CONSTRUCTING AND COMPARING TECHNIQUES
WHICH ASSIST IN LOCATING
A GLOBAL MAXIMUM

By

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CHAPTER I

INTRODUCTION

Purpose of Study

The literature on optimization techniques contains many different methods for locating a local maximum. There can hardly be one best optimization technique for every particular situation. Their performance must surely depend on the nature of the independent variables, characteristics of the response surface, as well as the criteria for judging "best".

Literature on the subject of global optimization devotes only a paragraph or two explaining that the global optimum (maximum) can be found with certainty only for a concave function over a convex region of the independent variables (1) (10) (13). Little can be said of the optimum conditions of a function (or process) over the entire region of the independent variables from a single attempt of a local optimization technique. However, repeated starts at different locations followed by convergence to the same point indicate with some "degree of certainty" that the global maximum has been found.

One of the purposes of this study is to construct and compare optimization techniques whose purpose it is to locate a global maximum. These techniques were constructed by using existing techniques which locate maxima together with alternative starting point strategies. This idea is suggested in some of the literature, but references to actual attempts to construct such techniques are not prevalent (1) (10) (13) (14). Zellnik, et. al. (14) did construct an optimization technique which searches for "alternate optima". The results from this publication were quite helpful in the formulation of the techniques used in this study. The techniques which were adopted for this study are the subject of Chapter II.

Data are then obtained on each of the global optimization techniques under a variety of experimental conditions which are discussed in Chapter III. Comparisons among the techniques are made in Chapter IV from the point of view that the potential user of such methods has no knowledge of the response surface.

Chapter V contains the summary and suggestions for further study.

Since it will be helpful, some of the terminology used throughout this dissertation follows.

Terminology

Independent Variables

The "independent variables" are those variables of the process being optimized which, when varied, may cause a change in the response. The symbols, $x_1, x_2, \ldots, x_n$, will represent n independent variables in a region X, where X contains all possible points of the vector $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$.

Range-R_i

When the independent variables cannot assume all values from $-\infty$, to ∞, they are usually said to be constrained or restricted. The

x_i, $i = 1, 2, \ldots, n$ will each be restricted to an interval of values between a specified lower bound, $x_{i\ell}$, and upper bound, x_{iu}. The range of each independent variable is

$$R_i = x_{iu} - x_{i\ell},\ i = 1, 2, \ldots, n.$$

This restricts the possible values of $\underset{\sim}{x}$ to be contained in at most an n-dimensional hypercube X.

Center-c_i

The "center" of the independent variable region, X, will be denoted by $\underset{\sim}{c} = (c_1, c_2, \ldots, c_n)$ where $c_i = (x_{iu} + x_{i\ell})/2$ and n is the number of dimensions.

Response

A "response" is a single observed realization (observation) of the process being optimized at a point $\underset{\sim}{x}$. This quantity might be in the units measured or as a function of the observed values. In either case, it is the response, $z(\underset{\sim}{x})$ at $\underset{\sim}{x}$, which provides a means for comparing the points in X. For example, in this study the point $\underset{\sim}{x}^{(2)}$ is said to be "better" than $\underset{\sim}{x}^{(1)}$ if the response $z(\underset{\sim}{x}^{(2)})$ is larger than the response $z(\underset{\sim}{x}^{(1)})$ since the optimization process is one of maximization.

Deterministic or Stochastic

Mathematical functions can be used to describe response surfaces. Functional values (responses) are said to be "deterministic" since separate evaluations at the same point, $\underset{\sim}{x}$, will yield identical values of, say, $z(\underset{\sim}{x})$.

Responses to separate evaluations of some processes may not be duplicated even though the point of evaluation is at the same $\underset{\sim}{x}$. In these cases the "true" response at a fixed setting of the independent variables is often masked by uncontrollable variables or noise in the response. These types of responses are called "stochastic".

Evaluation

An "evaluation" is the process necessary in obtaining a response $z(\underset{\sim}{x})$ at the point $\underset{\sim}{x}$. This can be a simple process if, for example in a deterministic situation, $z(\underset{\sim}{x}) = f(\underset{\sim}{x})$ where $f(\underset{\sim}{x})$ is a simple mathematical function.

A surface response may not only be a function of the point $\underset{\sim}{x}$, but also a function of unknown quantities (errors), say $e = (e_1, e_2, \ldots, e_m)$, as in a stochastic situation. A single evaluation at $\underset{\sim}{x}$ would yield $f(\underset{\sim}{x}; \underset{\sim}{e})$ where the contribution of $\underset{\sim}{e}$ to this value is unknown. In these instances a number of values $f(\underset{\sim}{x}; \underset{\sim}{e})_i$, $i = 1, \ldots, k$ might be made, and an average response surface value at $\underset{\sim}{x}$ calculated as

$$z(\underset{\sim}{x}) = \frac{\sum_{i=1}^{k} f(\underset{\sim}{x}; \underset{\sim}{e})_i}{k} .$$

A more lengthy process may result in this case as opposed to the simple mathematical function situation described above.

Optimization Techniques-Local and Global

Optimization techniques which are capable of locating local optima (maxima) are referred to simply as "local" techniques. These methods attempt to find the condition, $\underset{\sim}{x}$, which for any other conditions in the immediate neighborhood of $\underset{\sim}{x}$, render responses less than the local maximum.

The term "global" or "g-max" technique is applied to those optimization methods which attempt to find the condition $\underset{\sim}{x}$ which yields a larger response than any other point in the entire region X.

CHAPTER II

G-MAX PROCEDURES

A general procedure to approximate the global optimum is proposed in the first part of this chapter. The remainder of the chapter describes different local optimization techniques and restart procedures.

A General Technique to Approximate A Global Optimum

A general g-max technique can be constructed in two stages.

<u>Stage 1</u>

The global procedure begins by initializing a local technique, say L, at a point $\underset{\sim}{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ in X. Technique L terminates with an estimated local maximum, $z(\underset{\sim}{x}^{(1)})$.

<u>Stage 2</u>

A strategy is defined to sequentially select points $\underset{\sim}{x}$ from X. After each point is selected, the response $z(\underset{\sim}{x})$ is evaluated and compared to $z(\underset{\sim}{x}^{(1)})$. This process is continued until: (1) A successful point, $\underset{\sim}{x} = \underset{\sim}{x}^{(2)}$ say, is found such that $z(\underset{\sim}{x}^{(2)})$ is greater than $z(\underset{\sim}{x}^{(1)})$, or (2) the number of unsuccessful points equals a predetermined number, m. If condition 1 is satisfied, technique L is restarted at stage 1 with $\underset{\sim}{x}^{(0)} = \underset{\sim}{x}^{(2)}$. When condition 2 is satisfied, the entire procedure is terminated with $z(\underset{\sim}{x}^{(1)})$ declared the global optimum.

Six global techniques were constructed by combining three local optimization techniques and two alternative starting point strategies in all combinations into the general procedure in the above discussion. Three additional procedures consisted of employing the local techniques at a finite number of starting points. These techniques are discussed in the remainder of this chapter with emphasis on the starting point strategies. Particular local techniques will be discussed in Chapter III after the response surfaces are introduced. This was done so that the actual movements of each of the techniques could be observed during optimization on one of the response surfaces used in this study.

Local Optimization Techniques

Gradient

An often used optimization technique, the gradient ("steepest ascent"), still remains a useful procedure for finding local maxima. When compared to the one-variable-at-a-time, factorial, and random point procedures, Brooks (2) judged the gradient technique the best when confronted with stochastic as well as deterministic response surfaces. A gradient technique was involved in another comparison study by Leon (9) in which the "Variable Metric Method" by Davidon (5) was declared the superior technique over seven other methods. Comparisons among the techniques were made based on their performance on deterministic response surfaces of differentiable functions. The performance of the Variable Metric Method depends on the ability to obtain analytic derivatives as the technique involves computing the (Hessian) matrix of second order partial derivatives. It would not be fair to exclude the gradient method in this study based on the superior performance of such a technique, since the necessary assumptions underlying the Variable Metric Method are more stringent than the potential user in this study could assume.

Often compared to a blind person attempting to reach the top of the hill, the gradient technique begins by a local exploration of the response surface about an initial base point. From these explorations a gradient direction is calculated. A single step is made from the base point in the established direction to a new base point. An examination of the response surface is made about this new base point, the gradient direction recalculated, followed by another move. This process is repeated until a negligible change in the response surface at consecutive base points is observed.

A current working edition of this technique was adopted for this study as implemented by Scherich, et. al. (12).

Accelerated Gradient

The accelerated gradient technique begins as the gradient technique by computing a gradient direction relative to local response surface conditions at a specified starting point. Unlike the gradient technique which makes a single step along the gradient direction, the accelerated version locates an intermediate maximum (optimum) by a systematic scan of points in the computed direction. This intermediate maximum becomes the new base point for gradient re-evaluation. The purpose of the systematic scan or acceleration is to reduce, hopefully, the number of gradient calculations and increase the average improvement of each function evaluation.

Zellnik, et. al. (14) devised such a technique as an integral part of an optimization technique that also included an "alternative optimum" point selection procedure. This point selection procedure is discussed later in this chapter.

Pattern Search

Pattern search as developed by Hooke and Jeeves (7) is unlike the gradient techniques as it attempts to move closer to the maximum with each evaluation of the response surface. A portion of the evaluation in the gradient techniques is used only for determining gradient direction. Two types of moves made in the pattern search are called "basic" and "pattern".

The basic search begins at an initial base point searching by incremental amounts on a variable-at-a-time basis. Each search attempts to find a better response than the previous, relocating the base point each time to the last successful point. An unsuccessful search in one variable direction is followed by the same incremental amount in the opposite direction. Each successful search brings the base point closer to the maximum point.

After each independent variable has been searched, the basic search is temporarily terminated, noting the position of the last base point relative to the initial base point. The procedure then makes a pattern move.

A pattern move is so named because each variable is now varied simultaneously according in the direction indicated by the basic search, and by amounts proportional to the initial increments in the basic search. Moves are made in this direction until a failure is noted, wherein the basic search again starts at the last successful point.

Failure of a basic search to locate a better point causes a reduction in the variable increments. When reductions are made below a predetermined amount in each variable, the process is terminated. The maximum is stated to be at the last successful point.

Alternative Starting Point Procedures

Normal

Zellnik, et. al. (14) used a normal distribution to select points for locating "alternative optima". After a local maximum was found by the accelerated gradient technique, a finite number of points were chosen by selecting each component of a random point $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ according to

$$g(x_i | c_i, (.35R_i)^2) = \frac{1}{\sqrt{2\pi}\,(.35R_i)} e^{-\frac{1}{2}\left(\frac{x_i - c_i}{.35R_i}\right)^2}, \quad -\infty < x_i < \infty ,$$

which is the normal distribution with mean $\mu = c_i$ and variance $\sigma^2 = (.35R_i)^2$. (The standard abbreviated form will be used for the normal distribution, namely $n(\mu,\sigma^2) = n(c_i,(.35R_i)^2)$. The number of points selected depends on the number of dimensions, n. Zellnik and his colleagues chose $8n$ random points subsequent to convergence of the accelerated gradient technique. They felt that for $n \leq 3$ the $8n$ random points were sufficient to locate a value better than the one obtained by the accelerated gradient, if a better one exists.

For $n \geq 4$, possibly more than $8n$ random points were needed.

The Zellnik accelerated gradient technique with the normal point selection procedure was incorporated into the general global procedure with m (the maximum number of random points selected after convergence) equal to $8n$.

An intermediate procedure was also included in the Zellnik procedure to check for convergence to a saddle point. After the accelerated gradient converges to a point, say $\tilde{x}$, a search is made in the immediate vicinity of $\tilde{x}$ by selecting random points according to $n(x_i, (.05R_i)^2)$. This scan is to insure that $\tilde{x}$ is indeed a relative maximum and not a saddle point. While attempting to measure the effectiveness of this check, Zellnik and his colleagues observed that the accelerated gradient could not be forced to converge at the saddle point. The procedure proved beneficial only when the accelerated gradient was initialized exactly on the saddle point. Consequently, the Zellnik technique adopted for this study was without the saddle point check. It is refreshing to see an example of something of apparent analytic concern in the literature, which has little importance in practice.

It would seem, prior to experimentation at least, that the normal distribution point selection procedure may not be the most advantageous for two reasons. It would seem first of all that the preferred region to seek out alternative starts would be that region yet unexplored by the local optimization technique. This may not be the case, for example, if an optimization technique starts and converges near $\tilde{c}$, since the greater part of the probability mass of the $n(c_i, (.35R_i)^2)$ distribution for $i = 1, 2, \ldots, n$ is at the hypercube center. Secondly, the range of the random variables in a normal distribution is defined from

$-\infty$ to ∞. This implies that there is a non-zero probability of selecting points outside of X. In particular the probability that the random coordinate x_i is between its defined upper and lower limits is

$$\text{Prob.}\left[x_{i\ell} \le x_i \le x_{iu}\right] = \text{Prob.}\left[\frac{x_{i\ell}-c_i}{.35R_i} \le \frac{x_i-c_i}{.35R_i} \le \frac{x_{iu}-c_i}{.35R_i}\right]$$

$$= \text{Prob.}\left[\frac{-.5}{.35} \le \eta \le \frac{.5}{.35}\right]$$

$$= .8472$$

where $\eta \sim n(0,1)$ for $i = 1, 2, \ldots, n$. The probability that the random point (x_1,x_2) in two dimensions is outside the region X is $1 - (.8472)^2 = .2823$. Since there is a non-zero probability that a point is selected outside of X, logical checks must be made in the computerization of such a technique so that only the proper points are accepted.

For purposes of this study, points that are selected outside of X are moved to the boundary of X. The frequency of points selected on the boundary can be altered by the variance of the normals for a fixed n. Increasing the variance increases the number of points selected on the boundary. Decreasing the variance, decreases the frequency of points on the boundary, but causes a larger concentration of points about $\underset{\sim}{c}$. For this study a standard deviation of $.35R_i$ was maintained as suggested by Zellnik. This technique might prove useful for locating maxima on the boundary.

Uniform

Selecting the components of the random point $\underset{\sim}{x}$ according to the uniform distribution

$$U(x_i) = \frac{1}{x_{iu} - x_{i\ell}}, \quad x_{i\ell} \le x_i \le x_{iu}$$

where $i = 1, 2, \ldots, n$, will be referred to as the uniform alternative starting point selection procedure or simply the uniform method. Unlike the normal method, the ranges of the distributions can be restricted to the respective ranges of the independent variables. Like the normal method it cannot avoid selecting points from previously explored regions. However, the uniform method at least avoids showing favoritism for these regions regardless of the path of the optimization technique in X.

4-Point

The 4-point strategy is non-sequential in that four starts are made with an optimization technique and the global maximum taken as the largest of the four local maxima found. The relative positions of these four starting points in two dimensions is depicted in Figure 1. The four points are located far enough from the corners of the rectangular region to prevent initial calculations of the optimization technique from requesting points outside the region X. The 4-point technique was adopted to provide a comparison to the sequential normal and uniform methods.

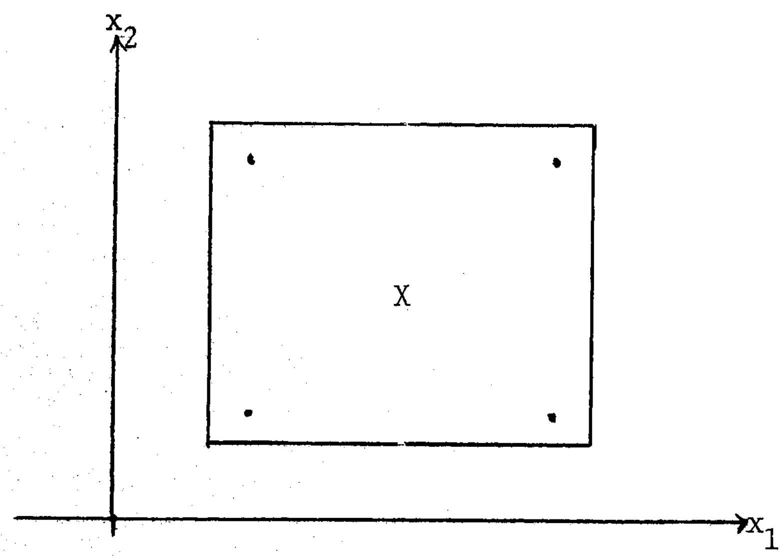


Figure 1. Starting Points of 4-Point Method

An extension of this method to n dimensions is not as amenable as the normal or uniform methods. One extension might be brought about through the use of factorial and/or fractional factorial arrangements of starting points. It is questionable that a "useful" technique can be found even beyond the case where $n = 3$. The number of desired starts may quickly become too large.

Summary of Methods

Nine techniques to approximate the global maximum result from all combinations of the three local optimization techniques and the three point selection procedures. These nine techniques are listed below in Table I together with their computer code number.

TABLE I

COMPUTER CODES FOR OPTIMIZATION TECHNIQUES TO APPROXIMATE THE GLOBAL OPTIMUM

Restart Procedure	Local Optimization Technique		
	1 Accelerated Gradient	2 Pattern Search	3 Gradient
1-Normal	1 1	1 2	1 3
2-4-Point	2 1	2 2	2 3
3-Uniform	3 1	3 2	3 3

CHAPTER III

EXPERIMENTAL CONDITIONS

Introduction

The nine optimization techniques described in Chapter I will be compared on an empirical basis from data obtained from a variety of experimental situations. The number of independent variables (factors) that should be included in such an experiment tends to boggle the mind. However, as with most experiments (there was only a limited amount of resources available) this meant that some variables must remain at fixed values. The purpose of this chapter is to discuss the experimental conditions which will provide a basis for comparing the g-max techniques.

Factors Held Constant

Number of Independent Variables

All optimizations will be on response surfaces which are a function of independent variables x_1 and x_2. Their ranges in all situations are $0 \leq x_1 \leq 6$ and $0 \leq x_2 \leq 5$. In other words, the region X has a rectangular shape. These intervals were selected mainly to alleviate scaling problems associated with a computer graphing procedure used later in this study.

Initial Starting Point

The global optimization techniques constructed with the 4-point strategy, obviously, have four starting points located in the four corners of X. Global techniques with the uniform and normal distributions of points all begin at the region center $\underset{\sim}{c} = (3.0, 2.5)$. Taking the approach that the eventual user of such procedures has no knowledge of the surface being optimized, this seems like a "reasonable" place to begin.

Number of Random Points

The normal and uniform procedures selected up to sixteen points in X subsequent to each convergence of a local technique. This is in keeping with the suggested number by Zellnik, et. al. (14).

Factors Varied

Response Surfaces

The Class of Functions $\mathcal{F}$. An important part or basis for an empirical comparison of optimization techniques is the type of response surfaces selected on which optima are to be sought. In this section, response surfaces will be constructed over a two-dimensional rectangular region X from a class of functions

$$\mathcal{F} = \{f(x_1,x_2;\underset{\sim}{\mu}^{(1)},\underset{\sim}{\mu}^{(2)}, \ldots,\underset{\sim}{\mu}^{(k)}, \underset{\sim}{\Sigma}^{(1)},\underset{\sim}{\Sigma}^{(2)}, \ldots,\underset{\sim}{\Sigma}^{(k)}) \mid (x_1, x_2) \text{ in } X\}$$

where $f(x_1,x_2; \underset{\sim}{\mu}^{(1)},\underset{\sim}{\mu}^{(2)}, \ldots,\underset{\sim}{\mu}^{(k)}, \underset{\sim}{\Sigma}^{(1)},\underset{\sim}{\Sigma}^{(2)}, \ldots,\underset{\sim}{\Sigma}^{(k)})$ is a sum of k bivariate normal probability density functions. A member of the class F will be referred to simply as $f(x_1,x_2)$ for brevity with functional form

$$f(x_1,x_2) = \sum_{i=1}^{k} (2\pi)^{-1}|\underset{\sim}{\Sigma}^{(i)}|^{-1/2} \exp[-1/2(\underset{\sim}{x} - \underset{\sim}{\mu}^{(i)})' \underset{\sim}{\Sigma}^{(i)^{-1}}(\underset{\sim}{x} - \underset{\sim}{\mu}^{(i)})]$$

$$= \sum_{i=1}^{k} BVN(\underset{\sim}{\mu}^{(i)}, \underset{\sim}{\Sigma}^{(i)}) \qquad (1)$$

where

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \underset{\sim}{\mu}^{(i)} = \begin{bmatrix} \mu_1^{(i)} \\ \mu_2^{(i)} \end{bmatrix},$$

$$\underset{\sim}{\Sigma}^{(i)} = \begin{bmatrix} \sigma_{11}^{(i)} & \sigma_{12}^{(i)} \\ \sigma_{12}^{(i)} & \sigma_{22}^{(i)} \end{bmatrix}.$$

Many different types of surfaces can be rapidly constructed from equation (1) due to the known relationships of the parameters to the surface shape of each term in the sum. Twelve such surfaces were constructed for later comparison of the optimization techniques discussed in Chapter II.

Motivation. Empirical comparisons of optimization techniques which have been made seem to be based on one of two types of surfaces. A study by Leon (9) consisted of examining the performance of techniques over functions of two independent variables. Although five functions were used, all five had a single minimum at the bottom of a surface valley surrounded by long and narrow surface contours. One of these surfaces is the function

$$ROSIE = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

which has contours similar to those in Figure 2 (11).

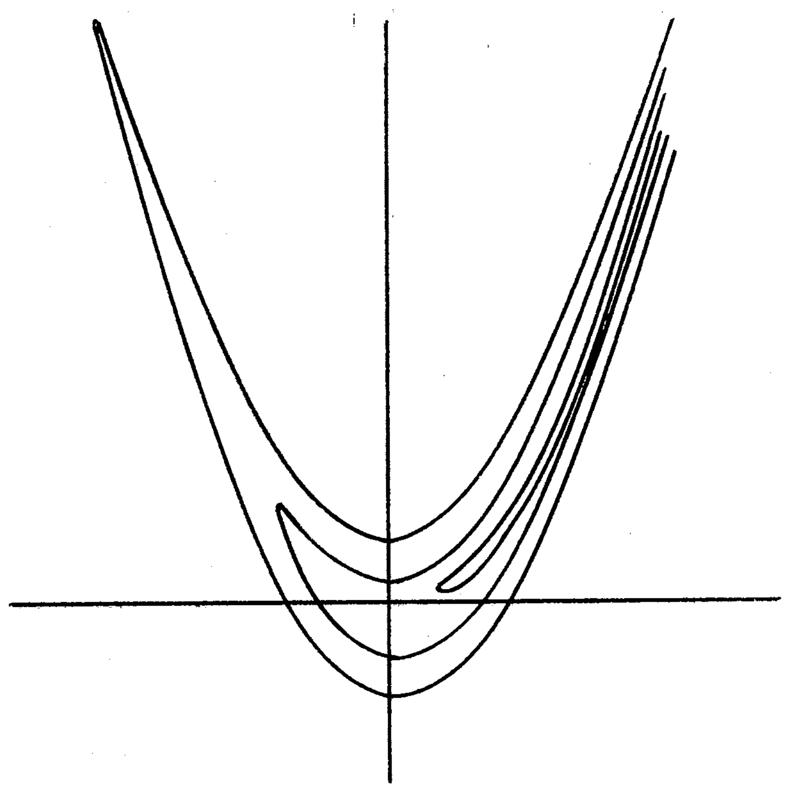


Figure 2: Contours of Function ROSIE

The user of optimization techniques should not be expected to make a selection from studies based only on such surfaces, but as Kempthorne (8) commented on this work; such studies "... help us to pick out some procedures as having greater robustness to the idiosyncrasies of particular problems than others." On the other end of the spectrum, Brooks (2) made technique comparisons employing surfaces with very simple looking contours. Contours of the four surfaces employed by Brooks are depicted in Figure 3.

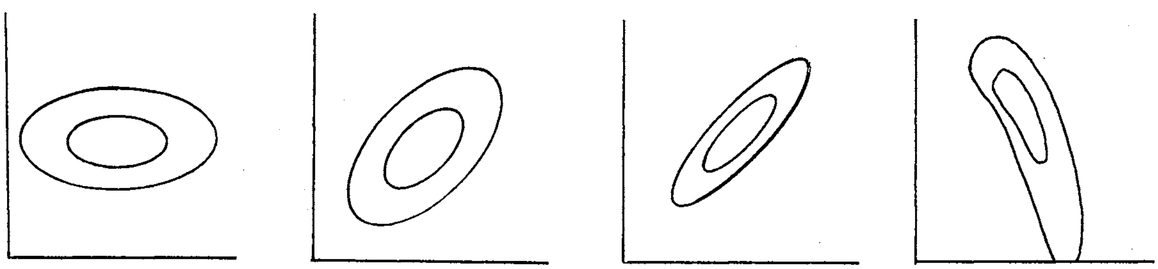


Figure 3. Four Surfaces from Brooks' Study

Studies which involve such surfaces are also of value, but by themselves are likely to be an oversimplification for giving guidance to the user.

A more meaningful study for the user who is confronted with an unknown surface might be one that evaluates performance based on a wide spectrum of shapes. Constructing such a wide variety of surfaces can be accomplished by identifying a class of functions

$$\mathcal{F} = \{f(x_1,x_2,\ldots,x_n;\ \theta_1,\theta_2,\ldots,\theta_m) \mid (\theta_1,\theta_2,\ldots,\theta_m) \text{ in } \Theta, (x_1,x_2,\ldots,x_m) \text{ in } X\}$$

which will generate surfaces with at least two modes over a rectangular region X.

Surface construction will be enhanced if the parameters $\underset{\sim}{\theta}$ in the elements of $\mathcal{F}$ have known relationships to the surface characteristics. Known alterations of shape of a particular surface can then be made by appropriate changes in the parameter values. The class of polynomials in two independent variables with coefficients $\underset{\sim}{\theta}$, for example, is a possible candidate for $\mathcal{F}$. However, for this author at least, this is not a desirable candidate since it is quite difficult to ascertain the values of the coefficients to construct specific surface shapes, particularly those which have two or more relative maxima.

<u>Ellipse of Dispersion (EOD)</u>. The parameters of probability density functions often have a known relationship to surface characteristics. The curve, for example, of the normal probability density function $n(\mu, \sigma^2)$ where

$$f(x;\ \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

has a maximum at μ and points of inflection at $\mu \pm \sigma$. It is well known that increasing σ^2 decreases the maximum value of the function and elongates the bell shaped curve.

Eisenberger (6) examined the bimodality conditions of a function formed by the weighted average of two normal densities; namely

$$g(x;\ p) = p \cdot n(\mu_1, \sigma_1^2) + (1-p) \cdot n(\mu_2, \sigma_2^2)$$

where $-\infty < x < \infty$ and $0 < p < 1$. Eisenberger found that:

1. if $\mu_1 = \mu_2$, $g(x; p)$ is unimodal for all p,
2. a sufficient condition that $g(x; p)$ be unimodal for all p is that
$$(\mu_2 - \mu_1)^2 < \frac{27}{4} \times \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} ,$$
3. a sufficient condition that there exists values of p for which $g(x; p)$ is bimodal is that
$$(\mu_2 - \mu_1)^2 > \frac{8\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} ,$$

and

4. for every set of values $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$, values of p exist for which $g(x; p)$ is unimodal.

If condition 3 is satisfied, it can only be said that a function of the form, say
$$f(x) = 2g(x; p = .5) = n(\mu_1, \sigma_1^2) + n(\mu_2, \sigma_2^2), \tag{2}$$
might be bimodal for fixed parameters (μ_1, σ_1^2) and (μ_2, σ_2^2).

Equation (2) can be extended to include two independent variables by
$$f(x_1,x_2) = \text{BVN}(\underset{\sim}{\mu}^{(1)}, \Sigma^{(1)}) + \text{BVN}(\underset{\sim}{\mu}^{(2)},\Sigma^{(2)}), \tag{3}$$
where $\text{BVN}(\underset{\sim}{\mu}^{(i)}, \Sigma^{(i)})$, $i = 1, 2$, represents the bivariate normal density described in equation (1). Like the univariate case, two summed bivariate normals may result in a unimodal or multimodal function depending on their parameter values. Guidelines for specifying $\underset{\sim}{\mu}^{(i)}$ and $\Sigma^{(i)}$ will be extended from a modification of Eisenberger's condition 3

to aid in the construction of bimodal surfaces by summing bivariate normal probability density functions.

For reasons that shall later become apparent, consider the sum of two univariate normals as in equation (2) where the parameters are chosen such that the two intervals $I_1 = [\mu_1 - \sigma_1, \mu_1 + \sigma_1]$ and $I_2 = [\mu_2 - \sigma_2, \mu_2 + \sigma_2]$ do not overlap. This restriction is more conservative towards the multimodal situation than the unimodal in the sense that the minimum distance between μ_1 and μ_2 as been increased from Eisenberger's restricted minimum of

$$(\mu_2 - \mu_1)^2 > \frac{8\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2} .$$

To illustrate, that the above is true, consider the situation where the intersection of I_1 and I_2, $I_1 \cap I_2 = \phi$, is the empty set, and without loss of generality $\mu_2 > \mu_1$. Then

$$(\mu_2 - \mu_1)^2 > \{\mu_2 - (\mu_2 - \sigma_2) + (\mu_1 + \sigma_1) - \mu_1\}^2 = (\sigma_1 + \sigma_2)^2. \quad (4)$$

It is also true by adding $2\sigma_1\sigma_2 > 0$ to both sides of the relationship

$$\sigma_1^2 + \sigma_2^2 \geq 2\sigma_1\sigma_2 \quad (5)$$

that

$$(\sigma_1 + \sigma_2)^2 \geq 4\sigma_1\sigma_1. \quad (6)$$

Multiplying (5) and (6) together results in

$$(\sigma_1^2 + \sigma_2^2)(\sigma_1 + \sigma_2)^2 \geq 8\sigma_1^2\sigma_2^2 ,$$

or

$$(\sigma_1 + \sigma_2)^2 \geq \frac{8\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \; . \tag{7}$$

The inequalities in (4) and (7) imply that

$$(\mu_2 - \mu_1)^2 > (\sigma_1 + \sigma_2)^2 \geq \frac{8\sigma_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2} \; .$$

This inequality suggests that (for a specified μ_1 and μ_2) if the parameters μ_1 and μ_2 in equation (2) are selected to satisfy $I_1 \cap I_2 = \phi$, then Eisenberger's restriction is also satisfied. The opportunities for bimodality in the resulting function $f(x)$ are at least as good since the minimal distance between μ_1 and μ_2 has been increased.

The criterion of nonoverlapping intervals I_1 and I_2 will now be extended by a rationalization to the case of two independent variables. Notice that the endpoints of the intervals I_1 and I_2 are the values at which the points of inflection of the respective normal curves occur. Similar values can also be found for the curve resulting from the intersection of a bivariate normal distribution with a plane perpendicular to the x_1, x_2-plane passing through the point (μ_1, μ_2). The resulting curve which "looks like a normal curve" also has points of inflection. There is a family of values (x_1, x_2) where points of inflection occur on the family of functions resulting from rotating the plane about the axis passing through (μ_1, μ_2) and $f(\mu_1, \mu_2)$. These points (x_1, x_2) will compose the boundary of a region called A.

To locate these boundary points, substitute

$$\frac{x_1-\mu_1}{\sigma_1} = r\cdot\cos(\theta) \quad \text{and} \quad \frac{x_2-\mu_2}{\sigma_2} = r\cdot\sin(\theta)$$

into the bivariate density form

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho)^2}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right\},$$

where $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$, to obtain the polar form

$$f(r,\ \theta) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{r^2}{2(1-\rho^2)}(1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))\right\} \tag{8}$$

where $-\infty < r < \infty$ and $0 \leq \theta < \pi$. The values of r will now be found where points of inflection occur on the curve of the bivariate normal surface above any line in the polar plane passing through the pole (0, 0). By taking the first and second partial derivatives of $f(r,\ \theta)$ in equation (8) with respect to r, holding θ fixed, one can compute

$$\frac{\partial f}{\partial r} = \frac{-r(1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))}{\left(2\pi\sigma_1\sigma_2(1-\rho^2)\right)^{3/2}} \exp\left\{-\frac{r^2}{2(1-\rho^2)}(1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))\right\},$$

and

$$\frac{\partial^2 f}{\partial r^2} = \left[- \frac{(1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))}{2\pi\sigma_1\sigma_2(1-\rho^2)^{3/2}} + \frac{r^2(1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))^2}{2\pi\sigma_1\sigma_2(1-\rho^2)^{5/2}} \right]$$

$$\times \exp\left\{- \frac{r^2}{2(1-\rho^2)} (1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))\right\} .$$

Setting the latter expression equal to zero and solving for r^2 results in

$$r^2 = \frac{1-2\rho\cdot\cos(\theta)\cdot\sin(\theta)}{2\pi\sigma_1\sigma_2(1-\rho^2)^{3/2}} \times \frac{2\pi\sigma_1\sigma_2(1-\rho^2)^{5/2}}{(1-2\rho\cdot\cos(\theta)\cdot\sin(\theta))^2}$$

$$= \frac{1-\rho^2}{1-2\rho\cdot\cos(\theta)\cdot\sin(\theta)} . \qquad (9)$$

As θ varies from 0 to π all the points of inflection found in the manner described above satisfy:

$$r = \pm\left(\frac{1-\rho^2}{1-2\rho\cdot\cos(\theta)\cdot\sin(\theta)}\right)^{1/2} .$$

By making the substitution

$$y_1 = \frac{x_1-\mu_1}{\sigma_1} = r\cdot\cos(\theta) \quad , \quad y_2 = \frac{x_2-\mu_2}{\sigma_2} = r\cdot\sin(\theta) \quad ,$$

and $r^2 = y_1^2 + y_2^2$ equation (9) becomes

$$y_1^2 + y_2^2 = \frac{1-\rho^2}{1-\frac{2\rho y_1 y_2}{y_1^2+y_2^2}}$$

which after a rearrangement of terms implies that "the points of inflection" in rectangular coordinates are points on the ellipse

$$\frac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\rho\frac{(x_1-\mu_1)}{\sigma_1}\frac{(x_2-\mu_2)}{\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}} = 1 - \rho^2 ,$$

or in matrix notation

$$(\underset{\sim}{x} - \underset{\sim}{\mu})' \, \Sigma^{-1} \, (\underset{\sim}{x} - \underset{\sim}{\mu}) = 1.$$

This ellipse will be referred to as the ellipse of dispersion (EOD). (This ellipse should not be confused with Cramér's (3) ellipse of concentration which he defines as $(\underset{\sim}{x} - \underset{\sim}{\mu})' \, \Sigma^{-1} \, (\underset{\sim}{x} - \underset{\sim}{\mu}) = 4$ in two dimensions.)

The class of functions used to construct surfaces over X in two dimensions was selected to be

$$\mathcal{F} = \left\{ f(x_1,x_2) = \sum_{i=1}^{k} \text{BVN}(\underset{\sim}{\mu}^{(i)}, \Sigma^{(i)}) \mid (x_1,x_2) \text{ in } X \right\}.$$

For the case where $k = 2$ an analogy can now be made to the one dimensional condition of $I_1 \cap I_2 = \phi$ to nonoverlapping regions A_1 and A_2 where

$$A_i = \{(x_1,x_2) | (\underset{\sim}{x} - \underset{\sim}{\mu}^{(i)})' \Sigma^{(i)^{-1}} (\underset{\sim}{x} - \underset{\sim}{\mu}^{(i)}) \leq 1\}, \; i = 1,2.$$

These regions will be used in the construction of multimodal surfaces.

To enhance multimodal surface construction from the sum of two bivariate normal densities, the conjecture is to specify the parameters $(\underset{\sim}{\mu}^{(1)}, \Sigma^{(1)})$ and $(\underset{\sim}{\mu}^{(2)}, \Sigma^{(2)})$ so that the resulting EOD's define A_1 and A_2 with the property that $A_1 \cap A_2 = \phi$. These guide lines will be very useful in the actual construction of a surface.

Surface Construction. Given the center, the half-length of the major and minor axes, and the angle of rotation ϕ of the EOD, the corresponding bivariate normal distributions may be easily determined. The center of the ellipse becomes (μ_1, μ_2). The variance-covariance matrix Σ is obtained by a simple orthogonal transformation.

To illustrate, let the positive u axis be the axis forming the angle ϕ with the x_1 axis as in Figure 4.

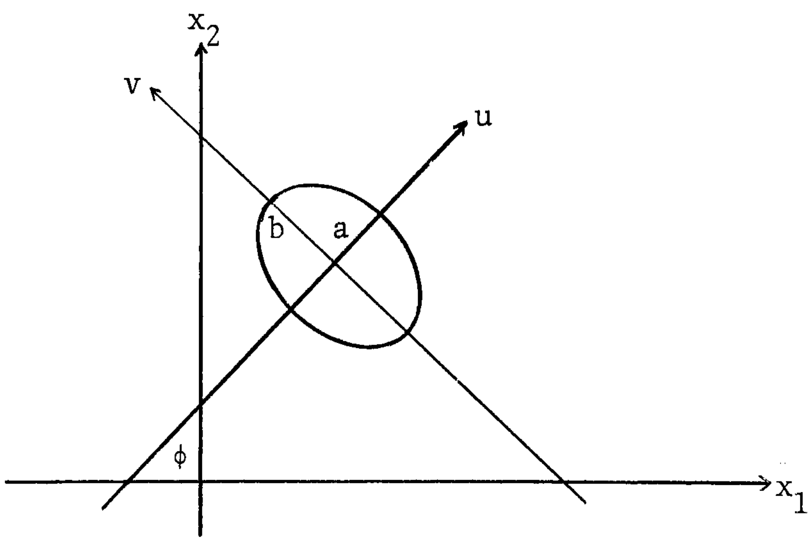


Figure 4. Constructing EOD

The equation of the EOD with respect to the u,v axes is

$$(u-\mu_1, v-\mu_2)\Delta^{-1}\begin{pmatrix} u - \mu_1 \\ v - \mu_2 \end{pmatrix} = 1 \quad \text{where} \quad \Delta = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} .$$

Making the orthogonal transformation

$$\begin{pmatrix} u - \mu_1 \\ v - \mu_2 \end{pmatrix} = P'\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \quad \text{where} \quad P' = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

we obtain

$$\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} P\Delta^{-1} P' \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} = 1.$$

Let $\Sigma^{-1} = P\Delta^{-1}P'$, then $\Sigma = (\Sigma^{-1})^{-1}$ is the required variance-covariance matrix to generate the desired EOD. So

$$\Sigma^{-1} = P\Delta^{-1}P'$$

$$= \begin{bmatrix} \frac{1}{a^2}\cos^2\phi + \frac{1}{b^2}\sin^2\phi & \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin\phi\cos\phi \\ \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin\phi\cos\phi & \frac{1}{a^2}\sin^2\phi + \frac{1}{b^2}\cos^2\phi \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} b^2\sin^2\phi + a^2\cos^2\phi & (a^2-b^2)\sin\phi\cos\phi \\ (a^2-b^2)\sin\phi\cos\phi & b^2\cos^2\phi + a^2\sin^2\phi \end{bmatrix}$$

To illustrate the usefulness of the EOD on the class of functions F, twelve surfaces will be constructed over the rectangular region $0 \leq x_1 \leq 6$ and $0 \leq x_2 \leq 5$. The orientation of the major and minor axes of the EOD's for each of the bivariate normals used in constructing each of the twelve surfaces

$$f(x_1, x_2) = \sum_{i=1}^{k} \mathrm{BVN}(\underset{\sim}{\mu}^{(i)}, \Sigma^{(i)})$$

is shown in Figures 5 and 6. The specifications of each of the EOD's are given in Table II together with the computed values of $\boldsymbol{\Sigma}^{(i)}$. The column labeled "a" is the half-length of the axis forming the acute angle ϕ with the positive x_1 axis in the first six surfaces. The column labeled "b" is the half-length of the axis of the EOD perpendicular to "a". The center of the respective EOD's are $\mu_1^{(i)}$ and $\mu_2^{(i)}$. The last six surfaces are merely 90° rotations of the first six about the region center (3.0, 2.5). This in effect is a rotation of the region X through 90° since x_1 and x_2 are arbitrary variables. The set of twelve surfaces also can be considered as two sets of six unique surfaces with a change in the ranges of the independent variables.

The resulting contours from the surfaces specified in Table II are illustrated in Figures 7 and 8. The locations of the relative maxima for each surface are listed in Table III and indicated on the surface figures by a dot. Surfaces 4 and 5 appear to have three maxima, but the "elliptical looking" contours near the center of these figures have no dot indicating these regions contain a minimum point. The values in Table III are approximated as the maxima could not be found analytically. The results were determined by different starts near the maxima of the accelerated gradient procedure for each surface. This process was repeated until the maxima listed in the table appeared to be accurate to at least five decimal places.

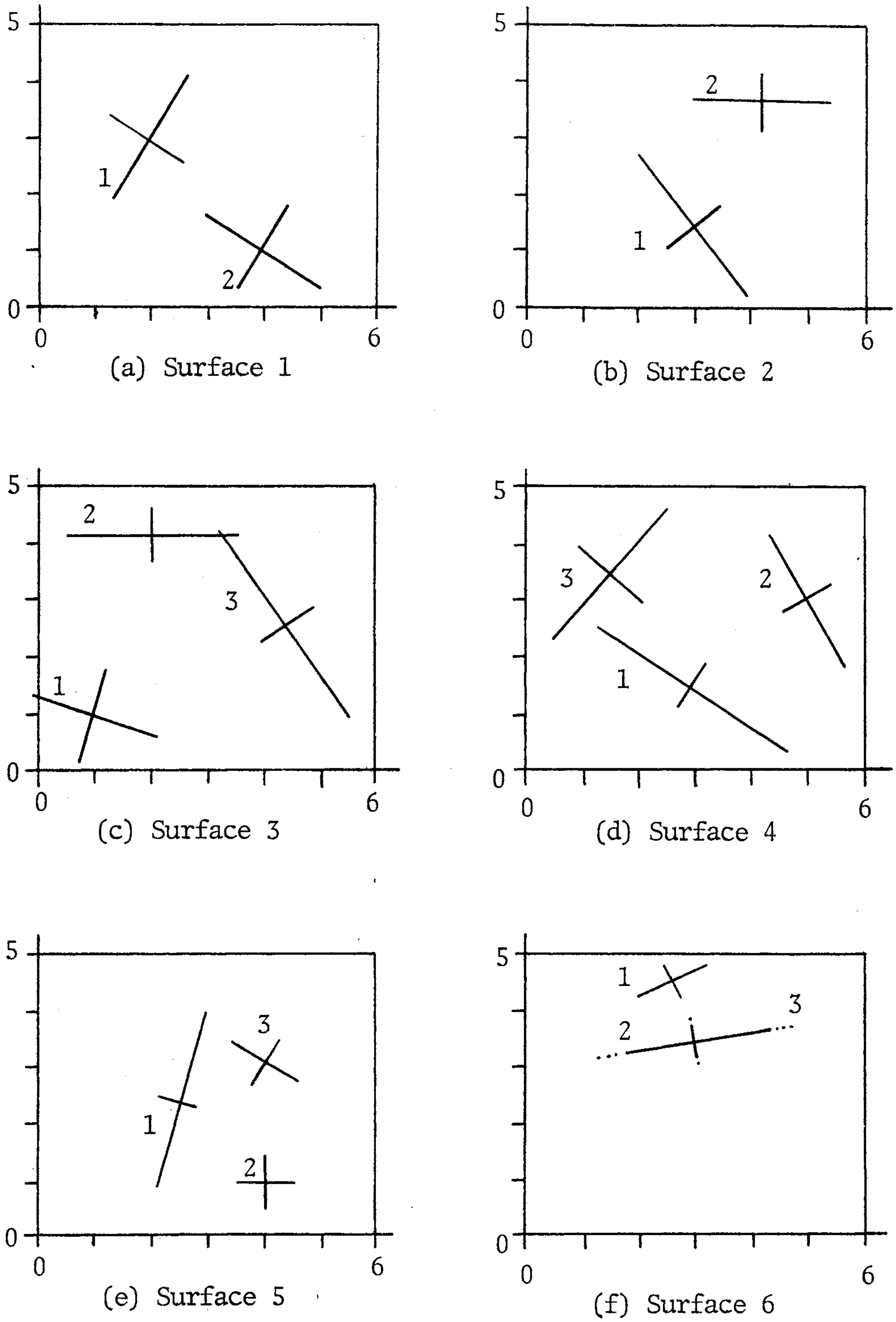


Figure 5. Major and Minor Axes of EOD's for Response Surfaces 1-6

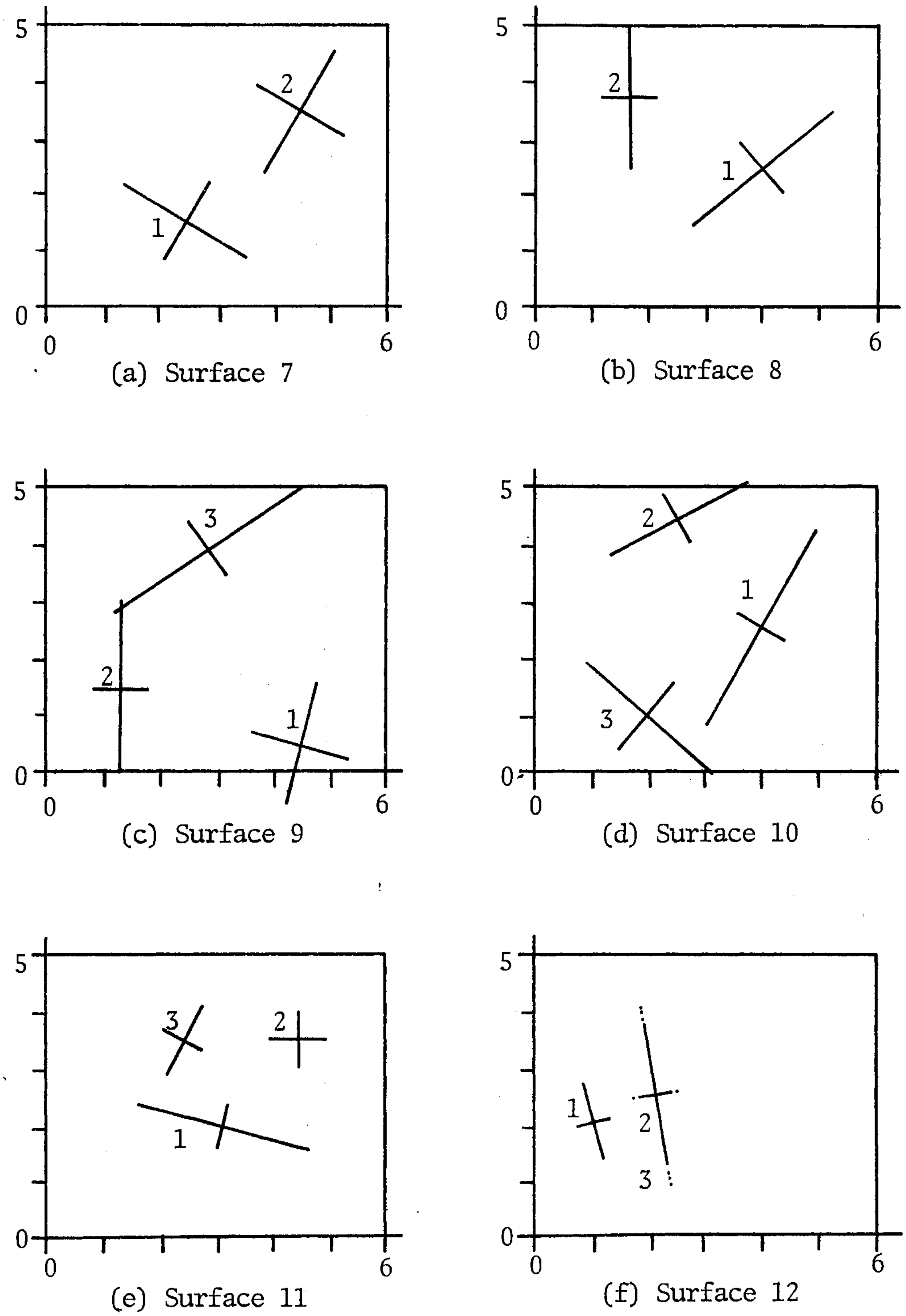


Figure 6. Major and Minor Axes of EOD's
for Response Surfaces 7-12

TABLE II

PARAMETER SPECIFICATIONS FOR THE TWELVE SURFACES

Surface	i	a	b	ϕ	μ_1	μ_2	σ_{11}	σ_{12}	σ_{22}
1	1	1.3	.8	60	2.0	3.0	.92	.49	1.49
	2	.9	1.3	60	4.0	1.0	1.37	- .33	1.00
2	1	.6	1.6	40	3.0	1.4	1.27	-1.27	1.65
	2	1.3	.5	0	4.2	3.8	1.56	0	.25
3	1	.9	1.1	75	1.0	1.0	1.18	- .10	.84
	2	1.5	.5	0	2.0	4.2	2.25	0	.25
	3	.6	2.0	35	4.4	2.6	1.56	-1.71	2.80
4	1	1.5	.8	50	1.5	3.5	1.26	.83	1.55
	2	.5	1.4	30	5.0	3.0	.68	- .74	1.53
	3	.5	2.0	60	3.0	1.5	3.06	-1.62	1.19
5	1	1.6	.4	75	2.5	2.4	.32	.60	2.40
	2	.5	.5	0	4.0	1.0	.25	0	.25
	3	.4	.7	60	4.0	3.1	.41	- .14	.24
6	1	.7	.3	15	2.6	4.5	.46	.10	.12
	2	1.3	.3	10	3.0	3.4	1.64	.27	.14
	3	1.8	.4	10	3.0	3.4	3.15	.53	.25
7	1	1.3	.8	150	2.5	1.5	1.49	- .49	.92
	2	.9	1.3	150	4.5	3.5	1.00	.33	1.37
8	1	.6	1.6	130	4.1	2.5	1.65	1.08	1.27
	2	1.3	.5	90	1.7	3.7	.25	0	1.56
9	1	.9	1.1	165	4.5	.5	.84	.10	1.18
	2	1.5	.5	90	1.3	1.5	.25	0	2.25
	3	.6	2.0	125	2.9	3.9	2.80	1.71	1.56
10	1	1.5	.8	140	2.0	1.0	1.55	- .83	1.26
	2	.5	1.4	120	2.5	4.5	1.53	.74	.68
	3	.5	2.0	150	4.0	2.5	1.19	1.62	3.06
11	1	1.6	.4	165	3.1	2.0	2.40	- .60	.32
	2	.5	.5	90	4.5	3.5	.25	0	.25
	3	.4	.7	150	2.4	3.5	.24	.14	.41
12	1	.7	.3	105	1.0	2.1	.12	- .10	.46
	2	1.3	.3	100	2.1	2.5	.14	- .27	1.64
	3	1.8	.4	100	2.1	2.5	.25	- .53	3.15

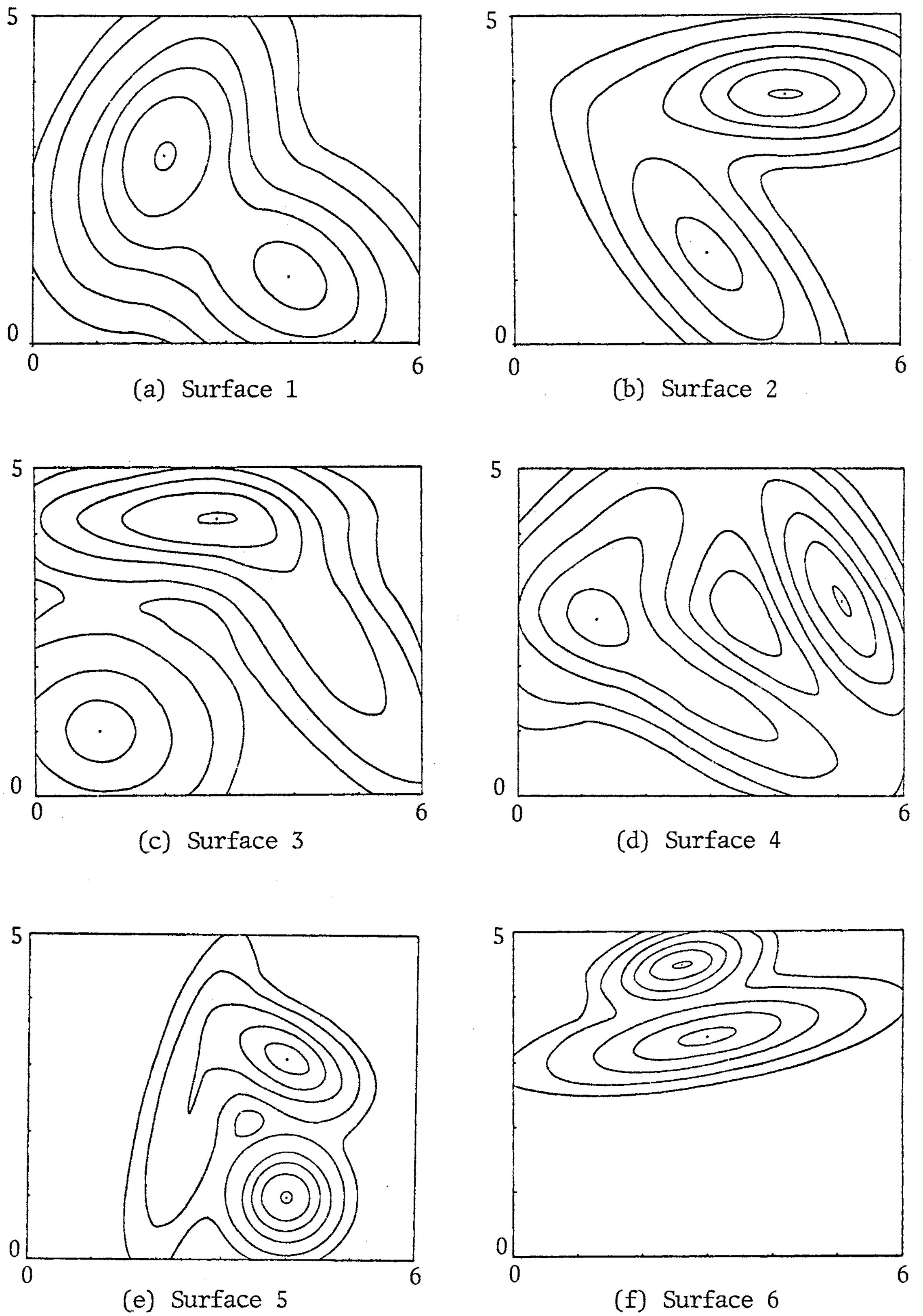
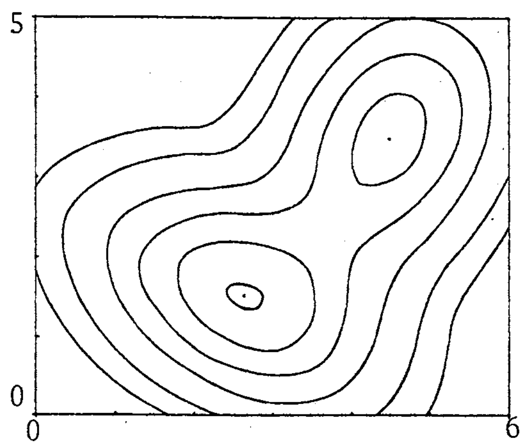
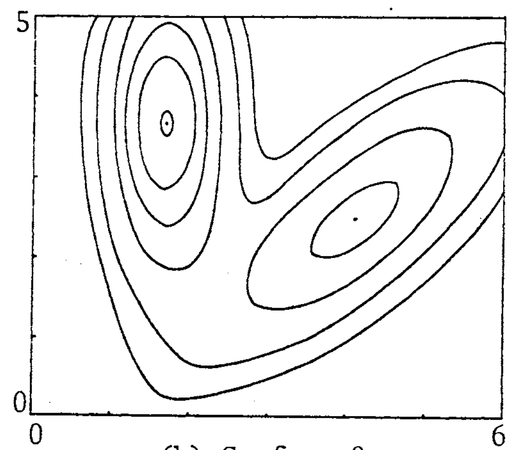


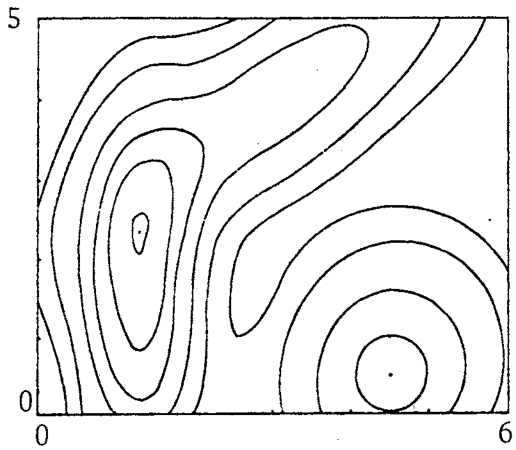
Figure 7. Contours for Response Surfaces 1-6



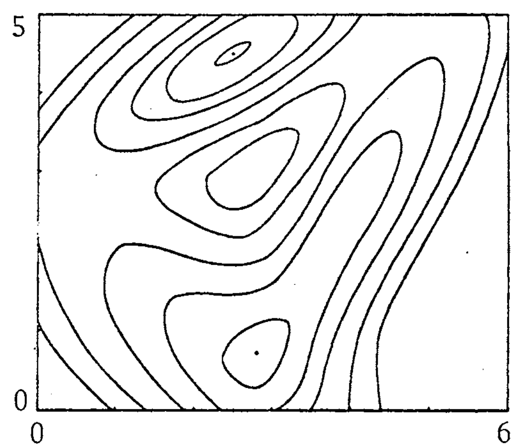
(a) Surface 7



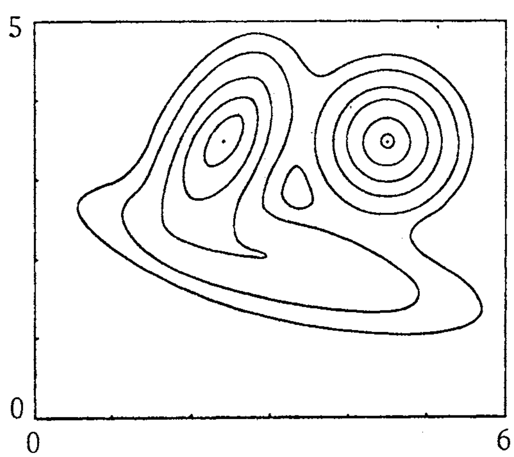
(b) Surface 8



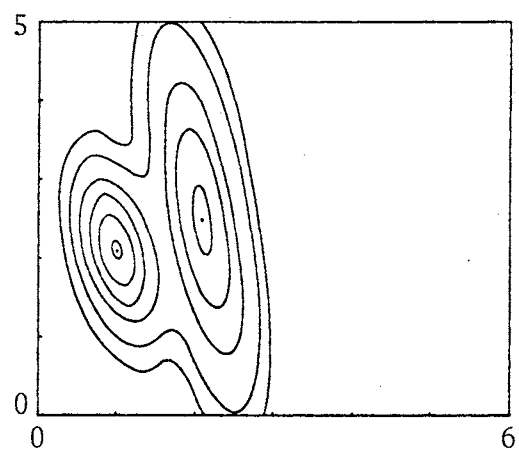
(c) Surface 9



(d) Surface 10



(e) Surface 11



(f) Surface 12

Figure 8. Contours for Response Surfaces 7-12

TABLE III

POINTS OF LOCAL MAXIMA FOR THE TWELVE SURFACES

	Global Maximum			Lower Maximum		
Surface	X_1	X_2	$f(x_1,x_2)$	X_1	X_2	$f(x_1,x_2)$
1	2.0250	2.8625	.16009	3.9846	1.0137	.14189
2	4.1992	3.7998	.25472	2.9690	1.4206	.16581
3	2.8200	4.2000	.24183	1.0000	1.0000	.16079
4	5.0000	3.0005	.22743	1.1867	2.7219	.19955
5	4.0000	1.0000	.63673	4.0000	3.1015	.56993
6	2.5986	4.4951	.76175	2.9820	3.4017	.62955
7	3.6377	1.5306	.16009	4.4855	3.4803	.14189
8	1.7001	3.6991	.25472	4.1075	2.5056	.16581
9	1.2900	2.3437	.24183	4.5000	0.5001	.16079
10	2.4993	4.4995	.22743	2.7756	0.6803	.19955
11	4.5000	3.5000	.63673	2.3995	3.4893	.56993
12	1.0035	2.1000	.76176	2.1075	2.4632	.62955

It is interesting to note the resulting surface shapes relative to the orientation of the EOD's in each case. Surfaces labeled 1 and 2 were constructed from only two bivariate normals (see Figure 5) forming two maxima with rather simple looking contours (see Figure 7). Surface 3 was formed by three bivariate normals as illustrated in Figure 5 with one of the two maxima formed by the bivariate densities associated with EOD's 2 and 3. The resulting shape shown in Figure 7 shows the more elongated ellipse, number 3, forming a rising ridge to the single mode.

Surface 5 illustrates the possibility of two non-overlapping EOD's to always provide two maxima. The ellipses numbered 1 and 3 for Surface 5 in Figure 5, obviously, do not overlap. However, the resulting surface from summing these bivariate normals gives only one local maximum.

Surface 6 was first constructed from EOD's 1 and 2. The lesser maximum of the two was smaller than desired so a third bivariate normal was super-imposed by EOD number 3 over the second. This is illustrated by the dotted lines for the larger EOD, number 3, for surfaces 6 and 12 respectively in Figures 4 and 5.

Comments. It appears that the types of surfaces generated by $f(x_1,x_2)$ are numerous. Surfaces with ridges, valleys, etc., can be generated merely by changing the parameter values. It was observed that even three maxima can result by summing only two bivariate normal density functions.

There is an obvious drawback to this class of functions that might restrict its usage to two independent variables. The algebraic expression of $f(x_1,x_2)$ is quite complex for obtaining analytic solutions for the points of local maxima. However, these problems are not believed to be insurmountable for usage in two dimensions. Contour plots of constant function values can help in identifying the relative positions of the maxima. Sufficiently accurate approximations can then be made of the maximum points by computational procedures for purposes of studies of this type.

Theoretically, the function values are all positive. Computationally, however, the function values can become so small that they caused underflows, at least on the IBM 360/65 computer. To remedy this problem an uncorrelated bivariate normal was centered at the mean of the bivariate normal having the EOD of smallest area in each response surface, and assigned very large variances of $\sigma_{11}^2 = \sigma_{22}^2 = 1 \times 10^8$. This was found to be adequate for preventing underflows and at the same time indicating the general direction of a maximum without significantly changing the function values as they were used in the study.

Local Optimization Techniques

The gradient, pattern search, and accelerated gradient local optimization techniques are three different basic approaches to locating a relative maximum which are basically different. However, within each method there are any number of variables which must assume fixed values before the techniques can be implemented, e.g. step sizes to determine local response surface characteristics, step sizes for computed directional moves, convergence criteria, etc. Surely those variables at different conditions would affect the performance of any of the techniques. Fortunately, the experiences in applications by Scherich, et. al. (12) and Zellnik, et. al. (14) provided insight to solving this problem for the gradient and accelerated gradient techniques.

Each of the three local techniques will now be outlined and the actual paths of optimization on Surface 1 will be plotted to demonstrate the actual process of optimization in each case.

Gradient. The gradient procedure as used by Scherich, et. al. (12) begins by evaluating the response surface at $\underset{\sim}{x}^{(0)} = \underset{\sim}{c} = (3.0, 2.5)$. The range of each variable is divided into ten units where each unit is $\Delta x_i = R_i/10$ in length for $i = 1,2$. Steps are made in the gradient direction by computing the coordinates of the point $\underset{\sim}{x}^{(m+1)}$ at the m^{th} step according to

$$x_i^{(m+1)} = x_i^{(m)} + 3(\Delta x_i)\ \alpha_i^{(m)}\ ,\ i = 1,2,$$

where $\alpha_i^{(m)}$ is the direction cosine of the i^{th} variable computed as

$$\alpha_i^{(m)} = \frac{\Delta f_i^{(m)}}{\sqrt{(\Delta f_1^{(m)})^2 + (\Delta f_2^{(m)})^2}}\ ,\quad i = 1,2,$$

and where

$$\Delta f_1 = f(x_1^{(m)} + \Delta x_1,\ x_2^{(m)}) - f(x_1^{(m)},\ x_2^{(m)}),$$

and Δf_2 is similarly defined. A reduction in x_i is made when $(\alpha_i^{(m)}/\alpha_i^{(m-1)}) < 0$, or when there is a reversal in the direction of the i^{th} variable. The new increment is then set to $\Delta x_i/2$.

Termination of the gradient procedure begins after the α_1 and α_2 have changed sign at least once for each variable. Final termination results when, for $m \geq 2$, the relationships

$$\left|\frac{f^{(m)} - f^{(m-1)}}{f^{(m-1)}}\right| < CC1 \quad \text{and} \quad \left|\frac{f^{(m+1)} - f^{(m)}}{f^{(m)}}\right| < CC2$$

are both satisfied for values of CC1 and CC2. The values CC1 and CC2 are proportional changes in the response surface at the $m - 1^{st}$ and m^{th} step. Specific values for CC1 and CC2 were respectively .02 and .01 as suggested by Scherich, et. al. (12).

Table IV and Figure 9 illustrate the path of the gradient technique on surface 1. The initial step is quite large going from coded point 0 to 1. The function evaluations needed for computing the direction cosines are not listed and are not included as candidates for the maximum value. From point 1 to 2 the direction of x_1 has changed causing Δx_1 to be halved, followed by gradient calculations, and a move to point 3. It appears from the figure that Δx_1 is again halved while Δx_2 has received its first reduction. This is likely the reason for the large step in the negative x_2 direction from a point that is very close to the maximum to point 4. The procedure recovers, terminating at point 7 where

$$\left|\frac{.15967 - .15837}{.15837}\right| = .0082 < .02$$

and

$$\left|\frac{.15973 - .15967}{.15967}\right| = .0004 < .01.$$

TABLE IV

TABULATED MOVES FOR GRADIENT ON SURFACE 1

Code	$f(x_1,x_2)$	x_1	x_2
0	.09783	3.0000	2.5000
1	.11091	1.2032	2.4104
2	.13754	2.0922	2.1766
3	.15884	1.9047	2.8584
4	.15375	1.9600	2.4949
5	.15837	1.9443	2.6805
6	.15967	1.9547	2.8649
7	.15973*	1.9803	2.7814

*-Denotes maximum found
Total number of evaluations = 22

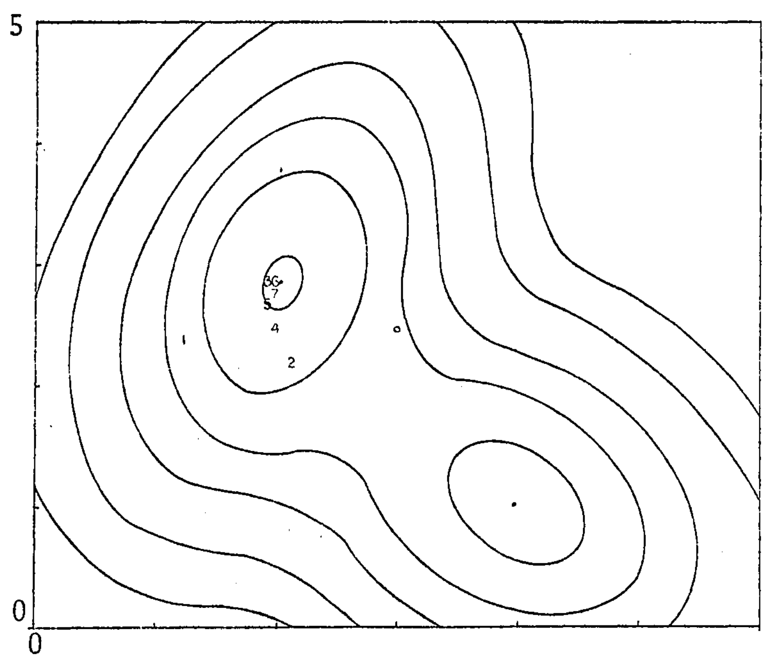


Figure 9. Plotted Moves of Gradient on Surface 1

Termination will also come about if after sixteen gradient calculations and moves the above criteria have not yet been satisfied.

Accelerated Gradient. The accelerated gradient in Zellnik, et. al. (14) begins by computing $\Delta f_i^{(0)}$ based on variable increments from the initial point $\underset{\sim}{x}^{(0)}$ of $\Delta x_i = R_i/50$, $i = 1, 2$. The first step in a gradient direction is computed as

$$\Delta x_i^{(0)} = \frac{\Delta f_i^{(0)}}{|\Delta f_i^{(0)}|_{max}} \times R_i \beta$$

where β is equal to .025 or 2.5% of the range of the variable indicating the most rapid change. In Figure 10 and Table V the relative positions of the initial point, 0, and the first step to point 1 are illustrated on the same surface as the gradient technique for comparison. There will be some variation in these plots due to the tracing of the computer output which is accomplished by plotting only at discrete points.

Since point 1 was a successful move (meaning a better point was found), a search is carried out along the gradient direction according to

$$\Delta x_i^{(m)} = (S.F.)\ \Delta x_i^{(m-1)} \tag{10}$$

$$x_i^{(m+1)} = x_i^{(m)} + \Delta x_i^{(m)} \ . \tag{11}$$

S.F. is a step factor initially set equal to 1.5 in order to "accelerate" the distance between successive points along the gradient

TABLE V

TABULATED MOVES FOR ACCELERATED GRADIENT ON SURFACE 1

Code	$f(x_1,x_2)$	x_1	x_2	Code	$f(x_1,x_2)$	x_1	x_2
0	.09783	3.0000	2.5000	C	.15730	1.9992	2.6222
1	.10855	2.8500	2.4864	D	.15628	2.2242	2.8622
2	.12436	2.6250	2.4661	E	.15811	2.0555	2.6822
3	.14341	2.2875	2.4355	F	.15823	2.1061	2.7362
4	.14771	1.7813	2.3897	G	.15784	2.1516	2.7848
5	.09270	1.0219	1.0219	H	.15847	2.0971	2.7507
6	.13967	1.5914	2.3726	I	.15890	2.0791	2.7411
7	.14618	1.7338	2.3854	J	.15953	2.0431	2.7699
8	.14817	1.7883	2.3972	K	.15982*	1.9711	2.8085
9	.14906	1.8023	2.4122	L	.15633	1.8271	2.8855
A	.15072	1.8305	2.4422	M	.15946	1.9351	2.8277
B	.15359	1.8867	2.5022	N	.15359	1.8867	2.5022

*-Denotes maximum found Total number of evaluations = 30

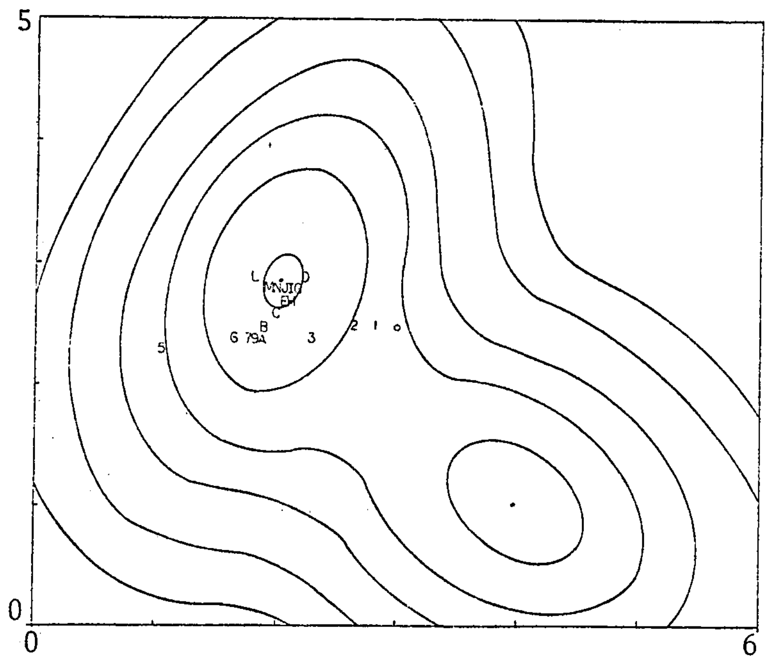


Figure 10. Plotted Moves of Accelerated Gradient on Surface 1

direction. These moves are illustrated in Figure 10 by the points 2, 3, 4 (which is later superimposed by 9), and 5, the first failure in the sequence. A subscan of the last interval is made by reducing the increment

$$\Delta x_i^{(m+1)} = (R.F.)\ \Delta x_i^{(m)}$$

by a reduction factor (R.F.) equal to 1/4. If this point is a success (point 6) a scan is made again with equations (10) and (11) where S.F. = .9. These steps are decelerated steps. This process continues until another failure occurs at which time the gradient direction is recomputed from the last successful point. The steps involved in the above discussion are referred to by Zellnik, et. al. (14) as the high range search.

In Figure 10, point 6 was not, however, a success. In this case another 1/4 reduction of the last interval is made resulting in point 7. Point 7 was also a failure. Two successive failures of the subscan cause the gradient to go into a low range search. The Δf_i are now calculated on independent variable increments of 0.02% of their respective ranges, and the basic step calculated with $\beta = .0015$. Zellnik points out that these successive failures are caused by (1) a maximum has been approached, or (2) a point is situated on a ridge. In order to recover from the "finer" search in low range that are not for the former reason, the S.F. value is increased to 2.0. Returning to Figure 10, the gradient direction is computed in low range and a basic step is made to point 8 followed by accelerated moves to 9, A, B, C, and D. Point D was a failure in this sequence of moves. A subscan produces points E, F, and G. Two failures call for re-evaluation of the gradient and the process continues. Points M and N result from two successive failures in a subscan. This is the second time this has occurred causing the procedure to terminate.

The maximum is taken to be the largest of all values excluding the function evaluations needed for computing gradient directions. The number of evaluations, however, includes the six values needed for this purpose.

Pattern Search. This technique consists of two types of searches. As developed by Hooke and Jeeves (7), the basic search is made in the vicinity of the initial point by changing one variable at a time. After the surface has been examined in this manner, pattern moves are made in the direction indicated by the successful moves in the first search.

Table VI and Figure 11 will be helpful in following this technique again on Surface 1. The search begins at 0 moving variable x_1 in the positive direction by an amount $\Delta x_1 = R_1/50$. This move resulted in a failure (and was not plotted) and consequently moved in the negative direction by the same amount to point 1, a success. From point 1 the search begins on x_2 to point 2 which was also a success. The direction of these moves is noted for two reasons. Since the successful moves were in the negative direction for x_1 and positive for x_2, future searches of this type will begin in these same directions, hopefully minimizing the number of function evaluations. The second reason is to make pattern moves. Using only the direction of the moves (and not incorporating the magnitude of the difference in function values as in the gradient routines) the procedure increments each variable that produced successful searches by $1.5\Delta x_i^{(m)}$ to the new coordinates

TABLE VI

TABULATED MOVES FOR PATTERN SEARCH ON SURFACE 1

Code	$f(x_1,x_2)$	x_1	x_2	Code	$f(x_1,x_2)$	x_1	x_2
0	.09783	3.0000	2.5000	8	.15415	2.1450	3.2125
1	.10639	2.8800	2.5000	9	.15190	2.3250	3.0625
2	.10650	2.8800	2.6000	A	.15669	2.1450	3.1125
3	.12278	2.7000	2.7500	B	.15861	2.1450	2.9625
4	.14616	2.4300	2.9750	C	.15735	2.1450	2.7375
5	.14960	2.0250	3.3125	D	.15958	2.0250	2.9625
6	.07505	1.4175	3.8188	E	.16009*	2.0250	2.8625
7	.15063	2.1450	3.2125	F	.15708	1.8450	2.7125

*-Denotes maximum found

Total number of evaluations = 33

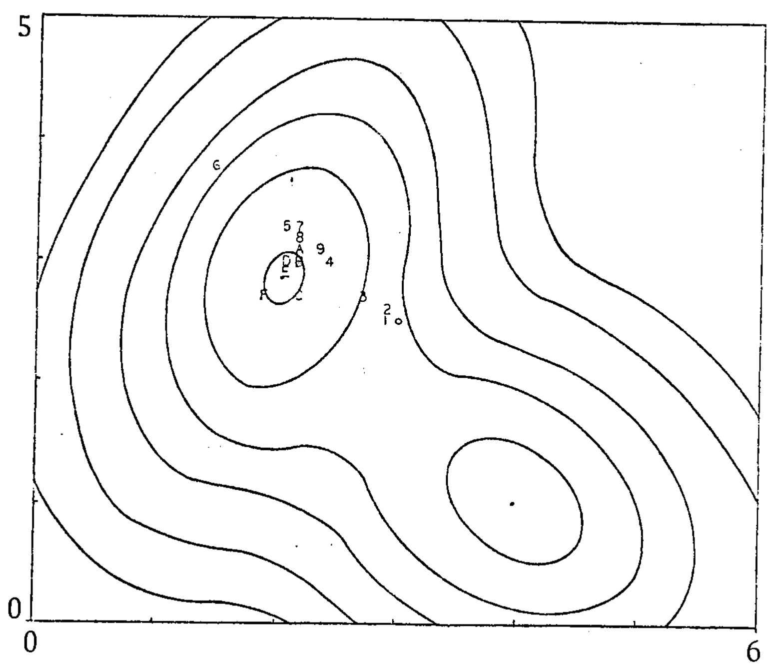


Figure 11. Plotted Moves of Pattern Search on Surface 1

$x_i^{(m+1)} = x_i^{(m)} + 1.5\Delta x_i^{(m)}$, $i = 1, 2$. The pattern moves give points 3, 4, 5 and 6. Point 6 was a failure. This causes the search process to restart from point 5, the new base point.

Termination of the procedure begins when the search about the base point in all variable directions does not produce a success. This causes the initiation of a new search with a new increment of $\Delta x_i/2$, $i = 1, 2$. If this brings about a success, the process of making pattern moves continues. If this does not locate a success, then the last increment is reduced by 1/4. The intervals are continually decreased by powers of 2 until the largest $\Delta x_i < .001$.

Restart Strategies

Introduction. The application of a local optimization technique and any of the normal, uniform, or 4-point strategies constitutes a technique to approximate the global maximum (g-max). The application of these global techniques will fall into one of the two procedures for global optimization as described below.

Normal or Uniform. Global techniques employing the normal or uniform restart strategies all follow this basic procedure:

1. A local optimization technique (accelerated gradient, pattern search, or gradient) is selected and initialized at the point $\tilde{c} = (3.0, 2.5)$. This process is allowed to converge to a point $\tilde{x}_1$.

2. Points, $\tilde{x}$, are selected from the chosen distribution (normal or uniform) sequentially comparing $z(\tilde{x})$ to $z(\tilde{x}_1)$. This

process continues until a point $\underset{\sim}{x}_2$ is found where $z(\underset{\sim}{x}_2)$ is greater than $z(\underset{\sim}{x}_1)$ or until 16 points have been chosen. If 16 points do not produce a better z, the procedure terminates. If an $\underset{\sim}{x}_2$ is found, the procedure restarts with the selected technique in step 1.

4-Point. The global procedure using the 4-point strategy begins by selecting one of the methods from the list of local optimization techniques. This selected local technique is initiated in the four corners of the region X with the approximation to the g-max resulting from the largest value observed during the four searches.

Experimental Error

The optimization techniques are applied to response surfaces that are deterministic and stochastic. This is accomplished by constructing response values as

$$z(\underset{\sim}{x}) = f(x_1, x_2) + e \quad ,$$

where $f(x_1, x_2)$ represents the function value of one of the twelve surfaces and "e" is a random deviate from a $n(0, \sigma^2)$ distribution. Three levels of "experimental error" were used by choosing the values of $\sigma = 0.0$, .01, and .05. A deterministic response surface results when $\sigma = 0.0$. The response is equal to $f(x_1, x_2)$ in this case. The other two values of σ represent stochastic situations where the observation is assumed to consist of a "true" average value plus a normally distributed random component.

Summary

The factors that were varied are response surfaces, response surface experimental error, local optimization techniques, and restart strategies.

CHAPTER IV

RESULTS

The Data

The purpose of this chapter is to present and analyze the data obtained from all combinations of the nine global optimization techniques, twelve response surfaces, and response surface error levels of σ = 0.0, .01, and .05 with replication. A computer program written by Collier (4) was very helpful in the acquisition of the data. This program was the options to implement any or all of the nine g-max procedures on any surface constructed by summing bivariate normal density functions. The program can also plot contours of surface function values and paths of search by any of the g-max procedures.

Replication

The number in the cells of Table VII indicates the number of replications obtained at the experimental conditions with σ = 0.0. Two starts of any procedure at the same point on a deterministic surface (σ = 0.0) will produce exactly the same results. Since the techniques involving the 4-point strategy have all starts determined *a priori*, replication is not necessary with σ = 0.0. Techniques with the normal and uniform strategies were replicated since alternative starts subsequent to the first start are randomly selected according to the specified

TABLE VII

NUMBER OF REPLICATES FOR σ=0.0

Restart Strategy		Normal			4-Point			Uniform		
Local Technique		Accel. Grad.	Pattern Search	Grad.	Accel. Grad.	Pattern Search	Grad.	Accel. Grad.	Pattern Search	Grad.
G-Max Code		1 1	1 2	1 3	2 1	2 2	2 3	3 1	3 2	3 3
Surface	1	3	3	3	1	1	1	3	3	3
	2	3	3	3	1	1	1	3	3	3
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	12	3	3	3	1	1	1	3	3	3

distribution. The number of replicates was determined by the available resources.

Three replications were obtained for the $\sigma = .01$ and $.05$ cases from all the g-max techniques and surfaces. Making the observations from response surfaces stochastic prevents even the techniques utilizing the 4-point method from producing identical experimental results.

The resulting data is listed in Appendix A where each experimental point (condition) is identified by a code for the appropriate g-max technique, surface number, experimental error (σ) level and rep (replication) number. The information given in each case consists of the number of "function" evaluations (NFE), the approximated global maximum (g-max $z(\underset{\sim}{x}) = z(\underset{\sim}{x}_a)$), the true response surface value $f(\underset{\sim}{x}_a)$ evaluated at the point $\underset{\sim}{x}_a$ determined by the approximated g-max $z(\underset{\sim}{x})$, and the true global maximum (g-max $f(\underset{\sim}{x})$). The value $f(\underset{\sim}{x}_a)$ at the point $\underset{\sim}{x}_a$ of g-max $z(\underset{\sim}{x}) = z(\underset{\sim}{x}_a)$ will be used in one of the two g-max technique performance measures.

Performance Measures

The results from each data point are reduced to two measures of performance. These are:

$y_{(1)}$ = the total number of response surface evaluations

and

$y_{(2)}$ = the proportion of the true global maximum attained

$$y_{(2)} = \frac{f(\underset{\sim}{x}_a)}{\text{g-max} f(\underset{\sim}{x})} .$$

The numerator is the function value at the point obtained from the global optimization of $z(\tilde{x})$, $\tilde{x}$ in X.

To illustrate the above, the performance from one replication of the normal/gradient (1 3) on surface 2 will be obtained when $\sigma = 0.0$ and when $\sigma = .01$.

Table VIII and Figure 12 illustrate, respectively, the numerical and graphical information from rep 1 of g-max technique 1 3 on surface 2 with $\sigma = 0.0$. Beginning at center $\tilde{c} = (3.0, 2.5)$, the gradient method locates a maximum at $\tilde{x} = (2.8008, 1.6105)$ with $z(\tilde{x}) = f(\tilde{x}) = .16307$. The number of evaluations was 28. The "random search" (normal distribution) process almost fails, but a better value of $z(3.0221, 3.7880) = .16640$ is found on the sixteenth point, the maximum number allowed. The gradient restarts at this point and terminates for the second time at $z(4.2117, 3.8122) = .25463$. This latter search required thirty-one evaluations. The next sixteen points selected from the normal distribution procedure all fail to find a better value, terminating the g-max procedure. The measures $y_{(1)}$ and $y_{(2)}$ are calculated to be

$$y_{(1)} = 28 + 16 + 31 + 16 = 91 \text{ NFE}$$

and

$$y_{(2)} = \frac{.25463}{.25472} = .99965 \doteq 1$$

TABLE VIII

COMPUTER OUTPUT OF NORMAL/GRADIENT ON SURFACE 2 WITH $\sigma = 0.0$

F(X)	X	
0.07755	3.0000	2.5000
0.03754	1.2084	2.3556
0.11336	2.0908	2.5029
0.13655	2.3299	2.1414
0.15754	2.7532	1.8104
0.13134	2.3497	1.6445
0.15323	2.5606	1.7097
0.16044	2.7842	1.7305
0.16164	2.7478	1.6418
0.16307	2.8008	1.6105

MAXIMUM NFE 28
F(X)= 0.16307 X= 2.8008 1.6105

RANDOM SEARCH

F(X)	X	
0.00323	5.3180	2.3891
0.08761	2.1035	3.4119
0.00188	5.0463	2.2407
0.04126	1.1144	2.7791
0.01741	0.9623	2.1901
0.07770	6.0000	3.5358
0.01055	0.3118	3.4045
0.01786	2.3962	4.7095
0.00361	5.0143	2.3663
0.00003	0.0000	1.2053
0.00253	6.0000	2.4628
0.00058	0.7341	1.0352
0.03010	5.2683	2.8591
0.13917	3.3603	0.6680
0.04777	1.6121	1.7449
0.16640	3.0221	3.7880

RANDOM SEARCH PRODUCED A CANDIDATE FOR ALTERNATE MAXIMA

F(X)	X	
0.16640	3.0221	3.7880
0.02564	4.2785	2.7139
0.20298	4.2368	3.4630
0.18302	4.1022	4.2046
0.25312	4.0803	3.8297

TABLE VIII (Continued)

F(X)	X	
0.19833	3.9728	3.4579
0.23999	4.0092	3.6444
0.25433	4.1564	3.8216
0.25157	4.1052	3.7304
0.25415	4.1467	3.7739
0.25463	4.2117	3.8122

MAXIMUM NFE 31
F(X)= 0.25463 X= 4.2117 3.8122

RANDOM SEARCH

F(X)	X	
0.04748	1.7199	1.5555
0.04923	4.9529	4.6550
0.01164	5.0018	5.0000
0.04200	2.3661	0.6128
0.03983	6.0000	3.1602
0.00896	1.2524	1.3134
0.07797	2.8420	3.0235
0.05147	3.5309	2.8749
0.01418	4.0421	5.0000
0.00438	1.8818	0.2184
0.01055	3.2245	5.0000
0.04540	3.0335	0.0000
0.00138	1.1971	0.6821
0.00009	1.1422	0.0000
0.00700	2.6954	5.0000
0.00024	0.0000	1.8695

RANDOM SEARCH PRODUCED NO
CANDIDATES FOR ALTERNATE MAX

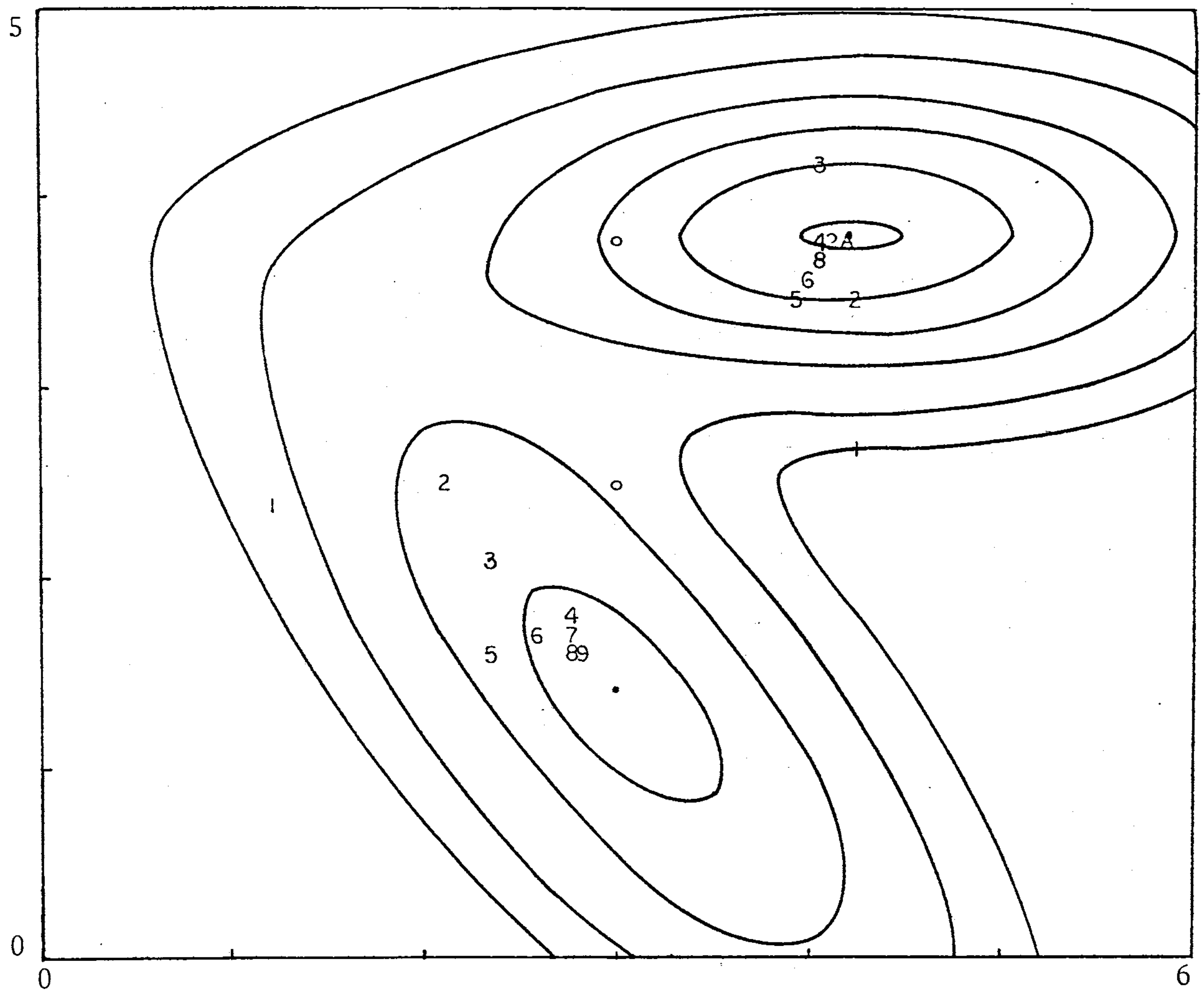


Figure 12. Graphic Results of Normal/Gradient on Surface 2 with σ = 0.0

The results of a similar experimental situation, but with $\sigma = .01$, is illustrated in Table IX and Figure 13. The effect of the $\sigma = .01$ "noise" level in the response is reflected in the z value at $\underset{\sim}{c} = (3.0, 2.5)$. A component of $-.00055$ was selected from the $n(0, (.01)^2)$ distribution and added to $f(\underset{\sim}{c})$ resulting in $z(\underset{\sim}{c}) = .07700$ (compare $z(\underset{\sim}{c})$ in Tables VIII and IX). The gradient technique terminates in this instance with NFE = 31 and a maximum surface response of $z(2.8078, 1.6090) = .16317$. Sixteen points from the normal distribution procedure all failed to locate a better value causing the entire procedure to terminate. The performance values in this instance are

$$y_{(1)} = 31 + 16 = 47 \quad \text{NFE}$$

and

$$y_{(2)} = \frac{f(2.8078, 1.6090)}{.25472} = \frac{.16320}{.25472} = .641$$

or the maximum value found was 64.1% of the true global maximum.

Graphical illustration of the paths of optimization for a sample of experimental conditions are contained in Appendix B. These figures are cross referenced with the proper data points in Appendix A so that the reader can get a better understanding of the numerical values in Appendix A.

An Analysis of the Data

The data in Appendix A are reduced to Tables X, XI, and XII by converting the original data to $y_{(1)}$ and $y_{(2)}$ and averaging over replication. For a comparison, the value of $y_{(2)}$ at the lower maximum of each of the bimodal surfaces is given in Table XIII.

TABLE IX

COMPUTER OUTPUT OF NORMAL/GRADIENT
ON SURFACE 2 WITH σ = .01

F(X)	X	
0.07700	3.0000	2.5000
0.03889	1.2066	2.3712
0.11337	2.0899	2.5154
0.12953	2.2137	2.1439
0.11325	3.0991	2.0766
0.15147	2.6791	1.9418
0.14341	2.4897	1.6017
0.16030	2.6987	1.6712
0.16098	2.9129	1.6425
0.16317	2.8078	1.6090
0.16286	2.8279	1.5215

MAXIMUM NFE 31
F(X)= 0.6317 X= 2.8078 1.6090

RANDOM SEARCH

F(X)	X	
0.02323	4.8434	2.7210
0.00626	2.5155	5.0000
0.06092	2.0702	4.0803
0.00719	5.0044	1.2518
0.05116	1.3403	3.6839
0.00636	6.0000	5.0000
0.02769	5.8695	2.9932
0.00534	4.4759	2.2841
0.00001	0.0000	0.0411
0.12733	2.3772	1.5627
0.06968	2.6548	0.5964
0.04693	2.1180	4.2631
0.01040	5.7651	4.8743
0.00201	0.3684	5.0000
0.02128	4.0642	2.0104
0.03237	0.9855	2.8924

RANDOM SEARCH PRODUCED NO
CANDIDATES FOR ALTERNATE MAX

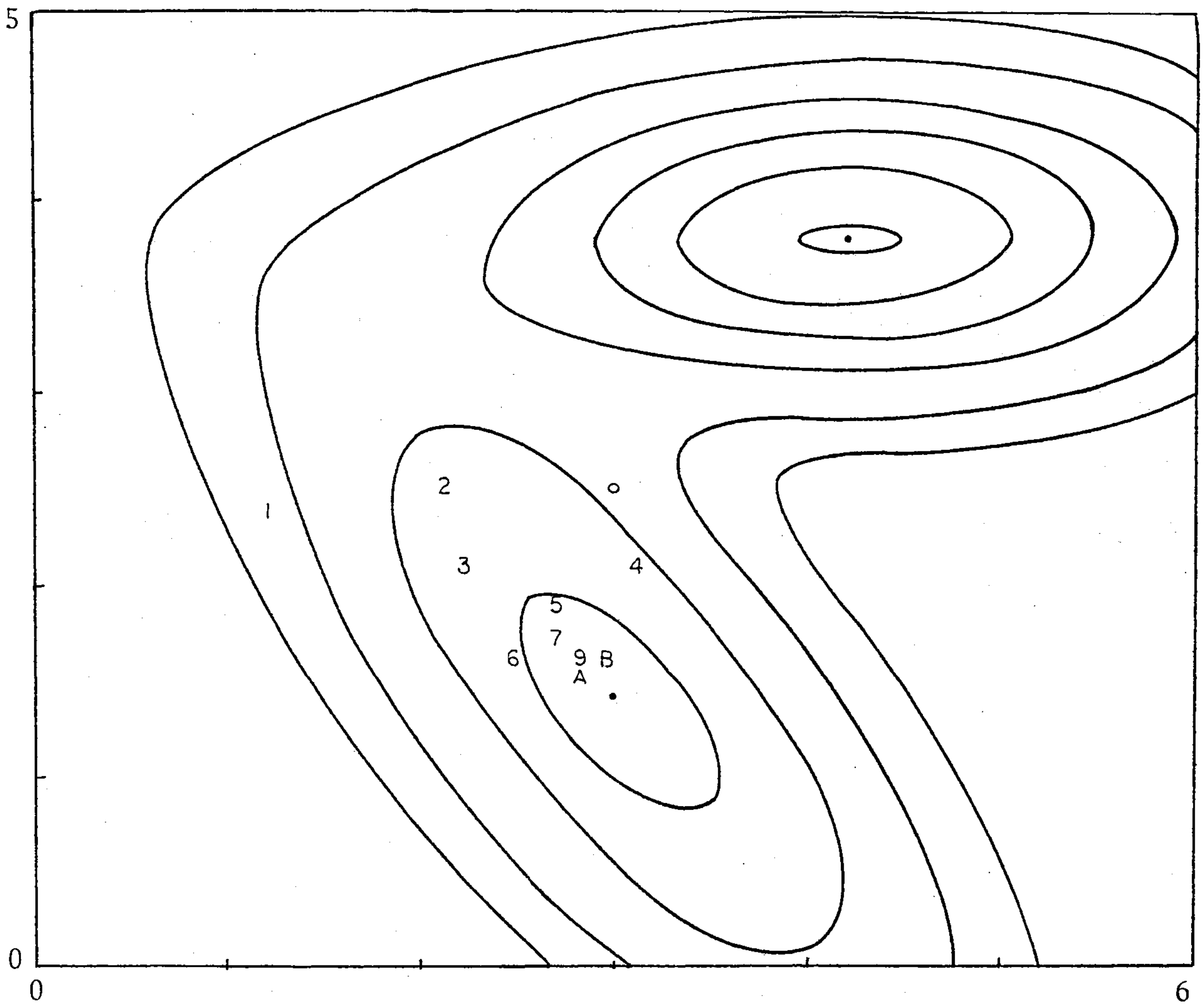


Figure 13. Graphic Results of Normal/Gradient on Surface 2 with σ = .01

TABLE X

G-MAX TECHNIQUE PERFORMANCE AVERAGES
OVER REPS FOR σ = 0.0

Opt. Code	S	$y_{(1)}$	$y_{(2)}$	Opt. Code	S	$y_{(1)}$	$y_{(2)}$	Opt. Code	S	$y_{(1)}$	$y_{(2)}$
1 1	1	46	0.998	1 2	1	49	1.000	1 3	1	38	0.998
1 1	2	71	0.984	1 2	2	99	1.000	1 3	2	78	0.997
1 1	3	66	0.976	1 2	3	48	1.000	1 3	3	50	0.999
1 1	4	54	0.807	1 2	4	60	0.877	1 3	4	41	0.877
1 1	5	76	0.892	1 2	5	88	0.895	1 3	5	41	0.892
1 1	6	50	0.825	1 2	6	71	0.826	1 3	6	53	0.997
1 1	7	37	0.995	1 2	7	52	1.000	1 3	7	35	0.990
1 1	8	55	0.840	1 2	8	66	0.883	1 3	8	66	0.879
1 1	9	48	0.887	1 2	9	62	1.000	1 3	9	50	0.998
1 1	10	69	0.889	1 2	10	93	0.901	1 3	10	63	0.874
1 1	11	76	0.898	1 2	11	78	0.930	1 3	11	59	0.930
1 1	12	99	0.994	1 2	12	115	1.000	1 3	12	50	0.999
AVERAGE		62	0.915	AVERAGE		73	0.943	AVERAGE		52	0.953
2 1	1	134	0.998	2 2	1	140	1.000	2 3	1	100	0.999
2 1	2	148	0.992	2 2	2	149	1.000	2 3	2	127	0.999
2 1	3	148	0.992	2 2	3	163	1.000	2 3	3	109	0.997
2 1	4	170	0.959	2 2	4	167	1.000	2 3	4	97	0.990
2 1	5	160	1.000	2 2	5	189	1.000	2 3	5	74	0.998
2 1	6	204	0.826	2 2	6	238	0.827	2 3	6	154	1.000
2 1	7	125	1.000	2 2	7	158	1.000	2 3	7	109	1.000
2 1	8	173	0.927	2 2	8	174	1.000	2 3	8	133	0.999
2 1	9	160	0.977	2 2	9	186	1.000	2 3	9	118	0.940
2 1	10	168	0.999	2 2	10	188	1.000	2 3	10	106	0.996
2 1	11	172	1.000	2 2	11	194	1.000	2 3	11	130	0.992
2 1	12	153	0.758	2 2	12	222	1.000	2 3	12	130	1.000
AVERAGE		160	0.952	AVERAGE		181	0.986	AVERAGE		116	0.992
3 1	1	46	0.998	3 2	1	49	1.000	3 3	1	38	0.998
3 1	2	70	0.860	3 2	2	89	0.884	3 3	2	67	0.879
3 1	3	60	0.836	3 2	3	51	1.000	3 3	3	50	0.999
3 1	4	54	0.807	3 2	4	60	0.877	3 3	4	41	0.877
3 1	5	75	0.923	3 2	5	101	0.930	3 3	5	59	0.902
3 1	6	58	0.883	3 2	6	84	0.884	3 3	6	53	0.997
3 1	7	37	0.995	3 2	7	52	1.000	3 3	7	44	0.993
3 1	8	85	0.998	3 2	8	57	0.884	3 3	8	64	0.879
3 1	9	73	0.910	3 2	9	62	1.000	3 3	9	50	0.998
3 1	10	100	0.900	3 2	10	106	0.901	3 3	10	84	0.890
3 1	11	87	0.949	3 2	11	78	0.930	3 3	11	59	0.902
3 1	12	51	0.825	3 2	12	78	0.827	3 3	12	50	0.999
AVERAGE		66	0.907	AVERAGE		72	0.926	AVERAGE		55	0.943

TABLE XI

G-MAX TECHNIQUE PERFORMANCE AVERAGES
OVER REPS FOR σ = .01

Opt. Code	S	$y_{(1)}$	$y_{(2)}$	Opt. Code	S	$y_{(1)}$	$y_{(2)}$	Opt. Code	S	$y_{(1)}$	$y_{(2)}$
1 1	1	49	0.936	1 2	1	68	0.972	1 3	1	65	0.913
1 1	2	54	0.791	1 2	2	98	0.816	1 3	2	85	0.984
1 1	3	72	0.681	1 2	3	110	0.845	1 3	3	109	0.868
1 1	4	52	0.769	1 2	4	60	0.707	1 3	4	69	0.802
1 1	5	86	0.961	1 2	5	137	0.928	1 3	5	59	0.889
1 1	6	45	0.853	1 2	6	123	0.884	1 3	6	74	0.989
1 1	7	52	0.832	1 2	7	66	0.848	1 3	7	65	0.957
1 1	8	49	0.971	1 2	8	83	0.952	1 3	8	109	0.988
1 1	9	77	0.926	1 2	9	168	0.973	1 3	9	103	0.985
1 1	10	67	0.754	1 2	10	81	0.780	1 3	10	88	0.722
1 1	11	69	0.865	1 2	11	111	0.889	1 3	11	55	0.872
1 1	12	59	0.921	1 2	12	124	0.927	1 3	12	60	0.941
AVERAGE		61	0.855	AVERAGE		102	0.877	AVERAGE		78	0.909
2 1	1	72	0.760	2 2	1	185	0.927	2 3	1	171	0.982
2 1	2	74	0.811	2 2	2	171	0.988	2 3	2	166	0.984
2 1	3	74	0.723	2 2	3	163	0.865	2 3	3	180	0.988
2 1	4	68	0.779	2 2	4	194	0.930	2 3	4	151	0.930
2 1	5	70	0.663	2 2	5	160	0.667	2 3	5	177	0.963
2 1	6	79	0.557	2 2	6	222	0.823	2 3	6	171	0.907
2 1	7	63	0.669	2 2	7	180	0.793	2 3	7	185	0.984
2 1	8	82	0.633	2 2	8	177	0.969	2 3	8	173	0.990
2 1	9	66	0.598	2 2	9	168	0.958	2 3	9	194	0.823
2 1	10	70	0.427	2 2	10	195	0.960	2 3	10	161	0.921
2 1	11	77	0.660	2 2	11	205	0.996	2 3	11	175	0.923
2 1	12	73	0.457	2 2	12	172	0.547	2 3	12	159	0.937
AVERAGE		72	0.645	AVERAGE		183	0.869	AVERAGE		172	0.944
3 1	1	62	0.924	3 2	1	61	0.984	3 3	1	61	0.955
3 1	2	73	0.972	3 2	2	103	0.869	3 3	2	90	0.834
3 1	3	72	0.874	3 2	3	109	0.958	3 3	3	74	0.971
3 1	4	78	0.871	3 2	4	77	0.789	3 3	4	76	0.842
3 1	5	64	0.771	3 2	5	84	0.724	3 3	5	59	0.914
3 1	6	74	0.818	3 2	6	101	0.880	3 3	6	50	0.996
3 1	7	42	0.979	3 2	7	95	0.956	3 3	7	82	0.995
3 1	8	79	0.822	3 2	8	96	0.846	3 3	8	93	0.860
3 1	9	50	0.959	3 2	9	106	0.844	3 3	9	76	0.790
3 1	10	102	0.895	3 2	10	98	0.868	3 3	10	109	0.887
3 1	11	42	0.554	3 2	11	87	0.877	3 3	11	69	0.885
3 1	12	53	0.813	3 2	12	112	0.882	3 3	12	66	0.928
AVERAGE		66	0.854	AVERAGE		94	0.873	AVERAGE		75	0.905

TABLE XII

G-MAX TECHNIQUE PERFORMANCE AVERAGES
OVER REPS FOR σ = .05

Opt. Code	S	$y_{(1)}$	$y_{(2)}$	Opt. Code	S	$y_{(1)}$	$y_{(2)}$	Opt. Code	S	$y_{(1)}$	$y_{(2)}$
1 1	1	66	0.971	1 2	1	82	0.994	1 3	1	40	1.000
1 1	2	54	0.710	1 2	2	107	0.766	1 3	2	59	0.754
1 1	3	63	0.946	1 2	3	75	0.997	1 3	3	51	0.985
1 1	4	59	0.858	1 2	4	79	0.796	1 3	4	52	0.893
1 1	5	51	0.594	1 2	5	77	0.727	1 3	5	43	0.893
1 1	6	59	0.866	1 2	6	116	0.884	1 3	6	53	0.995
1 1	7	49	0.979	1 2	7	73	0.999	1 3	7	36	0.992
1 1	8	56	0.866	1 2	8	105	0.883	1 3	8	70	0.875
1 1	9	62	0.992	1 2	9	74	0.996	1 3	9	53	0.993
1 1	10	77	0.912	1 2	10	124	0.914	1 3	10	77	0.905
1 1	11	65	0.785	1 2	11	88	0.894	1 3	11	47	0.889
1 1	12	45	0.826	1 2	12	72	0.825	1 3	12	56	0.937
AVERAGE		59	0.859	AVERAGE		89	0.889	AVERAGE		53	0.926
2 1	1	79	0.909	2 2	1	206	0.999	2 3	1	123	0.995
2 1	2	95	0.931	2 2	2	223	0.998	2 3	2	140	0.998
2 1	3	102	0.985	2 2	3	274	0.998	2 3	3	114	0.998
2 1	4	99	0.792	2 2	4	231	0.994	2 3	4	120	0.983
2 1	5	89	0.399	2 2	5	180	1.000	2 3	5	120	0.996
2 1	6	79	0.520	2 2	6	238	0.826	2 3	6	162	0.976
2 1	7	76	0.873	2 2	7	216	0.998	2 3	7	120	0.998
2 1	8	78	0.951	2 2	8	206	0.998	2 3	8	179	0.996
2 1	9	87	0.758	2 2	9	263	0.999	2 3	9	125	0.839
2 1	10	78	0.755	2 2	10	234	0.998	2 3	10	130	0.991
2 1	11	82	0.575	2 2	11	239	1.000	2 3	11	167	0.999
2 1	12	82	0.581	2 2	12	221	0.884	2 3	12	144	1.000
AVERAGE		85	0.753	AVERAGE		227	0.974	AVERAGE		137	0.981
3 1	1	41	0.970	3 2	1	69	0.988	3 3	1	43	0.997
3 1	2	74	0.991	3 2	2	130	0.995	3 3	2	83	0.990
3 1	3	51	0.816	3 2	3	81	0.996	3 3	3	51	0.879
3 1	4	65	0.867	3 2	4	112	0.917	3 3	4	46	0.867
3 1	5	56	0.596	3 2	5	116	0.762	3 3	5	46	0.892
3 1	6	46	0.697	3 2	6	88	0.826	3 3	6	53	0.998
3 1	7	81	0.954	3 2	7	71	0.993	3 3	7	41	0.995
3 1	8	64	0.901	3 2	8	128	0.997	3 3	8	98	0.996
3 1	9	65	0.944	3 2	9	67	0.992	3 3	9	53	0.991
3 1	10	53	0.939	3 2	10	107	0.953	3 3	10	103	0.955
3 1	11	42	0.492	3 2	11	92	0.894	3 3	11	51	0.891
3 1	12	47	0.819	3 2	12	85	0.826	3 3	12	48	0.998
AVERAGE		57	0.832	AVERAGE		96	0.928	AVERAGE		60	0.954

TABLE XIII

$y_{(2)}$ VALUES FOR LOWER MAXIMUM

Surfaces	$y_{(2)}$
1 and 7	.886
2 and 8	.651
3 and 9	.665
4 and 10	.878
5 and 11	.895
6 and 12	.826
Average	.800

One might be inclined to think that $y_{(1)}$ and $y_{(2)}$ are somehow related. It seems reasonable to expect that the more evaluations required by a procedure, the closer the approximation should be to 1.0 by making use of the more information concerning the response surface. A demonstration of this is the comparison between the results from Figure 12 and 13. However, in Table X, for example, there are some interesting cases that will now be considered.

Optimization technique 2 2 (4-point/pattern search) from Table X produced a value less than 1.0 only on surface 6. The interesting comparison is that on the rotated "cousin" of 6, surface 12, the global maximum was found in fewer evaluations. To explain this apparent anomaly, consider the results from one of the four starts of the 4-point strategy depicted in Figure 14. This was the only start that produced

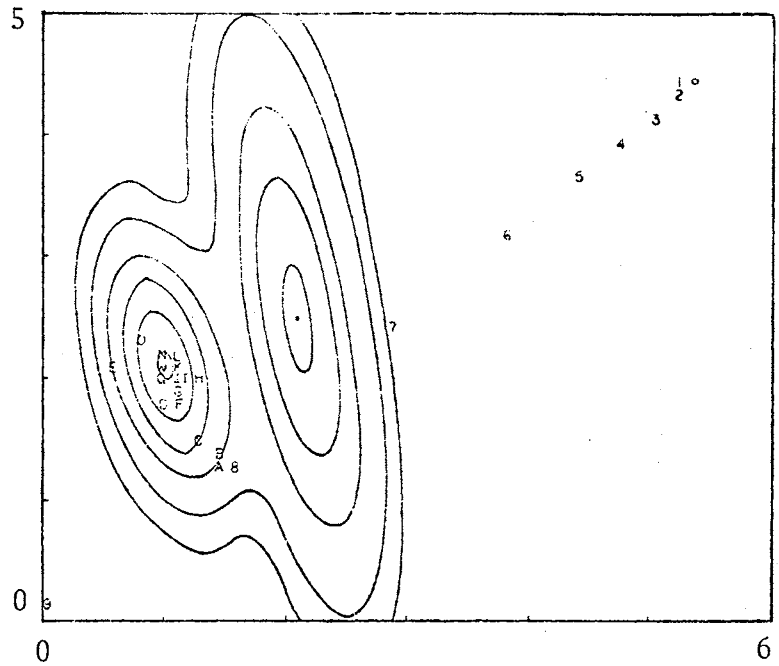


Figure 14. Results of 4-Point/Pattern Search, Surface 12, σ = 0.0

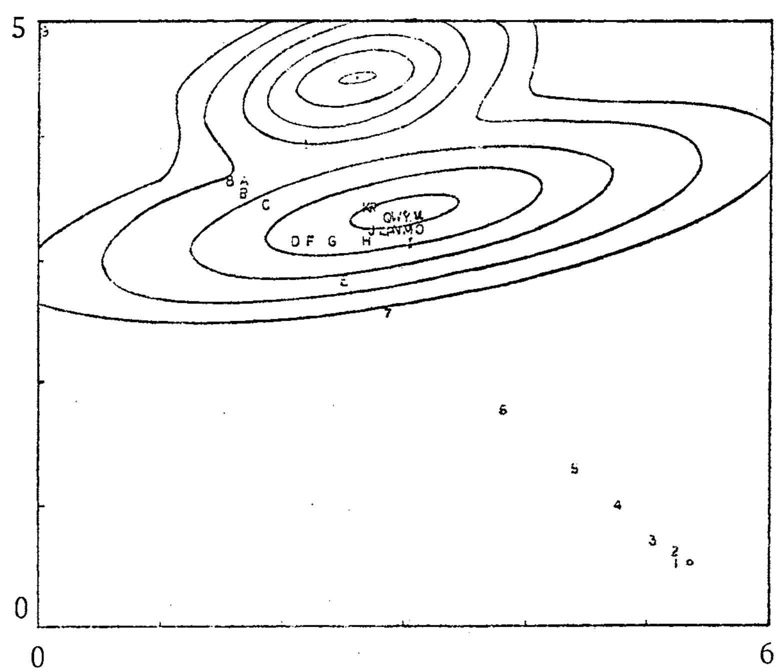


Figure 15. Results of 4-Point/Pattern Search, Surface 6, σ = 0.0

the global maximum, all others converged to the lower maximum. All four starts on surface 6 produced the lower value accounting for the value $y_{(2)} = .877$. The analogous starting point on surface 6, except for an apparently significant alteration in the coordinates due to the rotation and change in range, is located in the lower right hand corner of Figure 15. A comparison of the figures shows point 8 as the turning point. Point 8, the last successful point in the pattern move sequence from point 0, is different enough in location relative to the response surface contours, so that a search for new pattern moves from 8 leads to different maxima in the two cases.

Another interesting comparison in Table X is between the average responses attained for the three reps on surfaces 4 and 10 for the uniform/gradient (3 3). The averages over replicates for $y_{(1)}$, $y_{(2)}$ are 41, .877 for surface 4, and 84, .890 for surface 10. Again, these surfaces are similar except for the change in ranges achieved by rotation. It appears strange that forty-three more evaluations on the average were needed for surface 10 to achieve about the same average proportion of the g-max. Appendix A indicates that in this situation, the performance results were identical for all three replicates with $y_{(1)} = 41$ and $y_{(2)} = .887$, the lower maximum according to Table XIII. The uniform distribution selection procedure failed to locate a better value in all three replicates even though the random number generator had three separate starts.

The results of the three replicates of surface 10 are much more variable as shown in Table XIV.

TABLE XIV

THREE REPS OF UNIFORM/GRADIENT ON SURFACES 4 AND 10 WITH $\sigma = 0.0$

	Surface 4		Surface 10	
	$y_{(1)}$	$y_{(2)}$	$y_{(1)}$	$y_{(2)}$
1	41	.877	73	.994
2	41	.877	137	.974
	41	.877	41	.702
Averages	41	.877	84	.890
$\hat{\sigma}^2_{y(k)} =$	0	0	2389.3	.0266
$\hat{\sigma}_{y(k)}/\sqrt{3} =$	0	0	28.2	.094

Figure 16 shows how the values $y_{(1)} = 73$ and $y_{(2)} = .994$ for the first rep came about on surface 10. The convergence from the first start was at neither maximum, but on a ridge leading to the lower maximum. A better value was then located in the region of the g-max accounting for $y_{(2)} = .994$ in seventy-three evaluations.

The second rep was also successful in locating the g-max, but the path was quite different after the first convergence. In this case Figure 17 shows that the first restart occurred in the region of the lower maximum. After convergence to the lower optimum, another restart

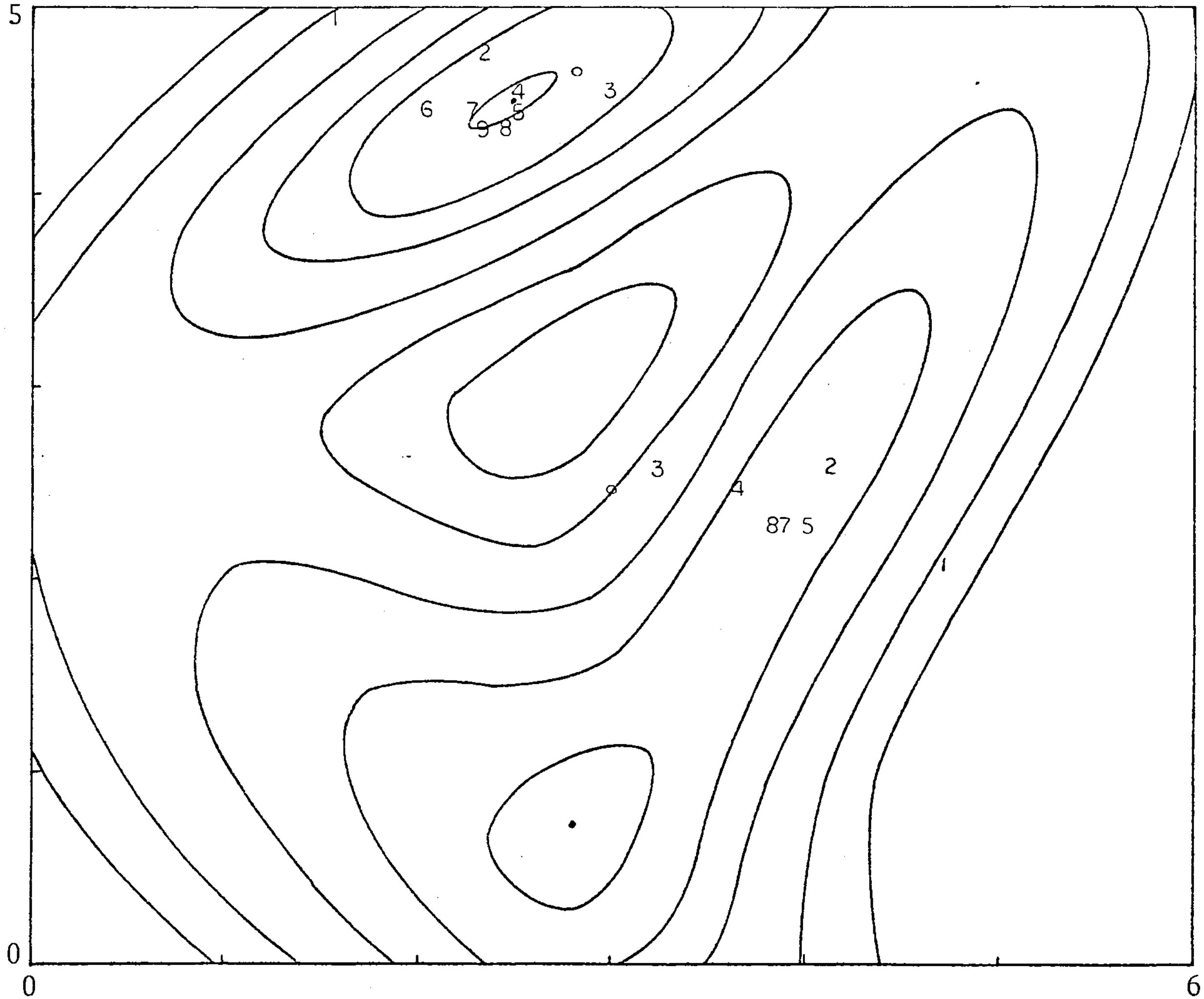


Figure 16. Graphic Results of Rep 1 of Uniform/Gradient on Surface 10
with $\sigma = 0.0$

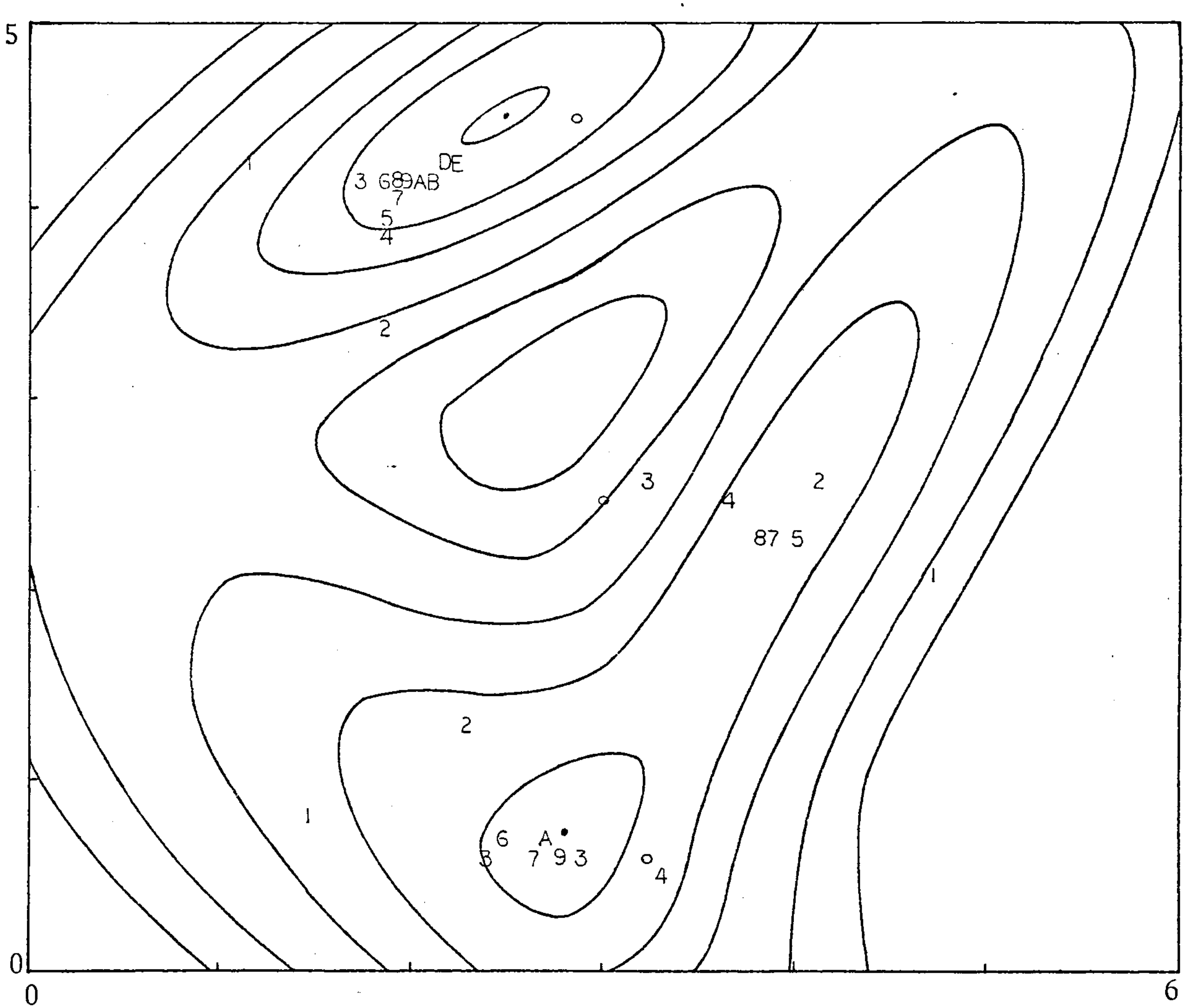


Figure 17. Graphic Results of Rep 2 of Uniform/Gradient on Surface 10 with σ = 0.0

is found in the g-max region accounting for $y_{(2)} = .974$ and $y_{(1)} = 137$ evaluations.

The third rep is not illustrated, but no restarts were found after the initial convergence to the ridge point owing to a $y_{(2)}$ value of only .702 with $y_{(1)} = 41$.

Also included in Table XIV are the respective estimates of the variance, $\hat{\sigma}^2_{y(k)}$, k = 1, 2, for $y_{(1)}$ and $y_{(2)}$ for each surface, and the estimates of the standard errors of the averages calculated as $\hat{\sigma}_{y(k)}/\sqrt{3}$. With only three replicates, the information on the uniform/gradient provided by these surfaces separately seems inconsistent with respect to the precision of the averages. Although the true variances of $y_{(k)}$ may not be identical for each of the surfaces, it is unlikely that additional replicates on surface 4 would fail to restart; or that the variance is as large as indicated for surface 10. (Recall these two cases were selected for an apparent anomaly in the first place.)

If the concern was only to measure the performance of the uniform/gradient on these two surfaces separately, then more replication would be desirable to obtain more accurate and/or precise estimates. If the user does not know whether he will encounter surface 4 or 10, say, then the average performance of a technique over both surfaces would be an informative measure, especially for the purpose of comparing another technique on the same surfaces.

To examine the precision of the performance of a technique by averaging over surfaces as well as replication, consider the representation of the true mean or expected value of $y_{i(k)}$ for $k = 1, 2$ and each surface $i = 1, 2, \ldots, 12$ as

$$E(y_{i(k)}) = \mu_{i(k)} = \bar{\mu}_{\cdot(k)} + (\mu_{i(k)} - \bar{\mu}_{\cdot(k)})$$

$$= \mu_{(k)} + S_{i(k)}$$

where $\bar{\mu}_{\cdot(k)} = \frac{\sum_i \mu_{i(k)}}{12}$. Each surface mean is decomposed into a constant $\mu_{(k)}$ plus a "fixed" deviation from the constant due to surface i. An observation from a single application of a g-max technique on surface i might be modeled as

$$y_{ij(k)} = \mu_{(k)} + S_{i(k)} + \varepsilon_{ij(k)}, \tag{12}$$

j = 1, 2, 3, where $\varepsilon_{ij(k)}$ is a random component with $E(\varepsilon_{ij(k)}) = 0$ for every i, j, and k and where

$$E(\varepsilon_{ij(k)}\,\varepsilon_{i'j'(k)}) = \begin{cases} \sigma^2_{i(k)} \text{ for } i = i' \text{ and } j = j' \\ 0, \text{ otherwise.} \end{cases}$$

The average

$$\bar{y}_{(k)} = \frac{\sum_{ij} y_{ij(k)}}{12(3)}$$

has variance

$$\delta^2_{(k)} = \frac{\bar{\sigma}^2_{\cdot(k)}}{36} \text{ where } \bar{\sigma}^2_{\cdot(k)} = \frac{\sum_i \sigma^2_{i(k)}}{12} .$$

The estimated variance of $\bar{y}_{(k)}$ is given by

$$\hat{\delta}^2_{(k)} = \frac{\hat{\bar{\sigma}}_{\cdot(k)}}{36} = \frac{\sum_i \hat{\sigma}^2_{i(k)}}{(12)36}$$

where

$$\hat{\sigma}^2_{i(k)} = \sum_j \frac{(\bar{y}_{ij(k)} - \bar{y}_{i\cdot(k)})^2}{2}$$

is a measure of the within surface variation.

Returning to the information provided in Table XIV, the average performance for the uniform/gradient as measured from these data is

$$\bar{y}_{(1)} = \frac{41 + 82}{2} = 63 \quad \text{and} \quad \bar{y}_{(2)} = \frac{.877 + .890}{2} = .936.$$

These quantities calculated by

$$\hat{\bar{\sigma}}^2_{\cdot(1)} = \frac{0 + 2389.3}{2} = 1194.7$$

are used in obtaining the estimates

$$\hat{\delta}_{(1)} = \sqrt{\frac{1194.7}{6}} = 14.1 \quad \text{and} \quad \hat{\delta}_{(2)} = \sqrt{\frac{.0133}{6}} = .047.$$

To summarize, the model described above in equation (12) provides a measure of the average performance of a g-max technique on all twelve surfaces by the average $\bar{y}_{(k)}$. An estimate of the precision of this estimate is provided by $\hat{\delta}_{(k)}$.

While $\hat{\delta}_{(k)}$ measures the precision of $\bar{y}_{(k)}$ for the surfaces examined, another purpose of this study is to gain some information on the best techniques for the surfaces not yet encountered. In other words, the average $\bar{y}_{(k)}$ is taken over a variety of response surfaces, but the standard error $\delta_{(k)}$ does not reflect this variability among these surfaces.

The performance measures might also be modeled as

$$y_{ij(k)} = \mu + S_i + \varepsilon_{ij}, \quad i = 1, 2, \ldots, 12, \; j = 1, 2, 3$$

where

$$E(S_i) = E(\varepsilon_{ij}) = 0,$$

and where

$$E(\varepsilon_{ij}\ \varepsilon_{i'j'}) = \begin{cases} \sigma_i^2 \text{ for } i = i', \; j = j' \\ 0, \text{ otherwise} \end{cases},$$

$$E(S_i\ S_{i'}) = \begin{cases} \sigma_S^2 \\ 0, \text{ otherwise} \end{cases},$$

$$E(S_{i'}\ \varepsilon_{ij}) = 0 \text{ for all } i', i, \text{ and } j.$$

The variance of $\bar{y}_{(k)}$ expressed as $\tau^2_{(k)}$ is given by

$$\tau^2_{(k)} = \frac{\bar{\sigma}^2_{\cdot(k)}}{36} + \frac{\sigma^2_{S(k)}}{12} = \delta^2_{(k)} + \frac{\sigma^2_{S(k)}}{12}.$$

The variance $\tau^2_{(k)}$ in this second model contains not only the variance from model 1, but also an additional component to reflect the added variability due to the response surfaces.

Variation among the observations "within" the surfaces and variation "among" the surface means may be used to estimate $\delta^2_{(k)}$ and $\sigma^2_{S(k)}$. The expected mean squares for "among" and "within" surfaces are

$$E[(\text{Sum of squares among surfaces})/11] = E\left[\sum_{ij} \frac{\left(\bar{y}_{i\cdot(k)} - \bar{y}_{\cdot\cdot(k)}\right)^2}{11}\right]$$

$$= \bar{\sigma}^2_{\cdot(k)} + 3\sigma^2_{S(k)}$$

and

$$E[(\text{Sum of squares within surfaces}/24)] = E\left[\sum_{ij} \frac{\left(y_{ij(k)} - \bar{y}_{i\cdot(k)}\right)^2}{24}\right]$$

$$= \bar{\sigma}^2_{\cdot(k)} \ .$$

These two expected values suggest the estimates

$$\hat{\bar{\sigma}}^2_{\cdot} = \sum \frac{\left(y_{ij(k)} - \bar{y}_{i\cdot(k)}\right)^2}{24}$$

and

$$\hat{\sigma}^2_S = 1/3\left(\sum_{ij} \frac{\left(y_{ij(k)} - \bar{y}_{\cdot\cdot(k)}\right)^2}{11} - \hat{\bar{\sigma}}^2_{\cdot(k)}\right).$$

These estimates in turn are used to form the estimate of $\tau^2_{(k)}$ as

$$\hat{\tau}^2_{(k)} = \frac{\hat{\bar{\sigma}}^2_{\cdot(k)}}{36} + \frac{\hat{\sigma}^2_{S(k)}}{12} \quad .$$

In the second model, the parameter, $\sigma^2_{S(k)}$, is included to give a better measure of the precision on the averages for the purpose for which they were obtained, namely, to obtain empirical evidence in order to select a g-max technique for "future" usage on any surface. Certainly the precision of the average performance of a g-max technique on the experimental response surfaces is measured by $\hat{\delta}_{(k)}$, but surface to surface variability becomes a relevant quantity when averages are to be used to infer about response surfaces that have not yet been sampled.

The quantity $\sigma^2_{S(k)}$ is a parameter reflecting the variability in the conceptual distribution of $y_{(k)}$ as imposed by the application of a g-max technique on surfaces encountered by the potential uses. This must be a quantity that in many situations can never by known and certainly no claim is made here that the surfaces employed in this study are a representative sample of possible response surfaces giving unbiased estimates of σ^2_S. What was attempted, however, was the construction of a wide variety of response surface shapes. The reader will recall that surface 1 had simple looking contours while surface 6 had flat regions and complex looking contours about the maxima. If this variety of response surface shapes has in turn produced a sample of surface averages that are at least as variable from surface to surface as in the conceptual situation, then $\hat{\sigma}^2_S$ will be over estimated relative to the true situation. This in turn will cause $\hat{\tau}_{(k)}$ to be an overestimate or an upper limit to the standard error of a g-max technique performance average. The $\hat{\delta}_{(k)}$ which does not incorporate surface to surface variability (or $\sigma_S^2 = 0$) can be considered as an estimated lower bound to $\tau_{(k)}$.

The averages over surfaces from Tables X, XI, and XII are listed in Tables XV, XVI, and XVII. The respective estimates of the standard errors, $\hat{\delta}_{(k)}$ and $\hat{\tau}_{(k)}$, of these averages are also listed. In a few instances the variation within surfaces was estimated to be larger than the among surfaces mean square. For these situations the two mean squares for within and between surfaces were pooled to give a single estimate of precision $\hat{\delta}_{(k)} = \hat{\tau}_{(k)} = \hat{\sigma}_{pooled}/\sqrt{36}$. Recall techniques 2 1, 2 2, and 2 3 were not replicated for the situation described in Table XV in which case $\bar{\sigma}_{\cdot}^2 = 0$. In these instances $\tau_{(k)}^2 = \sigma_S^2/12$ and an estimate of $\tau_{(k)}$ may be obtained from the surface to surface variability as

$$\hat{\tau}_{(k)} = \sqrt{\sum_i \frac{\left(y_{i(k)} - \bar{y}_{\cdot(k)}\right)^2}{11(12)}} .$$

TABLE XV

G-MAX TECHNIQUE AVERAGES AND PRECISION ESTIMATES FOR $\sigma = 0.0$

Opt. Code	$y_{(1)}$	$\hat{\delta}_{(1)}$	$\hat{\tau}_{(1)}$	$y_{(2)}$	$\hat{\delta}_{(2)}$	$\hat{\tau}_{(2)}$
1 1	62	2.4	4.9	.915	.0135	.0205
1 2	73	2.0	6.2	.943	.0130	.0185
1 3	52	1.7	3.7	.953	.0127	.0163
2 1	160		5.9	.952		.0229
2 2	181		8.3	.986		.0144
2 3	116		6.0	.992		.0050
3 1	66	4.6	5.4	.907	.0198	.0198
3 2	72	3.0	5.7	.926	.0172	.0174
3 3	56	3.3	3.7	.943	.0162	.0166

TABLE XVI

G-MAX TECHNIQUE AVERAGES AND PRECISION ESTIMATES FOR $\sigma = .01$

Opt. Code	$y_{(1)}$	$\hat{\delta}_{(1)}$	$\hat{\tau}_{(1)}$	$y_{(2)}$	$\hat{\delta}_{(2)}$	$\hat{\tau}_{(2)}$
1 1	59	2.9	2.9	.859	.0290	.0342
1 2	89	4.3	5.4	.889	.0220	.0276
1 3	53	1.8	3.4	.926	.0156	.0211
2 1	85	3.4	3.4	.753	.0350	.0554
2 2	227	5.2	7.3	.974	.0047	.0165
2 3	137	2.9	6.3	.981	.0076	.0130
3 1	57	3.0	3.7	.832	.0260	.0460
3 2	96	3.0	6.5	.928	.0163	.0240
3 3	60	2.2	6.3	.954	.0065	.0157

TABLE XVII

G-MAX TECHNIQUE AVERAGES AND PRECISION ESTIMATES FOR σ = .05

Opt. Code	$y_{(1)}$	$\hat{\delta}_{(1)}$	$\hat{\tau}_{(1)}$	$y_{(2)}$	$\hat{\delta}_{(2)}$	$\hat{\tau}_{(2)}$
1 1	61	2.9	3.8	.855	.0218	.0266
1 2	102	6.4	9.4	.877	.0222	.0235
1 3	78	4.7	5.8	.909	.0107	.0242
2 1	72	1.9	1.9	.645	.0500	.0500
2 2	183	4.3	5.4	.869	.0376	.0419
2 3	172	2.2	3.4	.944	.0113	.0140
3 1	66	3.1	5.0	.854	.0230	.0335
3 2	94	4.8	4.8	.873	.0247	.0247
3 3	75	4.5	4.7	.905	.0190	.0193

To illustrate the importance of the difference in the two precision estimates, consider, for example, the results of techniques 1 1 and 1 2 in Table XV. The difference of the $y_{(2)}$ means for these techniques, .943 - .915 = .028, with an estimated standard error of the difference (ESED) based on $\hat{\delta}^2_{(2)}$ of

$$\text{ESED} = \sqrt{(.0135)^2 + (.0130)^2} = .0187,$$

is $\frac{.028}{.0187} = 1.5$ standard deviations. Similar calculations with the larger estimates $\hat{\tau}_{(2)}$ makes the difference .028 only

$$\frac{.028}{(.9295)^2 + (.0185)^2} = 1.0$$ standard deviation. For purposes of selecting a one "best" g-max technique, the statistical evidence for comparing these two techniques as well as comparisons between most any other two techniques, is not overwhelming.

Fortunately, the apparent lack of statistical precision is far from being disasterous. There appear to be patterns in the data from which one can conclude that the "best" performing g-max techniques include only the gradient local optimization technique regardless of the performance measure $y_{(1)}$ or $y_{(2)}$. There is also evidence to indicate that the 4-point strategy is different in some sense from the uniform or normal restart procedures. More will be discussed about this later.

First consider the $y_{(2)}$ averages of the local techniques within the various starting point strategies. Without exception, the largest $y_{(2)}$ averages are produced by the appropriate restart strategy with the gradient in all three tables (XV, XVI, and XVII). In fact the accelerated gradient is always last with pattern search producing the middle averages.

Returning to Table XV, comparisons within restart strategies with respect to $y_{(1)}$ show gradient again the best with the normal, 4-point, and the uniform.

In Table XVI the accelerated gradient appears to be more desirable on surface responses which are stochastic ($\sigma = .01$). The accelerated gradient and the gradient appear to perform equally well with respect to $y_{(1)}$ on the normal and uniform restart procedures, but the accelerated gradient becomes the winner for its performance with the 4-point. The accelerated gradients' new position, however, is short lived.

Although the 4-point/accelerated gradient performs best with respect to $y_{(1)}$, the value of $y_{(2)} = .753$ is hardly acceptable. The .753 average is even below the average value of .8 for the lower maxima of all twelve of the response surfaces. Because of this the gradient is declared the better performer.

In the final table, Table XVIII, the superior performance of the gradient again appears to be challenged by the accelerated gradient with respect to $y_{(1)}$. In all three restart strategies the accelerated gradient requires the fewest number of evaluations. Its performance with respect to $y_{(2)}$ is, as mentioned earlier, the poorest of the three local techniques. In the 4-point strategy the performance of the accelerated gradient is completely unsatisfactory with an average of $y_{(2)} = .645$.

Since the gradient local optimization technique appears to give the best results, the restart strategies are now examined by comparing g-max techniques 1 3, 2 3, and 3 3 summarized in Table XVIII.

TABLE XVIII

GRADIENT PERFORMANCE AVERAGES

		Normal		4-Point		Uniform	
		$y_{(1)}$	$y_{(2)}$	$y_{(1)}$	$y_{(2)}$	$y_{(1)}$	$y_{(2)}$
Error Level (σ=)	0.0	52	.953	116	.992	56	.943
	.01	53	.926	137	.981	60	.954
	.05	78	.909	172	.944	75	.905
		61	.929	143	.972	64	.934

The data does not permit a clear ranking of these three procedures, but each may be placed in one of two categories depending on the needs of the potential user. The 4-point/gradient is the better technique if a premium is placed on the necessity of accurately locating the g-max. If response surface evaluations are a costly item, then either the normal/gradient or uniform/gradient is the better type of g-max technique. The data do not allow the latter two techniques to be further differentiated.

Discussion

The patterns in the $y_{(2)}$ averages displayed by the different optimization techniques within restart strategies can largely be attributed to the performance of the local optimization procedures. The following general observations on the performance of the local techniques seem to be consonant with the data. The reader is reminded that the observations given here might very well be characteristic only of the experimental conditions adopted for this study.

1. The accelerated gradient had the highest frequency of starts which resulted in termination at a non-maximum point with an increase in this frequency caused by an increase in the magnitude of the experimental error level. Pattern search and the gradient procedures were less guilty of this with the gradient giving the best performance of the three.
2. Given that a technique was in the "neighborhood" of a maximum, pattern search seemed to give the most accurate approximation with gradient second in this category. Accelerated gradient was the most inconsistent in this category.
3. The biggest disadvantage of pattern search was not its inability to attain a maximum, but that a large number of response surface evaluations were needed to both locate and then terminate the search at an optimum point.
4. Initial step sizes of the gradient were much larger than the accelerated gradient or pattern search which depended on "accelerated" moves in order to progress to a maximum. Consequently, the first steps of the gradient were observed to be away from a maximum point when the starting point was in its neighborhood.

CHAPTER V

SUMMARY AND SUGGESTIONS FOR FUTURE STUDY

Summary

The purpose of this study was to construct and compare optimization techniques which would provide an approximation to the global maximum of a response surface over a region of independent variables X.

Global optimization techniques were constructed by incorporating various restart strategies with existing optimization techniques which locate local maxima. Three local optimization techniques combined in all combinations with three restart strategies provided nine global optimization techniques. The three local techniques were the gradient, accelerated gradient, and a direct search procedure called pattern search. The restart strategies were the normal, uniform, and 4-point techniques. The normal or uniform strategies were so named to identify the probability distribution which provided the basis for selecting random points subsequent to the convergence of a local technique. Random points were selected for purposes of finding a larger response surface value than was previously located by a chosen local optimization technique. The 4-point strategy consisted of four starts in X chosen prior to the application of a selected local optimization technique.

Performance data on each of the nine global optimization were obtained for a variety of experimental conditions. Attempts were made to construct experimental conditions which would provide inferences from the experimental data to the situation where the potential user of such techniques does not know the response surface characteristics that he might encounter.

Each of the global optimization techniques was applied to response surfaces constructed by summing bivariate normal probability density functions. A criterion for specifying the parameters in the bivariate normal density functions were extended from results obtained from Eisenberger (6) on the weighted sum of two univariate normal probability densities. The criterion can provide rapid construction of response surface contours to conform to desired shapes. Each of the response surfaces contained two relative maxima.

The average performances of the global techniques were not obtained with a high degree of statistical precision. The consequences of this statistical imprecision, however, were not felt to be disastrous since there appear to be clear patterns in these data that point out the "best" techniques depending on the needs of the user. Good approximations to the global maximum were obtained by the gradient local optimization technique with points selected according to either the uniform or normal probability distributions. These two techniques, the normal/gradient and uniform/gradient, seem to be the "best" techniques if there is a high premium on the cost of response surface evaluations. The 4-point strategy with the gradient local optimization technique produces the most accurate average approximations to the global maximum. The consequence, however, of these improved approximations is that the 4-point/gradient requires about twice as many response surface evaluations as the normal/gradient or uniform/gradient.

Suggestions For Future Study

Topics of further investigation related to this study might consist of other local optimization techniques as well as different restart strategies. Extensions could be made to the situations where the global maximum is to be selected from response surfaces where there are three or more maxima. Extensions could also be made to the n-dimensional situation where response surfaces are a function of n independent variables. The following suggestions might be helpful in future studies in any of the general areas discussed above:

1. Aside from other local optimization techniques, the gradient procedure and possibly the pattern search might be considered as likely candidates for local optimization techniques to assist in locating global maxima. Of the accelerated gradient and pattern search procedures, pattern search is possibly the more likely competitor to the gradient. This can result if the total number of response surface evaluations is "significantly" reduced without "significantly" reducing the accuracy of maxima approximations. One way this might be done is to increase the Δx_i increments in the basic search portion of this procedure and increase the rate of acceleration in the pattern moves. Another way might be to increase the rate of reduction on the Δx_i increments in the final stages of convergence.

Presently the technique requires that a basic search must fail in both the positive and negative directions of all the independent variables before a reduction is made. This becomes difficult to achieve near maxima when the response surfaces are stochastic. These two alterations might bring about an improvement in this technique.

2. Introducing response surfaces with three or more maxima might produce results different from those observed in this study. With only two maxima the large initial step sizes of the gradient seemed in general to be advantageous. On responses surfaces with several maxima, restarts near a large maxima would more likely result in first steps to regions where surface contours would lead the gradient back to smaller maxima. The more cautious systematic techniques such as an accelerated or pattern search may well perform better in these situations than with surfaces having but two maxima unless, possibly, smaller step sizes were used with the gradient.

3. Characterizing response surfaces from summed multivariate normal probability distributions would be beneficial for constructing a "variety" of surface shapes. Graphical procedures are helpful for the bivariate case, but a characterization of the surfaces, if possible, might

provide some insight into the general case of n independent variables. The precision on averages of optimization technique performances will in part depend on the experimental response surfaces. Characteristics of surfaces that could be useful for assessing this precision prior to experimentation are the probability of a restart subsequent to the locations of a maximum, and the relative sizes of the maxima of each surface.

4. Further experimentation might be to investigate the effects of varying the maximum number of points selected in the random restart strategies.
5. Procedures might also be investigated where the initial starting point is selected from an initial search of the response surface. Points might be selected again according to some probability distribution of points from the independent variables (e.g. normal, uniform) or possibly points chosen from a statistical experimental design(e.g. factorial or fractional factorial).

Undoubtedly this study has produced more questions than it has answered. This was not unexpected in light of the relatively small amount of literature devoted to technique which approximate a global optimum. Hopefully the methods used in this study will be useful for further work in this area.

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APPENDIX A

LIST OF DATA

Table XIX contains a list of the data where the column headings represent the following:

DATA POINT - a. Restart strategy; 1 - Normal, 2 - 4-point, 3 - Uniform.

b. Local optimization technique; 1 - Accelerated gradient, 2 - Pattern Search, 3 - Gradient.

c. Experimental error level on response surface 1 - $\sigma = 0.0$, 2 - $\sigma = .01$, 3 - $\sigma = .05$.

d.

Surface Code	Surface Number
1 2	1
1 3	2
1 4	3
1 5	4
1 6	5
1 7	6
2 2	7
2 3	8
2 4	9
2 5	10
2 6	11
2 7	12

e. Replication Number - 1, 2, and 3

NFE - Total number of response surface (function) evaluations

$f(\underset{\sim}{x}_a)$ - True response surface value at point of approximated global maximum, $(\underset{\sim}{x}_a)$ as determined by the selected g-max procedure.

$z(\underset{\sim}{x}_a)$ - Maximum observed surface response for that particular optimization.

x_{1_a} - First coordinate of $\underset{\sim}{x}_a$, the point of the approximated global maximum.

x_{2_a} - Second coordinate of $\underset{\sim}{x}_a$.

$f(\underset{\sim}{x}_t)$ - The true global maximum.

x_{1_t} - First coordinate of $\underset{\sim}{x}_t$, the point of the true global maximum.

x_{2_t} - Second coordinate of $\underset{\sim}{x}_t$.

TABLE XIX

LIST OF DATA

Data Point a	b	c	d	e	NFE	$f(x_a)$	$z(x_a)$	x_{1_a}	x_{2_a}	$f(x_t)$	x_{1_t}	x_{2_t}
1	1	1	1 2	1	046	0.15982	0.15982	1.9711	2.8085	0.16009	2.0250	2.8625
1	1	1	1 2	2	046	.15982	.15982	1.9711	2.8085	0.16009	2.0250	2.8625
1	1	1	1 2	3	046	.15982	.15982	1.9711	2.8085	0.16009	2.0250	2.8625
1	1	1	1 3	1	080	0.25432	0.25432	4.2281	3.8255	0.25472	4.1992	3.7998
1	1	1	1 3	2	069	.24575	.24575	3.8948	3.7427	0.25472	4.1992	3.7998
1	1	1	1 3	3	065	.25181	.25181	4.3703	3.8327	0.25472	4.1992	3.7998
1	1	1	1 4	1	072	0.24182	0.24182	2.8408	4.2134	0.24183	2.8200	4.2000
1	1	1	1 4	2	072	.24107	.24107	2.9474	4.1762	0.24183	2.8200	4.2000
1	1	1	1 4	3	055	.22501	.22501	2.0423	4.2512	0.24183	2.8200	4.2000
1	1	1	1 5	1	054	0.18357	0.18357	1.4086	2.9803	0.22743	5.0000	3.0005
1	1	1	1 5	2	054	.18357	.18357	1.4086	2.9803	0.22743	5.0000	3.0005
1	1	1	1 5	3	054	.18357	.18357	1.4086	2.9803	0.22743	5.0000	3.0005
1	1	1	1 6	1	070	0.56942	0.56942	4.0036	3.1165	0.63673	4.0000	1.0000
1	1	1	1 6	2	080	.56993	.56993	3.9931	3.0989	0.63673	4.0000	1.0000
1	1	1	1 6	3	077	.56517	.56517	4.0766	3.0784	0.63673	4.0000	1.0000
1	1	1	1 7	1	050	0.62836	0.62836	2.9087	3.3861	0.76175	2.5986	4.4951
1	1	1	1 7	2	050	.62836	.62836	2.9087	3.3861	0.76175	2.5986	4.4951
1	1	1	1 7	3	050	.62836	.62836	2.9087	3.3861	0.76175	2.5986	4.4951
1	1	1	2 2	1	37	0.15934	0.15934	2.6388	1.4314	0.16009	2.6377	1.5206
1	1	1	2 2	2	037	.15934	.15934	2.6388	1.4314	0.16009	2.6377	1.5206
1	1	1	2 2	3	037	.15934	.15934	2.6388	1.4314	0.16009	2.6377	1.5206
1	1	1	2 3	1	35	0.15607	0.15607	3.6515	2.2048	0.25472	1.7001	3.6991
1	1	1	2 3	2	066	.25435	.25435	1.7104	3.6370	0.25472	1.7001	3.6991
1	1	1	2 3	3	065	.23164	.23164	1.8369	4.1235	0.25472	1.7001	3.6991
1	1	1	2 4	1	038	0.20915	0.20915	1.0606	2.7064	0.24183	1.2900	2.3437
1	1	1	2 4	2	067	.22513	.22513	1.2382	1.5572	0.24183	1.2900	2.3437
1	1	1	2 4	3	038	.20915	.20915	1.0606	2.7064	0.24183	1.2900	2.3437
1	1	1	2 5	1	79	0.22380	0.22380	2.2815	4.4138	0.22743	2.4993	4.4995
1	1	1	2 5	2	083	.22403	.22403	2.6433	4.6420	0.22743	2.4993	4.4995
1	1	1	2 5	3	045	.15892	.15892	3.7546	2.1102	0.22743	2.4993	4.4995
1	1	1	2 6	1	107	0.63673	0.63673	4.4994	3.5000	0.63673	4.5000	3.5000
1	1	1	2 6	2	060	.53946	.53946	2.4676	3.3610	0.63673	4.5000	3.5000
1	1	1	2 6	3	060	.53946	.53946	2.4676	3.3610	0.63673	4.5000	3.5000
1	1	1	2 7	1	110	0.76175	0.76175	1.0036	2.0989	0.76176	1.0035	2.1000
1	1	1	2 7	2	074	.74734	.74734	.9438	2.2057	0.76176	1.0035	2.1000
1	1	1	2 7	3	112	.76175	.76175	1.0042	2.0971	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	2	1	1	2	1	049	0.16009	0.16009	2.0250	2.8625	0.16009	2.0250	2.8625
1	2	1	1	2	2	049	.16009	.16009	2.0250	2.8625	0.16009	2.0250	2.8625
1	2	1	1	2	3	049	.16009	.16009	2.0250	2.8625	0.16009	2.0250	2.8625
1	2	1	1	3	1	115	0.25472	0.25472	4.2000	3.8000	0.25472	4.1992	3.7998
1	2	1	1	3	2	090	.25472	.25472	4.2000	3.8000	0.25472	4.1992	3.7998
1	2	1	1	3	3	091	.25472	.25472	4.2000	3.8000	0.25472	4.1992	3.7998
1	2	1	1	4	1	048	0.24176	0.24176	2.8200	4.2000	0.24183	2.8200	4.2000
1	2	1	1	4	2	048	.24176	.24176	2.8200	4.2000	0.24183	2.8200	4.2000
1	2	1	1	4	3	048	.24176	.24176	2.8200	4.2000	0.24183	2.8200	4.2000
1	2	1	1	5	1	060	0.19955	0.19955	1.1831	2.7141	0.22743	5.0000	3.0005
1	2	1	1	5	2	060	.19955	.19955	1.1831	2.7141	0.22743	5.0000	3.0005
1	2	1	1	5	3	060	.19955	.19955	1.1831	2.7141	0.22743	5.0000	3.0005
1	2	1	1	6	1	088	0.56989	0.56989	3.9994	3.0969	0.63673	4.0000	1.0000
1	2	1	1	6	2	088	.56989	.56989	3.9994	3.0969	0.63673	4.0000	1.0000
1	2	1	1	6	3	088	.56989	.56989	3.9994	3.0969	0.63673	4.0000	1.0000
1	2	1	1	7	1	071	0.62953	0.62953	3.0225	3.4062	0.76175	2.5986	4.4951
1	2	1	1	7	2	071	.62953	.62953	3.0225	3.4062	0.76175	2.5986	4.4951
1	2	1	1	7	3	071	.62953	.62953	3.0225	3.4062	0.76175	2.5986	4.4951
1	2	1	2	2	1	52	0.16010	0.16010	2.6400	1.5000	0.16009	2.6377	1.5206
1	2	1	2	2	2	052	.16005	.16005	2.6400	1.5000	0.16009	2.6377	1.5206
1	2	1	2	2	3	052	.16005	.16005	2.6400	1.5000	0.16009	2.6377	1.5206
1	2	1	2	3	1	44	0.16580	0.16580	4.1100	2.5000	0.25472	1.7001	3.6991
1	2	1	2	3	2	075	.25467	.25467	1.7100	3.7000	0.25472	1.7001	3.6991
1	2	1	2	3	3	078	.25466	.25466	1.7100	3.6875	0.25472	1.7001	3.6991
1	2	1	2	4	1	062	0.24175	0.24175	1.3050	2.3625	0.24183	1.2900	2.3437
1	2	1	2	4	2	062	.24175	.24175	1.3050	2.3625	0.24183	1.2900	2.3437
1	2	1	2	4	3	062	.24175	.24175	1.3050	2.3625	0.24183	1.2900	2.3437
1	2	1	2	5	1	108	0.22742	0.22742	2.5050	4.5000	0.22743	2.4993	4.4995
1	2	1	2	5	2	108	.22742	.22742	2.5050	4.5000	0.22743	2.4993	4.4995
1	2	1	2	5	3	064	.15986	.15986	3.8850	2.3125	0.22743	2.4993	4.4995
1	2	1	2	6	1	103	0.63673	0.63673	4.5000	3.5000	0.63673	4.5000	3.5000
1	2	1	2	6	2	066	.56991	.56991	2.4000	3.5000	0.63673	4.5000	3.5000
1	2	1	2	6	3	066	.56991	.56991	2.4000	3.5000	0.63673	4.5000	3.5000
1	2	1	2	7	1	112	0.76175	0.76175	1.0050	2.1000	0.76176	1.0035	2.1000
1	2	1	2	7	2	121	.76175	.76175	1.0050	2.1000	0.76176	1.0035	2.1000
1	2	1	2	7	3	111	.76175	.76175	1.0050	2.1000	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	3	1	1	2	1	038	0.15973	0.15973	1.9803	2.7814	0.16009	2.0250	2.8625
1	3	1	1	2	2	038	.15973	.15973	1.9803	2.7814	0.16009	2.0250	2.8625
1	3	1	1	2	3	038	.15973	.15973	1.9803	2.7814	0.16009	2.0250	2.8625
1	3	1	1	3	1	091	0.25463	0.25463	4.2117	3.8122	0.25472	4.1992	3.7998
1	3	1	1	3	2	071	.25399	.25399	4.1132	3.8158	0.25472	4.1992	3.7998
1	3	1	1	3	3	072	.25312	.25312	4.1943	3.7437	0.25472	4.1992	3.7998
1	3	1	1	4	1	050	0.24153	C.24153	2.8823	4.1821	0.24183	2.8200	4.2000
1	3	1	1	4	2	050	.24153	.24153	2.8823	4.1821	0.24183	2.8200	4.2000
1	3	1	1	4	3	050	.24153	.24153	2.8823	4.1821	0.24183	2.8200	4.2000
1	3	1	1	5	1	041	0.19947	0.19947	1.1807	2.7435	0.22743	5.0000	3.0005
1	3	1	1	5	2	041	.19947	.19947	1.1807	2.7435	0.22743	5.0000	3.0005
1	3	1	1	5	3	041	.19947	.19947	1.1807	2.7435	0.22743	5.0000	3.0005
1	3	1	1	6	1	041	0.56803	0.56803	3.9449	3.1386	0.63673	4.0000	1.0000
1	3	1	1	6	2	041	.56803	.56803	3.9449	3.1386	0.63673	4.0000	1.0000
1	3	1	1	6	3	041	.56803	.56803	3.9449	3.1386	0.63673	4.0000	1.0000
1	3	1	1	7	1	053	0.75952	0.75952	2.5660	4.4706	0.76175	2.5986	4.4951
1	3	1	1	7	2	053	.75952	.75952	2.5660	4.4706	0.76175	2.5986	4.4951
1	3	1	1	7	3	053	.75952	.75952	2.5660	4.4706	0.76175	2.5986	4.4951
1	3	1	2	2	1	35	0.15856	0.15856	2.5739	1.4161	0.16009	2.6377	1.5206
1	3	1	2	2	2	035	.15857	.15857	2.7987	1.4327	0.16009	2.6377	1.5206
1	3	1	2	2	3	035	.15857	.15857	2.7987	4.1327	0.16009	2.6377	1.5206
1	3	1	2	3	1	44	0.16302	0.16302	3.9125	2.2927	0.25472	1.7001	3.6991
1	3	1	2	3	2	077	.25440	.25440	1.6790	3.6653	0.25472	1.7001	3.6991
1	3	1	2	3	3	077	.25464	.25464	1.6925	3.6748	0.25472	1.7001	3.6991
1	3	1	2	4	1	050	0.24130	0.24130	1.2602	2.2624	0.24183	1.2900	2.3437
1	3	1	2	4	2	050	.24130	.24130	1.2602	2.2624	0.24183	1.2900	2.3437
1	3	1	2	4	3	050	.24130	.24130	1.2602	2.2624	0.24183	1.2900	2.3437
1	3	1	2	5	1	81	0.22439	0.22439	2.2965	4.3938	0.22743	2.4993	4.4995
1	3	1	2	5	2	068	.21243	.21243	2.0517	4.2363	0.22743	2.4993	4.4995
1	3	1	2	5	3	041	.15966	.15966	3.8497	2.2754	0.22743	2.4993	4.4995
1	3	1	2	6	1	82	0.63670	0.63670	4.5014	3.5046	0.63673	4.5000	3.5000
1	3	1	2	6	2	047	.56976	.56976	2.3970	3.4771	0.63673	4.5000	3.5000
1	3	1	2	6	3	047	.56976	.56976	2.3970	3.4771	0.63673	4.5000	3.5000
1	3	1	2	7	1	50	0.75992	0.75992	0.9798	2.1152	0.76176	1.0035	2.1000
1	3	1	2	7	2	050	.76103	.76103	1.0054	2.1248	0.76176	1.0035	2.1000
1	3	1	2	7	3	050	.76103	.76103	1.0054	2.1248	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	1	1	1	2	1	134	0.15981	0.15981	1.9632	2.8445	0.16009	2.0250	2.8625
2	1	1	1	3	1	148	0.25266	0.25266	4.3209	3.7589	0.25472	4.1992	3.7998
2	1	1	1	4	1	148	0.23991	0.23991	2.9954	4.2375	0.24183	2.8200	4.2000
2	1	1	1	5	1	170	0.21803	0.21803	4.8786	2.9196	0.22743	5.0000	3.0005
2	1	1	1	6	1	160	0.63665	0.63665	4.0081	0.9995	0.63673	4.0000	1.0000
2	1	1	1	7	1	204	0.62955	0.62955	3.0203	3.4054	0.76175	2.5986	4.4951
2	1	1	2	2	1	125	0.16009	0.16009	2.6352	1.5253	0.16009	2.6377	1.5206
2	1	1	2	3	1	173	0.23621	0.23621	1.8298	4.0607	0.25472	1.7001	3.6991
2	1	1	2	4	1	160	0.23621	0.23621	1.3464	2.0237	0.24183	1.2900	2.3437
2	1	1	2	5	1	168	0.22714	0.22714	2.5428	4.5405	0.22743	2.4993	4.4995
2	1	1	2	6	1	172	0.63673	0.63673	4.5000	3.4996	0.63673	4.5000	3.5000
2	1	1	2	7	1	153	0.57763	0.57763	2.1953	1.9141	0.76176	1.0035	2.1000
2	2	1	1	2	1	140	0.16008	0.16008	2.0325	2.8687	0.16009	2.0250	2.8625
2	2	1	1	3	1	149	0.25472	0.25472	4.2000	3.8000	0.25472	4.1992	3.7998
2	2	1	1	4	1	163	0.24183	0.24183	2.8425	4.2062	0.24183	2.8200	4.2000
2	2	1	1	5	1	167	0.22742	0.22742	4.9950	3.0000	0.22743	5.0000	3.0005
2	2	1	1	6	1	189	0.63668	0.63668	3.9975	1.0062	0.63673	4.0000	1.0000
2	2	1	1	7	1	238	0.62962	0.62962	3.0000	3.4031	0.76175	2.5986	4.4951
2	2	1	2	2	1	158	0.16009	0.16009	2.6325	1.5187	0.16009	2.6377	1.5206
2	2	1	2	3	1	174	0.25471	0.25471	1.7025	3.6937	0.25472	1.7001	3.6991
2	2	1	2	4	1	186	0.24183	0.24183	1.2900	2.3437	0.24183	1.2900	2.3437
2	2	1	2	5	1	188	0.22741	0.22741	2.5125	4.5062	0.22743	2.4993	4.4995
2	2	1	2	6	1	194	0.63661	0.63661	4.4925	3.4937	0.63673	4.5000	3.5000
2	2	1	2	7	1	222	0.76173	0.76173	1.0031	2.1047	0.76176	1.0035	2.1000
2	3	1	1	2	1	100	0.15996	0.15996	1.9920	2.8172	0.16009	2.0250	2.8625
2	3	1	1	3	1	127	0.25447	0.25447	4.2523	3.8055	0.25472	4.1992	3.7998
2	3	1	1	4	1	109	0.24109	0.24109	2.7702	4.1779	0.24183	2.8200	4.2000
2	3	1	1	5	1	097	0.22517	0.22517	4.8877	3.1572	0.22743	5.0000	3.0005
2	3	1	1	6	1	074	0.63570	0.63570	3.9718	0.9951	0.63673	4.0000	1.0000
2	3	1	1	7	1	154	0.76156	0.76155	2.5867	4.4894	0.76175	2.5986	4.4951
2	3	1	2	2	1	109	0.16005	0.16005	2.6264	1.5049	0.16009	2.6377	1.5206
2	3	1	2	3	1	133	0.25440	0.25440	1.6986	3.6370	0.25472	1.7001	3.6991
2	3	1	2	4	1	118	0.22734	0.22734	1.2961	2.8166	0.24183	1.2900	2.3437
2	3	1	2	5	1	106	0.22652	0.22652	2.5939	4.5716	0.22743	2.4993	4.4995
2	3	1	2	6	1	130	0.63144	0.63144	4.5485	3.4573	0.63673	4.5000	3.5000
2	3	1	2	7	1	130	0.76174	0.76174	1.0034	2.1041	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	1	1	1	2	1	046	0.15982	0.15982	1.9711	2.8085	0.16009	2.0250	2.8625
3	1	1	1	2	2	046	.15982	.15982	1.9711	2.8085	0.16009	2.0250	2.8625
3	1	1	1	2	3	046	.15982	.15982	1.9711	2.8085	0.16009	2.0250	2.8625
3	1	1	1	3	1	045	0.15391	0.15391	2.5661	1.7986	0.25472	4.1992	3.7998
3	1	1	1	3	2	097	.25469	.25469	4.1828	3.7981	0.25472	4.1992	3.7998
3	1	1	1	3	3	067	.24849	.24849	4.4374	3.8576	0.25472	4.1992	3.7998
3	1	1	1	4	1	079	0.23162	0.23162	2.3824	4.3135	0.24183	2.8200	4.2000
3	1	1	1	4	2	067	.24134	.24134	2.9379	4.1905	0.24183	2.8200	4.2000
3	1	1	1	4	3	034	.13322	.13322	4.1557	2.9812	0.24183	2.8200	4.2000
3	1	1	1	5	1	054	0.18357	0.18357	1.4086	2.9803	0.22743	5.0000	3.0005
3	1	1	1	5	2	054	.18357	.18357	1.4086	2.9803	0.22743	5.0000	3.0005
3	1	1	1	5	3	054	.18357	.18357	1.4086	2.9803	0.22743	5.0000	3.0005
3	1	1	1	6	1	075	0.56922	0.56922	4.0128	3.0785	0.63673	4.0000	1.0000
3	1	1	1	6	2	061	.55753	.55753	3.8568	3.1263	0.63673	4.0000	1.0000
3	1	1	1	6	3	090	.63673	.63673	3.9999	.9995	0.63673	4.0000	1.0000
3	1	1	1	7	1	075	0.76065	0.76065	2.6345	4.4988	0.76175	2.5986	4.4951
3	1	1	1	7	2	050	.62836	.62836	2.9087	3.3861	0.76175	2.5986	4.4951
3	1	1	1	7	3	050	.62836	.62836	2.9087	3.3861	0.76175	2.5986	4.4951
3	1	1	2	2	1	037	0.15934	0.15934	2.6388	1.4314	0.16009	2.6377	1.5206
3	1	1	2	2	2	037	.15934	.15934	2.6388	1.4314	0.16009	2.6377	1.5206
3	1	1	2	2	3	037	.15934	.15934	2.6388	1.4314	0.16009	2.6377	1.5206
3	1	1	2	3	1	088	0.25389	0.25389	1.7243	3.7797	0.25472	1.7001	3.6991
3	1	1	2	3	2	063	.25467	.25467	1.6969	3.7223	0.25472	1.7001	3.6991
3	1	1	2	3	3	104	.25412	.25412	1.7111	3.7800	0.25472	1.7001	3.6991
3	1	1	2	4	1	038	0.20915	0.20915	1.0606	2.7064	0.24183	1.2900	2.3437
3	1	1	2	4	2	143	.24183	.24183	1.2907	2.3384	0.24183	1.2900	2.3437
3	1	1	2	4	3	038	.20915	.20915	1.0606	2.7064	0.24183	1.2900	2.3437
3	1	1	2	5	1	111	0.22743	0.22743	2.4970	4.4980	0.22743	2.4993	4.4995
3	1	1	2	5	2	144	.22742	.22742	2.5040	4.5065	0.22743	2.4993	4.4995
3	1	1	2	5	3	045	.15892	.15892	3.7546	2.1102	0.22743	2.4993	4.4995
3	1	1	2	6	1	089	0.63670	0.63670	4.4947	3.5018	0.63673	4.5000	3.5000
3	1	1	2	6	2	111	.63673	.63673	4.5000	3.4996	0.63673	4.5000	3.5000
3	1	1	2	6	3	060	.53946	.53946	2.4676	3.3610	0.63673	4.5000	3.5000
3	1	1	2	7	1	048	0.62823	0.62823	2.1241	2.4172	0.76176	1.0035	2.1000
3	1	1	2	7	2	048	.62823	.62823	2.1241	2.4172	0.76176	1.0035	2.1000
3	1	1	2	7	3	058	.62823	.62823	2.1241	2.4172	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	2	1	1	2	1	049	0.16009	0.16009	2.0250	2.8625	0.16009	2.0250	2.8625
3	2	1	1	2	2	049	.16009	.16009	2.0250	2.8625	0.16009	2.0250	2.8625
3	2	1	1	2	3	049	.16009	.16009	2.0250	2.8625	0.16009	2.0250	2.8625
3	2	1	1	3	1	058	0.16579	0.16579	3.0225	1.3812	0.25472	4.1992	3.7998
3	2	1	1	3	2	102	.25472	.25472	4.2000	3.8000	0.25472	4.1992	3.7998
3	2	1	1	3	3	106	.25472	.25472	4.2000	3.8000	0.25472	4.1992	3.7998
3	2	1	1	4	1	048	0.24176	0.24176	2.8200	4.2000	0.24183	2.8200	4.2000
3	2	1	1	4	2	048	.24176	.24176	2.8200	4.2000	0.24183	2.8200	4.2000
3	2	1	1	4	3	058	.24176	.24176	2.8200	4.2000	0.24183	2.8200	4.2000
3	2	1	1	5	1	060	0.19955	0.19955	1.1831	2.7141	0.22743	5.0000	3.0005
3	2	1	1	5	2	060	.19955	.19955	1.1831	2.7141	0.22743	5.0000	3.0005
3	2	1	1	5	3	060	.19955	.19955	1.1831	2.7141	0.22743	5.0000	3.0005
3	2	1	1	6	1	088	0.56989	0.56989	3.9994	3.0969	0.63673	4.0000	1.0000
3	2	1	1	6	2	088	.56989	.56989	3.9994	3.0969	0.63673	4.0000	1.0000
3	2	1	1	6	3	126	.63670	.63670	4.0050	1.0000	0.63673	4.0000	1.0000
3	2	1	1	7	1	110	0.76166	0.76166	2.6100	4.5000	0.76175	2.5986	4.4951
3	2	1	1	7	2	071	.62953	.62953	3.0225	3.4062	0.76175	2.5986	4.4951
3	2	1	1	7	3	071	.62953	.62953	3.0225	3.4062	0.76175	2.5986	4.4951
3	2	1	2	2	1	052	0.16005	0.16005	2.6400	1.5000	0.16009	2.6377	1.5206
3	2	1	2	2	2	052	.16005	.16005	2.6400	1.5000	0.16009	2.6377	1.5206
3	2	1	2	2	3	052	.16005	.16005	2.6400	1.5000	0.16009	2.6377	1.5206
3	2	1	2	3	1	044	0.16580	0.16580	4.1100	2.5000	0.25472	1.7001	3.6991
3	2	1	2	3	2	062	.25470	.25470	1.6950	3.7000	0.25472	1.7001	3.6991
3	2	1	2	3	3	066	.25470	.25470	1.6950	3.7000	0.25472	1.7001	3.6991
3	2	1	2	4	1	062	0.24175	0.24175	1.3050	2.3625	0.24183	1.2900	2.3437
3	2	1	2	4	2	062	.24175	.24175	1.3050	2.3625	0.24183	1.2900	2.3437
3	2	1	2	4	3	062	.24175	.24175	1.3050	2.3625	0.24183	1.2900	2.3437
3	2	1	2	5	1	107	0.22742	0.22742	2.5050	4.5000	0.22743	2.4993	4.4995
3	2	1	2	5	2	146	.22738	.22738	2.4750	4.4875	0.22743	2.4993	4.4995
3	2	1	2	5	3	064	.15986	.15986	3.8850	2.3125	0.22743	2.4993	4.4995
3	2	1	2	6	1	101	0.63673	0.63673	4.5000	3.5000	0.63673	4.5000	3.5000
3	2	1	2	6	2	066	.56991	.56991	2.4000	3.5000	0.63673	4.5000	3.5000
3	2	1	2	6	3	066	.56991	.56991	2.4000	3.5000	0.63673	4.5000	3.5000
3	2	1	2	7	1	078	0.62963	0.62963	2.1000	2.5000	0.76176	1.0035	2.1000
3	2	1	2	7	2	078	.62963	.62963	2.1000	2.5000	0.76176	1.0035	2.1000
3	2	1	2	7	3	078	.62963	.62963	2.1000	2.5000	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	3	1	1	2	1	038	0.15973	0.15973	1.9803	2.7814	0.16009	2.0250	2.8625
3	3	1	1	2	2	038	.15973	.15973	1.9803	2.7814	0.16009	2.0250	2.8625
3	3	1	1	2	3	038	.15973	.15973	1.9803	2.7814	0.16009	2.0250	2.8625
3	3	1	1	3	1	044	0.16307	0.16307	2.8008	1.6105	0.25472	4.1992	3.7998
3	3	1	1	3	2	082	.25468	.25468	4.2125	3.7937	0.25472	4.1992	3.7998
3	3	1	1	3	3	075	.25416	.25416	4.1169	3.7957	0.25472	4.1992	3.7998
3	3	1	1	4	1	050	0.24153	0.24153	2.8823	4.1821	0.24183	2.8200	4.2000
3	3	1	1	4	2	050	.24153	.24153	2.8823	4.1821	0.24183	2.8200	4.2000
3	3	1	1	4	3	050	.24153	.24153	2.8823	4.1821	0.24183	2.8200	4.2000
3	3	1	1	5	1	041	0.19947	0.19947	1.1807	2.7435	0.22743	5.0000	3.0005
3	3	1	1	5	2	041	.19947	.19947	1.1807	2.7435	0.22743	5.0000	3.0005
3	3	1	1	5	3	041	.19947	.19947	1.1807	2.7435	0.22743	5.0000	3.0005
3	3	1	1	6	1	041	0.56803	0.56803	3.9449	3.1386	0.63673	4.0000	1.0000
3	3	1	1	6	2	041	.56803	.56803	3.9449	3.1386	0.63673	4.0000	1.0000
3	3	1	1	6	3	096	0.58770	0.58770	4.0772	0.8153	0.63673	4.0000	1.0000
3	3	1	1	7	1	053	0.75952	0.75952	2.5660	4.4706	0.76175	2.5986	4.4951
3	3	1	1	7	2	053	.75952	.75952	2.5660	4.4706	0.76175	2.5986	4.4951
3	3	1	1	7	3	053	.75952	.75952	2.5660	4.4706	0.76175	2.5986	4.4951
3	3	1	2	2	1	035	0.15857	0.15857	2.7987	1.4327	0.16009	2.6377	1.5206
3	3	1	2	2	2	035	.15857	.15857	2.7987	1.4327	0.16009	2.6377	1.5206
3	3	1	2	2	3	061	.15992	.15992	2.5866	1.5089	0.16009	2.6377	1.5206
3	3	1	2	3	1	044	0.16302	0.16302	3.9125	2.2927	0.25472	1.7001	3.6991
3	3	1	2	3	2	074	.25450	.25450	1.6868	3.6598	0.25472	1.7001	3.6991
3	3	1	2	3	3	075	.25424	.25424	1.6711	3.6737	0.25472	1.7001	3.6991
3	3	1	2	4	1	050	0.24130	0.24130	1.2602	2.2624	0.24183	1.2900	2.3437
3	3	1	2	4	2	050	.24130	.24130	1.2602	2.2624	0.24183	1.2900	2.3437
3	3	1	2	4	3	050	.24130	.24130	1.2602	2.2624	0.24183	1.2900	2.3437
3	3	1	2	5	1	073	0.22615	0.22615	2.3686	4.4418	0.22743	2.4993	4.4995
3	3	1	2	5	2	137	.22152	.22152	2.2269	4.3296	0.22743	2.4993	4.4995
3	3	1	2	5	3	041	.15966	.15966	3.8497	2.2754	0.22743	2.4993	4.4995
3	3	1	2	6	1	084	0.58379	0.58379	4.5501	3.2977	0.63673	4.5000	3.5000
3	3	1	2	6	2	047	.56976	.56976	2.3970	3.4771	0.63673	4.5000	3.5000
3	3	1	2	6	3	047	.56976	.56976	2.3970	3.4771	0.63673	4.5000	3.5000
3	3	1	2	7	1	050	0.76103	0.76103	1.0054	2.1248	0.76176	1.0035	2.1000
3	3	1	2	7	2	050	.76103	.76103	1.0054	2.1248	0.76176	1.0035	2.1000
3	3	1	2	7	3	050	.76103	.76103	1.0054	2.1248	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	1	2	1	2	1	070	0.14629	0.14785	2.3774	2.6994	0.16009	2.0250	2.8625
1	1	2	1	2	2	068	0.15993	0.16176	1.9800	2.8675	0.16009	2.0250	2.8625
1	1	2	1	2	3	059	0.16003	0.16224	2.0487	2.8678	0.16009	2.0250	2.8625
1	1	2	1	3	1	036	0.12925	0.13172	2.2869	2.2922	0.25472	4.1992	3.7998
1	1	2	1	3	2	082	0.25424	0.25505	4.2184	3.7702	0.25472	4.1992	3.7998
1	1	2	1	3	3	044	0.15898	0.16031	2.6965	1.5669	0.25472	4.1992	3.7998
1	1	2	1	4	1	061	0.23780	0.23948	3.0256	4.1120	0.24183	2.8200	4.2000
1	1	2	1	4	2	064	0.21451	0.21641	3.4894	4.1746	0.24183	2.8200	4.2000
1	1	2	1	4	3	063	0.23416	0.23557	2.4072	4.1638	0.24183	2.8200	4.2000
1	1	2	1	5	1	090	0.19912	0.20118	1.2503	2.6927	0.22743	5.0000	3.0005
1	1	2	1	5	2	035	0.15917	0.16046	2.1609	1.9709	0.22743	5.0000	3.0005
1	1	2	1	5	3	052	0.22707	0.22918	5.0056	2.9467	0.22743	5.0000	3.0005
1	1	2	1	6	1	034	0.24872	0.24957	2.5406	2.6657	0.63673	4.0000	1.0000
1	1	2	1	6	2	047	0.24951	0.25155	2.5563	2.6617	0.63673	4.0000	1.0000
1	1	2	1	6	3	072	0.63652	0.63698	4.0104	0.9919	0.63673	4.0000	1.0000
1	1	2	1	7	1	063	0.61503	0.61631	2.7829	3.3139	0.76175	2.5986	4.4951
1	1	2	1	7	2	032	0.60330	0.60394	3.1631	3.5171	0.76175	2.5986	4.4951
1	1	2	1	7	3	082	0.76165	0.76441	2.5881	4.4952	0.76175	2.5986	4.4951
1	1	2	2	2	1	049	0.15935	0.16046	2.7349	1.4448	0.16009	2.6377	1.5206
1	1	2	2	2	2	060	0.15960	0.15991	2.5850	1.4703	0.16009	2.6377	1.5206
1	1	2	2	2	3	039	0.15111	0.15218	3.0142	1.2861	0.16009	2.6377	1.5206
1	1	2	2	3	1	069	0.25378	0.25492	1.6592	3.7329	0.25472	1.7001	3.6991
1	1	2	2	3	2	037	0.15518	0.15661	3.7726	2.0904	0.25472	1.7001	3.6991
1	1	2	2	3	3	063	0.25281	0.25517	1.7339	3.8272	0.25472	1.7001	3.6991
1	1	2	2	4	1	077	0.24127	0.24215	1.2558	2.2744	0.24183	1.2900	2.3437
1	1	2	2	4	2	072	0.24061	0.24169	1.2565	2.4419	0.24183	1.2900	2.3437
1	1	2	2	4	3	037	0.23760	0.23975	1.3009	2.6098	0.24183	1.2900	2.3437
1	1	2	2	5	1	082	0.19896	0.20083	2.7315	0.7113	0.22743	2.4993	4.4995
1	1	2	2	5	2	081	0.19886	0.20075	2.8446	0.6832	0.22743	2.4993	4.4995
1	1	2	2	5	3	069	0.22424	0.22592	2.5349	4.6104	0.22743	2.4993	4.4995
1	1	2	2	6	1	037	0.29818	0.29939	1.9793	2.3388	0.63673	4.5000	3.5000
1	1	2	2	6	2	083	0.56461	0.56525	2.4116	3.4230	0.63673	4.5000	3.5000
1	1	2	2	6	3	074	0.63604	0.63691	4.4770	3.5049	0.63673	4.5000	3.5000
1	1	2	2	7	1	042	0.62911	0.62964	2.0843	2.5087	0.76176	1.0035	2.1000
1	1	2	2	7	2	047	0.62852	0.62950	2.0808	2.4912	0.76176	1.0035	2.1000
1	1	2	2	7	3	046	0.62899	0.62999	2.1042	2.4381	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	2	2	1	2	1	104	0.15972	0.16266	2.0325	2.9500	0.16009	2.0250	2.8625
1	2	2	1	2	2	070	0.15893	0.16137	1.9051	2.8000	0.16009	2.0250	2.8625
1	2	2	1	2	3	071	0.15859	0.16029	2.1450	2.8500	0.16009	2.0250	2.8625
1	2	2	1	3	1	088	0.16567	0.16749	2.9700	1.3969	0.25472	4.1992	3.7998
1	2	2	1	3	2	145	0.25425	0.25653	4.1550	3.7750	0.25472	4.1992	3.7998
1	2	2	1	3	3	089	0.16558	0.16764	2.9400	1.4562	0.25472	4.1992	3.7998
1	2	2	1	4	1	062	0.24150	0.24299	2.7600	4.2250	0.24183	2.8200	4.2000
1	2	2	1	4	2	085	0.24120	0.24261	2.8125	4.2500	0.24183	2.8200	4.2000
1	2	2	1	4	3	079	0.24059	0.24239	2.9250	4.2500	0.24183	2.8200	4.2000
1	2	2	1	5	1	080	0.15840	0.16076	2.3100	1.9133	0.22743	5.0000	3.0005
1	2	2	1	5	2	047	0.15841	0.16064	2.3100	1.9250	0.22743	5.0000	3.0005
1	2	2	1	5	3	110	0.22632	0.22845	4.9200	3.1062	0.22743	5.0000	3.0005
1	2	2	1	6	1	097	0.56988	0.57144	3.9900	3.0969	0.63673	4.0000	1.0000
1	2	2	1	6	2	054	0.24968	0.25136	2.5800	2.6000	0.63673	4.0000	1.0000
1	2	2	1	6	3	079	0.56898	0.57147	3.9600	3.1000	0.63673	4.0000	1.0000
1	2	2	1	7	1	100	0.62892	0.63070	2.9550	3.3812	0.76175	2.5986	4.4951
1	2	2	1	7	2	089	0.62924	0.63167	3.0450	3.4125	0.76175	2.5986	4.4951
1	2	2	1	7	3	160	0.76137	0.76371	2.6211	4.5000	0.76175	2.5986	4.4951
1	2	2	2	2	1	079	0.15972	0.16322	2.5500	1.5500	0.16009	2.6377	1.5206
1	2	2	2	2	2	058	0.15991	0.16191	2.5800	1.5500	0.16009	2.6377	1.5206
1	2	2	2	2	3	082	0.16008	0.16190	2.6550	1.5125	0.16009	2.6377	1.5206
1	2	2	2	3	1	143	0.25468	0.25659	1.6969	3.6766	0.25472	1.7001	3.6991
1	2	2	2	3	2	061	0.16564	0.16815	4.1381	2.4984	0.25472	1.7001	3.6991
1	2	2	2	3	3	111	0.25471	0.25665	1.7025	3.6875	0.25472	1.7001	3.6991
1	2	2	2	4	1	059	0.24059	0.24249	1.3350	2.2625	0.24183	1.2900	2.3437
1	2	2	2	4	2	071	0.24066	0.24286	1.3331	2.2625	0.24183	1.2900	2.3437
1	2	2	2	4	3	091	0.24110	0.24270	1.3050	2.4625	0.24183	1.2900	2.3437
1	2	2	2	5	1	151	0.19932	0.20152	2.7666	0.7125	0.22743	2.4993	4.4995
1	2	2	2	5	2	108	0.19777	0.19963	2.7019	0.7500	0.22743	2.4993	4.4995
1	2	2	2	5	3	113	0.22648	0.22772	2.4300	4.4250	0.22743	2.4993	4.4995
1	2	2	2	6	1	105	0.56969	0.57133	2.3831	3.4750	0.63673	4.5000	3.5000
1	2	2	2	6	2	081	0.56855	0.57050	2.4300	3.5000	0.63673	4.5000	3.5000
1	2	2	2	6	3	078	0.56907	0.56924	2.3775	3.4562	0.63673	4.5000	3.5000
1	2	2	2	7	1	059	0.62782	0.62939	2.1300	2.4250	0.76176	1.0035	2.1000
1	2	2	2	7	2	099	0.62953	0.63169	2.0944	2.4922	0.76176	1.0035	2.1000
1	2	2	2	7	3	057	0.62752	0.62899	2.1300	2.4750	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	3	2	1	2	1	041	0.16003	0.16077	1.9935	2.8541	0.16009	2.0250	2.8625
1	3	2	1	2	2	041	0.16004	0.16038	2.0251	2.8318	0.16009	2.0250	2.8625
1	3	2	1	2	3	038	0.15996	0.16129	2.0302	2.8162	0.16009	2.0250	2.8625
1	3	2	1	3	1	047	0.16320	0.16317	2.8078	1.6090	0.25472	4.1992	3.7998
1	3	2	1	3	2	085	0.25452	0.25532	4.1874	3.8194	0.25472	4.1992	3.7998
1	3	2	1	3	3	044	0.15826	0.15894	2.6600	1.7328	0.25472	4.1992	3.7998
1	3	2	1	4	1	053	0.23693	0.23689	3.0462	4.1019	0.24183	2.8200	4.2000
1	3	2	1	4	2	050	0.23796	0.23865	3.0592	4.2473	0.24183	2.8200	4.2000
1	3	2	1	4	3	050	0.24004	0.24040	3.0273	4.1961	0.24183	2.8200	4.2000
1	3	2	1	5	1	044	0.19427	0.19434	1.4169	2.6182	0.22743	5.0000	3.0005
1	3	2	1	5	2	035	0.19895	0.19807	1.2312	2.7590	0.22743	5.0000	3.0005
1	3	2	1	5	3	076	0.21610	0.21755	4.8416	2.9556	0.22743	5.0000	3.0005
1	3	2	1	6	1	044	0.56937	0.56976	4.0204	3.0990	0.63673	4.0000	1.0000
1	3	2	1	6	2	041	0.56637	0.56717	3.9392	3.1572	0.63673	4.0000	1.0000
1	3	2	1	6	3	044	0.56985	0.56965	3.9800	3.1046	0.63673	4.0000	1.0000
1	3	2	1	7	1	053	0.75341	0.75247	2.5361	4.5188	0.76175	2.5986	4.4951
1	3	2	1	7	2	053	0.75954	0.76026	2.5479	4.4817	0.76175	2.5986	4.4951
1	3	2	1	7	3	053	0.76054	0.76110	2.5650	4.4809	0.76175	2.5986	4.4951
1	3	2	2	2	1	035	0.15850	0.16045	2.5517	1.4257	0.16009	2.6377	1.5206
1	3	2	2	2	2	038	0.15986	0.16085	2.5901	1.4945	0.16009	2.6377	1.5206
1	3	2	2	2	3	035	0.15802	0.15748	2.8218	1.5565	0.16009	2.6377	1.5206
1	3	2	2	3	1	084	0.25308	0.25427	1.7298	3.5771	0.25472	1.7001	3.6991
1	3	2	2	3	2	050	0.16122	0.16372	3.8910	2.2336	0.25472	1.7001	3.6991
1	3	2	2	3	3	075	0.25423	0.25492	1.7042	3.6216	0.25472	1.7001	3.6991
1	3	2	2	4	1	056	0.24157	0.24076	1.3156	2.3855	0.24183	1.2900	2.3437
1	3	2	2	4	2	050	0.23766	0.23766	1.3984	2.4376	0.24183	1.2900	2.3437
1	3	2	2	4	3	053	0.24146	0.24123	1.3168	2.4087	0.24183	1.2900	2.3437
1	3	2	2	5	1	074	0.19929	0.19967	2.7769	0.6455	0.22743	2.4993	4.4995
1	3	2	2	5	2	081	0.19872	0.19890	2.7427	0.6043	0.22743	2.4993	4.4995
1	3	2	2	5	3	076	0.21977	0.22019	2.2116	4.2912	0.22743	2.4993	4.4995
1	3	2	2	6	1	047	0.56430	0.56408	2.3755	3.3995	0.63673	4.5000	3.5000
1	3	2	2	6	2	050	0.56883	0.56827	2.3662	3.4636	0.63673	4.5000	3.5000
1	3	2	2	6	3	044	0.56567	0.56626	2.3366	3.4693	0.63673	4.5000	3.5000
1	3	2	2	7	1	056	0.75722	0.75723	0.9884	2.0506	0.76176	1.0035	2.1000
1	3	2	2	7	2	059	0.62282	0.62361	2.1378	2.2915	0.76176	1.0035	2.1000
1	3	2	2	7	3	053	0.76096	0.76139	0.9880	2.1169	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	1	2	1	2	1	089	0.13766	0.13956	3.9073	1.2623	0.16009	2.0250	2.8625
2	1	2	1	2	2	076	0.15707	0.15840	1.8331	2.7931	0.16009	2.0250	2.8625
2	1	2	1	2	3	072	0.14185	0.14327	3.9585	1.0263	0.16009	2.0250	2.8625
2	1	2	1	3	1	074	0.20661	0.20621	5.0049	3.8309	0.25472	4.1992	3.7998
2	1	2	1	3	2	090	0.25159	0.25165	4.2682	3.7262	0.25472	4.1992	3.7998
2	1	2	1	3	3	120	0.25337	0.25515	4.3208	3.7825	0.25472	4.1992	3.7998
2	1	2	1	4	1	097	0.23922	0.24099	2.8698	4.2881	0.24183	2.8200	4.2000
2	1	2	1	4	2	102	0.23807	0.23950	2.5328	4.1956	0.24183	2.8200	4.2000
2	1	2	1	4	3	106	0.23718	0.23954	2.7422	4.1127	0.24183	2.8200	4.2000
2	1	2	1	5	1	123	0.20521	0.20673	4.7739	3.5562	0.22743	5.0000	3.0005
2	1	2	1	5	2	068	0.13597	0.13686	1.8097	3.8567	0.22743	5.0000	3.0005
2	1	2	1	5	3	105	0.19898	0.20049	1.1725	2.7857	0.22743	5.0000	3.0005
2	1	2	1	6	1	147	0.63155	0.63321	4.0531	0.9644	0.63673	4.0000	1.0000
2	1	2	1	6	2	067	0.11868	0.11917	4.1812	0.1016	0.63673	4.0000	1.0000
2	1	2	1	6	3	052	0.01282	0.01415	5.3962	1.0937	0.63673	4.0000	1.0000
2	1	2	1	7	1	063	0.32588	0.32704	4.4680	3.4769	0.76175	2.5986	4.4951
2	1	2	1	7	2	099	0.61971	0.62126	2.7487	3.3537	0.76175	2.5986	4.4951
2	1	2	1	7	3	074	0.24338	0.24372	4.0219	3.1741	0.76175	2.5986	4.4951
2	1	2	2	2	1	060	0.12180	0.12288	4.1813	3.9009	0.16009	2.6377	1.5206
2	1	2	2	2	2	089	0.15577	0.15771	2.8576	1.6319	0.16009	2.6377	1.5206
2	1	2	2	2	3	080	0.14186	0.14328	4.5111	3.4844	0.16009	2.6377	1.5206
2	1	2	2	3	1	077	0.24812	0.24882	1.8124	3.6397	0.25472	1.7001	3.6991
2	1	2	2	3	2	066	0.22434	0.22559	1.8262	4.2450	0.25472	1.7001	3.6991
2	1	2	2	3	3	092	0.25423	0.25551	1.7250	3.7456	0.25472	1.7001	3.6991
2	1	2	2	4	1	096	0.18769	0.18857	1.3125	0.7394	0.24183	1.2900	2.3437
2	1	2	2	4	2	078	0.17934	0.18027	1.3136	0.6188	0.24183	1.2900	2.3437
2	1	2	2	4	3	086	0.18308	0.18408	1.3125	0.6715	0.24183	1.2900	2.3437
2	1	2	2	5	1	076	0.17280	0.17355	3.2692	1.1962	0.22743	2.4993	4.4995
2	1	2	2	5	2	092	0.21717	0.21892	2.5601	4.3592	0.22743	2.4993	4.4995
2	1	2	2	5	3	066	0.12515	0.12654	1.4718	1.5231	0.22743	2.4993	4.4995
2	1	2	2	6	1	061	0.44012	0.44012	4.1813	3.7889	0.63673	4.5000	3.5000
2	1	2	2	6	2	110	0.44924	0.45160	4.1812	3.7704	0.63673	4.5000	3.5000
2	1	2	2	6	3	076	0.20931	0.21098	3.9224	3.9811	0.63673	4.5000	3.5000
2	1	2	2	7	1	102	0.61042	0.61090	2.1602	2.1484	0.76176	1.0035	2.1000
2	1	2	2	7	2	076	0.27623	0.27833	1.7238	4.3055	0.76176	1.0035	2.1000
2	1	2	2	7	3	068	0.44222	0.44366	2.2991	1.3101	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	2	2	1	2	1	210	0.15995	0.16255	2.0250	2.9172	0.16009	2.0250	2.8625
2	2	2	1	2	2	217	0.16007	0.16163	2.0025	2.8562	0.16009	2.0250	2.8625
2	2	2	1	2	3	190	0.15982	0.16151	2.0325	2.7953	0.16009	2.0250	2.8625
2	2	2	1	3	1	216	0.25439	0.25664	4.1550	3.7812	0.25472	4.1992	3.7998
2	2	2	1	3	2	207	0.25420	0.25688	4.1250	3.7875	0.25472	4.1992	3.7998
2	2	2	1	3	3	245	0.25372	0.25563	4.3069	3.7891	0.25472	4.1992	3.7998
2	2	2	1	4	1	250	0.24127	0.24335	2.9475	4.1937	0.24183	2.8200	4.2000
2	2	2	1	4	2	309	0.24172	0.24382	2.8125	4.2250	0.24183	2.8200	4.2000
2	2	2	1	4	3	262	0.24122	0.24342	2.9250	4.1750	0.24183	2.8200	4.2000
2	2	2	1	5	1	211	0.22424	0.22629	4.8750	3.2000	0.22743	5.0000	3.0005
2	2	2	1	5	2	259	0.22730	0.22934	4.9725	3.0250	0.22743	5.0000	3.0005
2	2	2	1	5	3	222	0.22672	0.22908	4.9350	3.0625	0.22743	5.0000	3.0005
2	2	2	1	6	1	170	0.63654	0.63873	4.0012	0.9875	0.63673	4.0000	1.0000
2	2	2	1	6	2	160	0.63651	0.63810	4.0050	1.0125	0.63673	4.0000	1.0000
2	2	2	1	6	3	210	0.63651	0.63884	4.0050	1.0125	0.63673	4.0000	1.0000
2	2	2	1	7	1	213	0.62907	0.63147	2.9400	3.3875	0.76175	2.5986	4.4951
2	2	2	1	7	2	250	0.62961	0.63152	2.9859	3.4000	0.76175	2.5986	4.4951
2	2	2	1	7	3	252	0.62935	0.63146	2.9625	3.4000	0.76175	2.5986	4.4951
2	2	2	2	2	1	181	0.15980	0.16330	2.6925	1.4687	0.16009	2.6377	1.5206
2	2	2	2	2	2	217	0.16007	0.16181	2.6625	1.5187	0.16009	2.6377	1.5206
2	2	2	2	2	3	250	0.15956	0.16207	2.6475	1.5937	0.16009	2.6377	1.5206
2	2	2	2	3	1	223	0.25470	0.25709	1.6931	3.7000	0.25472	1.7001	3.6991
2	2	2	2	3	2	238	0.25416	0.25598	1.6716	3.6562	0.25472	1.7001	3.6991
2	2	2	2	3	3	157	0.25363	0.25598	1.6875	3.5875	0.25472	1.7001	3.6991
2	2	2	2	4	1	228	0.24173	0.24326	1.2900	2.2937	0.24183	1.2900	2.3437
2	2	2	2	4	2	299	0.24141	0.24457	1.3116	2.2782	0.24183	1.2900	2.3437
2	2	2	2	4	3	261	0.24170	0.24377	1.2895	2.3922	0.24183	1.2900	2.3437
2	2	2	2	5	1	268	0.22689	0.22911	2.5800	4.5250	0.22743	2.4993	4.4995
2	2	2	2	5	2	197	0.22705	0.22858	2.4300	4.4750	0.22743	2.4993	4.4995
2	2	2	2	5	3	236	0.22730	0.22871	2.4600	4.4875	0.22743	2.4993	4.4995
2	2	2	2	6	1	230	0.63647	0.63774	4.4925	3.4875	0.63673	4.5000	3.5000
2	2	2	2	6	2	249	0.63644	0.63805	4.4869	3.5078	0.63673	4.5000	3.5000
2	2	2	2	6	3	238	0.63647	0.63807	4.5075	3.5125	0.63673	4.5000	3.5000
2	2	2	2	7	1	278	0.76113	0.76296	1.0087	2.0719	0.76176	1.0035	2.1000
2	2	2	2	7	2	206	0.62914	0.63130	2.0930	2.4691	0.76176	1.0035	2.1000
2	2	2	2	7	3	178	0.62931	0.63120	2.0925	2.5437	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	3	2	1	2	1	112	0.15972	0.16175	2.0225	2.9475	0.16009	2.0250	2.8625
2	3	2	1	2	2	124	0.15961	0.16150	1.9916	2.7617	0.16009	2.0250	2.8625
2	3	2	1	2	3	133	0.15866	0.16109	1.9201	2.7202	0.16009	2.0250	2.8625
2	3	2	1	3	1	148	0.25444	0.25622	4.1551	3.7842	0.25472	4.1992	3.7998
2	3	2	1	3	2	130	0.25462	0.25477	4.1783	3.8115	0.25472	4.1992	3.7998
2	3	2	1	3	3	142	0.25337	0.25479	4.1982	3.7482	0.25472	4.1992	3.7998
2	3	2	1	4	1	109	0.24119	0.24244	2.7648	4.1831	0.24183	2.8200	4.2000
2	3	2	1	4	2	103	0.24119	0.24004	2.8606	4.1675	0.24183	2.8200	4.2000
2	3	2	1	4	3	130	0.24155	0.24260	2.9135	4.2124	0.24183	2.8200	4.2000
2	3	2	1	5	1	130	0.22689	0.22688	4.9651	3.0853	0.22743	5.0000	3.0005
2	3	2	1	5	2	115	0.21888	0.21860	4.7745	3.2053	0.22743	5.0000	3.0005
2	3	2	1	5	3	115	0.22461	0.22469	4.9112	3.1961	0.22743	5.0000	3.0005
2	3	2	1	6	1	118	0.63507	0.63500	4.0025	1.0362	0.63673	4.0000	1.0000
2	3	2	1	6	2	112	0.63262	0.63494	3.9470	0.9790	0.63673	4.0000	1.0000
2	3	2	1	6	3	130	0.63430	0.63450	3.9809	0.9606	0.63673	4.0000	1.0000
2	3	2	1	7	1	157	0.75938	0.75974	2.5605	4.4707	0.76175	2.5986	4.4951
2	3	2	1	7	2	157	0.76141	0.76078	2.5794	4.4899	0.76175	2.5986	4.4951
2	3	2	1	7	3	172	0.70940	0.70953	2.3942	4.3800	0.76175	2.5986	4.4951
2	3	2	2	2	1	124	0.15952	0.16110	2.6682	1.4392	0.16009	2.6377	1.5206
2	3	2	2	2	2	124	0.15988	0.16188	2.6072	1.5704	0.16009	2.6377	1.5206
2	3	2	2	2	3	112	0.15979	0.16020	2.5617	1.5181	0.16009	2.6377	1.5206
2	3	2	2	3	1	136	0.25458	0.25508	1.6915	3.7356	0.25472	1.7001	3.6991
2	3	2	2	3	2	175	0.25443	0.25456	1.7161	3.6547	0.25472	1.7001	3.6991
2	3	2	2	3	3	227	0.25220	0.25313	1.6662	3.5442	0.25472	1.7001	3.6991
2	3	2	2	4	1	130	0.22457	0.22403	1.4706	2.8146	0.24183	1.2900	2.3437
2	3	2	2	4	2	103	0.16074	0.16126	4.4756	0.4941	0.24183	1.2900	2.3437
2	3	2	2	4	3	142	0.22358	0.22367	1.3899	2.8904	0.24183	1.2900	2.3437
2	3	2	2	5	1	124	0.22639	0.22680	2.5104	4.4510	0.22743	2.4993	4.4995
2	3	2	2	5	2	133	0.22557	0.22649	2.5999	4.6042	0.22743	2.4993	4.4995
2	3	2	2	5	3	133	0.22452	0.22667	2.6678	4.5326	0.22743	2.4993	4.4995
2	3	2	2	6	1	178	0.63648	0.63851	4.4876	3.4930	0.63673	4.5000	3.5000
2	3	2	2	6	2	154	0.63640	0.63530	4.4839	3.5027	0.63673	4.5000	3.5000
2	3	2	2	6	3	169	0.63625	0.63878	4.4842	3.4883	0.63673	4.5000	3.5000
2	3	2	2	7	1	133	0.76173	0.76196	1.0066	2.0958	0.76176	1.0035	2.1000
2	3	2	2	7	2	136	0.76083	0.76214	1.0103	2.0657	0.76176	1.0035	2.1000
2	3	2	2	7	3	163	0.76169	0.76353	1.0073	2.0911	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	1	2	1	2	1	046	0.15901	0.16180	1.9524	2.9487	0.16009	2.0250	2.8625
3	1	2	1	2	2	039	0.15578	0.15715	1.8253	2.6467	0.16009	2.0250	2.8625
3	1	2	1	2	3	037	0.15087	0.15217	1.7338	2.5440	0.16009	2.0250	2.8625
3	1	2	1	3	1	067	0.25351	0.25520	4.3077	3.7776	0.25472	4.1992	3.7998
3	1	2	1	3	2	058	0.25195	0.25280	4.0219	3.7778	0.25472	4.1992	3.7998
3	1	2	1	3	3	098	0.25159	0.25256	4.3266	3.7400	0.25472	4.1992	3.7998
3	1	2	1	4	1	032	0.13306	0.13417	4.2183	2.8458	0.24183	2.8200	4.2000
3	1	2	1	4	2	088	0.23917	0.24065	2.6852	4.2825	0.24183	2.8200	4.2000
3	1	2	1	4	3	032	0.21941	0.21967	3.4429	4.1484	0.24183	2.8200	4.2000
3	1	2	1	5	1	081	0.22688	0.22817	5.0568	2.9354	0.22743	5.0000	3.0005
3	1	2	1	5	2	054	0.20546	0.20663	4.6273	3.3859	0.22743	5.0000	3.0005
3	1	2	1	5	3	061	0.15923	0.16091	2.2048	2.0033	0.22743	5.0000	3.0005
3	1	2	1	6	1	034	0.24958	0.25021	2.6249	2.6912	0.63673	4.0000	1.0000
3	1	2	1	6	2	039	0.25439	0.25694	2.6606	2.6854	0.63673	4.0000	1.0000
3	1	2	1	6	3	094	0.63523	0.63649	3.9738	1.0224	0.63673	4.0000	1.0000
3	1	2	1	7	1	041	0.62484	0.62557	3.1300	3.3956	0.76175	2.5986	4.4951
3	1	2	1	7	2	060	0.62801	0.62790	2.9006	3.3907	0.76175	2.5986	4.4951
3	1	2	1	7	3	038	0.34027	0.34309	1.4824	3.2507	0.76175	2.5986	4.4951
3	1	2	2	2	1	076	0.14033	0.14243	1.9884	1.5974	0.16009	2.6377	1.5206
3	1	2	2	2	2	097	0.15852	0.16052	2.8188	1.4762	0.16009	2.6377	1.5206
3	1	2	2	2	3	069	0.15948	0.16136	2.5247	1.5538	0.16009	2.6377	1.5206
3	1	2	2	3	1	049	0.18492	0.18589	1.8694	2.7708	0.25472	1.7001	3.6991
3	1	2	2	3	2	086	0.25297	0.25452	1.7531	3.6345	0.25472	1.7001	3.6991
3	1	2	2	3	3	058	0.25086	0.25157	1.7598	3.8589	0.25472	1.7001	3.6991
3	1	2	2	4	1	080	0.24144	0.24427	1.2704	2.3970	0.24183	1.2900	2.3437
3	1	2	2	4	2	052	0.24109	0.24310	1.2749	2.2098	0.24183	1.2900	2.3437
3	1	2	2	4	3	063	0.20233	0.20473	1.6020	3.0380	0.24183	1.2900	2.3437
3	1	2	2	5	1	048	0.19926	0.20040	2.7436	0.6556	0.22743	2.4993	4.4995
3	1	2	2	5	2	046	0.22366	0.22484	2.3933	4.3567	0.22743	2.4993	4.4995
3	1	2	2	5	3	065	0.21787	0.21871	2.1398	4.2990	0.22743	2.4993	4.4995
3	1	2	2	6	1	043	0.34608	0.34738	2.3774	2.7988	0.63673	4.5000	3.5000
3	1	2	2	6	2	040	0.25062	0.25302	2.7595	2.1120	0.63673	4.5000	3.5000
3	1	2	2	6	3	044	0.34232	0.34285	2.0244	2.5516	0.63673	4.5000	3.5000
3	1	2	2	7	1	051	0.62426	0.62574	2.1144	2.3237	0.76176	1.0035	2.1000
3	1	2	2	7	2	043	0.61891	0.61998	2.1530	2.2419	0.76176	1.0035	2.1000
3	1	2	2	7	3	048	0.62896	0.63053	2.1089	2.5257	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	2	2	1	2	1	059	0.15688	0.15891	1.9200	2.6125	0.16009	2.0250	2.8625
3	2	2	1	2	2	074	0.15787	C.16000	2.0100	2.6500	0.16009	2.0250	2.8625
3	2	2	1	2	3	074	0.15974	0.16138	1.9650	2.8000	0.16009	2.0250	2.8625
3	2	2	1	3	1	118	0.25444	0.25612	4.1400	3.8000	0.25472	4.1992	3.7998
3	2	2	1	3	2	143	0.25447	0.25654	4.1437	3.8008	0.25472	4.1992	3.7998
3	2	2	1	3	3	130	0.25169	0.25378	4.0481	3.8484	0.25472	4.1992	3.7998
3	2	2	1	4	1	067	0.24061	0.24253	2.8650	4.2625	0.24183	2.8200	4.2000
3	2	2	1	4	2	095	0.24105	0.24337	2.7375	4.1875	0.24183	2.8200	4.2000
3	2	2	1	4	3	080	0.24060	0.24279	2.8669	4.2625	0.24183	2.8200	4.2000
3	2	2	1	5	1	091	0.19946	0.20109	1.1700	2.7000	0.22743	5.0000	3.0005
3	2	2	1	5	2	140	0.22695	0.22888	4.9496	3.0750	0.22743	5.0000	3.0005
3	2	2	1	5	3	104	0.19943	0.20140	1.1709	2.7492	0.22743	5.0000	3.0005
3	2	2	1	6	1	100	0.56981	0.57275	3.9787	3.1031	0.63673	4.0000	1.0000
3	2	2	1	6	2	079	0.25000	0.25161	2.5795	2.7000	0.63673	4.0000	1.0000
3	2	2	1	6	3	168	0.63543	0.63804	4.0200	0.9750	0.63673	4.0000	1.0000
3	2	2	1	7	1	085	0.62888	0.63144	3.0150	3.3875	0.76175	2.5986	4.4951
3	2	2	1	7	2	099	0.62936	0.63115	3.0151	3.4125	0.76175	2.5986	4.4951
3	2	2	1	7	3	079	0.62904	0.63158	2.9850	3.4125	0.76175	2.5986	4.4951
3	2	2	2	2	1	080	0.15981	0.16154	2.6100	1.4750	0.16009	2.6377	1.5206
3	2	2	2	2	2	064	0.15725	0.15939	2.8801	1.4500	0.16009	2.6377	1.5206
3	2	2	2	2	3	070	0.15983	0.16215	2.5800	1.5000	0.16009	2.6377	1.5206
3	2	2	2	3	1	132	0.25452	C.25511	1.6875	3.7375	0.25472	1.7001	3.6991
3	2	2	2	3	2	121	0.25372	0.25576	1.7400	3.6500	0.25472	1.7001	3.6991
3	2	2	2	3	3	132	0.25388	0.25541	1.6800	3.7875	0.25472	1.7001	3.6991
3	2	2	2	4	1	062	0.23770	0.24105	1.2150	2.4875	0.24183	1.2900	2.3437
3	2	2	2	4	2	081	0.24152	C.24337	1.2750	2.2625	0.24183	1.2900	2.3437
3	2	2	2	4	3	059	0.24059	0.24198	1.3350	2.2625	0.24183	1.2900	2.3437
3	2	2	2	5	1	108	0.19869	0.20123	2.7600	0.6000	0.22743	2.4993	4.4995
3	2	2	2	5	2	103	0.22536	0.22771	2.3419	4.4500	0.22743	2.4993	4.4995
3	2	2	2	5	3	110	0.22584	0.22822	2.3991	4.5000	0.22743	2.4993	4.4995
3	2	2	2	6	1	088	0.56965	0.57074	2.4000	3.5125	0.63673	4.5000	3.5000
3	2	2	2	6	2	105	0.56994	0.57157	2.3967	3.4871	0.63673	4.5000	3.5000
3	2	2	2	6	3	084	0.56797	0.57030	2.3699	3.5125	0.63673	4.5000	3.5000
3	2	2	2	7	1	079	0.62874	0.63014	2.1075	2.4250	0.76176	1.0035	2.1000
3	2	2	2	7	2	093	0.62963	0.63181	2.1000	2.4930	0.76176	1.0035	2.1000
3	2	2	2	7	3	083	0.62936	0.63185	2.1000	2.5312	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	3	2	1	2	1	047	0.15918	0.16121	2.0245	2.7302	0.16009	2.0250	2.8625
3	3	2	1	2	2	038	0.15945	0.16018	1.9384	2.8636	0.16009	2.0250	2.8625
3	3	2	1	2	3	044	0.16003	0.16038	2.0419	2.8469	0.16009	2.0250	2.8625
3	3	2	1	3	1	077	0.25034	0.25203	4.0021	3.8497	0.25472	4.1992	3.7998
3	3	2	1	3	2	078	0.25458	0.25316	4.2379	3.7938	0.25472	4.1992	3.7998
3	3	2	1	3	3	095	0.25175	0.25369	4.0802	3.7395	0.25472	4.1992	3.7998
3	3	2	1	4	1	053	0.20925	0.20862	3.5394	4.0003	0.24183	2.8200	4.2000
3	3	2	1	4	2	047	0.18787	0.18782	3.7611	4.0169	0.24183	2.8200	4.2000
3	3	2	1	4	3	053	0.24070	0.24000	2.9786	4.1739	0.24183	2.8200	4.2000
3	3	2	1	5	1	038	0.19353	0.19258	1.3875	2.5220	0.22743	5.0000	3.0005
3	3	2	1	5	2	044	0.19948	0.20143	1.2134	2.7132	0.22743	5.0000	3.0005
3	3	2	1	5	3	056	0.19873	0.20083	1.1917	2.6412	0.22743	5.0000	3.0005
3	3	2	1	6	1	050	0.56937	0.56959	4.0018	3.0809	0.63673	4.0000	1.0000
3	3	2	1	6	2	041	0.56485	0.56360	3.9029	3.1298	0.63673	4.0000	1.0000
3	3	2	1	6	3	047	0.56939	0.57058	4.0202	3.0879	0.63673	4.0000	1.0000
3	3	2	1	7	1	053	0.75977	0.75843	2.5575	4.4753	0.76175	2.5986	4.4951
3	3	2	1	7	2	053	0.76077	0.75968	2.5810	4.4788	0.76175	2.5986	4.4951
3	3	2	1	7	3	053	0.76004	0.76002	2.5539	4.4842	0.76175	2.5986	4.4951
3	3	2	2	2	1	038	0.15857	0.15800	2.7630	1.5882	0.16009	2.6377	1.5206
3	3	2	2	2	2	047	0.15951	0.15957	2.5493	1.5876	0.16009	2.6377	1.5206
3	3	2	2	2	3	038	0.15996	0.16072	2.6572	1.4805	0.16009	2.6377	1.5206
3	3	2	2	3	1	096	0.25400	0.25492	1.6957	3.6050	0.25472	1.7001	3.6991
3	3	2	2	3	2	091	0.25360	0.25377	1.6725	3.6037	0.25472	1.7001	3.6991
3	3	2	2	3	3	107	0.25360	0.25390	1.6534	3.6881	0.25472	1.7001	3.6991
3	3	2	2	4	1	050	0.23847	0.23892	1.3736	2.5321	0.24183	1.2900	2.3437
3	3	2	2	4	2	056	0.23960	0.24217	1.3223	2.5464	0.24183	1.2900	2.3437
3	3	2	2	4	3	053	0.24096	0.24150	1.3359	2.4238	0.24183	1.2900	2.3437
3	3	2	2	5	1	074	0.19842	0.19965	2.8438	0.6413	0.22743	2.4993	4.4995
3	3	2	2	5	2	151	0.22663	0.22620	2.4152	4.4310	0.22743	2.4993	4.4995
3	3	2	2	5	3	085	0.22686	0.22703	2.4170	4.4451	0.22743	2.4993	4.4995
3	3	2	2	6	1	053	0.56705	0.56786	2.3628	3.4266	0.63673	4.5000	3.5000
3	3	2	2	6	2	053	0.56840	0.56906	2.4078	3.4577	0.63673	4.5000	3.5000
3	3	2	2	6	3	047	0.56740	0.56631	2.3972	3.4368	0.63673	4.5000	3.5000
3	3	2	2	7	1	050	0.76003	0.76048	0.9972	2.1449	0.76176	1.0035	2.1000
3	3	2	2	7	2	047	0.76135	0.76182	1.0136	2.0811	0.76176	1.0035	2.1000
3	3	2	2	7	3	047	0.75987	0.76078	1.0106	2.0519	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	1	3	1	2	1	044	0.15290	0.16174	1.7333	2.6901	0.16009	2.0250	2.8625
1	1	3	1	2	2	057	0.15930	0.16220	2.0577	2.9908	0.16009	2.0250	2.8625
1	1	3	1	2	3	045	0.13722	0.15233	2.2048	2.2219	0.16009	2.0250	2.8625
1	1	3	1	3	1	048	0.13312	0.14709	2.2551	2.0995	0.25472	4.1992	3.7998
1	1	3	1	3	2	053	0.23364	0.24375	3.6772	3.8069	0.25472	4.1992	3.7998
1	1	3	1	3	3	060	0.23779	0.24299	4.6472	3.8485	0.25472	4.1992	3.7998
1	1	3	1	4	1	104	0.23626	0.24184	3.1363	4.2228	0.24183	2.8200	4.2000
1	1	3	1	4	2	059	0.12480	0.13292	4.8250	2.0476	0.24183	2.8200	4.2000
1	1	3	1	4	3	053	0.13310	0.14046	4.2361	2.8667	0.24183	2.8200	4.2000
1	1	3	1	5	1	035	0.15747	0.16988	2.3966	1.8008	0.22743	5.0000	3.0005
1	1	3	1	5	2	089	0.18196	0.19289	0.7948	2.8817	0.22743	5.0000	3.0005
1	1	3	1	5	3	032	0.18501	0.19158	1.0219	2.4875	0.22743	5.0000	3.0005
1	1	3	1	6	1	076	0.60384	0.60775	4.0314	1.1599	0.63673	4.0000	1.0000
1	1	3	1	6	2	108	0.60177	0.60691	3.9223	0.8510	0.63673	4.0000	1.0000
1	1	3	1	6	3	073	0.62930	0.63786	3.9346	0.9600	0.63673	4.0000	1.0000
1	1	3	1	7	1	037	0.58956	0.60030	3.4874	3.5160	0.76175	2.5986	4.4951
1	1	3	1	7	2	031	0.59730	0.59819	3.4098	3.5156	0.76175	2.5986	4.4951
1	1	3	1	7	3	066	0.76174	0.76992	2.6028	4.4951	0.76175	2.5986	4.4951
1	1	3	2	2	1	032	0.10728	0.11577	3.0298	2.3675	0.16009	2.6377	1.5206
1	1	3	2	2	2	048	0.15940	0.16363	2.5233	1.5723	0.16009	2.6377	1.5206
1	1	3	2	2	3	075	0.13310	0.14166	4.8517	3.6107	0.16009	2.6377	1.5206
1	1	3	2	3	1	056	0.25044	0.25932	1.6080	3.6969	0.25472	1.7001	3.6991
1	1	3	2	3	2	044	0.24057	0.24786	1.7113	3.2754	0.25472	1.7001	3.6991
1	1	3	2	3	3	047	0.25067	0.25261	1.6206	3.5959	0.25472	1.7001	3.6991
1	1	3	2	4	1	066	0.23567	0.24995	1.3549	2.0219	0.24183	1.2900	2.3437
1	1	3	2	4	2	071	0.23068	0.24445	1.1348	2.0605	0.24183	1.2900	2.3437
1	1	3	2	4	3	094	0.20565	0.21771	1.0813	1.4499	0.24183	1.2900	2.3437
1	1	3	2	5	1	078	0.19645	0.20418	2.9309	0.7166	0.22743	2.4993	4.4995
1	1	3	2	5	2	065	0.17453	0.18138	1.6502	4.2319	0.22743	2.4993	4.4995
1	1	3	2	5	3	058	0.14334	0.15330	1.7858	0.9995	0.22743	2.4993	4.4995
1	1	3	2	6	1	059	0.52821	0.53216	2.3436	3.2332	0.63673	4.5000	3.5000
1	1	3	2	6	2	062	0.55646	0.55811	2.4284	3.6325	0.63673	4.5000	3.5000
1	1	3	2	6	3	085	0.56802	0.57074	2.4334	3.4933	0.63673	4.5000	3.5000
1	1	3	2	7	1	050	0.60459	0.61155	2.1804	2.1030	0.76176	1.0035	2.1000
1	1	3	2	7	2	066	0.76001	0.76362	1.0262	2.0685	0.76176	1.0035	2.1000
1	1	3	2	7	3	062	0.74072	0.75025	0.9703	1.9944	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	2	3	1	2	1	060	0.15647	0.16867	1.9069	2.6000	0.16009	2.0250	2.8625
1	2	3	1	2	2	072	0.15750	0.16769	2.1900	2.8734	0.16009	2.0250	2.8625
1	2	3	1	2	3	072	0.15290	0.16598	2.3119	2.9752	0.16009	2.0250	2.8625
1	2	3	1	3	1	048	0.11761	0.13038	2.3100	2.5000	0.25472	4.1992	3.7998
1	2	3	1	3	2	132	0.25212	0.26303	4.0200	3.7937	0.25472	4.1992	3.7998
1	2	3	1	3	3	114	0.25411	0.26312	4.1119	3.8000	0.25472	4.1992	3.7998
1	2	3	1	4	1	148	0.23925	0.25067	2.9100	4.1250	0.24183	2.8200	4.2000
1	2	3	1	4	2	134	0.23781	0.24682	3.0600	4.2499	0.24183	2.8200	4.2000
1	2	3	1	4	3	047	0.13581	0.14238	4.0950	3.3125	0.24183	2.8200	4.2000
1	2	3	1	5	1	066	0.16022	0.17162	2.1301	2.0249	0.22743	5.0000	3.0005
1	2	3	1	5	2	058	0.16451	0.17348	1.9500	2.1250	0.22743	5.0000	3.0005
1	2	3	1	5	3	056	0.15783	0.16474	2.3100	1.9750	0.22743	5.0000	3.0005
1	2	3	1	6	1	128	0.56966	0.57965	3.9750	3.1008	0.63673	4.0000	1.0000
1	2	3	1	6	2	123	0.56685	0.57894	3.9900	3.1500	0.63673	4.0000	1.0000
1	2	3	1	6	3	161	0.63623	0.64765	4.0200	0.9992	0.63673	4.0000	1.0000
1	2	3	1	7	1	088	0.62935	0.64163	3.0149	3.4126	0.76175	2.5986	4.4951
1	2	3	1	7	2	069	0.62927	0.64129	2.9850	3.3875	0.76175	2.5986	4.4951
1	2	3	1	7	3	213	0.76117	0.77062	2.6212	4.4937	0.76175	2.5986	4.4951
1	2	3	2	2	1	085	0.11708	0.12594	3.6000	1.8500	0.16009	2.6377	1.5206
1	2	3	2	2	2	063	0.14998	0.15701	2.1750	1.6750	0.16009	2.6377	1.5206
1	2	3	2	2	3	049	0.14037	0.14805	3.3000	1.4375	0.16009	2.6377	1.5206
1	2	3	2	3	1	073	0.24659	0.25548	1.8000	3.5000	0.25472	1.7001	3.6991
1	2	3	2	3	2	073	0.25405	0.26987	1.6800	3.7750	0.25472	1.7001	3.6991
1	2	3	2	3	3	103	0.22680	0.24170	1.6800	4.3000	0.25472	1.7001	3.6991
1	2	3	2	4	1	123	0.24132	0.25167	1.3198	2.4250	0.24183	1.2900	2.3437
1	2	3	2	4	2	141	0.23087	0.24011	1.4400	2.7001	0.24183	1.2900	2.3437
1	2	3	2	4	3	239	0.23368	0.24312	1.3050	2.7125	0.24183	1.2900	2.3437
1	2	3	2	5	1	064	0.15793	0.16656	3.6937	1.9219	0.22743	2.4993	4.4995
1	2	3	2	5	2	109	0.21597	0.22814	2.1000	4.3000	0.22743	2.4993	4.4995
1	2	3	2	5	3	071	0.15836	0.16808	3.6900	2.0251	0.22743	2.4993	4.4995
1	2	3	2	6	1	113	0.56046	0.57124	2.4000	3.6000	0.63673	4.5000	3.5000
1	2	3	2	6	2	097	0.56863	0.58045	2.3700	3.5000	0.63673	4.5000	3.5000
1	2	3	2	6	3	124	0.56965	0.57566	2.4000	3.5125	0.63673	4.5000	3.5000
1	2	3	2	7	1	088	0.62711	0.63733	2.1300	2.4937	0.76176	1.0035	2.1000
1	2	3	2	7	2	145	0.74829	0.76221	0.9450	2.2000	0.76176	1.0035	2.1000
1	2	3	2	7	3	140	0.74235	0.75191	1.0800	2.0001	0.76176	1.0035	2.1000

TABLE XIX (Continued)

1	3	3	1	2	1	065	0.15459	0.16331	1.7841	2.8828	0.16009	2.0250	2.8625
1	3	3	1	2	2	065	0.15980	0.16733	1.9633	2.8539	0.16009	2.0250	2.8625
1	3	3	1	2	3	065	0.12397	0.11884	2.1498	3.7682	0.16009	2.0250	2.8625
1	3	3	1	3	1	065	0.24844	0.25970	4.0104	3.7170	0.25472	4.1992	3.7998
1	3	3	1	3	2	089	0.24955	0.26046	3.9532	3.8250	0.25472	4.1992	3.7998
1	3	3	1	3	3	102	0.25407	0.25514	4.1241	3.8196	0.25472	4.1992	3.7998
1	3	3	1	4	1	161	0.23337	0.24959	2.3228	4.2419	0.24183	2.8200	4.2000
1	3	3	1	4	2	102	0.21747	0.21894	1.8198	4.2396	0.24183	2.8200	4.2000
1	3	3	1	4	3	065	0.17906	0.18603	3.6055	3.7091	0.24183	2.8200	4.2000
1	3	3	1	5	1	065	0.16101	0.16140	2.0546	2.1986	0.22743	5.0000	3.0005
1	3	3	1	5	2	109	0.19693	0.20464	1.3244	2.7436	0.22743	5.0000	3.0005
1	3	3	1	5	3	032	0.18947	0.19157	1.2243	2.4500	0.22743	5.0000	3.0005
1	3	3	1	6	1	059	0.56451	0.56821	3.8998	3.1414	0.63673	4.0000	1.0000
1	3	3	1	6	2	056	0.56657	0.57980	3.9248	3.1091	0.63673	4.0000	1.0000
1	3	3	1	6	3	062	0.56782	0.56522	3.9715	3.1463	0.63673	4.0000	1.0000
1	3	3	1	7	1	059	0.74702	0.74688	2.5414	4.5395	0.76175	2.5986	4.4951
1	3	3	1	7	2	059	0.75404	0.75721	2.5918	4.4500	0.76175	2.5986	4.4951
1	3	3	1	7	3	105	0.76015	0.76508	2.5653	4.4759	0.76175	2.5986	4.4951
1	3	3	2	2	1	065	0.15526	0.16772	2.6534	1.7472	0.16009	2.6377	1.5206
1	3	3	2	2	2	065	0.14771	0.14760	3.0652	1.6772	0.16009	2.6377	1.5206
1	3	3	2	2	3	065	0.15673	0.16996	2.7756	1.3290	0.16009	2.6377	1.5206
1	3	3	2	3	1	100	0.25386	0.26160	1.6703	3.6283	0.25472	1.7001	3.6991
1	3	3	2	3	2	111	0.24854	0.25355	1.5927	3.6306	0.25472	1.7001	3.6991
1	3	3	2	3	3	117	0.25251	0.25205	1.6617	3.8338	0.25472	1.7001	3.6991
1	3	3	2	4	1	123	0.24128	0.24320	1.2634	2.2483	0.24183	1.2900	2.3437
1	3	3	2	4	2	123	0.23618	0.25060	1.1795	2.1413	0.24183	1.2900	2.3437
1	3	3	2	4	3	062	0.23719	0.24648	1.1835	2.3625	0.24183	1.2900	2.3437
1	3	3	2	5	1	065	0.15748	0.16904	3.8932	2.4958	0.22743	2.4993	4.4995
1	3	3	2	5	2	146	0.17571	0.18460	2.4554	0.3159	0.22743	2.4993	4.4995
1	3	3	2	5	3	053	0.15961	0.16808	3.9231	2.4406	0.22743	2.4993	4.4995
1	3	3	2	6	1	053	0.56210	0.57061	2.4794	3.5501	0.63673	4.5000	3.5000
1	3	3	2	6	2	062	0.56462	0.56880	2.3318	3.4246	0.63673	4.5000	3.5000
1	3	3	2	6	3	050	0.53967	0.54292	2.2726	3.2801	0.63673	4.5000	3.5000
1	3	3	2	7	1	041	0.62810	0.62817	2.0983	2.5810	0.76176	1.0035	2.1000
1	3	3	2	7	2	050	0.76111	0.76725	0.9993	2.1274	0.76176	1.0035	2.1000
1	3	3	2	7	3	088	0.76129	0.76518	0.9977	2.1233	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	1	3	1	2	1	081	0.11151	0.11933	4.1443	1.6296	0.16009	2.0250	2.8625
2	1	3	1	2	2	055	0.13204	0.13778	3.7172	1.3928	0.16009	2.0250	2.8625
2	1	3	1	2	3	080	0.12130	0.13130	3.9922	1.5536	0.16009	2.0250	2.8625
2	1	3	1	3	1	065	0.16905	0.18061	5.2901	3.9217	0.25472	4.1992	3.7998
2	1	3	1	3	2	083	0.22147	0.23032	4.8607	3.7933	0.25472	4.1992	3.7998
2	1	3	1	3	3	073	0.22953	0.23595	4.7604	3.7579	0.25472	4.1992	3.7998
2	1	3	1	4	1	084	0.13491	0.14548	0.5678	4.1875	0.24183	2.8200	4.2000
2	1	3	1	4	2	071	0.20171	0.21312	1.7658	4.3961	0.24183	2.8200	4.2000
2	1	3	1	4	3	066	0.18609	0.20030	1.3031	4.1005	0.24183	2.8200	4.2000
2	1	3	1	5	1	065	0.21944	0.22580	4.8116	3.3262	0.22743	5.0000	3.0005
2	1	3	1	5	2	077	0.09113	0.10374	4.6790	0.3515	0.22743	5.0000	3.0005
2	1	3	1	5	3	061	0.22122	0.22797	4.9564	2.8525	0.22743	5.0000	3.0005
2	1	3	1	6	1	062	0.00047	0.01046	0.6335	0.3750	0.63673	4.0000	1.0000
2	1	3	1	6	2	079	0.63195	0.64040	3.9557	0.9574	0.63673	4.0000	1.0000
2	1	3	1	6	3	069	0.63430	0.64399	3.9815	1.0398	0.63673	4.0000	1.0000
2	1	3	1	7	1	051	0.02156	0.03145	5.4087	4.5313	0.76175	2.5986	4.4951
2	1	3	1	7	2	104	0.62230	0.62906	2.7916	3.3806	0.76175	2.5986	4.4951
2	1	3	1	7	3	081	0.62807	0.63686	3.0973	3.4183	0.76175	2.5986	4.4951
2	1	3	2	2	1	063	0.12316	0.13074	4.7330	4.1084	0.16009	2.6377	1.5206
2	1	3	2	2	2	056	0.12915	0.13101	2.3530	2.1484	0.16009	2.6377	1.5206
2	1	3	2	2	3	069	0.06913	0.07164	5.4095	4.6719	0.16009	2.6377	1.5206
2	1	3	2	3	1	069	0.19015	0.20032	1.8251	4.6030	0.25472	1.7001	3.6991
2	1	3	2	3	2	095	0.14639	0.15154	1.6776	2.3500	0.25472	1.7001	3.6991
2	1	3	2	3	3	082	0.14736	0.15412	1.7564	5.0000	0.25472	1.7001	3.6991
2	1	3	2	4	1	072	0.15351	0.15952	1.3103	0.2874	0.24183	1.2900	2.3437
2	1	3	2	4	2	068	0.18698	0.19266	1.3090	0.7283	0.24183	1.2900	2.3437
2	1	3	2	4	3	058	0.09315	0.10114	5.4547	0.6701	0.24183	1.2900	2.3437
2	1	3	2	5	1	062	0.08698	0.09010	5.0342	4.4673	0.22743	2.4993	4.4995
2	1	3	2	5	2	076	0.06695	0.07405	5.4349	4.6142	0.22743	2.4993	4.4995
2	1	3	2	5	3	071	0.13737	0.14463	1.3129	3.7057	0.22743	2.4993	4.4995
2	1	3	2	6	1	090	0.62964	0.63735	4.4493	3.4448	0.63673	4.5000	3.5000
2	1	3	2	6	2	076	0.62363	0.63392	4.3993	3.5165	0.63673	4.5000	3.5000
2	1	3	2	6	3	066	0.00742	0.02164	5.5500	4.5602	0.63673	4.5000	3.5000
2	1	3	2	7	1	073	0.33543	0.34014	1.9038	4.0710	0.76176	1.0035	2.1000
2	1	3	2	7	2	080	0.70624	0.72016	0.9288	2.3618	0.76176	1.0035	2.1000
2	1	3	2	7	3	067	0.00239	0.01802	0.5284	4.5410	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	2	3	1	2	1	178	0.14156	0.15088	3.9450	0.9625	0.16009	2.0250	2.8625
2	2	3	1	2	2	180	0.14419	0.15539	2.3325	2.5219	0.16009	2.0250	2.8625
2	2	3	1	2	3	197	0.15932	0.16915	1.9875	2.9625	0.16009	2.0250	2.8625
2	2	3	1	3	1	199	0.25439	0.26671	4.1850	3.7750	0.25472	4.1992	3.7998
2	2	3	1	3	2	155	0.25204	0.26000	4.0275	3.7750	0.25472	4.1992	3.7998
2	2	3	1	3	3	158	0.24867	0.26060	4.3650	3.7125	0.25472	4.1992	3.7998
2	2	3	1	4	1	148	0.23920	0.25020	2.8725	4.1250	0.24183	2.8200	4.2000
2	2	3	1	4	2	134	0.16057	0.17109	1.0500	0.9734	0.24183	2.8200	4.2000
2	2	3	1	4	3	206	0.22742	0.23392	2.1300	4.1750	0.24183	2.8200	4.2000
2	2	3	1	5	1	184	0.19648	0.20722	1.3200	2.5780	0.22743	5.0000	3.0005
2	2	3	1	5	2	212	0.22139	0.23264	4.9275	3.2625	0.22743	5.0000	3.0005
2	2	3	1	5	3	186	0.21670	0.22967	4.7850	3.3812	0.22743	5.0000	3.0005
2	2	3	1	6	1	140	0.63127	0.64034	4.0200	0.9375	0.63673	4.0000	1.0000
2	2	3	1	6	2	168	0.00759	0.01861	5.4019	0.5000	0.63673	4.0000	1.0000
2	2	3	1	6	3	172	0.63579	0.64321	3.9750	0.9891	0.63673	4.0000	1.0000
2	2	3	1	7	1	219	0.62645	0.63826	2.8799	3.3625	0.76175	2.5986	4.4951
2	2	3	1	7	2	222	0.62727	0.63848	3.0825	3.3937	0.76175	2.5986	4.4951
2	2	3	1	7	3	225	0.62703	0.63567	3.0750	3.4375	0.76175	2.5986	4.4951
2	2	3	2	2	1	244	0.14103	0.15353	4.4400	3.3501	0.16009	2.6377	1.5206
2	2	3	2	2	2	135	0.09803	0.11178	4.8150	4.5000	0.16009	2.6377	1.5206
2	2	3	2	2	3	161	0.14158	0.15325	4.4250	3.4375	0.16009	2.6377	1.5206
2	2	3	2	3	1	186	0.24685	0.26050	1.5750	3.6812	0.25472	1.7001	3.6991
2	2	3	2	3	2	178	0.25370	0.26530	1.6950	3.5875	0.25472	1.7001	3.6991
2	2	3	2	3	3	168	0.23985	0.24919	1.5750	4.0000	0.25472	1.7001	3.6991
2	2	3	2	4	1	182	0.23087	0.23759	1.2909	1.7250	0.24183	1.2900	2.3437
2	2	3	2	4	2	148	0.23802	0.25121	1.2900	2.0187	0.24183	1.2900	2.3437
2	2	3	2	4	3	173	0.22614	0.23569	1.2750	1.5625	0.24183	1.2900	2.3437
2	2	3	2	5	1	219	0.22545	0.23188	2.6550	4.5984	0.22743	2.4993	4.4995
2	2	3	2	5	2	193	0.20311	0.21536	1.9087	4.1953	0.22743	2.4993	4.4995
2	2	3	2	5	3	174	0.22667	0.24664	2.4300	4.5000	0.22743	2.4993	4.4995
2	2	3	2	6	1	214	0.63374	0.64127	4.5469	3.4875	0.63673	4.5000	3.5000
2	2	3	2	6	2	212	0.63643	0.64833	4.5112	3.4891	0.63673	4.5000	3.5000
2	2	3	2	6	3	190	0.63245	0.64676	4.5525	3.4750	0.63673	4.5000	3.5000
2	2	3	2	7	1	197	0.62693	0.63609	2.0625	2.5500	0.76176	1.0035	2.1000
2	2	3	2	7	2	178	0.61911	0.62864	2.0550	2.7562	0.76176	1.0035	2.1000
2	2	3	2	7	3	140	0.00370	0.01525	0.7200	0.3984	0.76176	1.0035	2.1000

TABLE XIX (Continued)

2	3	3	1	2	1	196	0.15935	0.16522	1.9278	2.8315	0.16009	2.0250	2.8625
2	3	3	1	2	2	154	0.15838	0.16435	2.1128	2.7553	0.16009	2.0250	2.8625
2	3	3	1	2	3	163	0.15409	0.16309	1.7630	2.6809	0.16009	2.0250	2.8625
2	3	3	1	3	1	160	0.25311	0.25341	4.2977	3.8403	0.25472	4.1992	3.7998
2	3	3	1	3	2	160	0.24770	0.25178	4.3566	3.6998	0.25472	4.1992	3.7998
2	3	3	1	3	3	178	0.25096	0.25856	4.2863	3.8788	0.25472	4.1992	3.7998
2	3	3	1	4	1	172	0.23661	0.23494	2.5664	4.1456	0.24183	2.8200	4.2000
2	3	3	1	4	2	196	0.23919	0.25023	2.9485	4.1255	0.24183	2.8200	4.2000
2	3	3	1	4	3	172	0.24064	0.24495	2.6863	4.2408	0.24183	2.8200	4.2000
2	3	3	1	5	1	151	0.21405	0.20753	4.7969	3.4331	0.22743	5.0000	3.0005
2	3	3	1	5	2	142	0.22267	0.22351	4.8998	3.2515	0.22743	5.0000	3.0005
2	3	3	1	5	3	160	0.19784	0.20797	4.6716	3.6577	0.22743	5.0000	3.0005
2	3	3	1	6	1	175	0.63630	0.63978	3.9856	0.9882	0.63673	4.0000	1.0000
2	3	3	1	6	2	178	0.56741	0.56282	3.9680	3.1504	0.63673	4.0000	1.0000
2	3	3	1	6	3	178	0.63659	0.64077	3.9948	1.0094	0.63673	4.0000	1.0000
2	3	3	1	7	1	163	0.69077	0.68946	2.2977	4.4310	0.76175	2.5986	4.4951
2	3	3	1	7	2	181	0.62327	0.62773	2.9017	3.3430	0.76175	2.5986	4.4951
2	3	3	1	7	3	169	0.75913	0.76216	2.5436	4.4888	0.76175	2.5986	4.4951
2	3	3	2	2	1	196	0.15636	0.16489	2.6769	1.7130	0.16009	2.6377	1.5206
2	3	3	2	2	2	172	0.15878	0.16427	2.6540	1.3996	0.16009	2.6377	1.5206
2	3	3	2	2	3	187	0.15754	0.16384	2.8035	1.6067	0.16009	2.6377	1.5206
2	3	3	2	3	1	178	0.25158	0.25470	1.6351	3.5880	0.25472	1.7001	3.6991
2	3	3	2	3	2	157	0.25203	0.25743	1.6747	3.8703	0.25472	1.7001	3.6991
2	3	3	2	3	3	184	0.25265	0.25755	1.6616	3.8271	0.25472	1.7001	3.6991
2	3	3	2	4	1	196	0.20588	0.20825	1.4914	3.0842	0.24183	1.2900	2.3437
2	3	3	2	4	2	196	0.23081	0.23398	1.1755	1.8841	0.24183	1.2900	2.3437
2	3	3	2	4	3	190	0.16074	0.16292	4.4999	0.5269	0.24183	1.2900	2.3437
2	3	3	2	5	1	154	0.22652	0.23413	2.5310	4.5633	0.22743	2.4993	4.4995
2	3	3	2	5	2	196	0.20003	0.20164	1.8705	4.1710	0.22743	2.4993	4.4995
2	3	3	2	5	3	133	0.20158	0.20940	1.9019	4.1499	0.22743	2.4993	4.4995
2	3	3	2	6	1	181	0.56601	0.56845	2.3550	3.4167	0.63673	4.5000	3.5000
2	3	3	2	6	2	175	0.63501	0.63897	4.4663	3.4848	0.63673	4.5000	3.5000
2	3	3	2	6	3	169	0.56281	0.56759	2.3489	3.5358	0.63673	4.5000	3.5000
2	3	3	2	7	1	169	0.75787	0.76245	0.9844	2.0636	0.76176	1.0035	2.1000
2	3	3	2	7	2	157	0.62630	0.62864	2.1110	2.3611	0.76176	1.0035	2.1000
2	3	3	2	7	3	151	0.75712	0.75222	0.9862	2.0535	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	1	3	1	2	1	051	0.15800	0.16619	2.0061	2.6553	0.16009	2.0250	2.8625
3	1	3	1	2	2	062	0.14610	0.15616	1.7025	3.0522	0.16009	2.0250	2.8625
3	1	3	1	2	3	074	0.13959	0.14813	2.0201	3.4953	0.16009	2.0250	2.8625
3	1	3	1	3	1	098	0.24794	0.25133	3.9238	3.8377	0.25472	4.1992	3.7998
3	1	3	1	3	2	060	0.25355	0.27124	4.0808	3.7914	0.25472	4.1992	3.7998
3	1	3	1	3	3	060	0.24131	0.24856	4.4391	3.9336	0.25472	4.1992	3.7998
3	1	3	1	4	1	071	0.20062	0.20197	1.9375	3.9822	0.24183	2.8200	4.2000
3	1	3	1	4	2	059	0.21632	0.22547	2.0460	4.3669	0.24183	2.8200	4.2000
3	1	3	1	4	3	087	0.21750	0.23014	2.1452	4.0547	0.24183	2.8200	4.2000
3	1	3	1	5	1	066	0.15606	0.16117	2.7728	1.5057	0.22743	5.0000	3.0005
3	1	3	1	5	2	068	0.21734	0.22606	4.7872	3.3670	0.22743	5.0000	3.0005
3	1	3	1	5	3	101	0.22085	0.23264	5.1126	3.0468	0.22743	5.0000	3.0005
3	1	3	1	6	1	039	0.30191	0.30222	3.0686	3.7249	0.63673	4.0000	1.0000
3	1	3	1	6	2	060	0.53490	0.53826	3.7721	3.2486	0.63673	4.0000	1.0000
3	1	3	1	6	3	094	0.63587	0.64167	4.0205	1.0162	0.63673	4.0000	1.0000
3	1	3	1	7	1	077	0.76052	0.76404	2.6012	4.4788	0.76175	2.5986	4.4951
3	1	3	1	7	2	105	0.60140	0.60860	3.4238	3.4762	0.76175	2.5986	4.4951
3	1	3	1	7	3	039	0.50723	0.51623	3.8983	3.4967	0.76175	2.5986	4.4951
3	1	3	2	2	1	044	0.15839	0.16758	2.4804	1.4823	0.16009	2.6377	1.5206
3	1	3	2	2	2	048	0.15221	0.16095	2.3105	1.7703	0.16009	2.6377	1.5206
3	1	3	2	2	3	034	0.15945	0.16773	2.5220	1.5369	0.16009	2.6377	1.5206
3	1	3	2	3	1	066	0.21581	0.22217	1.4781	3.2404	0.25472	1.7001	3.6991
3	1	3	2	3	2	079	0.16437	0.16827	4.0269	2.3626	0.25472	1.7001	3.6991
3	1	3	2	3	3	091	0.24768	0.25426	1.6061	3.5192	0.25472	1.7001	3.6991
3	1	3	2	4	1	038	0.23443	0.24021	1.1506	2.2858	0.24183	1.2900	2.3437
3	1	3	2	4	2	058	0.22582	0.23078	1.3104	1.5610	0.24183	1.2900	2.3437
3	1	3	2	4	3	054	0.23560	0.24340	1.2520	1.9175	0.24183	1.2900	2.3437
3	1	3	2	5	1	080	0.18963	0.20065	2.7273	0.3837	0.22743	2.4993	4.4995
3	1	3	2	5	2	102	0.19435	0.20320	2.6319	0.7911	0.22743	2.4993	4.4995
3	1	3	2	5	3	125	0.22701	0.23716	2.5488	4.5492	0.22743	2.4993	4.4995
3	1	3	2	6	1	045	0.53359	0.54339	2.5621	3.5089	0.63673	4.5000	3.5000
3	1	3	2	6	2	042	0.26724	0.27065	2.2971	2.1905	0.63673	4.5000	3.5000
3	1	3	2	6	3	040	0.25810	0.26483	2.5401	2.1822	0.63673	4.5000	3.5000
3	1	3	2	7	1	047	0.60619	0.62039	2.0379	2.3538	0.76176	1.0035	2.1000
3	1	3	2	7	2	071	0.62963	0.63224	2.0982	2.5051	0.76176	1.0035	2.1000
3	1	3	2	7	3	040	0.62212	0.62845	2.1234	2.2856	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	2	3	1	2	1	062	0.15882	0.17060	2.0700	3.0250	0.16009	2.0250	2.8625
3	2	3	1	2	2	058	0.15531	0.16369	2.2500	3.0250	0.16009	2.0250	2.8625
3	2	3	1	2	3	062	0.15865	0.16797	2.1450	2.8625	0.16009	2.0250	2.8625
3	2	3	1	3	1	067	0.16335	0.17289	3.1198	1.1812	0.25472	4.1992	3.7998
3	2	3	1	3	2	113	0.24851	0.26620	4.3200	3.9000	0.25472	4.1992	3.7998
3	2	3	1	3	3	130	0.25247	0.26057	4.0350	3.8125	0.25472	4.1992	3.7998
3	2	3	1	4	1	187	0.23784	0.24305	3.1087	4.1594	0.24183	2.8200	4.2000
3	2	3	1	4	2	055	0.23278	0.24322	3.1200	4.2937	0.24183	2.8200	4.2000
3	2	3	1	4	3	085	0.22441	0.23288	2.0100	4.2000	0.24183	2.8200	4.2000
3	2	3	1	5	1	069	0.15824	0.17082	2.4300	1.8234	0.22743	5.0000	3.0005
3	2	3	1	5	2	102	0.22236	0.23811	4.9800	3.2016	0.22743	5.0000	3.0005
3	2	3	1	5	3	061	0.15767	0.17337	2.4450	1.7750	0.22743	5.0000	3.0005
3	2	3	1	6	1	063	0.25041	0.25988	2.5950	2.7500	0.63673	4.0000	1.0000
3	2	3	1	6	2	109	0.56356	0.57381	4.0800	3.1016	0.63673	4.0000	1.0000
3	2	3	1	6	3	079	0.56861	0.57793	3.9750	3.1375	0.63673	4.0000	1.0000
3	2	3	1	7	1	118	0.76079	0.76869	2.6250	4.5125	0.76175	2.5986	4.4951
3	2	3	1	7	2	091	0.62234	0.63521	3.1950	3.4129	0.76175	2.5986	4.4951
3	2	3	1	7	3	093	0.62811	0.63862	2.9545	3.4141	0.76175	2.5986	4.4951
3	2	3	2	2	1	097	0.15983	0.16773	2.5800	1.5000	0.16009	2.6377	1.5206
3	2	3	2	2	2	080	0.15916	0.16695	2.5500	1.4625	0.16009	2.6377	1.5206
3	2	3	2	2	3	108	0.14011	0.15173	4.6501	3.5249	0.16009	2.6377	1.5206
3	2	3	2	3	1	064	0.15158	0.16768	3.5700	2.0750	0.25472	1.7001	3.6991
3	2	3	2	3	2	135	0.24512	0.26269	1.7400	3.3656	0.25472	1.7001	3.6991
3	2	3	2	3	3	089	0.24946	0.25387	1.6800	3.9500	0.25472	1.7001	3.6991
3	2	3	2	4	1	126	0.12900	0.13675	3.2400	4.2000	0.24183	1.2900	2.3437
3	2	3	2	4	2	089	0.24152	0.25337	1.2750	2.2625	0.24183	1.2900	2.3437
3	2	3	2	4	3	103	0.24144	0.24732	1.2600	2.3500	0.24183	1.2900	2.3437
3	2	3	2	5	1	089	0.18956	0.20072	2.9999	0.5999	0.22743	2.4993	4.4995
3	2	3	2	5	2	097	0.21305	0.22324	2.9400	4.7500	0.22743	2.4993	4.4995
3	2	3	2	5	3	109	0.18992	0.20471	2.8800	1.0000	0.22743	2.4993	4.4995
3	2	3	2	6	1	106	0.55791	0.56900	2.4600	3.6250	0.63673	4.5000	3.5000
3	2	3	2	6	2	068	0.54819	0.55998	2.2800	3.3250	0.63673	4.5000	3.5000
3	2	3	2	6	3	088	0.56834	0.57980	2.4000	3.5375	0.63673	4.5000	3.5000
3	2	3	2	7	1	121	0.62756	0.63671	2.1300	2.4730	0.76176	1.0035	2.1000
3	2	3	2	7	2	145	0.76011	0.77218	1.0191	2.0566	0.76176	1.0035	2.1000
3	2	3	2	7	3	071	0.62837	0.63992	2.0850	2.5875	0.76176	1.0035	2.1000

TABLE XIX (Continued)

3	3	3	1	2	1	053	0.15605	0.16321	2.0880	3.1545	0.16009	2.0250	2.8625
3	3	3	1	2	2	065	0.15227	0.16079	2.3233	2.9307	0.16009	2.0250	2.8625
3	3	3	1	2	3	065	0.15054	0.15632	1.7306	2.5348	0.16009	2.0250	2.8625
3	3	3	1	3	1	065	0.16251	0.16759	2.8069	1.6546	0.25472	4.1992	3.7998
3	3	3	1	3	2	090	0.23149	0.24918	4.0479	4.0100	0.25472	4.1992	3.7998
3	3	3	1	3	3	116	0.24293	0.25020	4.5743	3.7647	0.25472	4.1992	3.7998
3	3	3	1	4	1	065	0.23964	0.24411	2.9704	4.1369	0.24183	2.8200	4.2000
3	3	3	1	4	2	059	0.22690	0.23061	3.3428	4.1420	0.24183	2.8200	4.2000
3	3	3	1	4	3	098	0.23817	0.24899	2.9773	4.1110	0.24183	2.8200	4.2000
3	3	3	1	5	1	065	0.19779	0.20413	1.1301	2.6359	0.22743	5.0000	3.0005
3	3	3	1	5	2	109	0.19933	0.21508	1.2138	2.6778	0.22743	5.0000	3.0005
3	3	3	1	5	3	053	0.17707	0.17986	1.7116	2.3491	0.22743	5.0000	3.0005
3	3	3	1	6	1	037	0.54561	0.54202	3.9140	3.2508	0.63673	4.0000	1.0000
3	3	3	1	6	2	099	0.63592	0.64189	3.9773	0.9888	0.63673	4.0000	1.0000
3	3	3	1	6	3	041	0.56348	0.57040	3.9198	3.0828	0.63673	4.0000	1.0000
3	3	3	1	7	1	056	0.75915	0.76008	2.6433	4.5222	0.76175	2.5986	4.4951
3	3	3	1	7	2	047	0.76028	0.76687	2.5997	4.5159	0.76175	2.5986	4.4951
3	3	3	1	7	3	047	0.75705	0.76216	2.5723	4.4578	0.76175	2.5986	4.4951
3	3	3	2	2	1	065	0.15834	0.16477	2.4586	1.6100	0.16009	2.6377	1.5206
3	3	3	2	2	2	065	0.15981	0.16793	2.6109	1.5780	0.16009	2.6377	1.5206
3	3	3	2	2	3	115	0.15958	0.16462	2.5607	1.5864	0.16009	2.6377	1.5206
3	3	3	2	3	1	065	0.15602	0.16595	3.8130	2.1111	0.25472	1.7001	3.6991
3	3	3	2	3	2	115	0.24803	0.24940	1.7273	3.9797	0.25472	1.7001	3.6991
3	3	3	2	3	3	098	0.25324	0.24894	1.6522	3.6363	0.25472	1.7001	3.6991
3	3	3	2	4	1	053	0.12893	0.13654	2.5454	3.4834	0.24183	1.2900	2.3437
3	3	3	2	4	2	128	0.22167	0.22979	1.3930	2.9167	0.24183	1.2900	2.3437
3	3	3	2	4	3	047	0.22242	0.21945	1.3179	2.8947	0.24183	1.2900	2.3437
3	3	3	2	5	1	095	0.19270	0.19860	2.5489	0.6965	0.22743	2.4993	4.4995
3	3	3	2	5	2	115	0.22455	0.23445	2.3201	4.4513	0.22743	2.4993	4.4995
3	3	3	2	5	3	118	0.18781	0.19568	2.7585	0.3609	0.22743	2.4993	4.4995
3	3	3	2	6	1	100	0.55241	0.54557	2.2774	3.3698	0.63673	4.5000	3.5000
3	3	3	2	6	2	056	0.56882	0.58060	2.3779	3.4503	0.63673	4.5000	3.5000
3	3	3	2	6	3	050	0.56927	0.56671	2.3721	3.4754	0.63673	4.5000	3.5000
3	3	3	2	7	1	047	0.60485	0.60597	2.1980	2.1338	0.76176	1.0035	2.1000
3	3	3	2	7	2	103	0.75649	0.76169	1.0242	2.1451	0.76176	1.0035	2.1000
3	3	3	2	7	3	047	0.75843	0.75724	0.9879	2.0623	0.76176	1.0035	2.1000

APPENDIX B

PLOTS OF SOME SELECTED OPTIMIZATION PATHS

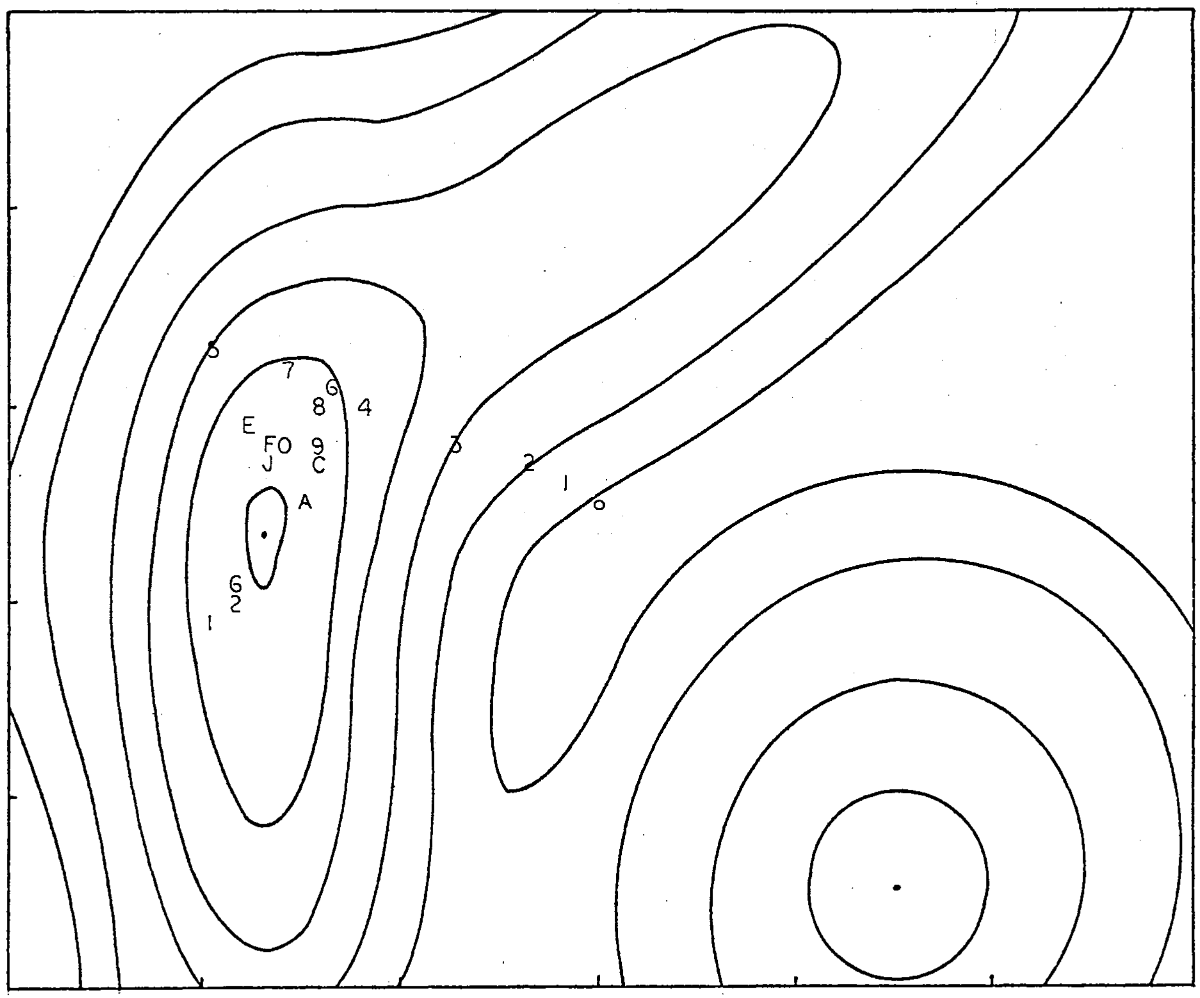


Figure 18. Graphic Results of Data Point 1 1 3 2 4 1

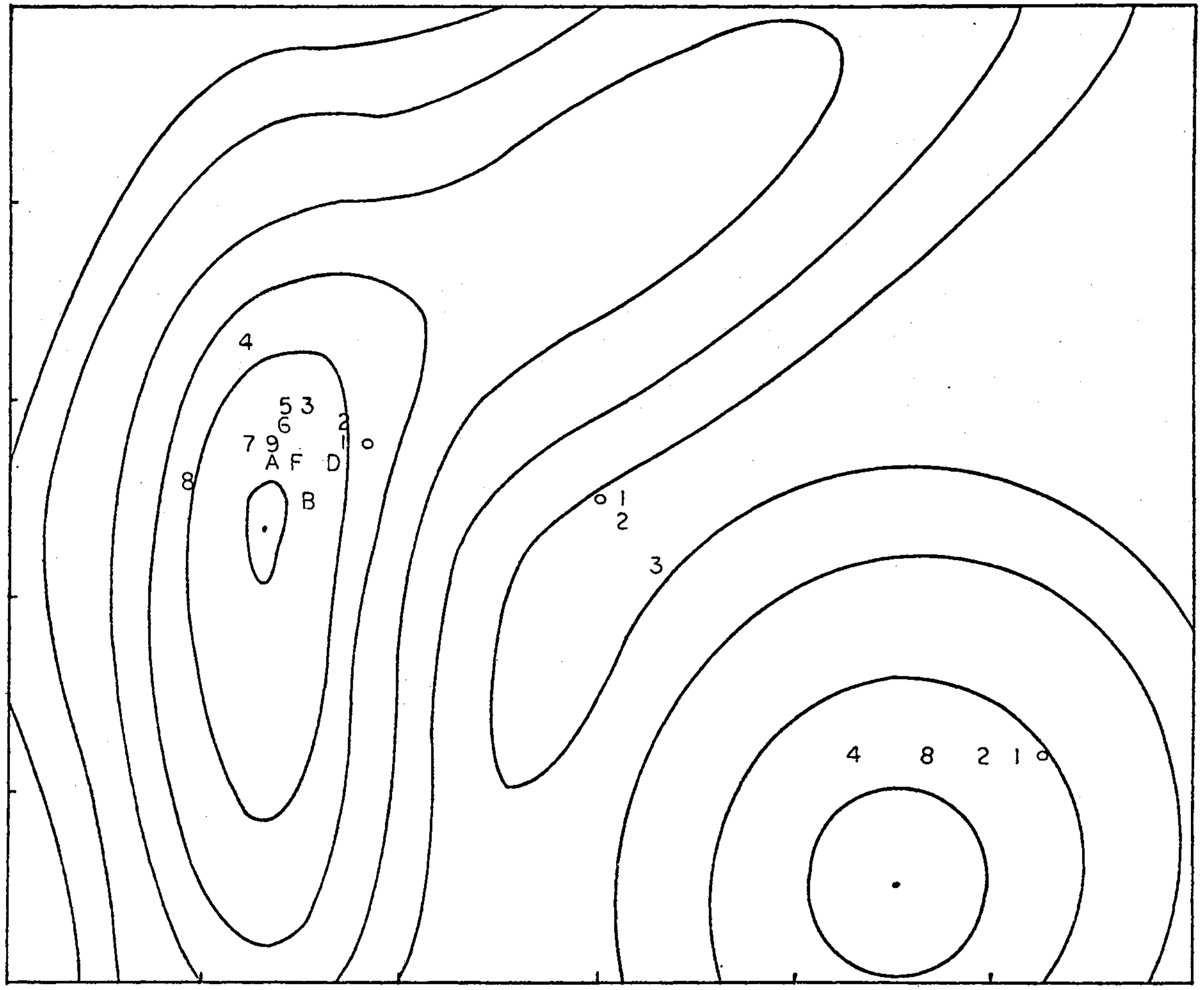


Figure 19. Graphic Results of Data Point 1 2 3 2 4 1

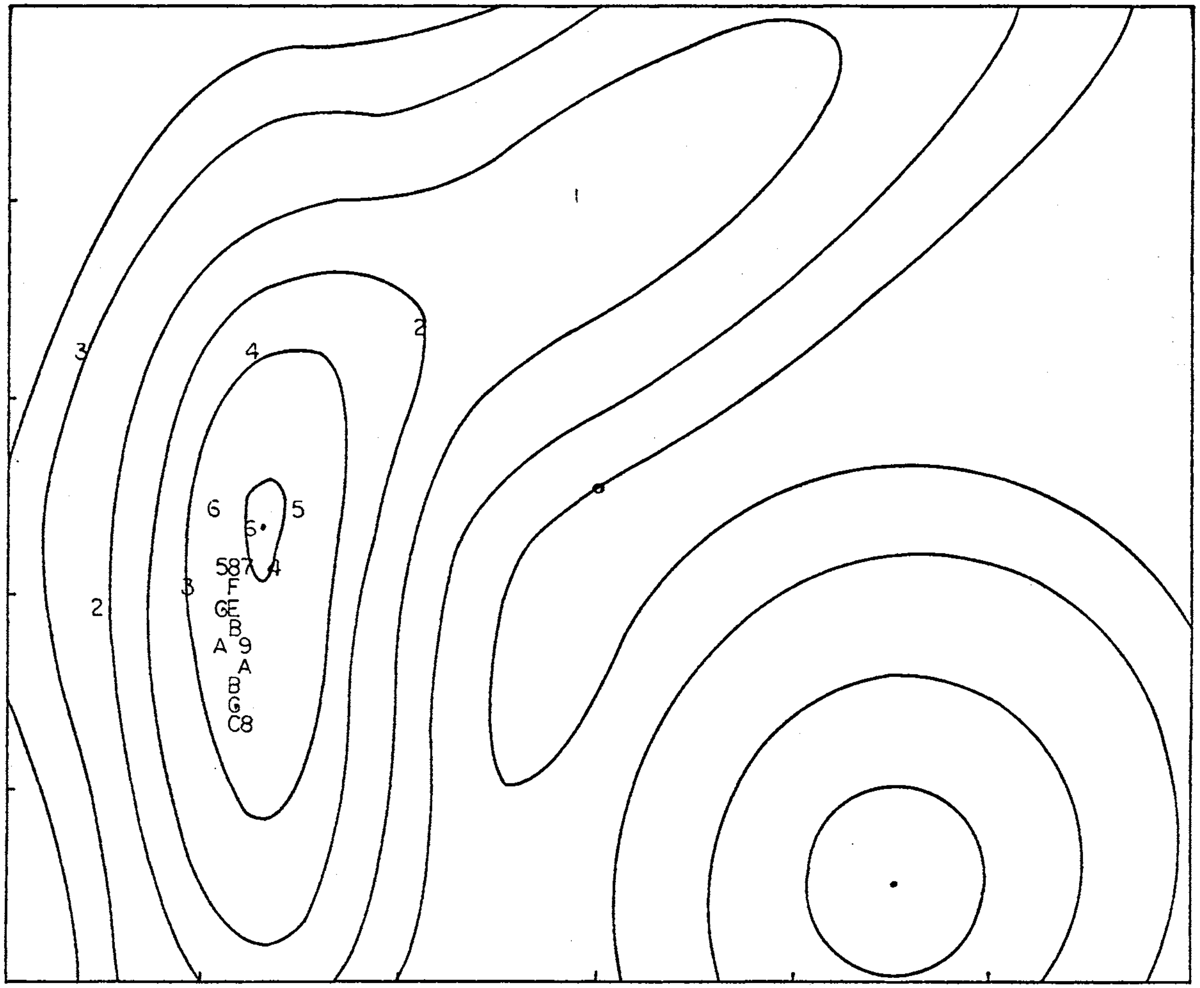


Figure 20. Graphic Results of Data Point 1 3 3 2 4 1

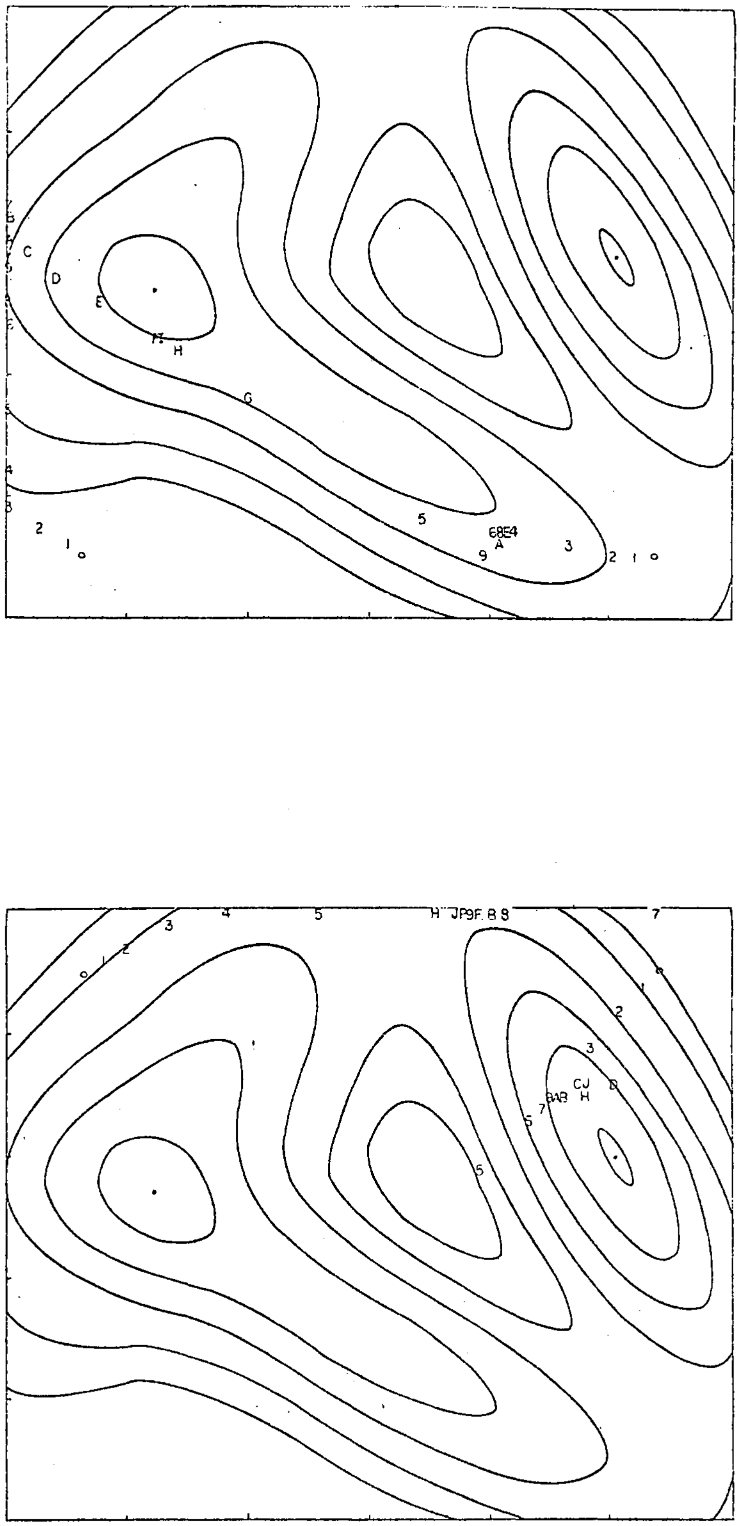


Figure 21. Graphic Results of Data Point 2 1 2 1 5 1

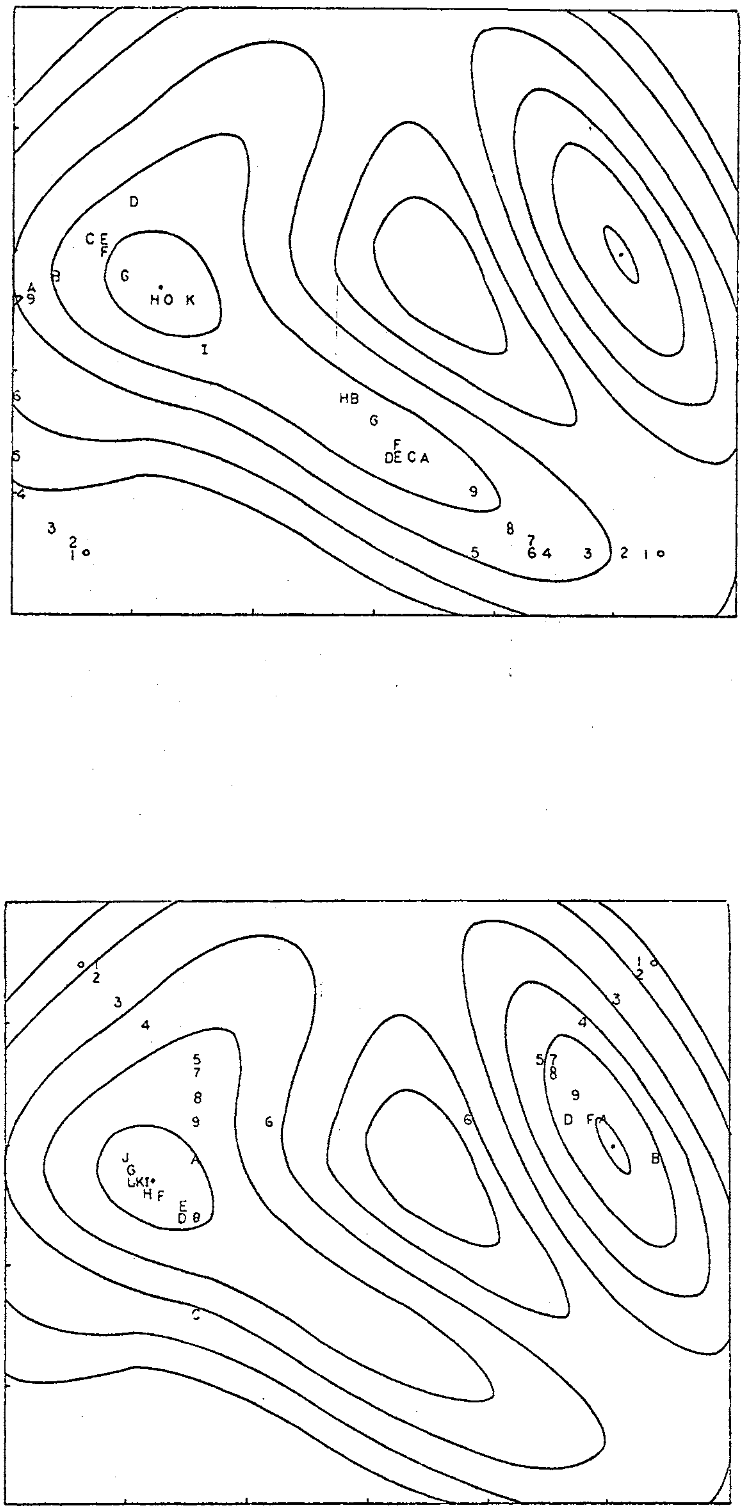


Figure 22. Graphic Results of Data Point
2 2 2 1 5 1

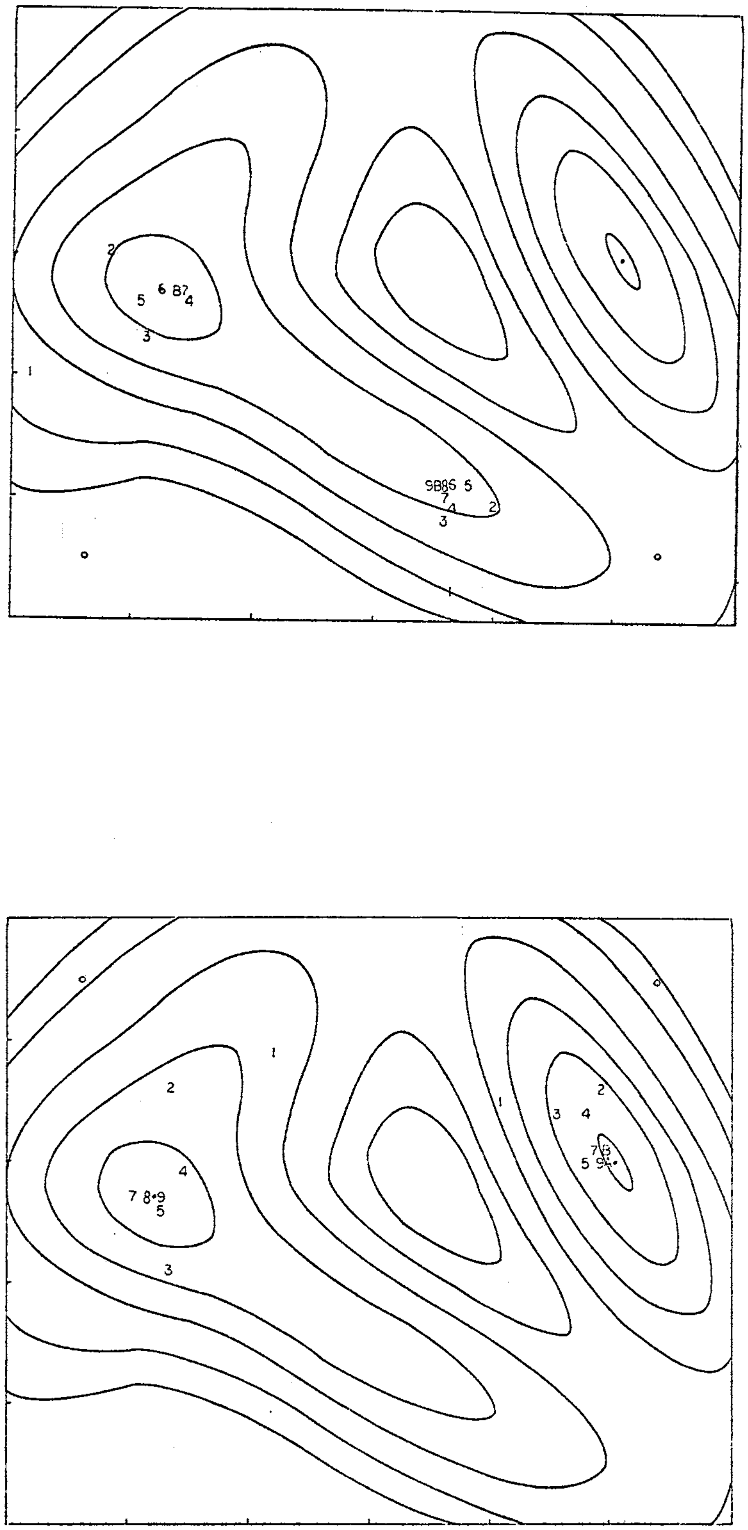


Figure 23. Graphic Results of Data Point
2 3 2 1 5 1

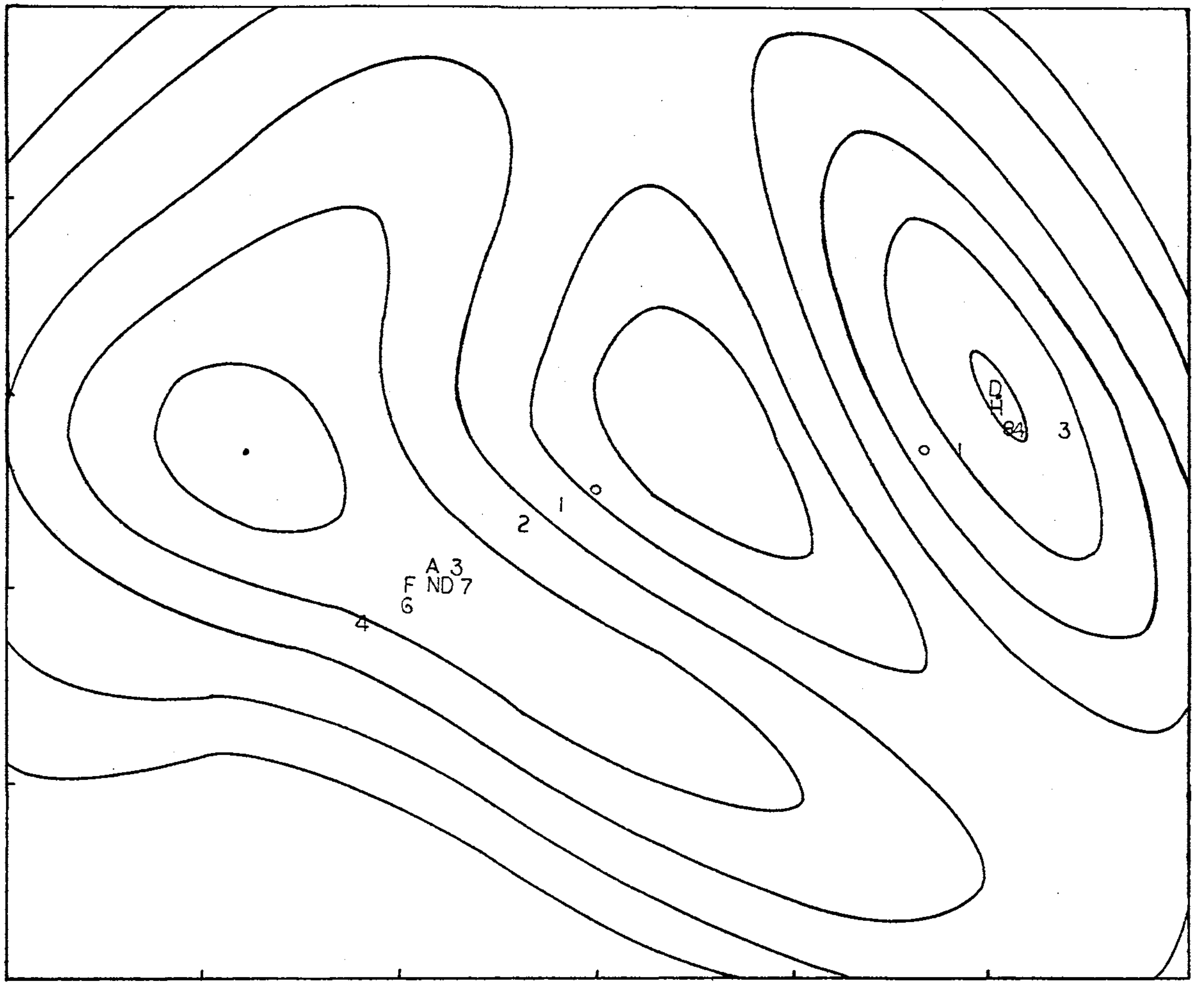


Figure 24. Graphic Results of Data Point 3 1 2 1 5 1

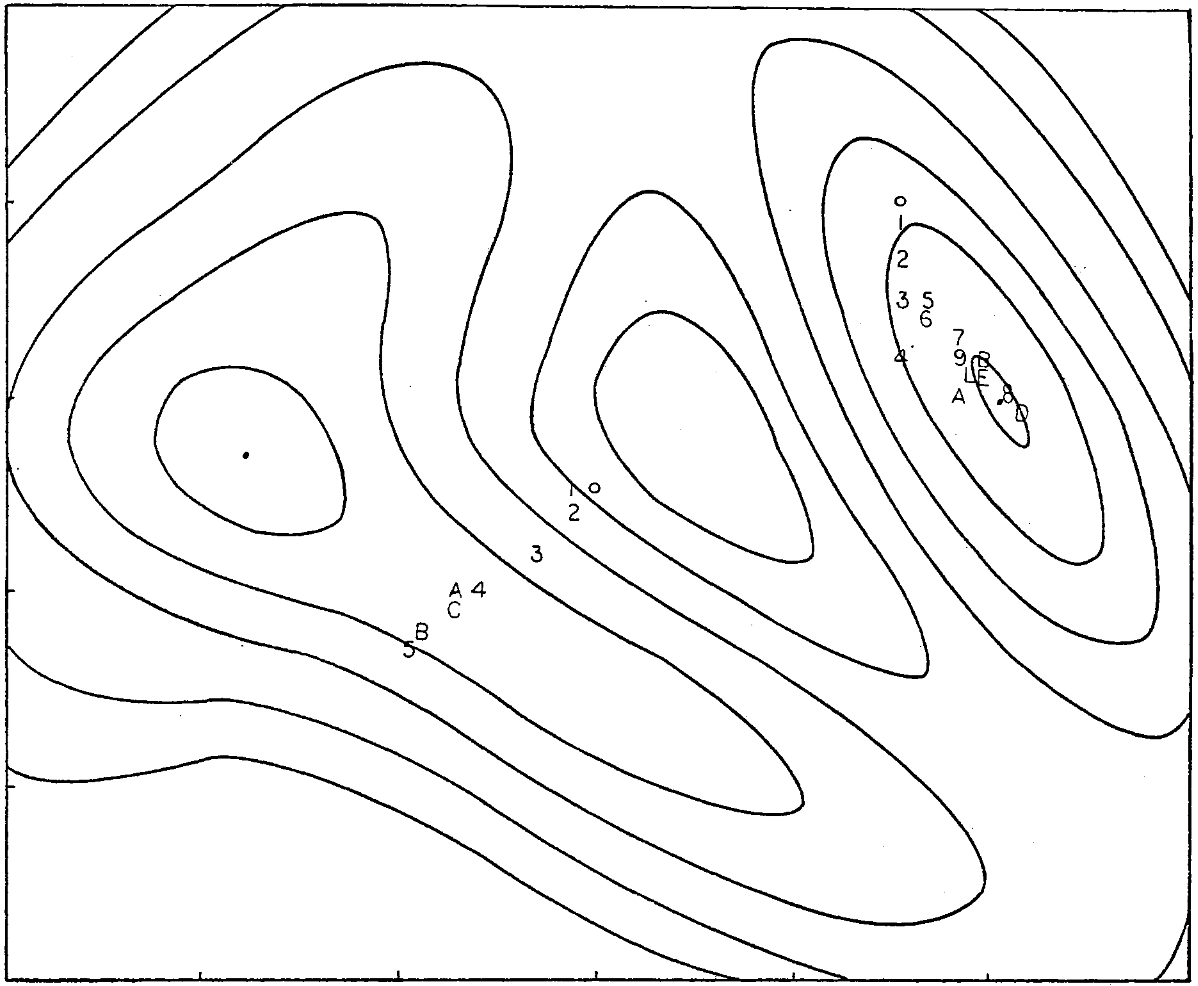


Figure 25. Graphic Results of Data Point 3 2 2 1 5 2

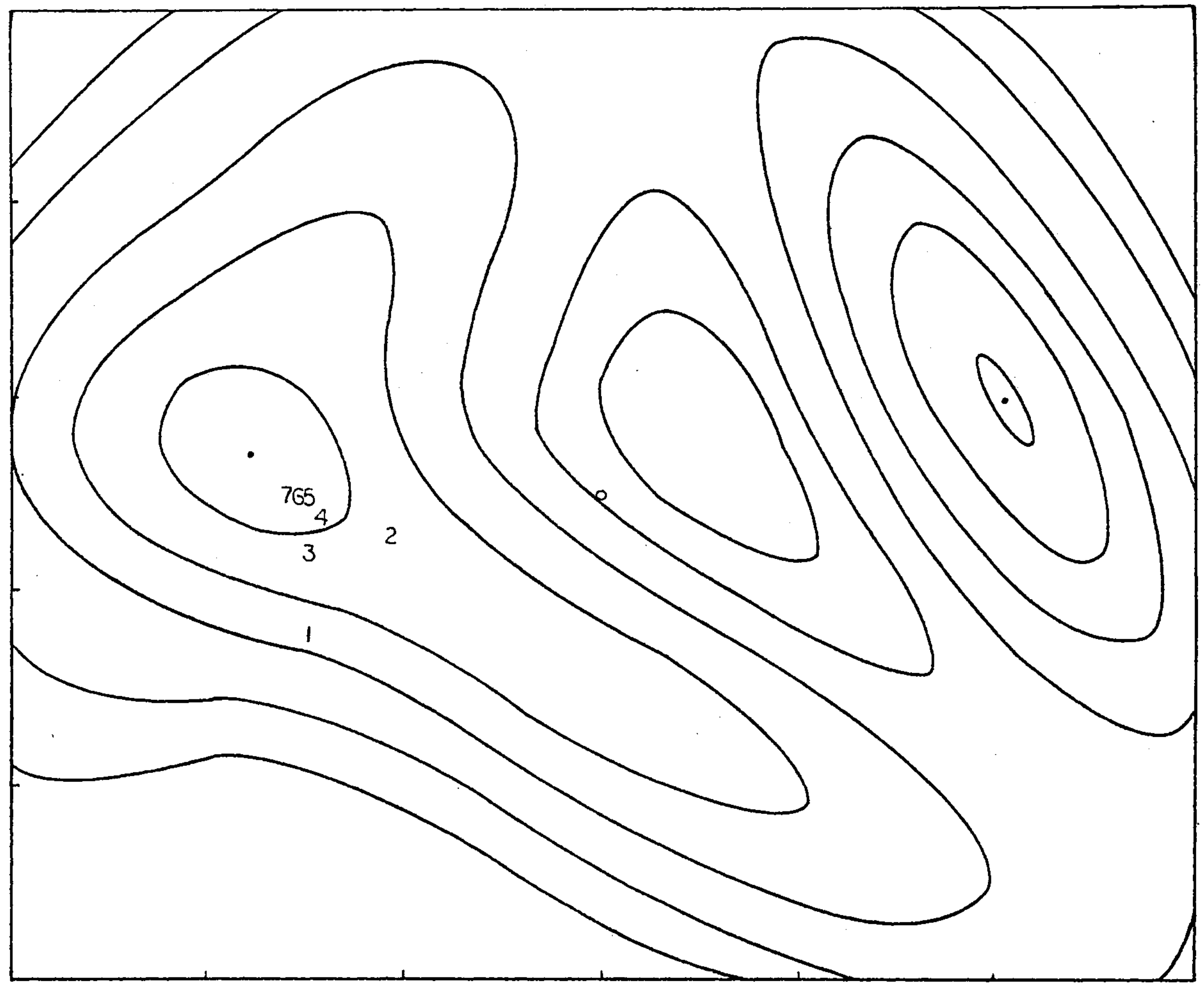


Figure 26, Graphic Results of Data Point 3 3 2 1 5 1

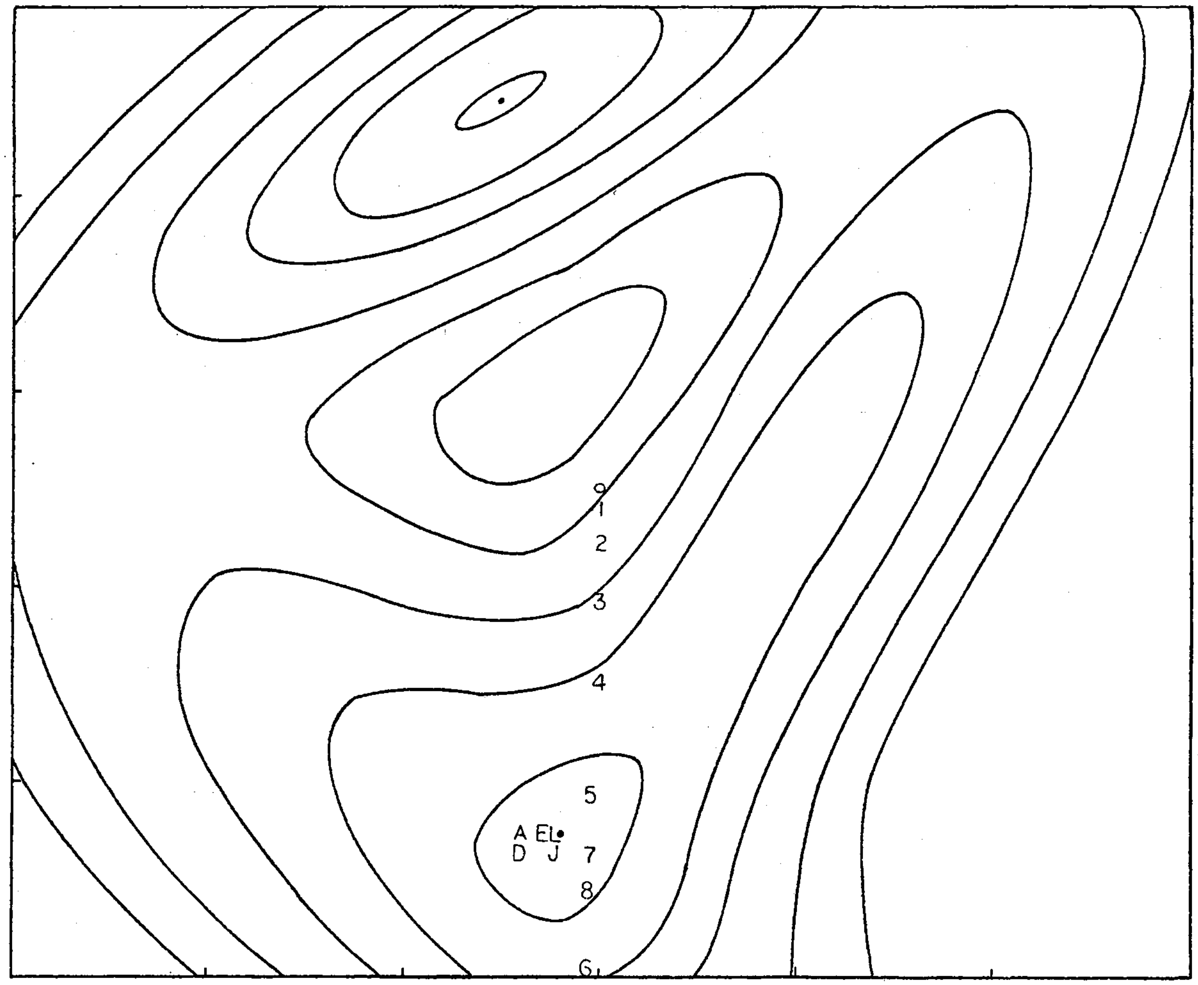


Figure 27. Graphic Results of Data Point 3 1 2 2 5 1

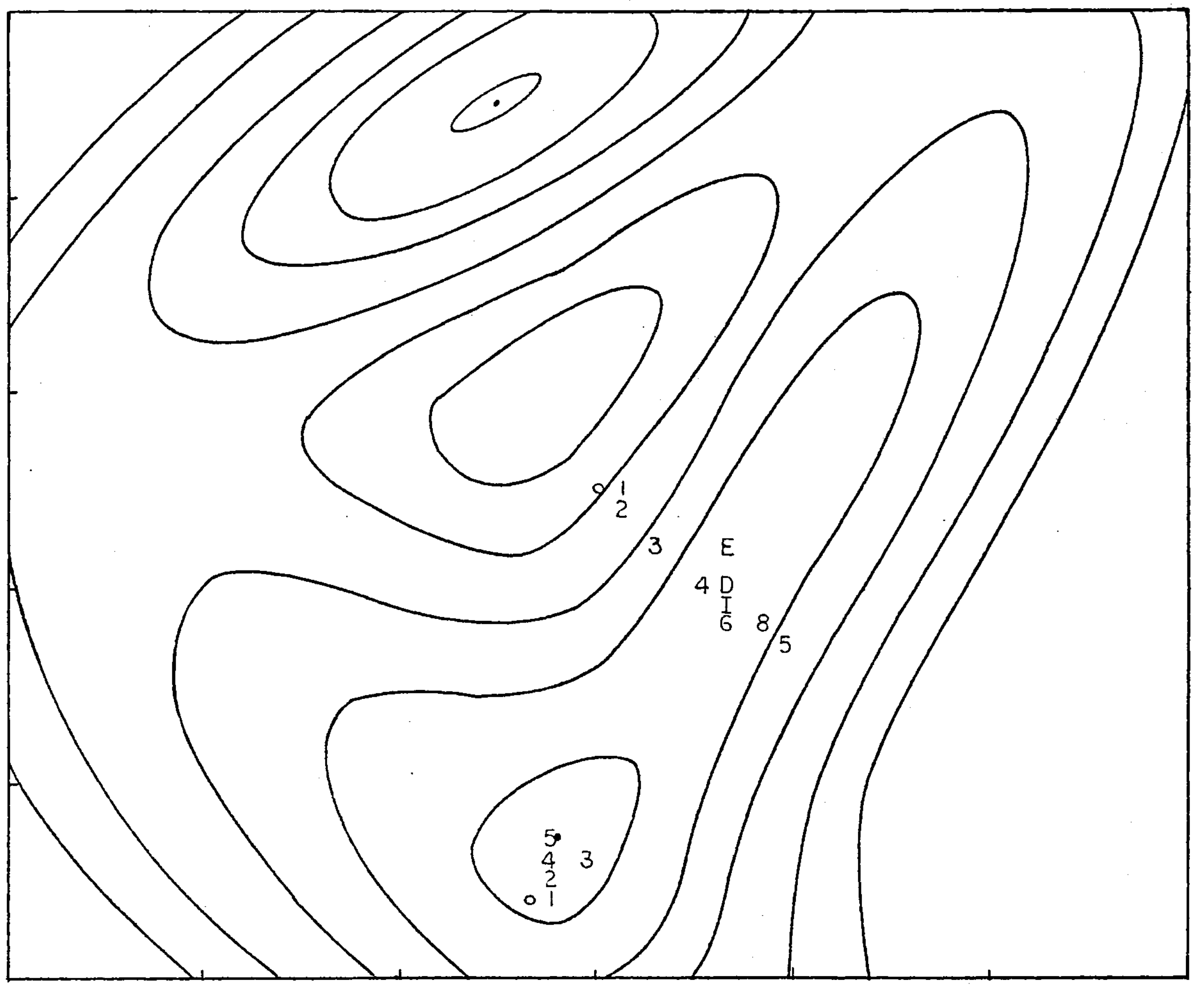


Figure 28. Graphic Results of Data Point 3 2 2 2 5 1

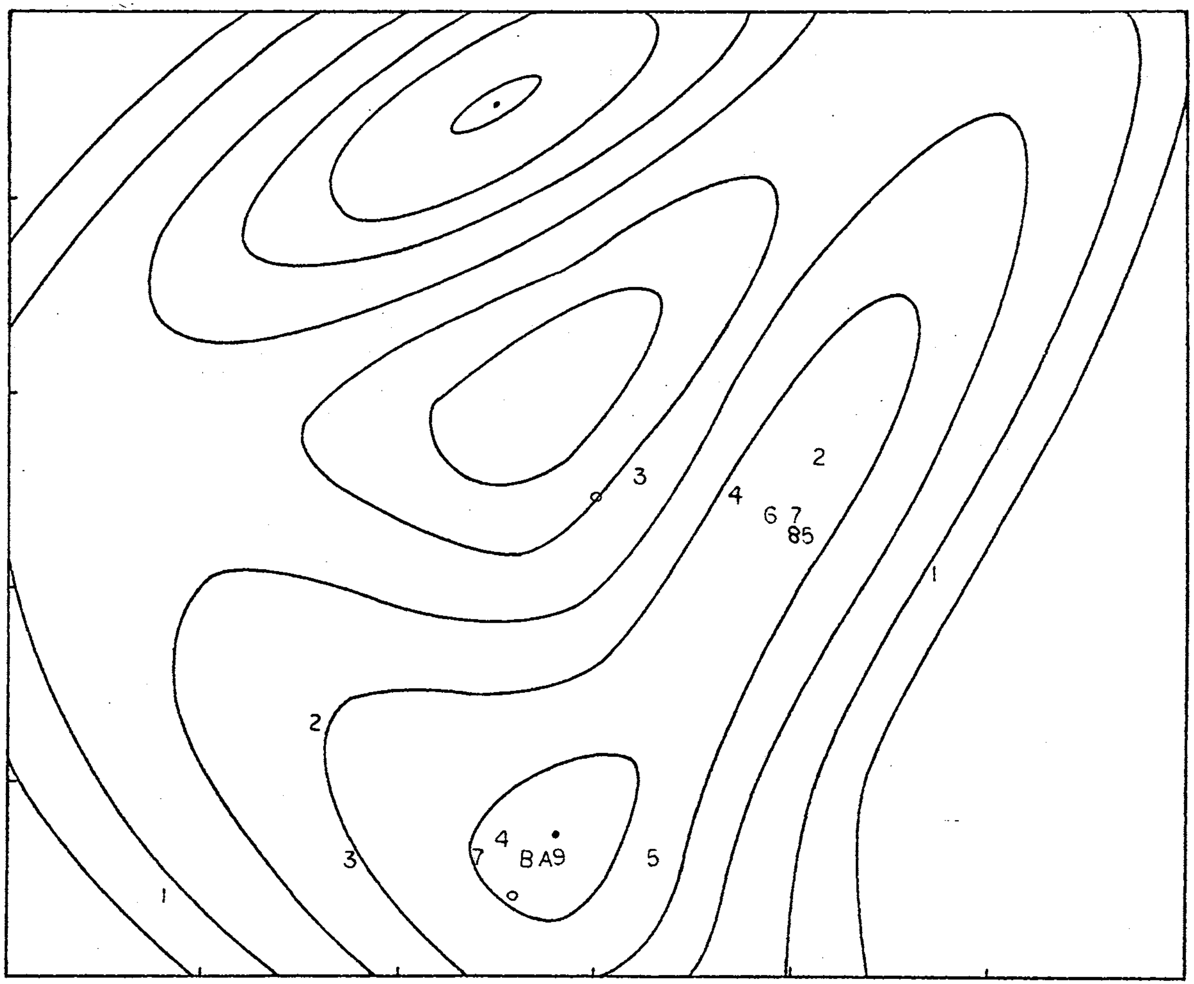


Figure 29. Graphic Results of Data Point 3 3 2 2 5 1

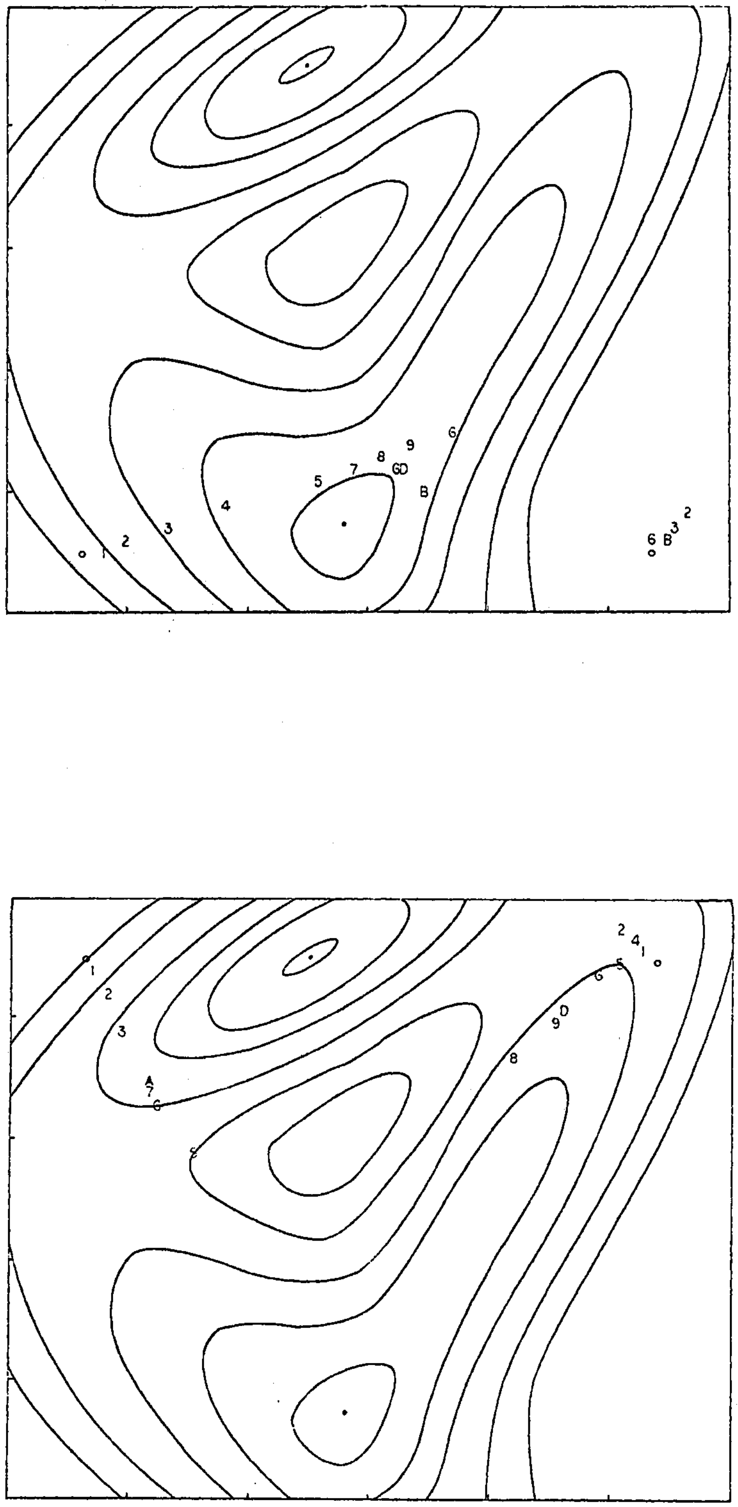


Figure 30. Graphic Results of Data Point
2 1 2 2 5 1

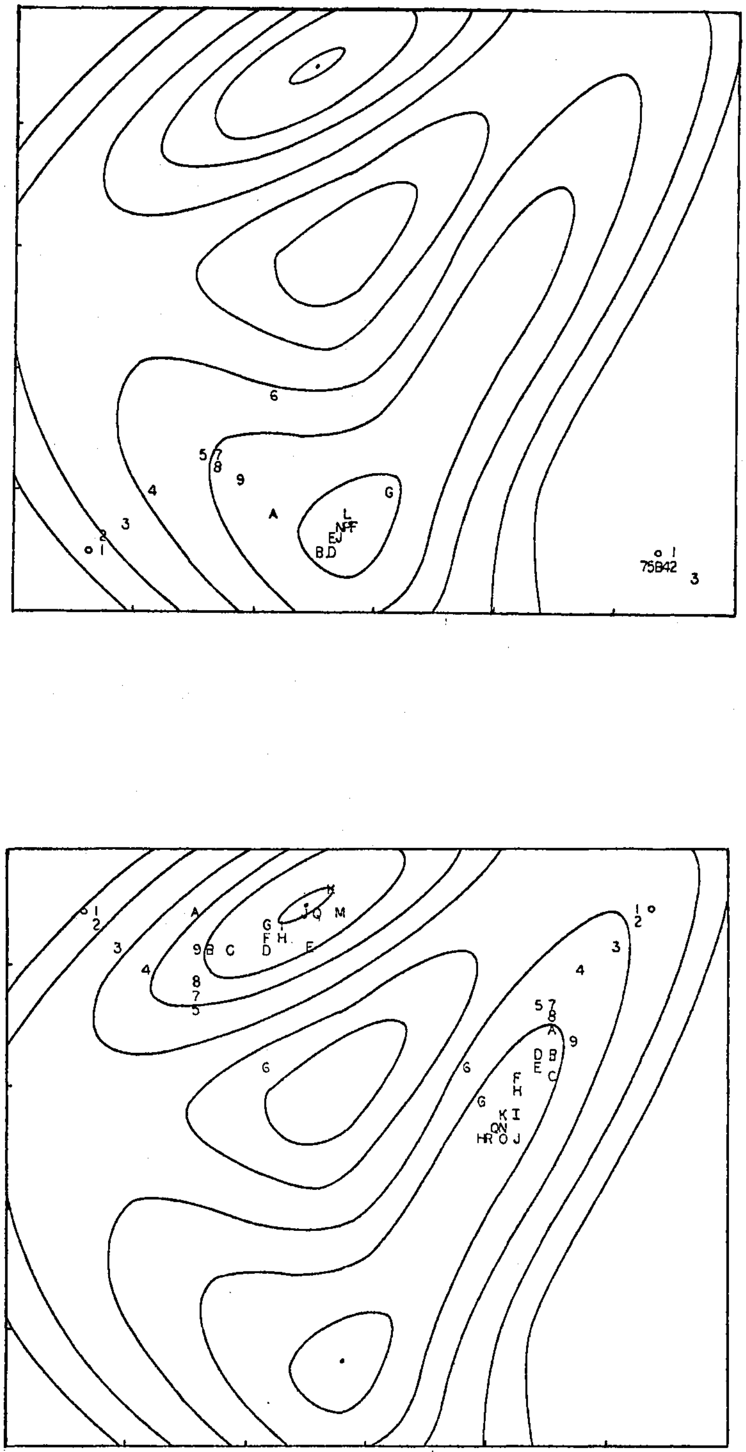
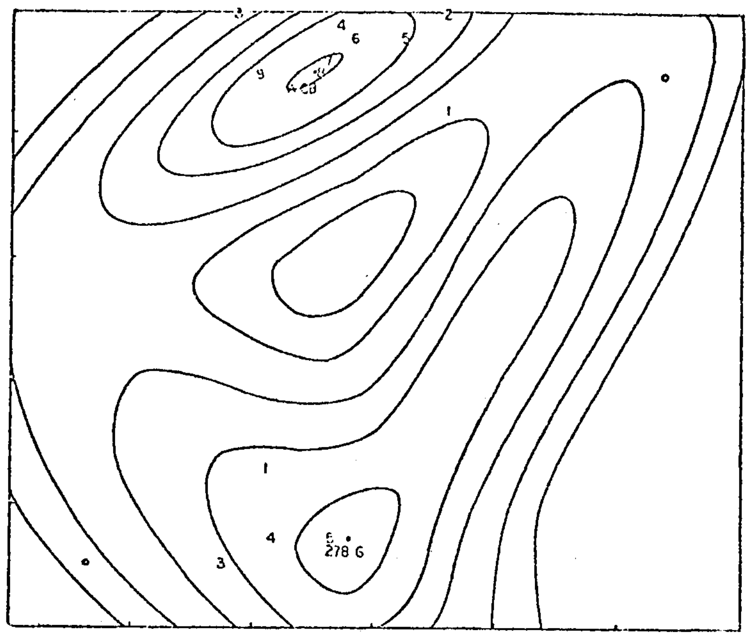


Figure 31. Graphic Results of Data Point
2 2 2 2 5 1



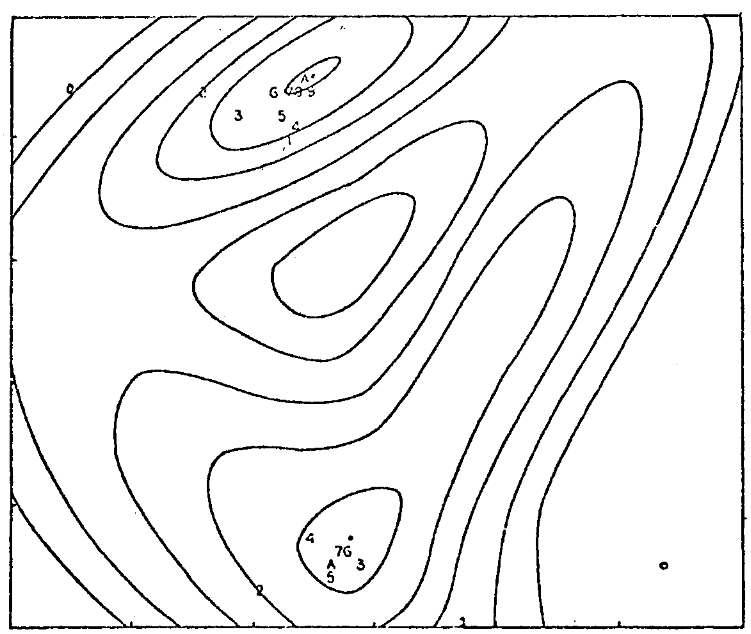


Figure 32. Graphic Results of Data Point
2 3 2 2 5 1

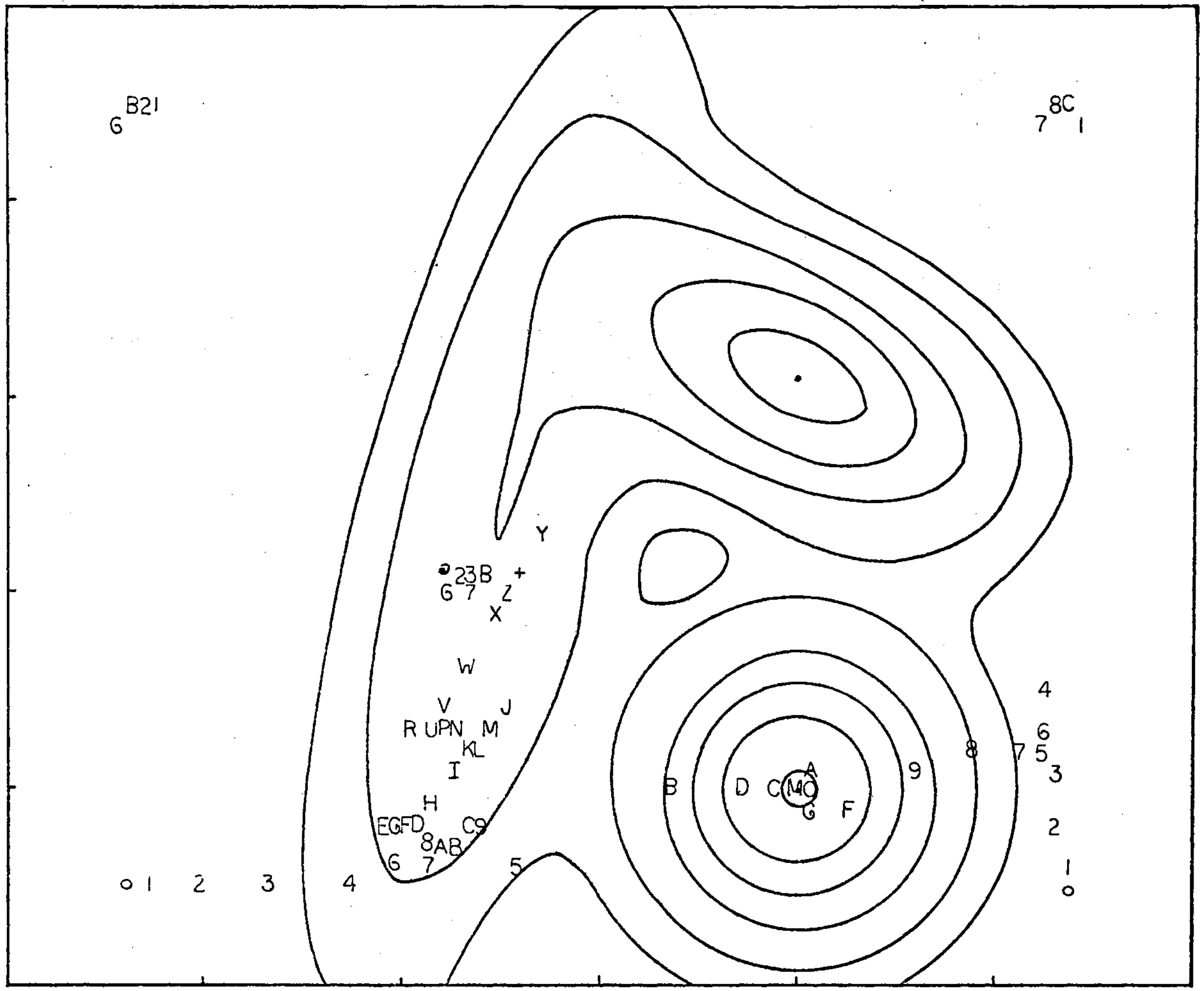


Figure 33. Graphic Results of Data Point 2 1 2 1 6 1

VITA

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Doctor of Philosophy

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