}{|x_n|}, 0, 0, \dots\} \in \ell_\infty .$$

(Note $x_i \neq 0$, $i = 1, 2, \dots$ since f is unique.) Clearly $f_n(x) \to \|x\| = 1$ and $f_n \in U(\ell_1^*)$, $n = 1, 2, \dots$. But

$$\|f_n - f\| = \max\{|\frac{x_1}{|x_1|} - t_1|, \dots, |\frac{x_n}{|x_n|} - t_n|, \sup_{k>n} |t_k|\}$$

which does not converge to 0 unless $t_n = \dfrac{x_n}{|x_n|}$ and $t_n \to 0$. Since this is impossible x is not a strongly smooth point of $U(\ell_1)$.

We shall now give a positive result concerning strongly smooth points.

Theorem 6.12 The boundary of the unit ball of a separable reflexive Banach space contains a dense subset of strongly smooth points.

Proof: (see [26]).

The last notion to be considered in this chapter is that of a support point.

Definition 6.13 Let K be a convex subset of a normed linear space E. $x \in K$ is a support point of K if there exists a hyperplane which supports K at x, that is, there is an $f \in E^*$ $(f \neq 0)$ such that $f(x) = \sup f(K)$.

It is easy to see that every boundary point of a compact convex subset K of the plane is a support point of K. It follows from Definitions 6.1 and 6.9 that exposed points and smooth points are support points. The next result shows that in the case of the unit ball of a normed linear space, every boundary point is a support point.

Theorem 6.14 Let E be a normed linear space. Then every $x \in E$ with $\|x\| = 1$ is a support point of $U(E)$.

Proof: Let $x \in U(E)$, $\|x\| = 1$. Since $U(E^*)$ is w*-compact, every $\hat{x} \in E^{**}$ attains its supremum for some $f \in U(E^*)$. Hence the hyperplane associated with f supports $U(E)$ at x. Thus x is a support point of $U(E)$. Q.E.D.

In general not every boundary point of a closed convex subset of a Banach space is a support point, but we do have the following result concerning the support points.

Theorem 6.15 Let K be a closed convex subset of a Banach space. Then the support points of K are dense in the boundary of K.

Proof: (see [4]).

Corollary 6.16 If K is a closed convex subset of a Banach space, then K is the intersection of all closed half-spaces which support it.

Proof: (see [4]).

As mentioned previously this chapter is a potpourri of results, examples and problems concerning other distinguished points in Banach spaces. The purpose of the chapter is to give the reader some ideas concerning the geometry of linear spaces. There are some notions which are not mentioned such as: strongly extreme points [29] and algebraically exposed points [22] because they are much less important than the others.

As is the case of extreme points, these other notions have useful applications in various areas. Information concerning exposed points can be applied to invariant means on locally compact groups and discrete semigroups (see [15]).

CHAPTER VII

SUMMARY AND CONCLUSIONS

This dissertation was written so that it is within the mathematical background of a second year graduate student. It could be used as reference material for a seminar on extreme points and their role in functional analysis. However the main purpose of the paper is to collect and present research in the literature in a readable and compact form. The main theme of the paper is to characterize the extreme points of the unit balls of five well known classes of Banach spaces. The reader should gain an appreciation of the importance of extreme points in the study of convex sets in functional analysis.

Chapter I is an introduction which explains the purpose of the paper and the background needed to read it. Chapter II gives the definition of an extreme point and some basic lemmas which are used later in the dissertation. The notation to be used later is also explained. Chapter III presents three important theorems concerning extreme points. They are given in chronological order so that the historical development of the subject, to a certain degree, can be followed. As a motivational device, some applications of extreme points are also given. The heart of the paper is in Chapter IV. The extreme points of the unit balls of the five chosen Banach spaces are characterized. In some cases this proved to be a lengthy proposition. It is the desire of the author that the material is presented in an understandable form. Conditions under which the unit ball is the closed convex hull of its extreme points are also given. The concern of Chapter V is with the extreme points of the unit ball of the duals of the spaces mentioned in Chapter IV.

Chapter VI gives some extended notions of extreme points. This should give the interested reader who wishes to do further study on these subjects a direction to proceed. The compilation of results in these areas into a comprehensive work could possibly be a worthwhile paper.

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