FLUID LINE DYNAMICS WITH THROUGH FLOW AND FINITE AMPLITUDE DISTURBANCES

By

RICHARD PAUL BRADEN

Bachelor of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia
1959

Master of Science
University of New Mexico
Albuquerque, New Mexico
1967

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
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Thesis Approved:

Karl N. Reid
Thesis Adviser

W. G. Tiederman, Jr.

Don F. Kincannon

J. E. Bose

D. D. Durham
Dean of the Graduate School

873228

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TABLE OF CONTENTS

Chapter | Page

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$\mathcal{B}_o$	Time-averaged bulk modulus of a liquid, lb_f/in^2
μ_o	Time-averaged absolute viscosity, $lb_f\ sec/in^2$
ν_o	Time-averaged kinematic viscosity, in^2/sec
ρ_o	Time-averaged fluid density, $lb_f\ sec^2/in^4$
σ_o	Prandtl number
p_o	Time-averaged static pressure, psia
T_o	Time-averaged fluid temperature, $^oRankine$
α	$j\sqrt{\frac{s a^2}{\nu_o}\left(1+\frac{F_{1*}}{s}\right)}$
γ	Ratio of specific heats C_p/C_v
δp	Pressure drop per unit length, lb_f/in^3
$\Gamma(s)$	$\frac{sL}{C_o}\sqrt{\frac{N_g}{D_g}}$
$\Gamma_b(s)$	$\frac{sL}{C_o}\sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{s}\right)}$
$\Gamma_d(s)$	$\frac{sL}{C_o}\sqrt{\frac{N_g\ (s+KF_{1*})}{D_a\ (s-[1-K]F_{1*})}}$ $\qquad 0 \leq K \leq 1$
Δ	$j\sqrt{\frac{s a^2 \sigma_o}{\nu_o}}$
Ψ_t	Viscous attenuation parameter, dimensionless

θ	Tangential coordinate, radians
μ	Instantaneous absolute viscosity, lb_f sec/in^2
ν	Instantaneous kinematic viscosity, in^2/sec
ρ	Instantaneous fluid density, lb_f sec^2/in^4
ψ	$j\sqrt{\frac{sa^2}{\nu_o}}$
ω	Frequency, radians/sec
a	Line inner radius, inches
j	$\sqrt{-1}$
p_a	Pressure, psig
p_b	Pressure, psig
p_c	Steady-state component of fluid axial pressure, psia
p_t	Transient axial pressure, psig
q	Volume flowrate, in^3
r	Radial coordinate, inches
t	Time, seconds
v_c	Steady-state component of axial velocity, in/sec
V_f	Dimensionless steady-state axial velocity, v_c/C_o
v_z	Axial velocity, in/sec
v_t	Transient axial velocity, in/sec
w_a	Mass flowrate, lb_f sec/in
w_b	Mass flowrate, lb_f sec/in
w_t	Mass flowrate, lb_f sec/in
A(s)	Polynomial in "s"

AM	Axial momentum equation
B(s)	Polynomial in "s"
C_o	Isentropic speed of sound in the fluid, $\sqrt{\frac{\gamma P_o}{\rho_o}}$ or $\sqrt{\gamma R_{gas} T_o}$
C_p	Specific heat at constant pressure, Btu/lb_m ^oR
C_v	Specific heat at constant volume, Btu/lb_m ^oR
$D_a(s)$	$\left(1 - \frac{2 J_1(\alpha)}{\alpha J_0(\alpha)}\right)$
$D_g(s)$	$\left(1 - \frac{2 J_1(\psi)}{\psi J_0(\psi)}\right)$
DN	Damping number, ν_o/a^2, 1/sec
EE	Energy equation
F_{1*}	$\frac{C_o}{L}\left(\frac{\partial V}{\partial Z}\right)_* = (\mathrm{sgn}\ P(t,0))\left(\frac{\partial P(t,0)}{\partial t}\right)_*$
IC	Integrated continuity equation
J_o	Bessel function of the first kind, zeroeth order
J_1	Bessel function of the first kind, first order
J_2	Bessel function of the first kind, second order
L	Line length, inches
$\mathcal{L}$	Laplace transform
$\mathcal{L}^{-1}$	Inverse Laplace transform
M	Mach number
M_b	Average through flow Mach number
M(s)	Arbitrary function
N_g	$\left(1 + \frac{2(\gamma-1) J_1(\Delta)}{\Delta J_0(\Delta)}\right)$

P	Nondimensional transient axial pressure, p_t/p_o
P_{out}	Nondimensional transient output pressure
P_{in}	Nondimensional transient input pressure
Q	Nondimensional transient flow, $W_t/\rho_o C_o \pi a^2$
R	Nondimensional radial coordinate, r/a
R_{gas}	Gas constant, $in^2/sec^2\ {}^oR$
S	Laplace variable
SE	Second order differential equation
SN	Solution to the second order differential equation
T	Nondimensional transient axial temperature, T_t/T_o
T_c	Steady-state axial temperature, ^oR
T_e	Isentropic delay time, L/C_o, seconds
T_t	Transient axial temperature, ^oR
TM	Transient mass flowrate equation
V	Nondimensional transient axial velocity, v_t/C_o
Y_o	Bessel function of the second kind, zeroeth order
Y_1	Shunt, admittance per unit length, in^5/lb_f
$Y_b(s)$	Nondimensional admittance, $D_g \sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{s}\right)}$
$Y_e(s)$	Nondimensional admittance, $\sqrt{N_g D_g}$
Z	Nondimensional axial coordinate, z/L
Z_1	Series impedance per unit length, lb_f/in^6
$Z_b(s)$	Nondimensional impedance, $\gamma \sqrt{\frac{\left(1+\frac{F_{1*}}{s}\right)}{N_g D_a}}$

$Z_c(s)$ Nondimensional impedance, $\dfrac{\gamma}{\sqrt{N_g D_g}}$

$Z_d(s)$ Nondimensional impedance, $\gamma \sqrt{\dfrac{(S + K F_{1*})(S - [1-K] F_{1*})}{S^2 N_g D_a}}$ $0 \leq K \leq 1$

$Z_e(s)$ Impedance, $\dfrac{C_o}{\pi a^2} \sqrt{\dfrac{1}{N_g D_g}}$

CHAPTER I

THE PROBLEM

Introduction

The transient solution for small, laminar disturbances in a fluid-filled line has been reported many times in the literature, as is shown below:

TABLE I

LITERATURE SUMMARY

Flow	Type of Transient Disturbance		
	Small Laminar Disturbances	Finite Amplitude Laminar Disturbances	Turbulent Disturbances
No Through Flow	Iberall (12) Nichols (13) Brown (3) Goodson(10) Zielke (22) Kantola (13)		
Laminar Incompressible Through Flow	Orner (17)		
Turbulent Through Flow	Brown, Margolis, Shah (6)	Brown, Margolis, Shah (6)	

The small laminar disturbance "models" of a fluid transmission line which have resulted from the anlyses shown above were sufficient to predict transients in instrumentation lines, most hydraulic systems, and selected pneumatic systems. In the simulation of hydraulic systems most of the transients occurred "slowly." The opening and closing of a valve or the movement of a control piston, for example, occurred over a relatively long period of time. The inputs to the hydraulic line were considered as a series of small disturbances, andthe small disturbance line model seemed to be adequate.

With the advent of fluid logic devices that change output from 14.7 psia to 18.7 psia in 4 or 5 milliseconds, hydraulic logic devices, and fast-response pneumatic control systems, the small disturbance line model often is inadequate - inadequate in the sense that the model can not predict transients accurately when it is subjected to these types of inputs:

1. inputs with both high frequency content and low frequency content;
2. inputs with or without through flow; and
3. inputs of small and finite amplitude.

The capability of the existing small disturbance of "acoustic" models for predicting high and low frequency behavior is excellent, providing the pressure disturbances are sufficiently small.

The small disturbance models do not include the effect of through flow. This is not due to any inherent deficiency in the small disturbance models, but rather to the general belief by engineers that the effect of through flow is negligible - that signal transmission in a fluid-filled line is not greatly altered by the addition of through flow unless the through flow velocity approaches the acoustic speed of sound in the fluid. The acoustic speed of sound in air is on the order of 1100 ft/sec, and in liquids is as high as 5000 ft/sec. In most practical applications through flow velocities are on the order of 100 ft/sec or less. Then the effect of through flow on dynamic behavior may be negligible, and the small disturbance model which neglects through flow may be completely adequate even when through flow is present.

The principal shortcoming of the small disturbance line model is its inability to meet requirement 3 above, that of predicting the response to both small and finite amplitude disturbances. The small disturbance ordinary differential equation line models are all linear models. Doubling the magnitude of the input doubles the magnitude of the output, and the output transients have the same percent of overshoot and rise time.

But experiments with pneumatic lines, such as the ones conducted by Kantola (13), show that when one increases the magnitude of a step input to the line, the output transient overshoot decreases and the rise time increases. Part of Kantola's experimental results are shown as Figure 1. Note the significant increase in apparent damping for the 1.0 psig step over the 0.1 psig step, and the accompanying increase in rise time.

No linear small disturbance line model will predict Kantola's results shown on Figure 1. A reexamination of the describing equations for the fluid-filled line is in order. By including the convective acceleration terms in the axial momentum and energy equations, it may be possible to predict the increase in apparent damping which occurs as the disturbance amplitude is increased. At least it may be possible to predict the trend in the output transient as disturbance amplitude

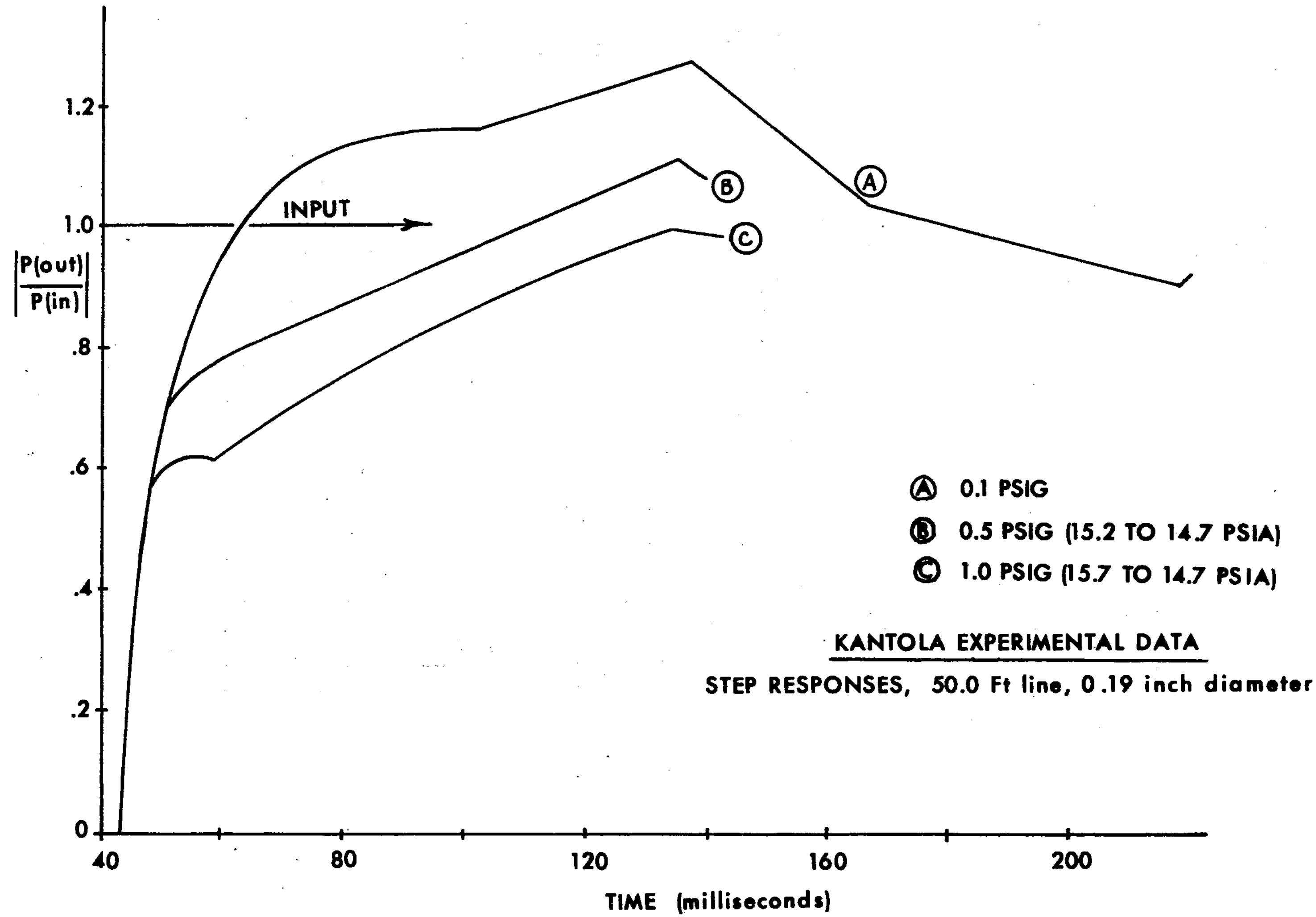

Figure 1. Kantola Experimental Data

increases.

Previous Investigations

Zielke(22) and Brown(5) investigated the problem of retaining the convective acceleration term $v_z \frac{\partial v_z}{\partial z}$ in the axial momentum equation, as shown below.

$$\frac{\partial v_z}{\partial t} - \frac{\nu_o}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{\rho_o}\frac{\partial p_z}{\partial z} = -\underline{v_z\frac{\partial v_z}{\partial z}} \tag{1.1}$$

They both concluded that the convective acceleration term should be evaluated as the solution progressed, making it a "weighting function" to force the left side of Equation (1.1). Their primary interest was in a highly accurate line model, and speed of computation was not essential. They solved Equation (1.1) by a method of characteristics, modified by the weighting function $\left(v_z \frac{\partial v_z}{\partial z}\right)$. The results were compared with data measured using small amplitude disturbances.

If speed of computation is not essential, Equation (1.1) may also be solved by finite difference methods.

When speed of computation is essential, the methods of characteristics and finite difference methods lead to accurate results but require significant storage and computational time. An ordinary differential equation model which approximates the true partial differential equation is less accurate, but is more compatible with the lumped parameter models or the ordinary differential equation models for the other components in the system. That is, the intended area of application of the line model is in simulation of complex hydraulic and pneumatic systems containing a wide variety of components. In a system simulation of this type, the high frequency portions of the input are normally greatly attenuated by system components other than the line, regardless of what type of transmission line model is being used. For this reason, most simulation schemes use an ordinary differential equation line model which is capable of predicting transients in the low to medium frequency range.

There are various types of ordinary differential equation models available, but the most common type used is the distributed parameter model. This model comes from a solution of the equations of motion, and the energy equation. The distributed parameter model is an infinite order ordinary differential equation system, and there is considerable literature ((9), (16), and (19), for example) that discusses the best ways to truncate the infinite order system to a finite order for efficient use in a system simulation.

Thesis Objective

The objective of this thesis is to develop a generalized line model which is suitable for system simulation, a model which includes the effects of finite amplitude disturbances and through flow. The model is intended to be used primarily in hydraulic and pneumatic system simulations where the high frequency portions of input disturbances are attenuated significantly. Therefore, primary consideration will be given to the accurate prediction of transients with low to middle-range frequency content.

Criteria for Judging Model Validity

The criteria used to judge the suitability of the model will be the following (listed in order of importance):

1. The model should predict an increase in apparent damping as the magnitude of the disturbance input to the line is increased. A real transmission line has this behavior, as is shown on Figure 1.

2. The model should be reducible to finite order by suitable approximations such that computational time and difficulty are reduced without severely limiting the accuracy of the model. Factors which may be considered in the suitability of a particular order model are rise time and apparent damping.

3. The model response should be in reasonable agreement with the apparent fundamental mode of corresponding experimental responses. (There appears to be no totally definitive way to compare model responses and experimental responses.)

Definition of Terms

The following terms are used in several places in the thesis:

1. Average Fluid Properties: The terms ρ_o, ν_o, μ_o, T_o, and p_o are time-averaged fluid properties about which the instantaneous variations ρ, ν, μ, T, p occur.

2. Laminar Disturbance: This is a disturbance in the transmission line of such a magnitude that the concentric layers of fluid retain their same relative radial position in the line.

3. Small Amplitude Disturbance: This is a disturbance of small enough magnitude that none of the instantaneous fluid properties vary from their average fluid properties by more than 10%.

4. Finite Amplitude Disturbance: This is a disturbance of such a magnitude that some of the instantaneous fluid properties vary from

their time-averaged values by more than 10%, but the disturbance is still laminar (see 2 above);

5. Laminar Through Flow: This is incompressible Poiseuille flow with the characteristic parabolic axial velocity profile. The Reynolds number of the through flow based on average axial velocity is less than 2000 and the centerline Mach number is less than about 0.4.

Related Literature

Goodson(10),(11) has published an excellent historical account and up-to-date summary of transmission line literature from the year 1808 to the present. Only that portion of the total literature which relates directly to this thesis is presented here.

Small Amplitude Disturbance Models

Iberall(12), 1950, developed the solution for viscous attenuation in instrument lines, including heat transfer effects. His primary objective was "to simplify the design of high-quality transmission lines for relatively low frequencies." The form of the axial momentum and energy equations which he used are shown below:

Axial Momentum.

$$\frac{\partial v_z}{\partial t} - \frac{\nu_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\rho_0}\frac{\partial p_z}{\partial z} \tag{1.2}$$

Energy Equation (and Continuity).

$$\frac{\partial T}{\partial t} - \frac{\gamma \nu_0}{\sigma_0 r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{T_0}{\rho_0}(\gamma-1)\frac{\partial \rho}{\partial t} \tag{1.3}$$

where v_z = axial velocity

r = tube radius $(0 \leq r \leq a)$

a = tube inner radius

p = transient pressure

T = transient temperature.

Iberall showed that the viscous attenuation parameter (Ψ_t) for the line is of the form:

$$\Psi_t = \sqrt{\frac{1 + \frac{2(\gamma - 1) J_1(\Delta)}{\Delta J_0(\Delta)}}{1 - \frac{2 J_1(\psi)}{\psi J_0(\psi)}}} \tag{1.4}$$

where $\Delta = j\sqrt{j\frac{\omega a^2 \sigma_0}{\nu_0}}$ and $\psi = j\sqrt{j\frac{\omega a^2}{\nu_0}}$ (1.5)

J_o and J_1 are Bessel Functions of the first kind, zeroeth and first order, respectively.

The basic restrictions on Iberall's solution are:

a) laminar axial disturbances,

b) constant diameter, rigid transmission line, and

c) mean flow velocity much less than the acoustic velocity in the fluid.

These same restrictions apply to all of the analyses discussed in this section.

Nichols(15), 1962, arrived at the same solution of the set of Equations (1.2) and (1.3), using small-signal analysis. He defined such terms as "shunt admittance" and "series impedance":

$$\text{Shunt admittance per unit length} = Y_1 = \frac{-\frac{\partial q}{\partial z}}{\delta p} \tag{1.6}$$

$$\text{Series impedance per unit length} = Z_1 = \frac{-\frac{\partial p}{\partial z}}{q} \tag{1.7}$$

where q = volume flowrate

δp = pressure drop per unit length.

Nichols concentrated on producing design curves and approximations for frequency response.

Brown(3), 1962, explored thoroughly the realm of step and impulse responses for the transmission line model which Iberall had solved in 1950. Iberall and Nichols used Fourier analysis techniques, but Brown employed the Laplace transform, and made the first investigations in the time domain. The Iberall-Nichols-Brown model, in two-port form, is shown below:

$$\begin{bmatrix} p_b \\ w_b \end{bmatrix} = \begin{bmatrix} \cosh \Gamma(S) & -Z_e(S) \sinh \Gamma(S) \\ \dfrac{-\sinh \Gamma(S)}{Z_e(S)} & \cosh \Gamma(S) \end{bmatrix} \begin{bmatrix} p_a \\ w_a \end{bmatrix} \tag{1.8}$$

where subscripts "a" and "b" represent the two ends of the transmission line,

$$\Gamma(S) = \frac{SL}{C_o} \sqrt{\frac{1 + \dfrac{2(\gamma-1) J_1(\Delta)}{\Delta J_o(\Delta)}}{1 - \dfrac{2 J_1(\psi)}{\psi J_o(\psi)}}} \tag{1.9}$$

and

$$Z_e(S) = \frac{C_o}{\pi a^2} \frac{1}{\sqrt{\left(1 + \dfrac{2(\gamma-1) J_1(\Delta)}{\Delta J_o(\Delta)}\right)\left(1 - \dfrac{2 J_1(\psi)}{\psi J_o(\psi)}\right)}} \tag{1.10}$$

It will be convenient in this thesis to write $\Gamma(S)$ and $Z_e(S)$ as:

$$\Gamma(S) = \frac{SL}{C_o}\sqrt{\frac{N_g}{D_g}} \quad \text{and} \quad Z_e(S) = \frac{\dfrac{C_o}{\pi a^2}}{\sqrt{N_g D_g}} \tag{1.11}$$

where

$$N_g = \left(1 + \frac{2(\gamma-1) J_1(\Delta)}{\Delta J_o(\Delta)}\right), \quad D_g = \left(1 - \frac{2 J_1(\psi)}{\psi J_o(\psi)}\right) \tag{1.12}$$

$$\Delta = j\sqrt{\frac{S a^2 \sigma_o}{\nu_o}}, \quad \text{and} \quad \psi = j\sqrt{\frac{S a^2}{\nu_o}} \tag{1.13}$$

Brown(3) considered both gases and liquids in his analysis. For the liquid case, $\gamma = 1.0$ and Equations (1.11) reduce to a simpler form.

Approximations for $\Gamma(S)$ and $Z_e(S)$

In the frictionless case, $\Gamma(S) = \frac{SL}{C_o}$ and $Z_e(S) = \frac{C_o}{\pi a^2}$. When friction is included however, $\Gamma(S)$ and $Z_e(S)$ take on the complex forms of Equations (1.9) and (1.10). In this case the Laplace domain model (Equation (1.8)) is very difficult to inverse transform.

Goodson(10), 1963, considered approximations for $\Gamma(S)$ and $Z_e(S)$ for liquids, that is, when $N_g = 1.0$:

$$\Gamma(S)_{liquids} = \frac{SL}{C_o}\sqrt{\frac{1}{(D_g)_{exact}}} \approx \frac{SL}{C_o}\sqrt{\frac{1}{(D_g)_{approx}}} \tag{1.14}$$

$$Z_e(S)_{liquids} = \frac{C_o}{\pi a^2}\sqrt{\frac{1}{(D_g)_{exact}}} \approx \frac{C_o}{\pi a^2}\sqrt{\frac{1}{(D_g)_{approx}}} \tag{1.15}$$

where $$(D_g)_{exact} = \left[1 - \frac{2J_1(\psi)}{\psi J_0(\psi)}\right] = \frac{J_2(\psi)}{J_0(\psi)} = -\frac{\psi^2}{8}\prod_{n=1}^{\infty}\left[\frac{1-\frac{\psi^2}{\alpha^2(2,n)}}{1-\frac{\psi^2}{\alpha^2(0,n)}}\right] \tag{1.16}$$

and $$(D_g)_{approx} = \frac{-\psi^2\left(1-\frac{\psi^2}{B_1}\right)}{8\left(1-\frac{\psi^2}{5.78}\right)\left(1-\frac{\psi^2}{B_2}\right)} = \frac{5.78 B_2 S(S+B_1 DN)}{8 B_1 (S+5.78DN)(S+B_2 DN)} \tag{1.17}$$

and $$\psi^2 = -\frac{Sa^2}{\nu_o}, \quad S = j\omega, \quad DN = \text{Damping Number} = \frac{\nu_o}{a^2} \tag{1.18}$$

The quantity $\alpha(0,n)$ is the n^{th} zero of $J_0(\psi)$ and the quantity $\alpha(2,n)$ is the n^{th} zero of $J_2(\psi)$.

To solve for B_1 and B_2 in Equation (1.17), Goodson first required that the limit of the approximate function equal the limit of the exact function as "S" approached (+) infinity.

$$\lim_{S\to\infty}(D_g)_{approx} = \lim_{S\to\infty}(D_g)_{exact} = 1.0 \Rightarrow \frac{B_2}{B_1} = \frac{8}{5.78} \qquad (1.19)$$

Then Goodson required that the value of B_1 be chosen so that "the magnitude at the value of $\frac{S}{DN}$ where the angle is maximum of the function involving B_2 coincides with the magnitude of the infinite product at the same value of $\frac{S}{DN}$." Goodson's results are $B_1 = 40.9$ and $B_2 = 56.6$. Then:

$$(D_g)_{approx} = \frac{S \ (S + 40.9 \ DN)}{(S + 5.78 \ DN) \ (S + 56.6 \ DN)} \qquad (1.20)$$

Equation (1.20) is equally valid when approximating $\Gamma(S)$ for an ideal gas, but the factor "N_g" of $\Gamma(S)$ is not equal to 1.0 in this case (see Equation (3.4)). Plots of $|D_g|$ exact and $|D_g|$ approx are shown on Figure 2. The development of a corresponding approximation for N_g is considered in Chapter III.

Small Amplitude Disturbance Studies

With Through Flow

Orner(17), 1969, used the same type of Fourier analysis as Iberall and Nichols, but he included the convective acceleration term in the axial momentum equation to account for through flow. That is:

$$\frac{\partial v_z}{\partial t} + \underline{\underline{v_z \frac{\partial v_z}{\partial z}}} - \frac{\nu_o}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = -\frac{1}{\rho_o}\frac{\partial p_z}{\partial z} \qquad (1.21)$$

Orner represented the axial velocity (v_z) as the sum of two components - a steady-state incompressible through flow component plus a compressible transient flow component:

$$v_z(t,r,z) = v_c(r) + v_t(t,r,z) \qquad (1.22)$$

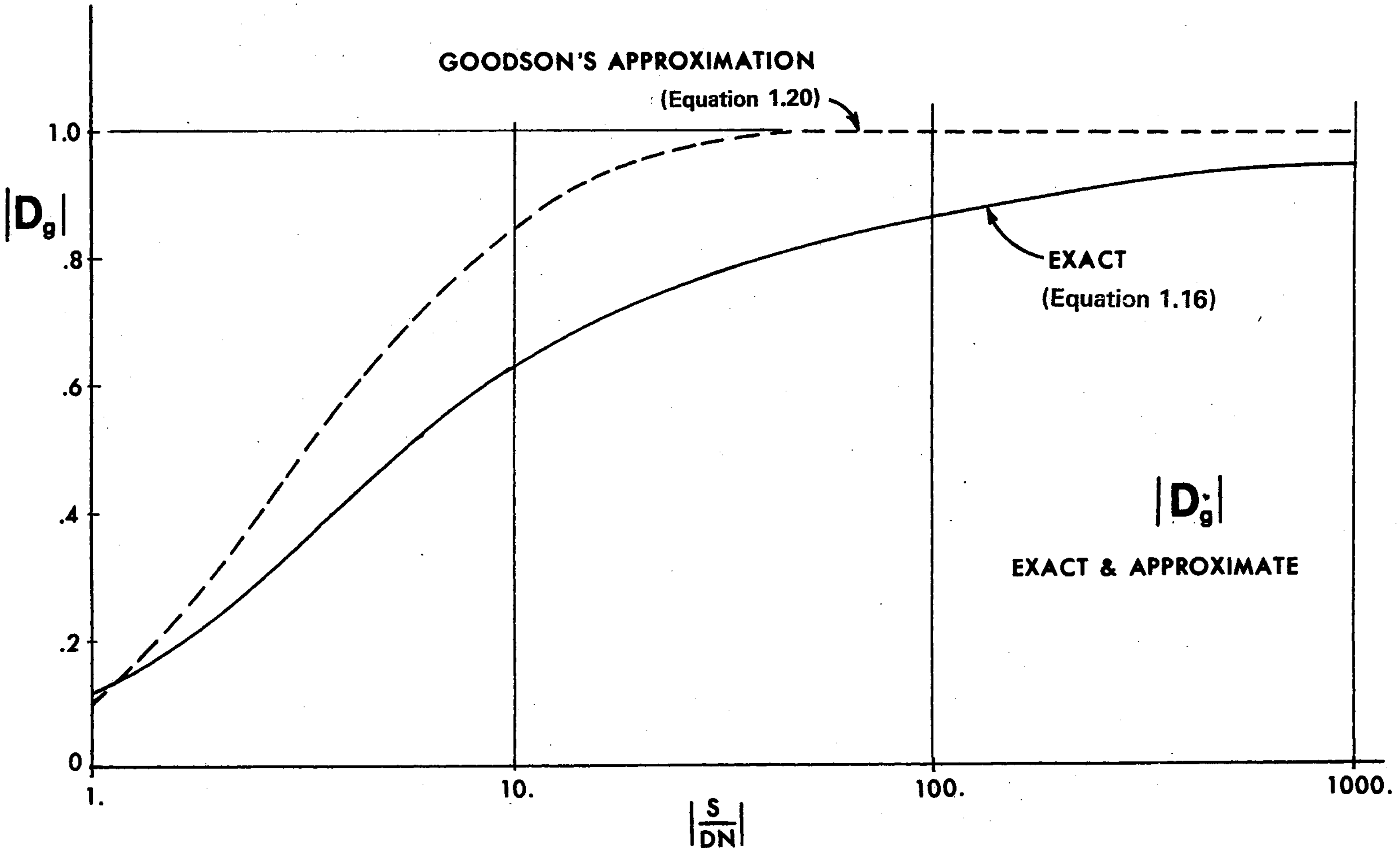

Figure 2. Goodson's Approximation for "D_g"

where $v_c(r)$ is the parabolic (Poiseuille) flow profile. Then Orner neglected the transient velocity (v_t) compared to (v_c), and approximated the convective acceleration term as follows:

$$v_z \frac{\partial v_z}{\partial z} \approx v_c \frac{\partial v_t}{\partial z} \tag{1.23}$$

Orner's solution is in terms of the confluent hypergeometric series, which "have not been tabulated to date" (1969). He performed a perturbation solution on his system of equations, but the solution did not compare well with the experimental data collected by his co-worker, Cooley(7). That is, Orner's analytical solution did not predict the large changes in frequency response with and without through flow which Cooley found by experiment.

Cooley(7), 1969, performed a series of experiments on a 0.125 inch diameter rigid line, 6.0 inches long. He measured frequency responses with various through flows (up to a Reynolds number of 2200), with a constant time-average line pressure of 3.0 psi absolute. Throughout the experiments, Cooley kept a constant ratio of transient flow to steady flow of 0.1, so the transient flow magnitude was increased as the through flow was increased. A portion of his results are shown on Figure 16 (Chapter VI).

Time Domain Studies

Kantola(13), 1969, measured a series of step responses for pneumatic lines of different diameters and lengths. He generated the "step" input by placing a metal diaphragm over the open end of the line, charging or evacuating the line to some pressure above or below ambient pressure, then bursting the diaphragm by mechanical means. Part of Kantola's results are shown on Figure 1, in the introduction to this thesis. The responses demonstrate the nonlinear characteristics of a pneumatic line when subjected to finite amplitude disturbances.

Organization of the Thesis

Chapter II

This chapter discusses the solution of a linearized form of the axial momentum and energy equations. The convective terms $v_z \frac{\partial v_z}{\partial z}$ and $v_z \frac{\partial T}{\partial z}$ are retained in these equations. The solution accounts for the effects of through flow and finite amplitude disturbances.

Chapter III

The model derived in Chapter II includes terms such as Cosh $\Gamma(S)$, Sinh $\Gamma(S)$, and $\Gamma(S)$. To use the model in the time domain for general cases, some approximations for these functions must be made. The approximations are listed in this chapter.

Chapter IV

Experimental procedures used to record small and large amplitude step responses for a blocked 60 ft, 0.40 inch diameter line are presented. The step responses were measured for positive-going and negative-going steps of $\pm$ 0.25, 1, 2, 4, 6, 8, and 10 psig with an ambient pressure of 11.2 psia. The experimental work was conducted at the U. S. Air Force Academy, Department of Aeronautics.

Chapter V

This chapter compares the experimental results of Chapter IV with the analytical model from Chapters II and III, in the time domain. Computed responses for 0.25 and 4.0 psig steps are shown and compared with experimental results. The experimental results show considerable high frequency content but the computed responses display only low frequency content, as would be expected (since the approximations used in the Laplace domain model are low frequency approximations.)

To compare the effect of finite amplitude disturbances in the model and in the experiment, the model damping was adjusted so that the computed response to a 0.25 psig step approximated the apparent fundamental mode (the low frequency mode) of the corresponding experimental response. Then it was possible to compare the effect of finite amplitude disturbances in the model and in the experiment.

Chapter VI

Available test data for the frequency response of a small pneumatic line with through flow is examined briefly. It is concluded that the solution offered in this thesis cannot predict the large changes reported by Cooley(7). A similar conclusion is reached about the Orner(17) solution.

Chapter VII

The basic model derivation in Chapter II assumed an ideal gas. This chapter simplifies the model for use with liquids. Computed step responses using the hydraulic (liquid) equations with both small and finite amplitude steps are shown.

Chapter VIII

This chapter includes a short summary, conclusions, and recommendations for further work.

CHAPTER II

ANALYTICAL MODEL

This chapter presents a solution to a nonlinear form of the axial momentum and energy equations for flow of a compressible fluid in a rigid, circular transmission line. The solution considers finite amplitude disturbances, with and without through flow inthe line.

The coordinate system for the line is illustrated in Figure 3 below.

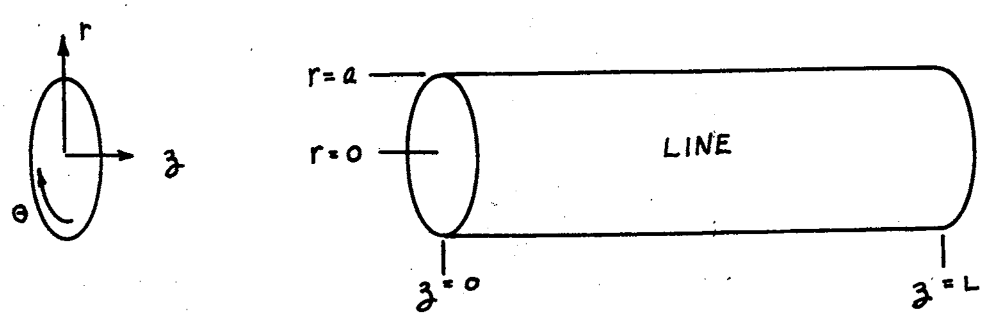

Figure 3. Coordinate System

Basic Assumptions

1. The line is rigid, circular in cross section, and has constant cross-sectional area.

2. The fluid is Newtonian, either an ideal gas or a liquid. The analysis in this chapter is valid for ideal gases; Chapter VII will consider the simpler case of a liquid.

3. The transient is "laminar" in nature (see "Definition of Terms," in Chapter I).

4. All fluid properties may be considered constant. These properties may be calculated at the average conditions in the line.

5. The through flow is laminar, incompressible Poiseuille flow (see "Definition of Terms", in Chapter I).

6. The time-varying pressure is uniform across any given cross section of the transmission line; i.e., pressure is not a function of the radial coordinate, (r).

7. $\frac{\partial^2 v_z}{\partial z^2} \ll \frac{\partial^2 v_z}{\partial r^2}$ and $\frac{\partial^2 T}{\partial z^2} \ll \frac{\partial^2 T}{\partial r^2}$ (D'Souza (8)).

8. The axial velocity, temperature, and pressure at any point within the line each may be represented as the sum of two components - an incompressible steady-state component (subscripted with a "c"), and a compressible, time-varying component (subscripted with a "t") which is superimposed onto the steady-state part. Thus:

$$v_z(t,r,z) = v_c(r) + v_t(t,r,z)$$

$$T_z(t,r,z) = T_c(r) + T_t(t,r,z)$$

$$p_z(t,z) = p_c(z) + p_t(t,z) \tag{2.1}$$

9. $v_r = v_\theta = 0$.

10. All partial derivatives with respect to θ are 0.

11. Isothermal walls

12. The line is long enough that radial end effects are negligible.

Derivation

The steps used in the derivation of the analytical model are summarized below:

1. Write the nonlinear Axial Momentum (AM) and Energy (EE) equations.

2. Solve the linear small-disturbance (AM) and (EE) equations for steady-state operation, and substitute the results into the non-linear (AM) and (EE) equations. The resulting (AM) and (EE) equations are "perturbations" about the steady-state.

3. Nondimensionalize (AM) and (EE).

4. Linearize the resulting dimensionless (AM) and (EE) equations.

5. Transform the linearized (AM) and (EE) equations, transient mass flowrate equation (TM), and integrated continuity equation (IC) to the Laplace Domain to eliminate the independent variable, "time."

6. Solve (AM) for the axial velocity profile V(S,R,Z), and substitute the solution into (TM). Solve (EE) for the axial temperature profile T(S,R,Z), and substitute the solution into (IC).

7. Integrate the (TM) and (IC) equations with respect to (R), and eliminate the independent variable (R).

8. Differentiate (TM) with respect to (Z), equate the result to (IC), and obtain a second order ordinary differential equation (SE) in P(S,Z).

9. Assume a solution for (SE) of the form:

$$P(s,z) = C_1 e^{\Gamma(s)z} + C_2 e^{-\Gamma(s)z} \quad (2.3)$$

Solve (SE) for P(S,Z); obtain the solution (SN).

10. Apply boundary conditions at Z = 0 and Z = 1 to the system of equations composed of (SN) and (TM). Solve for arbitrary constants (C_1) and (C_2) in 9 above.

11. Write the final solution (the trnasmission line model) in standard matrix form.

Basic Equations

With the assumptions listed at the beginning of this chapter, the describing equations may be written as shown below.

Axial Momentum

$$\frac{\partial v_t}{\partial t} + (v_c + v_t)\frac{\partial v_t}{\partial z} - \frac{\nu_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(v_c+v_t)}{\partial r}\right) = -\frac{1}{\rho_0}\frac{\partial(p_c+p_t)}{\partial z} \quad (2.4)$$

Energy Equation

$$\frac{\partial T_t}{\partial t} + (v_c+v_t)\frac{\partial T_t}{\partial z} - \frac{\gamma \nu_0}{\sigma_0 r}\frac{\partial}{\partial r}\left(r\frac{\partial(T_c+T_t)}{\partial r}\right) = -(\gamma-1)T_0\frac{\partial v_t}{\partial z} \quad (2.5)$$

Equation of State (Ideal Gases)

$$\frac{dp}{p_0} = \frac{d\rho}{\rho_0} + \frac{dT}{T_0} \Rightarrow \frac{\partial \rho}{\partial t} = \rho_0\left(\frac{1}{p_0}\frac{\partial p}{\partial t} - \frac{1}{T_0}\frac{\partial T}{\partial t}\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial z} = \rho_0\left(\frac{1}{p_0}\frac{\partial p}{\partial z} - \frac{1}{T_0}\frac{\partial T}{\partial z}\right) \quad (2.6)$$

Continuity Equation (Transient Flow)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_t)}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial v_t}{\partial z} = -\frac{1}{\rho_o}\left(\frac{\partial \rho}{\partial t} + v_t \frac{\partial \rho}{\partial z}\right) \tag{2.7}$$

Equations (2.6) and (2.7) combine to yield:

$$\frac{\partial v_t}{\partial z} = -\left(\frac{1}{p_o}\frac{\partial p}{\partial t} - \frac{1}{T_o}\frac{\partial T}{\partial t}\right) - v_t\left(\frac{1}{p_o}\frac{\partial p}{\partial z} - \frac{1}{T_o}\frac{\partial T}{\partial z}\right) \tag{2.8}$$

Integrated Continuity Equation (Transient Flow)

$$2\pi \int_{r=0}^{r=a} \frac{\partial(\rho v_t)}{\partial z} r\,dr = -2\pi \int_{r=0}^{r=a} \frac{\partial \rho}{\partial t} r\,dr \tag{2.9}$$

$$\Rightarrow \frac{\partial w(t,z)}{\partial z} \approx -2\pi \int_{r=0}^{r=a} \rho_o\left(\frac{1}{p_o}\frac{\partial p}{\partial t} - \frac{1}{T_o}\frac{\partial T}{\partial t}\right) r\,dr \tag{2.10}$$

where w(t,z) is the time-varying mass flowrate superimposed on the through flow in the transmission line. That is:

$$w(t,z) = 2\pi \int_{r=0}^{r=a} (\rho v_t)\, r\,dr \tag{2.11}$$

Steady-State Solutions

Equations (2.4) and (2.5) reduce to the linear (small amplitude) case when the convective terms are neglected. In the steady-state these equations become those listed below.

Steady-State Axial Momentum

$$\frac{\nu_o}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_c}{\partial r}\right) = -\frac{1}{\rho_o}\frac{\partial p_c}{\partial z} \tag{2.12}$$

Steady-State Energy Equation

$$\frac{\gamma \nu_0}{\sigma_0 r} \frac{\partial}{\partial r}\left(r \frac{\partial T_c}{\partial r} \right) = 0 \tag{2.13}$$

The solution to Equation (2.12) is:

$$v_c = v_{max}\left(1 - \frac{r^2}{a^2}\right) \tag{2.14}$$

where (v_{max}) is the centerline velocity ($r = 0$), and

$$\frac{\partial p_c}{\partial z} = \frac{-4 \mu_0 v_{max}}{a^2} \tag{2.15}$$

The solution to Equation (2.11) is

$$T_c = \text{constant} \tag{2.16}$$

Substitution of Equations (2.12) and (2.13) into Equations (2.4) and (2.5) yields the equations listed below.

Axial Momentum

$$\frac{\partial v_t}{\partial t} + (v_c + v_t)\frac{\partial v_t}{\partial z} - \frac{\nu_0}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_t}{\partial r} \right) = -\frac{1}{\rho_0}\frac{\partial p_t}{\partial z} \tag{2.17}$$

Energy Equation

$$\frac{\partial T_t}{\partial t} + (v_c + v_t)\frac{\partial T_t}{\partial z} - \frac{\gamma \nu_0}{\sigma_0 r}\frac{\partial}{\partial r}\left(r \frac{\partial T_t}{\partial r} \right) = -(\gamma - 1) T_0 \frac{\partial v_t}{\partial z} \tag{2.18}$$

Nondimensional Equations

Equations (2.17), (2.18), (2.10), and (2.11) may be nondimensionalized with the following substitutions:

$$R = \frac{r}{a} \;;\; Z = \frac{z}{L} \;;\; P = \frac{p_t}{p_o} \;;\; V = \frac{v_t}{C_o} \;;$$

$$T = \frac{T_t}{T_o} \;;\; V_f = \frac{v_c}{C_o} \;;\; Q = \frac{w(t,z)}{\rho_o C_o \pi a^2} \qquad (2.19)$$

where (C_o) is the isentropic speed of sound in the fluid, $\sqrt{\frac{\gamma p_o}{\rho_o}}$.

Axial Momentum

$$\frac{\partial V}{\partial t} - \frac{\nu_o}{a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) = -\frac{C_o}{L}\left(\frac{1}{\gamma}\frac{\partial P}{\partial Z} + (V_f + V)\frac{\partial V}{\partial Z}\right) \qquad (2.20)$$

Energy Equation (With Equation (2.8))

$$\frac{\partial T}{\partial t} - \frac{\nu_o}{\sigma_o a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) = \frac{(\gamma-1)}{\gamma}\left[\frac{\partial P}{\partial t} + \frac{C_o}{L}\left\{V\left(\frac{\partial P}{\partial Z} - \frac{\gamma}{(\gamma-1)}\frac{\partial T}{\partial Z}\right) - \frac{V_f}{(\gamma-1)}\frac{\partial T}{\partial Z}\right\}\right] \qquad (2.21)$$

Integrated Continuity Equation

$$\frac{\partial Q(t,Z)}{\partial Z} = \frac{-2L}{C_o}\int_0^1 \left(\frac{\partial P}{\partial t} - \frac{\partial T}{\partial t}\right) R\,dR \qquad (2.22)$$

Transient Mass Flowrate

$$Q(t,Z) = 2\int_0^1 V(t,R,Z)\,R\,dR \qquad (2.23)$$

Approximations and Linearization

An earlier investigation by Orner (17) neglected all the nonlinear terms on the right side of Equation (2.21). The order of magnitude of these terms may be examined by substituting the expressions for $\frac{\partial P}{\partial Z}$ and $\frac{\partial T}{\partial Z}$ which result from the small disturbance solution, Appendix A, into Equation (2.21). From Equations (A.51) and (A.52):

$$\frac{\partial P}{\partial Z} - \frac{\gamma}{(\gamma-1)}\frac{\partial T}{\partial Z} \approx 0 \tag{2.24}$$

The remaining term $\frac{V_f}{\gamma}\frac{\partial T}{\partial Z}$ was also neglected by Orner (17) since $|V_f| < 0.2$ and $\left|\frac{\partial T}{\partial Z}\right| \ll \left|\frac{\partial P}{\partial t}\right|$ (D'Souza (8)). With the above two approximations, Equation (2.21) reduces to the linear form:

$$\frac{\partial T}{\partial t} - \frac{\nu_0}{\sigma_0 a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) = \frac{(\gamma-1)}{\gamma}\frac{\partial P}{\partial t} \tag{2.25}$$

The order of magnitude of the right side of Equation (2.20) may also be examined by substituting in the known expressions for $\frac{\partial P}{\partial Z}$ and $\frac{\partial V}{\partial Z}$ from Appendix A. Using Equations (A.49) and (A.52), the right hand side of Equation(2.20) becomes:

$$\left[\frac{1}{\gamma}\frac{\partial P}{\partial Z} + (V_f+V)\frac{\partial V}{\partial Z}\right] \approx -\frac{L}{C_0}\left[\frac{\partial Q(t,0)}{\partial t} + (V_f+V)\frac{\partial P(t,0)}{\partial t}\right] \tag{2.26}$$

where Q(t,0) and P(t,0) are nondimensional boundary conditions at Z = 0. For fast transients it appears that the value of the term $(V_f+V)\frac{\partial P(t,0)}{\partial t}$ may be of the same order or larger than the term $\frac{\partial Q(t,0)}{\partial t}$, even though (V_f+V) may be small.

There are three independent variables, (t,R,Z), in the system of Equations (2.20), (2.25), (2.22), and (2.23). One way to eliminate the variable "time" is to apply the Laplace Transform to the system of equations. But Equation (2.20) must first be linearized.

The method of linearization used by Zielke (22) and Brown (5) when they solved Equation (2.20) by a modified method of characteristics was to make the term $V\frac{\partial V}{\partial Z}$ a "weighting function" which "forced" the homogeneous linear equation shown in Equation (2.27) below.

Axial Momentum

$$\frac{\partial V}{\partial t} - \frac{\nu_0}{a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) + \frac{C_0}{\gamma L}\frac{\partial P}{\partial Z} = -\frac{C_0}{L} V \frac{\partial V}{\partial Z} \tag{2.27}$$

The term (V_f) is missing on the right side of Equation (2.27) since Zielke and Brown did not consider through flow in their analyses. In effect, the term $V\frac{\partial V}{\partial Z}$ was assigned a constant value at some spatial coordinate (R,Z) at a particular time (t). This method of linearization, with some modification, will be used in this thesis.

The term $V\frac{\partial V}{\partial Z}$ in Equation (2.20) may be linearized by fixing either (V) or $\left(\frac{\partial V}{\partial Z}\right)$ at some particular time (t), but not both in the same term. That is, either (V) or $\left(\frac{\partial V}{\partial Z}\right)$ may be designated as a time-varying coefficient which must be recalculated and updated at intervals in the time domain solution. The time-varying coefficient will be designated in this thesis with a subscript (*).

This type of linearization is valid only for some small period of time (Δt), where (Δt) is much less than the reciprocal of the highest frequency of interest in the response of the line (ω_{max}). That is,

$$(\Delta t) << \frac{1}{\omega_{max}} \tag{2.28}$$

where ω_{max} is in radians per unit time.

The term $V_f \frac{\partial V}{\partial Z}$ in Equation (2.20) is already linear since V_f is not a function of time. To calculate the time-varying coefficients, the form of their solutions from the acoustic model (Appendix A) will be used. These forms are given as Equations (A.48) and (A.49).

By using Equations (A.48) and (A.49) the term $V\frac{\partial V}{\partial Z}$ may be represented in the linear forms shown below.

Method 1. Fix V for a given time increment.

$$V\frac{\partial V}{\partial Z} \approx V_* \frac{\partial V}{\partial Z} \tag{2.29}$$

where $V_* = \left[-\frac{LZ}{C_o} \frac{\partial P(t,0)}{\partial t} + Q(t,0) \right]_*$

When this method of linearization is used, both (V) and $\left(\frac{\partial V}{\partial Z}\right)$ must be averaged over (R). (V) is represented by a uniform axial velocity profile, and $\frac{\partial V}{\partial Z}$ must be averaged over (R) to make Equation (2.20) separable.

Method 2. Fix $\frac{\partial V}{\partial Z}$ for a given time increment.

$$V\frac{\partial V}{\partial Z} \approx V(t,R,Z)\left(\frac{\partial V}{\partial Z}\right)_* \tag{2.30}$$

where $\left(\frac{\partial V}{\partial Z}\right)_* = \left[-\frac{L}{C_o} \frac{\partial P(t,0)}{\partial t} \right]_*$

When this method of linearization is used, only $\frac{\partial V}{\partial Z}$ is averaged over (R). Thus, method 2 should be a more accurate method of linearization, and is the only method pursued in the body of this thesis. Appendix C shows the result obtained by combining both Method 1 and Method 2. This combination produced a model which was more stable numerically than the model which used the Method 2 linearization only, and may be useful under some circumstances as discussed in Chapter VIII (Summary and Conclusions).

One of the criteria for the transmission line model (as stated in Chapter I) is that the model should exhibit greater apparent damping as disturbance amplitude increased. This criterion is based on observation of actual experiments on pneumatic lines. The form for $\left(\frac{\partial V}{\partial Z}\right)_*$ shown as Equation (2.30) produced greater apparent damping as disturbance amplitude increased for negative-going step inputs, but produced less apparent damping for large disturbance amplitudes on positive-going step inputs. To correct this discrepancy the following form was used for $\left(\frac{\partial V}{\partial Z}\right)_*$:

$$\left(\frac{\partial V}{\partial Z}\right)_* = \left(\operatorname{sgn} P(t,0)\right)\left(\frac{L}{C_o}\frac{\partial P(t,0)}{\partial t}\right)_* \qquad (2.31)$$

This form for $\left(\frac{\partial V}{\partial Z}\right)_*$ produced a line model which exhibited greater apparent damping for larger disturbances regardless of the sign of the disturbance.

Rewriting the Axial Momentum Equation (2.20) using the second method of linearization yields:

$$\frac{\partial V(t,R,Z)}{\partial t} + \frac{C_o}{L}\left(\frac{\partial V}{\partial Z}\right)_* V(t,R,Z) - \frac{\nu_o}{a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial V(t,R,Z)}{\partial R}\right) = -\frac{C_o}{L}\left(\frac{1}{\gamma}\frac{\partial P(t,Z)}{\partial Z} + M_b\frac{\partial V(t,Z)}{\partial Z}\right) \qquad (2.32)$$

where $\left(\frac{\partial V}{\partial Z}\right)_*$ is given as Equation (2.31), and $M_b = (V_f)$ averaged over (R).

$$M_b = 2\int_0^1 V_f R\,dR = M_{avg} = \frac{M_{cl}}{2} = \frac{\nu_o Re}{2C_o a} \qquad (2.33)$$

where

M_{cl} = Mach number of the through flow based on centerline velocity,

M_{avg} = Mach number of the through flow based on average velocity,

Re = Reynolds number based on average through flow velocity.

Transformation Into the Laplace Domain

For the small increment of time (Δt) as defined in Equation (2.28), Equations (2.32), (2.25), (2.22), and (2.23) may be transformed into the Laplace domain. The results are shown below.

Axial Momentum

$$V(s,R,Z)\left(1+\frac{C_o}{sL}\left(\frac{\partial V}{\partial Z}\right)_*\right)-\frac{\nu_o}{sa^2R}\frac{\partial}{\partial R}\left(R\frac{\partial V(s,R,Z)}{\partial R}\right)=-\frac{C_o}{sL}\left(\frac{1}{\gamma}\frac{\partial P(s,Z)}{\partial Z}+M_b\frac{\partial V(s,Z)}{\partial Z}\right) \tag{2.34}$$

Energy Equation

$$T(s,R,Z)-\frac{\nu_o}{s\sigma_o a^2R}\frac{\partial}{\partial R}\left(R\frac{\partial T(s,R,Z)}{\partial R}\right)=\frac{(\gamma-1)}{\gamma}P(s,Z) \tag{2.35}$$

Integrated Continuity

$$\frac{\partial Q(s,Z)}{\partial Z}=\frac{-2sL}{C_o}\int_0^1\Big(P(s,Z)-T(s,R,Z)\Big)R\,dR \tag{2.36}$$

Transient Mass Flowrate

$$Q(s,Z)=2\int_0^1 V(s,R,Z)\,R\,dR \tag{2.37}$$

Solution of the Axial Momentum and Energy Equations

Equations (2.34) and (2.35) are made separable by assuming a product form of solution:

$$V(s,R,Z)=G_1(s,R)\;G_2(s,Z)$$

$$\text{and}\quad T(s,R,Z)=G_3(s,R)\;G_4(s,Z) \tag{2.38}$$

The term $\frac{\partial V(S,Z)}{\partial Z}$ on the right-hand side of Equation (2.34) may be approximated by its small disturbance solution, Equation (A.42). Rewriting Equations (2.34) and (2.35) with the substitution of Equations (2.38) and (A.42) yields the equations given below.

Axial Momentum

$$G_1\left(1+\frac{F_{1*}}{S}\right)-\frac{\nu_0}{S\alpha^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial G_1}{\partial R}\right)=-\frac{C_0}{G_2\gamma SL}\left(\frac{\partial P}{\partial Z}-\frac{C_0 D_g M_b}{SL}\frac{\partial^2 P}{\partial Z^2}\right) \tag{2.39}$$

where $D_g=\left(1-\frac{2J_1(\psi)}{\psi J_0(\psi)}\right)$ from Equations (A.40)

and $$F_{1*}=\frac{C_0}{L}\left(\frac{\partial V}{\partial Z}\right)_*=(\text{sgn } P(t,0))\left(\frac{\partial P(t,0)}{\partial t}\right)_* \tag{2.40}$$

Energy Equation

$$G_3-\frac{\nu_0}{S\sigma_0\alpha^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial G_3}{\partial R}\right)=\frac{(\gamma-1)}{G_4\gamma}P \tag{2.41}$$

Choose $$G_2=-\frac{C_0}{\gamma SL}\left(\frac{\partial P}{\partial Z}-\frac{C_0 D_g M_b}{SL}\frac{\partial^2 P}{\partial Z^2}\right) \tag{2.42}$$

and $$G_4=\frac{(\gamma-1)P}{\gamma} \tag{2.43}$$

Let $\alpha=j\sqrt{\frac{S a^2}{\nu_0}\left(1+\frac{F_{1*}}{S}\right)}$ and $\Delta=j\sqrt{\frac{S a^2\sigma_0}{\nu_0}}$ (2.44)

Substitution of Equations (2.42), (2.43), and (2.44) into Equations (2.39) and (2.41) yields:

$$G_1+\frac{1}{\alpha^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial G_1}{\partial R}\right)=1 \tag{2.45}$$

and $$G_3+\frac{1}{\Delta^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial G_3}{\partial R}\right)=1 \tag{2.46}$$

A homogeneous solution to Equation (2.45) is:

$$G_1=C_1\frac{J_0(\alpha R)}{J_0(\alpha)}+C_2\frac{Y_0(\alpha R)}{Y_0(\alpha)} \tag{2.47}$$

where J_o and Y_o are Bessel functions of the first and second kind, zeroeth order. A particular solution to Equation (2.45) is:

$$G_1 = 1 \tag{2.48}$$

Thenthe total solution to Equation (2.45) is:

$$G_1 = 1 + C_1 \frac{J_o(\alpha R)}{J_o(\alpha)} + C_2 \frac{Y_o(\alpha R)}{Y_o(\alpha)} \tag{2.49}$$

From the no-slip boundary condition $G_1\big|_{R=1} = 0$,

$$C_1 + C_2 = -1 \tag{2.50}$$

From the boundary condition $\left.\frac{\partial G_1}{\partial R}\right|_{R=0} = 0$

$$C_2 = 0 \tag{2.51}$$

Then $C_1 = -1$ and:

$$G_1 = -\left(\frac{J_o(\alpha R) - J_o(\alpha)}{J_o(\alpha)}\right) \tag{2.52}$$

Application of the boundary conditions $G_3\big|_{R=1} = 0$ and $\left.\frac{\partial G_3}{\partial R}\right|_{R=0} = 0$ yields the following solution for Equation (2.41):

$$G_3 = -\left(\frac{J_o(\Delta R) - J_o(\Delta)}{J_o(\Delta)}\right) \tag{2.53}$$

where (Δ) is defined in Equation (2.44).

The solution for the axial velocity profile becomes:

$$V(S,R,Z) = \frac{\left(\frac{J_o(\alpha R) - J_o(\alpha)}{J_o(\alpha)}\right) \frac{C_o}{\gamma SL} \left(\frac{\partial P}{\partial Z} - \frac{C_o D_g \dot{M}_b}{SL} \frac{\partial^2 P}{\partial Z^2}\right)}{\left(1 + \frac{F_{1*}}{S}\right)} \tag{2.54}$$

The axial temperature profile becomes:

$$T(S,R,Z) = \left(\frac{J_o(\Delta R) - J_o(\Delta)}{J_o(\Delta)}\right)\left(-\frac{(\gamma-1)}{\gamma} P(S,Z)\right) \tag{2.55}$$

Solutions of the Transient Mass Flowrate and

Integrated Continuity Equations

By substituting Equation (2.54) into Equation (2.37) and integrating with respect to (R), Equation (2.37) takes the form shown below.

Transient Mass Flowrate

$$Q(s,z) = \frac{-\frac{C_o D_a}{\gamma s L}\left(\frac{\partial P(s,z)}{\partial z} - \frac{C_o D_g M_b}{sL}\frac{\partial^2 P(s,z)}{\partial z^2}\right)}{\left(1+\frac{F_{1*}}{s}\right)} \tag{2.56}$$

where $D_a = \left(1 - \frac{2 J_1(\alpha)}{\alpha J_0(\alpha)}\right)$ (2.57)

(D_g) is given in Equations (A.40), (M_b) is Equation (2.33), and (F_{1*}) is Equation (2.40).

Substitution of Equation (2.55) into (2.36) and integrating with respect to (R) yields the equation given below.

Integrated Continuity

$$\frac{\partial Q(s,z)}{\partial z} = -\frac{sL\,N_g}{C_o}P(s,z) \tag{2.58}$$

where $N_g = \left(1 + \frac{2(\gamma-1)J_1(\Delta)}{\Delta J_0(\Delta)}\right)$ (2.59)

Differentiation of Equation (2.56) with respect to (Z) yields:

$$\frac{\partial Q(s,z)}{\partial z} = \frac{-\frac{C_o D_a}{\gamma s L}\left(\frac{\partial^2 P(s,z)}{\partial z^2} - \frac{C_o D_g M_b}{sL}\frac{\partial^3 P(s,z)}{\partial z^3}\right)}{\left(1+\frac{F_{1*}}{s}\right)} \tag{2.60}$$

The purpose of this thesis is to derive a systems model for a transmission line which predicts transients accurately at low and medium frequencies, in the range $0 < \left|\frac{sL}{C_o}\right| < 2\pi$. The term involving $\frac{\partial^3 P(S,Z)}{\partial Z^3}$ in Equation (2.60) is likely significant only at high frequencies, and will be neglected in the analysis which follows.

Ordinary Differential Equations

Neglecting the term $\frac{\partial^3 P(S,Z)}{\partial Z^3}$ in Equation (2.60), and equating Equation (2.60) with Equation (2.58) yields:

$$\frac{\partial^2 P(S,Z)}{\partial Z^2} = \left(\frac{SL}{C_o}\right)^2 \frac{N_g}{D_a}\left(1+\frac{F_{1*}}{S}\right) P(S,Z) \tag{2.61}$$

The solution to Equation (2.61) is of the form:

$$P(S,Z) = C_1 e^{\Gamma_b(S)Z} + C_2 e^{-\Gamma_b(S)Z} \tag{2.62}$$

where $$\Gamma_b(S) = \frac{SL}{C_o}\sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{S}\right)} \tag{2.63}$$

(D_a) is given as Equation (2.57) and (N_g) is Equation (2.59). The accompanying equation which describes flow Q(S,Z) as a function of pressure P(S,Z) is Equation (2.56). By substituting Equation (2.62) into Equations (2.61) and (2.56), this system of equations results:

$$P(S,Z) = C_1 e^{\Gamma_b(S)Z} + C_2 e^{-\Gamma_b(S)Z} \tag{2.64}$$

$$\frac{Q(S,Z)}{A(S)} = C_1(1+E(S))e^{\Gamma_b(S)Z} - C_2(1-E(S))e^{-\Gamma_b(S)Z} \tag{2.65}$$

where $$A(S) = -\frac{C_o D_a \Gamma_b(S)}{\gamma SL\left(1+\frac{F_{1*}}{S}\right)} \tag{2.66}$$

and $$E(S) = -\frac{C_o}{SL} D_g M_b \Gamma_b(S) \tag{2.67}$$

Solution Completion

To complete the solution of the system of Equations (2.64) and (2.65), the boundary conditions at Z=0 and Z=1 must be applied. That is:

$$\mathcal{L}(P(t,0)) = P(s,0) \quad ; \quad \mathcal{L}(Q(t,0)) = Q(s,0) \; ;$$

$$\mathcal{L}(P(t,1)) = P(s,1) \quad ; \quad \mathcal{L}(Q(t,1)) = Q(s,1) \tag{2.68}$$

Applying these boundary conditions to Equations (2.64) and (2.65) yields:

$$\begin{aligned} C_1 &= \frac{1}{2}\left(P(s,0)(1-E(s)) + \frac{Q(s,0)}{A(s)}\right) \\ C_2 &= \frac{1}{2}\left(P(s,0)(1+E(s)) - \frac{Q(s,0)}{A(s)}\right) \end{aligned} \tag{2.69}$$

A combination of Equations (2.64), (2.65), and (2.69) yields the final solution for the system of equations which are shown below.

Summary

$$\begin{bmatrix} P(s,1) \\ Q(s,1) \end{bmatrix} = \begin{bmatrix} \text{Cosh}\,\Gamma_b(s) + Y_b(s) M_b \,\text{Sinh}\,\Gamma_b(s) & -Z_b(s)\,\text{Sinh}\,\Gamma_b(s) \\ -\dfrac{\text{Sinh}\,\Gamma_b(s)}{Z_b(s)} & \text{Cosh}\,\Gamma_b(s) - Y_b(s) M_b\,\text{Sinh}\,\Gamma_b(s) \end{bmatrix} \begin{bmatrix} P(s,0) \\ Q(s,0) \end{bmatrix} \tag{2.70}$$

where

$$\Gamma_b(s) = \frac{sL}{C_o}\sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{s}\right)} \tag{2.71}$$

$$Y_b(s) = \frac{C_o}{sL} D_g \Gamma_b(s) = D_g\sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{s}\right)} \tag{2.72}$$

$$Z_b(s) = \frac{\gamma\, sL\left(1+\frac{F_{1*}}{s}\right)}{C_o \;\; D_a \;\; \Gamma_b(s)} = \gamma\sqrt{\frac{\left(1+\frac{F_{1*}}{s}\right)}{N_g \;\; D_a}} \tag{2.73}$$

$$N_g = \left(1 + \frac{2(\gamma-1)\,J_1(\Delta)}{\Delta\, J_0(\Delta)}\right) \quad ; \quad D_g = \left(1 - \frac{2\,J_1(\psi)}{\psi\, J_0(\psi)}\right) \tag{2.74}$$

$$D_a = \left(1 - \frac{2\,J_1(\alpha)}{\alpha\, J_0(\alpha)}\right)$$

$$\Delta = j\sqrt{\frac{S\,\sigma_0}{DN}} \quad ; \quad \psi = j\sqrt{\frac{S}{DN}} \quad ; \quad \alpha = j\sqrt{\frac{S}{DN}\left(1 + \frac{F_{1*}}{S}\right)} \tag{2.75}$$

$$DN = \frac{\nu_0}{a^2} \quad ; \quad F_{1*} = \frac{C_0}{L}\left(\frac{\partial V}{\partial Z}\right)_* = (\mathrm{sgn}\, P(t,0))\left(\frac{\partial P(t,0)}{\partial t}\right)_* \tag{2.76}$$

M_b = Average through flow mach number. (2.77)

Equations (2.70) represent the solution of the linearized axial momentum equation which includes the convective acceleration term $v_z \frac{\partial v_z}{\partial z}$ (Equation (2.4)), and the linear energy equation. This system of equations will be transformed to the time domain by using appropriate approximations for Γ(s), Cosh Γ(s), Sinh Γ(s), etc. The approximations are shown in Chapter III; transformation to the time domain is shown in Chapter V.

Comparison to Existing Models

Equations (2.70) reduce to the small disturbance solution of Appendix A when through flow and finite amplitude disturbance effects are deleted. That is:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \cosh\Gamma(S) & -Z_c(S)\sinh\Gamma(S) \\ -\dfrac{\sinh\Gamma(S)}{Z_c(S)} & \cosh\Gamma(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{2.78}$$

where $\Gamma(S) = \frac{SL}{C_0}\sqrt{\frac{N_g}{D_g}}$ and $Z_c(S) = \frac{\gamma}{\sqrt{N_g D_g}}$ (2.79)

By deleting the effects of finite amplitude disturbances, but retaining through flow, the result is:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \operatorname{Cosh}\Gamma(s) + Y_e(s) M_b \operatorname{Sinh}\Gamma(s) & -Z_c(s)\operatorname{Sinh}\Gamma(s) \\ -\dfrac{\operatorname{Sinh}\Gamma(s)}{Z_c(s)} & \operatorname{Cosh}\Gamma(s) - Y_e(s) M_b \operatorname{Sinh}\Gamma(s) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{2.80}$$

where

$$Y_e(s) = \sqrt{N_g D_g} \tag{2.81}$$

Orner (17) derived Equation (2.80) by using the Poincare Perturbation technique on the linearized axial momentum Equation (2.32), with $F_{1x} = 0$. This is a valid representation when the disturbance amplitude is small and through flow is large.

Orner's expression for $Y_e(s)$ is:

$$Y_e(s) = \frac{1}{\gamma}\left[1 - \frac{8(\gamma-1)}{\Delta^2}\left(1 - \frac{2J_1(\Delta)}{\Delta J_0(\Delta)}\right)\right] \tag{2.82}$$

where (Δ) is given in Equations (2.75). For $\left|\frac{SL}{C_o}\right| > \pi$, Equations (2.81) and (2.82) yield the same result; that is, $|Y_e(s)| \approx 1.0$. But as frequency approaches zero, Equation (2.82) approaches ∞, and Equation (2.81) approaches zero (since $D_g \rightarrow 0$ as $S \rightarrow 0$). Orner's result for $Y_e(s)$ and this thesis result differ because Orner represented the convective acceleration term as $M_b \frac{\partial V(t,R,Z)}{\partial Z}$ while this thesis used $M_b \frac{\partial V(t,z)}{\partial Z}$. That is, this thesis used an average value of $\frac{\partial V}{\partial Z}$ over the line cross section while Orner used an exact value of $\frac{\partial V}{\partial Z}$ at each point (t, R, Z).

For this reason, Orner's result should be more accurate. The matter seems rather inconsequential, however, since the entire term ($Y_e M_b \sinh \Gamma(s)$) in Equation (2.80) approaches zero so $S \rightarrow 0$, regardless of which form of (Y_e) is used.

CHAPTER III

APPROXIMATIONS FOR $\Gamma(s)$, COSH $\Gamma(s)$, SINH $\Gamma(s)$

To transform Equations (2.70) to the time domain, it is necessary to choose approximations for the functions which appear in these equations. These approximations are listed below.

Approximations for D_g, D_a, and N_g

The functions (D_g), (D_a), and (N_g) are monotonically increasing or decreasing functions as $S \to \infty$, so they may be approximated by relatively simple expressions. Goodson (10) suggested this approximation for (D_g), (see Figure 2):

$$D_g \approx \frac{S\ (S+40.9\,DN)}{(S+5.78\,DN)(S+56.6\,DN)} \tag{3.1}$$

where
$$DN = \text{Damping Number} = \frac{\nu_0}{a^2} \tag{3.2}$$

The basis for this approximation is given in Chapter I, "Related Literature." The Goodson approximation also applies to (D_a), by replacing (S) with $(S + F_{1*})$, where (F_{1*}) is defined as Equation (2.76).

$$D_a \approx \frac{(S+F_{1*})(S+40.9\,DN+F_{1*})}{(S+5.78\,DN+F_{1*})(S+56.6\,DN+F_{1*})} \tag{3.3}$$

There are no published approximations for (N_g), so this form was used (Prandtl number = 0.70):

$$N_g \approx \frac{(S+10\,DN)}{(S+7.14\,DN)} \tag{3.4}$$

This approximation meets the requirements that $|N_g|$ at S = 0 is 1.4, $|N_g|$ at $S = \infty$ is 1.0, and the differences between the approximate and exact magnitudes squared over the region $1 \leq \left|\frac{S}{DN}\right| \leq 1000$ is a minimum. The exact and approximate magnitudes of (N_g) are shown on Figure 4. The exact expression for (N_g) is shown below:

$$N_g = \left(1 + \frac{2(.4)J_1(\Delta)}{\Delta\ J_0(\Delta)}\right) \tag{3.5}$$

and (Δ) is given in Equations (2.75).

Approximations for Sinh Γ(s) and Cosh Γ(s)

The periodic functions Sinh Γ(s) and Cosh Γ(s) each may be represented by a power series expansion. For example, Cosh Γ(s) is given as Equation (3.6):

$$\text{Cosh}\ \Gamma(S) \approx 1 + \frac{\Gamma^2(s)}{2!} + \frac{\Gamma^4(s)}{4!} + \frac{\Gamma^6(s)}{6!} + \cdots \tag{3.6}$$

However, for such an expansion to be accurate when Γ(s) is large, an excessive number of terms must be retained. Also, improper truncation of such an expansion can lead to a numerical instability. Oldenburger (16) has shown that the product-term expansions shown below produce greater accuracy with fewer terms than the conventional power series expansions (like Equation (3.6)), and the resulting series is not as likely to lead to numerical instabilities.

Product-Term Expansions

$$\text{Sinh}\ \Gamma(s) \approx \Gamma(s) \prod_{K=1}^{\infty} \left(1 + \frac{\Gamma^2(s)}{K^2\pi^2}\right) \tag{3.7}$$

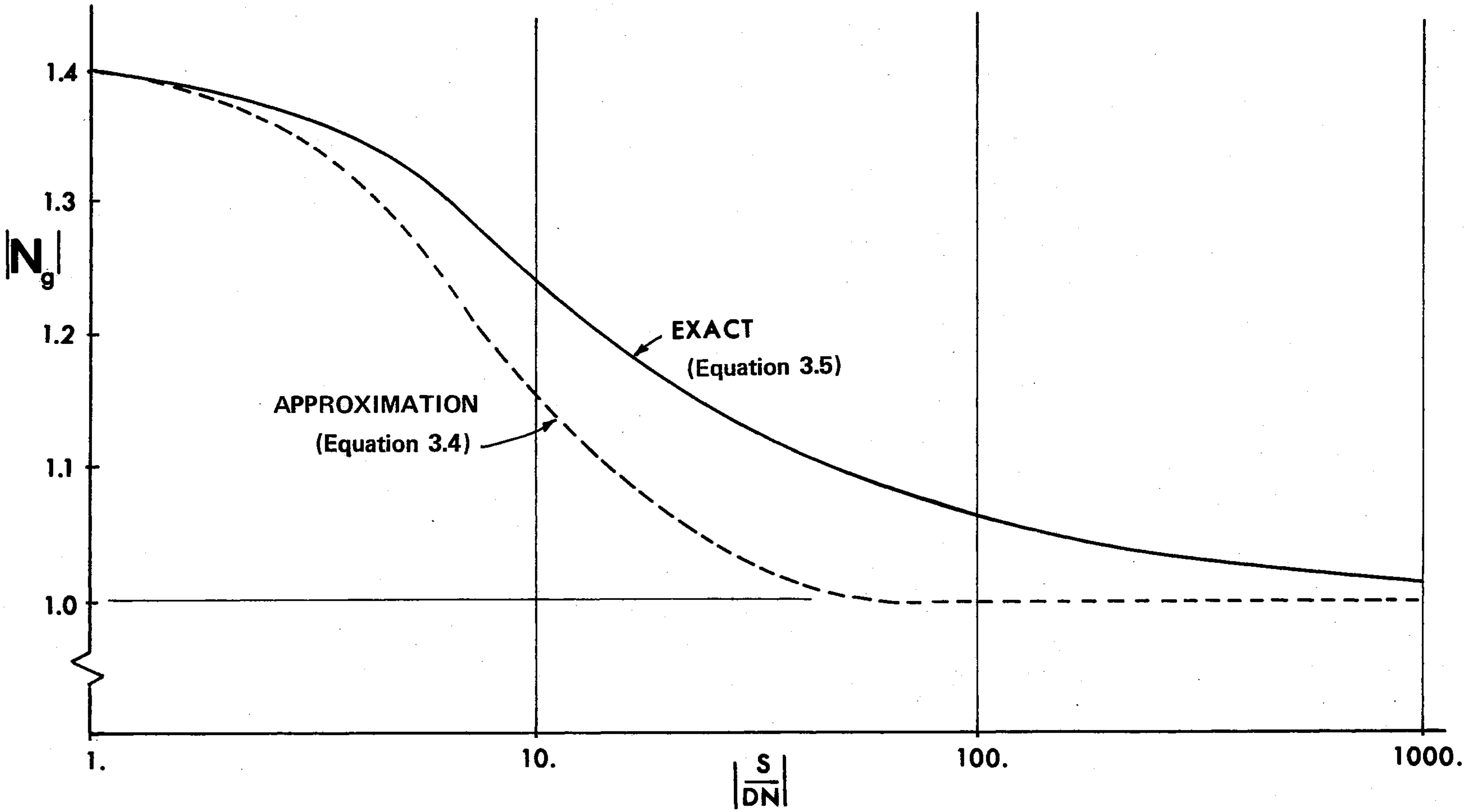

Figure 4. Approximation of "N_g"

$$\text{Cosh}\,\Gamma(s) \approx \prod_{k=1}^{\infty}\left(1 + \frac{4\,\Gamma^2(s)}{(2K-1)^2\pi^2}\right) \tag{3.8}$$

For the step responses in Chapter V of this thesis, Cosh $\Gamma(s)$ was approximated by both Equations (3.6) and (3.8). However, Equation (3.6) was numerically unstable for all but the smallest disturbance amplitudes, so it was discarded in favor of Equation (3.8). Figure 5 illustrates the relative accuracies of one, two, and four product term approximations for Cosh $\Gamma(s)$. For simplicity in plotting, $\Gamma(s)$ was approximated (for this plot only) by the simple lossless form:

$$\Gamma(s) = \frac{SL}{C_o} \tag{3.9}$$

The exact form of Cosh $\Gamma(s)$ is:

$$\text{Cosh}\,\Gamma(s) = \frac{1}{2}\left(e^{\Gamma(s)} + e^{-\Gamma(s)}\right) \tag{3.10}$$

The one, two, and four product-term expansions for Cosh $\Gamma(s)$ based on the lossy form of $\Gamma(s)$ are shown below:

Let $$\Gamma^2(S) = \left(\frac{L}{C_o}\right)^2 \frac{A(S)}{B(S)} \tag{3.11}$$

where $$\frac{A(S)}{B(S)} = S^2\frac{Ng}{Dg}$$

A(S) and B(S) are polynomials in "S" which are introduced to simplify the algebra.

One Product Term

$$\text{Cosh}\,\Gamma(s) = \frac{B(S) + .4053\left(\frac{L}{C_o}\right)^2 A(S)}{B(S)} \tag{3.12}$$

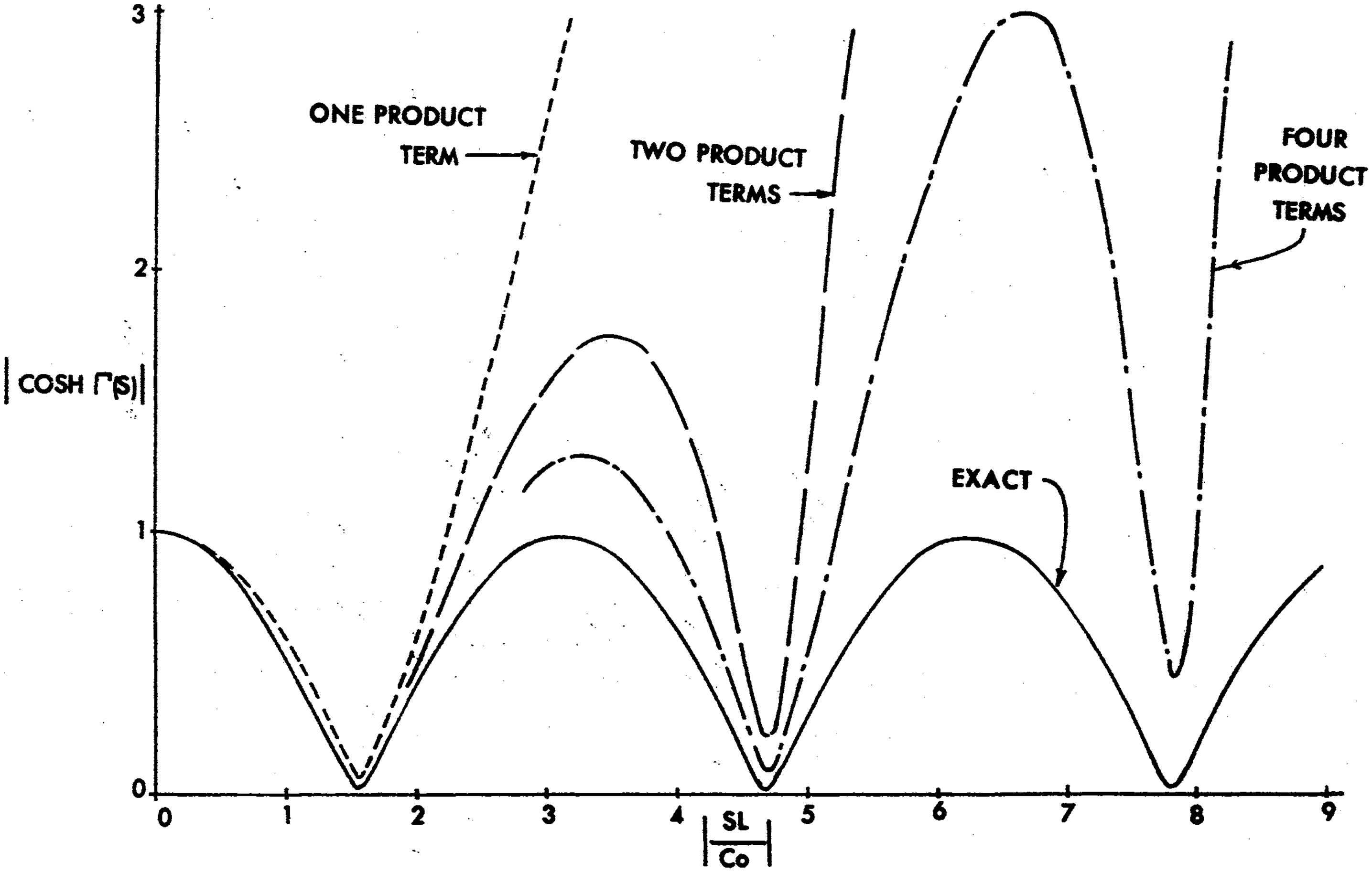

Figure 5. |Cosh Γ(s)| for One, Two, and Four Product Term

Two Product Terms

$$\operatorname{Cosh} \Gamma(s) = \frac{B(s)^2 + .4503\left(\frac{L}{C_o}\right)^2 A(s)B(s) + .01825\left(\frac{L}{C_o}\right)^4 A(s)^2}{B(s)^2} \quad (3.13)$$

Four Product Terms

$$\operatorname{Cosh} \Gamma(s) = \frac{B(s)^4 + K_1 A(s)B(s)^3 + K_2 A(s)^2 B(s)^2 + K_3 A(s)^3 B(s) + K_4 A(s)^4}{B(s)^4} \quad (3.14)$$

where $K_1 = .4748\left(\frac{L}{C_o}\right)^2$; $K_2 = .0294\left(\frac{L}{C_o}\right)^4$; $K_3 = .5067\times 10^{-3}\left(\frac{L}{C_o}\right)^6$;

and $K_4 = .2441\times 10^{-5}\left(\frac{L}{C_o}\right)^8$.

Approximation for $\Gamma_b(s)$

The exact expression for $\Gamma_b^2(s)$, from Equation (2.71), is:

$$\Gamma_b^2(s) = \left(\frac{sL}{C_o}\right)^2 \frac{N_g}{D_a}\left(1 + \frac{F_{1*}}{s}\right) \quad (3.15)$$

where (N_g), (D_a), and (F_{1*}) are given as Equations (2.74) and (2.76).

The approximation for Equation (3.15), using Equations (3.3) and (3.4) is:

$$\Gamma_b^2(s) \approx \left(\frac{L}{C_o}\right)^2 \frac{A(s)}{B(s)} = \left(\frac{L}{C_o}\right)^2 \frac{s(s+10DN)(s+5.78DN+F_{1*})(s+56.6DN+F_{1*})}{(s+7.14DN)(s+40.9DN+F_{1*})} \quad (3.16)$$

Plots of the magnitude of $\left(\frac{N_g}{D_g}\right)$ based on Equations (3.15) and (3.16) are shown on Figure 6 for the special case $F_{1*} = 0$. In this case $D_a = D_g$.

Equation (3.16) combined with Equations (3.12), (3.13) and (3.14) form the approximation "set" which will be used in Chapter V for numerical integration of step responses.

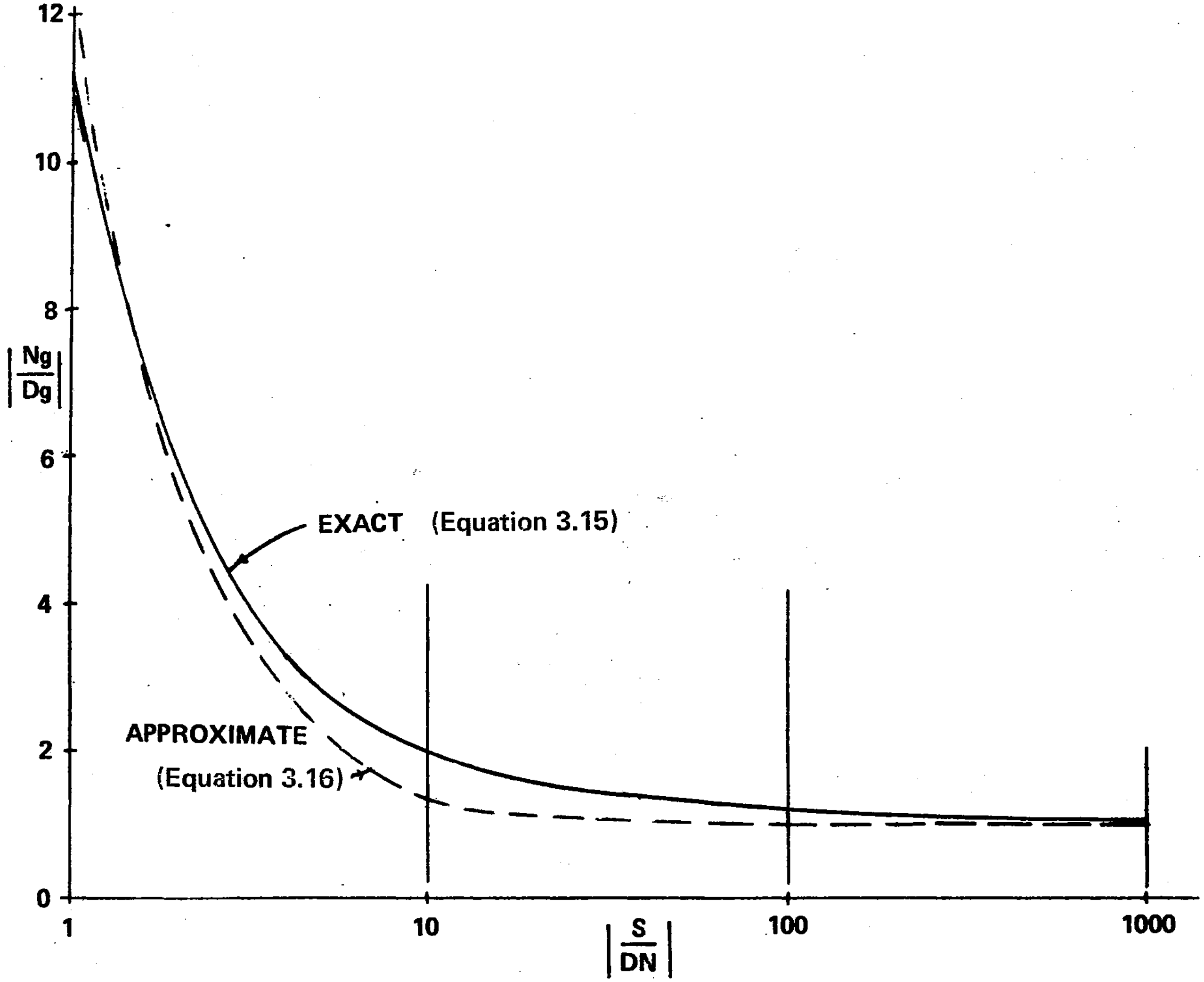

Figure 6. Exact and Approximate $|N_g/D_g|$

CHAPTER IV

EXPERIMENTAL PROCEDURES

The line model derived in Chapter II includes the effects of finite amplitude disturbances and through flow. Kantola's (13) experiments, as shown on Figure 1, were recorded for up to $\pm$ 1.0 psig steps, but for no larger disturbances. Cooley (7) reported frequency response experiments with through flow and small transient disturbances. To validate the model from Chapter II for predicting finite amplitude disturbance effects, it was necessary to perform experiments at much higher disturbance levels than that reported by Kantola (13). It was necessary to examine only finite amplitude effects since the addition of through flow into the experiment makes it difficult to separate through flow effects from finite disturbance effects.

For these reasons an experiment was set up to record pressure step responses of a pneumatic line blocked at one end. The experimental line was 60 ft long, 0.40 inch diameter, thick-walled copper tubing. The tubing remained in a roll about 20 inches in diameter.

The experiment was designed to record the pressure at the blocked end of the line while subjecting the open end to positive-going and negative-going pressure steps of magnitude 0.25, 1, 2, 4, 6, 8, and 10 psig. The 0.25 psig step was the smallest size step which produced consistent step responses. Since the atmospheric pressure at the Air Force Academy is approximately 11.2 psia, a positive-going step of

10 psig began with the line evacuated to 1.2 psia, and ended with the line pressure at 11.2 psia. A negative-going step of 10 psig began at 22.2 psia and ended at 11.2 psia.

The experiment was set up as shown in Figure 7. Two sets of two each pressure transducers were used, one set for the 0.25, 1, 2, and 4 psig steps, and the second set for the 4, 6, 8 and 10 psig steps. The pressure transducers were low output impedance, variable reluctance type, Pace Series CP51 and Validyne Series P40, ± 5 and ± 25 psi differential transducers.

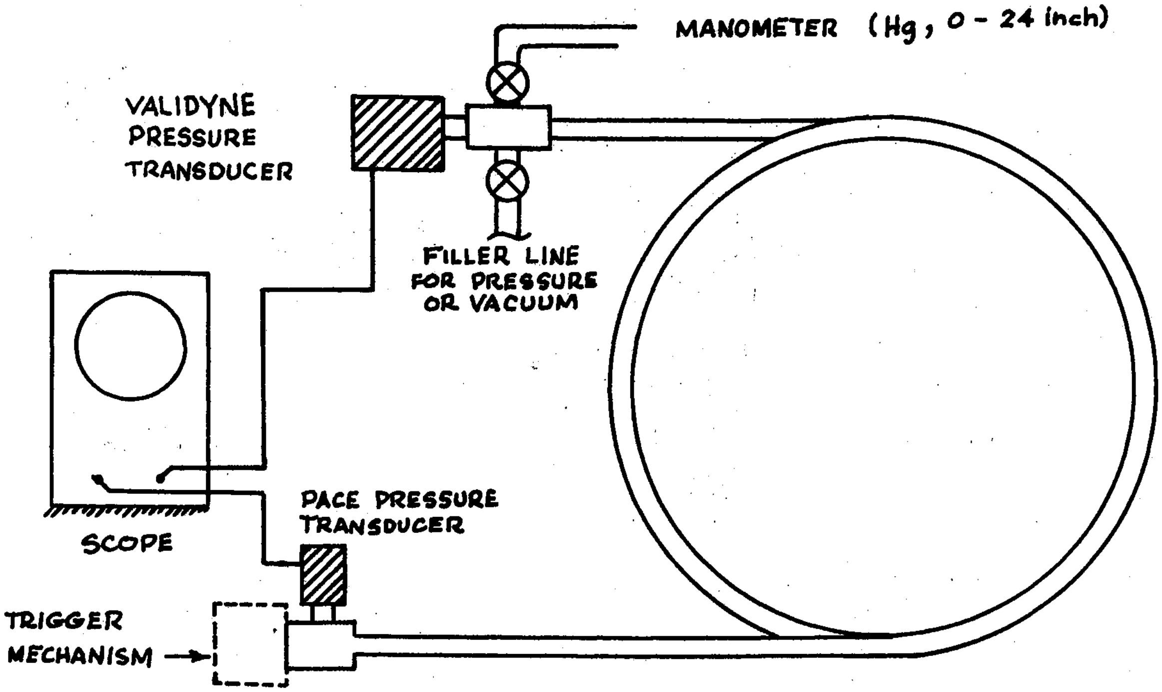

Figure 7. Experimental Apparatus

The pressure-time signals measured at the two ends of the line were recorded on polaroid film with a dual-beam Tektronix 555 oscilloscope.

Two types of mechanical trigger mechanisms were used. The first mechanism was a fast opening manually operated ball valve. It took six to ten milliseconds to open fully. The valve added some volume to the line in the closed position and, particularly at low magnitude pressure steps ($\pm$ 1/4 psig), it altered the wave front at the blocked end of the line. This is shown on Figure 8 as input-output set #1.

The second trigger mechanism added no volume to the line and opened fully in two to four milliseconds. It was a rubber stopper with a fishing line attached through the center. Even when the line was charged to +10 psig the stopper remained in the opening until a significant "jerk" was applied to the line. A typical result is shown on Figure 8 as input-output set #2.

The line was 60 ft. long, so the pressure signal took approximately 53 milliseconds to travel the length of the line. The results shown on Figure 8 are for a step input of $\pm$ 0.25 psig. All the experimental results shown in this thesis were initiated by trigger mechanism #2, the rubber stopper.

The Pace and Validyne pressure transducers have a flat frequency response from 0 to 1000 hertz. It is possible that some of the very high frequency content was lost, but the loss is not significant. At the first resonant frequency of the line $\omega T_e = \pi/2$ (where $T_e = L/C_o = 53$ milliseconds), $\omega \approx 30$ radians/sec, or 4.7 hertz. The second resonance occurred at 14.1 hertz, etc.

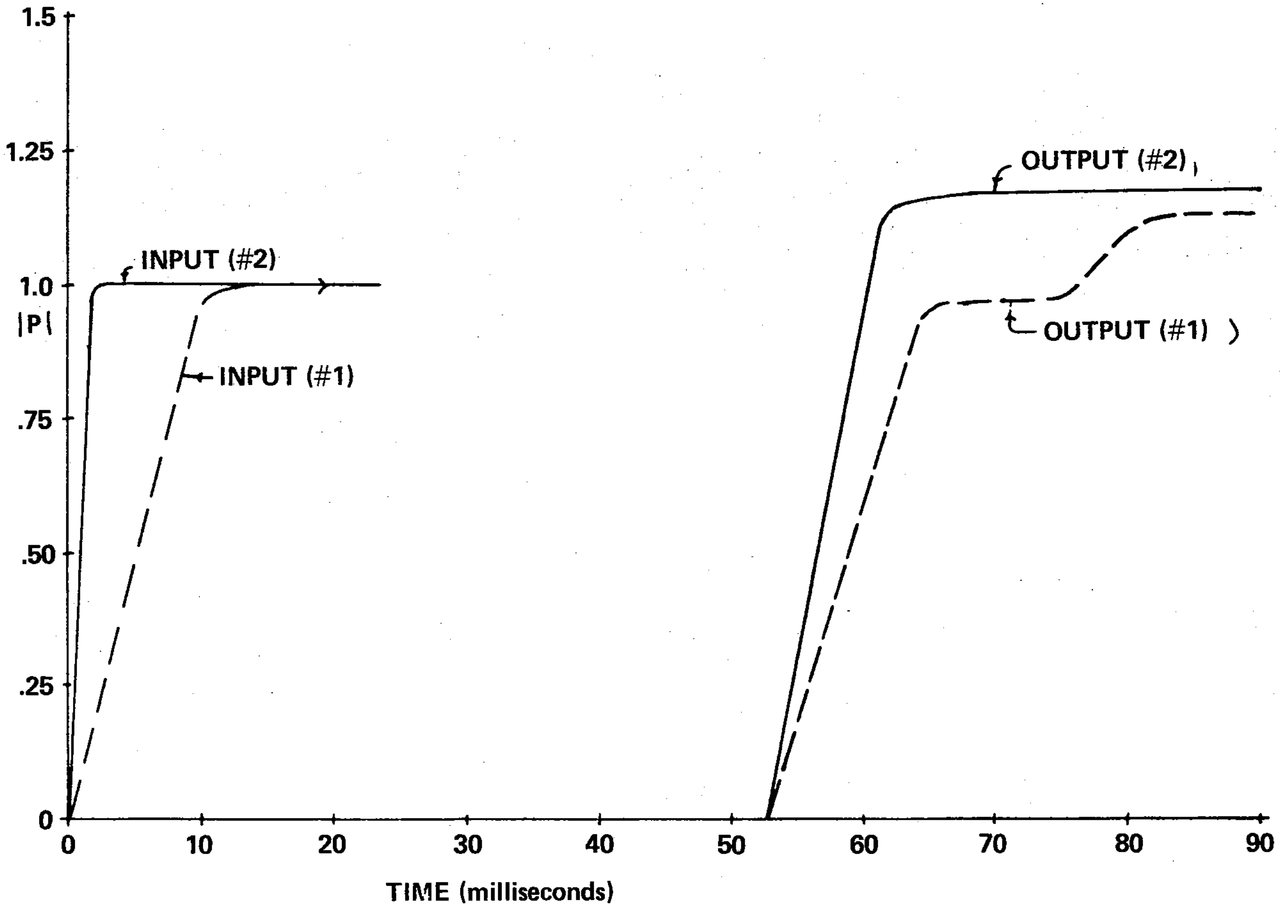

Figure 8. Relative Effects of Trigger Mechanisms

Figure 9 includes the total experimental results. These results will be shown again in Chapter V in conjunction with the computer integrated step responses.

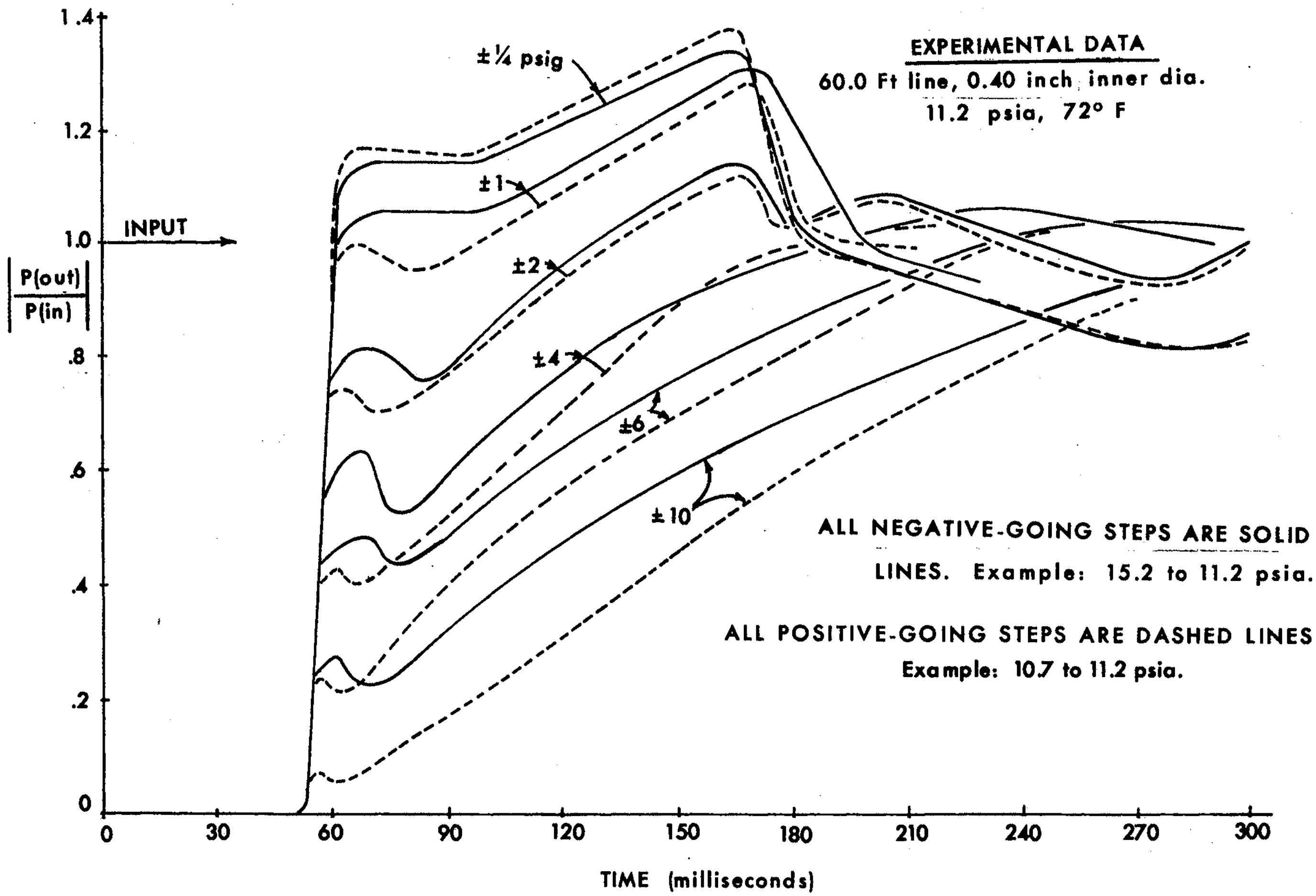

Figure 9. Experimental Results

CHAPTER V

TIME DOMAIN EVALUATION

The experimental results shown in Chapter IV include responses caused by both small and finite amplitude disturbances with no through flow. This chapter compares computed step responses based on the analytical results of Chapters II and III with the measured step responses presented in Chapter IV.

Preparation for Numerical Integration

With no through flow ($M_b = 0.$), Equations (2.70) may be written as:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \cosh \Gamma_b(S) & -Z_b(S) \sinh \Gamma_b(S) \\ \dfrac{-\sinh \Gamma_b(S)}{Z_b(S)} & \cosh \Gamma_b(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{5.1}$$

where
$$\Gamma_b(S) = \frac{SL}{C_o} \sqrt{\frac{N_g}{D_a}\left(1 + \frac{F_{1*}}{S}\right)} \tag{5.2}$$

$$Z_b(S) = \gamma \sqrt{\frac{\left(1 + \frac{F_{1*}}{S}\right)}{N_g D_a}} \tag{5.3}$$

and (N_g), (D_a), and (F_{1*}) are given as Equations (2.74) and (2.76).

The Chapter IV experiments were conducted by blocking both ends of a pneumatic line, charging or evacuating the line to a designated gage pressure, then opening one end of the line quickly to the atmosphere. The pressure transient at the end of the line which remained blocked was recorded as a function of time (see Figure 9).

In the computed model, the end of the line where Z = 0 is permanently blocked and the end of the line where Z = 1 will be opened suddenly to atmospheric pressure. Since Q(S,0) = 0, Equation (5.1) may be rewritten as:

$$P(S,0) = \frac{P(S,1)}{\text{Cosh}\,\Gamma_b(S)} \qquad (5.4)$$

where P(S,1) is the pressure input to the system and P(S,0) is the output.

A fourth-order Runge-Kutta integrator was selected for the numerical investigation. This example will show the preparation for integration when the one product term expansion for Cosh $\Gamma_b(S)$ was used. By substituting Equation (3.12) into Equation (5.4), the result is:

$$P(S,0) = \frac{P(S,1)}{\left(1 + .4053\left(\frac{L}{C_o}\right)^2 \Gamma_b^2(S)\right)} \qquad (5.5)$$

From Equation (3.16):

$$\Gamma^2{}_b(S) = \left(\frac{SL}{C_o}\right)^2 \frac{N_g}{D_a} = \left(\frac{L}{C_o}\right)^2 \frac{A(S)}{B(S)} \qquad (5.6)$$

where $A(S) = S\ (S + 10DN)\ (S + 5.78DN + F_{1*})\ (S + 56.6DN + F_{1*})$ (5.7)

and $B(S) = (S + 7.14DN)\ (S + 40.9DN + F_{1*})$ (5.8)

Equation (5.5) may be written in the alternate form:

$$P(S,0) = \frac{P(S,1)\ B(S)}{\left(B(S) + .4053\left(\frac{L}{C_o}\right)^2 A(S)\right)} \qquad (5.9)$$

or

$$P(S,0) = \frac{P(S,1)\ [\ G(1) + G(2)\ S + G(3)\ S^2\ ]}{[\ G(4) + G(5)\ S + G(6)\ S^2 + G(7)\ S^3 + G(8)\ S^4]} \qquad (5.10)$$

where G(1) through G(8) are functions of (DN), (L/C_o), and (F_{1*}). The damping number (DN) and the isentropic delay time (L/C_o) do not change during the numerical integration; the value of (F_{1*}) changes at every Runge-Kutta step.

For this problem, $(L/C_o) = .0532$ and DN = 0.8. These numbers are based on an average kinematic viscosity (ν_o) of 0.032 in^2/sec, at 72^oF and 11.2 psia. The tube inner radius (a) = 0.20 in, the tube length = 60 ft, and the isentropic speed of sound (C_o) = 1130 ft/sec.

$$\text{Let} \quad M(S) = \frac{P(S,1)}{[\, G(4) + \ldots + G(8)\, S^4]} \tag{5.11}$$

$$\text{Then} \quad P(S,0) = M(S)\, [\, G(1) + G(2)\, S + G(3)\, S^2] \tag{5.12}$$

$$\text{and} \quad S\, P(S,0) = M(S)\, [\, G(1)\, S + G(2)\, S^2 + G(3)\, S^3] \tag{5.13}$$

Let $Y(1) = \mathcal{L}^{-1}[M(S)\, S^0]$, $Y(2) = \mathcal{L}^{-1}[M(S)\, S]$, $Y(3) = \mathcal{L}^{-1}[M(S)\, S^2]$, $Y(4) = \mathcal{L}^{-1}[M(S)\, S^3]$, and $Y(10) = \mathcal{L}^{-1}[M(S)\, S^4]$. Then Equations (5.11), (5.12), and (5.13) may be written in the time domain as:

$$Y(10) = \frac{1}{G(8)} [P(t,1)-G(4)\, Y(1)-G(5)Y(2)-G(6)Y(3)-G(7)Y(4)] \tag{5.14}$$

$$P(t,0) = G(1)Y(1) + G(2)Y(2) + G(3)Y(3) \tag{5.15}$$

$$\frac{\partial P(t,0)}{\partial t_*} = G(1)Y(2) + G(2)Y(3) + G(3)Y(4) \tag{5.16}$$

Equations (5.14), (5.15), and (5.16) appear in the derivative function subroutine of the numerical integrator (see Appendix B).

Results

Figure 10 shows the computed step responses which result from Equations (5.14), (5.15), and (5.16) at step input levels of 0.25 and 4.0 psig. The experimental 0.25 and 4.0 psig step responses from Chapter IV are shown as dashed lines.

As shown on Figure 11, the one, two, and four product term expansions for Cosh $\Gamma_b(S)$ yield approximately the same overshoot for the same input step size. The computed responses do not have as much "apparent damping" as that shown by the real fluid system. This disparity is probably caused in part by the approximations used for $\Gamma_b(S)$ and Cosh $\Gamma_b(S)$ in the model, and in part by the restrictions on the model in the basic derivation. That is, the model neglects the effects of radial flows, developing flows at both ends of the line, and torroidal motion.

The experimental results shown on Figures 10 and 11 include significant high frequency content, as demonstrated by the sharp "corners" of the pressure response. The computed responses using a one product term expansion for Cosh $\Gamma_b(S)$ shows only the fundamental mode of the step response. Results using higher order approximations (two and four product terms) are dominated by the fundamental mode as well.

An unsuccessful attempt was made to "filter out" the high frequency content of the experimental step responses by a totally definitive mathematical method. However, one can still visualize a damped sinusoid which appears to be the effective fundamental mode of the experimental response. An approximate fundamental mode for the portion of the

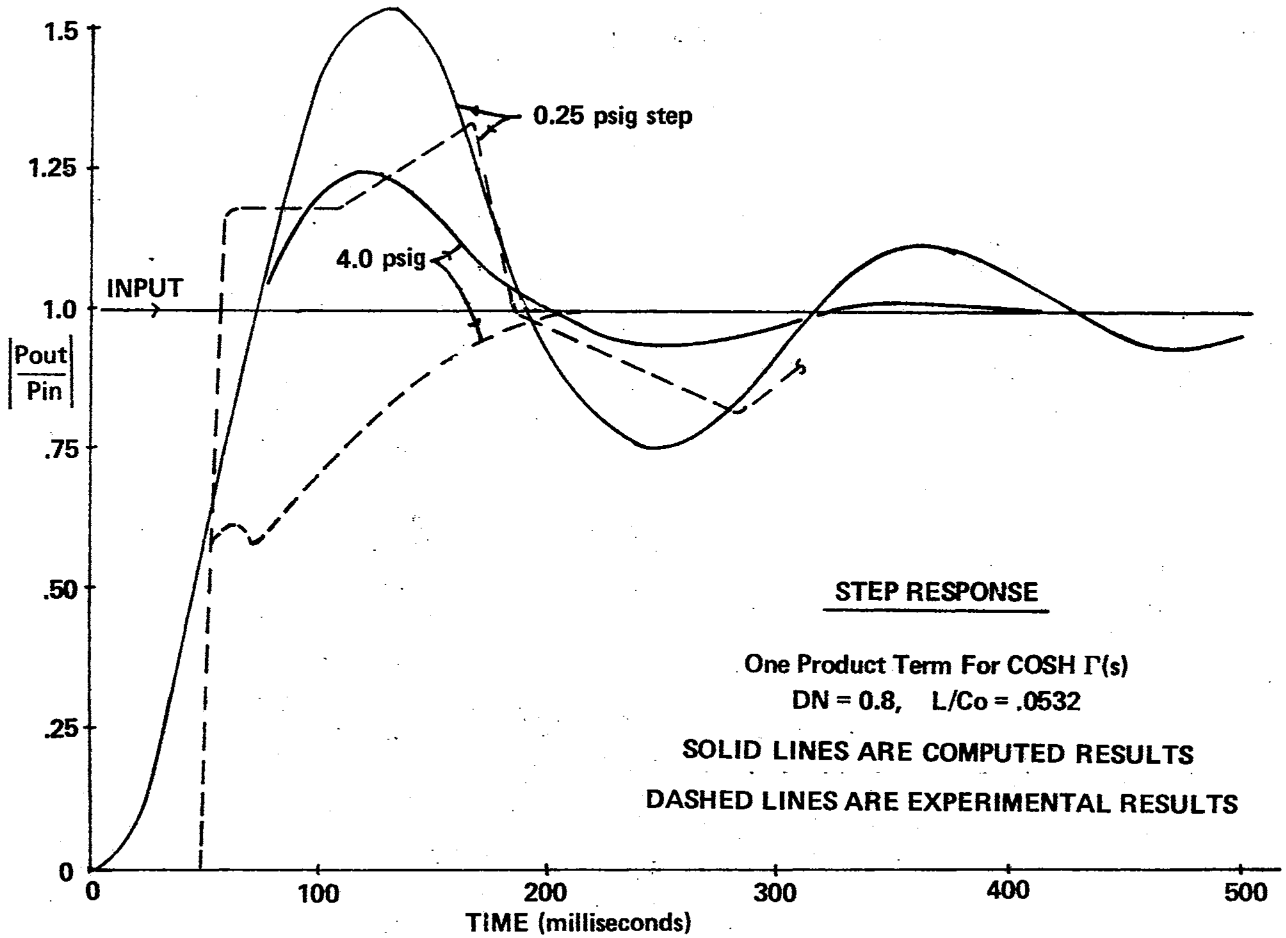

Figure 10. Computed Responses Versus Experimental Responses

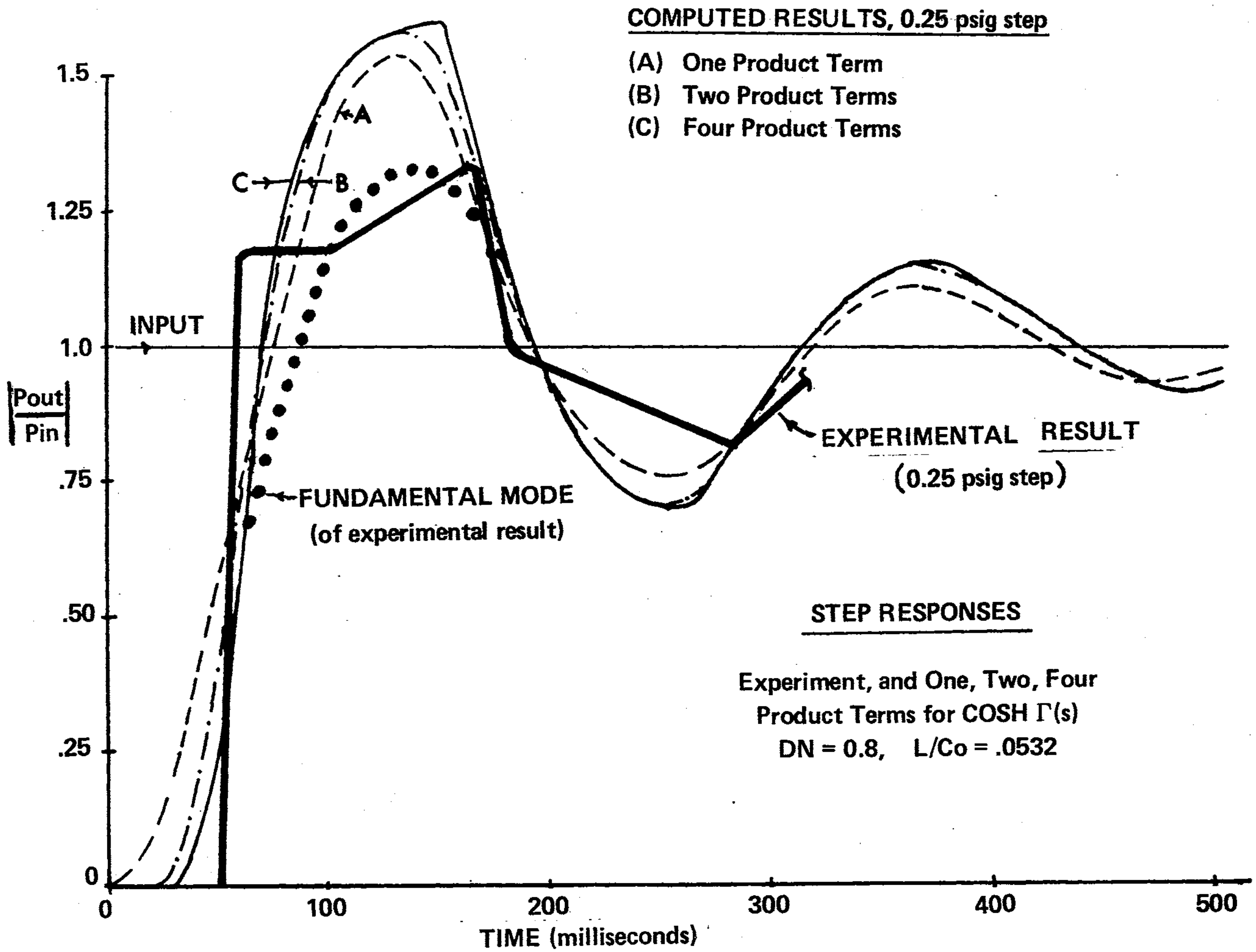

Figure 11. Experimental and Computed Traces at 0.25 psig Step

experimental result between 50 and about 175 milliseconds is shown on Figure 11. This fundamental mode was determined from the Fourier Analysis program, "Forit."

For purposes of comparison it is assumed (Criteria #3, p 7) that the damping associated with the model response for small amplitude inputs should closely agree with the damping of the approximate fundamental mode of the corresponding experimental response. As shown on Figure 12, a damping number of 2.0 yields the desired model response at a step of 0.25 psig. Comparison of the computed results with experimental results at step levels of $\pm$ 0.25, 2.0, 4.0, and 6.0 psig are made on Figure 13, based on a damping number of 2.0.

The model is able to predict the increase in apparent damping for the 2.0 psig step, but not for the 4.0 and 6.0 psig steps. Since the model is based on the assumption of laminar transient flow, and a pressure step of 4.0 or 6.0 psig may produce flow in the turbulent region, it is not surprising that the model cannot predict the large changes in apparent damping at the higher step levels.

Figure 14 is the computed result for a two product term expansion for Cosh $\Gamma_b(S)$. It is quite evident that this higher order model is experiencing some type of instability. The four product term expansion model is unstable for all steps greater than $\pm$ 0.25 psig also.

System Instability

Oldenburger(16) reported that the conventional power series expansion for Cosh $\Gamma(S)$, Equation (3.6), may introduce instabilities into an otherwise stable system of equations. But Oldenburger also showed that the infinite product term expansion for Cosh $\Gamma(S)$ and

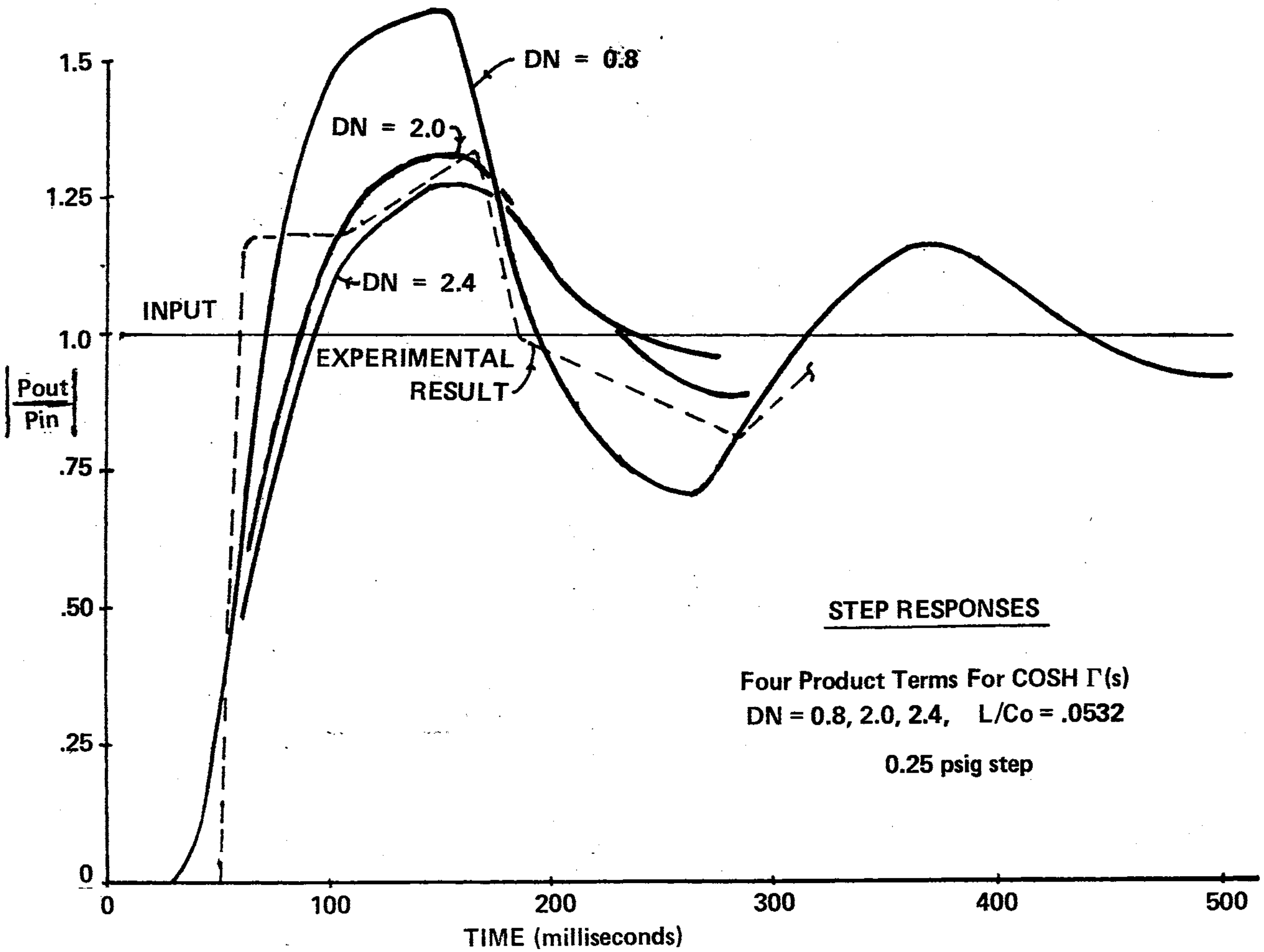

Figure 12. Computed Step Responses at Various Damping Numbers

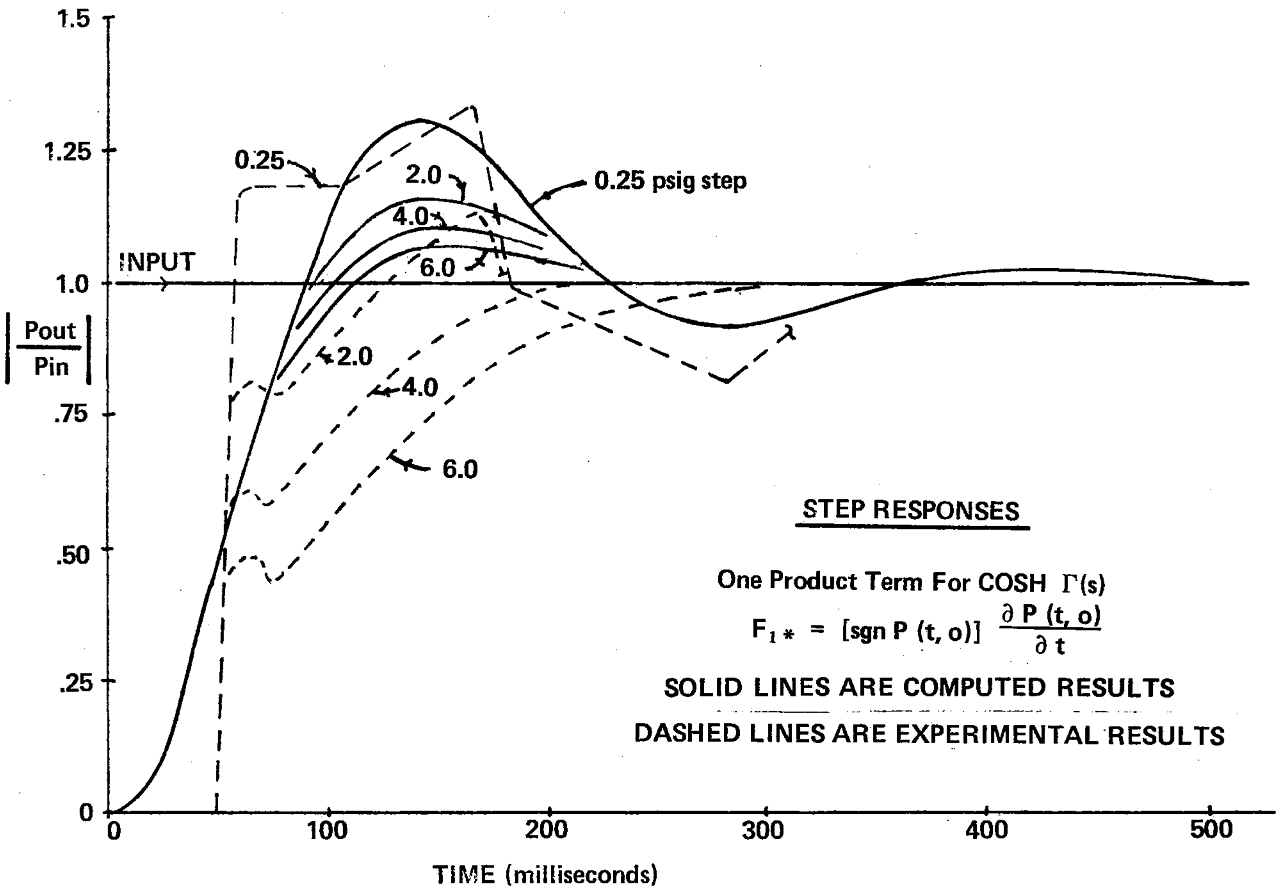

Figure 13. Step Responses, One Product Term

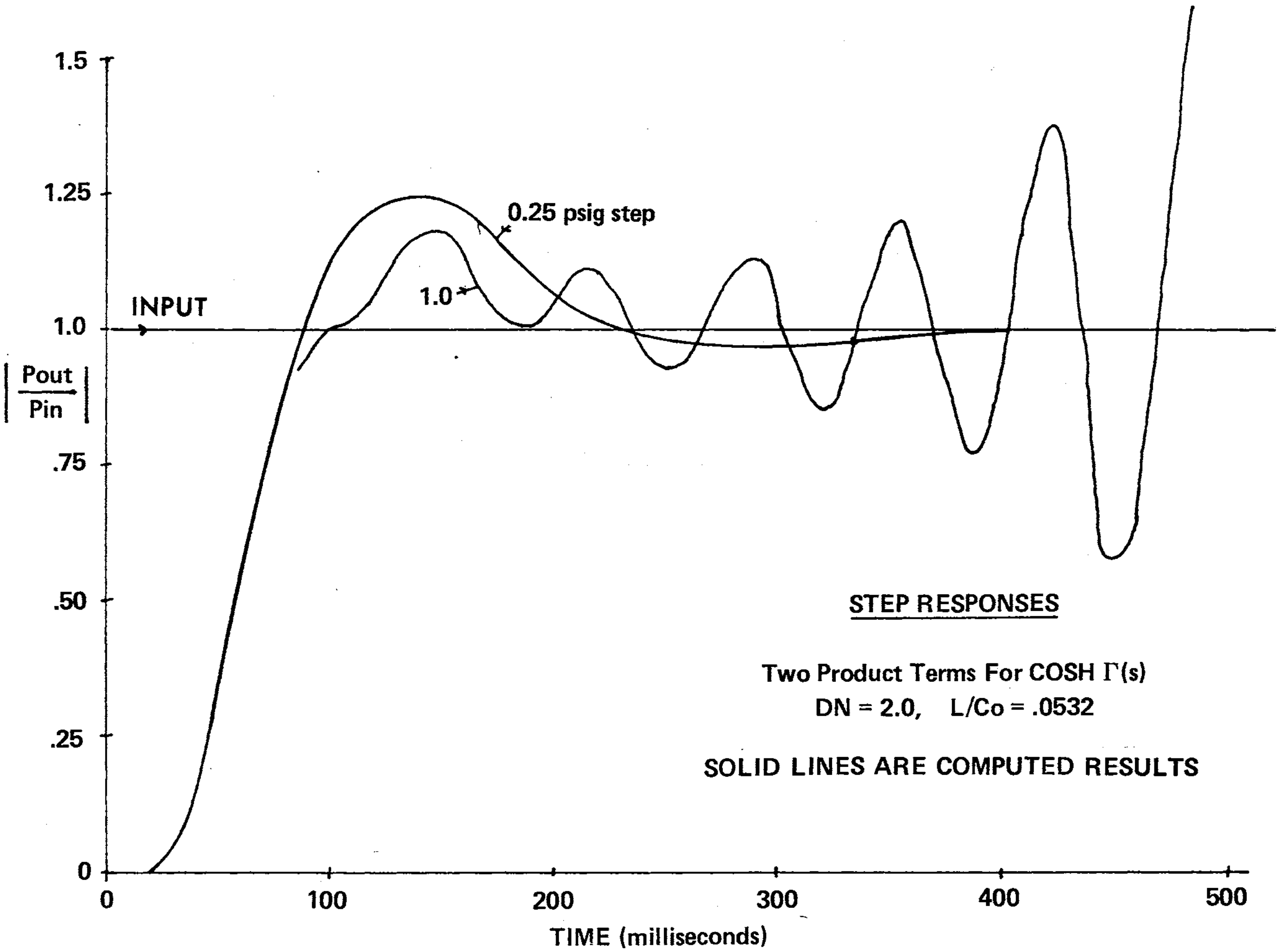

Figure 14. Step Responses, Two Product Terms

Sinh $\Gamma(S)$ are absolutely convergent. The computed step responses shown on Figure 14 clearly indicate an instability in the solution, caused by either numerical instability (accumulated error, round-off, etc.) or by the presence of positive real roots in the denominator of the transfer function, Equation (5.4), or both.

If the denominator of Equation (5.4) has positive real roots then the system of equations is unstable, regardless of the presence or absence of numerically induced instability. To examine the nature of the instability, Routh's Criterion was applied to the denominator of Equation (5.4) for one and two product term expansions for Cosh $\Gamma(S)$.

Routh's Criterion

For the one product term expansion for Cosh $\Gamma_b(S)$, the coefficients for Routh's Criterion are given as the denominator of Equation (5.10):

G(8)	G(6)	G(4)
G(7)	G(5)	
B1	B3	
C1		
D1		

(5.17)

$$\text{where } B1 = \frac{[\,G(6)\,G(7) - G(8)\,G(5)\,]}{G(7)} \text{ , etc.} \qquad (5.18)$$

The terms G(1) through G(8) are functions of (F_{1*}), (L/C_o), and (DN). Each time the terms B1, C1, or D1 change in sign, the denominator of Equation (5.10) has a positive real root and the system of equations is unstable. For the one product term expansion for Cosh $\Gamma_b(S)$ there is no change in sign for B1, C1, or D1 until $(F_{1*}) < 0$. F_{1*} is always greater than zero at the initial rise of the output to a step response, but it becomes negative as soon as the output reaches its maximum overshoot. If there is no overshoot, F_{1*} is never less than zero.

For the two product term expansion for Cosh $\Gamma_b(S)$, Equation (5.4) may be written as:

$$P(S,0) = \frac{P(S,1)\ [\ G(1) + G(2)\ S + \ldots + G(5)\ S^4]}{[\ G(6) + G(7)\ S + \ldots\ldots + G(14)\ S^8]} \qquad (5.19)$$

Routh's Criterion was applied to the denominator of Equation (5.19) using nine different combinations of (L/C_o) and (DN). The responses shown on Figure 14 are for $(L/C_o) = .0532$ and $(DN) = 2.0$. The regions where the system of equations is stable is shown on Table II below.

TABLE II

REGIONS OF STABILITY

DN	(L/C_o) .0266 (L=30ft)	.0532 (L=60ft)	.1064 (L=120ft)
1.0	$1.4 < F_{1*} < \infty$	$0 \le F_{1*} < 6$ and $16 < F_{1*} < \infty$	$0 \le F_{1*} < 4$ and $12 < F_{1*} < \infty$
2.0	$0 \le F_{1*} < 12$ and $30 < F_{1*} < \infty$	$1 < F_{1*} < 8$ and $30 < F_{1*} < \infty$	$0 \le F_{1*} < \infty$
4.0	$2 < F_{1*} < 18$ and $50 < F_{1*} < \infty$	$0 \le F_{1*} < \infty$	$0 \le F_{1*} < 16$ and $40 < F_{1*} < \infty$

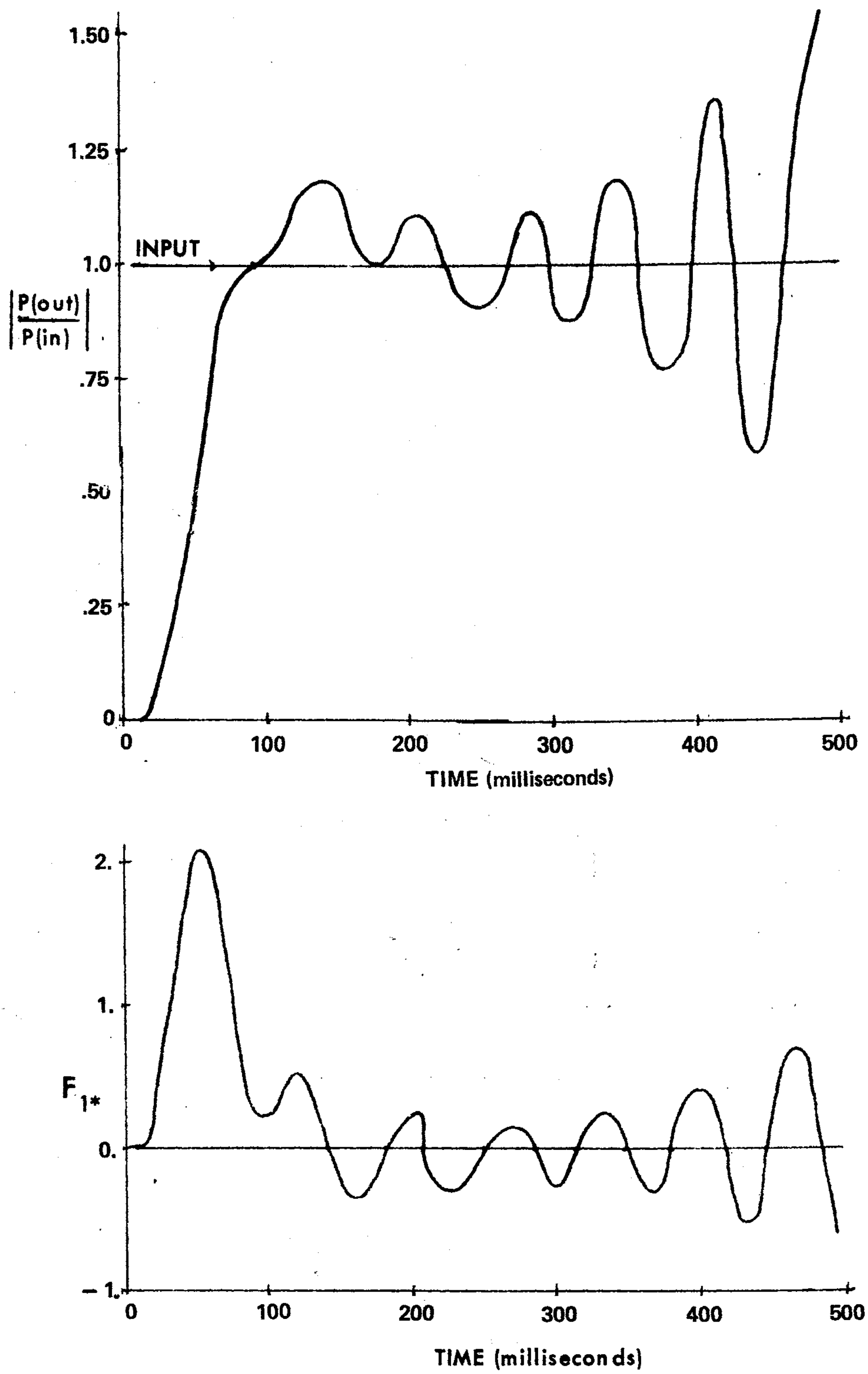

Figure 15. Step Response and F_{1*}

As was true for the one product term expansion for Cosh $\Gamma_b(S)$, the two product term expansion is unstable for all $F_{1*} < 0$. But Routh's Criterion also predicts instability for some combinations of (L/C_o) and (DN) when $F_{1*} \geq 0$. When $F_{1*} = 0$ the model reverts to the small disturbance "Acoustic" model of Appendix A, which is stable for all values of (L/C_o) and (DN).

There are some "grey areas" then where Routh's Criterion predicts the system of equations to be unstable, but the numerical integration of the equations proceeds in a stable manner. Figure 14 is one example. The system of equations is stable for a 0.25 psig step, but unstable for a 1.0 psig step input. This instability is probably caused by a large negative value of F_{1*} immediately after the output reaches its initial overshoot position (at 150 milliseconds.)

Figure 15 is a replot of the 1.0 psig step shown on Figure 14, but it also includes the magnitude of F_{1*} during the transient.

Routh's criterion demonstrates that the system of equations will be unstable for all $F_{1*} < 0$. However, in the case of one product term expansions the computed step responses are stable for all input step levels, even though $F_{1*} < 0$ for some portions of the transients. It must be concluded that the stabilizing influence when $F_{1*} > 0$ dominates over the unstabilizing influence when $F_{1*} < 0$. In the case of two product term expansions, all responses for step input levels greater than some small number (say 0.25 psig) are unstable.

The stability of the system of equations is dependent on the form and sign of F_{1*} as well as the approximations used for $\Gamma(S)$, Cosh $\Gamma(S)$, and Sinh $\Gamma(S)$. The example chosen in this thesis represents a worst case in the sense of the quality of the approximations for $\Gamma(S)$ (see

Figure 6 when S/DN = 10.) However, the main difficulty associated with system instability appears to result from the form of F_{1*}, rather than the quality of the approximations.

Unless an improved form for F_{1*} can be synthesized, it is recommended that only one product term expansions be used for Cosh $\Gamma(S)$ and Sinh $\Gamma(S)$ in this model.

CHAPTER VI

FREQUENCY DOMAIN EVALUATION

In this chapter frequency response computed from the analytical model, Equation (2.70), with through flow, is compared with the experimental results of Cooley(7).

Cooley's(7) experiments were conducted with small amplitude transient flow. Rewriting Equation (2.70) to meet these conditions ($M_b \neq 0$, but $F_{1*} = 0$) yields:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \cosh \Gamma(S) + Y_e(S) M_b \sinh \Gamma(S) & -Z_c(S) \sinh \Gamma(S) \\ -\dfrac{\sinh \Gamma(S)}{Z_c(S)} & \cosh \Gamma(S) - Y_e(S) M_b \sinh \Gamma(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{6.1}$$

where

$$\Gamma(S) = \frac{SL}{C_o} \sqrt{\frac{N_g}{D_g}} \tag{6.2}$$

$$Z_c(S) = \frac{\gamma}{\sqrt{N_g D_g}} = \frac{\gamma SL}{C_o} \frac{1}{D_g \, \Gamma(S)} \tag{6.3}$$

$$Y_e(S) = \sqrt{N_g D_g} = \frac{C_o}{SL} D_g \, \Gamma(S) \tag{6.4}$$

and (N_g), (D_g) are given as Equations (2.74).

If the end of the line Z = 1 is subjected to a constant pressure, P(S,1) = 0. Then Equation (6.1) may be rewritten as:

$$\frac{Q(S,0)}{P(S,0)} = \frac{\cosh \Gamma(S) + Y_e(S) M_b \sinh \Gamma(S)}{Z_c(S) \sinh \Gamma(S)} \tag{6.5}$$

Cooley(7) performed a series of frequency response experiments with a 6.0 inch line, 0.125 inches in inner diameter. He included through flow with an average Mach number, M_b, of 0.16. By substituting $M_b = 0.16$ and $S = j\omega$ into Equation (6.5), the "admittance" of the line, $\left|\frac{Q(S,0)}{P(S,0)}\right|$ may be calculated. In this case no approximations are used for (N_g) and (D_g) since their exact values may be computed from a Bessel Function subroutine.

Figure 16 shows Cooley's experimental data for $\left|\frac{Q(S,0)}{P(S,0)}\right|$ and Equation (6.5) for $M_b = 0.16$ and DN = 30.0. At the first resonance (1050 hertz) Cooley shows an increase in $\left|\frac{Q(S,0)}{P(S,0)}\right|$ from 3.2 without through flow to 5.21 with through flow, that is, an increase of 62% when through flow is included. Equation (6.5) predicts an increase in $\left|\frac{Q(S,0)}{P(S,0)}\right|$ from 3.2 to 3.3, a 3% increase.

Orner(17) examined the frequency response of a transmission line with through flow by applying the Poincare' perturbation technique to the axial momentum equation, including the convective acceleration term $\left(v_z \frac{\partial v_z}{\partial z}\right)$. He arrived at Equation (6.1) with identical expressions for $\Gamma(S)$ and $Z_c(S)$ as are shown in Equations (6.2) and (6.3). His expression for $Y_e(S)$ is as follows:

$$Y_e(S) = \frac{1}{\gamma}\left[1 - \frac{8(\gamma-1)}{\Delta^2}\left(1 - \frac{2J_1(\Delta)}{\Delta J_0(\Delta)}\right)\right] \qquad (6.6)$$

where $$\Delta = j\sqrt{\frac{S\sigma_0 a^2}{\nu_0}} \qquad (6.7)$$

The frequency response for Orner's first perturbation solution at $M_b = 0.16$ is approximately the same as this thesis result, as shown on Figure 16. His solution predicts a 3% increase in $\left|\frac{Q(S,0)}{P(S,0)}\right|$ at the first resonance (1050 hertz.)

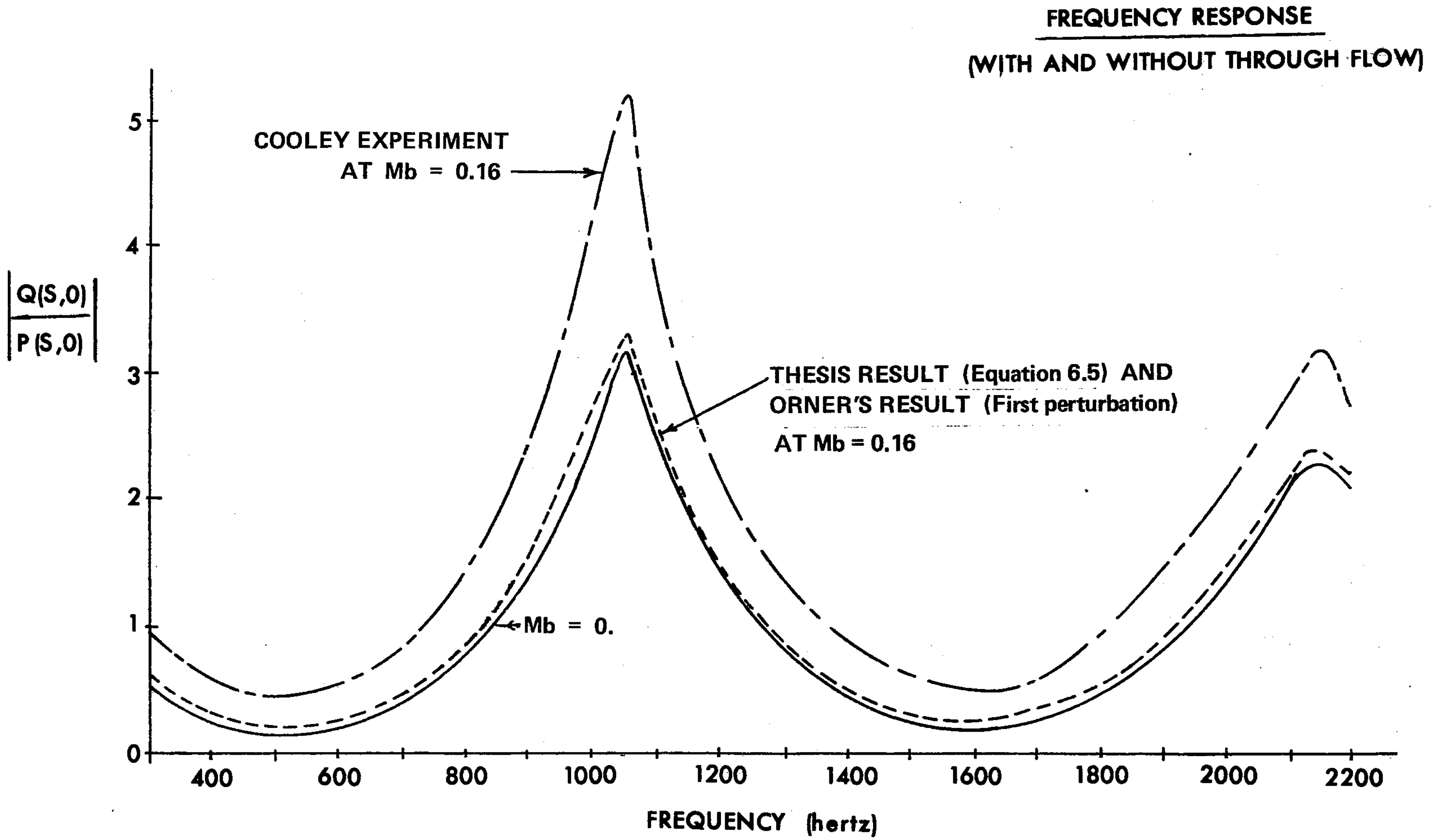

Figure 16. Experimental and Computed Frequency Response

Orner performed a second perturbation on the system of equations which predicted an additional increase in $\left|\frac{Q(S,0)}{P(S,0)}\right|$ of 9% at the first resonance, resulting in a final value of $\left|\frac{Q(S,0)}{P(S,0)}\right|$ of 3.6. Cooley's experiment shows $\left|\frac{Q(S,0)}{P(S,0)}\right|$ as 5.21 at this frequency.

Order of Magnitude Analysis for $Y_e(S)$

If the Cooley experiment is correct, and if the analyses of Orner and this thesis have included the significant terms in the axial momentum equation to account for through flow, then Equation (6.5) should be able to predict an admittance $\left|\frac{Q(S,0)}{P(S,0)}\right|$ approximately equal to 5.21 at 1050 hertz when $M_b = 0.16$.

At the first resonance (1050 hertz) the magnitude of Cosh $\Gamma(S)$ is approximately 1.0. The magnitude of Sinh $\Gamma(S)$ is approximately 0.22. Then $\left|\frac{Q(S,0)}{P(S,0)}\right|$ may be approximated as:

$$\left|\frac{Q(S,0)}{P(S,0)}\right| \approx \frac{1 + .22\,|M_b Y_e(s)|}{.22} \tag{6.8}$$

Equation (6.8) disregards the complex nature of Cosh $\Gamma(S)$, Sinh $\Gamma(S)$, and $Y_e(S)$, but it is acceptable for a rough bound on the term $(M_b Y_e(S))$. Given that $\left|\frac{Q(S,0)}{P(S,0)}\right|$ = 5.21 at 1050 hertz, then the minimum value for $(M_b\ Y_e(S))$ is 4.2. Since $M_b = 0.16$, the minimum magnitude of $Y_e(S)$ is 26.

Neither the Orner analysis nor this analysis could predict a magnitude of $Y_e(S)$ greater than 1.2 for any frequency (ω), $\frac{\omega L}{C_o} > \pi$. The first resonance of the Cooley experiment occurs at $\frac{\omega L}{C_o} = 9.3\,\pi$.

Clearly, the effect of through flow on the frequency response of a small diameter line as reported by Cooley cannot be predicted by the model offered in this thesis.

However, Equation (6.5) does predict a rather dramatic result when $\left|\frac{P(S,0)}{Q(S,0)}\right|$, the line "impedance" is plotted, rather than $\left|\frac{Q(S,0)}{P(S,0)}\right|$, the line "admittance." This is shown on Figure 17. Figure 17 is a reciprocal plot of Figure 16, showing the computed "impedance" of the line with through flow as a function of frequency, (ω). Figure 17 is based on the same relatively high through flow rate, ($M_b = 0.16$), which yields a through flow velocity on the order of 180 ft/sec.

Cooley(7) did not measure impedances in his experiment, and he reported that the signal-to-noise ratio of his instruments in the regions 400 to 800 hertz and 1400 to 1800 hertz was very low, negating the accuracy of the readings in these regions. So it would be inappropriate to take the reciprocal of the Cooley data from Figure 16 and plot it on Figure 17.

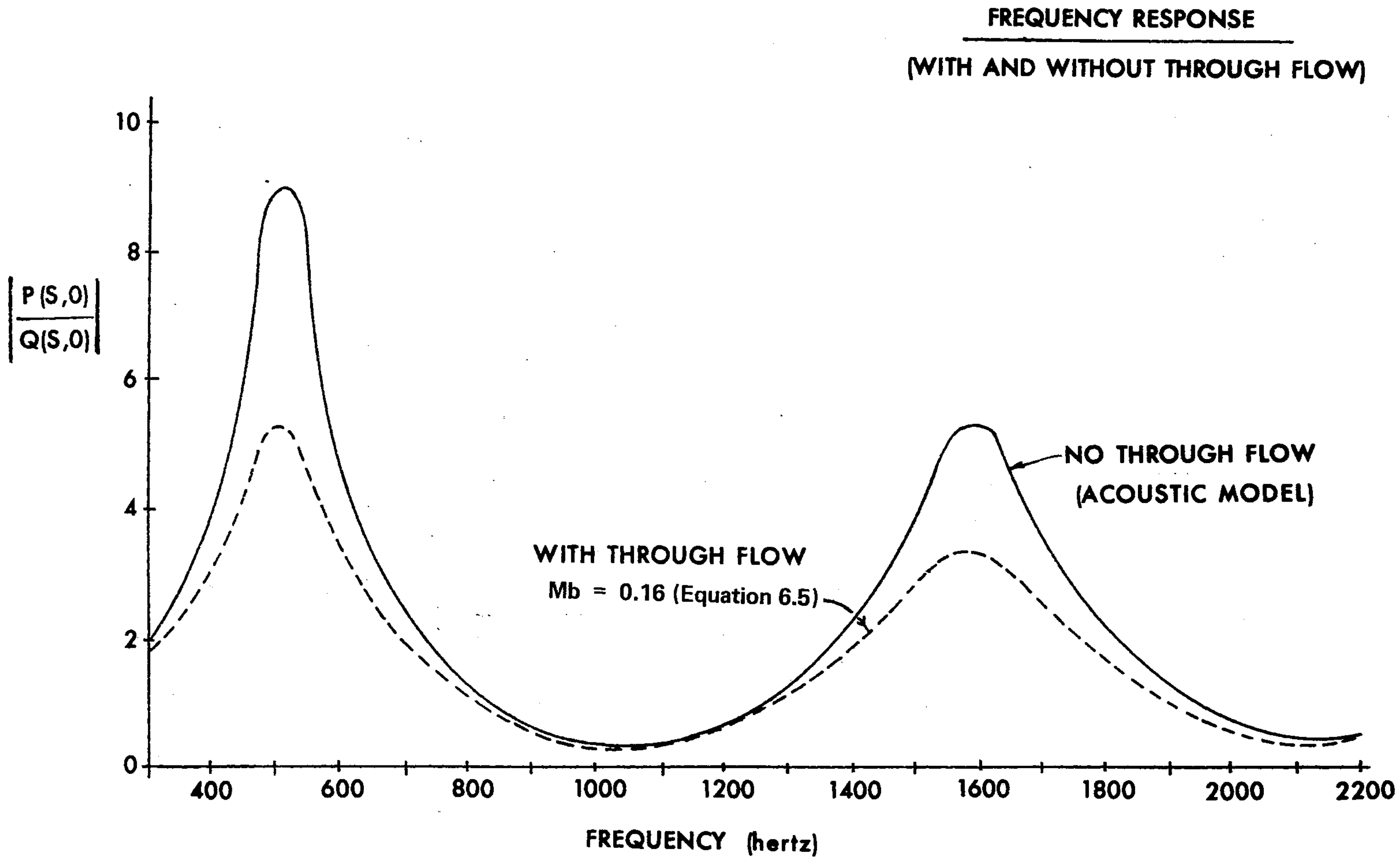

Figure 17. Computed Frequency Response With and Without Through Flow

CHAPTER VII

THE HYDRAULIC CASE

The basic line model, Equation (2.70), is applicable when the fluid is an ideal gas or a liquid. This chapter shows the simplification of the model when the fluid is a liquid.

To use Equation (2.70) the parameters (DN), (L/C_o), and (M_b) must be known. In the liquid case:

$$DN = \frac{\nu_o}{a^2} = \frac{\mu_o}{\rho_o a^2} \tag{7.1}$$

$$\frac{L}{C_o} = L\sqrt{\frac{\rho_o}{\beta_o}} \tag{7.2}$$

where (β_o) is the bulk modulus of the fluid, (μ_o) is the absolute viscosity, and (ρ_o) is the fluid density.

$$M_b = \frac{\text{Average through flow axial velocity}}{C_o} \tag{7.3}$$

The speed of sound in the fluid, C_o, is at least four or five times greater than the speed of sound in a pneumatic system, so for the same through flow axial velocity, M_b in the hydraulic case is only one fifth as large as M_b in the pneumatic case. In general, $M_b \ll 1.0$, and it may be neglected in the system of equations.

Writing Equations (2.70) with this simplification (M_b=0) yields:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \cosh \Gamma_b(S) & -Z_b(S) \sinh \Gamma_b(S) \\ \dfrac{-\sinh \Gamma_b(S)}{Z_b(S)} & \cosh \Gamma_b(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{7.4}$$

where $\Gamma_b(S)$ is given as Equation (2.71) and $Z_b(S)$ is Equation (2.72). When the fluid is a liquid, $\gamma = 1.0$, and the term (N_g) in $\Gamma_b(S)$ and $Z_b(S)$ is approximately equal to 1.0. From the approximations in Chapter III, Equations (3.16), $\Gamma_b^{\,2}(S)$ may be approximated as shown below for the liquid case:

$$\Gamma_b^2(S) \approx \left(\frac{L}{C_o}\right)^2 \times \frac{S \quad (S + 5.78DN + F_{1*}) \; (S + 56.6DN + F_{1*})}{(S + 40.9DN + F_{1*})} \qquad (7.5)$$

where F_{1*} is given as Equation (2.76).

Example

The hydraulic line is 60 ft long, 0.40 inch inner diameter. Other parameters are p_o = 11.2 psia, DN = 2.0/sec, L/C_o = 0.0137 sec. The line is subjected to pressure step inputs of 0.02 and 4.0 psig. Computed step responses based on approximations for Cosh $\Gamma(S)$ given in Chapter III and Equation (7.5) are shown on Figure 18. Note that the large disturbance; i.e., the 4.0 psig step, has a greatly damped response as compared to the small disturbance response.

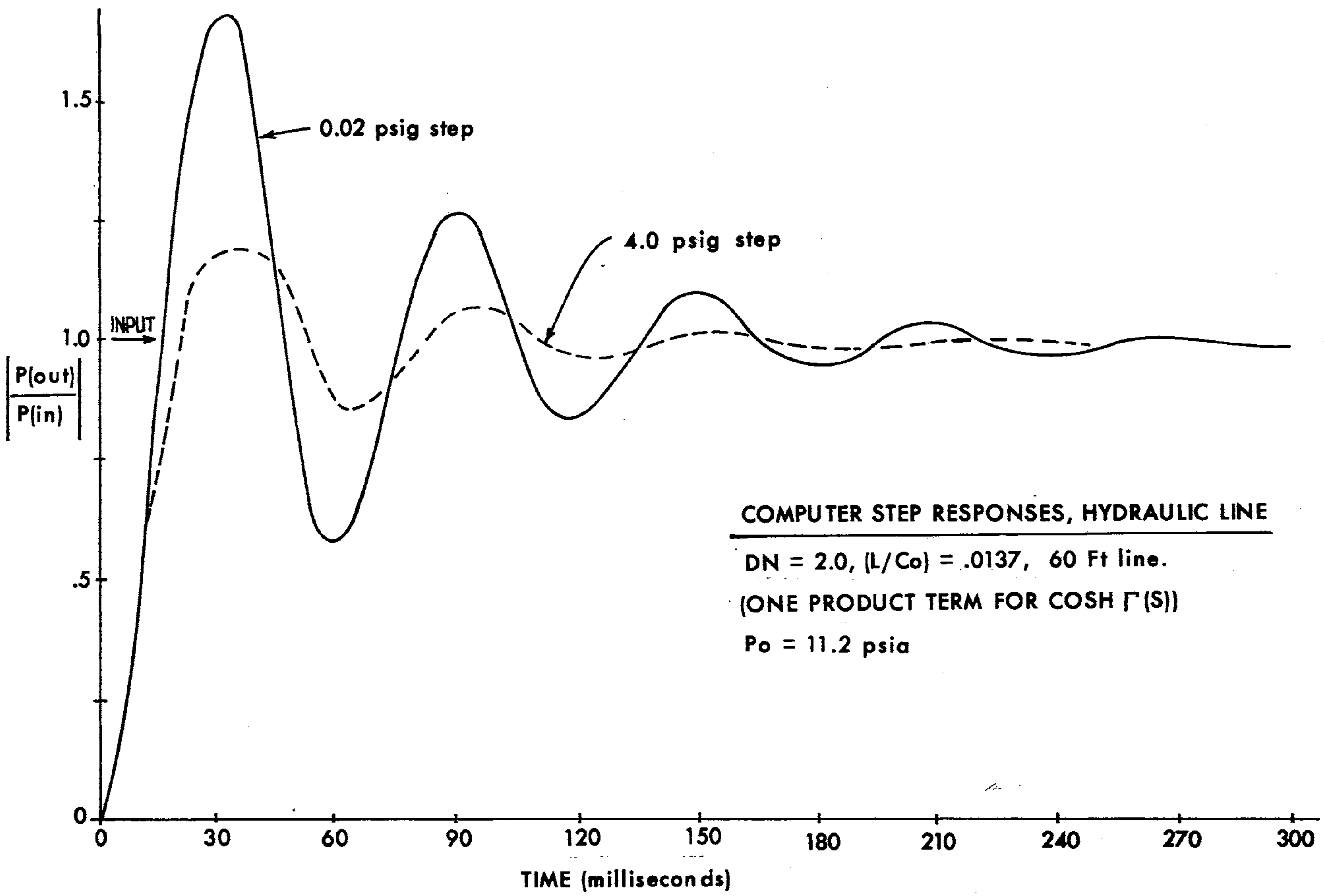

Figure 18. Computed Step Responses, 60 Ft Hydraulic Line

CHAPTER VIII

SUMMARY, CONCLUSION, AND RECOMMENDATIONS

Summary

The transmission line model developed in this thesis is an extension of the small amplitude (acoustic) model derived and utilized by Iberall (12), Nichols(15), and Brown(3). This model includes the effect of finite amplitude disturbances and through flow.

To include these effects, the nonlinear convective acceleration terms were retained in the axial momentum and energy equations:

Axial Momentum

$$\frac{\partial v_z}{\partial t} + \underline{v_z \frac{\partial v_z}{\partial z}} - \frac{\nu_0}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = -\frac{1}{\rho_0}\frac{\partial p_z}{\partial z} \qquad (8.1)$$

Energy Equation

$$\frac{\partial T_z}{\partial t} + \underline{v_z \frac{\partial T_z}{\partial z}} - \frac{\gamma \nu_0}{\sigma_0 r}\frac{\partial}{\partial r}\left(r \frac{\partial T_z}{\partial r}\right) = -(\gamma-1) T_0 \frac{\partial v_z}{\partial z} \qquad (8.2)$$

The nonlinear term $v_z \frac{\partial T_z}{\partial z}$ in the energy equation is of small order compared to the other terms in Equation (8.2), so it was neglected. But the term $v_z \frac{\partial v_z}{\partial z}$ in the axial momentum equation is not negligible when the disturbance is of finite amplitude.

The continuity equation and equation of state for ideal gases are used to express $\frac{\partial v_z}{\partial z}$ as a function of p_z and T_z. The initial development of the line model in Chapter II considers ideal gases as the working fluid. Chapter VII considers the simpler case where the fluid is a liquid.

The axial pressure, temperature, and velocity are separated into a steady-state incompressible through flow component subscripted with a "c" and a time-varying compressible component subscripted with a "t". That is:

$$v_z(t,r,z) = v_c(r) + v_t(t,r,z) \tag{8.3}$$

$$T_z(t,r,z) = T_c(r) + T_t(t,r,z) \tag{8.4}$$

$$p_z(t,z) = p_c(z) + p_t(t,z) \tag{8.5}$$

Equations (8.3), (8.4), (8.5) and the known steady-state solutions for (v_c) and (p_c) are substituted into Equations (8.1) and (8.2), resulting in these equations:

Axial Momentum

$$\frac{\partial v_t}{\partial t} + (v_c + v_t)\frac{\partial v_t}{\partial z} - \frac{\nu_o}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_t}{\partial r}\right) = -\frac{1}{\rho_o}\frac{\partial p_t}{\partial z} \tag{8.6}$$

Energy Equation

$$\frac{\partial T_t}{\partial t} - \frac{\gamma \nu_o}{\sigma_o r}\frac{\partial}{\partial r}\left(r\frac{\partial T_t}{\partial r}\right) = (\gamma - 1)\frac{T_o}{p_o}\frac{\partial p_t}{\partial t} \tag{8.7}$$

Equations (8.6) and (8.7) are nondimensionalized and the axial momentum equation is linearized by making the quantity $\frac{\partial V}{\partial Z}$ in the axial momentum equation a time-varying coefficient which is updated for each time increment (Δt). That is:

Axial Momentum

$$\frac{\partial V}{\partial t} + \frac{C_o}{L}\left(\frac{\partial V}{\partial Z}\right)_* V - \frac{\nu_o}{a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) = -\frac{C_o}{L}\left[\frac{1}{\gamma}\frac{\partial P}{\partial Z} + M_b\frac{\partial V}{\partial Z}\right] \tag{8.8}$$

where (M_b) is the Mach number of the average through flow. The time increment (Δt) must be much less than the reciprocal of the highest frequency of interest in the line. That is:

$$(\Delta t) \ll \frac{1}{\omega_{max}} \tag{8.9}$$

where (ω_{max}) is in radians per unit time.

To derive a form for the time-varying coefficient $\left(\frac{\partial V}{\partial Z}\right)_*$ the solution of the small disturbance or "acoustic" model is used. This model is shown as Appendix A in the thesis. The form used for $\left(\frac{\partial V}{\partial Z}\right)_*$ in the thesis, as taken in part from the acoustic model, is:

$$\left(\frac{\partial V}{\partial Z}\right)_* = [\ \mathrm{sgn}\ P(t,0)]\ \frac{L}{C_o}\left(\frac{\partial P(t,0)}{\partial t}\right)_* \tag{8.10}$$

The term $[\ \mathrm{sgn}\ P(t,0)\ ]$ is present to meet the criterion that the model must show an increase in apparent damping as disturbance amplitude increases, regardless of the sign of the disturbance (+ or -). This increase in apparent damping with increase in disturbance amplitude is an observed characteristic of real transmission lines, and it was necessary that the new model demonstrate the same characteristic.

By transforming the energy equation shown as Equation (8.7) and the axial momentum equation, Equation (8.8), into the Laplace domain, applying boundary conditions on (R) and (Z), this transmission line model resulted:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \cosh \Gamma_b(S) + Y_b(S) M_b \sinh \Gamma_b(S) & -Z_b(S) \sinh \Gamma_b(S) \\ \dfrac{-\sinh \Gamma_b(S)}{Z_b(S)} & \cosh \Gamma_b(S) - Y_b(S) M_b \sinh \Gamma_b(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{8.11}$$

where (P) and (Q) are nondimensional pressures and flow,

$$\Gamma_b(S) = \frac{SL}{C_o}\sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{S}\right)} \tag{8.12}$$

$$Y_b(S) = \frac{C_o}{SL} D_g \Gamma_b(S) = D_g\sqrt{\frac{N_g}{D_a}\left(1+\frac{F_{1*}}{S}\right)} \tag{8.13}$$

$$Z_b(S) = \frac{SL}{C_o}\frac{\gamma}{D_a \Gamma_b(S)}\left(1+\frac{F_{1*}}{S}\right) = \gamma\sqrt{\frac{\left(1+\frac{F_{1*}}{S}\right)}{N_g D_a}} \tag{8.14}$$

$$N_g = \left[1+\frac{2(\gamma-1)J_1(\Delta)}{\Delta J_0(\Delta)}\right], \quad D_g = \left[1-\frac{2J_1(\psi)}{\psi J_0(\psi)}\right], \quad D_a = \left[1-\frac{2J_1(\alpha)}{\alpha J_0(\alpha)}\right] \tag{8.15}$$

$$\Delta = j\sqrt{\frac{S\sigma_o}{DN}}, \quad \psi = j\sqrt{\frac{S}{DN}}, \quad \alpha = j\sqrt{\frac{S}{DN}\left(1+\frac{F_{1*}}{S}\right)} \tag{8.16}$$

$$DN = \frac{\nu_o}{a^2}, \quad F_{1*} = \frac{C_o}{L}\left(\frac{\partial V}{\partial Z}\right)_* = (\mathrm{sgn}\, P(t,0))\left(\frac{\partial P(t,0)}{\partial t}\right)_* \tag{8.17}$$

and (M_b) = average through flow Mach number. (8.18)

This model, Equation (8.11), simplifies to the small disturbance model of Appendix when $F_{1*} = 0.$ and $M_b = 0.$

Chapter IV presents the experimental step responses recorded from a 60 ft pneumatic line, 0.40 inch inner diameter. The step responses were initiated at gage pressures above and below atmospheric pressure, and terminated at atmospheric pressure, (11.2 psia). Experimental step responses are presented for ± 0.25, 1.0, 2.0, 4.0, 6.0, and 10.0 psig (Figure 9).

In Chapter V the experimental step responses of Chapter IV are compared with computed step responses from the analytical model. The computed step responses appeared too lightly damped, even at the smallest step size of ± 0.25 psig. The computer model damping was increased at this smallest step size so the computed step response and the approximate fundamental mode of the corresponding experimental response showed approximately the same percent of overshoot - indicating that approximately the same amount of damping was present in the computed and actual step responses. This increase in apparent damping was accomplished by changing the damping number (DN) of the computer model from its calculated value of 0.8 to a corrected value of 2.0. Then the transients predicted by the computer model with finite amplitude disturbances compared favorably with the experimental results of Chapter IV (see Figures 10 through 13).

When more than one product term was used to expand the term CoshΓ(S) in the model, instabilities appeared (Figure 14). The cause of the instabilities is examined in the last section of Chapter V.

Chapter VI is a brief look at frequency response data measured by Cooley(7) for a small pneumatic line with small amplitude sinusoidal disturbances and large through flow. Through flow is represented in the line model by the term (M_b), which is the average through flow Mach number.

Chapter VII presents the simplified model when the fluid is a liquid.

Conclusions

The purpose of this thesis was to derive a generalized time-domain, ordinary differential equation line model which will predict flow and pressure transients in a fluid-filled line subjected to both small and finite amplitude disturbances, with and without through flow. The line model should meet the basic criteria outlined on page 7 of this thesis.

That is:

1. The model should predict an increase in apparent damping as the magnitude of the disturbance input to the line is increased. As Figure 13 shows, the model meets this criterion.

2. The model should be reducible to finite order by suitable approximations such that computational time and difficulty are reduced without severely limiting the accuracy of the model. The approximations for the terms $\Gamma(S)$, $\mathrm{Cosh}\Gamma(S)$, and $\mathrm{Sinh}\Gamma(S)$ which appear in the Laplace domain model, Equation (2.70) and Equation (8.11), are given in Chapter III of this thesis. They enable the model to meet this criterion, but it is possible that the approximation for $\Gamma(S)$ could be improved (see Figure 6, where $\Gamma^2(S) = \left(\frac{SL}{C_o}\right)^2 \frac{N_g}{D_g}$).

3. The model response should be in reasonable agreement with the apparent fundamental mode of corresponding experimental responses. The line model in this thesis is a linearized model with a time-varying coefficient, F_{1*} (see Equations (8.17)). The model is designed primarily for numerical integration where F_{1*} is updated at every integration step. The low order polynomial approximations for $\mathrm{Cosh}\Gamma(S)$ and $\mathrm{Sinh}\Gamma(S)$ which facilitate inverse transformation of the Laplace domain form of the model result in a low order differential equation model. Consequently, the model should predict the fundamental (low frequency) mode of a transient response, but not the high frequency modes.

The model could be employed in applications requiring high frequency if suitable approximations for $\Gamma(S)$, $\mathrm{Cosh}\Gamma(S)$, and $\mathrm{Sinh}\Gamma(S)$ could be synthesized.

The model, with its approximations given in Chapter III, is a low frequency model. This low frequency model produced responses which appear to be too lightly damped, as shown on Figure 11. In this sense the model does not meet criterion #3 fully because the model responses (traces A, B, and C on Figure 11) are not in close agreement with the fundamental mode of the corresponding experimental result, which is also shown on Figure 11. It is possible that closer agreement between the computed traces and fundamental mode of the experimental trace could have been achieved by a better approximation for $\Gamma(S)$, but this is speculation.

The instability which occurred in the model when two or four product terms were used to expand $\cosh\Gamma(S)$ (see Figure 14) was not totally surprising. The two product term expansion for $\cosh\Gamma(S)$ yields a tenth-order differential equation and the four product term expansion yields a twentieth-order differential equation when step responses are computed (Equation 5.4). The tendancy toward numerical instability in the solution of high order differential equations containing a broad frequency spectrum is well known.

But this model added a new dimension for possible instability with its time-varying coefficient, F_{1*} (Equation 8.17). By applying Routh's Criterion to a two product term form of the model applicable to a special case (Equation 5.4) it was determined that the system of equations is unstable for all $F_{1*} < 0$, and may be unstable for some values of $F_{1*} > 0$, depending on the particular line length, diameter, fluid kinematic viscosity, etc. Routh's Criterion was applied to the approximations for $\Gamma(S)$ and $\cosh\Gamma(S)$, not their exact forms. So the approximations used for $\Gamma(S)$ and $\cosh\Gamma(S)$ may have contributed to the instability of the system of equations.

The transmission line model derived in the body of this thesis will predict an increase in apparent damping as disturbance amplitude increases, making it the first generalized line model that is sensitive to input disturbance level. At very small disturbance levels the model becomes the "acoustic" model of Appendix A.

If the user finds that the line model (Equation 2.70 or 8.11) tends to be unstable in his system simulation, he is referred to an alternate line model shown in Appendix C. The alternate line model does not predict as much increase in apparent damping with disturbance amplitude as does the primary model, but it is numerically stable for higher order approximations for $\mathrm{Cosh}\Gamma(S)$ and $\mathrm{Sinh}\Gamma(S)$ (see Figures 20, 21, and 22 in Appendix C).

The frequency response results given in Chapter VI show the following:

1. This line model, nor any other line model derived to date, can predict the large changes in frequency response behavior which one experimentalist, Cooley(7), has reported when through flow is introduced into a pneumatic line (see Figure 16).

2. The large discrepancy between analytical and experimental results in the through flow case merits further investigation.

Recommendations

Based on the analysis and findings of this thesis, it is recommended that additional work be conducted in these areas:

1. The synthesis of better forms for (F_{1*}) such that the resulting model is stable for high order approximations of CoshΓ(S) and SinhΓ(S), and such that the implicit instability which results when $F_{1*} < 0$ is eliminated.

2. The development of approximations for Γ(S), CoshΓ(S), and SinhΓ(S) which agree more closely with the exact forms, but which retain the mathematical simplicity of the forms used in this thesis.

3. Criteria #3, page 7 should be reexamined and a definitive procedure should be established for assessing the quality of the model.

4. A carefully planned experimental study should be made of the effect of through flow on the frequency response of a transmission line, to confirm the results of Cooley(7).

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APPENDIX A

SOLUTION FOR THE LINEAR PROBLEM

This appendix presents a solution to the linear axial momentum equation and linear energy equation for the flow of a compressible fluid in a rigid circular transmission line. This solution is identical to solutions presented by Iberall (12) and Brown (3).

Figure 19 identifies the line variables and coordinate system.

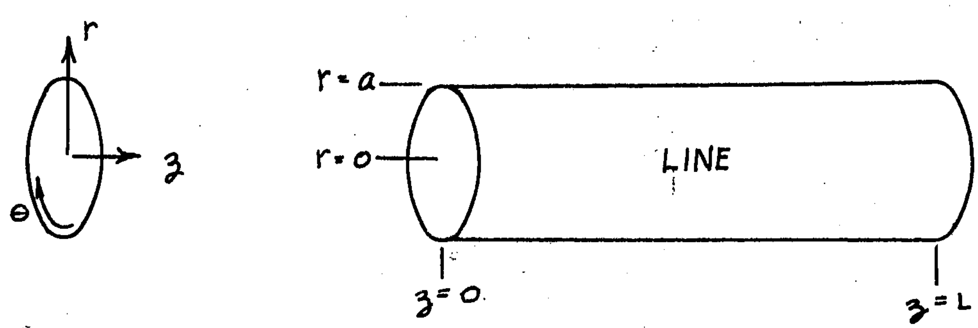

Figure 19. Coordinate System

Assumptions

1. $v_r = v_\theta = 0$.
2. All partials with respect to θ are zero.
3. Small amplitude, laminar perturbations.
4. No through flow.
5. $\partial p / \partial r \equiv 0$. (Pressure is constant across any given cross section of the line.)

Basic Equations

$$v_z = v_z(r,\ z,\ t)$$

$$p_z = p_z(z,\ t) \tag{A.1}$$

$$T_z = T_z(r,\ z,\ t)$$

Axial Momentum

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} - \frac{\nu_0}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = -\frac{1}{\rho_0} \frac{\partial p_z}{\partial z} \tag{A.2}$$

For small amplitude perturbations, the non-linear term $\left(v_z \frac{\partial v_z}{\partial z}\right)$ may be neglected (Brown (3), D'Souza (8)).

Energy Equation

$$\frac{\partial T_z}{\partial t} + v_z \frac{\partial T_z}{\partial z} - \frac{\gamma \nu_0}{\sigma_0 r} \frac{\partial}{\partial r}\left(r \frac{\partial T_z}{\partial r}\right) = -(\gamma - 1) T_0 \frac{\partial v_z}{\partial z}$$

For small amplitude perturbations, the term $v_z \frac{\partial T_z}{\partial z}$ neglected (Brown(3)).

Equation of State (Ideal gases)

$$-\frac{dp}{p_o} = \frac{d\rho}{\rho_o} + \frac{dT}{T_o} \Rightarrow \frac{\partial \rho}{\partial t} = \rho_o\left[\frac{1}{p_o}\frac{\partial p}{\partial t} - \frac{1}{T_o}\frac{\partial T}{\partial t}\right] \tag{A.4}$$

$$\Rightarrow \frac{\partial \rho}{\partial z} = \rho_o\left[\frac{1}{p_o}\frac{\partial p}{\partial z} - \frac{1}{T_o}\frac{\partial T}{\partial z}\right]$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_z)}{\partial z} = 0 \Rightarrow \frac{\partial v_z}{\partial z} = -\frac{1}{\rho_o}\left[\frac{\partial \rho}{\partial t} + v_z\frac{\partial \rho}{\partial z}\right] \tag{A.5}$$

For small amplitude perturbations, the term $\left(v_z \frac{\partial \rho}{\partial z}\right)$ may be neglected (Brown(3)). Combining Equations (A.4) and (A.5) yields:

$$\frac{\partial v_z}{\partial z} = -\left[\frac{1}{p_o}\frac{\partial p}{\partial t} - \frac{1}{T_o}\frac{\partial T}{\partial t}\right] \tag{A.6}$$

Integrated Continuity Equation

$$2\pi\int_{r=0}^{r=a} \frac{\partial(\rho v_z)}{\partial z}\, r\,dr = -2\pi\int_{r=0}^{r=a} \frac{\partial \rho}{\partial t}\, r\,dr \tag{A.7}$$

$$\Rightarrow \frac{\partial q(z,t)}{\partial z} = -2\pi\int_{r=0}^{r=a} \rho_o\left[\frac{1}{p_o}\frac{\partial p}{\partial t} - \frac{1}{T_o}\frac{\partial T}{\partial t}\right] r\,dr$$

where q(z,t) is the mass flow rate in the transmission line.

$$q(z,t) = 2\pi\int_{r=0}^{r=a} (\rho v_z)\, r\,dr \tag{A.8}$$

By non-dimensionalizing Equations (A.2) through (A.8) with these substitutions:

$$R = \frac{r}{a}\ , \quad Z = \frac{z}{l}\ , \quad P = \frac{p}{p_o}\ ,$$

$$V = \frac{v_z}{C_o}\ , \quad T = \frac{T_z}{T_o}\ , \quad Q = \frac{q(z,t)}{\rho_o C_o \pi a^2} \tag{A.9}$$

where $C_o = \sqrt{\gamma R_{gas} T_o}$ (A.10)

(isentropic speed of sound in the fluid), and by substituting the

Equations of State (A.4) and Continuity Equation (A.5) into Equations (A.2), (A.3), (A.7), and (A.8) the result is as follows.

Axial Momentum

$$\frac{\partial V}{\partial t} - \frac{\nu_0}{a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) = -\frac{p_0}{\rho_0 C_0 L}\frac{\partial P}{\partial Z} \tag{A.11}$$

Energy Equation

$$\frac{\partial T}{\partial t} - \frac{\nu_0}{\sigma_0 a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) = \frac{(\gamma-1)}{\gamma}\frac{\partial P}{\partial t} \tag{A.12}$$

Integrated Continuity Equation

$$\frac{\partial Q(t,Z)}{\partial Z} = \frac{-2L}{C_0}\int_0^1\left[\frac{\partial P}{\partial t} - \frac{\partial T}{\partial t}\right] R\,dR \tag{A.13}$$

Mass Flowrate Equation

$$Q(t,Z) = 2\int_0^1 V(t,R,Z)\,R\,dR \tag{A.14}$$

By transforming Equations (A.11) through (A.14) into the Laplace domain, the result is as follows.

Axial Momentum

$$s\,V(s) - \frac{\nu_0}{a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial V(s)}{\partial R}\right) = -\frac{p_0}{\rho_0 C_0 L}\frac{\partial P(s)}{\partial Z} \tag{A.15}$$

Energy Equation

$$s\,T(s) - \frac{\nu_0}{\sigma_0 a^2 R}\frac{\partial}{\partial R}\left(R\frac{\partial T(s)}{\partial R}\right) = \frac{(\gamma-1)}{\gamma}s\,P(s) \tag{A.16}$$

Integrated Continuity Equation

$$\frac{\partial Q(s,Z)}{\partial Z} = -\frac{2sL}{C_0}\int_0^1\left(P(s)-T(s)\right) R\,dR \tag{A.17}$$

Mass Flowrate

$$Q(S,Z) = 2\int_0^1 V(S,R,Z)\, R\, dR \tag{A.18}$$

Let Damping Number be $DN = \frac{\nu_0}{a^2}$, $\psi = j\sqrt{\frac{S}{DN}}$, and $\Delta = j\sqrt{\frac{S\sigma_0}{DN}}$. (A.19)

Rewriting Equations (A.15) and (A.16), the results are as follows.

Axial Momentum

$$V(S) + \frac{1}{\psi R}\frac{\partial}{\partial R}\left(R\frac{\partial V(S)}{\partial R}\right) = -\frac{C_0}{\gamma S L}\frac{\partial P(S)}{\partial Z} \tag{A.20}$$

Energy Equation

$$T(S) + \frac{1}{\Delta R}\frac{\partial}{\partial R}\left(R\frac{\partial T(S)}{\partial R}\right) = \frac{(\gamma-1)}{\gamma} P(S) \tag{A.21}$$

A solution to the Axial Momentum Equation, Equation (A.20) is:

$$V(S,R,Z) = \left(\frac{J_0(\psi R) - J_0(\psi)}{J_0(\psi)}\right)\frac{C_0}{\gamma S L}\frac{\partial P(S)}{\partial Z} \tag{A.22}$$

where J_o is the Bessel Function of the first kind, zeroeth order. This solution meets the boundary condition $V(S,R,Z)\Big]_{R=1} = 0$, the "no-slip" condition, and $\frac{\partial V(S,R,Z)}{\partial R}\Big]_{R=0} = 0$.

A solution to the Energy Equation, Equation (A.21) is:

$$T(S,R,Z) = -\left(\frac{J_0(\Delta R) - J_0(\Delta)}{J_0(\Delta)}\right)\frac{(\gamma-1)}{\gamma} P(S) \tag{A.23}$$

This solution meets the boundary condition $T(S,R,Z)\Big]_{R=1} = 0$, and $\frac{\partial T(S,R,Z)}{\partial R}\Big]_{R=0} = 0$. From Equation (A.18);

$$Q(S,Z) = \frac{2C_0}{\gamma S L}\frac{\partial P(S)}{\partial Z}\int_0^1\left(\frac{J_0(\psi R) - J_0(\psi)}{J_0(\psi)}\right) R\, dR \tag{A.24}$$

$$Q(S,Z) = -\frac{C_o D_g}{\gamma S L}\frac{\partial P(S)}{\partial Z} \tag{A.25}$$

where $D_g = \left(1 - \frac{2 J_1(\psi)}{\psi J_0(\psi)}\right)$ (A.26)

By substituting the solution to the Energy Equation, Equation (A.23) into the Integrated Continuity Equation, Equation (A.17), the result is:

$$\frac{\partial Q(S,Z)}{\partial Z} = -\frac{SL}{\gamma C_o} N_g P(S) \tag{A.27}$$

where $N_g = \left(1 + \frac{2(\gamma-1) J_1(\Delta)}{\Delta J_0(\Delta)}\right)$ (A.28)

By differentiating Equation (A.25) with respect to "Z", and equating the result to Equation (A.27), the result is:

$$-\frac{C_o}{\gamma S L} D_g \frac{\partial^2 P(S,Z)}{\partial Z^2} = -\frac{SL}{\gamma C_o} N_g P(S,Z) \tag{A.29}$$

or $$\frac{\partial^2 P(S,Z)}{\partial Z^2} = \left(\frac{SL}{C_o}\right)^2 \frac{N_g}{D_g} P(S,Z) = \Gamma(S)^2 P(S) \tag{A.30}$$

where $$\Gamma(S) = \frac{SL}{C_o}\sqrt{\frac{N_g}{D_g}} \tag{A.31}$$

A solution to Equation (A.30) is:

$$P(S,Z) = C_1 e^{\Gamma(S)Z} + C_2 e^{-\Gamma(S)Z} \tag{A.32}$$

The nondimensional flow Q(S,Z) is given by Equation (A.25):

$$Q(S,Z) = \frac{-C_o D_g \Gamma(S)}{\gamma S L}\left(C_1 e^{\Gamma(S)Z} - C_2 e^{-\Gamma(S)Z}\right) \tag{A.33}$$

Equations (A.32) and (A.33) may be solved for constants C_1 and C_2 by applying boundary conditions at Z = 0 and Z = 1:

$$\mathcal{L}(P(t,0)) = P(S,0) \;,\quad \mathcal{L}(Q(t,0)) = Q(S,0) \;,$$
$$\mathcal{L}(P(t,1)) = P(S,1) \;,\quad \mathcal{L}(Q(t,1)) = Q(S,1) \;. \tag{A.34}$$

The results are:

$$C_1 = \frac{1}{2}\left(P(S,0) - \frac{\gamma SL}{D_g C_o} Q(S,0)\right) \tag{A.35}$$
$$C_2 = \frac{1}{2}\left(P(S,0) + \frac{\gamma SL}{D_g C_o} Q(S,0)\right)$$

Since $\mathrm{Cosh}\,\Gamma(S)Z = \frac{1}{2}\left(e^{\Gamma(S)Z} + e^{-\Gamma(S)Z}\right)$ and $\mathrm{Sinh}\,\Gamma(S)Z = \frac{1}{2}\left(e^{\Gamma(S)Z} - e^{-\Gamma(S)Z}\right)$ (A.36)

Equations (A.32) and (A.33) may be rewritten as:

$$P(S,Z) = \mathrm{Cosh}\,\Gamma(S)Z\; P(S,0) - Z_c(S)\;\mathrm{Sinh}\,\Gamma(S)Z\; Q(S,0)$$
$$Q(S,Z) = \frac{-\mathrm{Sinh}\,\Gamma(S)Z}{Z_c(S)} P(S,0) + \mathrm{Cosh}\,\Gamma(S)Z\; Q(S,0) \tag{A.37}$$

where $$Z_c(S) = \frac{\gamma}{\sqrt{N_g D_g}} = \frac{SL\;\gamma}{C_o D_g \Gamma(S)} \tag{A.38}$$

Summary

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \mathrm{Cosh}\,\Gamma(S) & -Z_c(S)\;\mathrm{Sinh}\,\Gamma(S) \\ -\frac{\mathrm{Sinh}\,\Gamma(S)}{Z_c(S)} & \mathrm{Cosh}\,\Gamma(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{A.39}$$

where $Z_c(S)$ is given as Equation (A.38) and $\Gamma(S) = \frac{SL}{C_o}\sqrt{\frac{N_g}{D_g}}$.

$$N_g = \left[1 + \frac{2(\gamma-1)J_1(\Delta)}{\Delta J_o(\Delta)}\right] \;;\qquad D_g = \left[1 - \frac{2J_1(\psi)}{\psi J_o(\psi)}\right] \;; \tag{A.40}$$

$$\Delta = j\sqrt{\frac{S\sigma_o}{DN}} \;;\qquad \psi = j\sqrt{\frac{S}{DN}} \;;\qquad DN = \frac{\nu_o}{a^2} = \frac{\mu_o}{\rho_o a^2}$$

These important average values also come from this system of equations:

$$V(S,Z) = \frac{-C_o D_g}{\gamma S L} \frac{\partial P(S,Z)}{\partial Z} \tag{A.41}$$

$$\frac{\partial V(S,Z)}{\partial Z} = \frac{-C_o D_g}{\gamma S L} \frac{\partial^2 P(S,Z)}{\partial Z^2} \tag{A.42}$$

$$T(S,Z) = \frac{(\gamma-1)}{\gamma} P(S,Z)\left(1 - \frac{2 J_1(\Delta)}{\Delta J_o(\Delta)}\right) \tag{A.43}$$

$$\frac{\partial T(S,Z)}{\partial Z} = \frac{(\gamma-1)}{\gamma} \frac{\partial P(S,Z)}{\partial Z}\left(1 - \frac{2 J_1(\Delta)}{\Delta J_o(\Delta)}\right) \tag{A.44}$$

$$\frac{\partial P(S,Z)}{\partial Z} = \Gamma(S)\left[P(S,0)\ \mathrm{Sinh}\,\Gamma(S)Z - Z_c(S)\ \mathrm{Cosh}\,\Gamma(S)Z\ Q(S,0)\right] \tag{A.45}$$

Equations (A.41) through (A.45) may be inverse transformed to the time domain if suitable approximations are made for $\mathrm{Sinh}\,\Gamma(S)Z$ and $\mathrm{Cosh}\,\Gamma(S)Z$.

Let $\mathrm{Sinh}\,\Gamma(S)Z \approx \Gamma(S)Z$ (A.46)

$\mathrm{Cosh}\,\Gamma(S)Z \approx 1.$ (A.47)

Then

$$V(t,Z) \cong Q(t,0) - \frac{LZ}{C_o} \frac{\partial P(t,0)}{\partial t} \tag{A.48}$$

$$\frac{\partial V(t,Z)}{\partial Z} \cong \frac{-L}{C_o} \frac{\partial P(t,0)}{\partial t} \tag{A.49}$$

$$T(t,Z) \cong \frac{(\gamma-1)}{\gamma} P(t,0) - \frac{(\gamma-1) LZ}{C_o} \frac{\partial Q(t,0)}{\partial t} \tag{A.50}$$

$$\frac{\partial T(t,Z)}{\partial Z} \cong \frac{-(\gamma-1) L}{C_o} \frac{\partial Q(t,0)}{\partial t} \tag{A.51}$$

$$\frac{\partial P(t,Z)}{\partial Z} \cong \frac{-\gamma L}{C_o} \frac{\partial Q(t,0)}{\partial t} \tag{A.52}$$

Equations (A.41) through (A.52) will be used in the derivation in Chapter II.

APPENDIX B

COMPUTER PROGRAMS

There are five computer programs listed in this appendix. Three are written in Fortran IV and two are written in Algol.

1. Linear Frequency Response of a Transmission Line, with and without Through Flow: This program computes the ratio $\left|\frac{P(S,0)}{Q(S,0)}\right|$ and $\left|\frac{Q(S,0)}{P(S,0)}\right|$ for the pneumatic line of Cooley (7), which is 6.0 inches long and 0.125 inches in inner diameter. Damping Number of the air in the line is 30.18, and the term (Damping Number/ Prandtl Number) is 43.11. Average line pressure is approximately 3.0 psia.

This program calls one subroutine, "Bessel," which generates values for the complex Bessel Function of the first kind, zeroeth and first order.

2. Coefficients for Step Responses, Cne, Two, and Four Product Terms for Cosh $\Gamma(S)$, Pneumatic: This is a convenience program, written to supply the necessary coefficients for the "Step Response by Numerical Integration Program, Pneumatic." (See Chapter V) This program "NUMER" and "DENOM," where:

$$P(S,1) = \frac{P(S,0)}{\text{Cosh}\,\Gamma(S)} = P(S,0) \times \frac{\text{NUMER}}{\text{DENOM}} \tag{B.1}$$

where $\Gamma^2(S) = \left(\frac{L}{C_o}\right)^2 \frac{A(S)}{B(S)}$ (B.2)

and A(S) and B(S) are given as Equations (5.7) and (5.8).

3. Coefficients for Step Responses, One, Two, and Four Product Terms for Cosh $\Gamma(S)$, Hydraulic: This program is identical to (2) above, but uses expressions for A(S) and B(S) which are given as the numerator and denominator respectively of Equation (7.5). This program supplies the coefficients for "Step Response by Numerical Integration Program, Hydraulic."

4. Step Response by Numerical Integration, Pneumatic: This program is a numerical integrator which integrates Equations (B.1). The user selects the one, two, or four product term expansion for Cosh $\Gamma(S)$.

The coefficients for subroutine "Derfun," the derivative function generator, are read in from the punched card output of program (2) listed above. This program uses a fourth-order Runge-Kutta integrator, "Rkint," and has a built-in plot routine, "Xyplot."

5. Step Response by Numerical Integration, Hydraulic: This program reads in data cards for subroutine "Derfun" which have been generated from program (3) above. It is similar to program (4) above.

```
C---- THIS PROGRAM COMPUTES LINEAR FREQUENCY RESPONSE RATIO FOR THE
C     COOLEY LINE, WITH AND WITHOUT THROUGH-FLOW, JAN 73.
      COMPLEX CMPLX,CFN1,CFN2,CFN3,A1,A2,A3,A4,A5,A6,DGAM,AGAM,GAMMA,
     2COSH,SINH,CEXP,CSQRT,RATIO,ANSER2,A7,A8
      DIMENSION AHERTZ(30)
      ELOVCO=.4425E-3
      VA=.16
      DN1=30.18
      DN2=43.11
      READ(5,100) NVAL, (AHERTZ(J),J=1,14)
  100 FORMAT(I2,8X,14F5.1)
      IF(NVAL.GT.14) READ(5,200) (AHERTZ(J),J=15,30)
      WRITE(6,300)
  200 FORMAT(16F5.1)
  300 FORMAT(1H1,5X,'FREQUENCY RESPONSE, COOLEY LINE, WITH AND WITHOUT T
     2HROUGH-FLOW:',/,6X,63('='),//,11X,'FREQUENCY',17X,'RATIO: P(S)/Q(S
     3)',27X,'RATIO: Q(S)/P(S)',/,12X,'(HERTZ)',8X,'WITH THROUGH-FLOW
     4 NO THROUGH-FLOW',7X,'WITH THROUGH-FLOW    NO THROUGH-FLOW',/,11X,
     59('-'),7X,17('-'),4X,15('-'),7X,17('-'),4X,15('-'),/)
      DO 20 KK=1,NVAL
      W1=6.28318*AHERTZ(KK)/DN1
      W2=6.28318*AHERTZ(KK)/DN2
      W3=6.28318*AHERTZ(KK)
      CFN1=CMPLX(0.,-W1)
      CFN2=CMPLX(0.,-W2)
      CFN3=CMPLX(0.,W3)
      A1=CSQRT(CFN1)
      A2=CSQRT(CFN2)
      CALL BESSEL(A1,A3,A4,A5,N1)
      CALL BESSEL(A2,A6,A7,A8,N2)
      DGAM=(1.,0.)-2.*A3/A1
      AGAM=(1.,0.)+.8*A6/A2
      GAMMA=ELOVCO*CFN3*CSQRT(AGAM/DGAM)
      COSH=.5*(CEXP(GAMMA)+CEXP(-GAMMA))
      SINH=.5*(CEXP(GAMMA)-CEXP(-GAMMA))
      RATIO=1.4*ELOVCO*CFN3*SINH/(DGAM*GAMMA)/(COSH+VA*DGAM*GAMMA*SINH
     2/(CFN3*ELOVCO))
      ANSWER=CABS(RATIO)
      ANSER2=1.4*ELOVCO*CFN3*SINH/(DGAM*GAMMA*COSH)
      B3=CABS(ANSER2)
      B4=1./ANSWER
      B5=1./B3
   20 WRITE(6,400) AHERTZ(KK),ANSWER,B3, B4, B5
  400 FORMAT(10X,F10.1,4(10X,F10.4))
      STOP
      END
```

```
      SUBROUTINE BESSEL (Z,RJ,JO,J1,NTE)
C---- THIS SUBROUTINE COMPUTES VALUES FOR THE COMPLEX BESSEL  FUNCTIONS
C  5  "JO" AND "J1" FROM THE BASIC SERIES EXPANSION, "HANDBOOK OF MATH-
C     EMATICAL FUNCTIONS"-ABRAMOWITZ, PG 360, FORMULA 9.1.10. NEW TERMS
C     ARE ADDED IN THE SERIES UNTIL THE CHANGE IN "JO" AND "J1" IS LESS
C     THAN 0.01 %.  THE NO. OF TERMS IN THE SERIES IS GIVEN BY "NTE".
      COMPLEX CMPLX,Z,RJ,JO,J1,TERMO,TERM1,ZOVER2,ZOSQ
      NTE=0
      ZOVER2=.5*Z
      ZOSQ=-ZOVER2**2
      TERM1=ZOVER2
      J1=ZOVER2
      TERMO=(1.,0.)
      JO=(1.,0.)
      A=1.
   10 TERMO=TERMO*ZOSQ/A**2
      JO=JO+TERMO
      TERM1=TERM1*ZOSQ/(A*(A+1.))
      J1=J1+TERM1
      NTE=NTE+1
      BB=CABS(TERMO)/CABS(JO)
      CC=CABS(TERM1)/CABS(J1)
      IF(BB.LT..0001.AND.CC.LT..0001) GO TO 20
      A=A+1.
      GO TO 10
   20 RJ=J1/JO
      RETURN
      END
  DATA
30          200. 300. 400. 500. 505. 510. 600. 700. 800. 900.1000.1050.1055.1060.
1100.1200.1300.1400.1500.1595.1600.1605.1700.1800.1900.2000.2100.2145.2150.2155.
```

```
     BEGIN  COMMENT THIS PROGRAM COMPUTES COSH(GAMMA)=DENOM/NUMER FOR
ONE, TWO, AND FOUR PRODUCT TERMS [PNEUMATIC].  THE OUTPUT INCLUDES
PUNCHED CARDS WHICH MAY BE ENTERED DIRECTLY INTO THE "STEP RESPONSE BY
NUMERICAL INTEGRATION--PNEUMATIC" PROGRAM.  FOR EXAMPLE, THE "NUMERATOR,
ONE PRODUCT TERM" ARRAY HAS 5 ROWS AND 4 COLUMNS.  THE ARRAY IS PUNCHED
BY ROW, 5 PUNCHED CARDS WITH 4 NUMBERS ON EACH CARD;  COMMENT
     THIS PROGRAM READS IN ONE DATA CARD WITH PARAMETERS "L/CO", "DN",
"L", AND "RO":  [FORMAT 4F10]
     1)  COLUMNS 1-10   RATIO OF LINE LENGTH OVER ISEN. SPEED OF SOUND.
     2)  COLUMNS 11-20  DAMPING NUMBER (RATIO OF KINEMATIC VISCOSITY
OVER TUBE RADIUS**2).
     3)  COLUMNS 21-30  LINE LENGTH (IN FEET), FOR REFERENCE ONLY.
     4)  COLUMNS 31-40  LINE INNER DIAMETER (INCHES), FOR REF. ONLY.
     THE REMAINING DATA CARDS ARE THE COEFFICIENTS FOR ARRAYS [A] AND
[B], WHERE GAMMA=(S*L/CO)**2*[A]/[B];
     ARRAY A[0:6,0:3,0:3],B[0:4,0:2,0:3],BB[0:8,0:4,0:6],AB[0:10,0:5,0:6
],AA[0:12,0:6,0:6],BBBB[0:16,0:8,0:12],ABBB[0:18,0:9,0:12],AABB[0:20,0:1
0,0:12], AAAB[0:22,0:11,0:12], AAAA[0:24,0:12,0:12], DN[0:12], NUM[0:16,
0:12], DEN[0:24,0:12], NUMER[0: 8,0:6], DENOM[0:12,0:6],ABUF[0:6,0:3,
0:3], BBUF[0:4,0:2,0:3], APOT[0:6,0:3], BPOT[0:4,0:3];
FILE CARD(KIND=READER);FILE LINE(KIND=PRINTER);FILE PUNCH(KIND=PUNCH);
INTEGER I,J,K; REAL ELOVCO,EL2,EL4,EL6,EL8,M1,M2,M3,M4,M5,M6,M7,L,R;
     FORMAT TITL(X10,"COEFFICIENTS FOR STEP RESPONSES, ONE, TWO, AND FOU
R PRODUCT TERMS FOR COSH(GAMMA), [PNEUMATIC], W/PUNCHED OUTPUT:",/,X10,
113("="),/);
     FORMAT P1(/,X30,"ARRAY 'A':",/,X30,"----------",/);
     FORMAT P2(/,X30,"ARRAY 'B':",/,X30,"----------",/);
     FORMAT P3(/,X20,"NUMERATOR, TWO PRODUCT TERMS:",/,X20,29("-"),/);
     FORMAT P4(/,X20,"DENOMINATOR, TWO PRODUCT TERMS:",/,X20,31("-"),/);
     FORMAT P5(/,X20,"NUMERATOR, FOUR PRODUCT TERMS:",/,X20,30("-"),/);
    FORMAT P6(/,X20,"DENOMINATOR, FOUR PRODUCT TERMS:",/,X20,32("-"),/);
     FORMAT P7(X5,"S= ",I1," , DN= ",I1," , K=0 TO 3:",X10,4E20.4,  );
     FORMAT P8(X5,"S= ",I2," , K=0 TO 6:",X5,7E15.4,/ );
     FORMAT P9(X1,"S=",I2,", K=0-12:",X1,13E9.2,/ );
     FORMAT P10(/,X20,"NUMERATOR, ONE PRODUCT TERM:",/,X20,28("-"),/);
     FORMAT P11(/,X20,"DENOMINATOR, ONE PRODUCT TERM:",/,X20,30("-"),/);
     FORMAT P12(X5,"S= ",I1," , K=0 TO 3:",X5 ,4E25.4,/ );
     FORMAT P13(X5,"FOR THIS RUN, L/CO = ",R11.4," , DAMPING NO. = ",
R11.4," , L= ",R11.4," FEET, AND TUBE I.D.= ",R11.4," .",/);
     FORMAT P14(8E10.3);
     FORMAT P15(4F10.3);
     FORMAT P16("MAJOR BRADEN, DFAN, BOX CC, PNEUMATIC, DN=",F6.3,
", L/CO=" F8.5," :");
    PROCEDURE POLYMU(X,Y,Z,X1,X2,X3,Y1,Y2,Y3);
ARRAY X,Y,Z[0,0,0]; INTEGER X1,X2,X3,Y1,Y2,Y3; BEGIN INTEGER I,II,J,JJ,
K,KK,Z1,Z2,Z3; Z1:=X1+Y1; Z2:=X2+Y2; Z3:=X3+Y3; FOR I:=0 STEP 1 UNTIL Z1
 DO FOR J:=0 STEP 1 UNTIL Z2 DO FOR K:=0 STEP 1 UNTIL Z3 DO Z[I,J,K]:=0.
;  FOR I:=0 STEP 1 UNTIL X1 DO FOR II:=0 STEP 1 UNTIL Y1 DO FOR J:=0 STE
P 1 UNTIL X2 DO FOR JJ:=0 STEP 1 UNTIL Y2 DO FOR K:=0 STEP 1 UNTIL X3 DO
 FOR KK:=0 STEP 1 UNTIL Y3 DO Z[I+II,J+JJ,K+KK]:=Z[I+II,J+JJ,K+KK]+
X[I,J,K]*Y[II,JJ,KK]; END;
     PROCEDURE POLYAD(X,Y,Z,X1,X2,X3,Y1,Y2,Y3); ARRAY X,Y,Z[0,0,0];
INTEGER X1,X2,X3,Y1,Y2,Y3;  BEGIN INTEGER Z1,Z2,Z3,I,J,K; REAL XNUMB,
YNUMB; Z1:=X1; IF Y1 > Z1 THEN Z1:=Y1; Z2:=X2; IF Y2 > Z2 THEN Z2:=Y2;
Z3:=X3; IF Y3 > Z3 THEN Z3:=Y3;  FOR I:=0 STEP 1 UNTIL Z1 DO FOR J:=0
STEP 1 UNTIL Z2 DO FOR K:=0 STEP 1 UNTIL Z3 DO BEGIN IF I > X1 OR J > X2
 OR K > X3 THEN XNUMB:=0 ELSE XNUMB:=X[I,J,K]; IF I>Y1 OR J>Y2 OR K>Y3
THEN YNUMB:=0 ELSE YNUMB:=Y[I,J,K]; Z[I,J,K]:=XNUMB+YNUMB; END; END;
     READ(CARD,P15,ELOVCO,DN[1],L,R);
EL2:=ELOVCO*ELOVCO; EL4:=EL2*EL2; EL6:=EL2*EL4; EL8:=EL4*EL4;
     FOR I:=2 STEP 1 UNTIL 12 DO DN[I]:=DN[1]*DN[I-1];
     READ(CARD,/,FOR I:=0 STEP 1 UNTIL 6 DO FOR J:=0 STEP 1 UNTIL 3 DO
FOR K:=0 STEP 1 UNTIL 3 DO A[I,J,K]);
     READ(CARD,/,FOR I:=0,1,2,3,4 DO FOR J:=0,1,2 DO FOR K:=0,1,2,3 DO
B[I,J,K]);
     WRITE(LINE,TITL); WRITE(LINE,P13,ELOVCO,DN[1],L,R);
                       WRITE(LINE,P1); FOR I:=0 STEP 1 UNTIL 6 DO FOR J
:=0,1,2,3 DO WRITE(LINE,P7,I,J, FOR K:=0,1,2,3 DO A[I,J,K]);
     WRITE(LINE,P2); FOR I:=0,1,2,3,4 DO FOR J:=0,1,2 DO WRITE(LINE,P7,
I,J, FOR K:=0,1,2,3 DO B[I,J,K]);      WRITE(LINE[SKIP 1]);
     WRITE(PUNCH,P16,DN[1],ELOVCO);
     COMMENT  SOLVE FOR NUMER, 1 PRODUCT TERM, NUMER = ARRAY[B];
FOR I:=0 STEP 1 UNTIL 4 DO FOR J:=0,1,2   DO FOR K:=0,1,2,3 DO BBUF[I,J,
K]:=B[I,J,K];
  FOR I:=0,1,2,3,4 DO FOR J:=1,2 DO FOR K:=0,1,2,3 DO BBUF[I,J,K]:=DN[J]
*BBUF[I,J,K];  FOR I:=0,1,2,3,4 DO FOR J:=0,1,2 DO FOR K:=0,1,2,3 DO
BPOT[I,K]:=BPOT[I,K]+BBUF[I,J,K];  WRITE(LINE,P10);  FOR I:=0,1,2,3,4 DO
 WRITE(LINE,P12,I,FOR K:=0,1,2,3 DO BPOT[I,K]);
FOR I:=0,1,2,3,4 DO WRITE(PUNCH,P14, FOR K:=0,1,2,3 DO BPOT[I,K]);
     COMMENT  SOLVE FOR DENOM, ONE PRODUCT TERM; M7:=.4053*EL2;
FOR I:=0 STEP 1 UNTIL 6 DO FOR J:=0,1,2,3 DO FOR K:=0,1,2,3 DO ABUF[I,J,
K]:=A[I,J,K]*M7;  POLYAD(B,ABUF,ABUF,4,2,3,6,3,3);
     FOR I:=0 STEP 1 UNTIL 6 DO FOR J:=1,2,3 DO FOR K:=0,1,2,3 DO
ABUF[I,J,K]:=DN[J]*ABUF[I,J,K];  FOR I:=0 STEP 1 UNTIL 6 DO FOR J:=0,1,
2,3 DO FOR K:=0,1,2,3 DO APOT[I,K]:=APOT[I,K]+ABUF[I,J,K];
     WRITE(LINE,P11);  FOR I:=0 STEP 1 UNTIL 6 DO WRITE(LINE,P12,I, FOR
K:=0,1,2,3 DO APOT[I,K]);  FOR I:=0,1,2,3,4,5,6 DO WRITE(PUNCH,P14, FOR
K:=0,1,2,3 DO APOT[I,K]);
     COMMENT  SOLVE FOR NUMER, 2 PRODUCT TERMS;
     POLYMU(B,B,BB,4,2,3,4,2,3);
     FOR I:=0 STEP 1 UNTIL 8 DO FOR J:=1 STEP 1 UNTIL 4 DO FOR K:=0 STEP
 1 UNTIL 6 DO BB[I,J,K]:=DN[J]*BB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 8 DO FOR J:=0 STEP 1 UNTIL 4 DO FOR K:=0 STEP
 1 UNTIL 6 DO NUMER[I,K]:=NUMER[I,K]+BB[I,J,K];
     WRITE(LINE,P3); FOR I:=0 STEP 1 UNTIL 8 DO WRITE(LINE,P8,I, FOR K:=
0 STEP 1 UNTIL 6 DO NUMER[I,K]);  FOR I:=0 STEP 1 UNTIL 8 DO WRITE(PUNCH
,P14, FOR K:=0 STEP 1 UNTIL 6 DO NUMER[I,K]);
     COMMENT SOLVE FOR DENOM, 2 PRODUCT TERMS;
     POLYMU(B,B,BB,4,2,3,4,2,3); POLYMU(A,B,AB,6,3,3,4,2,3);
     POLYMU(A,A,AA,6,3,3,6,3,3);
     M1:=.450316*EL2; M2:=.0182506*EL4;
     FOR I:=0 STEP 1 UNTIL 10 DO FOR J:=0 STEP 1 UNTIL 5 DO FOR K:=0
STEP 1 UNTIL 6 DO AB[I,J,K]:=M1*AB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 12 DO FOR J:=0 STEP 1 UNTIL 6 DO FOR K:=0
STEP 1 UNTIL 6 DO AA[I,J,K]:=M2*AA[I,J,K];
     POLYAD(AA,AB,AA,12,6,6,10,5,6); POLYAD(AA,BB,AA,12,6,6,8,4,6);
     FOR I:=0 STEP 1 UNTIL 12 DO FOR J:=1 STEP 1 UNTIL 6 DO FOR K:=0
STEP 1 UNTIL 6 DO AA[I,J,K]:=DN[J]*AA[I,J,K];
     FOR I:=0 STEP 1 UNTIL 12 DO FOR J:=0 STEP 1 UNTIL 6 DO FOR K:=0
STEP 1 UNTIL 6 DO DENOM[I,K]:=DENOM[I,K]+AA[I,J,K];
     WRITE(LINE,P4); FOR I:=0 STEP 1 UNTIL 12 DO WRITE(LINE,P8,I, FOR K
:=0 STEP 1 UNTIL 6 DO DENOM[I,K]);  FOR I:=0 STEP 1 UNTIL 12 DO WRITE(
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```
PUNCH,P14, FOR K:=0 STEP 1 UNTIL 6 DO DENOM[I,K]);
     COMMENT  SOLVE FOR NUMERATOR, FOUR PRODUCT TERMS;
     POLYMU(B,B,BB,4,2,3,4,2,3); POLYMU(BB,BB,BBBB,8,4,6,8,4,6);
     FOR I:=0 STEP 1 UNTIL 16 DO FOR J:=1 STEP 1 UNTIL 8 DO FOR K:=0
STEP 1 UNTIL 12 DO BBBB[I,J,K]:=DN[J]*BBBB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 16 DO FOR J:=0 STEP 1 UNTIL 8 DO FOR K:=0
STEP 1 UNTIL 12 DO NUM[I,K]:=NUM[I,K]+BBBB[I,J,K];
     WRITE(LINE,P5); FOR I:=0 STEP 1 UNTIL 16 DO WRITE(LINE,P9, I, FOR K
:=0 STEP 1 UNTIL 12 DO NUM[I,K]);  FOR I:=0 STEP 1 UNTIL 16 DO WRITE(
PUNCH,P14, FOR K:=0 STEP 1 UNTIL 12 DO NUM[I,K]);
     COMMENT SOLVE FOR DENOMINATOR, FOUR PRODUCT TERMS;
     POLYMU(B,B,BB,4,2,3,4,2,3); POLYMU(A,B,AB,6,3,3,4,2,3);
     POLYMU(A,A,AA,6,3,3,6,3,3);
     POLYMU(BB,BB,BBBB,8,4,6,8,4,6); POLYMU(AB,BB,ABBB,10,5,6,8,4,6);
     POLYMU(AA,BB,AABB,12,6,6,8,4,6); POLYMU(AA,AB,AAAB,12,6,6,10,5,6);
     POLYMU(AA,AA,AAAA,12,6,6,12,6,6);
     M3:=.474778*EL2; M4:=.0294*EL4; M5:=.506679@-3*EL6; M6:=.24411@-5*
EL8;
     FOR I:=0 STEP 1 UNTIL 18 DO FOR J:=0 STEP 1 UNTIL 9 DO FOR K:=0
STEP 1 UNTIL 12 DO ABBB[I,J,K]:=M3*ABBB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 20 DO FOR J:=0 STEP 1 UNTIL 10 DO FOR K:=0
STEP 1 UNTIL 12 DO AABB[I,J,K]:=M4*AABB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 22 DO FOR J:=0 STEP 1 UNTIL 11 DO FOR K:=0
STEP 1 UNTIL 12 DO AAAB[I,J,K]:=M5*AAAB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 24 DO FOR J:=0 STEP 1 UNTIL 12 DO FOR K:=0
STEP 1 UNTIL 12 DO AAAA[I,J,K]:=M6*AAAA[I,J,K];
     POLYAD(AAAA,AAAB,AAAA,24,12,12,22,11,12);
     POLYAD(AAAA,AABB,AAAA,24,12,12,20,10,12);
     POLYAD(AAAA,ABBB,AAAA,24,12,12,18,9,12);
     POLYAD(AAAA,BBBB,AAAA,24,12,12,16,8,12);
     FOR I:=0 STEP 1 UNTIL 24 DO FOR J:=1 STEP 1 UNTIL 12 DO FOR K:=0
STEP 1 UNTIL 12 DO AAAA[I,J,K]:=DN[J]*AAAA[I,J,K];
     FOR I:=0 STEP 1 UNTIL 24 DO FOR J:=0 STEP 1 UNTIL 12 DO FOR K:=0
STEP 1 UNTIL 12 DO DEN[I,K]:=DEN[I,K]+AAAA[I,J,K];
     WRITE(LINE,P6); FOR I:=0 STEP 1 UNTIL 24 DO WRITE(LINE,P9,I, FOR K
:=0 STEP 1 UNTIL 12 DO DEN[I,K]);
     FOR I:=0 STEP 1 UNTIL 24 DO WRITE(PUNCH,P14, FOR K:=0 STEP 1 UNTIL
12 DO DEN[I,K]); WRITE(PUNCH,<"END OF DATA CARDS">); END.
  DATA
.0532       2.       60.       .4
 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
 0,0,0,0,0,0,0,  10., 0,0,623.8,0, 0, 3271.5,0,0,
 0,0,0, 1., 0,0,92.38,0, 0, 1575.,0,0, 3271.5,0,0,0,
 0,0,3.,0, 0, 154.76,0,0, 950.95,0,0,0, 0,0,0,0,
 0, 3.,0,0, 72.38,0,0,0,  0,0,0,0,  0,0,0,0,
 1.,0,0,0,  0,0,0,0,  0,0,0,0, 0,0,0,0
 0,0,0,0,  0,0,0,-7.14,  0,0,-292.03,0,
 0,0,0,-1.,0,0,-48.04,0,   0,0,0,0,
 0,0,-1.,0,  0, 7.14,0,0,  292.03,0,0,0,
 0, 1.,0,0,  48.04,0,0,0,   0,0,0,0,
 1.,0,0,0,   0,0,0,0,     0,0,0,0
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```
     BEGIN  COMMENT THIS PROGRAM COMPUTES COSH(GAMMA)=DENOM/NUMER FOR
ONE, TWO, AND FOUR PRODUCT TERMS [HYDRAULIC].  THE OUTPUT INCLUDES
PUNCHED CARDS WHICH MAY BE ENTERED DIRECTLY INTO THE "STEP RESPONSE BY
NUMERICAL INTEGRATION--HYDRAULIC" PROGRAM.  FOR EXAMPLE, THE "NUMERATOR,
ONE PRODUCT TERM" ARRAY HAS 4 ROWS AND 4 COLUMNS.  THE ARRAY IS PUNCHED
BY ROW, 4 PUNCHED CARDS WITH 4 NUMBERS ON EACH CARD:  COMMENT
     THIS PROGRAM READS IN ONE DATA CARD WITH PARAMETERS "L/CO", "DN",
"L", AND "RO":  [FORMAT 4F10]
     1)  COLUMNS 1-10   RATIO OF LINE LENGTH OVER ISEN. SPEED OF SOUND.
     2)  COLUMNS 11-20  DAMPING NUMBER (RATIO OF KINEMATIC VISCOSITY
OVER TUBE RADIUS**2).
     3)  COLUMNS 21-30  LINE LENGTH (IN FEET), FOR REFERENCE ONLY.
     4)  COLUMNS 31-40  LINE INNER DIAMETER (INCHES), FOR REF. ONLY.
     THE REMAINING DATA CARDS ARE THE COEFFICIENTS FOR ARRAYS [A] AND
[B], WHERE GAMMA=(S*L/CO)**2*[A]/[B];
     ARRAY A[0:5,0:2,0:3], B[0:3,0:1,0:3], BB[0:6,0:2,0:6], AB[0:8,0:3,
0:6], AA[0:10,0:4,0:6], BBBB[0:12,0:4,0:12], ABBB[0:14,0:5,0:12], AABB[
0:16,0:6,0:12],  AAAB[0:18,0:7,0:12], AAAA[0:20,0:8,0:12], DN[0:8],
NUM[0:12,0:12], DEN[0:20,0:12], NUMER[0:6,0:6], DENOM[0:10,0:6], ABUF[
0:5,0:2,0:3], BBUF[0:3,0:1,0:3], APOT[0:5,0:3], BPOT[0:3,0:3];
FILE CARD(KIND=READER);FILE LINE(KIND=PRINTER);FILE PUNCH(KIND=PUNCH);
INTEGER I,J,K; REAL ELOVCO,EL2,EL4,EL6,EL8,M1,M2,M3,M4,M5,M6,M7,L,R;
     FORMAT TITL(X10,"COEFFICIENTS FOR STEP RESPONSES, ONE, TWO, AND FOU
R PRODUCT TERMS FOR COSH(GAMMA), [HYDRAULIC], W/PUNCHED OUTPUT:",/,X10,
113("="),/);
    FORMAT P1(/,X30,"ARRAY 'A':",/,X30,"----------",/);
    FORMAT P2(/,X30,"ARRAY 'B':",/,X30,"----------",/);
     FORMAT P3(/,X20,"NUMERATOR, TWO PRODUCT TERMS:",/,X20,29("-"),/);
     FORMAT P4(/,X20,"DENOMINATOR, TWO PRODUCT TERMS:",/,X20,31("-"),/);
     FORMAT P5(/,X20,"NUMERATOR, FOUR PRODUCT TERMS:",/,X20,30("-"),/);
    FORMAT P6(/,X20,"DENOMINATOR, FOUR PRODUCT TERMS:",/,X20,32("-"),/);
     FORMAT P7(X5,"S= ",I1," , DN= ",I1," , K=0 TO 3:",X10,4E20.4,  );
     FORMAT P8(X5,"S= ",I2," , K=0 TO 6:",X5,7E15.4,/ );
     FORMAT P9(X1,"S=",I2,", K=0-12:",X1,13E9.2,/ );
     FORMAT P10(/,X20,"NUMERATOR, ONE PRODUCT TERM:",/,X20,28("-"),/);
     FORMAT P11(/,X20,"DENOMINATOR, ONE PRODUCT TERM:",/,X20,30("-"),/);
     FORMAT P12(X5,"S= ",I1," , K=0 TO 3:",X5 ,4E25.4,/ );
     FORMAT P13(X5,"FOR THIS RUN, L/CO = ",R11.4," , DAMPING NO. = ",
R11.4," , L= ",R11.4," FEET, AND TUBE I.D.= ",R11.4," .",/);
     FORMAT P14(8E10.3);
     FORMAT P15(4F10.3);
     FORMAT P16("MAJOR BRADEN, DFAN, BOX CC, HYDRAULIC, DN=",F6.3,
", L/CO=" F10.7," :");
    PROCEDURE POLYMU(X,Y,Z,X1,X2,X3,Y1,Y2,Y3);
ARRAY X,Y,Z[0,0,0]; INTEGER X1,X2,X3,Y1,Y2,Y3; BEGIN INTEGER I,II,J,JJ,
K,KK,Z1,Z2,Z3; Z1:=X1+Y1; Z2:=X2+Y2; Z3:=X3+Y3; FOR I:=0 STEP 1 UNTIL Z1
 DO FOR J:=0 STEP 1 UNTIL Z2 DO FOR K:=0 STEP 1 UNTIL Z3 DO Z[I,J,K]:=0.
; FOR I:=0 STEP 1 UNTIL X1 DO FOR II:=0 STEP 1 UNTIL Y1 DO FOR J:=0 STE
P 1 UNTIL X2 DO FOR JJ:=0 STEP 1 UNTIL Y2 DO FOR K:=0 STEP 1 UNTIL X3 DO
 FOR KK:=0 STEP 1 UNTIL Y3 DO Z[I+II,J+JJ,K+KK]:=Z[I+II,J+JJ,K+KK]+
X[I,J,K]*Y[II,JJ,KK]; END;
     PROCEDURE POLYAD(X,Y,Z,X1,X2,X3,Y1,Y2,Y3); ARRAY X,Y,Z[0,0,0];
```

```
INTEGER X1,X2,X3,Y1,Y2,Y3;  BEGIN INTEGER Z1,Z2,Z3,I,J,K; REAL XNUMB,
YNUMB; Z1:=X1; IF Y1 > Z1 THEN Z1:=Y1; Z2:=X2; IF Y2 > Z2 THEN Z2:=Y2;
Z3:=X3; IF Y3 > Z3 THEN Z3:=Y3;  FOR I:=0 STEP 1 UNTIL Z1 DO FOR J:=0
STEP 1 UNTIL Z2 DO FOR K:=0 STEP 1 UNTIL Z3 DO BEGIN IF I > X1 OR J > X2
 OR K > X3 THEN XNUMB:=0 ELSE XNUMB:=X[I,J,K]; IF I>Y1 OR J>Y2 OR K>Y3
THEN YNUMB:=0 ELSE YNUMB:=Y[I,J,K]; Z[I,J,K]:=XNUMB+YNUMB; END; END;
     READ(CARD,P15,ELOVCO,DN[1],L,R);
DN[0]:=1;FOR I:=2 STEP 1 UNTIL 8 DO DN[I]:=DN[1]*DN[I-1];
EL2:=ELOVCO*ELOVCO; EL4:=EL2*EL2; EL6:=EL2* EL4; EL8:=EL4*EL4;
     READ(CARD,/,FOR I:=0,1,2,3,4,5 DO FOR J:=0,1,2 DO FOR K:=0,1,2,3
DO A[I,J,K]);
     READ(CARD,/, FOR I:=0,1,2,3 DO FOR J:=0,1   DO FOR K:=0,1,2,3 DO
B[I,J,K]); WRITE(LINE,TITL); WRITE(LINE,P13,ELOVCO,DN[1],L,R);
     WRITE(LINE,P1); FOR I:=0,1,2,3,4,5 DO FOR J:=0,1,2 DO WRITE(LINE,P7
,I,J, FOR K:=0,1,2,3 DO A[I,J,K]);
     WRITE(LINE,P2); FOR I:=0,1,2,3 DO FOR J:=0,1 DO WRITE(LINE,P7,I,J,
FOR K:=0,1,2,3 DO B[I,J,K]); WRITE(LINE[SKIP 1]);
     WRITE(PUNCH,P16,DN[1],ELOVCO);
     COMMENT SOLVE FOR NUMER, 1 PRODUCT TERM. NUMER= ARRAY[B];
     FOR I:=0,1,2,3 DO FOR J:=0,1 DO FOR K:=0,1,2,3 DO BEGIN BBUF[I,J,K]
:=DN[J]*B[I,J,K]; BPOT[I,K]:=BPOT[I,K]+BBUF[I,J,K]; END;
     WRITE(LINE,P10); FOR I:=0,1,2,3 DO WRITE(LINE,P12,I, FOR K:=0,1,2,3
 DO BPOT[I,K]); FOR I:=0,1,2,3 DO WRITE(PUNCH,P14, FOR K:=0,1,2,3 DO
BPOT[I,K]);
     COMMENT SOLVE FOR DENOM, ONE PRODUCT TERM; M7:=.4053*EL2;
     FOR I:=0,1,2,3,4,5 DO FOR J:=0,1,2 DO FOR K:=0,1,2,3 DO ABUF[I,J,K]
:=M7*A[I,J,K]; POLYAD(B,ABUF,ABUF,3,1,3,5,2,3);
     FOR I:=0,1,2,3,4,5 DO FOR J:=0,1,2 DO FOR K:=0,1,2,3 DO BEGIN ABUF[
I,J,K]:=DN[J]*ABUF[I,J,K]; APOT[I,K]:=APOT[I,K]+ABUF[I,J,K]; END;
     WRITE(LINE,P11); FOR I:=0,1,2,3,4,5 DO WRITE(LINE,P12,I, FOR K:=0,1
,2,3 DO APOT[I,K]); FOR I:=0,1,2,3,4,5 DO WRITE(PUNCH,P14, FOR K:=0,1,2,
3 DO APOT[I,K]);
     COMMENT SOLVE FOR NUMER, 2 PRODUCT TERMS;
     POLYMU(B,B,BB,3,1,3,3,1,3);
     FOR I:=0,1,2,3,4,5,6 DO FOR J:=0,1,2 DO FOR K:=0,1,2,3,4,5,6 DO
NUMER[I,K]:=NUMER[I,K]+DN[J]*BB[I,J,K];
     WRITE(LINE,P3); FOR I:=0,1,2,3,4,5,6 DO WRITE(LINE,P8,I, FOR K:=0,
1,2,3,4,5,6 DO NUMER[I,K]); FOR I:=0,1,2,3,4,5,6 DO WRITE(PUNCH,P14, FOR
 K:=0,1,2,3,4,5,6 DO NUMER[I,K]);
     COMMENT SOLVE FOR DENOM, 2 PRODUCT TERMS;
     POLYMU(A,B,AB,5,2,3,3,1,3); POLYMU(A,A,AA,5,2,3,5,2,3);
M1:=.450316*EL2; M2:=.0182506*EL4;
     FOR I:=0 STEP 1 UNTIL 8 DO FOR J:=0,1,2,3 DO FOR K:=0 STEP 1 UNTIL
6 DO AB[I,J,K]:=M1*AB[I,J,K];
     FOR I:=0 STEP 1 UNTIL 10 DO FOR J:=0,1,2,3,4 DO FOR K:=0 STEP 1
UNTIL 6 DO AA[I,J,K]:=M2*AA[I,J,K];
     POLYAD(AA,AB,AA,10,4,6,8,3,6); POLYAD(AA,BB,AA,10,4,6,6,2,6);
     FOR I:=0 STEP 1 UNTIL 10 DO FOR J:=0 STEP 1 UNTIL 4 DO FOR K:=0
STEP 1 UNTIL 6 DO DENOM[I,K]:=DENOM[I,K]+DN[J]*AA[I,J,K];
     WRITE(LINE,P4); FOR I:=0 STEP 1 UNTIL 10 DO WRITE(LINE,P8,I, FOR
K:=0 STEP 1 UNTIL 6 DO DENOM[I,K]); FOR I:=0 STEP 1 UNTIL 10 DO WRITE(
PUNCH,P14, FOR K:=0 STEP 1 UNTIL 6 DO DENOM[I,K]);
     COMMENT SOLVE FOR NUMER, 4 PRODUCT TERMS;
     POLYMU(BB,BB,BBBB,6,2,6,6,2,6);
     FOR I:=0 STEP 1 UNTIL 12 DO FOR J:=0 STEP 1 UNTIL 4 DO FOR K:=0
STEP 1 UNTIL 12 DO NUM[I,K]:=NUM[I,K]+DN[J]*BBBB[I,J,K];
     WRITE(LINE,P5); FOR I:=0 STEP 1 UNTIL 12 DO WRITE(LINE,P9,I, FOR
K:=0 STEP 1 UNTIL 12 DO NUM[I,K]);  FOR I:=0 STEP 1 UNTIL 12 DO WRITE(
PUNCH,P14, FOR K:=0 STEP 1 UNTIL 12 DO NUM[I,K]);
     COMMENT SOLVE FOR DENOM, 4 PRODUCT TERMS;
     POLYMU(B,B,BB,3,1,3,3,1,3); POLYMU(A,B,AB,5,2,3,3,1,3);
POLYMU(A,A,AA,5,2,3,5,2,3); POLYMU(AB,BB,ABBB,8,3,6,6,2,6);
POLYMU(AA,BB,AABB,10,4,6,6,2,6); POLYMU(AA,AB,AAAB,10,4,6,8,3,6);
POLYMU(AA,AA,AAAA,10,4,6,10,4,6);
M3:=.474778*EL2; M4:=.0294*EL4; M5:=.506679@-3*EL6; M6:=.24411@-5*EL8;
     FOR I:=0 STEP 1 UNTIL 20 DO FOR J:=0 STEP 1 UNTIL 8 DO FOR K:=0
STEP 1 UNTIL 12 DO BEGIN IF I LEQ 14 AND J LEQ 5 THEN ABBB[I,J,K]:=M3*
ABBB[I,J,K]; IF I LEQ 16 AND J LEQ 6 THEN AABB[I,J,K]:=M4*AABB[I,J,K];
IF I LEQ 18 AND J LEQ 7 THEN AAAB[I,J,K]:=M5*AAAB[I,J,K]; AAAA[I,J,K]:=
M6*AAAA[I,J,K];  END;
     POLYAD(AAAA,AAAB,AAAA,20,8,12,18,7,12); POLYAD(AAAA,AABB,AAAA,20,8,
12,16,6,12); POLYAD(AAAA,ABBB,AAAA,20,8,12,14,5,12); POLYAD(AAAA,BBBB,
AAAA,20,8,12,12,4,12);
      FOR I:=0 STEP 1 UNTIL 20 DO FOR J:=0 STEP 1 UNTIL 8 DO FOR K:=0
STEP 1 UNTIL 12 DO DEN[I,K]:=DEN[I,K]+DN[J]*AAAA[I,J,K];
      WRITE(LINE,P6); FOR I:=0 STEP 1 UNTIL 20 DO WRITE(LINE,P9,I, FOR
K:=0 STEP 1 UNTIL 12 DO DEN[I,K]); FOR I:=0 STEP 1 UNTIL 20 DO WRITE(
PUNCH,P14, FOR K:=0 STEP 1 UNTIL 12 DO DEN[I,K]);  WRITE(PUNCH,<"END OF
DATA CARDS">);  END.
  DATA
 .0137       2.        60.       .40
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
 0,0,0, 1.,    0,0,62.38,0,  0, 327.15,0,0,
 0,0,3.,0,    0, 124.76,0,0,    327.15,0,0,0,
 0, 3.,0,0,    62.38,0,0,0,    0,0,0,0,
 1.,0,0,0,     0,0,0,0,      0,0,0,0
 0,0,0,-1.,  0,0,-40.9,0,     0,0,-1.,0,    0,0,0,0,
 0, 1.,0,0,    40.9,0,0,0,     1.,0,0,0,    0,0,0,0
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C---- STEP RESPONSES WITH TIME-DEPENDENT PARAMETERS, PNEUMATIC CASE.
C     A[6,3,3], B[4,2,3].
C
C---- THIS PROGRAM USES 4 OR 5 DATA CARDS TO PRESCRIBE PARAMETERS SUCH
C     AS STEP SIZE, LENGTH OF RUN (TIME), ETC.  THEN A SERIES OF DATA
C     CARDS WHICH HAVE BEEN GENERATED BY THE PROGRAM "PNEUMATIC" ARE
C     READ INTO ARRAYS [A],[B],[C],[D],[E],[F] TO PROVIDE THE NECESSARY
C     COEFFICIENTS FOR SUBROUTINE "DERFUN".
C
C        DATA CARD 1: THIS IS A HEADER CARD TO IDENTIFY THE RUN (A80).
C
C        DATA CARD 2:
C           1) NUMBER OF RUNS, IN COLUMN 1, FORMAT(I1). MAX=3.
C           2) IF RUN 1 USES ONE PRODUCT TERM FOR COSH(GAMMA), PUT A "1"
C     IN COLUMN 11. TWO PRODUCT TERMS, PUT A "2" IN 11.  FOUR PRODUCT
C     TERMS, PUT A "3" IN COLUMN 11.
C           3) PUT A 1, 2, OR 3 IN 21 FOR THE SECOND RUN, IF APPLICABLE.
C           4) PUT A 1, 2, OR 3 IN 31 FOR THE THIRD RUN, IF APPLICABLE.
C           5) NO. OF STEP SIZES (PSI) FOR EACH RUN, CLMS 41-42, (I2).
C           6) MAX ORDINATE FOR PLOTTER, CLMS 51-60, FORMAT F10.
C
C        DATA CARD 3:
C           1) RUNGE-KUTTA STEP SIZE FOR ONE-PRODUCT TERM RUN, CLMS 1-10
C     FORMAT F10, THEN NO. OF R-K STEPS IN CLMS 11-20, FORMAT I10.
C           2) STEP SIZE FOR TWO-PRODUCT TERMS, NO. OF R-K STEPS, 21-40.
C           3) STEP SIZE FOR FOUR-PRODUCT TERMS, NO. OF R-K STEPS, 41-60.
C           4) ATMOSPHERIC PRESSURE (PSIA), COLUMNS 61-70, FORMAT F10.
C
C        DATA CARDS 4 AND 5:
C           1) FIRST 8 STEP SIZES, IN PSIG, FORMAT 8F10.
C           2) IF MORE THAN 8 VALUES, PUT THEM ON DATA CARD 5.  IF NOT
C     MORE THAN 8 VALUES, LEAVE DATA CARD 5 OUT.
C
C        DATA CARDS 6 THROUGH 123 ARE AS FOLLOWS:
C           1) 6 THROUGH 10 GO INTO [A], NUMERATOR, ONE PRODUCT TERM.
C           2) 11 THROUGH 17 GO INTO [B], DENOMINATOR, ONE PRODUCT TERM.
C           3) 18 THROUGH 26 GO INTO [C], NUMERATOR, TWO PRODUCT TERMS.
C           4) 27 THRU 39 GO INTO [D], DENOMINATOR, TWO PRODUCT TERMS.
C           5) 40 THRU 73 GO INTO [E], NUMERATOR, FOUR PRODUCT TERMS.
C           6) 74 THRU 123 GO INTO [F], DENOMINATOR, FOUR PRODUCT TERMS.
C
C---- TO REVERT TO THE LINEAR "BROWN" MODEL, USE A VERY SMALL STEP SIZE.
C
C---- ALL PRESSURES ARE NORMALIZED BY DIVIDING BY "PATM".
C
      COMMON Y(102)
      COMMON/GOAL/PT(101,2),STEP(3),NRKS(3),NPM(3),K1,K2,K3,K4,CC,PA,PB,
     2PATM,NPV,NRUNS,AHEAD(80)
      COMMON/FORM/A(20),B(28),C(63),D(91),E(221)
      COMMON/SAGE/F(325)
      COMMON/BLOB/YMAXI
      DIMENSION AHOLD(16),IHOLD(3),IGO(3),ARY(6)
      DATA IGO/6,12,24/
      DATA ARY/'A','B','C','D','E','F'/
      READ(5,200) (AHEAD(J),J=1,80)
  200 FORMAT(80A1)
      READ(5,300) NRUNS,IHOLD(1),IHOLD(2),IHOLD(3),NPV,YMAXI
  300 FORMAT(4(I1,9X),I2,8X,F10.3)
      READ(5,400) STEP(1),NRKS(1),STEP(2),NRKS(2),STEP(3),NRKS(3),PATM
  400 FORMAT(4(F10.3,I10))
      READ(5,420) (AHOLD(J),J=1,8)
  420 FORMAT(8F10.3)
      IF(NPV.GT.8) READ(5,420) (AHOLD(J),J=9,16)
C---- READ IN ARRAYS [A] THROUGH [F].
      DO 20 J=1,5
      K=4*(J-1)
   20 READ(5,430) (A(K+L),L=1,4)
      DO 25 J=1,7
      K=4*(J-1)
   25 READ(5,430) (B(K+L),L=1,4)
  430 FORMAT(8E10.3)
      DO 30 J=1,9
      K=7*(J-1)
   30 READ(5,430) (C(K+L),L=1,7)
      DO 35 J=1,13
      K=7*(J-1)
   35 READ(5,430) (D(K+L),L=1,7)
      DO 40 J=1,17
      K=13*(J-1)
   40 READ(5,430) (E(K+L),L=1,13)
      DO 45 J=1,25
      K=13*(J-1)
   45 READ(5,430) (F(K+L),L=1,13)
      WRITE(6,450) (AHEAD(J),J=1,80)
  450 FORMAT (1H1,10X,80A1)
      WRITE(6,600) NRUNS,NPV, PATM, YMAXI
  600 FORMAT(16X,'THERE WILL BE ',I1,' RUNS OF ',I2,' PRESSURE VALUES EA
     2CH.  ATMOSPHERIC PRESSURE',/,16X,'IS ',F6.3,' PSIG.  FOR PLOTTING
     3PURPOSES, YMAX = ',F5.2,' .',/,11X,80('='),/)
C---- WRITE OUT ARRAYS [A] THROUGH [F].
      WRITE(6,700) ARY(1)
  700 FORMAT(20X,'ARRAY [',A1,']:',/,20X,'----------',/)
      DO 60 J=1,5
      K=4*(J-1)
   60 WRITE(6,720) (A(K+L),L=1,4)
  720 FORMAT(4E20.5)
      WRITE(6,700) ARY(2)
      DO 65 J=1,7
      K=4*(J-1)
   65 WRITE(6,720) (B(K+L),L=1,4)
      WRITE(6,700) ARY(3)
      DO 70 J=1,9
      K=7*(J-1)
   70 WRITE(6,740) (C(K+L),L=1,7)
      WRITE(6,700) ARY(4)
  740 FORMAT(7G15.4)
      DO 75 J=1,13
      K=7*(J-1)
   75 WRITE(6,740) (D(K+L),L=1,7)
      WRITE(6,700) ARY(5)
      DO 80 J=1,17
      K=13*(J-1)
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   80 WRITE(6,760) (E(K+L),L=1,13)
      WRITE(6,700) ARY(6)
      DO 85 J=1,25
      K=13*(J-1)
   85 WRITE(6,760) (F(K+L),L=1,13)
  760 FORMAT(13G10.3)
      DO 100 JJ=1,NRUNS
      DO 100 KK=1,NPV
      PA=AHOLD(KK)/PATM
C---- K3 CONTROLS THE RECOMPUTING OF THE [G] ARRAY IN DERFUN.
      K3=4
C---- K1=1 DENOTES ONE PRODUCT TERM, 2 DENOTES TWO PRODUCT TERMS,
C     AND 3 DENOTES FOUR PRODUCT TERMS.
C---- K2 IS THE ORDER OF THE DIFFERENTIAL EQUATION.
      K1=IHOLD(JJ)
      K2=IGO(K1)
      CALL GOTEAM
  100 CONTINUE
      STOP
      END
      SUBROUTINE DERFUN
C---- (PNEUMATIC CASE) A[6,3,3], B[4,2,3]
C     OUTPUT "PB" IS IN Y(102), "PBDOT" IS IN Y(100).
      COMMON Y(102)
      COMMON/GOAL/PT(101,2),STEP(3),NRKS(3),NPM(3),K1,K2,K3,K4,CC,PA,PB,
     2PATM,NPV,NRUNS,AHEAD(80)
      COMMON/FORM/A(20),B(28),C(63),D(91),E(221)
      COMMON/SAGE/F(325)
      DIMENSION G(42)
      GO TO(10,20,30), K1
C---- ONE PRODUCT TERM FOR COSH(GAMMA).
   10 K3=K3+1
      IF(K3.LT.4) GO TO 12
      K3=0
      B1=ABS(Y(100))
      B2=B1*B1
      B3=B1*B2
      G(1)=A(3)*B2+A(4)*B3
      G(2)=A(7)*B2+A(8)*B3
      G(3)=A(9)+A(10)*B1+A(11)*B2
      G(4)=A(13)+A(14)*B1
      G(5)=A(17)
      G(6)=B(3)*B2+B(4)*B3
      G(7)=B(7)*B2+B(8)*B3
      G(8)=B(9)+B(10)*B1+B(11)*B2+B(12)*B3
      G(9)=B(13)+B(14)*B1+B(15)*B2+B(16)*B3
      G(10)=B(17)+B(18)*B1+B(19)*B2
      G(11)=B(21)+B(22)*B1
      G(12)=B(25)
   12 DO 14 K=9,13
   14 Y(K)=Y(K-7)
      Y(14)=(PA-G(6)*Y(1)-G(7)*Y(2)-G(8)*Y(3)-G(9)*Y(4)-G(10)*Y(5)-G(11)
     2*Y(6))/G(12)
      Y(102)=G(1)*Y(1)+G(2)*Y(2)+G(3)*Y(3)+G(4)*Y(4)+G(5)*Y(5)
      Y(100)=G(1)*Y(2)+G(2)*Y(3)+G(3)*Y(4)+G(4)*Y(5)+G(5)*Y(6)
      RETURN
C---- TWO PRODUCT TERMS FOR COSH(GAMMA).
   20 K3=K3+1
      IF(K3.LT.4) GO TO 22
      K3=0
      B1=ABS(Y(100))
      B2=B1*B1
      B3=B1*B2
      B4=B1*B3
      B5=B1*B4
      B6=B1*B5
      G(1)=C(5)*B4+C(6)*B5+C(7)*B6
      G(2)=C(12)*B4+C(13)*B5+C(14)*B6
      G(3)=C(17)*B2+C(18)*B3+C(19)*B4+C(20)*B5+C(21)*B6
      G(4)=C(24)*B2+C(25)*B3+C(26)*B4+C(27)*B5
      G(5)=C(29)+C(30)*B1+C(31)*B2+C(32)*B3+C(33)*B4
      G(6)=C(36)+C(37)*B1+C(38)*B2+C(39)*B3
      G(7)=C(43)+C(44)*B1+C(45)*B2
      G(8)=C(50)+C(51)*B1
      G(9)=C(57)
      G(10)=D(5)*B4+D(6)*B5+D(7)*B6
      G(11)=D(12)*B4+D(13)*B5+D(14)*B6
      G(12)=D(17)*B2+D(18)*B3+D(19)*B4+D(20)*B5+D(21)*B6
      G(13)=D(24)*B2+D(25)*B3+D(26)*B4+D(27)*B5+D(28)*B6
      G(14)=D(29)+D(30)*B1+D(31)*B2+D(32)*B3+D(33)*B4+D(34)*B5+D(35)*B6
      G(15)=D(36)+D(37)*B1+D(38)*B2+D(39)*B3+D(40)*B4+D(41)*B5+D(42)*B6
      G(16)=D(43)+D(44)*B1+D(45)*B2+D(46)*B3+D(47)*B4+D(48)*B5+D(49)*B6
      G(17)=D(50)+D(51)*B1+D(52)*B2+D(53)*B3+D(54)*B4+D(55)*B5
      G(18)=D(57)+D(58)*B1+D(59)*B2+D(60)*B3+D(61)*B4
      G(19)=D(64)+D(65)*B1+D(66)*B2+D(67)*B3
      G(20)=D(71)+D(72)*B1+D(73)*B2
      G(21)=D(78)+D(79)*B1
      G(22)=D(85)
   22 DO 24 K=15,25
   24 Y(K)=Y(K-13)
      Y(26)=(PA-G(10)*Y(1)-G(11)*Y(2)-G(12)*Y(3)-G(13)*Y(4)-G(14)*Y(5)-
     2G(15)*Y(6)-G(16)*Y(7)-G(17)*Y(8)-G(18)*Y(9)-G(19)*Y(10)-G(20)*
     3Y(11)-G(21)*Y(12))/G(22)
      Y(102)=G(1)*Y(1)+G(2)*Y(2)+G(3)*Y(3)+G(4)*Y(4)+G(5)*Y(5)+G(6)*Y(6)
     2+G(7)*Y(7)+G(8)*Y(8)+G(9)*Y(9)
      Y(100)=G(1)*Y(2)+G(2)*Y(3)+G(3)*Y(4)+G(4)*Y(5)+G(5)*Y(6)+G(6)*Y(7)
     2+G(7)*Y(8)+G(8)*Y(9)+G(9)*Y(10)
      RETURN
C---- FOUR PRODUCT TERMS FOR COSH(GAMMA).
   30 K3=K3+1
      IF(K3.LT.4) GO TO 32
      K3=0
      B1=ABS(Y(100))
      B2=B1*B1
      B3=B1*B2
      B4=B1*B3
      B5=B1*B4
      B6=B1*B5
      B7=B1*B6
      B8=B1*B7
      B9=B1*B8
      B10=B1*B9
      B11=B1*B10
      B12=B1*B11
      G(1)=E(9)*B8+E(10)*B9+E(11)*B10+E(12)*B11+E(13)*B12
      G(2)=E(22)*B8+E(23)*B9+E(24)*B10+E(25)*B11+E(26)*B12
      G(3)=E(33)*B6+E(34)*B7+E(35)*B8+E(36)*B9+E(37)*B10+E(38)*B11+E(39)
     2*B12
      G(4)=E(46)*B6+E(47)*B7+E(48)*B8+E(49)*B9+E(50)*B10+E(51)*B11+E(52)
     2*B12
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      G(5)=E(57)*B4+E(58)*B5+E(59)*B6+E(60)*B7+E(61)*B8+E(62)*B9+E(63)*
     2B10+E(64)*B11+E(65)*B12
      G(6)=E(70)*B4+E(71)*B5+E(72)*B6+E(73)*B7+E(74)*B8+E(75)*B9+E(76)*
     2B10+E(77)*B11
      G(7)=E(81)*B2+E(82)*B3+E(83)*B4+E(84)*B5+E(85)*B6+E(86)*B7+E(87)*
     2B8+E(88)*B9+E(89)*B10
      G(8)=E(94)*B2+E(95)*B3+E(96)*B4+E(97)*B5+E(98)*B6+E(99)*B7+E(100)*
     2B8+E(101)*B9
      G(9)=E(105)+E(106)*B1+E(107)*B2+E(108)*B3+E(109)*B4+E(110)*B5+
     2E(111)*B6+E(112)*B7+E(113)*B8
      G(10)=E(118)+E(119)*B1+E(120)*B2+E(121)*B3+E(122)*B4+E(123)*B5+
     2E(124)*B6+E(125)*B7
      G(11)=E(131)+E(132)*B1+E(133)*B2+E(134)*B3+E(135)*B4+E(136)*B5+
     2E(137)*B6
      G(12)=E(144)+E(145)*B1+E(146)*B2+E(147)*B3+E(148)*B4+E(149)*B5
      G(13)=E(157)+E(158)*B1+E(159)*B2+E(160)*B3+E(161)*B4
      G(14)=E(170)+E(171)*B1+E(172)*B2+E(173)*B3
      G(15)=E(183)+E(184)*B1+E(185)*B2
      G(16)=E(196)+E(197)*B1
      G(17)=E(209)
      G(18)=F(9)*B8+F(10)*B9+F(11)*B10+F(12)*B11+F(13)*B12
      G(19)=F(22)*B8+F(23)*B9+F(24)*B10+F(25)*B11+F(26)*B12
      G(20)=F(33)*B6+F(34)*B7+F(35)*B8+F(36)*B9+F(37)*B10+F(38)*B11+
     2F(39)*B12
      G(21)=F(46)*B6+F(47)*B7+F(48)*B8+F(49)*B9+F(50)*B10+F(51)*B11+
     2F(52)*B12
      G(22)=F(57)*B4+F(58)*B5+F(59)*B6+F(60)*B7+F(61)*B8+F(62)*B9+
     2F(63)*B10+F(64)*B11+F(65)*B12
      G(23)=F(70)*B4+F(71)*B5+F(72)*B6+F(73)*B7+F(74)*B8+F(75)*B9+
     2F(76)*B10+F(77)*B11+F(78)*B12
      G(24)=F(81)*B2+F(82)*B3+F(83)*B4+F(84)*B5+F(85)*B6+F(86)*B7+
     2F(87)*B8+F(88)*B9+F(89)*B10+F(90)*B11+F(91)*B12
      G(25)=F(94)*B2+F(95)*B3+F(96)*B4+F(97)*B5+F(98)*B6+F(99)*B7+
     2F(100)*B8+F(101)*B9+F(102)*B10+F(103)*B11+F(104)*B12
      G(26)=F(105)+F(106)*B1+F(107)*B2+F(108)*B3+F(109)*B4+F(110)*B5+
     2F(111)*B6+F(112)*B7+F(113)*B8+F(114)*B9+F(115)*B10+F(116)*B11+
     3F(117)*B12
      G(27)=F(118)+F(119)*B1+F(120)*B2+F(121)*B3+F(122)*B4+F(123)*B5+
     2F(124)*B6+F(125)*B7+F(126)*B8+F(127)*B9+F(128)*B10+F(129)*B11+
     3F(130)*B12
      G(28)=F(131)+F(132)*B1+F(133)*B2+F(134)*B3+F(135)*B4+F(136)*B5+
     2F(137)*B6+F(138)*B7+F(139)*B8+F(140)*B9+F(141)*B10+F(142)*B11+
     3F(143)*B12
      G(29)=F(144)+F(145)*B1+F(146)*B2+F(147)*B3+F(148)*B4+F(149)*B5+
     2F(150)*B6+F(151)*B7+F(152)*B8+F(153)*B9+F(154)*B10+F(155)*B11+
     3F(156)*B12
      G(30)=F(157)+F(158)*B1+F(159)*B2+F(160)*B3+F(161)*B4+F(162)*B5+
     2F(163)*B6+F(164)*B7+F(165)*B8+F(166)*B9+F(167)*B10+F(168)*B11+
     3F(169)*B12
      G(31)=F(170)+F(171)*B1+F(172)*B2+F(173)*B3+F(174)*B4+F(175)*B5+
     2F(176)*B6+F(177)*B7+F(178)*B8+F(179)*B9+F(180)*B10+F(181)*B11
      G(32)=F(183)+F(184)*B1+F(185)*B2+F(186)*B3+F(187)*B4+F(188)*B5+
     2F(189)*B6+F(190)*B7+F(191)*B8+F(192)*B9+F(193)*B10
      G(33)=F(196)+F(197)*B1+F(198)*B2+F(199)*B3+F(200)*B4+F(201)*B5+
     2F(202)*B6+F(203)*B7+F(204)*B8+F(205)*B9
      G(34)=F(209)+F(210)*B1+F(211)*B2+F(212)*B3+F(213)*B4+F(214)*B5+
     2F(215)*B6+F(216)*B7+F(217)*B8
      G(35)=F(222)+F(223)*B1+F(224)*B2+F(225)*B3+F(226)*B4+F(227)*B5+
     2F(228)*B6+F(229)*B7
      G(36)=F(235)+F(236)*B1+F(237)*B2+F(238)*B3+F(239)*B4+F(240)*B5+
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     2F(241)*B6
      G(37)=F(248)+F(249)*B1+F(250)*B2+F(251)*B3+F(252)*B4+F(253)*B5
      G(38)=F(261)+F(262)*B1+F(263)*B2+F(264)*B3+F(265)*B4
      G(39)=F(274)+F(275)*B1+F(276)*B2+F(277)*B3
      G(40)=F(287)+F(288)*B1+F(289)*B2
      G(41)=F(300)+F(301)*B1
      G(42)=F(313)
   32 DO 34 K=27,49
   34 Y(K)=Y(K-25)
      Y(50)=(PA-G(18)*Y(1)-G(19)*Y(2)-G(20)*Y(3)-G(21)*Y(4)-G(22)*Y(5)-
     2G(23)*Y(6)-G(24)*Y(7)-G(25)*Y(8)-G(26)*Y(9)-G(27)*Y(10)-G(28)*Y(11
     3)-G(29)*Y(12)-G(30)*Y(13)-G(31)*Y(14)-G(32)*Y(15)-G(33)*Y(16)-
     4G(34)*Y(17)-G(35)*Y(18)-G(36)*Y(19)-G(37)*Y(20)-G(38)*Y(21)-G(39)
     5*Y(22)-G(40)*Y(23)-G(41)*Y(24))/G(42)
      Y(102)=G(1)*Y(1)+G(2)*Y(2)+G(3)*Y(3)+G(4)*Y(4)+G(5)*Y(5)+G(6)*Y(6)
     2+G(7)*Y(7)+G(8)*Y(8)+G(9)*Y(9)+G(10)*Y(10)+G(11)*Y(11)+G(12)*Y(12)
     3+G(13)*Y(13)+G(14)*Y(14)+G(15)*Y(15)+G(16)*Y(16)+G(17)*Y(17)
      Y(100)=G(1)*Y(2)+G(2)*Y(3)+G(3)*Y(4)+G(4)*Y(5)+G(5)*Y(6)+G(6)*
     2Y(7)+G(7)*Y(8)+G(8)*Y(9)+G(9)*Y(10)+G(10)*Y(11)+G(11)*Y(12)+G(12)
     3*Y(13)+G(13)*Y(14)+G(14)*Y(15)+G(15)*Y(16)+G(16)*Y(17)+G(17)*Y(18)
      RETURN
      END
      SUBROUTINE GOTEAM
C---- OUTPUT "PB" IS STORED IN Y(102).
      COMMON Y(102)
      COMMON/GOAL/PT(101,2),STEP(3),NRKS(3),NPM(3),K1,K2,K3,K4,CC,PA,PB,
     2PATM,NPV,NRUNS,AHEAD(80)
      COMMON/BLOB/YMAXI
      DIMENSION T(50),U(100)
   10 WRITE(6,100) (AHEAD(J),J=1,80)
  100 FORMAT(1H1,6X,80A1)
C---- ONE PRODUCT TERM FOR COSH(GAMMA).
      IF(K1.EQ.1) WRITE(6,200)
  200 FORMAT(15X,'THIS RUN USES THE ONE PRODUCT-TERM EXPANSION FOR COSH(
     2GAMMA).')
      IF(K1.EQ.2) WRITE(6,500)
  500 FORMAT(15X,'THIS RUN USES THE TWO-PRODUCT TERM EXPANSION FOR COSH(
     2GAMMA).')
      IF(K1.EQ.3) WRITE(6,600)
  600 FORMAT(15X,'THIS RUN USES THE FOUR-PRODUCT TERM EXPANSION FOR COSH
     2(GAMMA).')
      PAP=PA*PATM
      WRITE(6,300) STEP(K1),PAP
  300 FORMAT(15X,'TIME STEP IS ',F8.5,' ,  PRESSURE STEP INPUT = ',F12.5
     2,'  . ',/,15X,62('*'),/)
      WRITE(6,400)
  400 FORMAT(16X,' TIME      OUTPUT "PB/PA"        TIME     OUTPUT "PB/PA
     2"',/,16X,'(SEC)      (CONVERTED)        (SEC)      (CONVERTED)',
     3/,15X,' -----     --------------      -----     --------------')
   40 DO 42  J=1,102
   42 Y(J)=0.
C---- LOAD DELTA-TIME, NO. OF R-K STEPS, AND PRINT MULTIPLE.
      Y(K2+2)=STEP(K1)
      NRK=NRKS(K1)
      NP=NRK/100
C---- LOAD "TIME" INTO PT(K,1) AND ZEROES INTO PT(K,2).
      DP=Y(K2+2)
      DO 52 J=1,100
      PT(J,2)=0.
   52 PT(J,1)=DP*(J-1)*NP
```

```
C---- CALL THE INTEGRATOR.
      NPR=0
      IK=1
      DO 60 KK=1,NRK
      NPR=NPR+1
      CALL RKINT(KK,K2)
      CONVRT=Y(102)/PA
      IF(ABS(CONVRT).GT.YMAXI) GO TO 65
      IF(NPR.LT.NP) GO TO 60
      NPR=0
      IK=IK+1
      PT(IK,2)=CONVRT
   60 CONTINUE
   65 WRITE(6,700) ((PT(K,J),J=1,2),(PT(K+50,J),J=1,2),K=1,50)
  700 FORMAT(50(15X,F6.4,6X,G12.5,7X,F6.4,6X,G12.5,/))
C---- DO NOT CALL PLOTTER IF DATA IS BAD.
      IF(PT(10,2).EQ.0.) RETURN
C---- LOAD DATA FOR THE PLOTTER.
      DO 80 K=1,50
      J=2*K-1
      T(K)=PT(J,1)
      U(K)=1.
      U(K+50)=PT(J,2)
   80 IF(U(K+50).LT.0.) U(K+50)=0.
      CALL XYPLOT(T,U,50,100,8.17,8.)
      RETURN
      END
      SUBROUTINE  RKINT (LL,NSYS)
C     THIS SUBROUTINE SOLVES DIFFERENTIAL EQUATIONS BY USING A RUNGE KUTTA
C     METHOD
      DIMENSION DELY(4,50),BET(3),YU(50)
      COMMON Y(102)
      DOUBLE PRECISION YU
      IF (LL.NE.1) GO TO 1001
      BET(1)=0.5
      BET(2)=0.5
      BET(3)=1.0
      N2=NSYS+2
      NP1=NSYS+1
      XV=Y(NP1)
      CALL DERFUN
      DO 320 I=1,NSYS
  320 YU(I)=Y(I)
 1001 DO 1034 K=1,4
      IF (K.EQ.1) GO TO 1002
      CALL DERFUN
 1002 DO 1340 I=1,NSYS
      IPN2=I+N2
 1340 DELY(K,I)=Y(N2)*Y(IPN2)
      IF (K.EQ.4) GO TO 1034
      DO 1350 I=1,NSYS
 1350 Y(I)=YU(I)+BET(K)*DELY(K,I)
      Y(NP1)=XV+BET(K)*Y(N2)
 1034 CONTINUE
      DO 1039 I=1,NSYS
      DEL=(DELY(1,I)+2.0*DELY(2,I)+2.0*DELY(3,I)+DELY(4,I))/6.0
      YU(I)=YU(I)+DEL
      Y(I)=YU(I)
 1039 CONTINUE
      Y(NP1)=XV+Y(N2)
      CALL DERFUN
      XV=Y(NP1)
      RETURN
      END
      SUBROUTINE XYPLOT (XX,YY,NX,NY,XLINCH,YLINCH)
      COMMON/BLOB/YMAXI
      DIMENSION XX(1),YY(1),IY(10)
      DIMENSION IPLOT(100),IMINUS(100),ISYMBL(10)                       XY 0030
      DATA IBLANK,IAXIS/1H ,1HI/,IPLOT,IMINUS/100*1H ,100*1H_/          XY 0040
      DATA ISYMBL/1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,1H0/              XY 0050
      NXSIZE = XLINCH*6.0                                               XY 0120
      XSIZE = NXSIZE                                                    XY 0130
      YSIZE = YLINCH*10.0                                               XY 0140
      NYSIZE = YSIZE + 1                                                XY 0170
      YSIZE = NYSIZE - 1                                                XY 0180
      NPLOTS = NY/NX                                                    XY 0190
      XMIN=0.                                                           XY 0200
      XMAX =XX(NX)
      DX = XMAX - XMIN                                                  XY 0220
C---- USE A FIXED ABSCISSA, AS SHOWN BELOW.
C     YMIN=YY(1)
C     DO 8 I=1,NY
C   8 IF(YY(I).LT.YMIN) YMIN=YY(I)
C     YMAX =YY(1)
C     DO 10 I=1,NY                                                      XY 0250
C  10 IF(YY(I).GT.YMAX) YMAX=YY(I)
C     DY=YMAX-YMIN
C---- FIXED ABSCISSA
      YMIN=0.
      YMAX=YMAXI
      DY=YMAXI
      WRITE(6,6) YMIN,YMAX
      WRITE(6,1)(IMINUS(J),J=1,NYSIZE)                                  XY 0290
      IPLOT(1) = IAXIS                                                  XY 0300
      IPLOT(NYSIZE) = IAXIS                                             XY 0310
      NLINE = 0                                                         XY 0320
      DO 30 I=1,NX                                                      XY 0330
      IX = (XX(I)- XMIN)/DX*XSIZE
   32 IF(IX - NLINE) 30,33,34                                           XY 0350
   34 WRITE(6,4)(IPLOT(J),J=1,NYSIZE)                                   XY 0360
      NLINE = NLINE + 1                                                 XY 0370
      GO TO 32                                                          XY 0380
   33 NLINE = NLINE + 1                                                 XY 0390
      KI = I                                                            XY 0400
      DO 41 K=1,NPLOTS                                                  XY 0410
      IY(K) = (YY(KI)- YMIN)/DY*YSIZE + 1.5
      IYK = IY(K)                                                       XY 0430
      IPLOT(IYK) = ISYMBL(K)                                            XY 0440
   41 KI = KI + NX                                                      XY 0450
      WRITE(6,2)XX(I),(IPLOT(J),J=1,NYSIZE)
      DO 42 K=1,NPLOTS                                                  XY 0470
      IYK = IY(K)                                                       XY 0480
   42 IPLOT(IYK) = IBLANK                                               XY 0490
      IPLOT(1) = IAXIS                                                  XY 0500
      IPLOT(NYSIZE) = IAXIS                                             XY 0510
   30 CONTINUE                                                          XY 0520
      WRITE(6,3)(IMINUS(J),J=1,NYSIZE)                                  XY 0530
      RETURN                                                            XY 0540
    1 FORMAT( /,6X,' ABSCISSA ',5X,100A1)                               XY 0060
    2 FORMAT(6X,E10.3,5X,100A1)                                         XY 0070
    3 FORMAT(1H+,20X,100A1)                                             XY 0080
    4 FORMAT(21X,100A1)                                                 XY 0090
    6 FORMAT(1H1,6X,'MIN ORDINATE "PMIN" = ',G12.5,'  ,  MAX ORDINATE "PM
     2AX" = ',G12.5)
      END                                                               XY 0550
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C---- STEP RESPONSES WITH TIME-DEPENDENT PARAMETERS, HYDRAULIC CASE.
C      A[5,2,3], B[3,1,3].
C
C---- THIS PROGRAM USES 4 OR 5 DATA CARDS TO PRESCRIBE PARAMETERS SUCH
C     AS STEP SIZE, LENGTH OF RUN (TIME), ETC.  THEN A SERIES OF DATA
C     CARDS WHICH HAVE BEEN GENERATED BY THE PROGRAM "PNEUMATIC" ARE
C     READ INTO ARRAYS [A],[B],[C],[D],[E],[F] TO PROVIDE THE NECESSARY
C     COEFFICIENTS FOR SUBROUTINE "DERFUN".
C
C        DATA CARD 1: THIS IS A HEADER CARD TO IDENTIFY THE RUN (A80).
C
C        DATA CARD 2:
C           1) NUMBER OF RUNS, IN COLUMN 1, FORMAT(I1). MAX=3.
C           2) IF RUN 1 USES ONE PRODUCT TERM FOR COSH(GAMMA), PUT A "1"
C     IN COLUMN 11. TWO PRODUCT TERMS, PUT A "2" IN 11.  FOUR PRODUCT
C     TERMS, PUT A "3" IN COLUMN 11.
C           3) PUT A 1, 2, OR 3 IN 21 FOR THE SECOND RUN, IF APPLICABLE.
C           4) PUT A 1, 2, OR 3 IN 31 FOR THE THIRD RUN, IF APPLICABLE.
C           5) NO. OF STEP SIZES (PSI) FOR EACH RUN, CLMS 41-42, (I2).
C           6) MAX ORDINATE FOR PLUTTER, CLMS 51-60, FORMAT F10.
C
C        DATA CARD 3:
C           1) RUNGE-KUTTA STEP SIZE FOR ONE-PRODUCT TERM RUN, CLMS 1-10
C     FORMAT F10, THEN NO. OF R-K STEPS IN CLMS 11-20, FORMAT I10.
C           2) STEP SIZE FOR TWO-PRODUCT TERMS, NO. OF R-K STEPS, 21-40.
C           3) STEP SIZE FOR FOUR-PRODUCT TERMS, NO. OF R-K STEPS, 41-60.
C           4) ATMOSPHERIC PRESSURE (PSIA), COLUMNS 61-70, FORMAT F10.
C
C        DATA CARDS 4 AND 5:
C           1) FIRST 8 STEP SIZES, IN PSIG, FORMAT 8F10.
C           2) IF MORE THAN 8 VALUES, PUT THEM ON DATA CARD 5.  IF NOT
C     MORE THAN 8 VALUES, LEAVE DATA CARD 5 OUT.
C
C        DATA CARDS 6 THROUGH 101 ARE AS FOLLOWS:
C           1) 6 THROUGH  9 GO INTO [A], NUMERATOR, ONE PRODUCT TERM.
C           2) 10 THROUGH 15 GO INTO [B], DENOMINATOR, ONE PRODUCT TERM.
C           3) 16 THROUGH 22 GO INTO [C], NUMERATOR, TWO PRODUCT TERMS.
C           4) 23 THROUGH 33 INTO [D], DENOMINATOR, TWO PRODUCT TERMS.
C           5) 34 THROUGH 59 INTO [E], NUMERATOR, FOUR PRODUCT TERMS.
C           6) 60 THROUGH 101 INTO [F], DENOMINATOR, FOUR PRODUCT TERMS.
C
C---- TO REVERT TO THE LINEAR "BROWN" MODEL, USE A VERY SMALL STEP SIZE.
C
C---- ALL PRESSURES ARE NORMALIZED BY DIVIDING BY "PATM".
C
      COMMON Y(102)
      COMMON/GOAL/PT(101,2),STEP(3),NRKS(3),NPM(3),K1,K2,K3,K4,CC,PA,PB,
     2PATM,NPV,NRUNS,AHEAD(80)
      COMMON/FORM/A(20),B(28),C(63),D(91),E(221)
      COMMON/SAGE/F(325)
      COMMON/BLOB/YMAXI
      DIMENSION AHOLD(16),IHOLD(3),IGO(3),ARY(6)
      DATA IGO/5,10,20/
      DATA ARY/'A','B','C','D','E','F'/
      READ(5,200) (AHEAD(J),J=1,80)
  200 FORMAT(80A1)
      READ(5,300) NRUNS,IHOLD(1),IHOLD(2),IHOLD(3),NPV,YMAXI
  300 FORMAT(4(I1,9X),I2,8X,F10.3)
      READ(5,400) STEP(1),NRKS(1),STEP(2),NRKS(2),STEP(3),NRKS(3),PATM
  400 FORMAT(4(F10.3,I10))
      READ(5,420) (AHOLD(J),J=1,8)
  420 FORMAT(8F10.3)
      IF(NPV.GT.8) READ(5,420) (AHOLD(J),J=9,16)
C---- READ IN ARRAYS [A] THROUGH [F].
      DO 20 J=1,4
      K=4*(J-1)
   20 READ(5,430) (A(K+L),L=1,4)
      DO 25 J=1,6
      K=4*(J-1)
   25 READ(5,430) (B(K+L),L=1,4)
  430 FORMAT(5E10.3)
      DO 30 J=1,7
      K=7*(J-1)
   30 READ(5,430) (C(K+L),L=1,7)
      DO 35 J=1,11
      K=7*(J-1)
   35 READ(5,430) (D(K+L),L=1,7)
      DO 40 J=1,13
      K=13*(J-1)
   40 READ(5,430) (E(K+L),L=1,13)
      DO 45 J=1,21
      K=13*(J-1)
   45 READ(5,430) (F(K+L),L=1,13)
      WRITE(6,450) (AHEAD(J),J=1,80)
  450 FORMAT (1H1,10X,80A1)
      WRITE(6,600) NRUNS,NPV, PATM, YMAXI
  600 FORMAT(16X,'THERE WILL BE ',I1,' RUNS OF ',I2,' PRESSURE VALUES EA
     2CH.  ATMOSPHERIC PRESSURE',/,16X,'IS ',F6.3,' PSIG.  FOR PLOTTING
     3PURPOSES, YMAX = ',F5.2,' .',/,11X,80('='),/)
C---- WRITE OUT ARRAYS [A] THROUGH [F].
      WRITE(6,700) ARY(1)
  700 FORMAT(20X,'ARRAY [',A1,']:',/,20X,'----------',/)
      DO 60 J=1,4
      K=4*(J-1)
   60 WRITE(6,720) (A(K+L),L=1,4)
  720 FORMAT(4E20.5)
      WRITE(6,700) ARY(2)
      DO 65 J=1,6
      K=4*(J-1)
   65 WRITE(6,720) (B(K+L),L=1,4)
      WRITE(6,700) ARY(3)
      DO 70 J=1,7
      K=7*(J-1)
   70 WRITE(6,740) (C(K+L),L=1,7)
      WRITE(6,700) ARY(4)
  740 FORMAT(7G15.4)
      DO 75 J=1,11
      K=7*(J-1)
   75 WRITE(6,740) (D(K+L),L=1,7)
      WRITE(6,700) ARY(5)
      DO 80 J=1,13
      K=13*(J-1)
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   80 WRITE(6,760) (E(K+L),L=1,13)
      WRITE(6,700) ARY(6)
      DO 85 J=1,21
      K=13*(J-1)
   85 WRITE(6,760) (F(K+L),L=1,13)
  760 FORMAT(13G10.3)
      DO 100 JJ=1,NRUNS
      DO 100 KK=1,NPV
      PA=AHOLD(KK)/PATM
C---- K3 CONTROLS THE RECOMPUTING OF THE [G] ARRAY IN DERFUN.
      K3=4
C---- K1=1 DENOTES ONE PRODUCT TERM, 2 DENOTES TWO PRODUCT TERMS,
C     AND 3 DENOTES FOUR PRODUCT TERMS.
C---- K2 IS THE ORDER OF THE DIFFERENTIAL EQUATION.
      K1=IHOLD(JJ)
      K2=IGO(K1)
      CALL GOTEAM
  100 CONTINUE
      STOP
      END
      SUBROUTINE DERFUN
C---- [HYDRAULIC CASE] A[5,2,3], B[3,1,3]
C     OUTPUT "PB" IS IN Y(102), "PBDOT" IS IN Y(100).
      COMMON Y(102)
      COMMON/GOAL/PT(101,2),STEP(3),NRKS(3),NPM(3),K1,K2,K3,K4,CC,PA,PB,
     2PATM,NPV,NRUNS,AHEAD(80)
      COMMON/FORM/A(20),B(28),C(63),D(91),E(221)
      COMMON/SAGE/F(325)
      DIMENSION G(42)
      GO TO(10,20,30), K1
C---- ONE PRODUCT TERM FOR COSH(GAMMA).
   10 K3=K3+1
      IF(K3.LT.4) GO TO 12
      K3=0
      B1=ABS(Y(100))
      B2=B1*B1
      B3=B1*B2
      G(1)=A(3)*B2+A(4)*B3
      G(2)=A(7)*B2
      G(3)=A(9)+A(10)*B1
      G(4)=A(13)
      G(5)=B(3)*B2+B(4)*B3
      G(7)=B(7)*B2
      G(8)=B(9)+B(10)*B1+B(11)*B2+B(12)*B3
      G(9)=B(13)+B(14)*B1+B(15)*B2
      G(10)=B(17)+B(18)*B1
      G(11)=B(21)
   12 DO 14 K=8,11
   14 Y(K)=Y(K-6)
      Y(12)=(PA-G(6)*Y(1)-G(7)*Y(2)-G(8)*Y(3)-G(9)*Y(4)-G(10)*Y(5))/G(11
     2)
      Y(102)=G(1)*Y(1)+G(2)*Y(2)+G(3)*Y(3)+G(4)*Y(4)
      Y(100)=G(1)*Y(2)+G(2)*Y(3)+G(3)*Y(4)+G(4)*Y(5)
      RETURN
C---- TWO PRODUCT TERMS FOR COSH(GAMMA).
   20 K3=K3+1
      IF(K3.LT.4) GO TO 22
      K3=0
      B1=ABS(Y(100))
      B2=B1*B1
      B3=B1*B2
      B4=B1*B3
      B5=B1*B4
      B6=B1*B5
      G(1)=C(5)*B4+C(6)*B5+C(7)*B6
      G(2)=C(12)*B4+C(13)*B5
      G(3)=C(17)*B2+C(18)*B3+C(19)*B4
      G(4)=C(24)*B2+C(25)*B3
      G(5)=C(29)+C(30)*B1+C(31)*B2
      G(6)=C(36)+C(37)*B1
      G(7)=C(43)
      G(10)=D(5)*B4+D(6)*B5+D(7)*B6
      G(11)=D(12)*B4+D(13)*B5
      G(12)=D(17)*B2+D(18)*B3+D(19)*B4+D(20)*B5+D(21)*B6
      G(13)=D(24)*B2+D(25)*B3+D(26)*B4+D(27)*B5
      G(14)=D(29)+D(30)*B1+D(31)*B2+D(32)*B3+D(33)*B4+D(34)*B5+D(35)*B6
      G(15)=D(36)+D(37)*B1+D(38)*B2+D(39)*B3+D(40)*B4+D(41)*B5
      G(16)=D(43)+D(44)*B1+D(45)*B2+D(46)*B3+D(47)*B4
      G(17)=D(50)+D(51)*B1+D(52)*B2+D(53)*B3
      G(18)=D(57)+D(58)*B1+D(59)*B2
      G(19)=D(64)+D(65)*B1
      G(20)=D(71)
   22 DO 24 K=13,21
   24 Y(K)=Y(K-11)
      Y(22)=(PA-G(10)*Y(1)-G(11)*Y(2)-G(12)*Y(3)-G(13)*Y(4)-G(14)*Y(5)-
     2G(15)*Y(6)-G(16)*Y(7)-G(17)*Y(8)-G(18)*Y(9)-G(19)*Y(10))/G(20)
      Y(102)=G(1)*Y(1)+G(2)*Y(2)+G(3)*Y(3)+G(4)*Y(4)+G(5)*Y(5)+G(6)*Y(6)
     2+G(7)*Y(7)
      Y(100)=G(1)*Y(2)+G(2)*Y(3)+G(3)*Y(4)+G(4)*Y(5)+G(5)*Y(6)+G(6)*Y(7)
     2+G(7)*Y(8)
      RETURN
C---- FOUR PRODUCT TERMS FOR COSH(GAMMA).
   30 K3=K3+1
      IF(K3.LT.4) GO TO 32
      K3=0
      B1=ABS(Y(100))
      B2=B1*B1
      B3=B1*B2
      B4=B1*B3
      B5=B1*B4
      B6=B1*B5
      B7=B1*B6
      B8=B1*B7
      B9=B1*B8
      B10=B1*B9
      B11=B1*B10
      B12=B1*B11
      G(1)=E(9)*B8+E(10)*B9+E(11)*B10+E(12)*B11+E(13)*B12
      G(2)=E(22)*B8+E(23)*B9+E(24)*B10+E(25)*B11
      G(3)=E(33)*B6+E(34)*B7+E(35)*B8+E(36)*B9+E(37)*B10
      G(4)=E(46)*B6+E(47)*B7+E(48)*B8+E(49)*B9
      G(5)=E(57)*B4+E(58)*B5+E(59)*B6+E(60)*B7+E(61)*B8
      G(6)=E(70)*B4+E(71)*B5+E(72)*B6+E(73)*B7
      G(7)=E(81)*B2+E(82)*B3+E(83)*B4+E(84)*B5+E(85)*B6
      G(8)=E(94)*B2+E(95)*B3+E(96)*B4+E(97)*B5
      G(9)=E(105)+E(106)*B1+E(107)*B2+E(108)*B3+E(109)*B4
      G(10)=E(118)+E(119)*B1+E(120)*B2+E(121)*B3
      G(11)=E(131)+E(132)*B1+E(133)*B2
      G(12)=E(144)+E(145)*B1
      G(13)=E(157)
```

```
      G(18)=F(9)*B8+F(10)*B9+F(11)*B10+F(12)*B11+F(13)*B12
      G(19)=F(22)*B8+F(23)*B9+F(24)*B10+F(25)*B11
      G(20)=F(33)*B6+F(34)*B7+F(35)*B8+F(36)*B9+F(37)*B10+F(38)*B11+
     2F(39)*B12
      G(21)=F(46)*B6+F(47)*B7+F(48)*B8+F(49)*B9+F(50)*B10+F(51)*B11
      G(22)=F(57)*B4+F(58)*B5+F(59)*B6+F(60)*B7+F(61)*B8+F(62)*B9+
     2F(63)*B10+F(64)*B11+F(65)*B12
      G(23)=F(70)*B4+F(71)*B5+F(72)*B6+F(73)*B7+F(74)*B8+F(75)*B9+
     2F(76)*B10+F(77)*B11
      G(24)=F(81)*B2+F(82)*B3+F(83)*B4+F(84)*B5+F(85)*B6+F(86)*B7+
     2F(87)*B8+F(88)*B9+F(89)*B10+F(90)*B11+F(91)*B12
      G(25)=F(94)*B2+F(95)*B3+F(96)*B4+F(97)*B5+F(98)*B6+F(99)*B7+
     2F(100)*B8+F(101)*B9+F(102)*B10+F(103)*B11
      G(26)=F(105)+F(106)*B1+F(107)*B2+F(108)*B3+F(109)*B4+F(110)*B5+
     2F(111)*B6+F(112)*B7+F(113)*B8+F(114)*B9+F(115)*B10+F(116)*B11+
     3F(117)*B12
      G(27)=F(118)+F(119)*B1+F(120)*B2+F(121)*B3+F(122)*B4+F(123)*B5+
     2F(124)*B6+F(125)*B7+F(126)*B8+F(127)*B9+F(128)*B10+F(129)*B11
      G(28)=F(131)+F(132)*B1+F(133)*B2+F(134)*B3+F(135)*B4+F(136)*B5+
     2F(137)*B6+F(138)*B7+F(139)*B8+F(140)*B9+F(141)*B10
      G(29)=F(144)+F(145)*B1+F(146)*B2+F(147)*B3+F(148)*B4+F(149)*B5+
     2F(150)*B6+F(151)*B7+F(152)*B8+F(153)*B9
      G(30)=F(157)+F(158)*B1+F(159)*B2+F(160)*B3+F(161)*B4+F(162)*B5+
     2F(163)*B6+F(164)*B7+F(165)*B8
      G(31)=F(170)+F(171)*B1+F(172)*B2+F(173)*B3+F(174)*B4+F(175)*B5+
     2F(176)*B6+F(177)*B7
      G(32)=F(183)+F(184)*B1+F(185)*B2+F(186)*B3+F(187)*B4+F(188)*B5+
     2F(189)*B6
      G(33)=F(196)+F(197)*B1+F(198)*B2+F(199)*B3+F(200)*B4+F(201)*B5
      G(34)=F(209)+F(210)*B1+F(211)*B2+F(212)*B3+F(213)*B4
      G(35)=F(222)+F(223)*B1+F(224)*B2+F(225)*B3
      G(36)=F(235)+F(236)*B1+F(237)*B2
      G(37)=F(248)+F(249)*B1
      G(38)=F(261)
   32 DO 34 K=23,41
   34 Y(K)=Y(K-21)
      Y(42)=(PA-G(18)*Y(1)-G(19)*Y(2)-G(20)*Y(3)-G(21)*Y(4)-G(22)*Y(5)-
     2G(23)*Y(6)-G(24)*Y(7)-G(25)*Y(8)-G(26)*Y(9)-G(27)*Y(10)-G(28)*Y(11
     3)-G(29)*Y(12)-G(30)*Y(13)-G(31)*Y(14)-G(32)*Y(15)-G(33)*Y(16)-
     4G(34)*Y(17)-G(35)*Y(18)-G(36)*Y(19)-G(37)*Y(20))/G(38)
      Y(102)=G(1)*Y(1)+G(2)*Y(2)+G(3)*Y(3)+G(4)*Y(4)+G(5)*Y(5)+G(6)*Y(6)
     2+G(7)*Y(7)+G(8)*Y(8)+G(9)*Y(9)+G(10)*Y(10)+G(11)*Y(11)+G(12)*Y(12)
     3+G(13)*Y(13)
      Y(100)=G(1)*Y(2)+G(2)*Y(3)+G(3)*Y(4)+G(4)*Y(5)+G(5)*Y(6)+G(6)*
     2Y(7)+G(7)*Y(8)+G(8)*Y(9)+G(9)*Y(10)+G(10)*Y(11)+G(11)*Y(12)+G(12)
     3*Y(13)+G(13)*Y(14)
      RETURN
      END
      SUBROUTINE GOTEAM
C---- OUTPUT "PB" IS STORED IN Y(102).
      COMMON Y(102)
      COMMON/GOAL/PT(101,2),STEP(3),NRKS(3),NPM(3),K1,K2,K3,K4,CC,PA,PB,
     2PATM,NPV,NRUNS,AHEAD(80)
      COMMON/FORM/A(20),B(28),C(63),D(91),E(221)
      COMMON/SAGE/F(325)
      COMMON/BLOB/YMAXI
      DIMENSION T(50),U(100)
   10 WRITE(6,100) (AHEAD(J),J=1,80)
  100 FORMAT(1H1,6X,80A1)
C---- ONE PRODUCT TERM FOR COSH(GAMMA).
      IF(K1.EQ.1) WRITE(6,200)
  200 FORMAT(15X,'THIS RUN USES THE ONE PRODUCT-TERM EXPANSION FOR COSH(
     2GAMMA).')
      IF(K1.EQ.2) WRITE(6,500)
  500 FORMAT(15X,'THIS RUN USES THE TWO-PRODUCT TERM EXPANSION FOR COSH(
     2GAMMA).')
      IF(K1.EQ.3) WRITE(6,600)
  600 FORMAT(15X,'THIS RUN USES THE FOUR-PRODUCT TERM EXPANSION FOR COSH
     2(GAMMA).')
      PAP=PA*PATM
      WRITE(6,300) STEP(K1),PAP
  300 FORMAT(15X,'TIME STEP IS ',F8.5,' ,  PRESSURE STEP INPUT = ',F12.5
     2,'  . ',/,15X,62('*'),/)
      WRITE(6,400)
  400 FORMAT(16X,' TIME      OUTPUT "PB/PA"         TIME      OUTPUT "PB/PA
     2"',/,16X,'(SEC)      (CONVERTED)           (SEC)      (CONVERTED)',
     3/,15X,' -----      --------------        -----      --------------')
   40 DO 42  J=1,102
   42 Y(J)=0.
C---- LOAD DELTA-TIME, NO. OF R-K STEPS, AND PRINT MULTIPLE.
      Y(K2+2)=STEP(K1)
      NRK=NRKS(K1)
      NP=NRK/100
C---- LOAD "TIME" INTO PT(K,1) AND ZEROES INTO PT(K,2).
      DP=Y(K2+2)
      DO 52 J=1,100
      PT(J,2)=0.
   52 PT(J,1)=DP*(J-1)*NP
C---- CALL THE INTEGRATOR.
      NPR=0
      IK=1
      DO 60 KK=1,NRK
      NPR=NPR+1
      CALL RKINT(KK,K2)
      CONVRT=Y(102)/PA
      IF(ABS(CONVRT).GT.YMAXI) GO TO 65
      IF(NPR.LT.NP) GO TO 60
      NPR=0
      IK=IK+1
      PT(IK,2)=CONVRT
   60 CONTINUE
   65 WRITE(6,700) ((PT(K,J),J=1,2),(PT(K+50,J),J=1,2),K=1,50)
  700 FORMAT(50(15X,F6.4,6X,G12.5,7X,F6.4,6X,G12.5,/))
C---- DO NOT CALL PLOTTER IF DATA IS BAD.
      IF(PT(10,2).EQ.0.) RETURN
C---- LOAD DATA FOR THE PLOTTER.
      DO 80 K=1,50
      J=2*K-1
      T(K)=PT(J,1)
      U(K)=1.
      U(K+50)=PT(J,2)
   80 IF(U(K+50).LT.0.) U(K+50)=0.
      CALL XYPLOT(T,U,50,100,8.17,8.)
      RETURN
      END
      SUBROUTINE  RKINT (LL,NSYS)
C     THIS SUBROUTINE SOLVES DIFFERENTIAL EQUATIONS BY USING A RUNGE KUTTA
C     METHOD
      DIMENSION DELY(4,50),BET(3),YU(50)
      COMMON Y(102)
```

```
      DOUBLE PRECISION YU
      IF (LL.NE.1) GO TO 1001
      BET(1)=0.5
      BET(2)=0.5
      BET(3)=1.0
      N2=NSYS+2
      NP1=NSYS+1
      XV=Y(NP1)
      CALL DERFUN
      DO 320 I=1,NSYS
  320 YU(I)=Y(I)
 1001 DO 1034 K=1,4
      IF (K.EQ.1) GO TO 1002
      CALL DERFUN
 1002 DO 1340 I=1,NSYS
      IPN2=I+N2
 1340 DELY(K,I)=Y(N2)*Y(IPN2)
      IF (K.EQ.4) GO TO 1034
      DO 1350 I=1,NSYS
 1350 Y(I)=YU(I)+BET(K)*DELY(K,I)
      Y(NP1)=XV+BET(K)*Y(N2)
 1034 CONTINUE
      DO 1039 I=1,NSYS
      DEL=(DELY(1,I)+2.0*DELY(2,I)+2.0*DELY(3,I)+DELY(4,I))/6.0
      YU(I)=YU(I)+DEL
      Y(I)=YU(I)
 1039 CONTINUE
      Y(NP1)=XV+Y(N2)
      CALL DERFUN
      XV=Y(NP1)
      RETURN
      END
      SUBROUTINE XYPLOT (XX,YY,NX,NY,XLINCH,YLINCH)
      COMMON/BLOB/YMAXI
      DIMENSION XX(1),YY(1),IY(10)
      DIMENSION IPLOT(100),IMINUS(100),ISYMBL(10)                       XY 0030
      DATA IBLANK,IAXIS/1H ,1HI/,IPLOT,IMINUS/100*1H ,100*1H_/          XY 0040
      DATA ISYMBL/1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,1H0/              XY 0050
      NXSIZE = XLINCH*6.0                                               XY 0120
      XSIZE = NXSIZE                                                    XY 0130
      YSIZE = YLINCH*10.0                                               XY 0140
      NYSIZE = YSIZE + 1                                                XY 0170
      YSIZE = NYSIZE - 1                                                XY 0180
      NPLOTS = NY/NX                                                    XY 0190
      XMIN=0.                                                           XY 0200
      XMAX =XX(NX)
      DX = XMAX - XMIN                                                  XY 0220
C---- USE A FIXED ABSCISSA, AS SHOWN BELOW.
C     YMIN=YY(1)
C     DO 8 I=1,NY
C   8 IF(YY(I).LT.YMIN) YMIN=YY(I)
C     YMAX =YY(1)
C     DO 10 I=1,NY                                                      XY 0250
C  10 IF(YY(I).GT.YMAX) YMAX=YY(I)
C     DY=YMAX-YMIN
C---- FIXED ABSCISSA
      YMIN=0.
      YMAX=YMAXI
      DY=YMAXI
```

```
      WRITE(6,6) YMIN,YMAX
      WRITE(6,1)(IMINUS(J),J=1,NYSIZE)                                  XY 0290
      IPLOT(1) = IAXIS                                                  XY 0300
      IPLOT(NYSIZE) = IAXIS                                             XY 0310
      NLINE = 0                                                         XY 0320
      DO 30 I=1,NX                                                      XY 0330
      IX = (XX(I)- XMIN)/DX*XSIZE
   32 IF(IX - NLINE) 30,33,34                                           XY 0350
   34 WRITE(6,4)(IPLOT(J),J=1,NYSIZE)                                   XY 0360
      NLINE = NLINE + 1                                                 XY 0370
      GO TO 32                                                          XY 0380
   33 NLINE = NLINE + 1                                                 XY 0390
      KI = I                                                            XY 0400
      DO 41 K=1,NPLOTS                                                  XY 0410
      IY(K) = (YY(KI)- YMIN)/DY*YSIZE + 1.5
      IYK = IY(K)                                                       XY 0430
      IPLOT(IYK) = ISYMBL(K)                                            XY 0440
   41 KI = KI + NX                                                      XY 0450
      WRITE(6,2)XX(I),(IPLOT(J),J=1,NYSIZE)
      DO 42 K=1,NPLOTS                                                  XY 0470
      IYK = IY(K)                                                       XY 0480
   42 IPLOT(IYK) = IBLANK                                               XY 0490
      IPLOT(1) = IAXIS                                                  XY 0500
      IPLOT(NYSIZE) = IAXIS                                             XY 0510
   30 CONTINUE                                                          XY 0520
      WRITE(6,3)(IMINUS(J),J=1,NYSIZE)                                  XY 0530
      RETURN                                                            XY 0540
    1 FORMAT( /,6X,' ABSCISSA ',5X,100A1)                               XY 0060
    2 FORMAT(6X,E10.3,5X,100A1)                                         XY 0070
    3 FORMAT(1H+,20X,100A1)                                             XY 0080
    4 FORMAT(21X,100A1)                                                 XY 0090
    6 FORMAT(1H1,6X,'MIN ORDINATE "PMIN" = ',G12.5,' ,  MAX ORDINATE "PM
     2AX" = ',G12.5)
      END                                                               XY 0550
```

APPENDIX C

AN ALTERNATE MODEL, WITHOUT THROUGH FLOW

This appendix outlines an alternate solution to the nonlinear axial momentum equation, Equation (2.20), and the linear energy equation, Equation (2.25). This model is recommended for use only when the primary model, Equations (2.70), tends to be unstable in a particular system simulation.

The linearized, nondimensional axial momentum equation may be written in the form shown below when through flow is neglected:

$$\frac{\partial V(t,R,Z)}{\partial t} + \frac{C_o}{L} K \left(\frac{\partial V}{\partial Z}\right)_* V(t,R,Z) - \frac{\nu_o}{a^2 R} \frac{\partial}{\partial R}\left(R \frac{\partial V(t,R,Z)}{\partial R}\right) = - \frac{C_o}{L}\left[\frac{1}{\gamma} \frac{\partial P(t,Z)}{\partial Z} + (1-K) V_* \frac{\partial V(t,Z)}{\partial Z}\right] \tag{C.1}$$

where $(V)_* = \left(\operatorname{sgn} P(t,0)\right)\left(\frac{LZ}{C_o} \frac{\partial P(t,0)}{\partial t} - Q(t,0)\right)_*$ (C.2)

from Equation (A.48), and $\left(\frac{\partial V}{\partial Z}\right)_* = \left(\operatorname{sgn} P(t,0)\right)\left(\frac{L}{C_o} \frac{\partial P(t,0)}{\partial t}\right)_*$ (C.3)

from Equation (A.49).

By transforming Equations (C.1) and (2.25) to the Laplace domain and solving these equations, the solutions for the transient axial velocity and transient axial temperature profiles result:

Transient Axial Velocity

$$V(s,R,Z) = \frac{\left(\frac{J_o(\alpha R) - J_o(\alpha)}{J_o(\alpha)}\right) \frac{C_o}{sL}\left(\frac{1}{\gamma} \frac{\partial P}{\partial Z} + (1-K) V_* \frac{\partial V}{\partial Z}\right)}{\left(1 + \frac{C_o}{L} K \left(\frac{\partial V}{\partial Z}\right)_*\right)} \tag{C.4}$$

Transient Axial Temperature

$$T(S,R,Z) = \left(\frac{J_0(\Delta R) - J_0(\Delta)}{J_0(\Delta)}\right)\left(-\frac{(\gamma-1)}{\gamma} P(S,Z)\right) \tag{C.5}$$

These equations correspond to Equations (2.54) and (2.55) in the main body of the thesis.

By substituting Equations (C.4) and (C.5) into Equations (2.37) and (2.36) respectively, and integrating Equations (2.37) and (2.36) with respect to (R), the results are:

$$Q(S,Z) = \frac{-\frac{C_o D_a}{\gamma S L}\left[\frac{\partial P}{\partial Z} - \frac{D_g C_o}{SL}(1-K) V_* \frac{\partial^2 P}{\partial Z^2}\right]}{\left[1 + \frac{C_o K}{SL}\left(\frac{\partial V}{\partial Z}\right)_*\right]} \tag{C.6}$$

and

$$\frac{\partial Q(S,Z)}{\partial Z} = -\frac{SL}{C_o} N_g P(S,Z) \tag{C.7}$$

where (D_a), (D_g), and (N_g) are given as Equations (2.74).

By differentiating Equation (C.6) with respect to (Z), neglecting the higher order term $\frac{\partial^3 P(S,Z)}{\partial Z^3}$, and equating the result to Equation (C.7), this ordinary differential equation results:

$$\frac{\partial^2 P(S,Z)}{\partial Z^2} = \left(\frac{SL}{C_o}\right)^2 \frac{N_g}{D_a} \frac{(S + K F_{1*})}{(S - [1-K] D_g F_{1*})} P(S,Z) \tag{C.8}$$

$$0 \leq K \leq 1$$

where F_{1*} is given as Equation (2.76).

The solution to Equation (C.8) is of the form:

$$P(S,Z) = C_1 e^{\Gamma_d(S) Z} + C_2 e^{-\Gamma_d(S) Z} \tag{C.9}$$

where $$\Gamma_d(S) = \left(\frac{SL}{C_o}\right)\sqrt{\frac{N_g (S + K F_{1*})}{D_a (S - [1-K] D_g F_{1*})}} \tag{C.10}$$

Equations (C.9) and (C.6) form a system of equations in the spatial coordinate (Z). By applying the boundary conditions at Z = 0 and Z = 1, this transmission line model results:

$$\begin{bmatrix} P(S,1) \\ Q(S,1) \end{bmatrix} = \begin{bmatrix} \cosh \Gamma_d(S) & -Z_d(S) \sinh \Gamma_d(S) \\ \dfrac{-\sinh \Gamma_d(S)}{Z_d(S)} & \cosh \Gamma_d(S) \end{bmatrix} \begin{bmatrix} P(S,0) \\ Q(S,0) \end{bmatrix} \tag{C.11}$$

$$\text{where } \Gamma_d(S) \cong \frac{SL}{C_o} \sqrt{\frac{N_g\,(S+K F_{1*})}{D_a\,(S-[1-K]F_{1*})}} \tag{C.12}$$

$$Z_d(S) \cong \frac{SL\ \gamma \qquad (S+K F_{1*})}{C_o\, D_a\, \Gamma_d(S)\,(S-[1-K]F_{1*})} = \gamma \sqrt{\frac{(S+KF_{1*})(S-[1-K]F_{1*})}{S^2 N_g\, D_a}} \tag{C.13}$$

The terms (N_g) and (D_a) are given as Equations (2.74), and

$$F_{1*} = (\operatorname{sgn} P(t,0))\left(\frac{\partial P(t,0)}{\partial t}\right)_* \tag{C.14}$$

from Equation (2.76).

In the special case where K = 1.0 above, this model becomes the same as the model in the main text, Equations (2.70).

Using the approximations for (N_g), (D_a), and Cosh $\Gamma(S)$ given in Chapter III, Equation (C.11) may be rewritten in the same form as Equation (5.4) to compute step responses. That is,

$$P(S,1) = \frac{P(S,0)}{\cosh \Gamma_d(S)} \tag{C.15}$$

The step responses which result from the one, two, and four product term expansions for Cosh $\Gamma_d(S)$ are shown as Figures 20, 21, and 22. The computed step responses and the experimental step responses are shown for step inputs of 0.25, 2.0, 4.0, and 6.0 psig. The computed

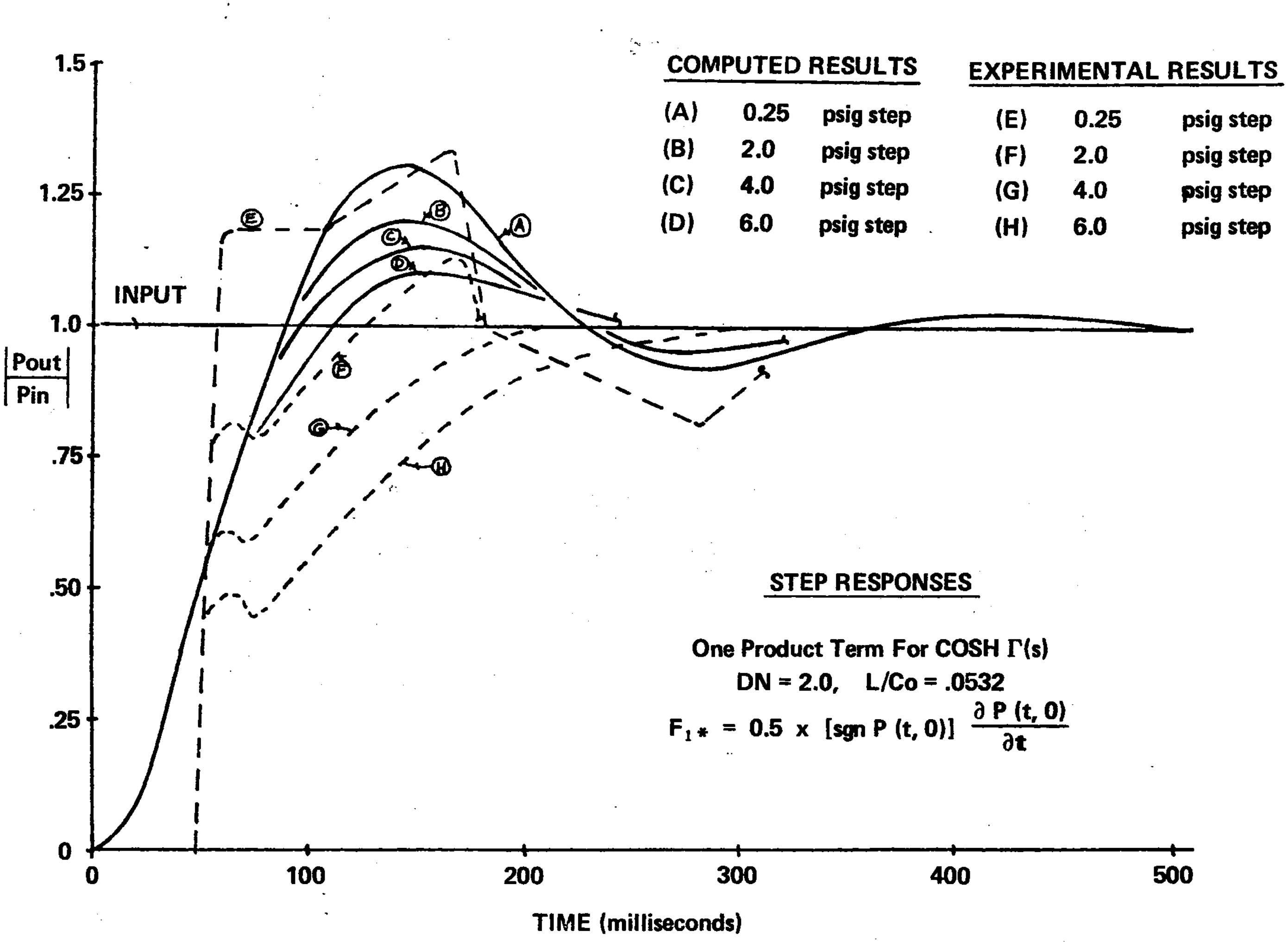

Figure 20. Alternate Model Step Responses, One Product Term

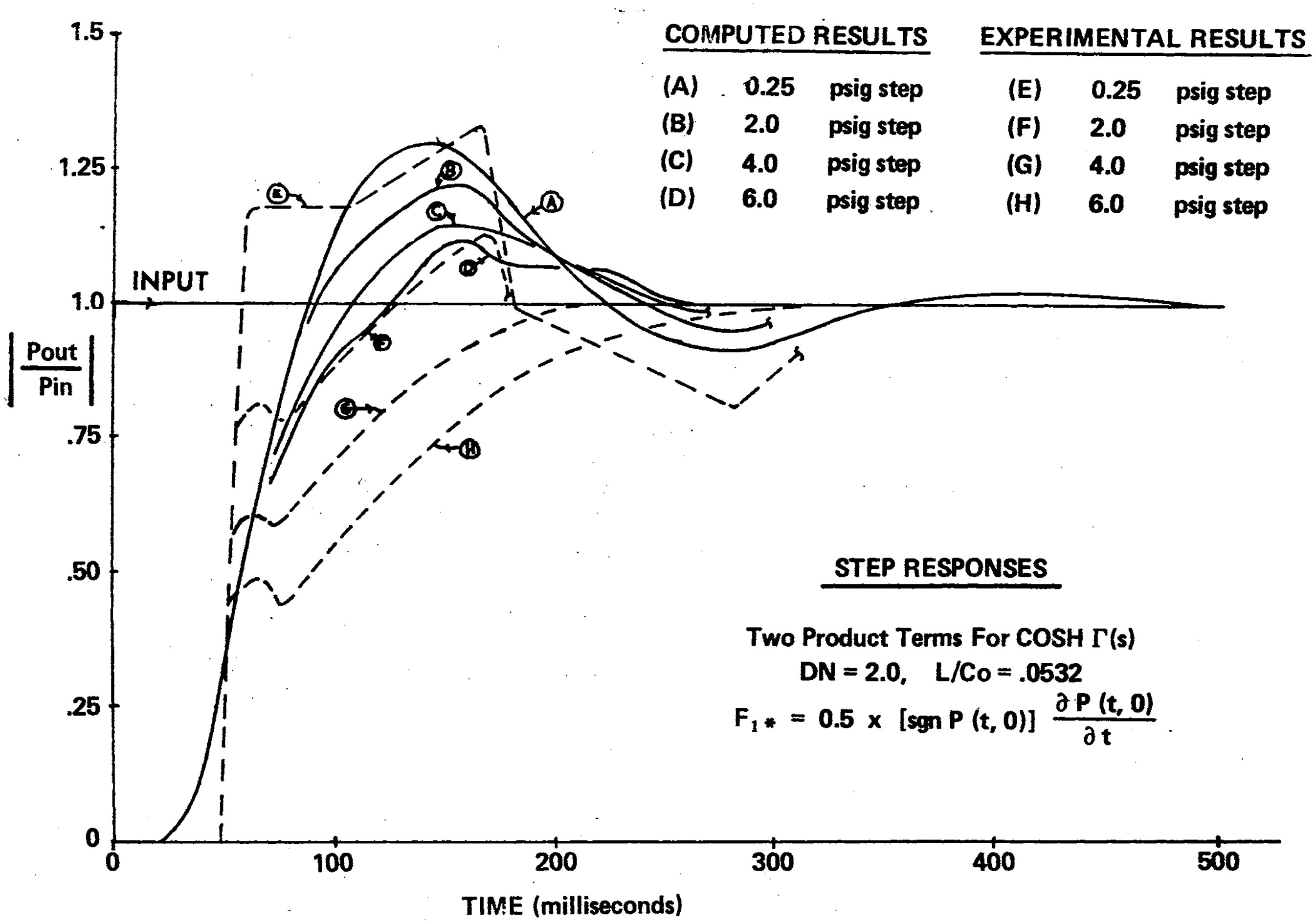

Figure 21. Alternate Model Step Responses, Two Product Terms

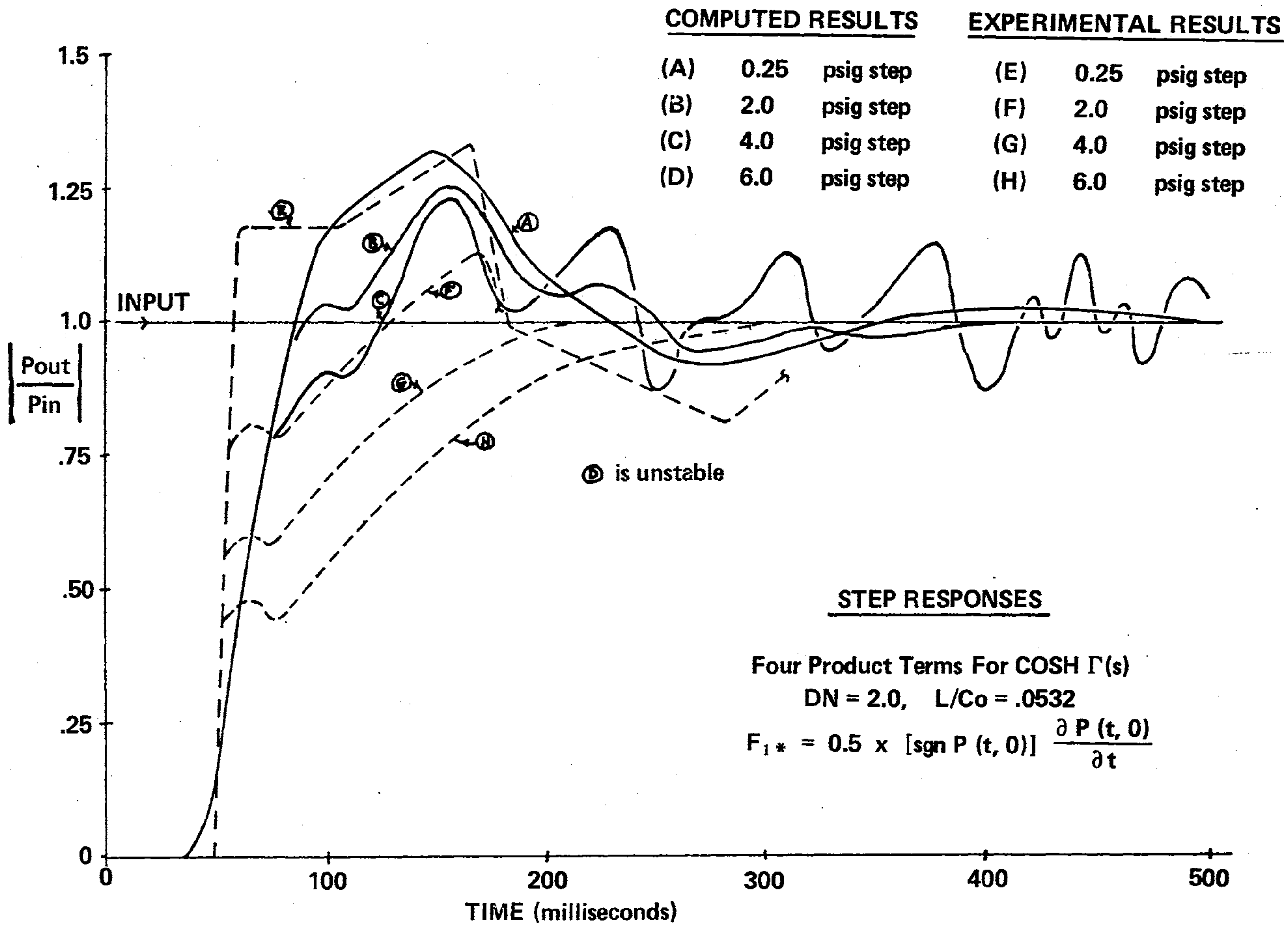

Figure 22. Alternate Model Step Responses, Four Product Terms

step responses are based on parameters $K = 0.5$, $DN = 2.0$, $L/C_o = .0532$ (the 60 ft pneumatic line discussed in Chapter V), 0.40 inch inner diameter, at an ambient pressure (p_o) of 11.2 psia.

This model does not predict as great an increase in apparent damping as disturbance amplitude increases as that predicted by the model in the main text, Equations (2.70). (Compare Figures 20, 21, and 22 with Figure 13.) But this model is more stable than Equations (2.70).

VITA

Richard Paul Braden

Candidate for the Degree of

Doctor of Philosophy

Thesis: FLUID LINE DYNAMICS WITH THROUGH FLOW AND FINITE AMPLITUDE DISTURBANCES

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born in Warren, Ohio, March 17, 1937, the son of Harriet Pawlowski Braden and Paul Braden. Married to Patricia Ligon Braden on August 9, 1959. Three children: Lisa, Annalori, and Jill.

Education: Graduated from High School, Zephyrhills, Florida in May, 1954. Received Bachelor of Mechanical Engineering degree from Georgia Institute of Technology, 1959, and Master of Science degree in Mechanical Engineering from University of New Mexico, 1967. Completed requirements for the Doctor of Philosophy degree in Mechanical and Aerospace Engineering from Oklahoma State University in July, 1973.

Professional Experience: Propulsion Systems Engineer with Lockheed Aircraft Corporation, Marietta, Georgia, 1955 through 1959. Associated with U. S. Army missile program, 1959 to 1963. Schooled in Air Force weapons systems, then assigned to Air Force Weapons Laboratory, Albuquerque, New Mexico, 1964 to 1967. Assigned to U. S. Air Force Academy, Department of Mathematics, 1967 to 1969. Began coursework for Ph. D. in 1969. Presently assigned to Department of Aeronautics, USAFA, as an Assistant Professor of Aeronautics.