AN EVALUATION OF ANALYTICAL METHODS USING THE RAYLEIGH-RITZ APPROACH FOR THE FREE VIBRATIONS OF STIFFENED NONCIRCULAR CYLINDRICAL SHELLS

By

ROBERT LESLIE BRUGH

Bachelor of Science
Virginia Polytechnic Institute
Blacksburg, Virginia
1962

Master of Science
Wichita State University
Wichita, Kansas
1970

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of the Oklahoma State University
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Dean of the Graduate College

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TABLE OF CONTENTS

Chapter | Page

LIST OF TABLES

Table		Page

LIST OF FIGURES

LIST OF SYMBOLS

a	length of the shell
$A_{s\ell}$, A_{rk}	cross-sectional area of the ℓ^{th} stringer, k^{th} ring
D_x, D_θ	orthotropic flexural stiffness of the shell in longitudinal direction, circumferential direction
E_x, E_θ	Modulus of Elasticity of the shell in the longitudinal direction, circumferential direction
$E_{s\ell}$, E_{rk}	Modulus of Elasticity of the ℓ^{th} stringer, k^{th} ring
$G_{x\theta}$	Modulus of Rigidity of the shell
$(GJ)_{s\ell}$, $(GJ)_{rk}$	torsional stiffness of the ℓ^{th} stringer, k^{th} ring
h	thickness of the shell
IRA_1 to IRA_{13}, IRB_1 to IRB_{15}	circumferential integrals in the ring equations
ISA_1 to ISA_9, ISB_1 to ISB_5	circumferential integrals in the shell equations

IX_1 to IX_5	longitudinal integrals in the shell and stringer equations
$I_{yy_{s\ell}}$, $I_{xx_{rk}}$	moment of inertia of the ℓ^{th} stringer, k^{th} ring cross-sectional area about y′ and x′ axes, respectively
$I_{zz_{s\ell}}$, $I_{zz_{rk}}$	moment of inertia of the ℓ^{th} stringer, k^{th} ring cross-sectional area about z′ axis
$I_{yz_{s\ell}}$, $I_{xz_{rk}}$	product of inertia of the ℓ^{th} stringer, k^{th} ring cross-sectional area about y′ and z′, x′ and z′ axes, respectively
K	stiffness matrix
$K11_{mn,\bar{m}\bar{n}}$, $K12_{mn,\bar{m}\bar{n}}$, $K13_{mn,\bar{m}\bar{n}}$, $K22_{mn,\bar{m}\bar{n}}$, $K23_{mn,\bar{m}\bar{n}}$, $K33_{mn,\bar{m}\bar{n}}$	elements of the stiffness matrix
m	a longitudinal term in the assumed displacement series
M^*	the final longitudinal term in the assumed displacement series
M	mass matrix
$M11_{mn,\bar{m}\bar{n}}$, $M12_{mn,\bar{m}\bar{n}}$, $M13_{mn,\bar{m}\bar{n}}$, $M22_{mn,\bar{m}\bar{n}}$, $M23_{mn,\bar{m}\bar{n}}$, $M33_{mn,\bar{m}\bar{n}}$	elements of the mass matrix
n	a circumferential term in the assumed displacement series

N^*	the final circumferential term in the assumed displacement series
$\{q\}_o$, $\{q\}_s$, $\{q\}_r$	displacement vector of the shell, ℓ^{th} stringer, k^{th} ring
R	radius of the shell
R_c	radius of the centroid of the k^{th} ring
$RF_{1,k}$ to $RF_{4,k}$	longitudinal functions in ring equations
$RNG1_k$ to $RNG22_k$	constants in ring equations
$SF_{1,\ell}$ to $SF_{8,\ell}$	circumferential functions in stringer equations
$STR1_\ell$ to $STR14_\ell$	constants in stringer equations
SX_1 to SX_8, ST_1 to ST_4, SXT_1 to SXT_4, SPC	constants in shell equations
T	kinetic energy
δT_o, $\delta T_{s\ell}$, δT_{rk}	first variation of the kinetic energy of the shell, ℓ^{th} stringer, k^{th} ring
u, v, w	longitudinal, circumferential, and radial displacements of the shell median surface
δu, δv, δw	first variation of the longitudinal circumferential and radial displacements of the shell median surface

$\bar{u}_{mn}, \bar{v}_{mn}, \bar{w}_{mn}$	generalized coordinates for the symmetric mode displacements u, v and w, respectively
$\bar{u}'_{mn}, \bar{v}'_{mn}, \bar{w}'_{mn}$	generalized coordinates for the antisymmetric mode displacements u, v and w, respectively
U	strain energy
$\delta U_o, \delta U_{s\ell}, \delta U_{rk}$	first variation of the strain energy of the shell, ℓ^{th} stringer, k^{th} ring
x, θ, z	longitudinal, circumferential and radial coordinates of the shell
x′, y′, z′	longitudinal, circumferential and radial coordinates of the stiffeners
$\bar{x}_{rk}, \bar{y}_{s\ell}$	x distance of the centroid of the k^{th} ring, y distance of the centroid of the ℓ^{th} stringer from the z axis passing through its point of attachment
$\bar{z}_{s\ell}, \bar{z}_{rk}$	z distance of the centroid of the ℓ^{th} stringer, k^{th} ring from the middle surface of the shell
$\alpha_{\ell j}, \alpha_{kj}$	constant of proportionality between the discrete and conventionally smeared strain energies of ℓ^{th} stringer, k^{th} ring for the j^{th} mode shape

$\beta_{\ell j}$, β_{kj}	constant of proportionality between the discrete and conventionally smeared kinetic energies of ℓ^{th} stringer, k^{th} ring for the j^{th} mode shape
$\{\epsilon\}_o$, $\{\epsilon\}_s$, $\{\epsilon\}_r$	strain of the shell, ℓ^{th} stringer, k^{th} ring
$\nu_{x\theta}$, $\nu_{\theta x}$	Poisson's Ratios for the orthotropic shell
ρ_o, $\rho_{s\ell}$, ρ_{rk}	mass density of the shell, ℓ^{th} stringer, k^{th} ring
$\{\sigma_R\}_o$, $\{\sigma_R\}_{s\ell}$, $\{\sigma_R\}_{rk}$	stress resultants of the shell, ℓ^{th} stringer, k^{th} ring
ω	circular frequencies

Subscripts

c	centroid
k	kth ring
ℓ	ℓ^{th} stringer
o	shell
r	ring
s	stringer

Notes:

(1) A comma before a subscript denotes partial differentiation with respect to that subscript:

e.g., $u_{,x}$ denotes $\frac{\partial u}{\partial x}$ and $w_{,\theta\theta}$ denotes $\frac{\partial^2 w}{\partial \theta^2}$.

(2) Superscript T denotes transpose of a matrix.

CHAPTER I

INTRODUCTION

Discussion

The free-vibrational characteristics of unstiffened or stiffened circular or noncircular cylindrical shells are of interest to designers of aircraft fuselages and submarine hulls. Noncircular cross sections are due either to special internal storage requirements or to imperfections occurring during manufacture. The purpose of this study is to develop and evaluate two methods of analysis for determining the free-vibrational characteristics of ring- and/or stringer-stiffened, singly symmetric, noncircular cylinders. For the first method, the stiffeners are treated as being located at discrete locations. The second method considers the stiffeners to be "smeared" over the surface of the shell, thereby transforming the stiffened shell into a somewhat equivalent orthotropic cylinder.

Background

Methods for vibrations analyses of unstiffened, circular, isotropic cylinders with specialized boundary conditions have been available for many years. With the advent of the digital computer, the general case of stiffened, noncircular, anisotropic cylinders with arbitrary boundary conditions could be studied. A brief review of some historically important studies which have contributed to the solution of this complicated problem is presented in the following paragraphs.

The equations which govern the static and dynamic behavior of noncircular cylinders have been derived by extending the classical theory of thin circular cylindrical shells to include the effects of noncircularity. Kempner (1) presented energy expressions and differential equations useful in the analysis of arbitrary cylindrical shells. Kempner and his associates have used these equations to study a wide range of problems dealing primarily with linear and nonlinear statics problems (2, 3, 4, 5, 6, 7). Using the same class of ovals as Kempner, Klosner and Pohle (8, 9, 10) studied the free vibrations of infinitely long oval cylinders utilizing an approximate method. Culberson and Boyd (11) obtained exact solutions of the free vibration equations of motion for the same class of oval cylinders studied by Klosner and Pohle.

Based on solution functions used by Boyd (12) in static analyses of noncircular panels subjected to uniform pressures, studies have been performed considering linear buckling (13) and free vibration (14) of noncircular cylindrical shell panels. Finite-difference (15) and Rayleigh-Ritz (16) analyses of the small deflection static behavior of oval cylinders and noncircular panels also have been conducted.

Sewall (17) used the Rayleigh-Ritz technique to study the free vibrations of elliptical cylindrical shells having free-free, fixed-free, and freely supported ends. A free vibration study of noncircular cylinders was also performed by Malkina (18).

McElman, Mikulas and Stein (19) presented a theoretical analysis of the vibration and stability characteristics of eccentrically stiffened circular cylinders and flat plates. In this study, the stiffeners were not considered as discrete members, but their effects were averaged or "smeared out" over the plate or shell.

Egle et al. (20, 21) utilized the Rayleigh-Ritz technique to determine the free vibration frequencies and mode shapes of stiffened circular cylinders. The rings and stringers in this study were treated as discrete elements, and a number of practical end conditions were employed.

Bushnell and Almroth (22) have developed a computer program to determine the vibration and stability characteristics of stiffened shells of a general shape under general loadings. The numerical solution is based upon a two-dimensional, finite-difference approximation. The shell surface is covered with mesh lines parallel to the coordinate lines and the mesh spacing is variable over the surface.

Boyd and Rao (23) studied the free vibrations of ring- and stringer-stiffened elliptical cylindrical shell structures, treating the stiffeners as discrete elements. Their study utilized the Rayleigh-Ritz technique to find the free vibration frequencies and mode shapes for a wide range of cross section eccentricities, numbers of stiffeners and end conditions. The Flügge strain-displacement relations were used for the cylindrical shell. In addition, the shell was limited to isotropic materials and symmetric rings. The method of analysis was complicated by the effects of noncircularity and the compatibility requirements between the shell and the stiffeners. These effects increased the computer storage requirements and the computation time for these highly complex problems.

Even with the research accomplished to date, a need still exists for a method of analysis that has:

- increased computational efficiencies,
- orthotropic material properties for the shell,
- improved compatibility relations, and
- arbitrary stiffener cross sections.

Furthermore, a need exists for additional understanding of the vibrational characteristics of stiffened shell structures. This study was initiated to provide these needs.

Approach to the Problem

The objective of the present study is to develop a method of analysis that satisfies the above needs and to obtain additional understanding of the vibrational characteristics of stiffened shell structures. Two methods of analysis are developed here to study the free-vibrational characteristics of unstiffened or stiffened arbitrary cylindrical shells.

Both methods use the Rayleigh-Ritz approach. In the first method of analysis, the stiffeners are considered to be discretely located. The second method "smears" the stiffeners over the surface of the shell based on an equivalent energy approach. These methods are evaluated for their computational efficiencies.

Orthotropic material properties for the shell are included in this analysis. With orthotropic material properties, the vibrational characteristics of filament wound pressure vessels can be studied. Also, this allows for the use of composite materials. This method could be easily modified to include sandwich-type materials. Kraus (24) showed a good correlation between the numerical results obtained from Love's First Approximation Theory of thin elastic shells and those from higher-order theories (e.g., Flügge's Theory) and theory of elasticity solutions. Therefore, the theory of Love was selected for this study.

An improved set of compatibility relations for the stiffeners was developed for this study (Appendix A). These compatibility relations are consistent with the shell and beam theories and are preferred to those used in References 21 and 23. In addition, the stiffeners can have arbitrary cross sections. For example, shell structures using zee or channel stiffeners can be analyzed with this method of analysis. Stiffeners with arbitrary cross sections are quite commonly used in many structures.

The derivation of the energy expressions for the shell structures is described in Chapter II. Expressions for the stiffener energies are presented in Appendixes B and C. The elements of the mass and stiffness matrices are given in Appendix D. The computer program developed is discussed in Chapter III. An evaluation of the methods and numerical results are discussed in Chapter IV. Finally, Chapter V summarizes the results of this study, contains conclusions and makes recommendations for future investigations.

CHAPTER II

FORMULATION OF THE SOLUTION

The equations of motion in this analysis were developed using Hamilton's Principle by minimizing the Action Integral with respect to the undetermined coefficients of assumed displacement functions, i.e.,

$$\delta A = \int_{t_o}^{t_1} (\delta T - \delta U)\, dt = 0 \quad . \tag{2.1}$$

The first variation of the strain and kinetic energies of the shell, stringers, and rings (with respect to their own coordinate systems) was formulated. Compatibility relations were derived to express the displacements of the stiffeners in terms of the displacements of the shell median surface. Using these relations, the first variations of the strain and kinetic energies for the stiffeners were expressed in terms of the displacements of the shell median surface. The energies for the shell, stringers and rings were combined to obtain the first variation of the total strain and kinetic energies for the stiffened cylinder. Finite series satisfying the kinematic boundary conditions were assumed for each component of point displacements on the shell median surface. The assumed displacement functions with undetermined coefficients were substituted into the expressions for the first variation of the total energies and combined to form an eigenvalue problem.

Geometry

The geometry of a typical noncircular shell is shown in Figure 1. The three orthogonal coordinates x, θ and z locate points on the reference surface of the shell and u, v and w are the corresponding displacement components. The variable radius of curvature of the shell cross section is expressed as a function of the θ coordinate. The geometries of a typical stringer and ring are shown in Figures 2 and 3, respectively. The local coordinates of the stiffeners x', y' and z' are measured from the centroid of the stiffener.

Compatibility Relations

The compatibility equations relate the displacements $\{q\}_i$ of any point in the i^{th} stiffener to those $\{q\}_o$ of the median surface of the shell. (The derivation of the equations is presented in Appendix A.) These compatibility relations can be expressed as

$$\{q\}_i = [C]_i \{q\}_o \qquad (2.2)$$

where i = s for the stringers and i = r for the rings. The matrix $[C]_s$ may be expressed as

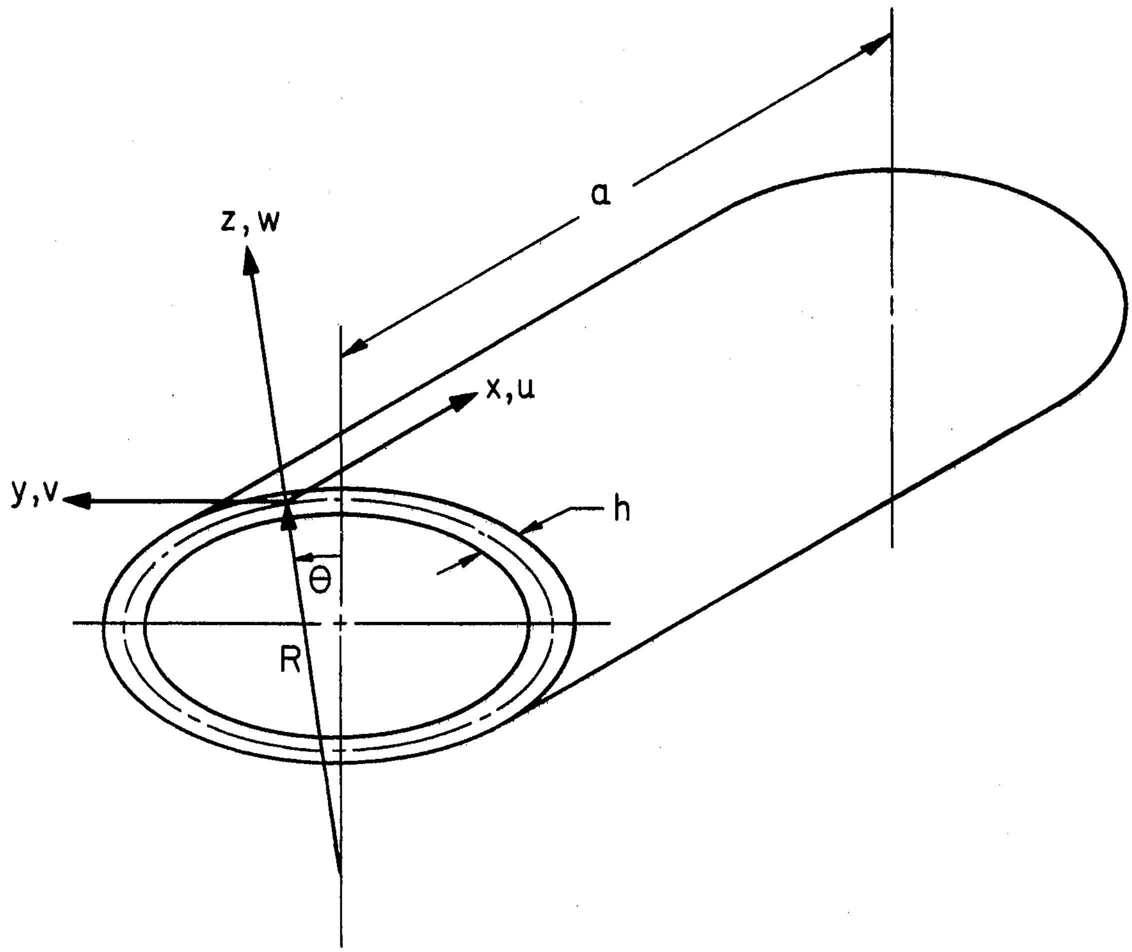

Figure 1. Geometry of an Elliptical Shell

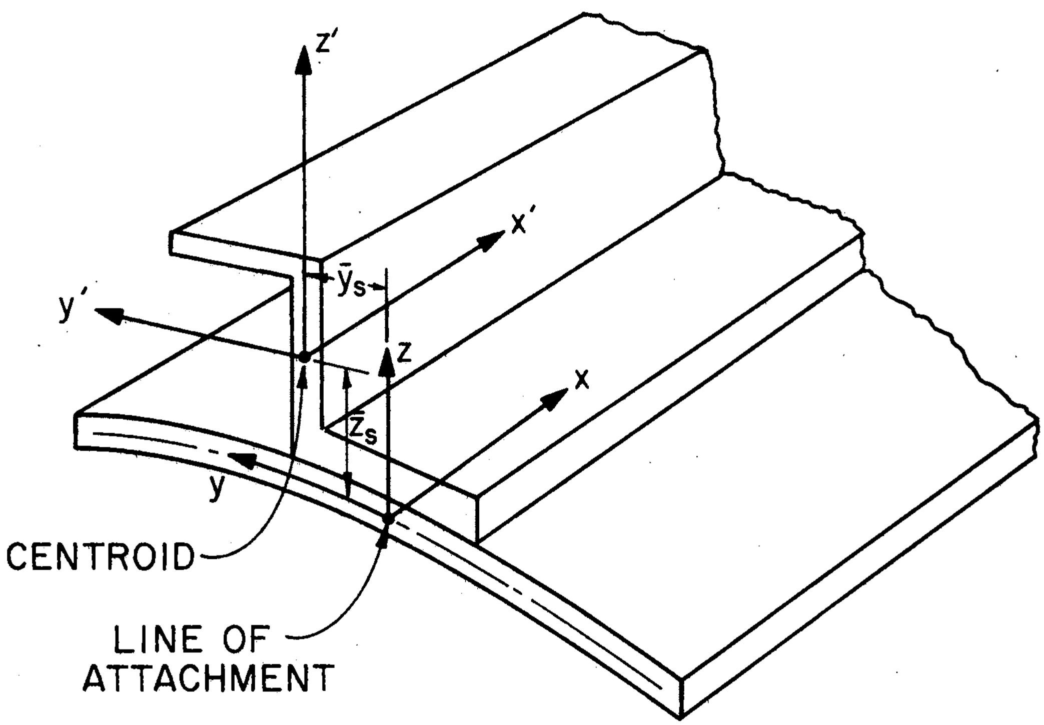

Figure 2. Geometry of a Typical Stringer

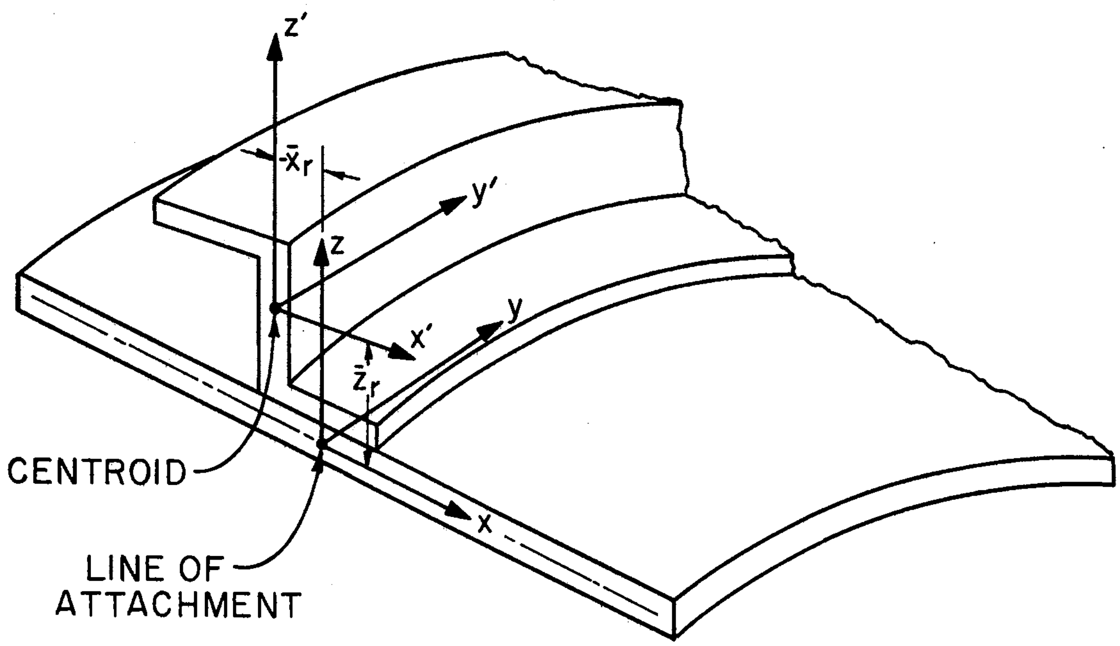

Figure 3. Geometry of a Typical Ring

$$[C]_s = \begin{bmatrix} 1 & -(\bar{y}_s + y_s')\frac{\partial}{\partial x} & -(\bar{z}_s + z_s')\frac{\partial}{\partial x} \\ 0 & 1+\frac{1}{R}(\bar{z}_s + z_s') & -\frac{1}{R}(\bar{z}_s + z_s')\frac{\partial}{\partial \theta} \\ 0 & -\frac{1}{R}(\bar{y}_s + y_s') & 1+\frac{1}{R}(\bar{y}_s + y_s')\frac{\partial}{\partial \theta} \end{bmatrix}$$

and, the matrix $[C]_r$ may be expressed as

$$[C]_r = \begin{bmatrix} 1 & 0 & -(\bar{z}_r + z_r')\frac{\partial}{\partial x} \\ -\frac{1}{R}(\bar{x}_r + x_r')\frac{\partial}{\partial \theta} & 1+\frac{1}{R}(\bar{z}_r + z_r') & -\frac{1}{R}(\bar{z}_r + z_r')\frac{\partial}{\partial \theta} \\ 0 & 0 & 1+(\bar{x}_r + x_r')\frac{\partial}{\partial x} \end{bmatrix}$$

<u>Strain-Displacement Relations</u>

The strain at any point in the shell can be expressed in terms of six middle surface strain components; i.e., two normal strains (ϵ_x and ϵ_θ), one shearing strain ($\epsilon_{x\theta}$); two curvature changes (κ_{xz} and $\kappa_{\theta z}$) and one twist (τ). These may be expressed in matrix form as

$$\{\epsilon\}_o = \begin{Bmatrix} \epsilon_x \\ \epsilon_\theta \\ \epsilon_{x\theta} \\ \kappa_{xz} \\ \kappa_{\theta z} \\ \tau \end{Bmatrix}_o$$

Likewise, the strains at the centroid of the stringer are

$$\{\epsilon\}_s = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \\ \varkappa_{xz} \\ \varkappa_{xy} \\ \tau \end{Bmatrix}_s$$

and the strains at the centroid of the ring are

$$\{\epsilon\}_r = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \\ \varkappa_{yz} \\ \varkappa_{yx} \\ \tau \end{Bmatrix}_r .$$

The strain-displacement equations relate the strain at any point in the structure to the displacements of the shell median surface. These may be written for the i^{th} component in matrix form as

$$\{\epsilon\}_i = [B]_i \{q\}_o \quad . \qquad (2.3)$$

The strain-displacement relations used in this study reflect the postulates of Love's First Approximation Theory for thin elastic shells. The matrix $[B]_o$ may be expressed as

$$[B]_o = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{1}{R}\frac{\partial}{\partial \theta} & \frac{1}{R} \\ \frac{1}{R}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} \\ 0 & \frac{1}{R^2}\frac{\partial}{\partial \theta}+\frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{1}{R}\right) & -\frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}-\frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{1}{R}\right)\frac{\partial}{\partial \theta} \\ 0 & \frac{1}{R}\frac{\partial}{\partial x} & -\frac{2}{R}\frac{\partial^2}{\partial x \partial \theta} \end{bmatrix}.$$

The Bernoulli theory of bending was used for the stringers and rings. Thus, the matrix $[B]_s$ for the stringers is

$$[B]_s = \begin{bmatrix} \frac{\partial}{\partial x} & -\bar{y}_s\frac{\partial^2}{\partial x^2} & -\bar{z}_s\frac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} \\ 0 & \frac{\partial^2}{\partial x^2} & 0 \\ 0 & -\frac{1}{R}\frac{\partial}{\partial x} & \frac{1}{R}\frac{\partial^2}{\partial \theta \partial x} \end{bmatrix}$$

and the matrix $[B]_r$ for the rings is shown in Equation (2.4) on page 15.

$$
[B]_r = \begin{bmatrix}
0 & 0 & 0 \\
-\frac{\bar{x}_r}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\frac{\partial}{\partial\theta} - \frac{\bar{x}_r}{R_c R}\frac{\partial^2}{\partial\theta^2} & \frac{1}{R_c}\left(1+\frac{\bar{z}_r}{R}\right)\frac{\partial}{\partial\theta} + \frac{\bar{z}_r}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right) & -\frac{\bar{z}_r}{R_c R}\frac{\partial^2}{\partial\theta^2} - \frac{\bar{z}_r}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\frac{\partial}{\partial\theta} + \frac{1}{R_c} + \frac{\bar{x}_r}{R_c}\frac{\partial}{\partial x} \\
0 & 0 & 0 \\
0 & -\frac{1}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right) - \frac{1}{R R_c}\frac{\partial}{\partial\theta} & \frac{1}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\frac{\partial}{\partial\theta} + \frac{1}{R R_c}\frac{\partial^2}{\partial\theta^2} \\
-\frac{1}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R_c}\right)\frac{\partial}{\partial\theta} - \frac{1}{R_c^2}\frac{\partial^2}{\partial\theta^2} & 0 & \frac{w_{,x}}{R_c} + \frac{\bar{z}_r}{R_c}\frac{\partial}{\partial\theta}\left(\frac{1}{R_c}\right)\frac{\partial^2}{\partial x\partial\theta} + \frac{\bar{z}_r}{R_c^2}\frac{\partial^3}{\partial x\partial\theta^2} \\
-\frac{1}{R_c^2}\frac{\partial}{\partial\theta} & 0 & \left(-\frac{1}{R_c}+\frac{\bar{z}_r}{R_c^2}\right)\frac{\partial^2}{\partial x\partial\theta}
\end{bmatrix}
\tag{2.4}
$$

Strain and Kinetic Energies

The strain energy of the shell, stringer or ring can each be expressed in terms of the middle surface strain components and stress resultants as

$$U_i = \frac{1}{2} \int_S \{\sigma_R\}_i^T \{\epsilon\}_i \, dS \tag{2.5}$$

where the six stress resultants consist of two normal stress resultants, one shearing stress resultant, two bending stress resultants and one torsional stress resultant. The stress resultants for the shell are

$$\{\sigma_R\}_o = \begin{Bmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_\theta \\ M_x \\ M_{x\theta} \end{Bmatrix}_o$$

and for the stringer

$$\{\sigma_R\}_s = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_y \\ M_z \\ M_x \end{Bmatrix}_s$$

and for the ring

$$\{\sigma_R\}_r = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_z \\ M_y \end{Bmatrix}_r .$$

The stress resultants can be expressed in terms of the middle surface strain components as

$$\{\sigma_R\}_i = [D]_i \{\varepsilon\}_i \quad . \tag{2.6}$$

After Equations (2.3) and (2.6) are substituted into Equation (2.5), the first variation of the strain energy is

$$\delta U_i = \int_S \{\delta q\}_o^T [B]_i^T [D]_i [B]_i \{q\}_o \, dS \quad . \tag{2.7}$$

The kinetic energy of the shell, stringer or ring can be expressed as

$$T_i = \frac{1}{2} \int_S m_i \{\dot{q}\}_i^T \{\dot{q}\}_i \, dS \quad . \tag{2.8}$$

where m_i is the mass of the i^{th} element. After Equation (2.2) is substituted into Equation (2.8), the first variation of the kinetic energy is

$$\delta T_i = \int_S m_i \{\delta\dot{q}\}_o^T [C]_i^T [C]_i \{\dot{q}\}_o \, dS \quad . \tag{2.9}$$

Shell Energies

The first variation of the shell strain energy obtained from specializing Equation (2.7) is expressed as

$$\delta U_o = \int_o^a \int_o^{2\pi} \{\delta q\}_o^T [B]_o^T [D]_o [B]_o \{q\}_o R \, d\theta \, dx \qquad (2.10)$$

where for orthotropic material, the matrix $[D]_o$ is

$$[D]_o = \begin{bmatrix} \frac{E_x h}{1-\nu_{x\theta}\nu_{\theta x}} & \frac{\nu_{\theta x} E_\theta h}{1-\nu_{x\theta}\nu_{\theta x}} & 0 & 0 & 0 & 0 \\ \frac{\nu_{x\theta} E_x h}{1-\nu_{x\theta}\nu_{\theta x}} & \frac{E_\theta h}{1-\nu_{x\theta}\nu_{\theta x}} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{x\theta} h & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E_x h^3}{12(1-\nu_{x\theta}\nu_{\theta x})} & \frac{\nu_{\theta x} E_\theta h^3}{12(1-\nu_{x\theta}\nu_{\theta x})} & 0 \\ 0 & 0 & 0 & \frac{\nu_{x\theta} E_x h^3}{12(1-\nu_{x\theta}\nu_{\theta x})} & \frac{E_\theta h^3}{12(1-\nu_{x\theta}\nu_{\theta x})} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{G_{x\theta} h^3}{12} \end{bmatrix}.$$

After Equation (2.10) is expanded, the first variation of the strain energy for the noncircular cylindrical shell is

$$\delta U_o = \int_o^a \int_o^{2\pi} \left\{ \frac{E_x h}{1-\nu_{x\theta}\nu_{\theta x}} \left[u,_x \delta u,_x + \frac{\nu_{x\theta}}{R} \left(u,_x \delta v,_\theta \right.\right.\right.$$

$$\left.\left. + \, u,_x \delta w \right) \right] + \frac{E_\theta h}{1-\nu_{x\theta}\nu_{\theta x}} \left[\frac{\nu_{\theta x}}{R} \left(v,_\theta \delta u,_x + w \delta u,_x \right) \right.$$

$$+ \frac{1}{R^2}\left(v,_\theta \delta v,_\theta + w\delta v,_\theta + v,_\theta \delta w + w\delta w\right)\Big]$$

$$+ G_{x\theta} h\Big[\frac{1}{R^2} u,_\theta \delta u,_\theta + \frac{1}{R} v,_x \delta u,_\theta + \frac{1}{R} u,_\theta \delta v,_x$$

$$+ v,_x \delta v,_x\Big] + \frac{E_x h^3}{12(1-\nu_{x\theta}\nu_{\theta x})}\Big[w,_{xx}\delta w,_{xx}$$

$$- \frac{\nu_{x\theta}}{R^2}\left(w,_{xx}\delta v,_\theta - w,_{xx}\delta w,_{\theta\theta}\right) - \frac{\nu_{x\theta}}{R}\left(\frac{1}{R}\right),_\theta \Big(w,_{xx}\delta v$$

$$- w,_{xx}\delta w,_\theta\Big)\Big] + \frac{E_\theta h^3}{12(1-\nu_{x\theta}\nu_{\theta x})}\Big[\frac{\nu_{\theta x}}{R^2}\Big(v,_\theta \delta w,_{xx}$$

$$- w,_{\theta\theta}\delta w,_{xx}\Big) + \frac{\nu_{\theta x}}{R}\left(\frac{1}{R}\right),_\theta \left(v\delta w,_{xx} - w,_\theta \delta w,_{xx}\right)$$

$$- \frac{1}{R^4}\left(v,_\theta \delta v,_\theta - w,_{\theta\theta}\delta v,_\theta - v,_\theta \delta w,_{\theta\theta} + w,_{\theta\theta}\delta w,_{\theta\theta}\right)$$

$$- \frac{1}{R^3}\left(\frac{1}{R}\right),_\theta \Big(v\delta v,_\theta - w,_\theta \delta v,_\theta + v,_\theta \delta v - w,_{\theta\theta}\delta v$$

$$- v\delta w,_{\theta\theta} + w,_\theta \delta w,_{\theta\theta} - v,_\theta \delta w,_\theta + w,_{\theta\theta}\delta w,_\theta\Big)$$

$$- \frac{1}{R^2}\left\{\left(\frac{1}{R}\right),_\theta\right\}^2 \left(v\delta v - w,_\theta \delta v - v\delta w,_\theta + w,_\theta \delta w,_\theta\right)\Big]$$

$$+ \frac{G_{x\theta}h^3}{12}\Big[\frac{1}{R^2} v,_x \delta v,_x - \frac{2}{R^2}\left(w,_{x\theta}\delta v,_x + v,_x \delta w,_{x\theta}\right)$$

$$+ \frac{4}{R^2} w,_{x\theta}\delta w,_{x\theta}\Big]\Big\} R\, d\theta\, dx \quad . \qquad (2.11)$$

The first variation of the shell kinetic energy obtained from specializing Equation (2.9) to the case of harmonic motion is

$$\delta T_o = \omega^2 \int_o^a \int_o^{2\pi} m_o \{\delta q\}_o^T [C]_o^T [C]_o \{q\}_o R\, d\theta\, dx \; . \quad (2.12)$$

Expanding this expression, the first variation of the kinetic energy for the noncircular cylindrical shell is

$$\delta T_o = \omega^2 \int_o^a \int_o^{2\pi} \rho_o h\{u\delta u + v\delta v + w\delta w\} \, R \, d\theta \, dx \quad . \qquad (2.13)$$

Stringer Energies

The first variation of the strain energy for the ℓ^{th} stringer located at θ_ℓ is expressed as

$$\delta U_{s\ell} = \int_o^a \{\delta q\}_o^T [B]_{s\ell}^T [D]_{s\ell} [B]_{s\ell} \{q\}_o \, dx \Big|_{\theta=\theta_\ell} \qquad (2.14)$$

where

$$[D]_{s\ell} = \begin{bmatrix} E_{s\ell}A_{s\ell} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{s\ell}I_{yy_{s\ell}} & -E_{s\ell}I_{yz_{s\ell}} & 0 \\ 0 & 0 & 0 & -E_{s\ell}I_{yz_{s\ell}} & E_{s\ell}I_{zz_{s\ell}} & 0 \\ 0 & 0 & 0 & 0 & 0 & (GJ)_{s\ell} \end{bmatrix} .$$

Equation (2.14) is expanded and presented in Appendix B.

Two methods of analysis were used in this research. The first method is called the "discrete" method. In this approach, the first variation of the stringer strain energy (Appendix B) is evaluated at the specific θ (i.e., θ_ℓ) where each stringer is located. Thus, the total contribution of all the stringers is expressed, for a discrete analysis, as

$$\delta U_{D_s} = \sum_{\ell=1}^{ns} \delta U_{s\ell} \Big|_{\theta = \theta_\ell} \quad . \tag{2.15}$$

The second method employs a "modified smearing" technique. In a "conventional smearing" approach, the first variation of the strain energy (Appendix B) is integrated over a region wherein the stringer is assumed to be "smeared" and divided by the length of the region, 2Δ. Thus, the total contribution of all the stringers for a conventional smearing approach is

$$\delta U_{CS_s} = \sum_{\ell=1}^{ns} \frac{1}{2\Delta} \int_{\theta_\ell - \Delta}^{\theta_\ell + \Delta} \delta U_{s\ell} \, d\theta \quad . \tag{2.16}$$

In this study, the conventional smearing technique is modified by introducing a constant of proportionality defined as the ratio of the "discrete" energy to the "conventionally smeared" energy. This is expressed for the j^{th} mode as

$$\delta U_{s\ell} \Big|_{\theta=\theta_\ell} = \alpha_{\ell j} \frac{1}{2\Delta} \int_{\theta_\ell - \Delta}^{\theta_\ell + \Delta} \delta U_{s\ell} \, d\theta \quad . \tag{2.17}$$

Thus, the total contribution of all the stringers for the modified smearing approach may be expressed as

$$\delta U_{MS_s} = \sum_{\ell=1}^{ns} \alpha_{\ell j} \frac{1}{2\Delta} \int_{\theta_\ell - \Delta}^{\theta_\ell + \Delta} \delta U_{s\ell} \, d\theta \quad . \tag{2.18}$$

(It should be noted, for $\alpha_{\ell j} = 1.0$, Equation (2.18) is identical to Equation (2.16).)

The first variation of the kinetic energy for the ℓ^{th} stringer, located at θ_ℓ, is expressed as

$$\delta T_{s\ell} = \omega^2 \int_0^a m_\ell \{\delta q\}_o^T [C]_{s\ell}^T [C]_{s\ell} \{q\}_o \, dx \,\Big|_{\theta=\theta_\ell} \qquad (2.19)$$

This expression is expanded and presented in Appendix B. The equations for the kinetic energy are developed in the same manner as those for the strain energy. Therefore, the total contribution of all the stringers, for a discrete analysis, is

$$\delta T_{D_s} = \sum_{\ell=1}^{ns} \delta T_{s\ell} \,\Big|_{\theta=\theta_\ell} \qquad (2.20)$$

and, for a conventional smearing analysis, is

$$\delta T_{CS_s} = \sum_{\ell=1}^{ns} \frac{1}{2\Delta} \int_{\theta_\ell - \Delta}^{\theta_\ell + \Delta} \delta T_{s\ell} \, d\theta \quad . \qquad (2.21)$$

Furthermore, for a modified smearing analysis,

$$\delta T_{MS_s} = \sum_{\ell=1}^{ns} \beta_{\ell j} \frac{1}{2\Delta} \int_{\theta_i - \Delta}^{\theta_i + \Delta} \delta T_{s\ell} \, d\theta \qquad (2.22)$$

where

$$\beta_{\ell j} = \frac{\delta T_{s\ell}\big|_{\theta=\theta_\ell}}{\frac{1}{2\Delta}\int_{\theta_\ell-\Delta}^{\theta_\ell+\Delta} \delta T_{s\ell}\, d\theta} \quad . \tag{2.23}$$

The values of $\alpha_{\ell j}$ and $\beta_{\ell j}$ are determined by the following iteration sequence. The constants $\alpha_{\ell j}$ and $\beta_{\ell j}$ are first assumed to have a value of one (thus formulating a conventional smearing analysis). The eigenvector associated with the particular eigenvalue of interest is substituted into Equations (2.17) and (2.23) and new values of $\alpha_{\ell j}$ and $\beta_{\ell j}$ are determined from these equations. The improved values are substituted into Equations (2.18) and (2.22) to formulate the modified smeared problem. From the solution of this problem, new values of $\alpha_{\ell j}$ and $\beta_{\ell j}$ are determined. This iterative process is continued until the difference between the values of the j^{th} eigenvalue obtained from two consecutive iterations is less than 0.01.

Ring Energies

The first variation of the strain energy for the k^{th} ring (located at x_k) is

$$\delta U_{rk} = \int_0^{2\pi} \{\delta q\}_o^T [B]_{rk}^T [D]_{rk} [B]_{rk} \{q\}_o R\, d\theta \Big|_{x=x_k} \tag{2.24}$$

where

$$[D]_{rk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{rk}E_{rk} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{rk}I_{xx_{rk}} & -E_{rk}I_{xz_{rk}} & 0 \\ 0 & 0 & 0 & -E_{rk}I_{xz_{rk}} & E_{rk}I_{zz_{rk}} & 0 \\ 0 & 0 & 0 & 0 & 0 & (GJ)_{rk} \end{bmatrix} .$$

This expression is expanded and presented in Appendix C. The equations for the ring strain energy are developed in the same manner as those for the stringer. Thus, the total contribution of all the rings, for a discrete analysis, is expressed as

$$\delta U_{D_r} = \sum_{k=1}^{nr} \delta U_{rk}\Big|_{x=x_k} \tag{2.25}$$

and, for a conventional smearing analysis, as

$$\delta U_{CS_r} = \sum_{k=1}^{nr} \frac{1}{2\Delta} \int_{x_k-\Delta}^{x_k+\Delta} \delta U_{rk}\, dx \tag{2.26}$$

and, for a modified smearing analysis, as

$$\delta U_{MS_r} = \sum_{k=1}^{nr} \alpha_{kj} \frac{1}{2\Delta} \int_{x_k-\Delta}^{x_k+\Delta} \delta U_{rk}\, dx \tag{2.27}$$

where

$$\alpha_{kj} = \frac{\delta U_{rk} \big|_{x=x_k}}{\frac{1}{2\Delta} \int_{x_k-\Delta}^{x_k+\Delta} \delta U_{rk} \, dx} \quad . \tag{2.28}$$

The first variation of the kinetic energy for the k^{th} ring (located at x_k) is

$$\delta T_{rk} = \omega^2 \int_0^{2\pi} m_r \{\delta q\}_o^T [C]_{rk}^T [C]_{rk} \{q\}_o R \, d\theta \big|_{x=x_k} . \tag{2.29}$$

This expression is expanded and presented in Appendix C. The equations for the ring kinetic energy are developed in the same manner as those for the stringer. Therefore, the total contribution of all the rings, for a discrete analysis, is expressed as

$$\delta T_{D_r} = \sum_{k=1}^{nr} \delta T_{rk} \big|_{x=x_k} \tag{2.30}$$

and, for a conventional smearing analysis, as

$$\delta T_{CS_r} = \sum_{k=1}^{nr} \frac{1}{2\Delta} \int_{x_k-\Delta}^{x_k+\Delta} \delta T_{rk} \, dx \tag{2.31}$$

and, for a modified smearing analysis, as

$$\delta T_{MS_r} = \sum_{k=1}^{nr} \beta_{kj} \frac{1}{2\Delta} \int_{x_k-\Delta}^{x_k+\Delta} \delta T_{rk} \, dx \tag{2.32}$$

where

$$\beta_{kj} = \frac{\delta T_{rk} \big|_{x=x_k}}{\frac{1}{2\Delta}\int_{x_k-\Delta}^{x_k+\Delta} \delta T_{rk}\, dx} \quad , \qquad (2.33)$$

The modified smearing analysis of rings is carried out in the same manner as in the analysis of stringer energies.

Displacement Functions

The displacements u, v and w are assumed to be double finite series. Conventionally, each term of the series is a product of a circumferential and an axial function weighted by a time-dependent, generalized coordinate. The assumed displacement functions are:

$$u(x, \theta, t) = \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} (\bar{u}_{mn} \cos n\theta + \bar{u}'_{mn} \sin n\theta)\, U_m(x) e^{i\omega t}$$

$$v(x, \theta, t) = \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} (\bar{v}_{mn} \sin n\theta - \bar{v}'_{mn} \cos n\theta)\, V_m(x) e^{i\omega t} \qquad (2.34)$$

$$w(x, \theta, t) = \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} (\bar{w}_{mn} \cos n\theta + \bar{w}'_{mn} \sin n\theta)\, W_m(x) e^{i\omega t}$$

where $U_m(x)$, $V_m(x)$ and $W_m(x)$ are the axial mode functions which satisfy at least the kinematic boundary conditions of the stiffened shell; $\bar{u}_{mn}$, $\bar{v}_{mn}$ and $\bar{w}_{mn}$ are unknown amplitude coefficients of the symmetric circumferential modes and $\bar{u}'_{mn}$, $\bar{v}'_{mn}$ and $\bar{w}'_{mn}$ are those associated with the antisymmetric modes.

In this analysis the axial mode functions $U_m(x)$, $V_m(x)$ and $W_m(x)$ were expressed by a single function $\Phi_m(x)$ such that

$$U_m(x) = \frac{d}{dx}\Phi_m(x)$$

$$V_m(x) = \Phi_m(x) \tag{2.35}$$

$$W_m(x) = \Phi_m(x) \quad .$$

The following functions were used in this analysis:

Boundary Condition	Function Used	Eqn. No.
Freely supported:	$\Phi_m(x) = \sqrt{2}\sin\frac{m\pi x}{a}$	(2.36a)
Clamped-free:	$\Phi_m(x) = X_m$, Characteristic function of a Clamped-free beam.	(2.36b)
Clamped-clamped:	$\Phi_m(x) = X_m$, Characteristic function of a Clamped-clamped beam.	(2.36c)
Free-free:	$\Phi_0(x) = 1$ $\Phi_1(x) = x/a - \frac{1}{2}$ $\Phi_m(x) = X_{m-1}$, Characteristic function of a Free-free beam. $(m \geq 2)$	(2.36d)

The characteristic functions X_m, their derivatives and eigenvalue properties are tabulated in Reference 25.

Frequency Equation

The first variation of the total strain energy of the stiffened shell is obtained, for the discrete analysis, by combining Equations (2.11), (2.15) and (2.25). Similarly, the first variation of the total kinetic energy is obtained by combining Equations (2.13), (2.22) and (2.32). Substituting the first variations of the total strain and kinetic energies into Equation (2.1) we obtain

$$\int_{t_0}^{t_1} \{\delta q\}^T \Big[[K] - \omega^2 [M]\Big] \{q\} \; dt = 0 \quad .$$

Since the $\{\delta q\}$ are arbitrary, the following eigenvalue problem is obtained

$$\Big[[K] - \omega^2 [M]\Big] \{q\} = 0 \quad .$$

This equation can be rearranged in the following form

$$\left[\begin{bmatrix} K_{ss} & K_{sa} \\ K_{sa}^T & K_{aa} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{ss} & M_{sa} \\ M_{sa}^T & M_{aa} \end{bmatrix} \right] \begin{Bmatrix} q_s \\ q_a \end{Bmatrix} = 0 \qquad (2.37)$$

where $\{q_s\}$ and $\{q_a\}$ denote the symmetric and antisymmetric mode vectors.

In Equation (2.37) the off-diagonal submatrices of both the stiffness and mass matrices vanish if the cross section of the stiffened shell is symmetric with respect to the vertical axis. Thus, the above equation is uncoupled into two equations - one for symmetric, and the other for antisymmetric modes. The equation for the symmetric modes may be written as

$$\left[\begin{bmatrix} K11 & K12 & K13 \\ K12^T & K22 & K23 \\ K13^T & K23^T & K33 \end{bmatrix} - \omega^2 \begin{bmatrix} M11 & M12 & M13 \\ M12^T & M22 & M23 \\ M13^T & M23^T & M33 \end{bmatrix}\right] \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = 0. \quad (2.38)$$

The frequency equation for the smeared analysis is formulated in the same manner as in the case of the discrete analysis. Equations (2.11), (2.18) and (2.27) are combined for the first variation of the total strain energy and Equations (2.13), (2.22) and (2.32) are combined for the first variation of the total kinetic energy.

CHAPTER III

COMPUTER SOLUTION

General

A computer program was developed to determine the modes of free vibration and the corresponding frequencies for an arbitrary stringer- and ring-stiffened, orthotropic noncircular cylindrical shell. The mass and stiffness matrices of the structure are computed in the program and the eigenvalues and eigenvectors are calculated using the subroutine EIGENP (26). The material properties of the shell may be either isotropic or orthotropic. The program allows the stiffeners to have arbitrary cross sections and to be treated as either discrete or smeared. Circumferential integrals and smearing integrals are evaluated using an eight point Gaussian quadrature method with four subintervals. Figure 4 shows a flow chart of this program. The Oklahoma State University IBM Model 360/65 computer was employed for this project.

The input data for the program is categorized into four kinds. The first is general data, e.g., number of terms considered in the assumed displacement series, the number of stringers, the number of rings, etc. The other three kinds of data are shell data, stringer

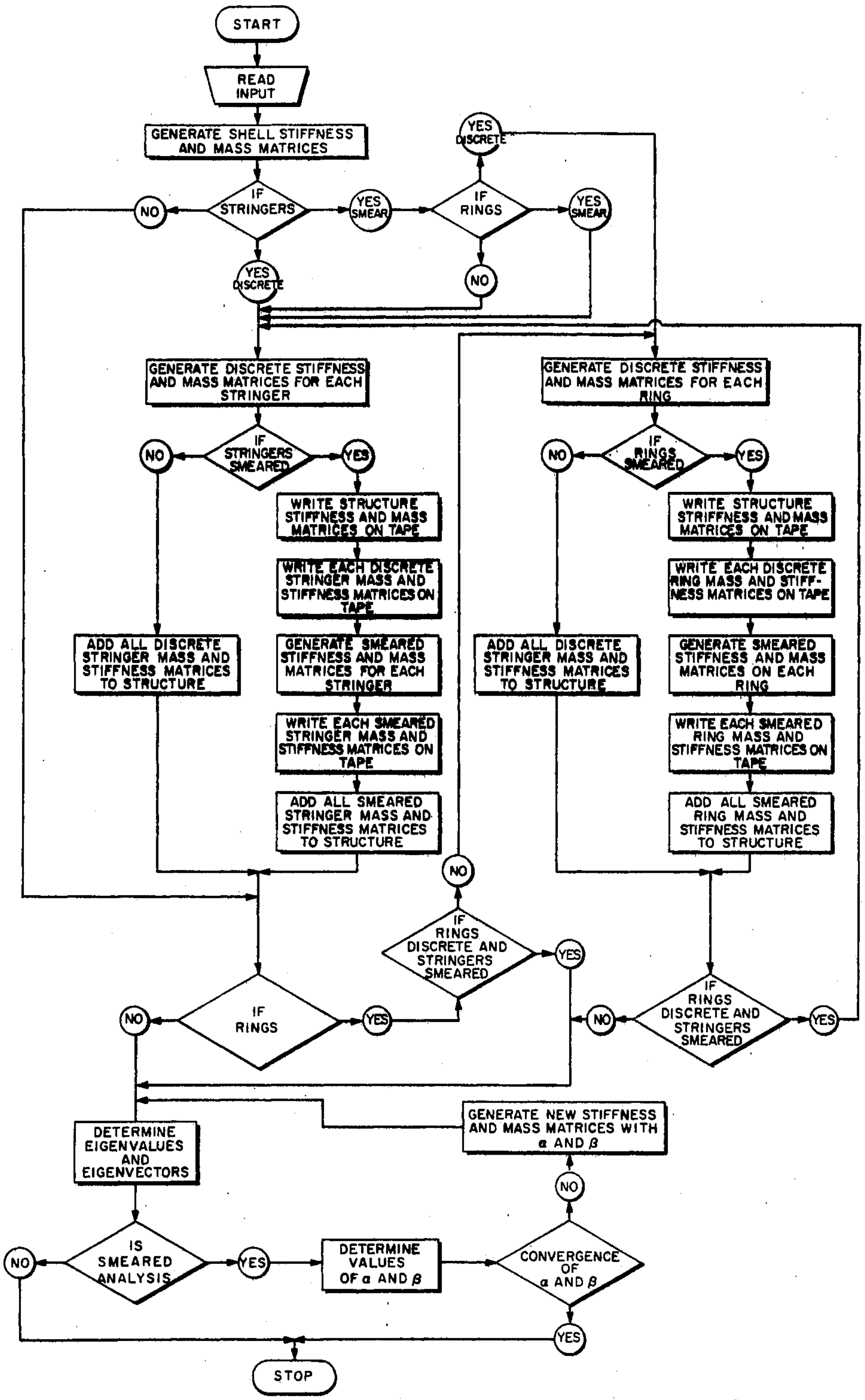

Figure 4. Flow Chart of the Computer Program

data, and ring data. The radius of curvature, R, of the shell is considered to be a tabulated function of the θ coordinate. The expressions for R, $\left(\frac{1}{R}\right)_{,\theta}$, $R_{,\theta}$ are calculated in function subprograms. These subprograms must be supplied for each problem.

Natural Frequencies and Mode Shapes

The order of the stiffness and mass matrices of Equation (2.38) depends on the total number of circumferential and axial terms in the assumed displacement series. If M^* circumferential and N^* axial terms are used, then the order of the matrices is $3M^*N^*$ and Equation (2.38) may be written as

$$\left[[K] - \omega^2 [M]\right] \begin{Bmatrix} \bar{u} \\ --- \\ \bar{v} \\ --- \\ \bar{w} \end{Bmatrix} = 0 \tag{3.1}$$

where the generalized coordinates are

$$\bar{u} = \begin{Bmatrix} \bar{u}_{00} \\ \bar{u}_{01} \\ \bar{u}_{02} \\ \vdots \\ \bar{u}_{0N^*} \\ \bar{u}_{10} \\ \bar{u}_{11} \\ \bar{u}_{12} \\ \vdots \\ \bar{u}_{M^*N^*} \end{Bmatrix} \qquad \bar{v} = \begin{Bmatrix} \bar{v}_{00} \\ \bar{v}_{01} \\ \bar{v}_{02} \\ \vdots \\ \bar{v}_{0N^*} \\ \bar{v}_{10} \\ \bar{v}_{11} \\ \bar{v}_{12} \\ \vdots \\ \bar{v}_{M^*N^*} \end{Bmatrix} \qquad \bar{w} = \begin{Bmatrix} \bar{w}_{00} \\ \bar{w}_{01} \\ \bar{w}_{02} \\ \vdots \\ \bar{w}_{0N^*} \\ \bar{w}_{10} \\ \bar{w}_{11} \\ \bar{w}_{12} \\ \vdots \\ \bar{w}_{M^*N^*} \end{Bmatrix}$$

and

K = stiffness matrix,

M = mass matrix,

ω = the natural frequencies in radians/second.

If the matrices K and M become singular due to rows and columns of zeroes for certain boundary conditions, the matrices are condensed by eliminating those rows and columns of zeroes. The subroutine called EIGENP (26) is used to calculate the frequencies (ω) of Equation (3.1) and the resulting generalized coordinates.

After the natural frequencies and associated generalized coordinates are obtained, the corresponding mode shapes are determined from another program based on Equation (2, 34). As in conventional free vibration problems, only the normalized displacements can be found. The displacement mode shapes can be calculated at any x or θ value.

CHAPTER IV

NUMERICAL RESULTS

Introduction

The discrete method of analysis described in this dissertation was substantiated by comparing results of this study with some of those obtained by previous investigators. A comparison of the discrete and smearing methods of analysis was made and is presented in this chapter.

Comparison of Discrete Analysis With Known Solutions

The natural frequencies of (1) unstiffened circular cylinders, (2) unstiffened noncircular cylinders and (3) stringer- and/or ring-stiffened circular cylinders obtained using the discrete method of analysis were compared with known solutions and presented in this section.

Comparison of Results for Unstiffened Circular Shells

Forsberg (27) obtained the exact frequencies for freely supported, unstiffened, circular cylinders by solving the differential equations of motion. Boyd and Rao (23) solved the same problem using the Rayleigh-Ritz approach. Both analyses used the Flügge shell theory. The results of these analyses and those obtained from this study are listed in Table I for the longitudinal mode m = 1. A good correlation between the frequencies was obtained from the three analyses. The frequencies obtained from the present analysis are slightly higher than those obtained using the higher-order theory of Flügge, References 23 and 27.

Sewall et al. (17) studied freely supported circular cylinders using Sander's shell theory and the Rayleigh-Ritz approach. The same structure was studied (23) using Flügge shell theory. The results obtained from this study and those from References 17 and 23 are shown in Table II for the longitudinal modes m = 1 and 2. The correlation of the results from the three studies was excellent.

The experimental and analytical results of Reference 28 for a clamped-free, unstiffened, circular shell are compared to those of the present analysis (for the longitudinal modes m = 1 and 2) in Table III. Comparisons were also made with the experimental results of Park et al. (29), the analytical results of Egle and Soder (21), and the

TABLE I

COMPARISON OF ANALYTICAL FREQUENCIES OF A FREELY SUPPORTED UNSTIFFENED CIRCULAR CYLINDER[a]

n	m	Present Analysis	Boyd & Rao[b]	Forsberg[c]
2	1	779	778	778
	2	2450	2449	2449
	3	4254	4253	4253
3	1	628	628	627
	2	1459	1458	1458
	3	2683	2682	2681
4	1	975	974	974
	2	1305	1304	1303
	3	2023	2021	2020
	4	2950	2947	2946

a) The geometry of the shell is given in Reference 27.

b) Reference 23.

c) Reference 27.

TABLE II

COMPARISON OF FREQUENCIES OF A FREELY SUPPORTED CIRCULAR CYLINDER[a]

n	m = 1			m = 2		
	Present Analysis	Boyd & Rao[b]	Sewall[c]	Present Analysis	Boyd & Rao[b]	Sewall[c]
1	1565.3	1565.3	1565.0	2309.3	2309.3	2309.0
2	894.1	894.1	894.1	1782.4	1782.4	1782.0
3	529.9	529.8	529.8	1314.9	1314.9	1315.0
4	338.7	338.6	338.6	968.4	968.4	968.4
5	235.6	235.6	235.6	726.3	726.3	726.3
6	182.2	182.1	182.1	560.4	560.3	560.3
7	162.2	162.2	162.2	448.6	448.6	448.6
8	167.0	166.9	166.9	377.2	377.2	377.2
9	188.6	188.6	188.6	338.2	338.1	338.1
10	221.4	221.3	221.3	325.8	325.7	325.1
11	261.7	261.7	261.7	335.1	335.0	335.0
12	308.1	308.0	308.0	361.2	361.0	361.0
13	359.5	359.5	359.5	399.6	399.6	399.5
14	415.6	415.6	415.6	447.5	447.5	447.5

a) The geometry of this shell is given in Reference 17.

b) Reference 23.

c) Reference 17.

TABLE III

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE UNSTIFFENED CIRCULAR CYLINDER[a]

n	m = 1			m = 2		
	Present Analysis	Sewall[b] Analysis	Sewall[b] Exp.	Present Analysis	Sewall[b] Analysis	Sewall[b] Exp.
1	787.2	791.6	-	2017.5	>2000.0	-
2	394.2	395.9	-	1311.0	1314.0	-
3	218.5	219.3	201.8 206.2	875.0	876.8	-
4	137.8	138.2	131.7	612.4	613.6	-
5	103.1	103.3	100.8	449.0	449.7	429.1
6	97.8	97.9	96.9	345.8	346.3	326.3 334.4
7	112.1	112.1	113.0	283.5	283.8	273.5
8	138.1	138.1	140.4	252.4	252.6	247.4
9	171.4	171.3	174.3	246.7	246.8	244.2
10	210.1	210.1	214.6	260.8	260.9	261.5
11	253.7	253.7	258.4	289.6	289.6	292.1
12	301.7	301.7	307.6	328.8	328.8	335.2
13	354.1	354.1	361.2	375.8	375.8	381.1 393.2
14	410.8	410.8	416.0 424.9	429.1	429.1	437.8

a) The geometry of the shell is given in Reference 28.

b) Reference 28.

analytical results of Boyd and Rao (23). These are presented in Table IV. Generally, the comparisons are good. The slight differences might be attributed to the difference in shell theories and to inexact boundary conditions in the experiment.

The validity of the present analysis for the free-free boundary-conditions case was established by comparing the results of this analysis with the experimental and analytical results of Reference 17. Table V shows good agreement between the results of the two studies.

Comparison of Results for Stringer-Stiffened Circular Shells

Egle and Sewall (20) presented frequencies obtained for freely supported, circular cylinders, with and without stringers, using the Donnell shell theory and neglecting the insurface inertias. Reference 23 presented results for the same structures using the Donnell theory but including the insurface inertias. Table VI shows the results (for the longitudinal modes m = 1 and 2) obtained from this analysis and those presented in References 20 and 23. The frequencies obtained by Egle and Soder are slightly higher than those of Boyd and Rao. This discrepancy is evidently attributable to the neglect of the insurface inertias. The results of Boyd and Rao are higher than those of the present analysis. The differences might be attributed to differences in compatibility relations used.

TABLE IV

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE UNSTIFFENED CIRCULAR CYLINDER[a]

n	m = 1				m = 2			
	Present Analysis	Boyd[b] & Rao Anal.	Egle &[c] Soder Anal.	Park[d] et al. Exp.	Present Analysis	Boyd[b] & Rao Anal.	Egle &[c] Soder Anal.	Park[d] et al. Exp.
2	104.1	104.4	104.4	87.2 95.1	507.2	508.2	-	-
3	55.6	55.6	55.6	51.5	280.9	281.3	-	-
4	52.0	52.0	52.0	50.4	177.7	177.9	177.9	168.5 170.2
5	71.6	71.6	-	70.9	135.3	135.4	-	132.8
6	101.8	101.8	-	101.4	132.0	132.0	-	128.8 130.1
7	139.2	139.1	139.1	138.8	154.2	154.2	154.2	153.6
8	182.6	182.6	182.6	182.2	191.2	191.2	191.2	191.3

a) The geometry of the shell is given in Reference 29.

b) Reference 23.

c) Reference 21.

c) Reference 29.

TABLE V

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A FREE-FREE UNSTIFFENED CIRCULAR CYLINDER[a]

	m = 1				m = 2			
n	Present Anal.	Boyd[b] & Rao Anal.	Sewall[c] Anal.	Sewall[c] Exp.	Present Anal.	Boyd[b] & Rao Anal.	Sewall[c] Anal.	Sewall[c] Exp.
1	2013.1	2012.0	2014.0	-	2289.6	2288.0	2293.0	-
2	8.2	7.5	7.5	7.7	1614.0	1613.0	1616.0	-
3	19.4	19.0	19.0	18.9	1066.8	1066.0	1068.0	-
4	34.4	34.2	34.2	35.7	717.3	716.9	717.8	-
5	53.5	53.4	53.4	53.0	504.6	504.4	504.8	-
6	76.8	76.6	76.7	76.4	375.6	375.4	375.6	377.3
7	104.2	104.1	104.1	103.8	300.0	299.8	299.9	299.1
8	135.8	135.7	135.7	135.3	262.3	262.6	262.2	257.4 262.1
9	171.5	171.4	171.5	170.7	253.5	253.6	253.4	248.8 249.3
10	211.5	211.4	211.5	210.2	266.5	266.5	266.3	268.8
11	255.7	255.6	255.7	253.0	294.9	294.8	294.7	290.9
12	304.1	303.9	304.1	305.5	334.1	333.9	334.0	327.6
13	356.7	356.5	356.7	352.0	381.2	381.2	381.1	-
14	413.5	413.3	413.5	412.5	434.7	434.7	434.7	436.6

a) The geometry of the shell is given in Reference 17.

b) Reference 23.

c) Reference 17.

TABLE VI

COMPARISON OF FREQUENCIES OF A FREELY SUPPORTED CIRCULAR CYLINDER WITH AND WITHOUT STRINGERS[a]

m	n	Stringer Stiffened						Unstiffened		
		Sym. Mode			Antisym. Mode					
		Present[b] Discrete Analysis	Boyd[c] & Rao	Egle &[d] Sewall	Present[b] Discrete Analysis	Boyd[c] & Rao	Egle &[d] Sewall	Pres.[b] Anal.	Boyd[c] & Rao	Egle &[d] Sewall
1	3	160	-	169	160	-	169	-	-	171
	4	99	-	103	103	-	108	-	-	108
	5	91	-	95	91	-	95	-	-	98
	6	106	-	109	112	-	116	-	-	117
	7	140	-	145	140	-	145	-	-	151
	8	179	-	183	187	-	192	-	-	194
	9	231	-	236	231	-	236	-	-	243
	10	273	-	278	291	-	297	-	-	300
	11	345	-	350	345	-	350	-	-	362
	12	403	-	408	419	-	425	-	-	431

TABLE VI (Continued)

m	n	Stringer Stiffened						Unstiffened		
		Sym. Mode			Antisym. Mode					
		Present[b] Discrete Analysis	Boyd & Rao[c]	Egle & Sewall[d]	Present[b] Discrete Analysis	Boyd & Rao[c]	Egle & Sewall[d]	Pres. Anal.[b]	Boyd & Rao[c]	Egle & Sewall[d]
2	3	557	555	591	557	555	591	568	568	602
	4	336	337	346	349	348	365	353	353	365
	5	235	236	241	235	236	241	245	246	251
	6	190	192	194	196	197	202	198	200	203
	7	187	189	191	187	189	191	192	194	196
	8	205	208	209	211	213	217	214	216	218
	9	252	254	256	252	254	256	253	256	258
	10	292	295	297	300	303	306	305	308	309
	11	353	355	358	353	355	358	364	367	369
	12	418	421	424	424	427	430	432	435	436

a) The geometry of the shell is given in Reference 20.
b) Love Theory and insurface inertias included.
c) Donnell Theory and insurface inertias included, Reference 23.
d) Donnell Theory and insurface inertias neglected, Reference 20.

Comparison of Results for Ring-Stiffened Circular Shells

Al-Najafi and Warburton (30) presented both analytical and experimental results for ring-stiffened circular shells with freely supported and free-free boundary conditions. Their analytical results were obtained using a finite element technique in which the insurface inertias were neglected. Reference 23 presented results for the same structure including insurface inertias. A comparison is shown in Table VII between the results of this study and those of References 23 and 30. The frequencies obtained from the present analysis for the freely supported case are lower than the analytical values of References 23 and 30. This might be attributed to either the difference in shell theories or the compatibility relations used. For the free-free case, the finite element results were, in general, observed to be closer to the experimental values than the results obtained from either the present analysis or Reference 23. In general, the results of this analysis and those of References 23 and 30 show a good correlation.

Comparison of Results for Ring- and Stringer-Stiffened Circular Shells

Park et al. (29) presented the results of an experimental study of ring- and stringer-stiffened circular shells with clamped-free boundary conditions. Egle and Soder (21) compared their analytical results with

TABLE VII

COMPARISON OF FREQUENCIES OF A RING-STIFFENED CIRCULAR SHELL[a]

m	Freely Supported				Free-Free			
	Present Analysis	Boyd & Rao Anal.[b]	War-burton Anal.[c]	War-burton Exp.[c]	Present Analysis	Boyd & Rao Anal.[b]	War-burton Anal.[c]	War-burton Exp.[c]
0	-	-	-	-	1540	1550	1547	1551
1	1848	1867	1873	1867	1536	1538	1537	1539
2	2070	2089	2091	2076	1765	1889	1895	1890
3	2630	2651	2650	2600	2290	2303	2290	2287
4	3391	3415	3429	3355	3055	3075	3044	3044
5	4215	4239	4270	-	3935	3955	3920	3916
6	4945	4925	5022	-	4776	4910	-	-
7	5805	5846	-	-	5546	5548	-	-
8	6548	6585	-	-	6303	6349	-	-
9	7294	7330	-	-	7080	7103	-	-
10	8046	8079	-	-	-	-	-	-

a) The geometry of the shell is given in Reference 30, (5 rings, d = 0.25 inches).

b) Reference 23.

c) Reference 30.

the experimental results of Reference 29 for a circular cylinder with three internal rings and sixteen internal stringers. Boyd and Rao (23) studied the same stiffened shell. The results of the present analysis are compared in Table VIII with those of References 21, 23 and 29. The results of this analysis and those of Boyd and Rao (23) are consistently lower than those of Egle and Soder (21). In general, the results of this analysis compared favorably with those of Boyd and Rao (23). The difference might be attributed to the difference in compatibility relations used.

Comparison of Results for Unstiffened Noncircular Shells

Elliptical shells were studied to substantiate the validity of the analysis for noncircular shells. Sewall et al. (17) presented analytical and experimental results for elliptical shells with various end conditions. Analytical results for these shells were presented by Boyd and Rao (23). Tables IX and X show the results of this study and those of References 17 and 23 for freely supported, elliptical shells having eccentricities of 0.526 and 0.760, respectively. It is evident from Tables IX and X that the results of this analysis are in excellent agreement with those of Reference 23 and with those for the lower circumferential modes of Reference 17.

TABLE VIII

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE RING- AND STRINGER-STIFFENED CIRCULAR SHELL[a]

n	m	Present Analysis	Boyd[b] & Rao Anal.	Egle &[c] Soder Anal.	Park[d] et al. Exp.
2	1	100.5	100.2	105.8	80.2 88.2
	2	433.9	432.2	433.9	-
	3	915.9	907.0	-	-
4	1	207.1	207.6	216.9	184.6
	2	276.3	276.0	285.9	251.5
	3	440.0	437.2	447.1	397.0 430.4
6	1	313.4	308.5	315.0	-
	2	350.7	345.9	353.8	-
	3	411.4	402.6	414.0	-

a) The geometry of the shell is given in Reference 29, (Model 1S).

b) Reference 23.

c) Reference 21.

d) Reference 29.

TABLE IX

COMPARISON OF FREQUENCIES OF A FREELY SUPPORTED ELLIPTICAL CYLINDER,[a]

$\epsilon = 0.526,\ m = 1$

n	Symmetric			Antisymmetric		
	Present Analysis	Boyd & Rao[b]	Sewall[c]	Present Analysis	Boyd & Rao[b]	Sewall[c]
0	2550.3	2550.2	2550.2	-	-	-
1	1439.7	1439.7	1440.0	1685.7	1685.7	1686.0
2	876.6	876.6	876.6	888.9	888.9	888.9
3	524.1	524.1	524.1	524.2	524.2	524.2
4	335.6	335.5	335.5	335.6	335.6	335.5
5	234.3	234.3	234.3	234.4	234.3	234.2
6	184.3	184.2	184.2	184.3	184.2	184.2
7	156.9	156.9	157.1	157.0	156.9	157.0
8	160.1	160.1	160.2	160.2	160.2	160.2
9	189.7	189.7	189.8	189.4	189.4	189.8
10	221.5	221.5	221.9	221.8	221.8	221.9
11	260.8	260.8	261.9	261.7	261.7	261.9
12	307.7	307.6	308.1	307.9	307.9	308.1
13	348.9	348.9	359.5	355.8	355.8	359.5
14	405.7	405.7	415.6	413.9	413.9	415.6

a) The geometry of the shell is given in Reference 17.

b) Reference 23.

c) Reference 17.

TABLE X

COMPARISON OF FREQUENCIES OF A FREELY SUPPORTED ELLIPTICAL CYLINDER,[a]
$\epsilon = 0.760$, $m = 1$

n	Symmetric			Antisymmetric		
	Present Analysis	Boyd & Rao[b]	Sewall[c]	Present Analysis	Boyd & Rao[b]	Sewall[c]
0	2611.8	2611.8	2612.0	-	-	-
1	1237.8	1237.7	1238.0	1855.7	1855.7	1856.0
2	785.1	785.1	785.2	858.5	858.5	858.5
3	491.1	491.1	491.1	492.5	492.5	492.4
4	319.8	319.8	319.4	318.9	318.9	319.4
5	-	-	-	226.6	226.6	226.9
6	-	-	-	-	-	-
7	138.5	138.5	138.5	138.6	138.5	138.5
8	140.0 177.8	140.0 177.8	140.1 178.3	140.1 178.5	140.1 178.5	140.1 178.3
9	182.3 226.1	182.3 226.1	184.1 226.9	184.1	184.0	184.1
10	221.7	221.7	223.9	223.5	223.5	223.9
11	261.6	261.6	263.6	259.2	259.2	263.6
12	310.6	310.6	307.3	296.9	296.9	307.3
13	378.4	378.4	359.4	338.7	338.6	359.4
14	464.9	464.8	417.1	399.6	399.6	417.1

a) The geometry of the shell is given in Reference 17.

b) Reference 23.

c) Reference 17.

The results for a free-free elliptical shell are presented in Table XI. The results of this analysis give a good correlation with those of References 17 and 23.

Comparison of Discrete and Smearing Methods of Analysis

Having shown the validity of the discrete method of analysis, solutions for stiffened circular shells were obtained from the smearing and discrete methods of analysis and are compared in this section.

Effects of Smearing Stringers

The shell used by Egle and Soder (21) was selected as the case for the comparison. Table XII presents the frequencies obtained for the doubly symmetric (i.e., with respect to vertical and horizontal axes through the centroid of the cylinder cross section) modes of this shell stiffened with 4, 8, 16 and 60 stringers using both the discrete and the conventional smearing methods of analysis. Indistinguishable results were obtained from the two methods for the shell with 60 stringers. Even for the shells stiffened by 4, 8 and 16 stringers, the lower frequencies obtained from the smearing method of analysis were in good agreement with those obtained from the discrete analysis.

The effect of the ratio of stringer-mass to shell-mass (m_s/m_o) was studied. The shell with four stringers was used for this study to determine the effects for widely spaced stringer cases. Using both the

TABLE XI

COMPARISON OF FREQUENCIES OF A FREE-FREE ELLIPTICAL CYLINDER,[a]
$\epsilon = 0.526$, $m = 0$

n	Symmetric				Antisymmetric			
	Pres. Anal.	Boyd[b] & Rao Anal.	Sewall[c] Anal.	Sewall[c] Exp.	Pres. Anal.	Boyd[b] & Rao Anal.	Sewall[c] Anal.	Sewall[c] Exp.
2	5.62	5.62	5.62	5.6	5.68	5.68	5.68	5.6
3	15.89	15.89	15.89	16.1	15.89	15.89	15.89	16.2
4	30.52	30.52	30.52	30.9	30.52	30.52	30.52	30.8
5	49.42	49.42	49.41	50.1	49.42	49.42	49.41	50.1
6	72.54	72.54	72.54	74.8	72.54	72.54	72.54	74.4
7	99.87	99.87	99.87	102.4	99.87	99.87	99.87	102.4
8	131.42	131.42	131.40	134.6	131.42	131.42	134.40	-
9	167.18	167.18	167.20	171.5	167.17	167.17	167.20	171.7
10	207.13	207.13	207.10	212.5	207.10	207.10	207.10	212.8
11	251.34	251.34	251.30	258.8	251.26	251.26	251.30	258.4
12	299.46	299.46	299.60	312.1	299.72	299.72	299.60	-
13	351.95	351.95	352.20	363.8	353.19	353.19	352.20	362.3 363.0
14	408.14	408.14	409.00	423.2	411.17	411.17	409.00	-

a) The geometry of the shell is given in Reference 17.

b) Reference 23.

c) Reference 17.

TABLE XII

COMPARISON OF FREQUENCIES OBTAINED FROM DISCRETE AND CONVENTIONAL SMEARED ANALYSES FOR STIFFENED CIRCULAR SHELLS[a]

m = 1

Number of Stringers	Symmetric Modes n	Discrete Analysis	Conventional Smearing Analysis	Error
4	4	98.81	100.99	2.21%
	6	105.56	108.55	2.84%
	8	178.81	181.87	1.71%
8	4	95.57	98.57	3.14%
	6	104.81	104.98	0.16%
	8	166.98	175.46	5.10%
16	4	94.30	94.46	0.17%
	6	98.67	98.90	0.23%
	8	151.88	164.52	8.35%
60	4	78.76	78.76	0.00%
	6	80.01	81.01	0.00%
	8	128.39	128.39	0.00%

a) The shell and stringer properties are given in Reference 21.

discrete and conventional smearing methods of analysis, the frequencies for the even (symmetric) modes were obtained for various mass ratios. Figure 5 shows the ratio of the lowest frequencies (ω_o/ω_s) versus the ratio of the masses. As would be expected, the difference between the frequencies obtained from the two methods increases as the mass-ratio increases.

The "modified smearing" method of analysis was studied considering the shell stiffened by four stringers. Table XIII presents the results of this study. The frequencies obtained for the double symmetric modes converged monotonically to values close to the discrete values. For the singly symmetric (i.e., with respect to the vertical axis only) modes, the frequencies converged to values further from the discrete values. Investigating the assumed mode shapes, it was observed that if a stringer was located precisely at a nodal point of the assumed displacement function, the modified smearing method evidently added more energy to the system than was present in the physical problem. This may explain the divergence of the singly symmetric modes of Table XIII from the frequencies obtained by the discrete analysis.

The effect of mass-ratio was also studied using the modified smearing method of analysis. Figure 6 shows the ratio of the lowest frequencies (ω_o/ω_s) associated with the doubly symmetric modes versus the mass ratio for both smearing methods of analysis. For all mass-ratios, the frequencies obtained using the modified smearing

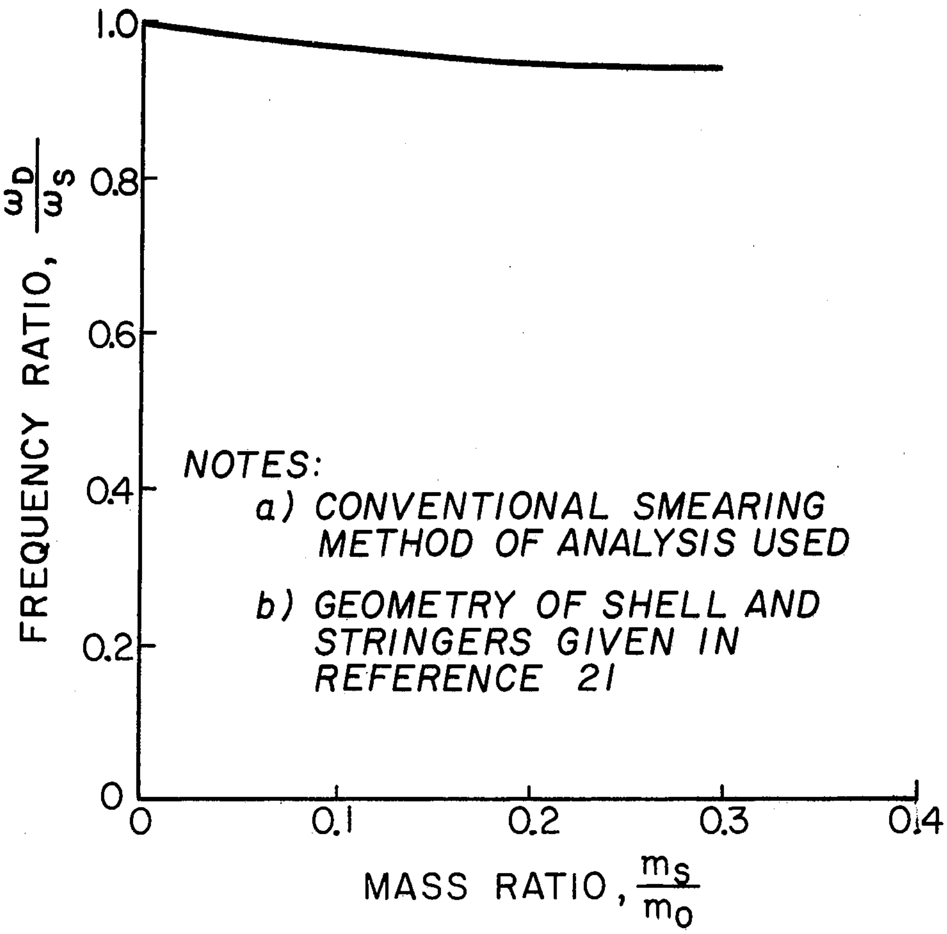

Figure 5. The Effect on Frequency of Varying Mass Ratio of a Shell Stiffened by Four Stringers

TABLE XIII

COMPARISON OF FREQUENCIES OBTAINED FROM DISCRETE AND MODIFIED SMEARED ANALYSES FOR STIFFENED CIRCULAR SHELLS[a]

$m = 1$

n	Discrete Analysis	Modified Smearing Method of Analysis				
		Iteration 0	Iteration 1	Iteration 2	Iteration 3	Iteration 4
			Even Modes			
4	98.81	100.99	98.59	98.69	98.66	98.67
6	105.56	108.55	105.41	105.43	105.43	105.43
			Odd Modes			
5	90.64	90.84	90.86	90.93	90.93	-
7	139.71	140.86	141.72	141.79	141.78	-

a) The shell and stringer properties are given in Reference 21.

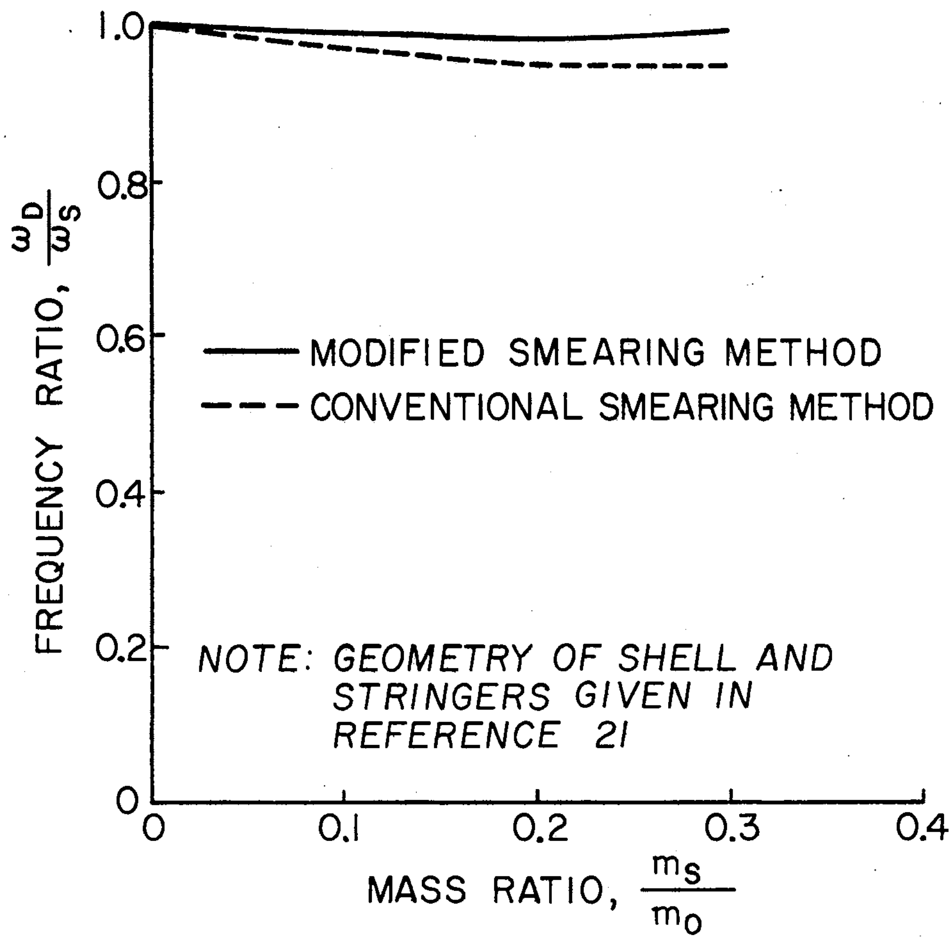

Figure 6. The Effect on Frequency of Varying Mass Ratio of a Shell Stiffened by Four Stringers

method were in closer agreement with those obtained from the discrete method of analysis.

Results for Smearing Rings

Freely supported shells, stiffened by rings having various cross-sectional areas, were studied. A shell stiffened with one ring having a square cross section and located at the center of the shell was studied first. Frequencies were obtained for five cases by varying the cross-sectional area of the ring. A shell having two rings with identical square cross sections and located at a/4 and 3a/4 was also studied. A range of frequencies was obtained by varying the cross-sectional area of the rings. Figure 7 shows the ratios of frequencies obtained by the two methods versus the ratios of ring-mass to shell-mass. The difference between the frequencies obtained from the discrete and smearing methods of analysis increases as the mass-ratio increases. For a shell stiffened by only one ring, the frequencies obtained from the conventional smearing method were closer to the discrete values than those of the modified smearing method. However, for two or more rings, the reverse occurred.

The longitudinal mode shapes associated with the fundamental frequency are shown in Figure 8, having two different mass ratios, and two rings for simply supported shells stiffened by one ring. These were obtained using the discrete method of analysis. The smearing methods of analysis used the sine function (in the longitudinal direction)

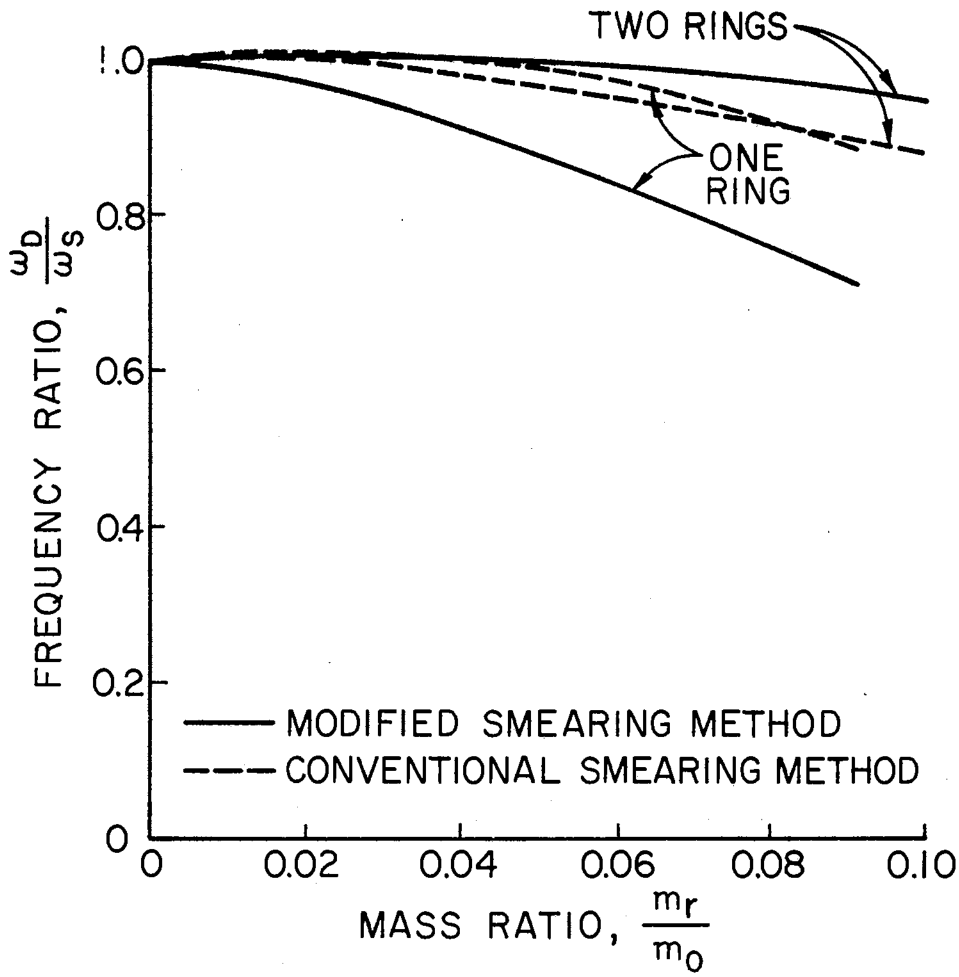

Figure 7. The Effect on Frequency of Varying Mass Ratio of a Shell Stiffened by Rings

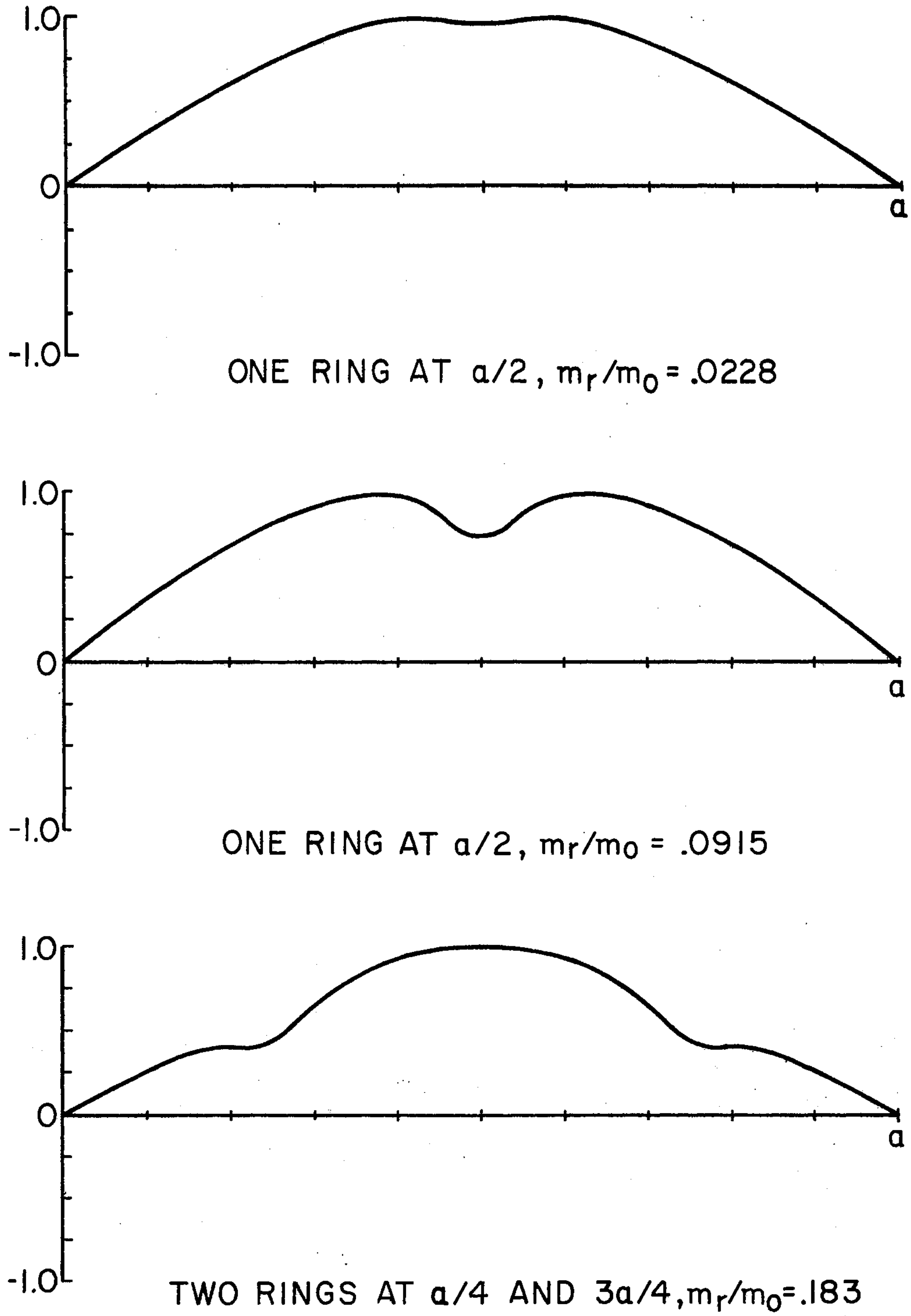

Figure 8. Longitudinal Mode Shapes, n = 4, m = 1

to represent the w displacements of the fundamental mode. As the number of rings increased for a given mass ratio, the sine function better approximated the fundamental longitudinal mode shape. Therefore, the difference between the frequencies obtained from the smearing methods of analysis and the discrete analysis decreases.

Table XIV presents the frequencies obtained using the three methods of analysis for the simply supported shell stiffened by one ring having a mass ratio of 0.0228. For the even longitudinal mode shapes, the frequencies obtained using the modified smearing method of analysis are less than the discrete values. The torsional strain energy is the predominant part of the strain energy for the case in which the ring is located at the node point of the mode shape in the physical problem. However, the smearing method of analysis uses the same constant of proportionality for both the extensional and torsional strain energies. Therefore, the rings evidently have more extensional strain energy theoretically than is physically present. The frequencies obtained will be lower as a result. This characteristic is typical of shells with stiffeners located at (or near) node points.

TABLE XIV

COMPARISON OF FREQUENCIES OBTAINED FROM DISCRETE AND SMEARING METHODS OF ANALYSIS FOR A SHELL WITH ONE RING[a]

n	m	Discrete Analysis	Conventional Smearing Analysis	Modified Smearing Analysis
4	1	630.2	615.0	650.1
	2	1362.3	1371.4	1345.9
	3	2182.9	2189.6	2191.5
	4	2857.2	2858.9	2822.6
	5	3393.1	3400.3	3403.9
	6	3878.4	3881.3	3831.2

a) The shell and ring properties are given in Reference 30, $d = 0.25$, $R = 7.567$ inches.

CHAPTER V

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Two methods of analysis have been presented in this study to determine the natural frequencies and mode shapes of unstiffened or stiffened, circular or noncircular, cylindrical shells. One method of analysis considered the stiffeners to be discretely located and the other considered the stiffeners to be "smeared" over the surface of the shell (using an equivalent energy approach). Both methods employ the Rayleigh-Ritz energy method. Cases for stiffened and unstiffened, circular and noncircular cylindrical shells having various boundary conditions were investigated using both methods of analysis. A summary of the results of this study, conclusions made from this study and recommendations for future investigations are given in the following paragraphs.

Summary of the Results

The following results were obtained.

1. The natural frequencies obtained for discretely stiffened or unstiffened, circular or noncircular cylinders using Love's

First Approximation Theory for thin elastic shells were in good agreement with those obtained by other investigators using higher-order theories.

2. The discrete method of analysis presented in this study was shown to be valid for simply supported, free-free and clamped-free boundary conditions by comparisons with results obtained by previous investigators.

3. For the discretely stiffened shells, the natural frequencies obtained using the set of improved compatibility relations were in good agreement with those obtained by other investigators using other compatibility relations.

4. The computation time required for problem solution and the computer storage requirements for the discrete method of analysis were about the same as required by other discrete methods also using the Rayleigh-Ritz approach.

5. As the number of stringers increased, the differences between the frequencies obtained using the conventional smearing method and the discrete method of analysis decreased.

6. As the stringer mass-ratios increased, for a constant number of stringers, the differences between the lowest frequencies obtained from the conventional smearing method and the discrete method increased.

7. For stringer-stiffened cylinders, the frequencies obtained for the doubly symmetric modes using the modified smearing method converged to values close to the discrete values. However, for the singly symmetric modes, the iterative procedure did not improve the calculated frequencies.

8. In general, the frequencies obtained from both smearing methods of analysis were in good agreement with those obtained using the discrete method of analysis.

9. As the number of rings increased, the difference between the frequencies obtained from either smearing method and the discrete method of analysis decreased.

10. For shells stiffened by two or more rings, the fundamental frequencies obtained from the modified smearing method were closer to the discrete values than those obtained from the conventional smearing method.

11. As the mass-ratios increased for a constant number of rings, the differences between the frequencies obtained from the smearing methods and the discrete method of analysis increased.

12. For ring-stiffened shells, the frequencies obtained for the even longitudinal modes using the modified smearing method converged to values less than the discrete values.

Conclusions

The major conclusions made from this study are listed.

1. For shells stiffened by either stringers or rings located at the nodal points of the terms in the assumed displacement series, the best agreement between the smearing methods and the discrete method of analysis can be obtained using the conventional smearing method of analysis. However, if the stiffeners are not located at the nodal points, the best agreement is obtained using the modified smearing method of analysis. Background of this conclusion is discussed in Chapter IV.

2. Because stringers produce weak circumferential coupling, comparable results can be obtained from either the discrete or the smearing methods of analysis assuming the same number of terms in the displacement series is used. Therefore, for a shell stiffened by only stringers, the computation time required for problem solution and the computer storage requirements for the smearing method of analysis are nearly equal to those of the discrete method of analysis.

3. For a ring-stiffened shell, the difference between the values of the frequencies obtained from the discrete and smearing methods of analysis depends not only on the mass-ratio but on the number of rings as well. When the results obtained from the smearing and discrete methods are in good

agreement, the smearing methods of analysis required fewer terms in the displacement series than was required by the discrete method of analysis. Thus, the computer storage requirement is less for the smearing methods of analysis than for the discrete methods of analysis. Sufficient data was not available to compare the computer times required by the three methods of analysis.

Recommendations

The following recommendations are made for further study:

1. The smearing method of analysis for ring-stiffened shells should be studied further to define the limitations of this method. The study should consider such parameters as number of rings, mass ratios, and stiffener cross section and shell stiffness parameters.

2. The modified smearing method of analysis should be studied using three constants of proportionality instead of the two (i.e., β and α) used in this study. One should be associated with the kinetic energy of the stiffener and two others with the extensional and torsional strain energies. This approach should improve the modified smearing method of analysis, particularly for those shells having stiffeners located at (or near) the node points of the displacement functions.

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APPENDIXES

APPENDIX A

DERIVATION OF THE COMPATIBILITY RELATIONS

The compatibility relations for the stiffeners were derived based on the assumption that the stiffeners are attached to the shell along a single line of attachment. The displacement vector of any point in the cross section of the i^{th} stiffener can be written as

$$\{q\}_i = \{q\}_{ci} + \{\phi\}_{i/ci} \times \{\Omega\}_{i/ci} \quad , \tag{A1}$$

where

$$i = \begin{cases} s \text{ for the stringer} \\ r \text{ for the ring,} \end{cases}$$

$\{q\}_i$ = the displacement vector of an arbitrary point in the cross section of the stiffener,

$\{q\}_{ci}$ = the displacement vector of the centroid of the stiffener,

$\{\phi\}_{i/ci}$ = the rotation-vector at an arbitrary point in the cross section with respect to axes through the centroid of the stiffener,

$\{\Omega\}_{i/ci}$ = the position vector of an arbitrary point in the cross section measured from the centroid of the stiffener.

These vectors may be expanded as follows:

$$\{q\}_i = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_i$$

$$\{q\}_{ci} = \begin{Bmatrix} u_c \\ v_c \\ w_c \end{Bmatrix}_i$$

$$\{\phi\}_{i/ci} = \begin{Bmatrix} \phi_x \\ \phi_\theta \\ \phi_z \end{Bmatrix}_{i/ci} \qquad \text{(A2)}$$

$$\{\Omega\}_{s/cs} = \begin{Bmatrix} 0 \\ y'_s \\ z'_s \end{Bmatrix} \qquad \text{(see Figure 2)}$$

$$\{\Omega\}_{r/cr} = \begin{Bmatrix} x'_r \\ 0 \\ z'_r \end{Bmatrix} \qquad \text{(see Figure 3)}$$

The displacement vector of the centroid of the i^{th} stiffener can be written as:

$$\{q\}_{ci} = \{q\}_o + \{\phi\}_{ci/o} \times \{\Omega\}_{ci/o} , \qquad \text{(A3)}$$

where

$\{q\}_o$ = the displacement vector of an arbitrary point on the middle surface of the shell along the line of attachment,

$\{\phi\}_{ci/o}$ = the angle of rotation vector through the centroid of the i^{th} stiffener measured from the middle surface of the shell axes,

$\{\Omega\}_{ci/o}$ = the position vector of the centroid of the i^{th} stiffener measured from the line of attachment to the shell.

These vectors may be expanded as follows:

$$\{q\}_o = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_o$$

$$\{\phi\}_{ci/o} = \begin{Bmatrix} \phi_x \\ \phi_\theta \\ \phi_z \end{Bmatrix}_{ci/o}$$

$$\{\Omega\}_{cs/o} = \begin{Bmatrix} 0 \\ \bar{y}_s \\ \bar{z}_s \end{Bmatrix} \qquad \text{(A4)}$$

$$\{\Omega\}_{cr/o} = \begin{Bmatrix} \bar{x}_r \\ 0 \\ \bar{z}_r \end{Bmatrix} .$$

After Equation (A3) is substituted into Equation (A1), the following compatibility relations can be written:

$$\{q\}_i = \{q\}_o + \{\phi\}_{ci/o} \times \{\Omega\}_{ci/o} + \{\phi\}_{i/ci} \times \{\Omega\}_{i/ci} . \qquad \text{(A5)}$$

Assuming the Bernoulli theory of bending for the stiffeners and Love's first approximation to the theory of thin elastic shells, the angle of rotation vectors can be equated as follows:

$$\{\phi\}_{ci/o} = \{\phi\}_{i/ci} = \{\phi\}_o \tag{A6}$$

where

$\{\phi\}_o$ = the angle of rotation vector of the middle surface of the shell.

This vector may be expanded as:

$$\{\phi\}_o = \begin{Bmatrix} \phi_x \\ \phi_\theta \\ \phi_z \end{Bmatrix}_o \tag{A7}$$

where

$$\phi_x = \frac{w,_\theta}{R} - \frac{v}{R}$$

$$\phi_\theta = -w,_x$$

$$\phi_z = \begin{cases} \dfrac{-u,_\theta}{R} & \text{for rings} \\ v,_x & \text{for stringers} \end{cases}$$

After Equations (A2), (A4) and (A7) are substituted into Equation (A5), the compatibility relations of the stringer can be written in terms of displacements of the shell as follows:

$$\{q\}_s = \begin{Bmatrix} u - (\bar{z}_s + z_s')\,w,_x - (\bar{y}_s + y_s')\,v,_x \\ v - (\bar{z}_s + z_s')\left(\dfrac{w,_\theta}{R} - \dfrac{v}{R}\right) \\ w + (\bar{y}_s + y_s')\left(\dfrac{w,_\theta}{R} - \dfrac{v}{R}\right) \end{Bmatrix} \tag{A8}$$

Likewise, the compatibility relations of the ring can be written as:

$$\{q\}_r = \begin{Bmatrix} u - (\bar{z}_r + z_r')\,w,_x \\ v - (\bar{x}_r + x_r')\dfrac{u,_\theta}{R} - (\bar{z}_r + z_r')\left(\dfrac{w,_\theta}{R} - \dfrac{v}{R}\right) \\ w + (\bar{x}_r + x_r')\,w,_x \end{Bmatrix} \tag{A9}$$

APPENDIX B

ENERGY EXPRESSIONS FOR STRINGERS

The first variation of the kinetic energy of the ℓ^{th} stringer located at θ_ℓ can be expressed as:

$$\delta T_{s\ell} = \rho_{s\ell}\omega^2 \int_0^a \Big\{ A_{s\ell}(u\delta u + v\delta v + w\delta w) - \bar{y}_{s\ell} A_{s\ell}(v,_x \delta u$$

$$+ u\delta v,_x + \frac{1}{R} v\delta w - \frac{1}{R} w,_\theta \delta w + \frac{1}{R} w\delta v - \frac{1}{R} w\delta w,_\theta)$$

$$- \bar{z}_{s\ell} A_{s\ell}(w,_x \delta u + u\delta w,_x - \frac{2}{R} v\delta v + \frac{1}{R} w,_\theta \delta v$$

$$+ \frac{1}{R} v\delta w,_\theta) + (\bar{y}^2_{s\ell} A_{s\ell} + I_{zz_{s\ell}})(v,_x \delta v,_x + \frac{1}{R^2} v\delta v$$

$$- \frac{1}{R^2} w,_\theta \delta v - \frac{1}{R^2} v\delta w,_\theta + \frac{1}{R^2} w,_\theta \delta w,_\theta) + (\bar{z}^2_{s\ell} A_{s\ell}$$

$$+ I_{yy_{s\ell}})(w,_x \delta w,_x + \frac{1}{R^2} v\delta v - \frac{1}{R^2} w,_\theta \delta v - \frac{1}{R^2} v\delta w,_\theta$$

$$+ \frac{1}{R^2} w,_\theta \delta w,_\theta) + (\bar{y}_{s\ell} \bar{z}_{s\ell} A_{s\ell} + I_{yz_{s\ell}})(w,_x \delta v,_x +$$

$$+ v,_x \delta w,_x) \Big\} \Big|_{\theta = \theta_\ell} dx \; .$$

The first variation of the strain energy of the ℓ^{th} stringer located at θ_ℓ can be expressed as:

$$\delta U_{s\ell} = \int_0^a \Big\{ E_{s\ell} A_{s\ell} (u,_x \delta u,_x) - \bar{y}_{s\ell} E_{s\ell} A_{s\ell} (u,_x \delta v,_{xx}$$

$$+ v,_{xx} \delta u,_x) - \bar{z}_{s\ell} E_{s\ell} A_{s\ell} (u,_x \delta w,_{xx} + w,_{xx} \delta u,_x)$$

$$+ E_{s\ell} (\bar{y}^2_{s\ell} A_{s\ell} + I_{zz_{s\ell}}) (v,_{xx} \delta v,_{xx}) + E_{s\ell} (\bar{z}^2_{s\ell} A_{s\ell}$$

$$+ I_{yy_{s\ell}}) (w,_{xx} \delta w,_{xx}) + E_{s\ell} (\bar{y}_{s\ell} \bar{z}_{s\ell} A_{s\ell}$$

$$+ I_{yz_{s\ell}}) (v,_{xx} \delta w,_{xx} + w,_{xx} \delta v,_{xx})$$

$$+ (GJ)_{s\ell} \frac{1}{R^2} (w,_{\theta x} \delta w,_{\theta x} - w,_{\theta x} \delta v,_x - v,_x \delta w,_{\theta x}$$

$$+ v,_x \delta v,_x) \Big\} \Big|_{\theta = \theta_\ell} dx \ .$$

APPENDIX C

ENERGY EXPRESSIONS FOR RINGS

The first variation of the kinetic energy of the k^{th} ring located at x_k can be expressed as:

$$\delta T_{rk} = \rho_{rk}\omega^2 \int_0^{2\pi} \Big\{ A_{rk}(u\delta u + v\delta v + w\delta w) - \bar{z}_{rk}A_{rk}(u\delta w,_x$$

$$+ w,_x\delta u - \frac{2}{R}v\delta v + \frac{1}{R}v\delta w,_\theta + \frac{1}{R}w,_\theta\delta v)$$

$$-\bar{x}_{rk}\, A_{rk}(\frac{1}{R}u,_\theta\delta v + \frac{1}{R}v\delta u,_\theta - w\delta w,_x - w,_x\delta w)$$

$$+ (\bar{z}^2_{rk}\, A_{rk} + I_{xx_{rk}})\,(w,_x\delta w,_x + \frac{1}{R^2}v\delta v - \frac{1}{R^2}v\delta w,_\theta$$

$$- \frac{1}{R^2}w,_\theta\delta v + \frac{1}{R^2}w,_\theta\delta w,_\theta) + (\bar{x}^2_{rk}A_{rk} + I_{zz_{rk}})\,(\frac{1}{R^2}u,_\theta\delta u,_\theta$$

$$+ w,_x\delta w,_x) + (\bar{x}_{rk}\,\bar{z}_{rk}\,A_{rk} + I_{xz_{rk}})\,(-\frac{1}{R^2}u,_\theta\delta v$$

$$+ \frac{1}{R^2}u,_\theta\delta w,_\theta - \frac{1}{R^2}v\delta u,_\theta + \frac{1}{R^2}w,_\theta\delta u,_\theta)\Big\}\Big|_{x=x_k} R_c\, d\theta$$

The first variation of the strain energy of the k^{th} ring located at x_k can be expressed as:

$$\delta U_{rk} = \int_0^{2\pi} \Big\{ \frac{E_{rk}\,A_{rk}}{R_c}\,(v,_\theta\delta v,_\theta + w\delta v,_\theta + v,_\theta\delta w + w\delta w)$$

$$+ \frac{E_{rk} A_{rk} \bar{z}_{rk}}{R_c} \Big(\frac{2}{R} v,_\theta \delta v,_\theta + \left(\frac{1}{R}\right),_\theta v \delta v,_\theta$$

$$- \frac{1}{R} w,_{\theta\theta} \delta v,_\theta - \left(\frac{1}{R}\right),_\theta w,_\theta \delta v,_\theta + \frac{1}{R} w \delta v,_\theta$$

$$+ \left(\frac{1}{R}\right),_\theta v,_\theta \delta v + \left(\frac{1}{R}\right),_\theta w \delta v - \frac{1}{R} v,_\theta \delta w,_{\theta\theta}$$

$$- \frac{1}{R} w \delta w,_{\theta\theta} - \left(\frac{1}{R}\right),_\theta v,_\theta \delta w,_\theta - \left(\frac{1}{R}\right),_\theta w \delta w,_\theta$$

$$+ \frac{1}{R} v,_\theta \delta w + \left(\frac{1}{R}\right),_\theta v \delta w - \frac{1}{R} w,_{\theta\theta} \delta w - \left(\frac{1}{R}\right),_\theta w,_\theta \delta w \Big)$$

$$+ \frac{E_{rk} A_{rk} \bar{z}^2_{rk}}{R_c} \Big(\frac{1}{R^2} v,_\theta \delta v,_\theta + \frac{1}{R}\left(\frac{1}{R}\right),_\theta v \delta v,_\theta$$

$$- \frac{1}{R^2} w,_{\theta\theta} \delta v,_\theta - \frac{1}{R}\left(\frac{1}{R}\right),_\theta w,_\theta \delta v,_\theta + \frac{1}{R}\left(\frac{1}{R}\right),_\theta v,_\theta \delta v$$

$$+ \left\{\left(\frac{1}{R}\right),_\theta\right\}^2 v \delta v - \frac{1}{R}\left(\frac{1}{R}\right),_\theta w,_{\theta\theta} \delta v$$

$$- \left\{\left(\frac{1}{R}\right),_\theta\right\}^2 w,_\theta \delta v - \frac{1}{R^2} v,_\theta \delta w,_{\theta\theta} - \frac{1}{R}\left(\frac{1}{R}\right),_\theta v \delta w,_{\theta\theta}$$

$$+ \frac{1}{R^2} w,_{\theta\theta} \delta w,_{\theta\theta} + \frac{1}{R}\left(\frac{1}{R}\right),_\theta w,_\theta \delta w,_{\theta\theta}$$

$$- \frac{1}{R}\left(\frac{1}{R}\right),_\theta v,_\theta \delta w,_\theta - \left\{\left(\frac{1}{R}\right),_\theta\right\}^2 v \delta w,_\theta$$

$$+ \frac{1}{R}\left(\frac{1}{R}\right),_\theta w,_{\theta\theta} \delta w,_\theta + \left\{\left(\frac{1}{R}\right),_\theta\right\}^2 w,_\theta \delta w,_\theta \Big)$$

$$+ \frac{E_{rk} I_{xx_{rk}}}{R_c} \Big(\left\{\left(\frac{1}{R}\right),_\theta\right\}^2 v \delta v + \frac{1}{R}\left(\frac{1}{R}\right),_\theta v,_\theta \delta v$$

$$- \left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 w_{,\theta}\delta v - \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} w_{,\theta\theta}\delta v + \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} v\delta v_{,\theta}$$

$$+ \frac{1}{R^2} v_{,\theta}\delta v_{,\theta} - \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} w_{,\theta}\delta v_{,\theta} - \frac{1}{R^2} w_{,\theta\theta}\delta v_{,\theta}$$

$$- \left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 v\delta w_{,\theta} - \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} v_{,\theta}\delta w_{,\theta}$$

$$+ \left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 w_{,\theta}\delta w_{,\theta} + \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} w_{,\theta\theta}\delta w_{,\theta}$$

$$- \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} v\delta w_{,\theta\theta} - \frac{1}{R^2} v_{,\theta}\delta w_{,\theta\theta} + \frac{1}{R}\left(\frac{1}{R}\right)_{,\theta} w_{,\theta}\delta w_{,\theta\theta}$$

$$+ \frac{1}{R^2} w_{,\theta\theta}\delta w_{,\theta\theta}\Big) + \frac{E_{rk} I_{zz_{rk}}}{R_c}\Big(\left\{\left(\frac{1}{R_c}\right)_{,\theta}\right\}^2 u_{,\theta}\delta u_{,\theta}$$

$$+ \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta\theta}\delta u_{,\theta} - \left(\frac{1}{R_c}\right)_{,\theta} w_{,x}\delta u_{,\theta}$$

$$+ \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta u_{,\theta\theta} + \frac{1}{R_c^2} u_{,\theta\theta}\delta u_{,\theta\theta} - \frac{1}{R_c} w_{,x}\delta u_{,\theta\theta}$$

$$- \left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta w_{,x} - \frac{1}{R_c} u_{,\theta\theta}\delta w_{,x} + w_{,x}\delta w_{,x}\Big)$$

$$+ \frac{E_{rk} I_{zz_{rk}} \bar{z}_{rk}}{R_c}\Big(-\left\{\left(\frac{1}{R_c}\right)_{,\theta}\right\}^2 w_{,x\theta}\delta u_{,\theta}$$

$$- \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta\theta}\delta u_{,\theta} - \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta}\delta u_{,\theta\theta}$$

$$- \frac{1}{R_c^2} w_{,x\theta\theta}\delta u_{,\theta\theta} + \left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta}\delta w_{,x}$$

$$+ \frac{1}{R_c} w_{,x\theta\theta}\delta w_{,x} - \left\{\left(\frac{1}{R_c}\right)_{,\theta}\right\}^2 u_{,\theta}\delta w_{,x\theta}$$

$$- \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta\theta}\delta w_{,x\theta} + \left(\frac{1}{R_c}\right)_{,\theta} w_{,x}\delta w_{,x\theta}$$

$$- \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta w_{,x\theta\theta} - \frac{1}{R_c^2} u_{,\theta\theta}\delta w_{,x\theta\theta}$$

$$+ \frac{1}{R_c} w_{,x}\delta w_{,x\theta\theta}\Big)$$

$$+ \frac{E_{rk} I_{zz_{rk}} \bar{z}^2_{rk}}{R_c}\left(\left\{\left(\frac{1}{R_c}\right)_{,\theta}\right\}^2 w_{,x\theta}\delta w_{,x\theta}\right.$$

$$+ \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta\theta}\delta w_{,x\theta} + \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta}\delta w_{,x\theta\theta}$$

$$\left.+ \frac{1}{R_c^2} w_{,x\theta\theta}\delta w_{,x\theta\theta}\right) + \frac{(GJ)_{rk}}{R_c}\left(\frac{1}{R_c^2} u_{,\theta}\delta u_{,\theta}\right.$$

$$\left.+ \frac{1}{R_c} w_{,x\theta}\delta u_{,\theta} + \frac{1}{R_c} u_{,\theta}\delta w_{,x\theta} + w_{,x\theta}\delta w_{,x\theta}\right)$$

$$- \frac{(GJ)_{rk}\bar{z}_{rk}}{R_c}\left(\frac{1}{R_c^2} w_{,x\theta}\delta u_{,\theta} + \frac{1}{R_c^2} u_{,\theta}\delta w_{,x\theta}\right.$$

$$\left.+ \frac{2}{R_c} w_{,x\theta}\delta w_{,x\theta}\right) + \frac{(GJ)_{rk}\bar{z}^2_{rk}}{R_c^3} w_{,x\theta}\delta w_{,x\theta}\Big\}\Big|_{x=x_k} d\theta$$

$$+ \int_0^{2\pi}\left\{ \frac{E_{rk} A_{rk} \bar{x}_{rk}}{R_c}\left(- \left(\frac{1}{R}\right)_{,\theta} v_{,\theta}\delta u_{,\theta} - \left(\frac{1}{R}\right)_{,\theta} w\delta u_{,\theta}\right.\right.$$

$$- \frac{1}{R} v_{,\theta}\delta u_{,\theta\theta} - \frac{1}{R} w\delta u_{,\theta\theta} - \left(\frac{1}{R}\right)_{,\theta} u_{,\theta}\delta v_{,\theta}$$

$$- \frac{1}{R} u_{,\theta\theta}\delta v_{,\theta} + w_{,x}\delta v_{,\theta} - \left(\frac{1}{R}\right)_{,\theta} u_{,\theta}\delta w - \frac{1}{R} u_{,\theta\theta}\delta w$$

$$+ w,_x \delta w + v,_\theta \delta w,_x + w \delta w,_x \Big)$$

$$+ \frac{E_{rk} A_{rk} \bar{x}_{rk}^2}{R_c} \Big(\Big\{ \Big(\frac{1}{R}\Big),_\theta \Big\}^2 u,_\theta \delta u,_\theta$$

$$+ \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta u,_{\theta\theta} \delta u,_\theta - \Big(\frac{1}{R}\Big),_\theta w,_x \delta u,_\theta$$

$$+ \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta u,_\theta \delta u,_{\theta\theta} + \frac{1}{R^2} u,_{\theta\theta} \delta u,_{\theta\theta} - \frac{1}{R} w,_x \delta u,_{\theta\theta}$$

$$- \Big(\frac{1}{R}\Big),_\theta u,_\theta \delta w,_x - \frac{1}{R} u,_{\theta\theta} \delta w,_x + w,_x \delta w,_x \Big)$$

$$+ \frac{E_{rk} A_{rk} \bar{x}_{rk} \bar{z}_{rk}}{R_c} \Big(- \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta v,_\theta \delta u,_\theta$$

$$- \Big\{ \Big(\frac{1}{R}\Big),_\theta \Big\}^2 v \delta u,_\theta + \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta w,_{\theta\theta} \delta u,_\theta$$

$$+ \Big\{ \Big(\frac{1}{R}\Big),_\theta \Big\}^2 w,_\theta \delta u,_\theta - \frac{1}{R^2} v,_\theta \delta u,_{\theta\theta} - \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta v \delta u,_{\theta\theta}$$

$$+ \frac{1}{R^2} w,_{\theta\theta} \delta u,_{\theta\theta} + \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta w,_\theta \delta u,_{\theta\theta}$$

$$- \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta u,_\theta \delta v,_\theta - \frac{1}{R^2} u,_{\theta\theta} \delta v,_\theta + \frac{1}{R} w,_x \delta v,_\theta$$

$$- \Big\{ \Big(\frac{1}{R}\Big),_\theta \Big\}^2 u,_\theta \delta v - \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta u,_{\theta\theta} \delta v + \Big(\frac{1}{R}\Big),_\theta w,_x \delta v$$

$$+ \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta u,_\theta \delta w,_{\theta\theta} + \frac{1}{R^2} u,_{\theta\theta} \delta w,_{\theta\theta} - \frac{1}{R} w,_x \delta w,_{\theta\theta}$$

$$+ \Big\{ \Big(\frac{1}{R}\Big),_\theta \Big\}^2 u,_\theta \delta w,_\theta + \frac{1}{R}\Big(\frac{1}{R}\Big),_\theta u,_{\theta\theta} \delta w,_\theta$$

$$- \left(\frac{1}{R}\right)_{,\theta} w_{,x}\delta w_{,\theta} + \frac{1}{R} v_{,\theta}\delta w_{,x} + \left(\frac{1}{R}\right)_{,\theta} v\delta w_{,x}$$

$$- \frac{1}{R} w_{,\theta\theta}\delta w_{,x} - \left(\frac{1}{R}\right)_{,\theta} w_{,\theta}\delta w_{,x}\Big)$$

$$- \frac{E_{rk} I_{xz_{rk}}}{R_c} \Big(\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta v$$

$$+ \left(\frac{1}{R}\right)_{,\theta}\frac{1}{R_c} u_{,\theta\theta}\delta v - \left(\frac{1}{R}\right)_{,\theta} w_{,x}\delta v + \frac{1}{R}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta v_{,\theta}$$

$$+ \frac{1}{R}\frac{1}{R_c} u_{,\theta\theta}\delta v_{,\theta} - \frac{1}{R} w_{,x}\delta v_{,\theta} - \left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta w_{,\theta}$$

$$- \left(\frac{1}{R}\right)_{,\theta}\frac{1}{R_c} u_{,\theta\theta}\delta w_{,\theta} + \left(\frac{1}{R}\right)_{,\theta} w_{,x}\delta w_{,\theta}$$

$$- \frac{1}{R}\left(\frac{1}{R_c}\right)_{,\theta} u_{,\theta}\delta w_{,\theta\theta} - \frac{1}{R}\frac{1}{R_c} u_{,\theta\theta}\delta w_{,\theta\theta} + \frac{1}{R} w_{,x}\delta w_{,\theta\theta}$$

$$+ \left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_c}\right)_{,\theta} v\delta u_{,\theta} + \frac{1}{R}\left(\frac{1}{R_c}\right)_{,\theta} v_{,\theta}\delta u_{,\theta}$$

$$- \left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_c}\right)_{,\theta} w_{,\theta}\delta u_{,\theta} - \frac{1}{R}\left(\frac{1}{R_c}\right)_{,\theta} w_{,\theta\theta}\delta u_{,\theta}$$

$$+ \left(\frac{1}{R}\right)_{,\theta}\frac{1}{R_c} v\delta u_{,\theta\theta} + \frac{1}{R}\frac{1}{R_c} v_{,\theta}\delta u_{,\theta\theta}$$

$$- \left(\frac{1}{R}\right)_{,\theta}\frac{1}{R_c} w_{,\theta}\delta u_{,\theta\theta} - \frac{1}{R}\frac{1}{R_c} w_{,\theta\theta}\delta u_{,\theta\theta}$$

$$- \left(\frac{1}{R}\right)_{,\theta} v\delta w_{,x} - \frac{1}{R} v_{,\theta}\delta w_{,x} + \left(\frac{1}{R}\right)_{,\theta} w_{,\theta}\delta w_{,x}$$

$$+ \frac{1}{R} w_{,\theta\theta}\delta w_{,x}\Big)$$

$$
\begin{aligned}
&- \frac{E_{rk}\, I_{xz_{rk}}\, \overline{z}_{rk}}{R_c} \Big(- \left(\frac{1}{R}\right)_{,\theta} \left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta}\, \delta v \\
&- \left(\frac{1}{R}\right)_{,\theta} \frac{1}{R_c} w_{,x\theta\theta}\, \delta v - \left(\frac{1}{R_c}\right)_{,\theta} \frac{1}{R} w_{,x\theta}\, \delta v_{,\theta} \\
&- \frac{1}{R} \frac{1}{R_c} w_{,x\theta\theta}\, \delta v_{,\theta} + \left(\frac{1}{R}\right)_{,\theta} \left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta}\, \delta w_{,\theta} \\
&+ \left(\frac{1}{R}\right)_{,\theta} \frac{1}{R_c} w_{,x\theta\theta}\, \delta w_{,\theta} + \frac{1}{R} \left(\frac{1}{R_c}\right)_{,\theta} w_{,x\theta}\, \delta w_{,\theta\theta} \\
&+ \frac{1}{R} \frac{1}{R_c} w_{,x\theta\theta}\, \delta w_{,\theta\theta} - \left(\frac{1}{R}\right)_{,\theta} \left(\frac{1}{R_c}\right)_{,\theta} v\, \delta w_{,x\theta} \\
&- \frac{1}{R} \left(\frac{1}{R_c}\right)_{,\theta} v_{,\theta}\, \delta w_{,x\theta} + \left(\frac{1}{R}\right)_{,\theta} \left(\frac{1}{R_c}\right)_{,\theta} w_{,\theta}\, \delta w_{,x\theta} \\
&+ \frac{1}{R} \left(\frac{1}{R_c}\right)_{,\theta} w_{,\theta\theta}\, \delta w_{,x\theta} - \left(\frac{1}{R}\right)_{,\theta} \frac{1}{R_c} v\, \delta w_{,x\theta\theta} \\
&- \frac{1}{R} \frac{1}{R_c} v_{,\theta}\, \delta w_{,x\theta\theta} + \left(\frac{1}{R}\right)_{,\theta} \frac{1}{R_c} w_{,\theta}\, \delta w_{,x\theta\theta} \\
&+ \frac{1}{R} \frac{1}{R_c} w_{,\theta\theta}\, \delta w_{,x\theta\theta} \Big) \Big\} \Big|_{x=x_k} d\theta
\end{aligned}
$$

APPENDIX D

ELEMENTS OF THE MASS AND STIFFNESS MATRICES

The matrix elements of Equation (2.38) are presented in this appendix. The circumferential and longitudinal integrals (ISA_i, ISB_i, IRA_i, IRB_i, and IX_i), the stringer circumferential functions ($SF_{i,\ell}$) and the ring longitudinal functions ($RF_{i,\ell}$) used in these expressions are defined in Appendix E. Appendix F defines the remaining constants which are various combinations of the material properties of the shell, stringers and rings.

The elements of the mass and stiffness matrices of a stringer and ring stiffened noncircular shell may be written as follows:

Contribution of the noncircular shell:

$$K11_{mn,\bar{m}\bar{n}} = SX_3\, ISA_1\, IX_1 + SXT_1\, n\bar{n}\, ISA_2\, IX_2$$

$$K12_{mn,\bar{m}\bar{n}} = ST_4\, n\, ISA_5\, IX_4 - SXT_1\, \bar{n}\, ISA_6\, IX_2$$

$$K13_{mn,\bar{m}\bar{n}} = ST_4\, ISA_5\, IX_4$$

$$K22_{mn,\bar{m}\bar{n}} = \{ST_3\, n\bar{n}\, ISA_8 + ST_1(n\bar{n}\, ISA_4 + \bar{n}\, ISB_2 + n\, ISB_4 + ISB_1)\}\, IX_5 + \{SXT_1\, ISA_9 + SXT_2\, ISA_2\}\, IX_2$$

$$K23_{mn,\bar{m}\bar{n}} = \{ST_3\, \bar{n}\, ISA_8 + ST_1(n^2\bar{n}\, ISA_4 + n\bar{n}\, ISB_2 + n^2\, ISB_4$$

$$+ n\ ISB_1)\} \ IX_5 - SX_2\ (\bar{n}\ ISA_8 + ISB_5)\ IX_3$$

$$+ SXT_3\ n\ ISA_2\ IX_2$$

$$K33_{mn,\bar{m}\bar{n}} = SX_1\ ISA_1\ IX_1 + SXT_4\ n\bar{n}\ ISA_2\ IX_2 - SX_2(\bar{n}^2\ ISA_8$$

$$+ \bar{n}\ ISB_5)\ IX_3 - ST_2(n^2\ ISA_8 + n\ ISB_3) IX_4$$

$$+ \{ ST_3\ ISA_8 + ST_1(n^2\bar{n}^2\ ISA_4 + n\bar{n}^2\ ISB_2$$

$$+ n^2\bar{n}\ ISB_4 + n\bar{n}\ ISB_1\} \ IX_5$$

$$M11_{mn,\bar{m}\bar{n}} = SPC\ ISA_1\ IX_2$$

$$M22_{mn,\bar{m}\bar{n}} = SPC\ ISA_9\ IX_5$$

$$M33_{mn,\bar{m}\bar{n}} = SPC\ ISA_1\ IX_5$$

Contribution of the ℓ^{th} stringer:

$$K11^{\ell}_{mn,\bar{m}\bar{n}} = STR8_{\ell}\ SF_{1,\ell}\ IX_1$$

$$K12^{\ell}_{mn,\bar{m}\bar{n}} = -STR9_{\ell}\ SF_{4,\ell}\ IX_1$$

$$K13^{\ell}_{mn,\bar{m}\bar{n}} = -STR10_{\ell}\ SF_{1,\ell}\ IX_1$$

$$K22^{\ell}_{mn,\bar{m}\bar{n}} = STR11_{\ell}\ SF_{3,\ell}\ IX_1 + STR12_{\ell}\ SF_{6,\ell}\ IX_2$$

$$K23^{\ell}_{mn,\bar{m}\bar{n}} = STR13_{\ell}\ SF_{2,\ell}\ IX_1 + STR12_{\ell}\ SF_{6,\ell}\ IX_2$$

$$K33^{\ell}_{mn,\bar{m}\bar{n}} = STR14_{\ell}\ SF_{1,\ell}\ IX_1 + STR12_{\ell}\ SF_{6,\ell}\ IX_2$$

$$M11^{\ell}_{mn,\bar{m}\bar{n}} = STR1_{\ell}\ SF_{1,\ell}\ IX_2$$

$$M12^{\ell}_{mn,\bar{m}\bar{n}} = -STR2_{\ell}\ SF_{4,\ell}\ IX_2$$

$$M13^{\ell}_{mn,\bar{m}\bar{n}} = -STR3_{\ell}\ SF_{1,\ell}\ IX_2$$

$$M22^{\ell}_{mn,\bar{m}\bar{n}} = \{ STR1_{\ell} SF_{3,\ell} + STR4_{\ell} SF_{5,\ell} + STR5_{\ell} SF_{6,\ell} + STR6_{\ell} SF_{6,\ell} \} IX_5 + STR5_{\ell} SF_{3,\ell} IX_2$$

$$M23^{\ell}_{mn,\bar{m}\bar{n}} = \{ -STR2_{\ell} SF_{7,\ell} + STR5_{\ell}\, n\, SF_{6,\ell} + STR3_{\ell}\, n\, SF_{5,\ell} + STR6_{\ell}\, n\, SF_{6,\ell} \} IX_5 + STR7_{\ell} SF_{2,\ell} IX_2$$

$$M33^{\ell}_{mn,\bar{m}\bar{n}} = \{ STR1_{\ell} SF_{1,\ell} - STR2_{\ell}\, n\, SF_{8,\ell} - STR2_{\ell}\, \bar{n}\, SF_{7,\ell} + STR5_{\ell}\, n\bar{n}\, SF_{6,\ell} + STR6_{\ell}\, n\bar{n}\, SF_{6,\ell} \} IX_5 + STR6_{\ell} SF_{1,\ell} IX_2$$

Contribution of the k^{th} ring:

$$K11^{k}_{mn,\bar{m}\bar{n}} = \{ RNG17_k (n\bar{n}\, IRB_1 + n^2\bar{n}\, IRB_2 + n\bar{n}^2\, IRB_3 + n^2\bar{n}^2\, IRA_1) + RNG14_k\, n\bar{n}\, IRA_2 + RNG12_k (n\bar{n}\, IRB_6 + n^2\bar{n}\, IRB_7 + n\bar{n}^2\, IRB_8 + n^2\bar{n}^2\, IRA_{11}) \} RF_{1,k}$$

$$K12^{k}_{mn,\bar{m}\bar{n}} = \{ RNG11_k (n\bar{n}\, IRB_9 + n\bar{n}^2\, IRA_{12}) + RNG13_k (n\bar{n}\, IRB_7 + \bar{n}\, IRB_6 + n\bar{n}^2\, IRA_{11} + \bar{n}^2\, IRB_8) + RNG21_k (\bar{n}\, IRB_{10} + n\bar{n}\, IRB_{11} + \bar{n}^2\, IRB_{12} + n\bar{n}^2\, IRA_{13}) \} RF_{3,k}$$

$$K13^{k}_{mn,\bar{m}\bar{n}} = \{ RNG17_k (\bar{n}\, IRB_4 + \bar{n}^2\, IRA_3) - RNG18_k (n\bar{n}\, IRB_1 + n^2\bar{n}\, IRB_2 + n\bar{n}^2\, IRB_3 + n^2\bar{n}^2\, IRA_1)$$

$$+ RNG14_k\, n\bar{n}\, IRA_4 - RNG15_k\, n\bar{n}\, IRA_2$$

$$+ RNG12_k\, (\bar{n}\, IRB_9 + \bar{n}^2\, IRA_{12})\}\, RF_{1,k}$$

$$+ \{ RNG13_k(n^2\bar{n}\, IRB_7 + n\bar{n}\, IRB_6 + n^2\bar{n}^2\, IRA_{11}$$

$$+ n\bar{n}^2\, IRB_8) + RNG11_k(\bar{n}\, IRB_9 + \bar{n}^2\, IRA_{12})$$

$$+ RNG21_k(n\bar{n}\, IRB_{10} + n^2\bar{n}\, IRB_{11} + n\bar{n}^2\, IRB_{12}$$

$$+ n^2\bar{n}^2\, IRA_{13})\}\, RF_{3,k}$$

$$K22^k_{mn,\bar{m}\bar{n}} = \{ RNG8_k\, n\bar{n}\, IRA_5 + RNG9_k(2\, n\bar{n}\, IRA_{12} + \bar{n}\, IRB_5$$

$$+ n\, IRB_9) + RNG10_k(n\bar{n}\, IRA_{11} + \bar{n}\, IRB_8 + n\, IRB_7$$

$$+ IRB_6) + RNG20_k(IRB_6 + n\, IRB_7 + \bar{n}\, IRB_8$$

$$+ n\bar{n}\, IRA_{11})\}\, RF_{2,k}$$

$$K23^k_{mn,\bar{m}\bar{n}} = \{ RNG8_k\, \bar{n}\, IRA_5 + RNG9_k(n^2\bar{n}\, IRA_{12} + n\bar{n}\, IRB_5$$

$$+ \bar{n}\, IRA_{12} + IRB_9) + RNG10_k(n^2\bar{n}\, IRA_{11} + n\bar{n}\, IRB_8$$

$$+ n^2\, IRB_7 + n\, IRB_6) + RNG20_k(n\, IRB_6 + n^2\, IRB_7$$

$$+ n\bar{n}\, IRB_8 + n^2\bar{n}\, IRA_{11})\}\, RF_{2,k} + \{RNG11_k\, \bar{n}\, IRA_5$$

$$+ RNG13_k(\bar{n}\, IRA_{12} + IRB_9) + RNG21_k\, (IRB_9$$

$$+ \bar{n}\, IRA_{12}) - RNG22_k(n\, IRB_{10} + n^2\, IRB_{14}$$

$$+ n\bar{n}\, IRB_{15} + n^2\bar{n}\, IRA_{13})\}\, RF_{4,k}$$

$$K33^k_{mn,\bar{m}\bar{n}} = \{ RNG8_k\, IRA_5 + RNG9_k\, (\bar{n}^2\, IRA_{12} + \bar{n}\, IRB_9$$

$$+ n^2\, IRA_{12} + n\, IRB_5) + RNG10_k(n^2\bar{n}^2\, IRA_{11}$$

$$+ n\bar{n}^2\, IRB_8 + n^2\bar{n}\, IRB_7 + n\bar{n}\, IRB_6)$$

$$+ RNG20_k(n\bar{n}\, IRB_6 + n^2\bar{n}\, IRB_7 + n\bar{n}^2\, IRB_8$$

$$+ n^2\bar{n}^2\, IRA_{11})\}\, RF_{2,k} + \{ RNG17_k\, IRA_5$$

$$-RNG18_k(n\, IRB_{13} + n^2\, IRA_3 + \bar{n}\, IRB_4 + n^2\, IRA_3)$$

$$+ RNG19_k(n\bar{n}\, IRB_1 + n^2\bar{n}\, IRB_2 + n\bar{n}^2\, IRB_3$$

$$+ n^2\bar{n}^2\, IRA_1) + RNG14_k\, n\bar{n}\, IRA_6$$

$$-2\, RNG15_k\, n\bar{n}\, IRA_4 + RNG16_k\, n\bar{n}\, IRA_2\}\, RF_{1,k}$$

$$+ \{ RNG11_k\, IRA_5 + RNG13_k\, (\bar{n}^2\, IRA_{12} + \bar{n}\, IRB_9)$$

$$+ RNG21_k(\bar{n}\, IRB_9 + \bar{n}^2\, IRA_{12}) - RNG22_k(n\bar{n}\, IRB_{10}$$

$$+ n^2\bar{n}\, IRB_{14} + n\bar{n}^2\, IRB_{15} + n^2\bar{n}^2\, IRA_{13})\}\, RF_{4,k}$$

$$+ \{ RNG11_k\, IRA_5 + RNG12_k\, IRA_5$$

$$+ RNG13_k(n^2\, IRA_{12} + n\, IRB_5) + RNG21_k(n\, IRB_5$$

$$+ n^2\, IRA_{12}) - RNG22_k\, (n\bar{n}\, IRB_{10} + n^2\bar{n}\, IRB_{11}$$

$$+ n\bar{n}^2\, IRB_{12} + n^2\bar{n}^2\, IRA_{13})\}\, RF_{3,k}$$

$$M11^k_{mn,\bar{m}\bar{n}} = \{ RNG1_k\, IRA_7 + RNG5_k\, n\bar{n}\, IRA_9\}\, RF_{1,k}$$

$$M12^k_{mn,\bar{m}\bar{n}} = \{ RNG2_k\, \bar{n}\, IRA_{10} + RNG7_k\, \bar{n}\, IRA_9\}\, RF_{3,k}$$

$$M13^k_{mn,\bar{m}\bar{n}} = - RNG3_k\, IRA_7\, RF_{1,k} + RNG7_k\, n\bar{n}\, IRA_9\, RF_{3,k}$$

$$M22^k_{mn,\bar{m}\bar{n}} = \{ RNG1_k\, IRA_8 + RNG4_k\, IRA_{10} + RNG6_k IRA_9\}\, RF_{2,k}$$

$$M23^{k}_{mn,\bar{m}\bar{n}} = \{ RNG3_{k}\ IRA_{10} + RNG6_{k}\ n\ IRA_{9} \}\ RF_{2,k}$$

$$M33^{k}_{mn,\bar{m}\bar{n}} = \{ RNG5_{k} + RNG6_{k} \}\ IRA_{7}\ RF_{1,k} + \{ RNG1_{k}\ IRA_{7} + n\bar{n}\ RNG6_{k}\ IRA_{9} \}\ RF_{2,k} + RNG2_{k}\ IRA_{7}\ \{ RF_{3,k} + RF_{4,k} \}$$

APPENDIX E

FUNCTIONS AND INTEGRALS OF THE MATRIX ELEMENTS

This appendix contains the functions and integrals used in the equations in Appendix D. The circumferential integrals are evaluated numerically using the eight-point Gaussian quadrature method with four subintervals. The circumferential integrals are divided into four groups (ISA_i, ISB_i, IRA_i, IRB_i) and defined as follows:

$$ISA_1 = \int_0^{\pi} R \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$ISA_2 = \int_0^{\pi} \frac{1}{R} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$ISA_3 = \int_0^{\pi} \frac{1}{R^3} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$ISA_4 = \int_0^{\pi} \frac{1}{R^3} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$ISA_5 = \int_0^{\pi} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$ISA_6 = \int_0^{\pi} \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$ISA_7 = \int_0^{\pi} \frac{1}{R^2} \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$ISA_8 = \int_0^{\pi} \frac{1}{R} \cos n\theta \ \cos \bar{n}\theta \ d\theta$$

$$ISA_9 = \int_0^{\pi} R \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$ISB_1 = \int_0^{\pi} \frac{1}{R}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$ISB_2 = \int_0^{\pi} \frac{1}{R^2}\left(\frac{1}{R}\right)_{,\theta} \sin n\theta \ \cos \bar{n}\theta \ d\theta$$

$$ISB_3 = \int_0^{\pi} \left(\frac{1}{R}\right)_{,\theta} \sin n\theta \ \cos \bar{n}\theta \ d\theta$$

$$ISB_4 = \int_0^{\pi} \frac{1}{R^2}\left(\frac{1}{R}\right)_{,\theta} \cos n\theta \ \sin \bar{n}\theta \ d\theta$$

$$ISB_5 = \int_0^{\pi} \left(\frac{1}{R}\right)_{,\theta} \cos n\theta \ \sin \bar{n}\theta \ d\theta$$

$$IRA_1 = \int_0^{\pi} \frac{1}{R_c^3} \cos n\theta \ \cos \bar{n}\theta \ d\theta$$

$$IRA_2 = \int_0^{\pi} \frac{1}{R_c^3} \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$IRA_3 = \int_0^{\pi} \frac{1}{R_c^2} \cos n\theta \ \cos \bar{n}\theta \ d\theta$$

$$IRA_4 = \int_0^{\pi} \frac{1}{R_c^2} \sin n\theta \ \sin \bar{n}\theta \, d\theta$$

$$IRA_5 = \int_0^{\pi} \frac{1}{R_c} \cos n\theta \ \cos \bar{n}\theta \ d\theta$$

$$IRA_6 = \int_0^{\pi} \frac{1}{R_c} \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$IRA_7 = \int_0^{\pi} R_c \cos n\theta \ \cos \bar{n}\theta \ d\theta$$

$$IRA_8 = \int_0^{\pi} R_c \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$IRA_9 = \int_0^{\pi} \frac{R_c}{R^2} \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$IRA_{10} = \int_0^{\pi} \frac{R_c}{R} \sin n\theta \ \sin \bar{n}\theta \ d\theta$$

$$IRA_{11} = \int_0^{\pi} \frac{1}{R_c} \frac{1}{R^2} \cos n\theta \ \cos \bar{n}\theta \ d\theta$$

$$IRA_{12} = \int_0^{\pi} \frac{1}{R_c} \frac{1}{R} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IRA_{13} = \int_0^{\pi} \frac{1}{R_c^2} \frac{1}{R} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IRB_1 = \int_0^{\pi} \frac{1}{R_c} \left\{ \left(\frac{1}{R_c} \right)_{,\theta} \right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IRB_2 = \int_0^{\pi} \frac{1}{R_c^2} \left(\frac{1}{R_c} \right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IRB_3 = \int_0^{\pi} \frac{1}{R_c^2} \left(\frac{1}{R_c} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IRB_4 = \int_0^{\pi} \left(\frac{1}{R_c} \right)_{,\theta} \frac{1}{R_c} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IRB_5 = \int_0^{\pi} \frac{1}{R_c} \left(\frac{1}{R} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IRB_6 = \int_0^{\pi} \frac{1}{R_c} \left\{ \left(\frac{1}{R} \right)_{,\theta} \right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IRB_7 = \int_0^{\pi} \frac{1}{R} \frac{1}{R_c} \left(\frac{1}{R} \right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IRB_8 = \int_0^{\pi} \frac{1}{R} \frac{1}{R_c} \left(\frac{1}{R} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IRB_9 = \int_0^{\pi} \frac{1}{R_c}\left(\frac{1}{R}\right)_{,\theta} \cos n\theta \; \sin \bar{n}\theta \; d\theta$$

$$IRB_{10} = \int_0^{\pi} \frac{1}{R_c}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_c}\right)_{,\theta} \sin n\theta \; \sin \bar{n}\theta \; d\theta$$

$$IRB_{11} = \int_0^{\pi} \frac{1}{R}\,\frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} \cos n\theta \; \sin \bar{n}\theta \; d\theta$$

$$IRB_{12} = \int_0^{\pi} \frac{1}{R_c^2}\left(\frac{1}{R}\right)_{,\theta} \sin n\theta \; \cos \bar{n}\theta \; d\theta$$

$$IRB_{13} = \int_0^{\pi} \frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} \sin n\theta \; \cos \bar{n}\theta \; d\theta$$

$$IRB_{14} = \int_0^{\pi} \frac{1}{R_c^2}\left(\frac{1}{R}\right)_{,\theta} \cos n\theta \; \sin \bar{n}\theta \; d\theta$$

$$IRB_{15} = \int_0^{\pi} \frac{1}{R}\,\frac{1}{R_c}\left(\frac{1}{R_c}\right)_{,\theta} \sin n\theta \; \cos \bar{n}\theta \; d\theta$$

The closed-form expressions for the longitudinal integrals were obtained from the table of formulas for integrals derived by Felgar (31). These integrals may be defined by a general axial mode function, Φ, as follows:

$$IX_1 = \int_0^{a} \Phi''_m \, \Phi''_{\bar{m}} \, dx$$

$$IX_2 = \int_0^{a} \Phi'_m \, \Phi'_{\bar{m}} \, dx$$

$$IX_3 = \int_0^a \Phi''_m \Phi_{\bar{m}} \, dx$$

$$IX_4 = \int_0^a \Phi_m \Phi''_{\bar{m}} \, dx$$

$$IX_5 = \int_0^a \Phi_m \Phi_{\bar{m}} \, dx$$

After Equations (2.35a to 2.35d) are substituted into the above equations, the longitudinal integrals for various boundary conditions may be written as:

For freely supported cylinders:

$$\left.\begin{aligned} IX_1 &= \frac{m^4\pi^4}{a^3} \\ IX_2 &= -IX_3 = -IX_4 = \frac{m^2\pi^2}{a} \\ IX_5 &= a \end{aligned}\right\} \quad \text{For } m = \bar{m}$$

$$IX_1 \text{ to } IX_5 = 0 \quad \Big\} \quad \text{For } m \neq \bar{m}$$

For clamped-free cylinders:

$$IX_1 = \begin{cases} \beta_m^4 \, a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}$$

$$IX_2 = \begin{cases} \alpha_m \beta_m (2 + \alpha_m \beta_m a) & m = \bar{m} \\ \dfrac{4\beta_m \beta_{\bar{m}}}{\beta_m^4 - \beta_{\bar{m}}^4} \left[(-1)^{m+\bar{m}} (\alpha_{\bar{m}} \beta_m^3 - \alpha_m \beta_{\bar{m}}^3) \right. & \\ \qquad \left. - \beta_{\bar{m}} \beta_m (\alpha_m \beta_m - \alpha_{\bar{m}} \beta_{\bar{m}})\right] & m \neq \bar{m} \end{cases}$$

$$IX_3 = \begin{cases} \alpha_m \beta_m (2 - \alpha_m \beta_m a) & m = \bar{m} \\ \dfrac{4\beta_m^2 (\alpha_{\bar{m}} \beta_{\bar{m}} - \alpha_m \beta_m)}{\beta_{\bar{m}}^4 - \beta_m^4} [(-1)^{m+\bar{m}} \beta_m^2 + \beta_{\bar{m}}^2] & m \neq \bar{m} \end{cases}$$

$$IX_4 = \begin{cases} \alpha_m \beta_m (2 - \alpha_m \beta_m a) & m = \bar{m} \\ \dfrac{4\beta_{\bar{m}}^2 (\alpha_m \beta_m - \alpha_{\bar{m}} \beta_{\bar{m}})}{\beta_m^4 - \beta_{\bar{m}}^4} [(-1)^{m+\bar{m}} \beta_{\bar{m}}^2 + \beta_m^2] & m \neq \bar{m} \end{cases}$$

$$IX_5 = \begin{cases} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}$$

For clamped-clamped cylinders:

$$IX_1 = \begin{cases} \beta_m^4 a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}$$

$$IX_2 = -IX_3 = -IX_4 = \begin{cases} \alpha_m \beta_m (\alpha_m \beta_m a - 2) & m = \bar{m} \\ \dfrac{4\beta_{\bar{m}}^2 \beta_m^2 (\alpha_{\bar{m}} \beta_{\bar{m}} - \alpha_m \beta_m)}{\beta_m^4 - \beta_{\bar{m}}^4} [(-1)^{m+\bar{m}} + 1] & m \neq \bar{m} \end{cases}$$

$$IX_5 = \begin{cases} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}$$

For free-free cylinders:

<u>m = 0</u>

$$\left. \begin{aligned} IX_1 &= IX_2 = IX_3 = IX_4 = 0 \\ IX_5 &= a \end{aligned} \right\} \quad \bar{m} = 0$$

$$\left. IX_1 = IX_2 = IX_3 = IX_4 = IX_5 = 0 \right\} \quad \bar{m} = 1$$

$$IX_1 = IX_2 = IX_3 = IX_5 = 0 \qquad \Big\} \ \bar{m} \geq 2$$

$$IX_4 = \begin{cases} 4\alpha_{\bar{m}-1}\beta_{\bar{m}-1} & \bar{m} \geq 2 \text{ even only} \\ 0 & \bar{m} > 2 \text{ odd only} \end{cases}$$

<u>m = 1</u>

$$IX_1 = IX_2 = IX_3 = IX_4 = IX_5 = 0 \qquad \Big\} \ \bar{m} = 0$$

$$\left.\begin{array}{l} IX_1 = IX_3 = IX_4 = 0 \\ IX_2 = 1/a \,; \quad IX_5 = a/12 \end{array}\right\} \ \bar{m} = 1$$

$$IX_1 = IX_3 = IX_5 = 0 \qquad \Big\} \ \bar{m} \geq 2$$

$$IX_2 = \begin{cases} -4/a & \bar{m} > 2 \text{ odd only} \\ 0 & \bar{m} \geq 2 \text{ even only} \end{cases}$$

$$IX_4 = \begin{cases} 4/a - 2\alpha_{\bar{m}-1}\beta_{\bar{m}-1} & \bar{m} > 2 \text{ odd only} \\ 0 & \bar{m} \geq 2 \text{ even only} \end{cases}$$

<u>m ≥ 2</u>

$$\left.\begin{array}{l} IX_1 = IX_2 = IX_4 = IX_5 = 0 \\ IX_3 = \begin{cases} 4\beta_{m-1}\alpha_{m-1} & m \geq 2 \text{ even only} \\ 0 & m > 2 \text{ odd only} \end{cases} \end{array}\right\} \ \bar{m} = 0$$

$$\left.\begin{array}{l} IX_1 = IX_4 = IX_5 = 0 \\ IX_2 = \begin{cases} -4/a & m > 2 \text{ odd only} \\ 0 & m \geq 2 \text{ even only} \end{cases} \\ IX_3 = \begin{cases} 4/a - 2\alpha_{m-1}\beta_{m-1} & m > 2 \text{ odd only} \\ 0 & m \geq 2 \text{ even only} \end{cases} \end{array}\right\} \ \bar{m} = 1$$

$$
\left.
\begin{array}{ll}
IX_1 = \begin{cases} \beta^4_{m-1} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases} \\[2ex]
IX_2 = \begin{cases} \alpha_{m-1}\beta_{m-1}(\alpha_{m-1}\beta_{m-1} a + 6) & m = \bar{m} \\ \dfrac{4\beta_{m-1}\beta_{\bar{m}-1}(\alpha_{m-1}\beta^3_{\bar{m}-1} - \alpha_{\bar{m}-1}\beta^3_{m-1})}{\beta^4_{\bar{m}-1} - \beta^4_{m-1}}\left[1 + (-1)^{m+\bar{m}-2}\right] & m \neq \bar{m} \end{cases} \\[2ex]
IX_3 = \begin{cases} \alpha_{m-1}\beta_{m-1}(2 - \alpha_{m-1}\beta_{m-1} a) & m = \bar{m} \\ \dfrac{4\beta^4_{m-1}(\alpha_{\bar{m}-1}\beta_{\bar{m}-1} - \alpha_{m-1}\beta_{m-1})}{\beta^4_{\bar{m}-1} - \beta^4_{m-1}}\left[1 + (-1)^{m+\bar{m}-2}\right] & m \neq \bar{m} \end{cases} \\[2ex]
IX_4 = \begin{cases} \alpha_{m-1}\beta_{m-1}(2 - \alpha_{m-1}\beta_{m-1} a) & m = \bar{m} \\ \dfrac{4\beta^4_{\bar{m}-1}(\alpha_{m-1}\beta_{m-1} - \alpha_{\bar{m}-1}\beta_{\bar{m}-1})}{\beta^4_{m-1} - \beta^4_{\bar{m}-1}}\left[1 + (-1)^{m+\bar{m}-2}\right] & m \neq \bar{m} \end{cases} \\[2ex]
IX_5 = \begin{cases} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}
\end{array}
\right\} \bar{m} \geq 2
$$

The stringer circumferential functions are defined as follows:

$$SF_{1,\ell} = \cos n\theta_\ell \cos \bar{n}\theta_\ell$$

$$SF_{2,\ell} = \cos n\theta_\ell \sin \bar{n}\theta_\ell$$

$$SF_{3,\ell} = \sin n\theta_\ell \sin \bar{n}\theta_\ell$$

$$SF_{4,\ell} = \sin n\theta_\ell \cos \bar{n}\theta_\ell$$

$$SF_{5,\ell} = \frac{1}{R} \sin n\theta_\ell \sin \bar{n}\theta_\ell$$

$$SF_{6,\ell} = \frac{1}{R^2} \sin n\theta_\ell \sin \bar{n}\theta_\ell$$

$$SF_{7,\ell} = \frac{1}{R} \cos n\theta_\ell \sin \bar{n}\theta_\ell$$

$$SF_{8,\ell} = \frac{1}{R} \sin n\theta_\ell \cos \bar{n}\theta_\ell$$

The ring longitudinal functions for different boundary conditions are defined as follows:

For freely supported cylinders:

$$RF_{1,k} = 2\,\frac{m\bar{m}\pi^2}{a^2} \cos\frac{m\pi x_k}{a} \cos\frac{\bar{m}\pi x_k}{a}$$

$$RF_{2,k} = 2 \sin\frac{m\pi x_k}{a} \sin\frac{\bar{m}\pi x_k}{a}$$

$$RF_{3,k} = 2\,\frac{\bar{m}\pi}{a} \sin\frac{m\pi x_k}{a} \cos\frac{\bar{m}\pi x_k}{a}$$

$$RF_{4,k} = 2\,\frac{m\pi}{a} \cos\frac{m\pi x_k}{a} \sin\frac{\bar{m}\pi x_k}{a}$$

For clamped-free cylinders:

$$RF_{1,k} = \beta_m\beta_{\bar{m}} \{\sinh \beta_m x_k + \sin \beta_m x_k - \alpha_m(\cosh \beta_m x_k - \cos \beta_m x_k)\}\{\sinh \beta_{\bar{m}} x_k + \sin \beta_{\bar{m}} x_k - \alpha_{\bar{m}}(\cosh \beta_{\bar{m}} x_k - \cos \beta_{\bar{m}} x_k)\}$$

$$RF_{2,k} = \{\cosh \beta_m x_k - \cos \beta_m x_k - \alpha_m(\sinh \beta_m x_k - \sin \beta_m x_k)\}\{\cosh \beta_{\bar{m}} x_k - \cos \beta_{\bar{m}} x_k - \alpha_{\bar{m}}(\sinh \beta_{\bar{m}} x_k - \sin \beta_{\bar{m}} x_k)\}$$

$$RF_{3,k} = \beta_{\bar{m}}\{\cosh \beta_m x_k - \cos \beta_m x_k - \alpha_m(\sinh \beta_m x_k - \sin \beta_m x_k)\}\{\sinh \beta_{\bar{m}} x_k + \sin \beta_{\bar{m}} x_k - \alpha_{\bar{m}}(\cosh \beta_{\bar{m}} x_k - \cos \beta_{\bar{m}} x_k)\}$$

$$RF_{4,k} = \beta_m\{\sinh \beta_m x_k + \sin \beta_m x_k - \alpha_m(\cosh \beta_m x_k - \cos \beta_m x_k)\}\{\cosh \beta_{\bar{m}} x_k - \cos \beta_{\bar{m}} x_k - \alpha_{\bar{m}}(\sinh \beta_{\bar{m}} x_k - \sin \beta_{\bar{m}} x_k)\}$$

For clamped-free cylinders:

The expressions are the same as clamped-free except the values of α_m, $\alpha_{\bar{m}}$, β_m and $\beta_{\bar{m}}$ are different.

For free-free cylinders:

<u>m = 0</u>

$$\left.\begin{aligned} RF_{1,k} &= RF_{3,k} = RF_{4,k} = 0 \\ RF_{2,k} &= 1 \end{aligned}\right\} \quad \bar{m} = 0$$

$$\left.\begin{aligned} RF_{1,k} &= RF_{4,k} = 0 \\ RF_{2,k} &= x_k/a - \tfrac{1}{2} \\ RF_{3,k} &= 1/a \end{aligned}\right\} \quad \bar{m} = 1$$

$$\left.\begin{aligned} RF_{1,k} &= RF_{4,k} = 0 \\ RF_{2,k} &= \cosh \beta_{\bar{m}-1} x_k + \cos \beta_{\bar{m}-1} x_k \\ &\quad - \alpha_{\bar{m}-1}(\sinh \beta_{\bar{m}-1} x_k + \sin \beta_{\bar{m}-1} x_k) \end{aligned}\right\} \quad \bar{m} \geq 2$$

$$\left.\begin{aligned} RF_{3,k} &= \beta_{\bar{m}-1}\{\sinh \beta_{\bar{m}-1}x_k - \sin \beta_{\bar{m}-1}x_k \\ &\quad - \alpha_{\bar{m}-1}(\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k)\} \end{aligned}\right\} \quad \bar{m} \geq 2$$

<u>m = 1</u>

$$\left.\begin{aligned} RF_{1,k} &= RF_{3,k} = 0 \\ RF_{2,k} &= x_k/a - \tfrac{1}{2} \\ RF_{4,k} &= 1/a \end{aligned}\right\} \quad \bar{m} = 0$$

$$\left.\begin{aligned} RF_{1,k} &= 1/a^2 \\ RF_{2,k} &= x_k^2/a^2 + \tfrac{1}{4} - x_k/a \\ RF_{3,k} &= RF_{4,k} = x_k/a^2 - 1/2a \end{aligned}\right\} \quad \bar{m} = 1$$

$$\left.\begin{aligned} RF_{1,k} &= \frac{\beta_{\bar{m}-1}}{a}\{\sinh \beta_{\bar{m}-1}x_k - \sin \beta_{\bar{m}-1}x_k \\ &\quad - \alpha_{\bar{m}-1}(\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k)\} \\ RF_{2,k} &= \left\{\frac{x_k}{a} - \tfrac{1}{2}\right\}\{\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k \\ &\quad - \alpha_{\bar{m}-1}(\sinh \beta_{\bar{m}-1}x_k + \sin \beta_{\bar{m}-1}x_k)\} \\ RF_{3,k} &= \beta_{\bar{m}-1}\left\{\frac{x_k}{a} - \tfrac{1}{2}\right\}\{\sinh \beta_{\bar{m}-1}x_k - \sin \beta_{\bar{m}-1}x_k \\ &\quad - \alpha_{\bar{m}-1}(\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k)\} \\ RF_{4,k} &= \frac{1}{a}\{\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k \\ &\quad - \alpha_{\bar{m}-1}(\sinh \beta_{\bar{m}-1}x_k + \sin \beta_{\bar{m}-1}x_k)\} \end{aligned}\right\} \quad \bar{m} \geq 2$$

m ≥ 2

$\bar{m} = 0$:

$$RF_{1,k} = RF_{3,k} = 0$$

$$RF_{2,k} = \cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k - \alpha_{m-1}(\sinh \beta_{m-1}x_k + \sin \beta_{m-1}x_k)$$

$$RF_{4,k} = \beta_{m-1}\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k - \alpha_{m-1}(\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k)\}$$

$\bar{m} = 1$:

$$RF_{1,k} = \frac{\beta_{m-1}}{a}\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k - \alpha_{m-1}(\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k)\}$$

$$RF_{2,k} = \left\{\frac{x_k}{a} - \tfrac{1}{2}\right\}\{\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k - \alpha_{m-1}(\sinh \beta_{m-1}x_k + \sin \beta_{m-1}x_k)\}$$

$$RF_{3,k} = \frac{1}{a}\{\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k - \alpha_{m-1}(\sinh \beta_{m-1}x_k + \sin \beta_{m-1}x_k)\}$$

$$RF_{4,k} = \left\{\frac{x_k}{a} - \tfrac{1}{2}\right\}\beta_{m-1}\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k - \alpha_{m-1}(\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k)\}$$

$\bar{m} \geq 2$:

$$RF_{1,k} = \beta_{m-1}\beta_{\bar{m}-1}\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k - \alpha_{m-1}(\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k)\}\{\sinh \beta_{\bar{m}-1}x_k - \sin \beta_{\bar{m}-1}x_k - \alpha_{\bar{m}-1}(\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k)\}$$

$$
\left.
\begin{aligned}
RF_{2,k} &= \{\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k \\
&\quad - \alpha_{m-1}(\sinh \beta_{m-1}x_k \\
&\quad + \sin \beta_{m-1}x_k)\}\{\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k \\
&\quad - \alpha_{\bar{m}-1}(\sinh \beta_{\bar{m}-1}x_k + \sin \beta_{\bar{m}-1}x_k)\} \\
RF_{3,k} &= \beta_{\bar{m}-1}\{\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k \\
&\quad - \alpha_{m-1}(\sinh \beta_{m-1}x_k \\
&\quad + \sin \beta_{m-1}x_k)\}\{\sinh \beta_{\bar{m}-1}x_k - \sin \beta_{\bar{m}-1}x_k \\
&\quad - \alpha_{\bar{m}-1}(\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k)\} \\
RF_{4,k} &= \beta_{m-1}\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k \\
&\quad - \alpha_{m-1}(\cosh \beta_{m-1}x_k \\
&\quad + \cos \beta_{m-1}x_k)\}\{\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k \\
&\quad - \alpha_{\bar{m}-1}(\sinh \beta_{\bar{m}-1}x_k + \sin \beta_{\bar{m}-1}x_k)\}
\end{aligned}
\right\} \quad \bar{m} \geq 2
$$

APPENDIX F

CONSTANTS OF THE MATRIX ELEMENTS

This appendix contains the constants used in the equations in Appendix D. These expressions contain various combinations of the material properties of the shell, stringers and rings.

Constants of the Shell

$$D_1 = \frac{h^3}{12(1 - \nu_{x\theta}\nu_{\theta x})}$$

$$D_x = E_x D_1$$

$$D_\theta = E_\theta D_1$$

$$SX_1 = 2D_x$$

$$SX_2 = 2D_x \nu_{x\theta}$$

$$SX_3 = \frac{24D_x}{h^2}$$

$$ST_1 = 2D_\theta$$

$$ST_2 = 2D_\theta \nu_{\theta x}$$

$$ST_3 = \frac{24D_\theta}{h^2}$$

$$ST_4 = \frac{24D_\theta \nu_{\theta x}}{h^2}$$

$$SXT_1 = 2G_{x\theta} h$$

$$SXT_2 = \frac{G_{x\theta} h^3}{6}$$

$$SXT_3 = \frac{G_{x\theta} h^3}{3}$$

$$SXT_4 = \frac{2G_{x\theta} h^3}{3}$$

$$SPC = 2\rho_o h$$

Constants of the Stringers

$$STR1_\ell = \rho_{s\ell} A_{s\ell}$$

$$STR2_\ell = \rho_{s\ell} A_{s\ell} \bar{y}_{s\ell}$$

$$STR3_\ell = \rho_{s\ell} A_{s\ell} \bar{z}_{s\ell}$$

$$STR4_\ell = 2 \rho_{s\ell} A_{s\ell} \bar{z}_{s\ell}$$

$$STR5_\ell = \rho_{s\ell} (\bar{y}^2_{s\ell} A_{s\ell} + I_{zz_{s\ell}})$$

$$STR6_\ell = \rho_{s\ell} (\bar{z}^2_{s\ell} A_{s\ell} + I_{yy_{s\ell}})$$

$$STR7_\ell = \rho_{s\ell} (\bar{y}_{s\ell} \bar{z}_{s\ell} A_{s\ell} + I_{yz_{s\ell}})$$

$$STR8_\ell = E_{s\ell} A_{s\ell}$$

$$STR9_\ell = E_{s\ell} A_{s\ell} \bar{y}_{s\ell}$$

$STR10_{\ell} = E_{s\ell} A_{s\ell} \bar{z}_{s\ell}$

$STR11_{\ell} = E_{s\ell} (\bar{y}^2_{s\ell} A_{s\ell} + I_{zz_{s\ell}})$

$STR12_{\ell} = (GJ)_{s\ell}$

$STR13_{\ell} = E_{s\ell} (\bar{z}_{s\ell} \bar{y}_{s\ell} A_{s\ell} + I_{yz_{s\ell}})$

$STR14_{\ell} = E_{s\ell} (\bar{z}^2_{s\ell} A_{s\ell} + I_{yy_{s\ell}})$

Constants of the Rings

$RNG1_k = 2\rho_{rk} A_{rk}$

$RNG2_k = 2\rho_{rk} A_{rk} \bar{x}_{rk}$

$RNG3_k = 2\rho_{rk} A_{rk} \bar{z}_{rk}$

$RNG4_k = 4\rho_{rk} A_{rk} \bar{z}_{rk}$

$RNG5_k = 2\rho_{rk} (\bar{x}^2_{rk} A_{rk} + I_{zz_{rk}})$

$RNG6_k = 2\rho_{rk} (\bar{z}^2_{rk} A_{rk} + I_{xx_{rk}})$

$RNG7_k = 2\rho_{rk} (\bar{x}_{rk} \bar{z}_{rk} A_{rk} + I_{xz_{rk}})$

$RNG8_k = 2E_{rk} A_{rk}$

$RNG9_k = 2E_{rk} A_{rk} \bar{z}_{rk}$

$RNG10_k = 2E_{rk} A_{rk} \bar{z}^2_{rk}$

$RNG11_k = 2E_{rk} A_{rk} \bar{x}_{rk}$

$$RNG12_k = 2E_{rk} A_{rk} \overline{x}^2_{rk}$$

$$RNG13_k = 2E_{rk} A_{rk} \overline{x}_{rk} \overline{z}_{rk}$$

$$RNG14_k = 2(GJ)_{rk}$$

$$RNG15_k = 2(GJ)_{rk} \overline{z}_{rk}$$

$$RNG16_k = 2(GJ)_{rk} \overline{z}^2_{rk}$$

$$RNG17_k = 2E_{rk} I_{zz_{rk}}$$

$$RNG18_k = 2E_{rk} I_{zz_{rk}} \overline{z}_{rk}$$

$$RNG19_k = 2E_{rk} I_{zz_{rk}} \overline{z}^2_{rk}$$

$$RNG20_k = 2E_{rk} I_{xx_{rk}}$$

$$RNG21_k = 2E_{rk} I_{xz_{rk}}$$

$$RNG22_k = 2E_{rk} I_{xz_{rk}} \overline{z}_{rk}$$

VITA

Robert Leslie Brugh

Candidate for the Degree of

Doctor of Philosophy

Thesis: AN EVALUATION OF ANALYTICAL METHODS USING THE RAYLEIGH-RITZ APPROACH FOR THE FREE VIBRATIONS OF STIFFENED NONCIRCULAR CYLINDRICAL SHELLS

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born in Petersburg, Virginia, August 1, 1939, the son of Mr. and Mrs. L. C. Brugh.

Education: Graduated from Dinwiddie High School, Dinwiddie, Virginia, in June, 1956; received Associate of Science degree in Aeronautical Engineering from Bluefield College in 1959; received Bachelor of Science degree in Aerospace Engineering from Virginia Polytechnic Institute in 1962; received Master of Science degree in Engineering Mechanics from Wichita State University in 1970; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1973.

Professional Experience: Aeronautical Engineer, U. S. Army Transportation Research Command, 1962-1964, 1966-1967; Stress Analyst, Boeing Company, 1964-1966, 1967-1969; Graduate Teaching Assistant, Oklahoma State University, 1969-1973.

Professional Organizations: Member of the American Institute of Aeronautics and Astronautics; Member of the American Helicopter Society; Student Member of American Society of Mechanical Engineers.