ISOPERIMETRIC PROBLEMS WITH SIDE CONDITIONS INVOLVING CONVEX BODIES IN E_n

By

JAMES VERNON BALCH, JR.

Bachelor of Arts
Arkansas College
Batesville, Arkansas
1964

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1969

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
December, 1973

ISOPERIMETRIC PROBLEMS WITH SIDE CONDITIONS

INVOLVING CONVEX BODIES IN E_n

Thesis Approved:

E. K. McLachlan
Thesis Adviser

Robert S Brown

P. L. Claypool

D B Ciabele

D D Durham
Dean of the Graduate College

PREFACE

This thesis is a study of isoperimetric problems involving certain subclasses of the class C_n of convex bodies in E_n, where E_n denotes n-dimensional Euclidean space. An isoperimetric problem is one which involves the maximization of the ratio of the ith root of the ith dimensional measure and the jth root of the jth dimensional measure of a body in E_n, where $1 \leq i, j \leq n$. For example, the (2,1)-isoperimetric ratios associated with a square of side length s is $\sqrt{s^2}/4s = \frac{1}{4}$. This ratio is independent of size, as the preceding example illustrates. When $i > j$, attention may be restricted without loss of generality to convex bodies in E_n since it can be shown that an optimal body must be convex.

The first three chapters concern themselves with variations of the "classical" isoperimetric problem, i.e., maximizing the ratio of the nth root of the full n-dimensional volume and the (n-1)st root of the (n-1)-dimensional surface area of a convex body. In Chapter I the methods of Steiner are used to show that the n-dimensional ball is the solution on C_n. In Chapter II classical isoperimetric problems are investigated which involve convex bodies whose boundaries are unions of line segments and arcs of a fixed length or whose boundaries contain certain fixed points. The convex bodies under consideration in Chapter III are the subsets of E_2 with polygonal boundaries, that is, polygons. Finally, in Chapter IV the "classical" isoperimetric ratio is generalized to the (i,j)-isoperimetric ratio defined above and problems related to its maximization on polytopes in E_n are investigated.

I would like to express appreciation to all those who assisted me in the preparation of this thesis. In particular, I would like to thank my advisor Dr. E. K. McLachlan for his advice, assistance and patience. For their encouragement and cooperation while serving as members of my committee, thanks goes to Professors Claypool and Brown. I also want to thank my typists Cynthia Wise and Janet Jones for their skill with an intricate text. I would also like to publicly thank my wife Ann for her encouragement and patience during some discouraging times.

Finally, I am indebted to Dr. L. Wayne Johnson and Dr. John Jewett, successive heads of the Department of Mathematics and Statistics, for their financial assistance in the form of Graduate Teaching Assistantships and an Instructorship during my time as a graduate student.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Figure | Page

CHAPTER I

THE CLASSICAL ISOPERIMETRIC PROBLEM

In this chapter, attention is given to the following "classical" isoperimetric problem: Maximize the full n-dimensional volume $V_n(A)$ on the class C_n of compact convex bodies A of a fixed (n-1)-dimensional surface area $S_{n-1}(A) = s$ in E_n. The definition of a convex body intended here is that found in Valentine [16], i.e., a closed, bounded, convex set with a nonempty interior.

The equivalence of alternative formulations of the problem is first demonstrated; then examples from nature are given to experimentally illustrate its solution. Finally, it is proved that the solution is a closed disc for $n = 2$, and a closed ball for $n = 3$. The method of proof employed in the solution for $n = 3$ involves a process known as Steiner symmetrization. This method may be generalized to prove that the solution for $n \geq 3$ is the closed n-ball.

1. Equivalent Formulations of the Problem

The <u>classical isoperimetric problem</u> for E_n can be stated as follows: From among those sets A in the colection C_n of a fixed (n-1)-dimensional surface area s, which maximize(s) the n-dimensional volume $V_n(A)$? The name associated with the problem is derived from this formulation for $n = 2$, where area is to be maximized as perimeter is held constant. An <u>alternate formulation</u> is also possible: Among those sets A in the collection C_n of a fixed n-dimensional volume v, which minimize(s) the (n-1)-dimensional surface area $S_{n-1}(A)$?

The equivalence of the two formulations above will be demonstrated by showing that each of the formulations is equivalent to a third. Some preliminary definitions are necessary:

For a closed body A in C_n, the isoperimetric ratio associated with A is defined as

$$I(A) = \frac{\sqrt[n]{V_n(A)}}{\sqrt[n-1]{S_{n-1}(A)}},$$

where $V_n(A)$ and $S_{n-1}(A)$ are the n-dimensional volume and (n-1)-dimensional surface area of A, respectively.

Let λ be a real positive constant and let z be an arbitrary point in E_n. The homothety H_λ with ratio $\lambda > 0$ and centered at z is the correspondence which associates with an arbitrary point x in E_n the point $y = z + \lambda(x - z)$ in E_n (cf. Figure 1). The image $H_\lambda(A)$ of A by the above homothety has the same shape as A. In particular, line segments are mapped to line segments, the angles between them remain constant, and the ratios of their lengths are preserved under homothetic mappings (cf. [2], pp. 86ff.). If $H_\lambda(A)$ is the image of a compact convex set A under the homothety H_λ then from [2], p. 89,

$$S_{n-1}(H_\lambda(A)) = \lambda^{n-1}S_{n-1}(A)$$

and

$$V_n(H_\lambda(A)) = \lambda^n V_n(A). \qquad (1.1.1)$$

It follows from the definitions and properties above that:

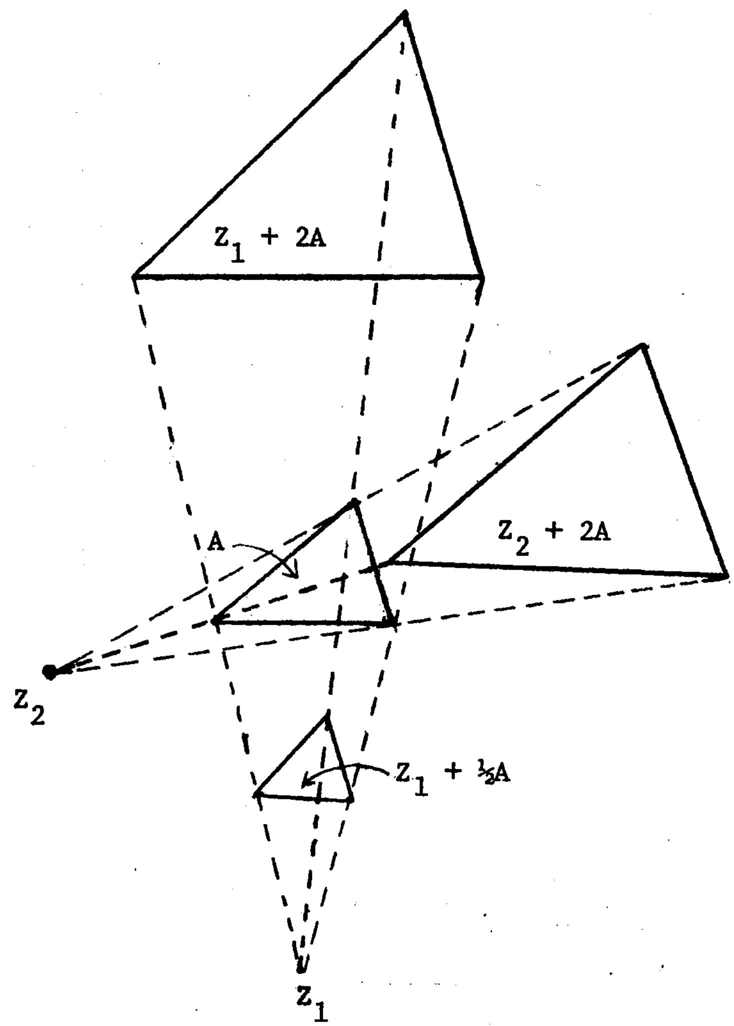


Figure 1. Three Homotheties of A.

$$I[H_\lambda(A)] = \frac{\sqrt[n]{V_n(H_\lambda(A))}}{\sqrt[n-1]{S_{n-1}(H_\lambda(A))}} = \frac{\sqrt[n]{\lambda^n V_n(A)}}{\sqrt[n-1]{\lambda^{n-1} S_{n-1}(A)}}$$

$$= \frac{\lambda \sqrt[n]{V_n(A)}}{\lambda \sqrt[n-1]{S_{n-1}(A)}} = \frac{\sqrt[n]{V_n(A)}}{\sqrt[n-1]{S_{n-1}(A)}}$$

$$= I(A). \tag{1.1.2}$$

Thus, it is clear that the isoperimetric ratio associated with a convex body A is also preserved under a homothety.

The equivalence of the first formulation of the isoperimetric problem to the problem of maximizing the isoperimetric ratio may now be demonstrated: Let A^* be a member of C_n which maximizes the isoperimetric ratio, then $I(A^*) \geq I(A)$ for every set A in C_n. Also let A_1 be a solution to the original formulation of the isoperimetric problem, i.e., $V_n(A_1) \geq V_n(A)$ for every set A in C_n such that $S_{n-1}(A) = S_{n-1}(A_1) = s$.

(a). To show that A_1 maximizes the isoperimetric ratio, it will suffice, from the definition of A^*, to show that $I(A_1) \geq I(A^*)$: If

$$\lambda = \sqrt[n-1]{\frac{S_{n-1}(A_1)}{S_{n-1}(A^*)}}$$

and H_λ is centered at the origin, then $H_\lambda(A^*) = \lambda(A^*)$. Also from (1.1.1),

$$S_{n-1}(H_\lambda(A^*)) = S_{n-1}(\lambda A^*) = \lambda^{n-1} S_{n-1}(A^*)$$

$$= \frac{S_{n-1}(A_1)}{S_{n-1}(A^*)} S_{n-1}(A^*) = S_{n-1}(A_1). \qquad (1.1.3)$$

Hence $V_n(H(A^*)) \le V_n(A_1)$ from the choice of A_1. It now follows from (1.1.2) and (1.1.3) that

$$I(A^*) = I(H_\lambda(A^*)) = \frac{\sqrt[n]{V_n(H_\lambda(A^*))}}{\sqrt[n-1]{S_{n-1}(H_\lambda(A^*))}}$$

$$= \frac{\sqrt[n]{V_n(H_\lambda(A^*))}}{\sqrt[n-1]{S_{n-1}(A_1)}} \le \frac{\sqrt[n]{V_n(A_1)}}{\sqrt[n-1]{S_{n-1}(A_1)}}$$

$$= I(A_1). \qquad (1.1.4)$$

It remains to be shown that: (b) If A^* maximizes the isoperimetric ratio, then it is a solution, up to a homothety, of the original formulation of the isoperimetric problem. It will suffice, due to (1.1.3) and the choice of A_1, to show that $V_n(H_\lambda(A^*)) \ge V_n(A_1)$. Now recalling that $I(A^*) \ge I(A_1)$, (1.1.3) clearly implies that

$$\frac{\sqrt[n]{V_n(H_\lambda(A^*))}}{\sqrt[n-1]{S_{n-1}(A_1)}} \ge \frac{\sqrt[n]{V_n(A_1)}}{\sqrt[n-1]{S_{n-1}(A_1)}},$$

and, therefore, that $V_n(H_\lambda(A^*)) \ge V_n(A_1)$.

Similarly, the equivalence of the alternate formulation of the isoperimetric problem to that of maximizing the isoperimetric ratio may be established: Again let A^* be a member of C_n which maximizes the isoperimetric ratio, and let A_2 be a solution to the alternate formulation of the isoperimetric problem. Then $S_{n-1}(A_2) \leq S_{n-1}(A)$ for every set A in C_n such that $V_n(A) = V_n(A_2) = v$.

First show: (a) If A_2 is a solution to the alternate formulation of the isoperimetric problem, then A_2 also maximizes the isoperimetric ratio. Clearly, from the definition of A^*, it will suffice to show that $I(A_2) \geq I(A^*)$. If

$$\lambda = \sqrt[n]{\frac{V_n(A_2)}{V_n(A^*)}}$$

and H_λ is centered at the origin, then

$$\begin{aligned} V_n(H_\lambda(A^*)) &= V_n(\lambda A^*) = \lambda^n V_n(A^*) \\ &= \frac{V_n(A_2)}{V_n(A^*)} V_n(A^*) = V_n(A_2). \end{aligned} \tag{1.1.5}$$

Hence, from the choice of A_2 it is clear that $S_{n-1}(H_\lambda(A^*)) \geq S_{n-1}(A_2)$. Now from (1.1.2)

$$\begin{aligned} I(A^*) = I(H_\lambda(A^*)) &= \frac{\sqrt[n]{V_n(H_\lambda(A^*))}}{\sqrt[n-1]{S_{n-1}(H_\lambda(A^*))}} \\ &= \frac{\sqrt[n]{V_n(A_2)}}{\sqrt[n-1]{S_{n-1}(H_\lambda(A^*))}} \leq \frac{\sqrt[n]{V_n(A_2)}}{\sqrt[n-1]{S_{n-1}(A_2)}} \\ &= I(A_2). \end{aligned} \tag{1.1.6}$$

It remains to be shown that : (b) If A^* maximizes the isoperimetric ratio, then A^* is a solution, up to a homothety, of the alternate formulation of the isoperimetric problem. From (1.1.5) and the choice of A_2, it clearly will suffice to show that $S_{n-1}(H_\lambda(A^*)) \leq S_{n-1}(A_2)$. Recalling (1.1.5) and (1.1.6), clearly

$$\frac{\sqrt[n]{V_n(A_2)}}{\sqrt[n-1]{S_{n-1}(H_\lambda(A^*))}} \gtrsim \frac{\sqrt[n]{V_n(A_2)}}{\sqrt[n-1]{S_{n-1}(A_2)}}$$

and, therefore, $S_{n-1}(H_\lambda(A^*)) \leq S_{n-1}(A_2)$.

These arguments demonstrate the equivalence of a <u>third formulation</u> of the classical isoperimetric problem: From among those sets in the collection C_n, which maximize(s) the isoperimetric ratio?

2. Examples from Nature

The solutions to the classical isoperimetric problem for $n = 2$ and $n = 3$ are suggested by certain natural phenomena observable in nature. In still air, for example, soap bubbles are spherical. Surface tension minimizes their surface area for the constant volume of their enclosed air. Animals also seem to be aware of the spherical solution of the classical isoperimetric problem. In cold weather, they reduce their exposed surface area by curling up to as nearly a spherical shape as possible in order to minimize their surface areas and, hence, the amount of heat escaping from their bodies. These examples illustrate the solution to the alternate formulation of the problem for $n = 3$.

The disc is the solution to the isoperimetric problem for $n = 2$. Although somewhat difficult to perform in practice, the following experiment will demonstrate this fact: Draw a simple closed curve on a plane surface S and place a flexible right cylinder C on this curve. When a liquid is poured into the cylinder, it will seek to minimize its associated potential energy by lowering its center of mass. The volume of a cylinder is the product of its height and the area of its base. The height of the liquid in the cylinder and, hence, the height of its center of gravity will therefore be minimized for a fixed volume when the area of its base is maximized. The introduction of any fixed amount of liquid causes the cylinder to meet the surface in a circle. Since the perimeter of the base of the cylinder is the same as the length of the original curve, the greatest surface area for a fixed perimeter is obtained when the base region is in the shape of a disc (cf. Figures 2(a) and 2(b)).

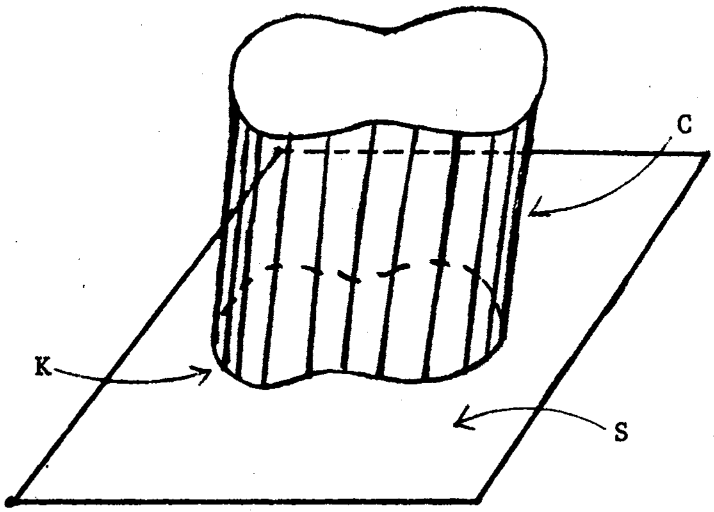
A less graphic but more practical experimental illustration of the same phenomena occurs when a drop of a viscous fluid (oil or paint) is allowed to fall onto a horizontal plane surface. The fluid will spread in the shape of a disc on the surface under the influence of gravity, thus increasing the surface area. This process will continue until equilibrium is reached with the opposing force tending to minimize surface tension by minimizing the surface area covered. For a fixed surface area, the surface tension is minimized with the perimeter. The isoperimetric ratio is, in this case, the square root of surface area divided by perimeter; hence, it will clearly be maximized when the system is in equilibrium. Since the fluid spreads into the shape of a disc, the disc must maximize the isoperimetric ratio for $n = 2$.

These last two examples illustrate the solution to the original formulation of the problem for $n = 2$.

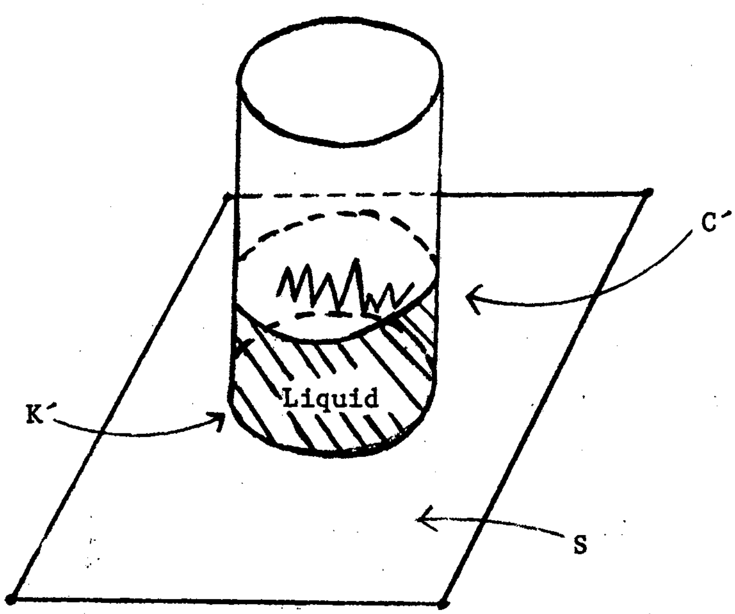
3. Proof of the Existence of a Solution

Although the isoperimetric problem was known to the ancient Greeks, the proofs presented here for its solution in the cases $n = 2$ and $n = 3$ are based on those first suggested by Steiner in the early part of the 19th Century. He offered five simple proofs that the disc is the solution for the case $n = 2$, but his most significant contribution was a process of symmetrization. It provided a method for altering the shape of a nonspherical convex body in such a way that its volume is preserved and its surface area is decreased. This process was the basis for one of Steiner's proofs for $n = 2$, but its more important applications were to the cases in which $n \geq 3$. In each case, however, Steiner's proof was based on an unproved assertion. He neglected to show that a solution to the problem existed. In one of his proofs for $n = 2$, for example, he supposed that it was sufficient to show that any noncircular simple closed plane curve could be transformed into a curve of equal length which encloses a larger area. This would mean that a circle is the only possible candidate for a curve of fixed length which encloses maximum area. In order to complete Steiner's argument, a proof of the existence of a solution is necessary. The existence theorem will be stated with sufficient generality to make it applicable to the classical problem for $n \geq 3$. Some preliminaries are necessary, however.

In order to prove the existence theorem, a definition of the distance between compact convex sets is required. Secondly, a theorem



(a) Empty Flexible Cylinder



(b) Flexible Cylinder Containing a Liquid

Figure 2. Experimental Illustration of the Classical Isoperimetric Problem

reminiscent of the familiar Bolzano-Weierstrass theorem is needed. It is its natural analogue where the convergence dealt with is that of convex sets rather than that of points.

For real $\rho \geq 0$ A_ρ, the ρ-<u>parallel set</u> of A, is $\{x: a \in A$ and $||x - a|| \leq \rho\}$, where $||\ \ ||$ denotes the usual norm for E_n. The set A_ρ is then the collection of all points which lie within a distance of ρ from some point in A (cf. Figure 3(a)). For nonempty sets A and B in C_n, the <u>Hausdorff metric</u> $d(A,B)$ is defined as follows: $d(A,B) = \inf\{\rho: A \subset B_\rho, B \subset A_\rho\}$ (cf. Figure 3(b)). A sequence $\{A_i\}$ of convex sets is said to <u>converge</u> if and only if there exists a nonempty convex set A such that $d(A_i,A) \to 0$ as $i \to \infty$.

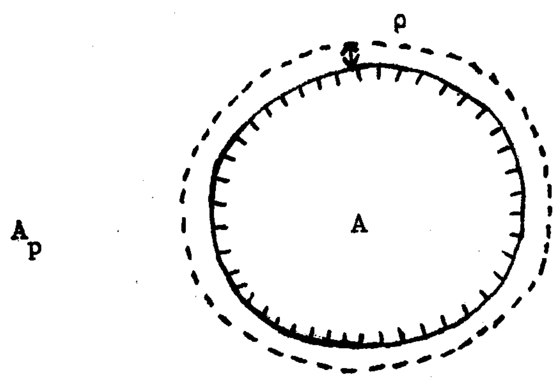
The following theorem is now meaningful:

<u>Theorem</u> 1.3.1 (Blaschke Convergence Theorem) A uniformly bounded infinite collection of sets from C_n contains a sequence which converges to a nonempty member of C_n.

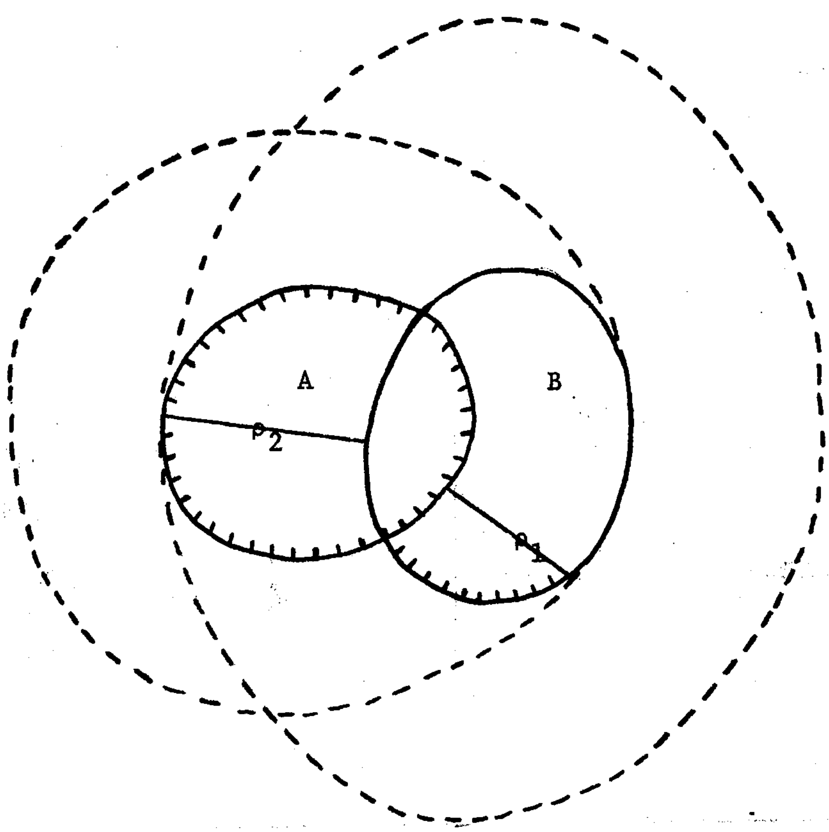
For a new proof of this theorem, the reader is referred to Valentine [16], pp. 37-39. The missing step in Steiner's argument is provided by the following theorem:

<u>Theorem</u> 1.3.2 (Existence Theorem) Let F be the collection of those sets A in C_n which are contained in the n-sphere $B(0,r)$, centered at the origin 0 with radius $r > 1$, and which have a prescribed (n-1)-dimensional surface area. Then F contains a member A^* such that $V_n(A^*) \geq V_n(A)$, for all A in F.

Proof: Since the members of F are contained in the n-sphere $B(0,r)$, their volumes are bounded above by that of $B(0,r)$. Let u be the



(a) The p-parallel Set of A



(b) Distance d(A,B) is Defined as $\min(\rho_1,\rho_2)$

Figure 3. The Hausdorff Metric

least upper bound of the volumes of the members of $\mathcal{F}$. Now construct a sequence $\{A_k\}$ of members from the family $\mathcal{F}$ so that for each positive integer k, $V_n(A_k) > u - 1/k$. Clearly, $V_n(A_k) \to u$ as $k \to \infty$. The infinite sequence $\{A_k\}$ must contain a convergent subsequence. This is evident in case $\{A_k\}$ is convergent or contains only finitely many distinct terms. If $\{A_k\}$ is not convergent and contains infinitely many distinct terms, the Blaschke Convergence Theorem implies $\{A_k\}$ has a convergent subsequence $\{B_k\}$ whose limit B is a member of C_n. Since the (n-1)-dimensional surface area function S_{n-1} is continuous on C_n, B must have the same (n-1)-dimensional surface area as do the terms B_k of the sequence of sets converging to B. Furthermore, B is compact, convex, and is contained in $B(0,r)$, so $B \in \mathcal{F}$. Finally, since $\{B_k\}$ is a subsequence of $\{A_k\}$ and the n-dimensional volume function V_n is continuous on C_n, it follows that

$$u = \lim_{k \to \infty} \left(V_n(B_k) \right) = V_n\left(\lim_{k \to \infty} B_k \right) = V_n(B) . \blacksquare$$

4. The Classical Isoperimetric Problem for n = 2

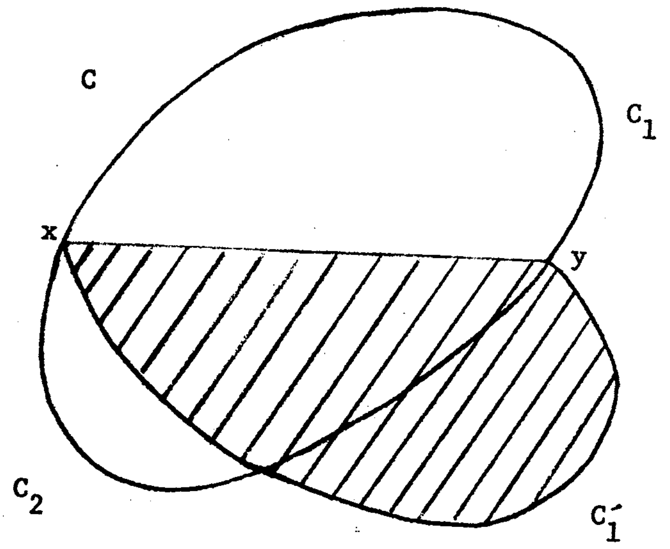
In view of (1.1.2) and the equivalence of its three formulations, the solution to the classical isoperimetric problem for $n = 2$ is provided by the following:

<u>Theorem</u> 1.4.1. Among the sets in C_2 of perimeter one, the closed disc is the unique set of maximum area.

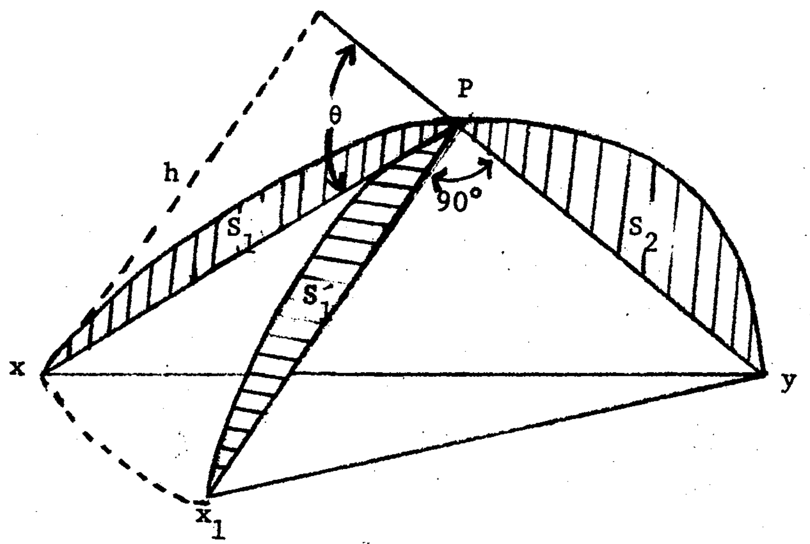
Proof: Since the translate of a solution to the problem is also a solution, it will suffice to consider only those sets in C_2 which are bounded by a disc of radius one-half centered at the origin. Hence, the Existence Theorem implies that a solution must exist. It will suffice, therefore, to show that no set other than a closed disc can be a solution. This is accomplished by the following argument due to Benson, [2]:

Suppose that the curve C of length one bounds a solution to the classical isoperimetric problem for $n = 2$. Assume further that x and y are points on C, chosen in such a way that the chord xy divides C into arcs C_1 and C_2 of length one-half. If the area enclosed by C_1 and the chord xy were greater than that enclosed by C_2 and the chord xy, the reflection C_1' of C_1 about the line $L(x,y)$ would yield the curve $C_1'' = C_1 \cup C_1'$. Obviously C_1'' has length one (as does C) but encloses a region of greater area than that enclosed by C (cf. Figure 4(a)). This contradicts the assumption that C bounds a solution. It follows, therefore, that a chord of C which bisects the curve must also divide the region enclosed by C into two regions of equal area.

It will now be shown that the arc C_1 is a semicircle with the chord xy as its diameter. If C_1 is not a semicircular arc, there must exist at least one point p of C_1 such that angle xpy is not a right angle (cf. Figure 4(b)). The region bounded by C_1 and xy consists of three parts: the triangle Δxpy, the sector S_1 bounded by C_1 and the chord xp, and the sector S_2 bounded by C_1 and the chord yp. Now regard S_1 as being hinged at p. There exists a rotation of S_1 about p such that x coincides with x_1, where the angle x_1py is a right angle. Denote S_1 in that new position by S_1'. Let S^* be $S_1' \cup S_2 \cup (\Delta x_1py)$. If the mirror image of S^* about



(a) Reflection of C_1 about $L(x,y)$



(b) Arc C_1 is Semicircular.

Figure 4. Solution of the Isoperimetric Problem for $n = 2$

$L(x,y)$ is denoted by T^*, the perimeter of $S^* \cup T^*$ is equal to the length of C. Since the area of S_1 is invariant under rotation, the difference between the area of S^* and the area enclosed by the arc C_1 and the chord xy is the difference between the areas of triangles Δxpy and $\Delta x_1 py$. The area of triangle $\Delta xpy = (1/2)|py||px| \sin\theta$, where the angle θ is supplementary to the vertex angle at p. Clearly, $\text{area}(\Delta x_1 py) = (1/2)|py||px_1|$. But since $|px| = |px_1|$ and $\theta \neq \pi/2$, $\text{area}(\Delta x_1 py) > \text{area}(\Delta xpy)$. Hence, $\text{area}(S^* \cup T^*) > \text{area}(R(C))$, where $R(C)$ is the region bounded by C. This contradicts the assumption, however, that C bounds a solution to the isoperimetric problem. Hence C is a circle. ■

5. The Classical Isoperimetric Problem for n = 3

For $n = 3$ the classical isoperimetric problem may be stated as follows: Among the sets in C_3 of volume one, which minimize(s) surface area?

Steiner Symmetrization

The methods employed in the solution of the isoperimetric problem for $n = 2$ are not applicable in the cases $n \geq 3$. It is necessary, therefore, to define a basic new concept on which the solution of the isoperimetric problem for $n \geq 3$ will largely depend. This is the concept of Steiner symmetrization, a process whereby a nonspherical convex body may be transformed into another convex body of equal volume and decreased surface area. The definition of Steiner symmetrization and the verification of the above mentioned properties follow:

Let K be a convex body and let π be any plane. Through each point ρ of K construct the line ℓ_ρ perpendicular to π. Denote $\ell_\rho \cap \pi$ by $\overline{P}_\pi$ and suppose $\ell_\rho \cap K = [\rho_1, \rho_2]$, a closed interval. Translate this interval along ℓ_ρ to coincidence with $[\overline{\rho}_1, \overline{\rho}_2]$, where $\overline{\rho}_1$ and $\overline{\rho}_2$ are translates of ρ_1 and ρ_2, respectively, so that $\overline{P}_\pi$ is the midpoint of $[\overline{\rho}_1, \overline{\rho}_2]$. The union of all such intervals $[\overline{\rho}_1, \overline{\rho}_2]$ as ρ varies over K is called the <u>(Steiner) symmetrical</u> $\overline{K}_\pi$ of K about the plane π (cf. Figure 5). The process of obtaining $\overline{K}_\pi$ from K is called <u>symmetrization</u> of K with respect to π.

The following theorem shows that the process of symmetrization as defined above preserves set inclusion.

<u>Theorem</u> 1.5.1. Let A and B be convex bodies in E_3 such that $A \subset B$ and let π be an arbitrary plane, then $\overline{A}_\pi \subset \overline{B}_\pi$.

The proof of the above theorem follows easily from the observation that the symmetrization of $\ell_x \cap A$ is contained in the symmetrization of $\ell_x \cap B$ for each $x \in A$.

The next theorem shows that the process of Steiner symmetrization preserves convex bodies.

<u>Theorem</u> 1.5.2. If K is a convex body and π is any plane, then $\overline{K}_\pi$ is also a convex body.

The proof of this theorem consists of three basic steps: (i) show that $\overline{K}_\pi$ is bounded with a nonempty interior; (ii) show that $\overline{K}_\pi$ is convex; and (iii) show that $\overline{K}_\pi$ is closed.

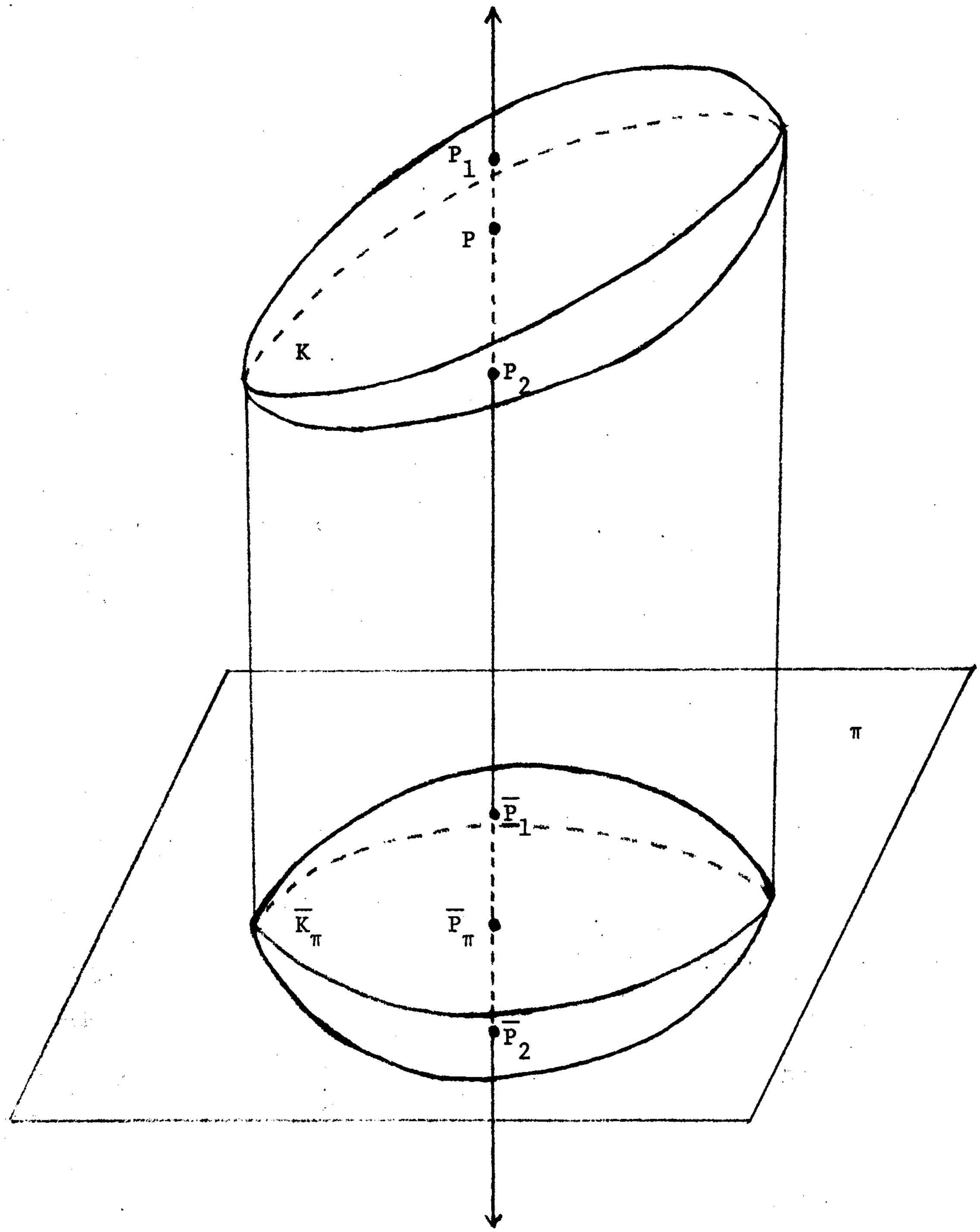


Figure 5. Steiner Symmetrization of K

(i). Let K be a bounded set with a nonempty interior. Then there exist balls $B(x_1,r_1)$ and $B(x_2,r_2)$ where $B(x_i,r_i)$ is the ball centered at x_i with radius r_i for $i = 1,2$, such that $B(x_1,r_1) \subset K \subset B(x_2,r_2)$. Theorem 1.5.1 implies, therefore, that $\overline{B(x_1,r_1)_\pi} \subset \overline{K}_\pi \subset \overline{B(x_2,r_2)_\pi}$. Furthermore, since $\overline{B(x_i,r_i)_\pi} = \overline{B(P_\pi(x_i)r_i)\pi}$, where $P_\pi(x_i)$ is the projection of x_i onto π, then $B(P_\pi(x_i),r_i) \subset \overline{K}_\pi \subset B(P_\pi(x_2),r_2)$. The set $\overline{K}_\pi$ is, therefore, bounded and has a nonempty interior.

(ii). Establish a coordinate system in E_3 so that π is the xy-coordinate plane and denote by $P_\pi(K)$ the projection of K onto π. The set K_π is convex since a projection is a linear transformation and the linear transform of a convex set is convex. Now for every $(x,y,0) \in P_\pi(K)$, define

$$\phi(x,y) = \sup\{z \mid (x,y,z) \in K\}$$

and

$$\psi(x,y) = \inf\{z \mid (x,y,z) \in K\}.$$

Since K is convex, the following argument will show that ψ is convex on $P_\pi(K)$:

Let $z_1 = \psi(\overline{V}_1)$, $z_2 = \psi(\overline{V}_2)$, where $\overline{V}_1,\overline{V}_2 \in P_\pi(K)$, and let λ be a real number such that $0 \leq \lambda \leq 1$. Then

$$\psi(\lambda\overline{V}_1 + (1-\lambda)\overline{V}_2) = \inf\{z \mid (\lambda\overline{V}_1 + (1-\lambda)\overline{V}_2;z) \in K\}.$$

Since $(\overline{V}_1;z_1) \in K$, $(\overline{V}_2;z_2) \in K$, and K is convex, then

$$\lambda(\overline{V}_1;z_1) + (1-\lambda)(\overline{V}_2;z_2) = (\lambda\overline{V}_1 + (1-\lambda)\overline{V}_2;\lambda z_1 + (1-\lambda)z_2) \in K.$$

This implies that

$$\psi(\lambda\overline{V}_1 + (1-\lambda)\overline{V}_2) \geq \lambda z_1 + (1-\lambda)z_2 = \lambda\psi(\overline{V}_1) + (1-\lambda)\psi(\overline{V}_2);$$

hence, ψ is convex on $P_\pi(K)$.

A similar argument shows that ϕ is concave on $P_\pi(K)$. Now, since the negative of a concave function is a convex function, the sum of two convex functions is a convex function, and a positive multiple of a convex function is a convex function, it follows that $(\psi - \phi)/2$ is a convex function. Similarly, $(\phi - \psi)/2$ may be shown to be a concave function.

Clearly,

$$\overline{K}_\pi = \{(x,y,z) \mid (x,y,0) \in P_\pi(K),\ (\psi - \phi)/2 \leq z \leq (\phi - \psi)/2\}.$$

The above set of a convex function and the below set of a concave function defined on a convex domain (such as K_π) are convex (cf. [16], p. 129). Since $\overline{K}_\pi$ is the intersection of two such sets, then $\overline{K}_\pi$ must be convex.

(iii). Since P_π is a linear transformation in E_3, and K is closed, it follows that $P_\pi(K)$ is closed. A convex function is continuous on the relative interior of its domain (cf. [2], p. 156-7). Since, by part (a), the interior of $P_\pi(K)$ is not empty, ϕ and ψ are continuous on the interior of $P_\pi(K)$. It will now be shown that ϕ and ψ are continuous on the boundary of $P_\pi(K)$:

For each $V = (x',y',0)$ on the boundary of $P_\pi(K)$ and each $U = (x,y,0)$ in the interior of $P_\pi(K)$, the interior of the interval UV is contained in the interior of $P_\pi(K)$ (cf. [16], p. 10). There exists a collinear sequence $U_n = (x_n,y_n,0)$ of points from the interval

UV such that $U_n \to V$. If $\phi(x_n,y_n) \to z'$ as $n \to \infty$, then, since K is closed, $(x',y',z') \in K$. Further, from the definition of ϕ, it follows that $\phi(x',y') \geq z'$. Suppose that $\phi(x',y') > z'$ and let B be a ball centered at (x',y',z') with a small enough radius so that B contains no points of the form $\lambda(x',y',\phi(x',y')) + (1-\lambda)(x,y,\phi(x,y))$ (cf. Figure 6). For sufficiently large n, $(x_n,y_n,\phi(x_n,y_n))$ lies inside B due to the convergence of the sequence $\{(x_n,y_n,\phi(x_n,y_n))\}$ to the center of B. From the definition of $\phi(x_n,y_n)$, however, it is clear that the segment joining $(x',y',\phi(x',y'))$ to $(x,y,\phi(x,y))$ fails to lie entirely inside K. This contradicts the convexity of K since K closed implies that K contains $(x,y,\phi(x,y))$ and $(x',y',\phi(x',y'))$. Hence, $\phi(x_n,y_n) \to z'$ as $n \to \infty$ and ϕ must, therefore, be continuous on $P_\pi(K)$. Now since

$$\overline{K}_\pi = \{(x,y,z) \mid (x,y,0) \in P_\pi(K),\ (\psi - \phi)/2 \leq z \leq (\phi - \psi)/2\},$$

where $P_\pi(K)$ is closed and $(\psi - \phi)/2$ and $(\phi - \psi)/2$ are continuous functions, it follows that $\overline{K}_\pi$ is closed. ∎

The next theorem shows that Steiner symmetrization preserves volume and does not increase surface area.

Theorem 1.5.3. If $\overline{K}_\pi$ is the symmetrical of the convex body K about the plane π, then (a) $V_3(\overline{K}_\pi) = V_3(K)$ and (b) $S_2(\overline{K}_\pi) \leq S_2(K)$, where (b) is a strict inequality if K is symmetric to no translate of π.

Proof: (a). If

$$K = \{(x,y,z) \mid (x,y,0) \in P_\pi(K),\ \psi(x,y) \leq z \leq \phi(x,y)\},$$

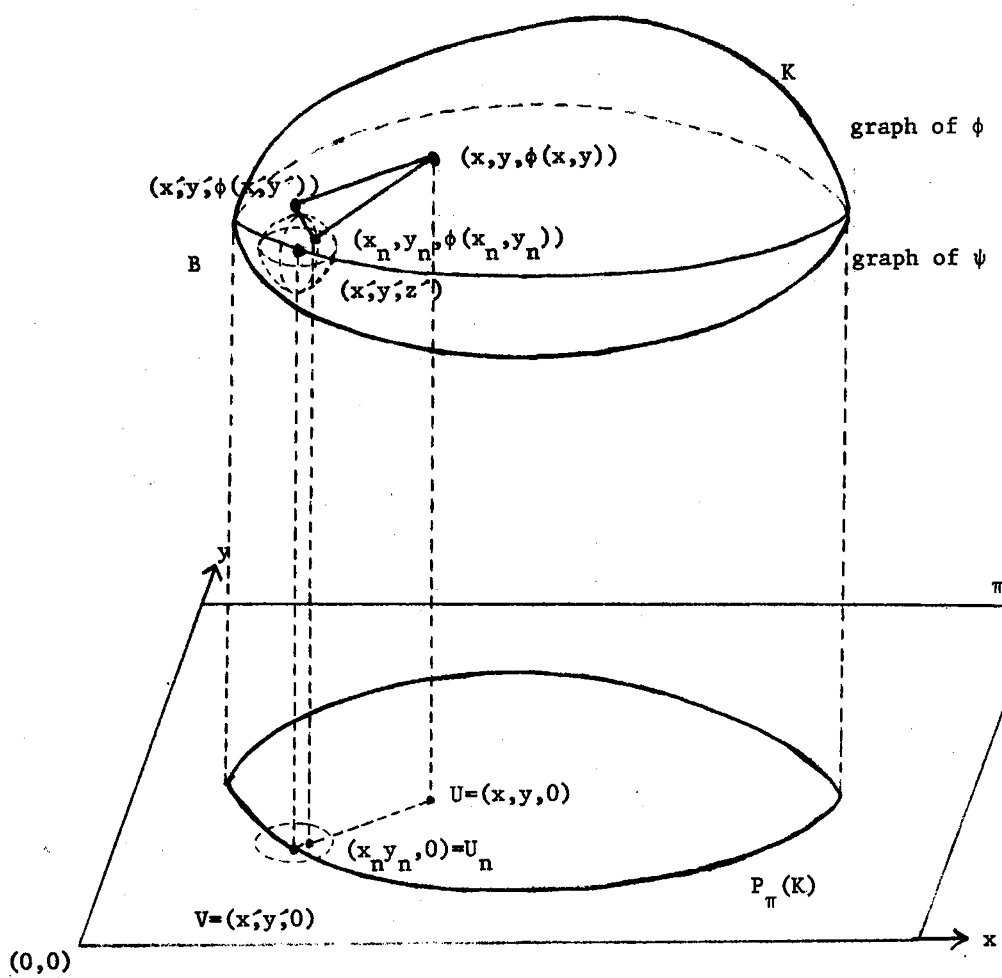


Figure 6. Continuity of ϕ on the Boundary of $P_\pi(K)$

then

$$\overline{K}_\pi = \{(x,y,z) \mid (x,y,0) \in P_\pi(K),$$
$$(\psi(x,y)-\phi(x,y))/2 \leq z \leq (\phi(x,y)-\psi(x,y))/2\}.$$

Therefore,

$$V_3(K) = \iint_{P_\pi(K)} (\phi - \psi)dxdy,$$

but

$$V_3(\overline{K}_\pi) = \iint_{P_\pi(K)} [(\phi - \psi)/2 - (\psi - \phi)/2]dxdy$$
$$= \iint_{P_\pi(K)} (\phi - \psi)dxdy,$$

so $V_3(\overline{K}_\pi) = V_3(K)$.

(b). In view of the above representations for K and $\overline{K}_\pi$, it follows that:

$$S_2(K) = \iint_{P_\pi(K)} \left[\sqrt{1 + \phi_x^2 + \phi_y^2} + \sqrt{1 + \psi_x^2 + \psi_y^2}\right]dxdy + \int_\Gamma (\phi - \psi)ds,$$

where $\Gamma = \text{bd } P_\pi(K)$ and s = arc length along Γ;

$$S_2(\overline{K}_\pi) = 2 \iint_{P_\pi(K)} \sqrt{1 + (\phi_x - \psi_x)^2/4 + (\phi_y - \psi_y)^2/4}\ dxdy$$
$$+ \int_\Gamma \left[(\phi - \psi)/2 - (\psi - \phi)/2\right] ds,$$

where Γ and s are defined as above. Then,

$$S_2(K) - S_2(\overline{K}_\pi) =$$

$$\iint_{P_\pi(K)} \left[\sqrt{1 + \phi_x^2 + \phi_y^2} + \sqrt{1 + \psi_x^2 + \psi_y^2} - 2\sqrt{1 + (\phi_x - \psi_x)^2/4 + (\phi_y - \psi_y)^2/4} \right] dA,$$

which is nonnegative if the integrand is nonnegative. To establish this, begin with the following inequality:

$$(p_1 + q_1)^2 + (p_2 + q_2)^2 + (p_1q_2 - p_2q_1)^2 \geq 0. \qquad (1.5.1)$$

So

$$p_1^2 + 2p_1q_1 + q_1^2 + p_2^2 + 2p_2q_2 + q_2^2 + p_1^2q_2^2 - 2p_1p_2q_1q_2 + p_2^2q_1^2 \geq 0,$$

and hence,

$$1 + q_1^2 + q_2^2 + p_1^2 + p_1^2q_1^2 + p_1^2q_2^2 + p_2^2 + p_2^2q_1^2 + p_2^2q_2^2 \geq 1 + 2p_1q_1 + 2p_2q_2 + p_1^2q_1^2 + 2p_1p_2q_1q_2 + p_2^2q_2^2.$$

Factoring yields

$$(1 + p_1^2 + p_2^2)(1 + q_1^2 + q_2^2) \geq (1 + p_1q_1 - p_2q_2)^2.$$

Hence,

$$2\sqrt{(1 + p_1^2 + p_2^2)(1 + q_1^2 + q_2^2)} \geq 2(1 - p_1q_1 - p_2q_2).$$

Adding $2 + p_1^2 + p_2^2 + q_1^2 + q_2^2$ to both sides gives:

$$2 + p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2\sqrt{(1 + p_1^2 + p_2^2)(1 + q_1^2 + q_2^2)}$$

$$\geq 4 + p_1^2 - 2p_1q_1 + q_1^2 + p_2^2 - 2p_2q_2 + q_2^2 ,$$

or

$$(1 + p_1^2 + p_2^2) + (1 + q_1^2 + q_2^2) + 2\sqrt{(1 + p_1^2 + p_2^2)(1 + q_1^2 + q_2^2)}$$

$$\geq 4 + (p_1 - q_1)^2 + (p_2 - q_2)^2 ,$$

or equivalently

$$\left(\sqrt{1 + p_1^2 + p_2^2} + \sqrt{1 + q_1^2 + q_2^2}\right)^2 \geq \left(2\sqrt{1 + (p_1 - q_1)^2/4 + (p_2 - q_2)^2/4}\right)^2.$$

Therefore,

$$\sqrt{1 + p_1^2 + p_2^2} + \sqrt{1 + q_1^2 + q_2^2} \geq 2\sqrt{1 + (p_1 - q_1)^2/4 + (p_2 - q_2)^2/4} \quad .$$

If $p_1 = \phi_x$, $p_2 = \phi_y$, $q_1 = \psi_x$ and $q_2 = \psi_y$, then it is readily seen that the integral associated with $S_2(K) - S_2(\overline{K}_\pi)$ has a nonnegative integrand. From the remarks above it is clear, therefore, that

$$S_2(K) \geq S_2(\overline{K}_\pi). \tag{1.5.2}$$

The equality (1.5.2) holds only in the case where (1.5.1) is an equality, that is, when $p_1 = -q_1$ and $p_2 = -q_2$, or, equivalently, when $\phi_x = -\psi_x$ and $\phi_y = -\psi_y$. A stronger result may, therefore, be derived from (1.5.1). The integral of a continuous nonnegative function is zero only when the function is identically zero. Hence, $S_2(K) = S_2(\overline{K}_\pi)$ precisely when $\phi_x = -\psi_x$ and $\phi_y = -\psi_y$. However, if

$\phi_x = -\psi_x$ then $\phi = -\psi + f(y)$. Furthermore, $\phi_y = -\psi_y$ implies that $f'(y) = 0$ and $f(y) = 2c$ for some choice of the constant c. Therefore, $\phi = -\psi + 2c$ or $\phi - c = -(\psi - c)$; hence, the boundary of K is symmetric with respect to the plane $z = c$, a translate of π. ■

Solution of the Isoperimetric Problem for $n = 3$

Since the plane of symmetrization π in Theorem 1.5.3 was arbitrarily chosen, it follows that only those convex bodies which are symmetric with respect to some translate of each plane may never have their surface areas decreased under the process of Steiner symmetrization. The following lemma asserts that each cross section containing interior points of such a body must be a disc.

Lemma 1.5.4. If K is a compact convex body which is symmetric with respect to some translate of each plane, then each cross section of K containing interior points of K must be a closed disc.

Proof: Let K_π denote $K \cap \pi$ for any plane π meeting the interior of K. If L is a line in π, then K must be symmetric with respect to a translate π_L' of the plane π_L (cf. Figure 7). Let $L' = \pi \cap \pi_L'$ and let c be the midpoint of $L' \cap K_\pi$. Then K_π is symmetric with respect to L'. Furthermore, K_π, hence also $K_\pi \cap L'$, is symmetric with respect to some translate M' in π of any line M in π which is perpendicular to L'. Since c must belong to $M' \cap K_\pi$ and M' is perpendicular to L', then $L' \cap M' = c$ and c is also the midpoint of $M' \cap K_\pi$ since $M' \cap K_\pi$ must be symmetric with respect to L'.

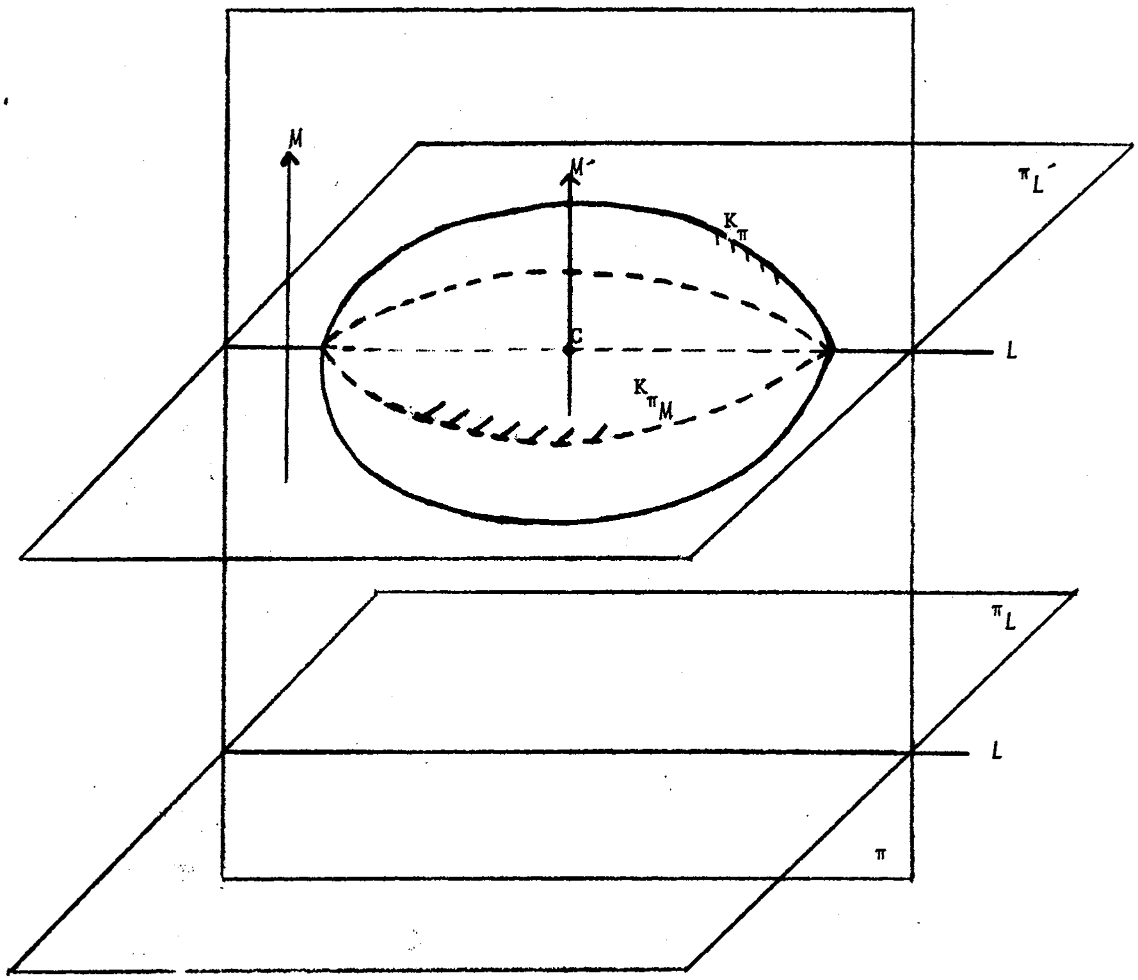
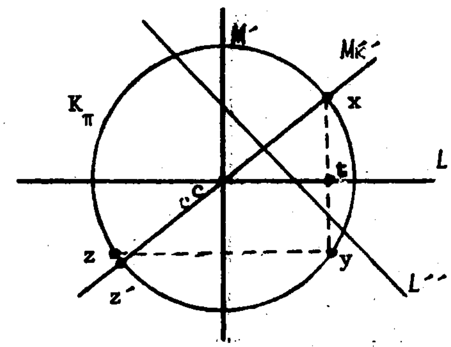


Figure 7. Compact Convex Body K Is Symmetric with respect to Plane $\pi_L{}'$ and K_π Is Symmetric with respect to Lines $L^{\perp}$ and M'^{π}

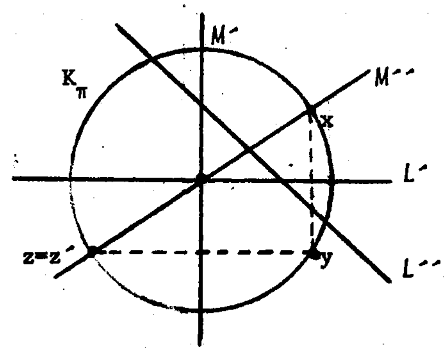
The following arguments will show that (i) any line L'' in π is a line of symmetry for K_π if and only if c is the midpoint of $L'' \cap K_\pi$; and (ii) length$(K_\pi \cap L_1)$ = length$(K_\pi \cap L_2)$, where L_1 and L_2 are arbitrary lines of symmetry for K_π:

(i). Let L'' be a third line of symmetry for K_π and construct a line M'' through c and perpendicular to L''. Let x and z′ be the endpoints of the segment $L \cap \mathrm{bd}(K_\pi)$. Denote by y the reflection of x through L' and by z the reflection of y through M' (cf. Figure 8(a)). Let t be the intersection of the line L' with the line segment xy. Then, from the similarity of triangles Δxct and Δxzy, c is the midpoint of the segment xz. The points x and c determine a unique line M'', hence z = z′ and c is the midpoint of $M'' \cap K_\pi$ (cf. Figure 8(b)). Since z is the symmetric image of x with respect to L'', it follows that $c \in L_3$. Also since some translate of any line in π is a line of symmetry for K_π and any line of symmetry for K_π must pass through c, then any line L'' in π is a line of symmetry for K_π if and only if c is the midpoint of $L'' \cap K_\pi$ (cf. Figure 8(c)).

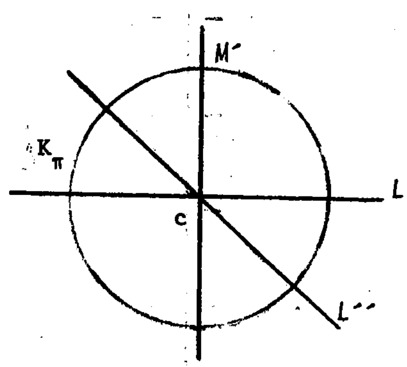
(ii). Construct K_π with distinct lines of symmetry L_1 and L_2. Clearly, from (i), $L_1 \cap L_2$ = c. Let p be an endpoint of $L_1 \cap \mathrm{bd}(K_\pi)$; let q be an endpoint of $L_2 \cap \mathrm{bd}(K_\pi)$, and let t be the midpoint of segment pq. Denote by L_3 the perpendicular bisector of pq (cf. Figure 9). Then $t \in L_3$ and K_π is symmetric with respect to some translate of L_3. Hence, K_π is symmetric with respect to L_3 since L_3 is the only member of its family of parallel lines with respect to which q is the symmetric image of p. Therefore, by part



(a) Points y and z Are the Reflections of Points x and y through L' and M', respectively



(b) Point c is the Midpoint of $M'' \cap K_\pi$



(c) Point c is the Midpoint of $L^* \cap K_\pi$

Figure 8. Convex Body K_π Is Symmetric with respect to Line L

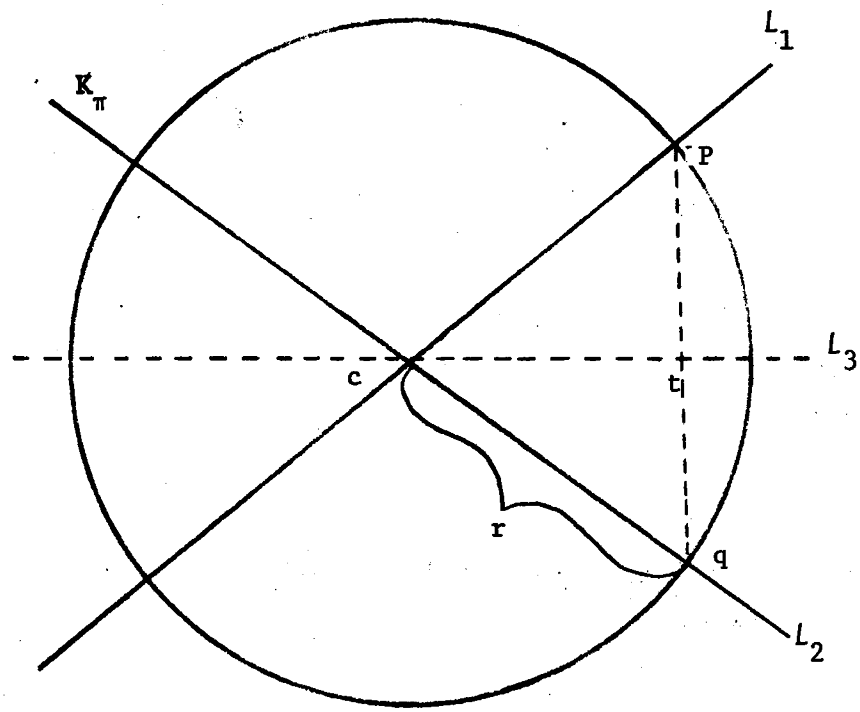


Figure 9. Each Cross-section K_π Containing Interior Points of K Is a Disc

(i) $c \in L_3$. Now, by (i), the line perpendicular to L_1 at c is a line of symmetry so

$$\text{length}(K_\pi \cap L_1) = 2|cp| = 2\sqrt{|ct|^2 + |tp|^2}$$

and similarly,

$$\text{length}(K_\pi \cap L_2) = 2|cq| = 2\sqrt{|ct|^2 + |tq|^2}.$$

However, by assumption $|tp| = |tq|$, so

$$\text{length}(K_\pi \cap L_1) = \text{length}(K_\pi \cap L_2).$$

Denote their common value by $2r$.

Hence, since L_1 and L_2 are arbitrary lines of symmetrization,

$$K_\pi = \{p \in E_3\colon\ p \in \pi,\ \ d(p,c) \leq r\},\ \ \text{a disc.}\ \blacksquare$$

The next theorem will show that K is, in fact, a ball.

<u>Theorem</u> 1.5.5. If K is a compact convex body which is symmetric with respect to some translate of each plane in E_3, then K is a ball.

Proof: Let K be as in the theorem and let π be a plane symmetrization for K. Each line in π determines a plane containing that line which is perpendicular to π. The convex body K is, by hypothesis, symmetric with respect to a translate π_L of such a plane. Then K_π is symmetric with respect to the line $L = \pi_L \cap \pi$. Since π is a plane of symmetrization of K, π contains interior points of K. Hence, by Lemma 1.5.4, since K_π is symmetric with respect to the translate L of an arbitrary line in π, K_π is a disc (cf. Figure 10). Also, from the proof of (i) in Lemma 1.5.4, the center of K_π is the

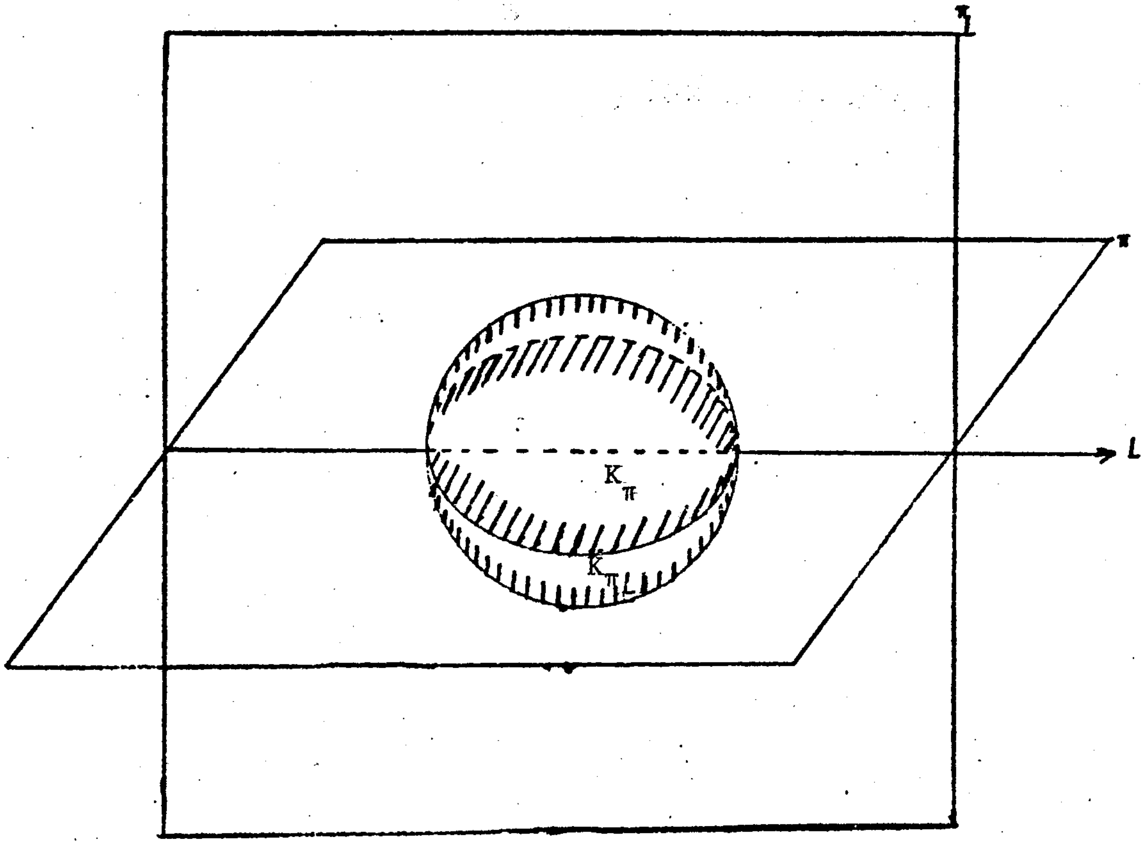


Figure 10. A Ball Is the Solution for n = 3

midpoint c of $L \cap K_\pi$.

Since K_{π_L} is, by assumption, a plane of symmetrization for K, the roles of π and π_L may be reversed in the preceding argument. The resulting conclusion is that K_{π_L} is also a disc whose center is the midpoint of $L \cap K_{\pi_L}$. Now, since $L \cap K_\pi = L \cap K_{\pi_L}$, K_π and K_{π_L} are discs centered at c with a common diameter. Hence, they must have equal radii r and must, therefore, be congruent. Now, since exactly one translate of a given plane contains c, an arbitrary plane of symmetry for K perpendicular to π contains c. Also, since K is symmetric with respect to some translate of each plane, then K is symmetric with respect to a plane π_M if and only if $c \in \pi_M$. Hence, if π_M is a plane perpendicular to π through c, then K_{π_M} is a disc of radius r centered at c. It now follows that K is a ball of radius r centered at c. ■

In view of equation (1.1.2) and the equivalence of the various formulations of the classical isoperimetric problem, the next theorem combines the Existence Theorem (Theorem 1.3.2) with the results of this section into a single summary result: A ball is the solution to the classical isoperimetric problem for $n = 3$.

<u>Theorem</u> 1.5.5. Among the compact convex bodies of volume one in E_3, a set whose surface area has the minimum possible measure must be a ball.

Proof: Denote by C_r the collection of all compact convex bodies of volume one in E_3 bounded by the sphere S_r of radius r and centered at the origin. In view of the equivalent formulations of the classical isoperimetric problem, the Existence Theorem implies that a member K of each nonempty collection $\mathcal{C}_r$ exists which has the least possible surface area. Suppose that K is not a ball, then Theorem 1.5.5 implies that a plane π containing the center of S_r exists none of whose translates is a plane of symmetrization for K. Theorems 1.5.2 and 1.5.3 then imply that the Steiner symmetrical $\overline{K}_\pi$ of K with respect to π is again a convex body of volume one but with a smaller surface area than that of K. Then Theorem 1.5.1 implies that $\overline{K}_\pi \subset (\overline{S}_r)_\pi = S_r$, so K cannot be a minimizing body in S_r. Hence, a ball of volume one is the unique (up to a translation) minimizing body in each of the collections $\mathcal{C}_r$ to which it belongs. Now, since every compact convex body is bounded by some sphere and the same convex body (up to a translation) is minimal for each collection $\mathcal{C}_r$ which contains it, clearly this minimizing body must be minimal on the entire unbounded collection. ■

Appropriate generalizations of the above theorems lead to the result that the solution to the isoperimetric problem in E_n is an n-dimensional ball.

CHAPTER II

ISOPERIMETRIC PROBLEMS WITH CONSTRAINTS

This chapter will concern itself with some applications of the results from the preceding chapter to problems which bear a close relationship to the classical isoperimetric problem. In each case, the problem may be viewed as maximizing the area (volume) enclosed by a curve (surface) of fixed length (area). The problems differ from the classical isoperimetric problem, however, in that certain additional constraints are imposed on the bounding curve (surface).

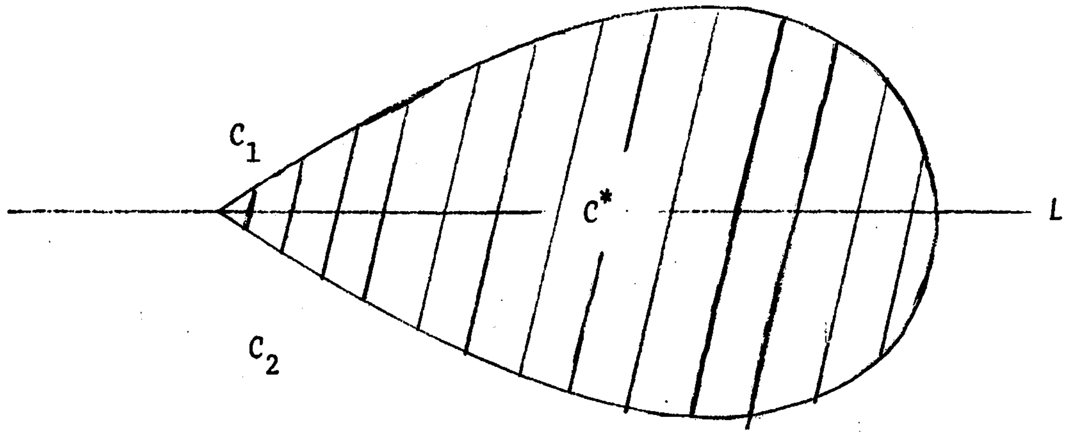
1. Dido's Problem

Dido's problem is suggested by the following story concerning the mythological Queen Dido of Carthage, from whom the problem derives its name: A North African tribal chieftan promised Dido "as much land as might lie within the bounds of a bull's hide." Being somewhat greedy, Dido was determined to enclose as much land with the bull's hide as she possibly could. Dido proceeded, therefore, to cut the bullhide into narrow strips and join them together end to end, thereby forming a long rope of a fixed length ℓ. She was then faced with the problem of placing her rope so as to maximize the area bounded by the rope and the (straight) seashore. History does not record her decision, but the next theorem will show that the optimal placement is in the shape of a semicircle of radius ℓ/π whose diameter lies along the seashore.

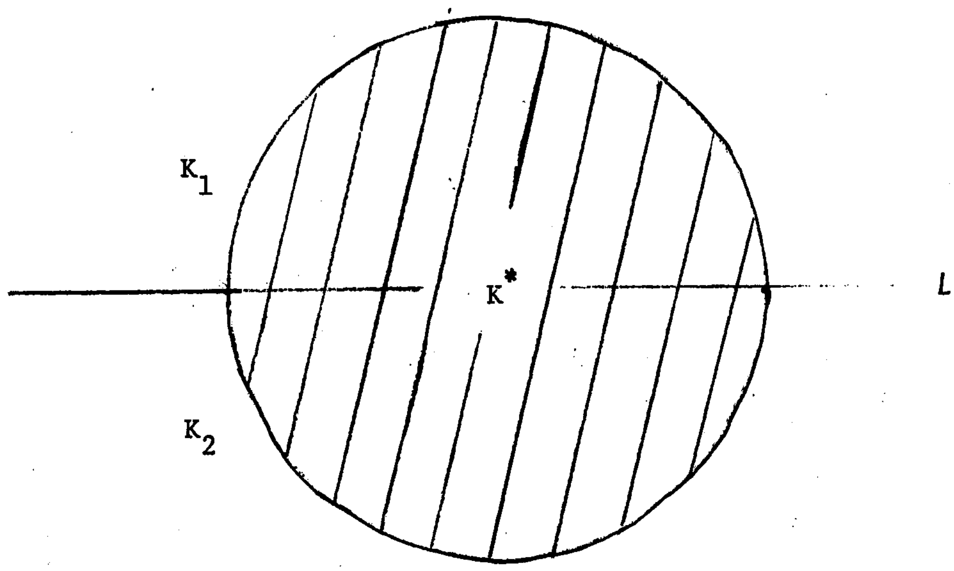
Theorem 2.1.1. The maximum area bounded by a given line L and a simple curve C of length ℓ with endpoints on the line L is attained when and only when C is in the shape of a semicircle of radius ℓ/π whose diameter lies along L.

Proof: Let C_1 be a curve of length ℓ having its endpoints on the line L. Denote by C_2 the reflection of C_1 about L. Similarly, let K_1 be a semicircular arc of length ℓ whose diameter lies on L, and denote by K_2 its reflection about L. Let C^* and K^* be the regions bounded by $C_1 \cup C_2$ and $K_1 \cup K_2$, respectively (cf. Figure 11). Then the isoperimetric property of a disc implies that $\text{area}(K^*) \geq \text{area}(C^*)$, with equality if and only if $C_1 \cup C_2$ is a circle. Since the area bounded by K_1 and L is half that of K^* and the area bounded by C_1 and L is half that of C^*, the maximum area is obtained when $C = K_1$, a semicircle of radius ℓ/π whose diameter lies along L. ■

A related problem for $n = 3$ is suggested by the consideration of the following set of circumstances: A farmer desires to construct a closed grain storage bin from an unlimited supply of non-rustproof materials which may be shaped into any form. He has only a limited supply of rustproofing paint, however, which must be applied at a given minimal thickness on all but the flat, unexposed base of the completed bin in order to insure its resistance to rust. A question then arises concerning the shape of the weather-resistant bin of greatest volume which the farmer may construct. That the optimal shape of the surface to be painted is hemispherical follows from the theorem below.



(a) The Region C^* Is Bounded By C_1 and Its Reflection C_2 About Line L



(b) The Disc K^* Is Bounded by K_1 and Its Reflection K_2 About the Line L

Figure 11. The Solution to Dido's Problem

Theorem 2.1.2. The volume bounded by a given plane π and a surface S of fixed area a whose boundary lies in π is maximized when S is hemispherical.

The proof of this theorem is the exact analogue, for $n = 3$, of the proof given for Theorem 2.1.1, in which $n = 2$. This argument may be generalized to show that the solution to Dido's problem in the n-dimensional setting is an n-hemisphere of the required (n-1)-dimensional surface area. Alternate formulations of the isoperimetric problem imply that analogus alternate formulations of Dido's problem are also possible. For example, which among the (n-1)-dimensional surfaces bounding a region of volume v with the plane π containing their boundaries, has (have) least surface area?

The solution to Dido's problem for $n = 3$ is undoubtedly accountable for the hemispherical shape of a bowl, since the contents of the bowl (assumed full to its top, a level plane) is maximized for a given surface area when the surface is hemispherical. This phenomenon also motivates the hemispherical shape of greenhouses, large sports arenas, Eskimo igloos, Navajo hogans, geodesic domes, etc.

A new, but related problem results when the endpoints of the curve Γ described in the formulation of Dido's problem are regarded as fixed on the line L. The following theorem will show that its solution is not, in general, the same as that for Dido's problem.

Theorem 2.1.3. Let p_1 and p_2 be points in E_2 with respective coordinates $(-a,0)$ and $(a,0)$ and let C be the collection of simple, convex, plane curves Γ of fixed length $\ell > 2a$ whose endpoints are p_1 and p_2. Then any member Γ_0 of C which bounds the maximum possible area with p_1p_2 is the arc of length ℓ cut off by chord p_1p_2 from the circle C_0 determined by p_1p_2 and ℓ (cf. Figure 12(a)).

Proof: Let $C(b) = \{(x,y) \in E_2\colon x^2 + (y-b)^2 = a^2 + b^2\}$, and let $C^+(b) = \{(x,y) \in C(b)\colon y \geq 0\}$ and $C^-(b) = \{(x,y) \in C(b)\colon y < 0\}$. Denote by $P(b)$ and $N(b)$ the regions whose boundaries are $C^+(b) \cup I(a)$ and $C^-(b) \cup I(a)$, respectively, where $I(a) = \{(x,0) \in E_2\colon -a < x < a\}$ (cf. Figure 12(a)).

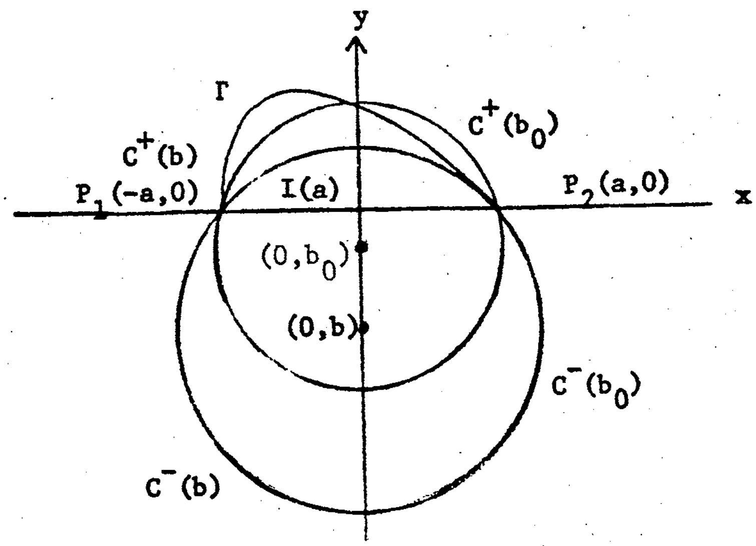
Since

$$\text{length}(C^+(b)) = \begin{cases} 2\sqrt{a^2+b^2}\,\arctan(a/|b|)\,, & \text{if } b < 0, \\ \pi a \quad \text{if } b = 0, & \\ 2\sqrt{a^2+b^2}\,(\pi - \arctan(a/b))\,, & \text{if } b > 0, \end{cases}$$

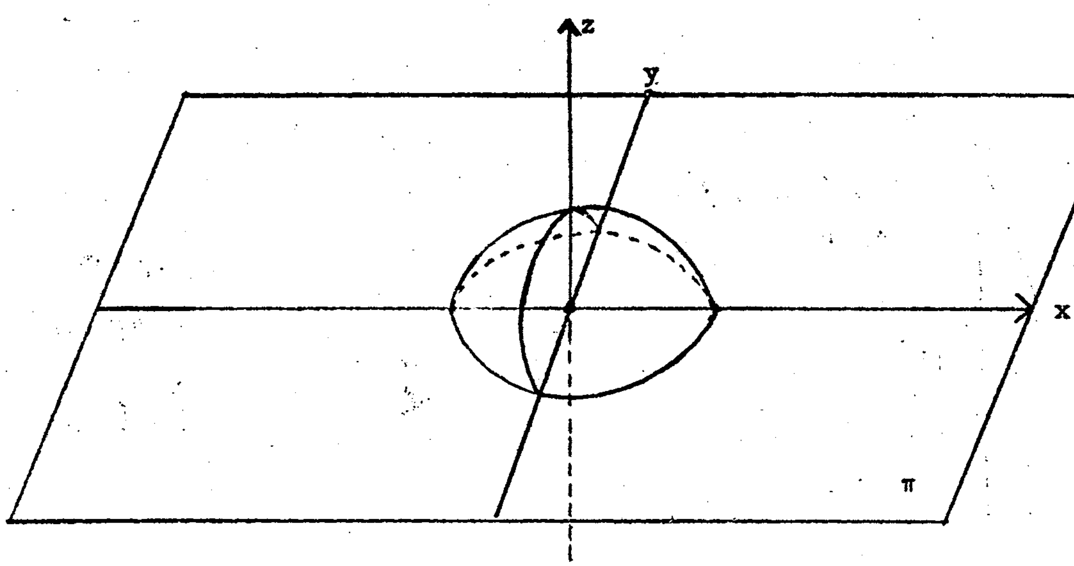
then length $(C^+(b))$ is an increasing function bounded below by $2a$ but unbounded above. Hence, the intermediate value theorem guarantees the existence of a number b_0 such that $\text{length}(C^+(b_0)) = \text{length}(\Gamma_0) = \ell > 2a$. Now, since $P(b_0) \cup N(b_0)$ is a disc whose circumference is $\text{length}(\Gamma_0) + \text{length } C^-(b_0)$, the isoperimetric property of a circle implies that the area of $P(b_0)$ is greater than or equal to the area of the region bounded by Γ_0 and $I(a)$, with equality only when Γ_0 coincides with $C^+(b_0)$. Hence, Γ_0 is the arc of length ℓ cut off from the circle centered at $(0,b_0)$ by the chord p_1p_2. ∎

The solution to an analogus problem for $n = 3$ is provided by the following theorem.

<u>Theorem</u> 2.1.4. Let $C = \{(x,y,z) \in E_3\colon x^2 + y^2 = r^2,\ z = 0\}$ and let π be the plane containing C; hence, $\pi = \{(x,y,z) \in E_3\colon z = 0\}$. Let



(a) Dido's Problem for n=2 with Fixed Endpoints



(b) Dido's Problem for n=3 with Fixed Circular Boundary

Figure 12. Dido's Problem with Fixed Boundary

$\mathcal{C}$ be the collection of simple, convex surfaces in E_3 of area a where $a > \pi r^2$, whose common boundary is the circle C. Then any member Γ_0 of $\mathcal{C}$ which bounds with π the maximum possible volume is that portion of the sphere determined by C and a which is cut off by π (cf. Figure 12(b)).

The obvious analogy between Theorem 2.1.3 and Theorem 2.1.4 carries over into their proofs. The proof of Theorem 2.1.4 will, therefore, be omitted.

2. Dido's Problem with Intervening Points of Attachment

In this section, variations of Dido's problem for $n = 2$ with fixed endpoints are considered. The curves in the collection $\mathcal{C}$ under consideration are required to contain intervening points of attachment between the endpoints. In particular, the following problem will ultimately be solved in this section: Among the curves in E_2 with endpoints at p_1 and p_k, containing the intervening points $p_2, p_3, \ldots, p_{k-1}$ lying in that order on the line L determined by p_1 and p_k, and having length $\ell > |p_1 p_k|$, which bound(s) with L the greatest possible area? The following less general problem will lead to its solution.

<u>Problem</u> 2.2.1. A rope (idealized curve) of length ℓ is attached to the ends (idealized points p_1 and p_3) of a beam (idealized segment of the line L) and must pass through a ring (idealized point p_2 on the segment $p_1 p_3$) attached to the beam. If the rope must lie in a given

halfplane bounded by L, which placement of the rope bounds the greatest possible area with the beam?

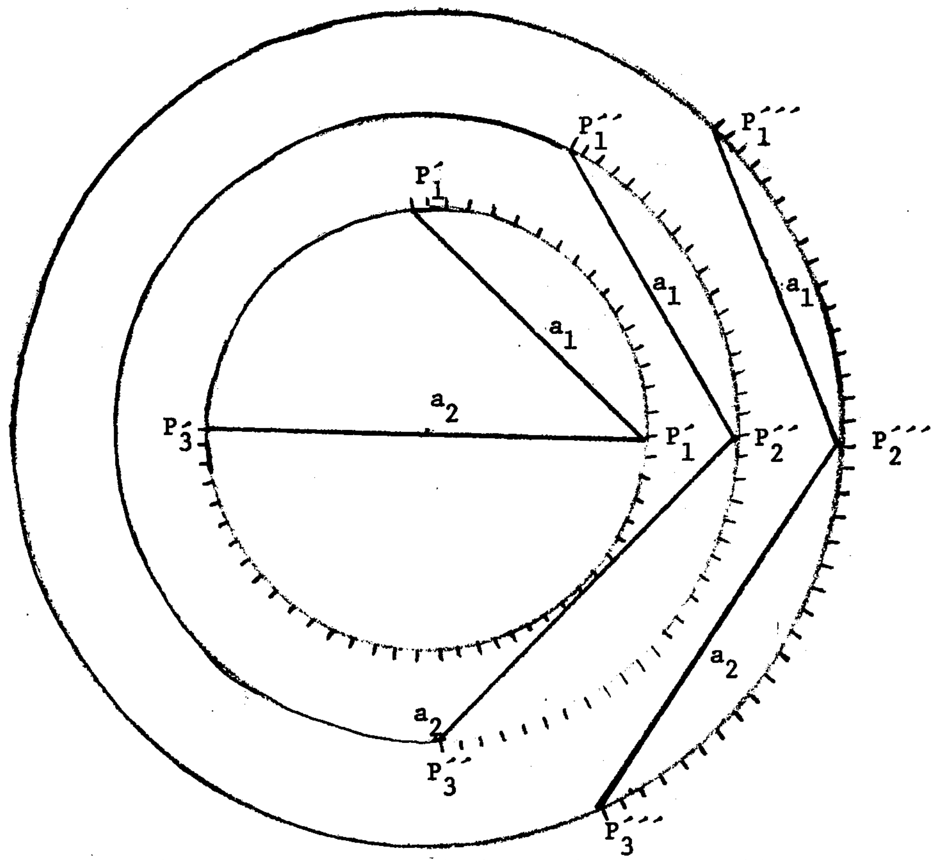
Solution: The argument below will show that the solution to the problem is the union of two circular arcs of the same radius.

(i). The first step of the argument is to show that there exists a traingle $T(d_0) = p_1'p_2'p_3'$ and a circle $C(d_0)$ (of diameter d_0) such that (a) $T(d_0)$ is inscribed in $C(d_0)$, (b) $|p_1'p_2'| = |p_1p_2|$ and $|p_2'p_3'| = |p_2p_3|$, and (c) $|\widehat{p_1'p_2'}| + |\widehat{p_2'p_3'}| = \ell$, where $\widehat{p_i'p_j'}$ is an arc of $C(d_0)$ joining p_i' to p_j' and $|\widehat{p_i'p_j'}|$ is its length.

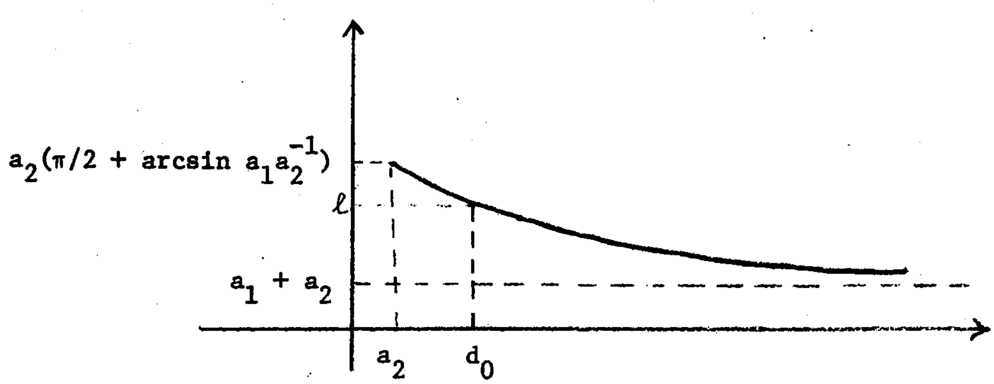
Clearly, if $d \geq \max\{|p_1'p_2'|,|p_2'p_3'|\}$ then a triangle $T(d) = p_1'p_2'p_3'$ satisfying (b) may be inscribed in a circle $C(d)$ (of diameter d). It will now be shown that a number d_0 exists so that $T(d_0)$ and $C(d_0)$ satisfy (b) and (c) simultaneously.

Let $a_1 = |p_1'p_2'|$ and $a_2 = |p_2'p_3'|$ and assume, without loss of generality, that $a_2 \geq a_1$.

Case 1. If $\ell \leq a_2(\pi/2 + \arcsin a_1/a_2)$ then for $d \geq a_2$ define the function $s(d) = d(\sum_{i=1}^{2} \arcsin a_i/d)$, i.e., $s(d)$ is the sum of the lengths of the minor arcs of $C(d)$ cut off by the chords of length a_1 and a_2. Clearly, the sum of these arc lengths decreases as d increases; hence, s is a decreasing function of d (cf. Figure 13(a)). Also note that $s(a_2) = a_2(\pi/2 + \arcsin a_1a_2)$ and $\lim_{d \to \infty} s(d) = a_1 + a_2$. Hence, since s is continuous on $[a_2,\infty)$ and $a_1 + a_2 < \ell \leq (\pi/2 + \arcsin a_1/a_2)$, the intermediate value theorem guarantees the existence of a number $d_0 \geq a_2$ such that $s(d_0) = \ell$ (cf. Figure 13(b)). Therefore, conditions (a) and (b) are simultaneously satisfied by the triangle $T(d_0)$ and the circle $C(d_0)$.



(a) The Function $s(d)$ is a Decreasing Function of d



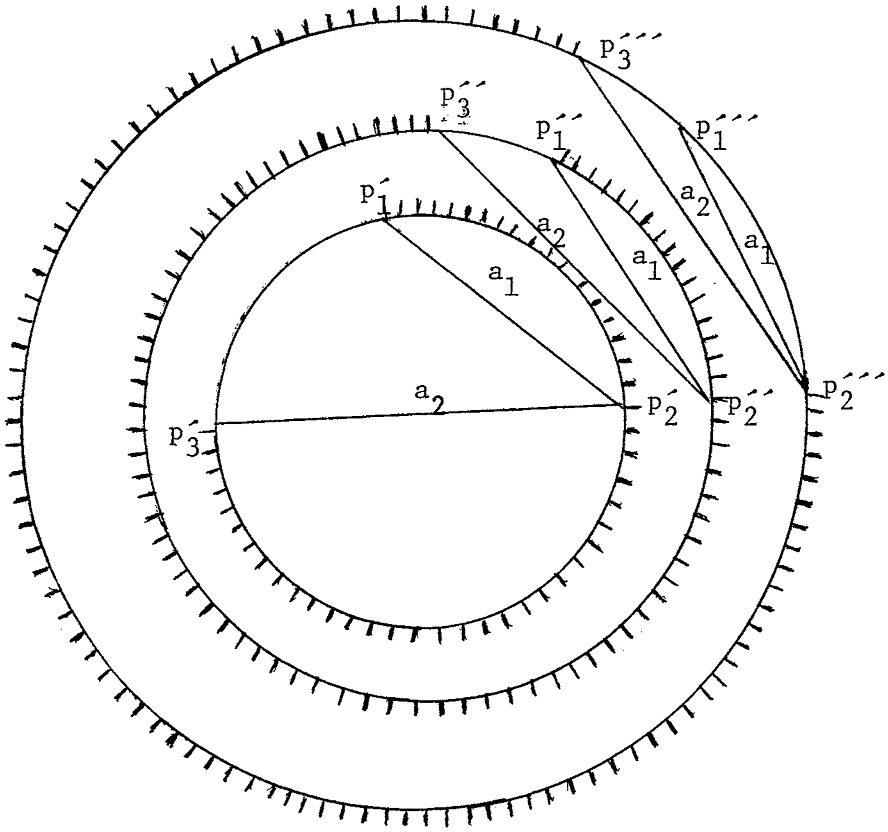
(b) The Graph of $s(d)$ on $[a_2,\infty)$

Figure 13. Existence of the Number b_0 for Case 1

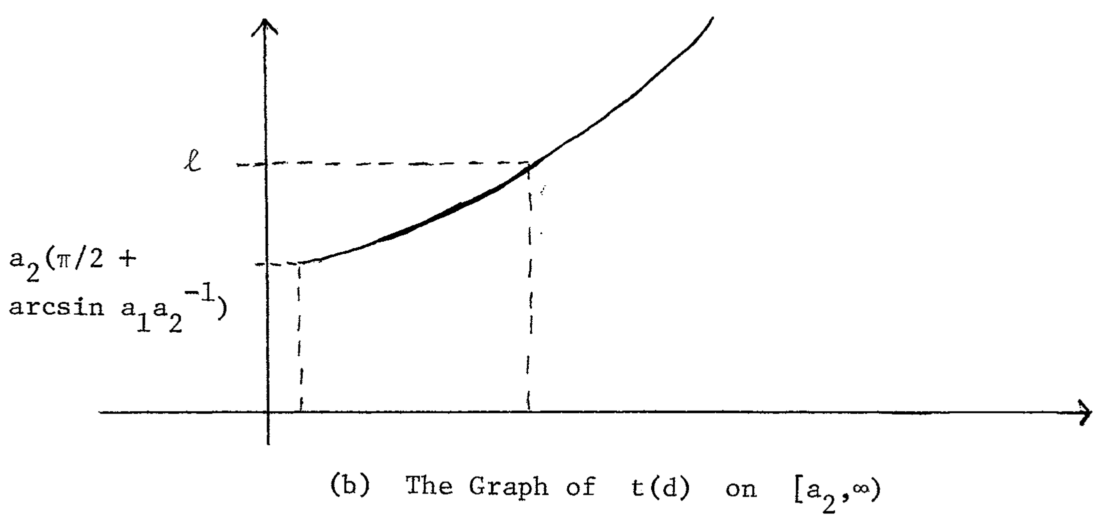
Case 2. If $\ell > a_2(\pi/2 + \arcsin a_1/a_2)$ then for $d \geq a_2$, define the function $t(d) = d(\arcsin a_1/d + \pi - \arcsin a_2/d)$; i.e., $t(d)$ is the sum of the lengths of the minor arc of C_d cut off by a chord of length a_1 and the major arc of C_d cut of by a chord of length a_2. Clearly, the sum of these arc lengths increases as d increases, so t is an increasing function of d (cf. Figure 14(a)). Also, note that $t(a_2) = a_2(\pi/2 + \arcsin a_1/a_2)$ and $t(d) \to \infty$ as $d \to \infty$. Hence, since t is continuous on $[a_2,\infty)$ and $\ell > a_2(\pi/2 + \arcsin a_1/a_2)$, the intermediate value theorem guarantees the existence of a number $d_0 \geq a_2$ such that $t(d_0) = \ell$ (cf. Figure 14(b)). Therefore, conditions (a) and (b) are simultaneously satisfied by the triangle $T(d_0)$ and the circle $C(d_0)$.

(ii). The following is the second and final step of the argument: Let p_1, p_2 and p_3 be collinear points for which arcs $\widehat{p_1p_2}$ and $\widehat{p_2p_3}$ are congruent to arcs $\widehat{p_1'p_2'}$ and $\widehat{p_2'p_3'}$ from $C(d_0)$, respectively. Now it will be shown that the curve $(\widehat{p_1p_2}) \bigcup (\widehat{p_2p_3})$ bounds with p_1p_3 greater area than does any other simple curve Γ of length ℓ, containing p_2, and with p_1 and p_3 as endpoints.

Suppose Γ is such a curve and is hinged at p_2 (cf. Figure 15(a)). Rigidly move Γ so that p_1 and p_2 coincide with p_1' and p_2', respectively, and rotate the arc of Γ between p_2 and p_3 about p_2 until p_3 coincides with p_3'. Denote Γ in this new position by Γ'. Let $\widehat{p_3'p_1'}$ be the arc of $C(d_0)$ joining p_3' to p_1' and not containing p_2' (cf. Figure 15(b)). Then the length of $\Gamma' \bigcup (\widehat{p_3'p_1'})$ is $\ell + |\widehat{p_3'p_1'}|$ (the circumference of $C(d_0)$, where circle $C(d_0) =$ $(\widehat{p_1'p_2'}) \bigcup (\widehat{p_2'p_3'}) \bigcup (\widehat{p_3'p_1'})$). The isoperimetric property of the circle

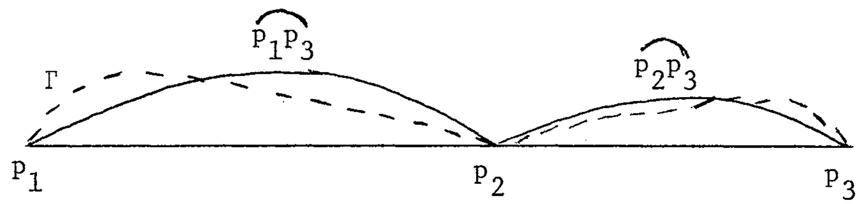


(a) The Function $t(d)$ Is an Increasing Function of d

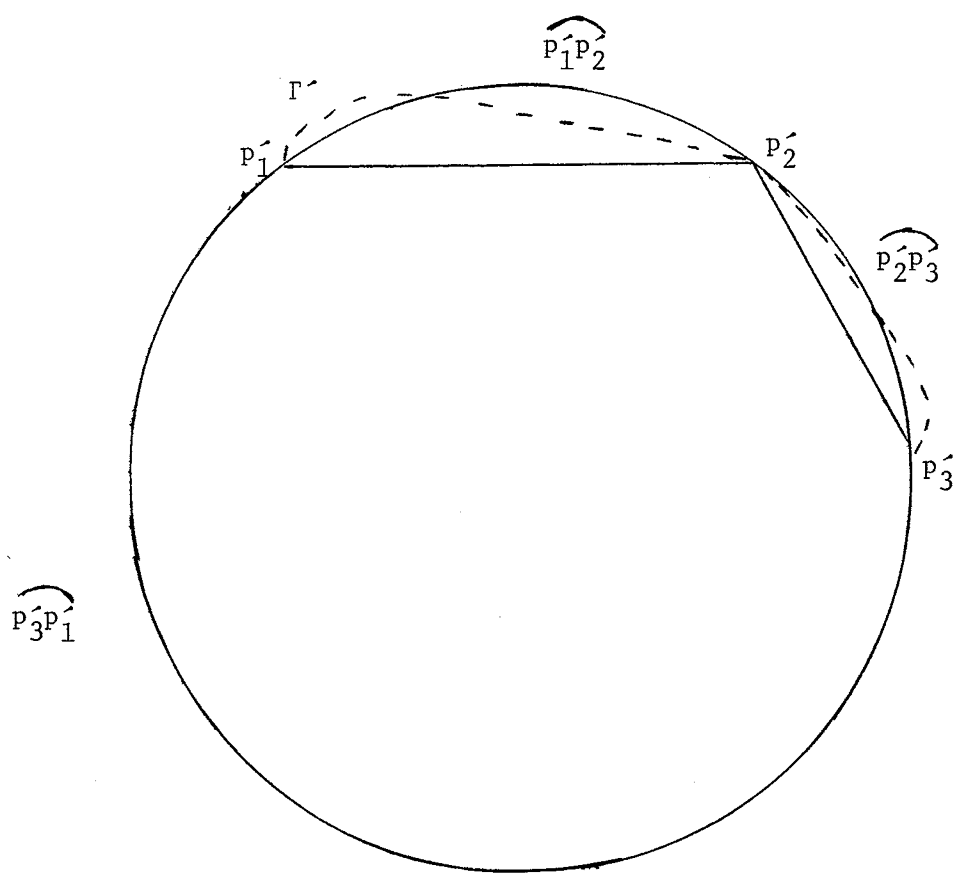


(b) The Graph of $t(d)$ on $[a_2, \infty)$

Figure 14. Existence of the Number b_0 for Case 2



(a) The Optimal Curve is $(\widehat{p_1p_2}) \bigcup (\widehat{p_2p_3})$



(b) The Isoperimetric Property of a Circle

Figure 15. The Optimal Curve Is the Union of Circular Arcs of Radius d_0

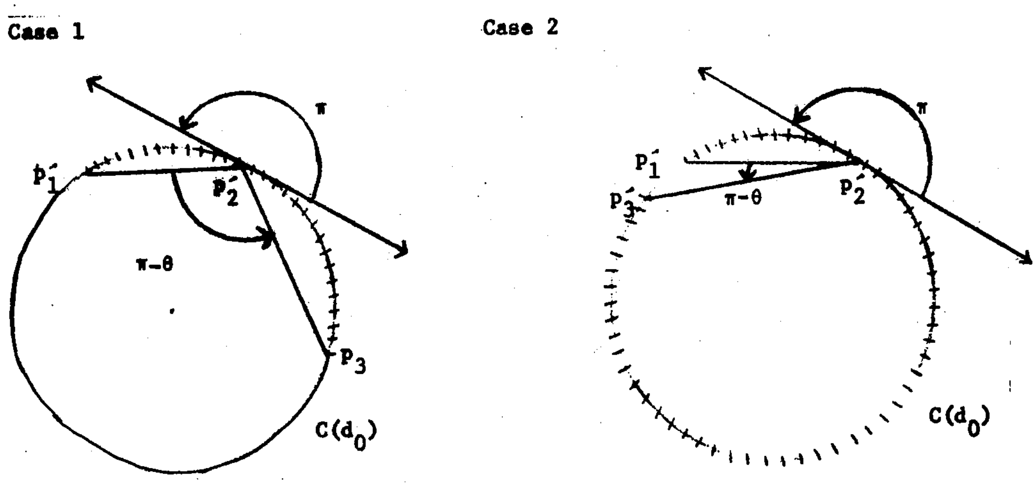
implies, therefore, that the area bounded by $C(d_0)$ is greater than that bounded by $\Gamma' \cup (p_3'p_1')$ if $\Gamma' \neq (\widehat{p_1'p_2'}) \cup (\widehat{p_2'p_3'})$. Hence, the area bounded between the arcs $\widehat{p_1'p_2'}$ and $\widehat{p_2'p_3'}$ of $C(d_0)$ and their respective chords is greater than the area bounded between Γ' and the chords $p_1'p_2'$ and $p_2'p_3'$. Now regard $(\widehat{p_1'p_2'}) \cup (\widehat{p_2'p_3'})$ as hinged at p_2' and rotate the chord $p_2'p_3'$ until p_1', p_2', and p_3' are collinear. Rigidly move this configuration into coincidence with $(\widehat{p_1p_2}) \cup (\widehat{p_2p_3}) \cup p_1p_3$ (cf. Figure 15(a)). Since rigid transformations preserve areas, the sum of the areas bounded between arcs $\widehat{p_1p_2}$ and $\widehat{p_2p_3}$ and their respective chords is greater than that bounded between Γ and the chord p_1p_3. Therefore, $\widehat{p_1p_2} \cup \widehat{p_2p_3}$ is the unique optimal placement of the rope.

It is interesting to note that $(\widehat{p_1p_2}) \cup (\widehat{p_2p_3})$ is also a simple curve, as the following discussion will show.

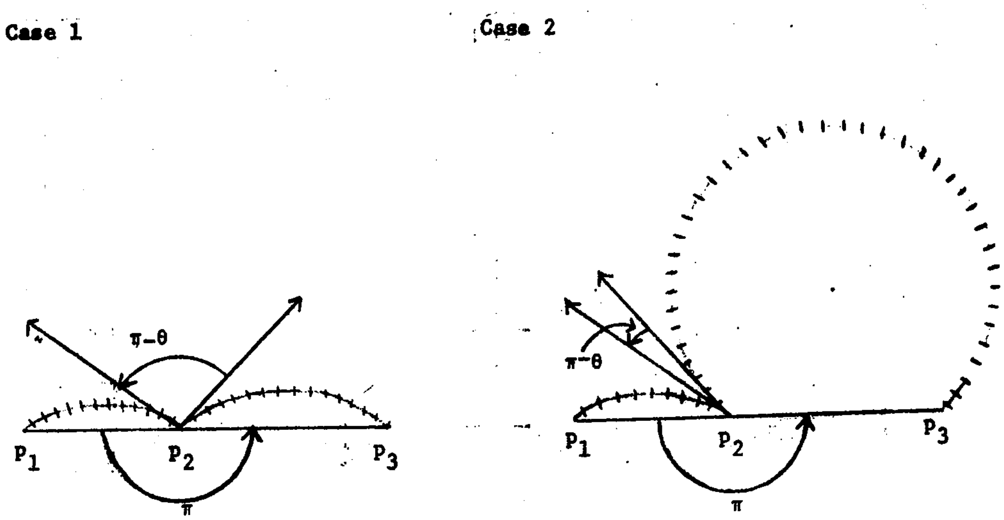
A tangent line to $C(d_0)$ constructed at p_2' is divided into rays from p_2' with π radians between their angles of inclination (cf. Figure 16(a)). Clearly then, the segment $p_2'p_3'$ may undergo a rotation of π radians before these rays intersect (hence, also before the arcs $\widehat{p_1'p_2'}$ and $\widehat{p_2'p_3'}$ intersect) at a point other than p_2' (cf. Figure 16(b)). Since from Figures 13(a) and 14(a) it is also clear that $|\widehat{p_1'p_2'}| + |\widehat{p_2'p_3'}| \leq \pi d_0)$, the circumference of $C(d_0)$, $\widehat{p_1p_2}$ and $\widehat{p_2p_3}$ must intersect only at p_2. ■

The above argument is useful in the solution of the following problem.

Problem 2.2.2. Let C be the collection of curves Γ in E_2 containing the intervening points of attachment $p_2, p_3, \ldots, p_{k-1}$ lying in that



(a) Tangent to C_{d_0} at p_2'



(b) Arcs $\widehat{p_1'p_2'}$ and $\widehat{p_2'p_3'}$ Are Rotated about P_2'

Figure 16. Arcs $\widehat{P_1P_2}$ and $\widehat{P_2P_3}$ Intersect only at P_2

order on the line L determined by their endpoints p_1 and p_k and having a fixed length $\ell > |p_1 p_k|$. Among those members of C lying entirely in one of the closed halfplanes determined by L, which bound(s) with L the greatest possible area?

The following argument will show that again the solution is the union of circular arcs of the same radius.

<u>Solution</u>: For $i = 1,2, \ldots, n-1$, let ℓ_i be a real number such that $\ell_i \geq |p_i p_{i+1}|$ and $\sum_{i=1}^{n-1} \ell_i = \ell$. Among the curves satisfying the conditions of the problem and whose arcs Γ_i between the points p_i and p_{i+1} have length ℓ_i for $1 \leq i \leq n - 1$, Dido's problem with fixed endpoints implies that the unique convex curve whose arcs are circular arcs is the only one which bounds the maximum possible area $a(\ell_1, \ell_2, \ldots, \ell_{n-1})$ with L. The number $a(\ell_1, \ell_2, \ldots, \ell_{n-1})$ is the sum of the areas bounded by Γ_i and $p_i p_{i+1}$, $1 \leq i \leq n - 1$, where it is understood that $\Gamma_i \cup (p_i p_{i+1})$ is a simple closed arc but it is not required that Γ be a simple arc. Since a is a continuous function defined on $\{(\ell_1, \ell_2, \ldots, \ell_{n-1}) \in E_{n-1}:$ $\ell_i \geq |p_i p_{i+1}|$, $\sum_{i=1}^{n-1} \ell_i = \ell\}$, a closed and bounded set in E_{n-1}, then a assumes its maximum value.

Let Γ^* be a curve satisfying the conditions of the problem and whose bounded area with the line L is the maximum possible. Denote the arc which is the restriction of Γ^* joining p_i to p_{i+1} by Γ_i^*, for $1 \leq i \leq n - 1$. Let C_i be a circular arc with endpoints p_i and p_{i+1} such that $\text{length}(C_1) + \text{length}(C_i) = \text{length}(\Gamma_1^*) + \text{length}(\Gamma_i^*)$, for $i = 1,2, \ldots, n-1$. Since Γ^* is optimal, the sum of the areas bounded by $\Gamma_1^* \cup (p_1 p_2)$ and $\Gamma_i^* \cup (p_i p_{i+1})$ must be greater than or equal to the sum of the areas bounded by $C_1 \cup (p_1p_2)$ and $C_i \cup (p_ip_{i+1})$. Problem 2.2.1 implies, therefore, that the arcs Γ_i^* are each circular arcs of the same radius as that of the circular arc Γ_1^*. Hence, Γ^* is the union of circular arcs of the same radius.

Now let $a_i = |p_ip_{i+1}|$, for $i = 1,2, \ldots, n-1$, and assume $a_k \geq a_i$, for all $i \neq k$. For $d \geq a_k$, define

$$s(d) = d \sum_{i=1}^{n-1} \arcsin a_i d^{-1}$$

and

$$t(d) = d\Big(\sum_{\substack{i=1 \\ i\neq k}}^{n-1} \arcsin a_i d^{-1} + \pi - \arcsin a_k d^{-1}\Big).$$

The replacement of $b(\pi/2 + \arcsin a_1/a_2)$ by

$$a_k(\pi/2 + \sum_{\substack{i=1 \\ i\neq k}}^{n-1} a_i a_k^{-1})$$

in part (i) of the solution to Problem 2.2.1 implies that a circle $C(d_0)$ exists for which $\sum_{i=1}^{n-1} |\widehat{p_i'p_{i+1}'}| = \ell$, where $\widehat{p_i'p_{i+1}'}$ is the arc of $C(d_0)$ joining p_i' to p_{i+1}' and $|\widehat{p_i'p_{i+1}'}|$ is its length. Hence, d_0 is the common radius of the circular arcs in Γ^*. Then Γ^* contains no more than one major arc and it must be cut off by a chord p_kp_{k+1} of greatest length among $\{p_1p_2, p_2p_3, \ldots, p_{n-1}p_n\}$. ∎

It is interesting to note that since $|\widehat{p_i'p_{i+1}'}| < \pi d_0/2$ for $i \neq k$, then $|\widehat{p_i'p_{i+1}'}| + |\widehat{p_j'p_{j+1}'}| \leq \pi d_0$ for $i,j = 1,2, \ldots, n-1$. Therefore, no two of the arcs $\widehat{p_ip_{i+1}}$ and $\widehat{p_jp_{j+1}}$ of Γ^* may intersect at points other than endpoints if $i \neq j$. Hence, the solution curve Γ^* is again simple.

3. Optimal Curves Bounding Polygons

In this section, the results lead to a solution of the following general problem: Among those closed convex curves of a fixed length ℓ bounding a fixed n-gon $P_n = p_1p_2 \dots p_n$, which bound(s) a region of greatest possible area? The solution of the following Dido related problem is a first step in the solution of the more general problem above.

Problem 2.3.1. Let p_1 and p_3 be fixed points on a line L and let p_2 be a fixed point not on L which determines with L the plane π. Let C be the collection of convex curves Γ in π of a fixed length $\ell > |p_1p_2| + |p_2p_3|$, having p_1 and p_3 as endpoints, and bounding with L the point p_2. Which member(s) of C bound(s) the greatest area with L?

The following argument will show that again the solution curve is the union of circular arc(s) of the same radius.

Case 1. If ℓ is sufficiently large so that p_2 is bounded by L and the solution Γ_0 to Dido's problem with fixed endpoints p_1 and p_2 for the length ℓ, then Γ_0, a circular arc, must also be the solution to Problem 2.3.1.

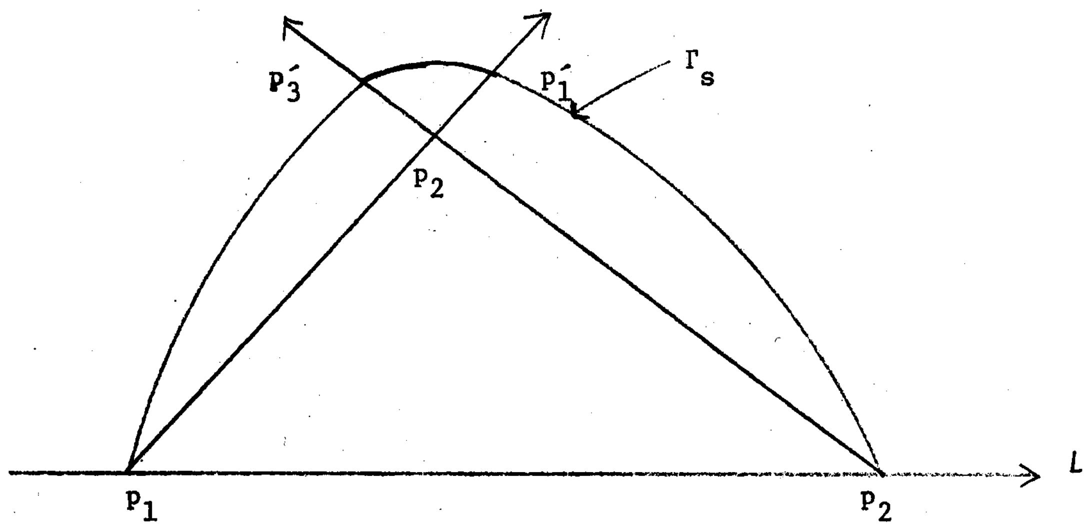
Case 2. If ℓ is not large enough so that p_2 is bounded by L and Γ_0, the following discussion leads to the solution. The sets bounded by L and the members Γ of C are compact convex bodies.

The Blaschke Convergence Theorem can, therefore, be used as in the proof of the Existence Theorem in Chapter I to imply the existence of a solution Γ_s.

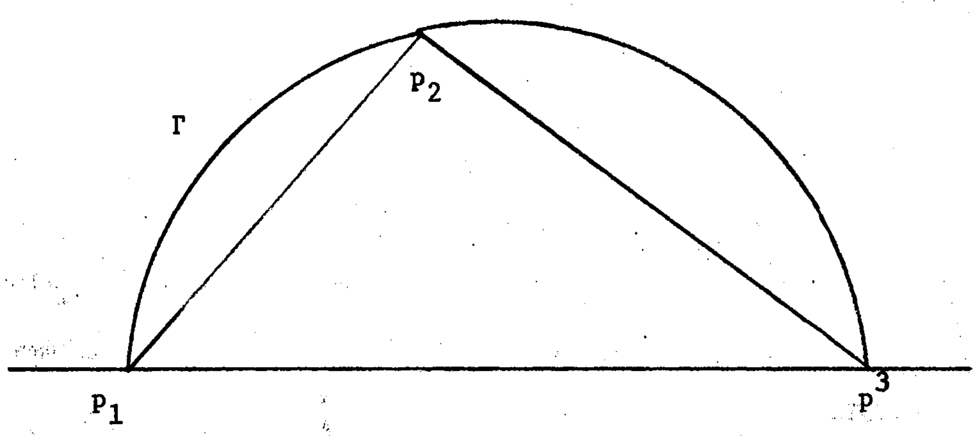
Assume that a solution curve Γ_s does not contain the point p_2. Denote the points of intersection of Γ_s with the rays $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_3p_2}$ by p_1' and p_3', respectively. Then arcs Γ_1 and Γ_3 of Γ_s between p_1 and p_1' and between p_3 and p_3', respectively, must be solutions to Dido's problem with fixed endpoints and, hence, they are circular arcs (cf. Figure 17(a)). But the arc of Γ_s between p_3' and p_1' is common to Γ_1 and Γ_3; hence, Γ_1 and Γ_3 must be circular arcs of the same radius and Γ_s must itself be a circular arc. This contradicts the assumption, however, that ℓ was not sufficiently large to allow a circular arc of length ℓ with endpoints at p_1 and p_3 to bound p_2. Any solution must, therefore, contain the point p_2 (cf. Figure 17(b)). Hence, the unique solution for Case 2 is obtained by rigidly moving and properly rotating about p_2, the solution obtained from Problem 2.2.1 for collinear points p_1, p_2, and p_3 (where $|p_1'p_2'| = |p_1p_2|$ and $|p_2'p_3'| = |p_2p_3|$), until p_1, p_2, and p_3 coincide with p_1', p_2', and p_3', respectively.■

The solution of the preceding problem may now be combined with the results of the last section to provide the solution of the problem referred to in the beginning of this section for the case $k = 3$. The following lemma facilitates the solution of that problem.

<u>Lemma</u> 2.3.2-1. Let p_1, p_2, and p_3 be points in E_2. Then there exists a unique disc of minimum radius containing p_1, p_2, and p_3.



(a) Arc Γ_s Is A Circular Arc



(b) Solution Curve Γ Must Contain the Point p_2

Figure 17. Solution to Problem 2.3.1 for Case 2

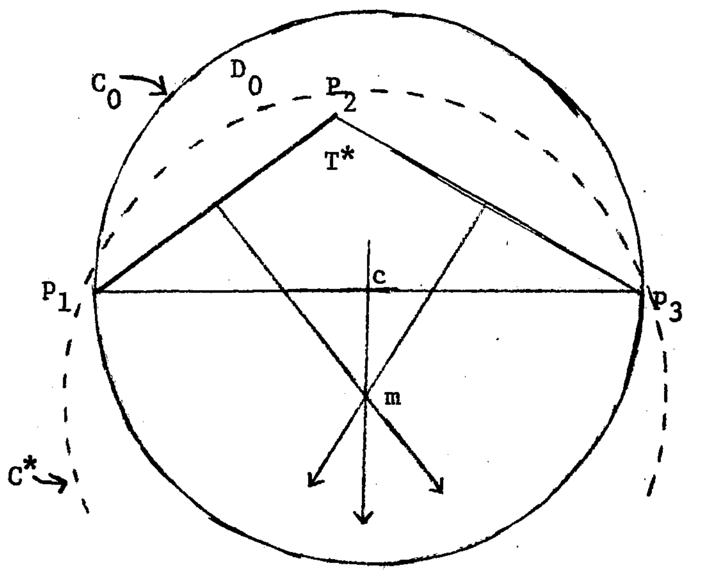
Proof: Assume, without loss of generality, that $|p_1p_3| \geq \max\{|p_1p_2|, |p_2p_3|\}$. Let r and m be the radius and center point, respectively, of the unique disc D^* whose boundary C^* contains p_1, p_2, and p_3, i.e., C^* is the circumscribing circle of p_1, p_2, and p_3. Then segments p_1p_2, p_2p_3, and p_3p_1 are chords of C^*; hence, their perpendicular bisectors must meet at the center m of D^*. Also let T^* be the triangle determined by p_1, p_2, and p_3.

Case 1. Suppose that T^* fails to contain m. Then the disc D_0 (with center c) whose diameter is the interval p_1p_3 contains p_2. Hence, since D_0 is clearly the unique disc of minimum radius containing p_1 and p_3, it is also the disc of minimum radius containing p_1, p_2, and p_3 (cf. Figure 18(a)).

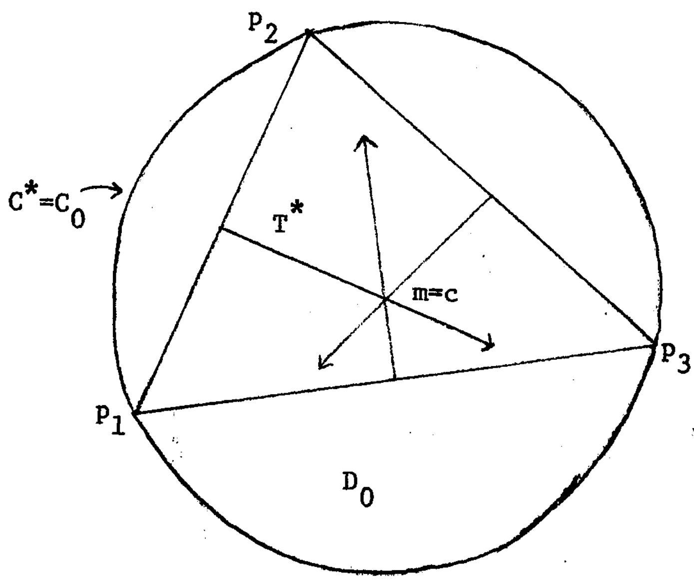
Case 2. Suppose that T^* contains m. Than any circle not concentric with C^* but with radius less than or equal to r must fail to contain a semicircular arc of C^*; hence, also at least one of the points p_1, p_2, and p_3. Therefore, D^* is the unique circle D_0 of minimum radius which contains the points p_1, p_2, and p_3 (cf. Figure 18(b)).

Problem 2.3.2. Among the simple closed convex plane curves bounding the fixed noncollinear points p_1, p_2, and p_3 and of a fixed length ℓ, where $\ell > |p_1p_2| + |p_2p_3| + |p_3p_1|$, which bound(s) the greatest possible area?

Several cases are distinguishable but in each, the solution is again the union of circular arcs of the same radius. Let $|C_0|$ be the circumference of the circle of minimum radius bounding the points p_1, p_2, and p_3.



(a) Case 1



(b) Case 2

Figure 18. The Disc D_0 of Minimum Radius Containing P_1, P_2, and P_3

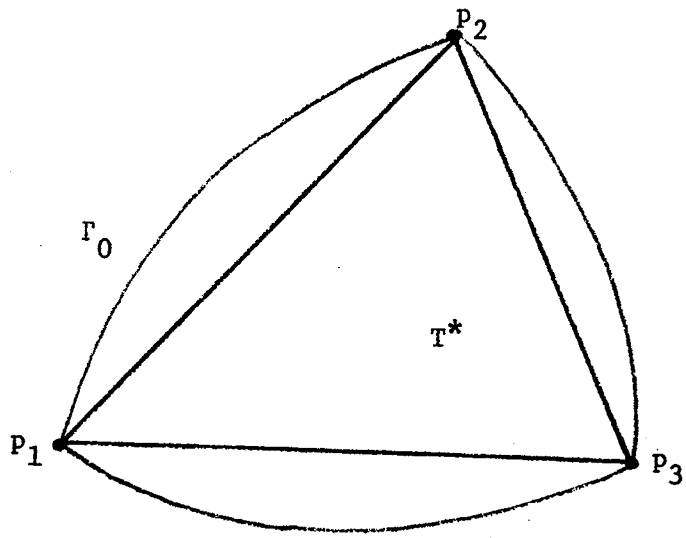
Case 1. If $\ell \geq |C_0|$ then the solution to the classical isoperimetric problem implies that any curve bounding the points p_1, p_2, and p_3 and enclosing the maximum area is a circle of radius $\ell/2\pi$.

Case 2. Suppose $\ell < |C_0|$. Since the curves Γ under consideration are convex, each must bound the triangle T^* determined by p_1, p_2, and p_3. Triangle T^* may be translated so that it is bounded by Γ and a vertex lies on Γ. From the convexity of C, T^* may then be rotated about that vertex until a second vertex lies on Γ. Hence, it may be assumed, without loss of generality, that the points p_1 and p_3 lie on each curve Γ satisfying the conditions of the problem.

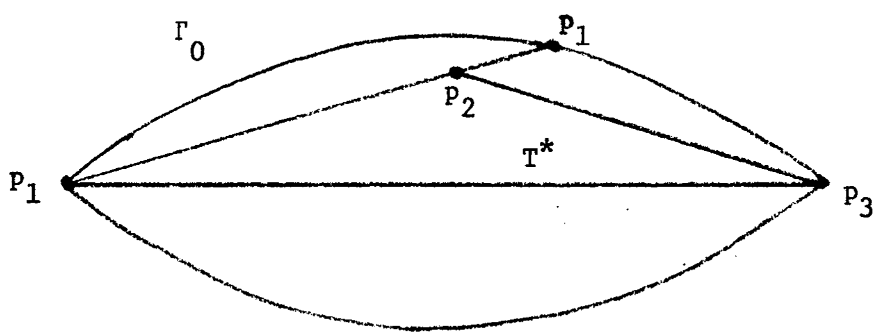
The Blaschke Convergence Theorem may again be used as in Section 3 of Chapter I to show that an optimal curve Γ_0 exists.

(i). If p_2 also lies on Γ_0, then Γ_0 is the union of circular arcs whose common radius is determined by the solution of Problem 2.2.2 for collinear points p_1', p_2', p_3' and p_4', where $p_1'p_2' = p_1'p_2'$, $p_2'p_3' = p_2p_3$ and $p_3'p_4' = p_3p_1$ (cf. Figure 19(a)).

(ii). Suppose p_2 does not lie on Γ_0. The arc of Γ_0 between p_1 and p_3 bounding p_2 with the chord p_1p_3 must also bound the greatest area with the segment p_1p_3 among all such curves of its length. Hence, (from Case 1 of Problem 2.3.1) it must be a circular arc. Let p_1' be the point of intersection of Γ_0 with the ray $\overrightarrow{p_1p_2}$. The arc of Γ_0 containing p_3 and with endpoints p_1' and p_1 is (from Case 2 of Problem 2.3.1) the union of two circular arcs of the same radius (cf. Figure 19(b)). Since these arcs of Γ_0 have in common the arc of Γ_0 between p_1' and p_3 and not containing p_1, Γ_0 must be the union of arcs all with the same radius. Since $\ell < |C_0|$ by



(a) Case 2(i)



(b) Case 2(ii)

Figure 19. The Curve Γ_0 of Length $\ell < |C_0|$ Bounding p_1, p_2, and p_3 and the Maximum Area

assumption, Γ_0 is the union of two minor arcs of the same radius cut off by the chord p_1p_3. ■

The next problem is the generalization of Problem 2.3.2 obtained by allowing the number k of bounded points to assume any positive integral value. The following lemmas will provide results concerning the smallest circle bounding an n-gon P_n.

Lemma 2.3.3-1. For each n-gon P_n, there exists a unique circle C_0 of smallest radius which bounds P_n.

Proof: No procedure for finding a circle C_0 of smallest radius containing P_n will be given here (as in Lemma 2.2.4-1). The Blaschke Convergence Theorem implies (as in Section 3 of Chapter I) the existence of at least one such circle. If two distinct circles C_1 and C_2 of the smallest possible radius r_0 containing P_n exist, the intersection of their associated discs must also contain P_n. Therefore, if $C_1 \cap C_2 = \{q_1, q_2\}$ then the circle C_0 centered at $(q_1 + q_2)/2$ with radius $||q_1 - q_2||/2$ must also bound P_n. Since $||q_1 - q_2||/2$ is less than the common radius r_0 of C_1 and C_2, there exists exactly one circle C_0 of the minimum radius r_0 and bounding P_n. ■

Lemma 2.3.3-2. If C_0 is the circle of minimum radius r_0 bounding the n-gon P_n, then the center c_0 of C_0 is contained in P_n.

Proof: Let $P_n = p_1p_2 \cdots p_n$ be an n-gon and let C_0 be the circle of smallest radius r_0 bounding P_n. Without loss of generality, as before, it may be assumed that C_0 contains at least two vertices of

P_n. If each major arc of C_0 (an arc of length greater than or equal to πr_0) intersects P_n, then clearly P_n contains c_0, the center of C_0. If a major arc of C_0 exists which fails to intersect P_n, then let $|p_i p_j| = \max\{|p_h p_k| : 1 \leq h,\ k \leq n\}$ and let m_0 be the midpoint of $p_i p_j$. Then there exists a circle C_0' of radius $r_0' < r_0$ centered at a point c_0' on the segment $c_0 m_0$ and bounding the n-gon P_n since each circle centered on $c_0 m_0$ bounding p_i and p_j must also bound the minor arc $\widehat{p_i p_j}$ of C_0. This is a contradiction. Hence, each major arc of C_0' intersects P_n so P_n contains c_0' (cf. Figure 20). $\blacksquare$

<u>Problem</u> 2.3.3. Let $\mathcal{C}$ be the collection of simple closed convex curves in E_2 of a fixed length ℓ and bounding the points $p_1, p_2, \ldots, p_k$. Which member(s) of $\mathcal{C}$ bound(s) the greatest area?

<u>Solution</u>: It may be assumed, without loss of generality, that the points $p_1, p_2, \ldots, p_k$ are the vertices of a convex polygon P_0 since the curves in $\mathcal{C}$ are convex. The following discussion will again show that the solution curve is the union of circular arcs with the same radius.

Case 1. If $\ell \geq |C_0|$, the circumference of C_0, the minimum circle, then again (from the classical isoperimetric problem) the solution is the family of circles containing P_0 with radius $\ell/(2\pi)$.

Case 2. If $\ell < |C_0|$, let C_r be the circle of radius r concentric with C_0. Regard the boundary of P_0 as being severed at p_1 and having a new endpoint p_{n+1} (p_8 in Figure 21) replacing p_1 as an endpoint of the line segment $p_n p_1$. Some of the vertices of P_0 (at least two) must lie on C_0 and will determine a (possibly degenerate)

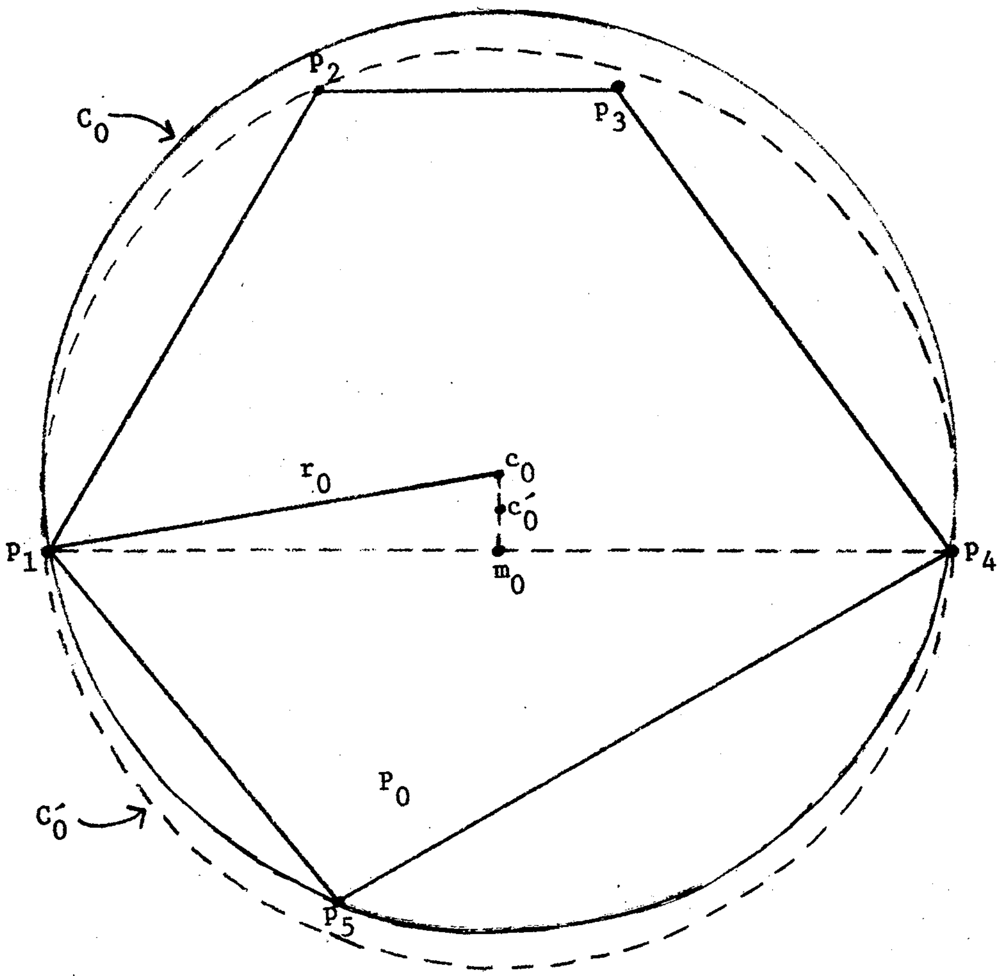


Figure 20. The Minimum Circle C_0 and the Polygon P_0

polygon P_0' (cf. Figure 21). By Lemma 2.3.3-2, P_0' contains the center c_0 of P_0'. Suppose $p_{i-1}p_i$ and p_ip_{i+1} are hinged at p_i, $2 \le i \le n$, such that the interior angle at p_i is not allowed to decrease from its value in P_0. Increase r while keeping the vertices of P_0' on the expanding circle C_r and allowing p_{n+1} to move away from p_1 but keeping p_1 on a fixed ray whose endpoint is the center of C_0 as in Figure 21. That part of the boundary of P_0 between the vertices of P_0' is kept rigid until the increasing radius brings a new vertex (or several new vertices simultaneously) into contact with C_r. Figure 21 shows the circle with radius $r' = r$ where p_4 has just touched C_r. For $r = r''$, the vertex p_3 has just touched C_r, but p_7 has not yet touched C_r. After a vertex p_i, $2 \le i \le n$, belongs to C_r, it remains on C_r for larger r. Recall that the abutting sides are hinged at p_i so as to allow the interior angle to increase after p_i comes into contact with C_r. For all $r > r_0$, define $s(r)$ as the sum of the lengths of the arcs of C_r cut off by the altered boundary of P_0 as described above (cf. Figure 21). Since $s(r)$ is a continuous function of r such that $s(r_0) = |C_0|$ and $\lim_{r \to \infty} s(r) = |P_0|$, the perimeter of P_0, $(|P_0| < \ell < |C_0|)$, there exists a circle C_{r_ℓ} such that $s(r_\ell) = \ell$.

The solution Γ_0 is obtained from P_0 by constructing arcs from C_{r_ℓ} joining the vertices of P_0 which lie on C_{r_ℓ}. Figure 22 illustrates this for P_0 of Figure 21 where $r_\ell = r''$. The optimality of this curve follows easily from the preceding problems. $\blacksquare$

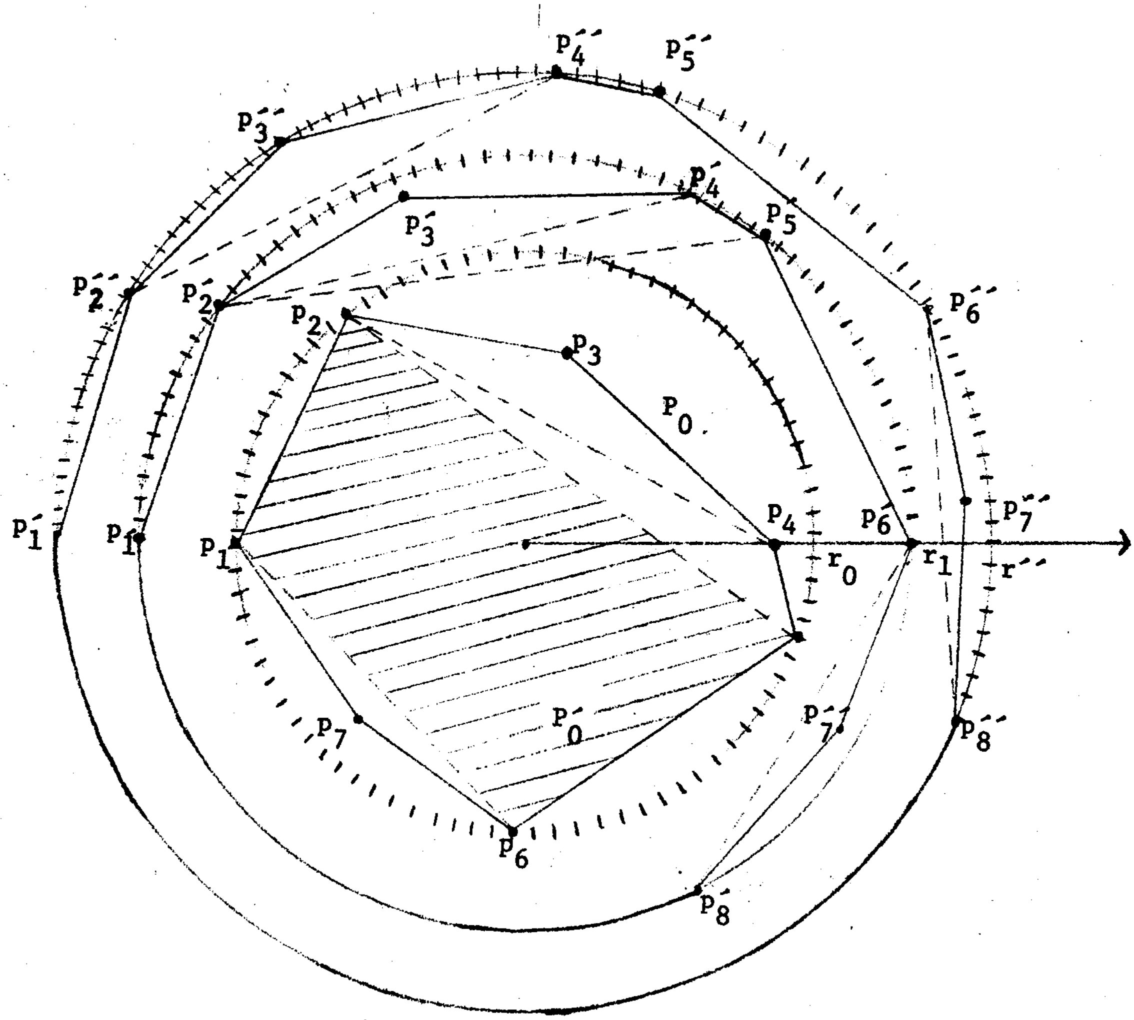


Figure 21. The Subtended Arc Function $s(r)$

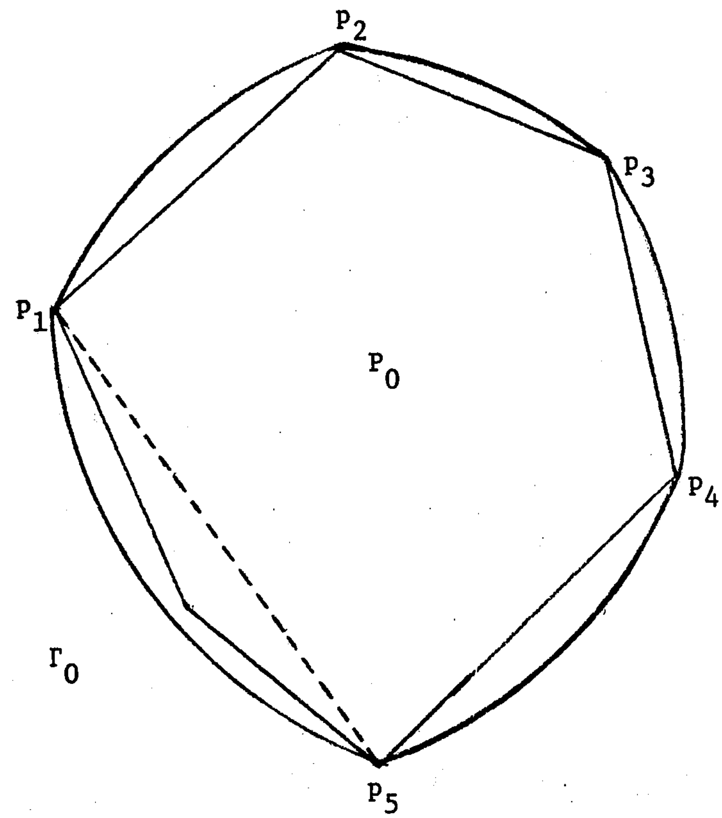


Figure 22. The Polygon P_0 and the Solution Curve Γ_0

CHAPTER III

POLYGONAL ISOPERIMETRIC PROBLEMS

This chapter focuses on the isoperimetric problem as applied to various classes of polygons. More specificly, general polygons, Reuleaux polygons, and honeycombs consisting of polygonal cells are considered and optimal curves and surfaces are obtained.

1. The General Polygonal Isoperimetric Problem

The isoperimetric-related problem of concern in this section is that of maximizing area on the class $C(n)$ of convex n-gons of a fixed perimeter ℓ.

Optimal n-gons with Fixed Characteristics

An n-gon, hence also its area, is completely determined by the specification of its angles and the lengths of its sides. Let $P_n = p_1 p_2 \ldots p_n$ be an n-gon with interior angle α_i at the vertex p_i and such that $\ell_i = |p_i p_{i+1}|$ for $i = 1, 2, \ldots, n$, where p_{n+1} is p_1. By assumption,

$$\sum_{i=1}^{n} \ell_i = \ell.$$

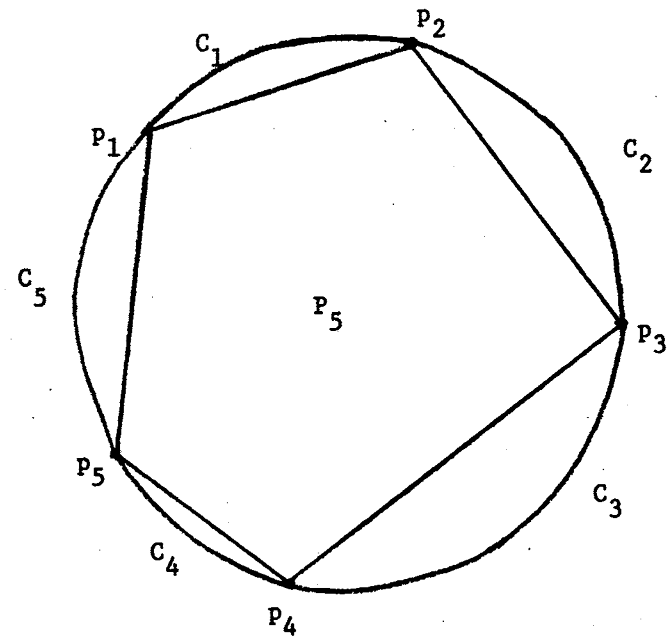
The following argument will show that a similar relationship must hold among the angles of P_n. If the exterior angle formed by extending one of the sides of P_n at p_i is denoted by β_i, then $\alpha_i + \beta_i = \pi$. Since the amount of rotation of the normal to a convex polygon is 2π, then $\sum_{i=1}^{n} \beta_i = 2\pi$. It is clear, therefore, that

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} (\pi - \beta_i) = n\pi - \sum_{i=1}^{n} \beta_i = (n - 2)\pi.$$

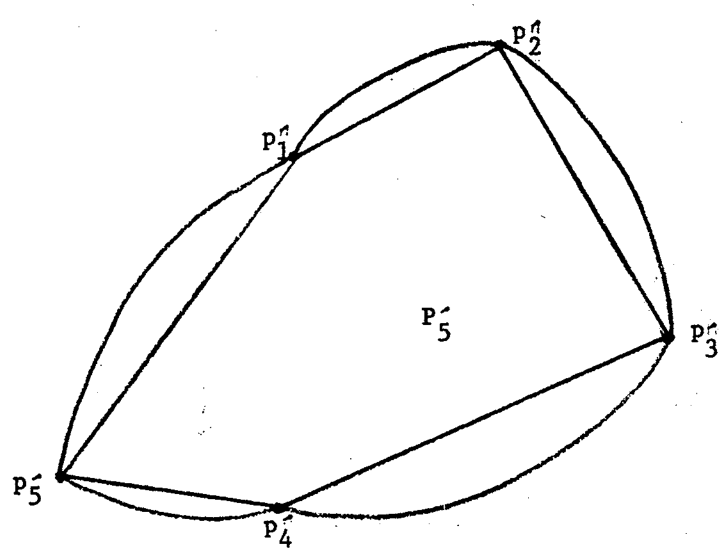
The angles and side lengths of P_n are not independent. For example, a triangle whose side lengths are determined admits only one possible specification for its angles. When $n > 3$, this relationship is more flexible. In the following theorem, for example, side lengths are held fixed as the effects of angular specifications are considered.

Theorem 3.1.1. The area of an n-gon which is inscribed in a circle is greater than that of other n-gons with sides of the same length and in the same order of succession.

Proof: The following argument, involving the isoperimetric property of the circle, was offered by Steiner: Let $P_n = p_1 p_2 \dots p_n$ be an n-gon inscribed in a circle C. Regard the arcs C_i of C cut off by the sides $p_i p_{i+1}$ of P_n, where $p_{n+1} = p_1$, as rigid and connected by flexible joints at the vertices p_i of P_n. Now deform this configuration by changing the angles at the vertices of the polygon so that the joined circular arcs C_i no longer form a circle. The resulting figure has the same perimeter as does C (cf. Figure 23). The circular solution to the isoperimetric problem implies, therefore, that its area is decreased. This area is the sum of the area of the deformed polygon, and the areas of the sectors bounded by the sides of the polygon with the arcs C_i of C which they cut off. Since the



(a) The Polygon P_5 is Inscribed in a Circle



(b) The Polygon P_5 is Deformed into P_5'
By Changing its Vertex Angles

Figure 23. The Optimal Vertex Angles for Fixed Sides

latter areas are unchanged under the deformation, it follows that the decrease in total area bounded by $\bigcup_{i=1}^{n} C_i$ is due to a decrease in the area of the polygon. ■

For each specification of its side lengths, an n-gon P_n may be assigned interior vertex angles in such a way that its vertices all lie on a circle. To see this, note that a circle C_{r*} may be chosen with a large enough radius $r*$ so that the lengths of the minor arcs cut off by chords equal in length to the sides of P_n sum to less than the circumference of C_{r*}. As the radius r decreases, the sum of these arc lengths continuously increases while the circumference continuously decreases. Therefore, there exists a circle C_{r_n} of radius $r_n < r*$, whose circumference is equal to the sum of the lengths of the arcs cut off by chords equal in length to the sides of P_n. Hence, this circle circumscribes an n-gon P_n whose sides are equal in length to those of P_n. Theorem 3.1.1 now implies that if the sides of an n-gon P_n are fixed, area is maximized when its angles are such that its vertices lie on a circle.

The following sequence of lemmas leads to a theorem in which the relationship between area and side lengths is investigated. To understand the precise meaning of the first lemma, it should be noted that an n-gon is said to be circumscribed about a circle if each side of the n-gon is tangent to the circle.

<u>Lemma</u> 3.1.2-1. For each specification of its angles, an n-gon P_n of perimeter ℓ may be assigned side lengths in such a way that it is circumscribed about a circle.

Proof: Construct an n-gon Q_n whose interior vertex angles are those specified. Let C_r be a circle contained in Q_n and translate the sides of Q_n into tangency to C_r. Denote the resulting n-gon by Q_n'. Then P_n, the polygon of the required perimeter, may be obtained from Q_n' by a positive homothety. ■

<u>Lemma</u> 3.1.2-2. If an n-gon P_n of perimeter ℓ is circumscribed about a circle C_p (of radius r) then the area of P_n is given by $r\ell/2$.

Proof: Let $P_n = p_1p_2 \ldots p_n$ be an n-gon circumscribed about a circle C_r of radius r and centered at 0. The segments $0p_1, 0p_2, \ldots, 0p_n$ divide P_n into triangles with 0 as a common vertex (cf. Figure 24). By assumption, the sides of the n-gon are each tangent to C_r. Hence, if the sides p_ip_{i+1} are considered as bases for these triangles (where $p_{n+1} = p_1$), then their altitudes are the segments $0p_i'$ (where p_i' is the point of tangency of the segment p_ip_{i+1} to the circle C_r). The area of P_n is, therefore, $(|p_1p_2||0p_1'| + |p_2p_3||0p_2| + \ldots + |p_np_1||0p_n'|)/2$. But

$$\sum_{i=1}^{n} |p_ip_{i+1}| = \ell$$

and $|0p_i'| = r$ for $i = 1, 2, \ldots, n$, so the area of P_n is $r\ell/2$. ■

<u>Lemma</u> 3.1.2-3. The rhombus has greater area than does any other quadrilateral with the same angles and perimeter.

Proof: Let $P_4 = p_1p_2p_3p_4$ be a quadrilateral parallelogram with perimeter ℓ and having angle α_i at p_i. The area of P_4 is $\ell_1\ell_2 \sin \alpha_2$, where $\ell_i = |p_ip_{i+1}|$. But

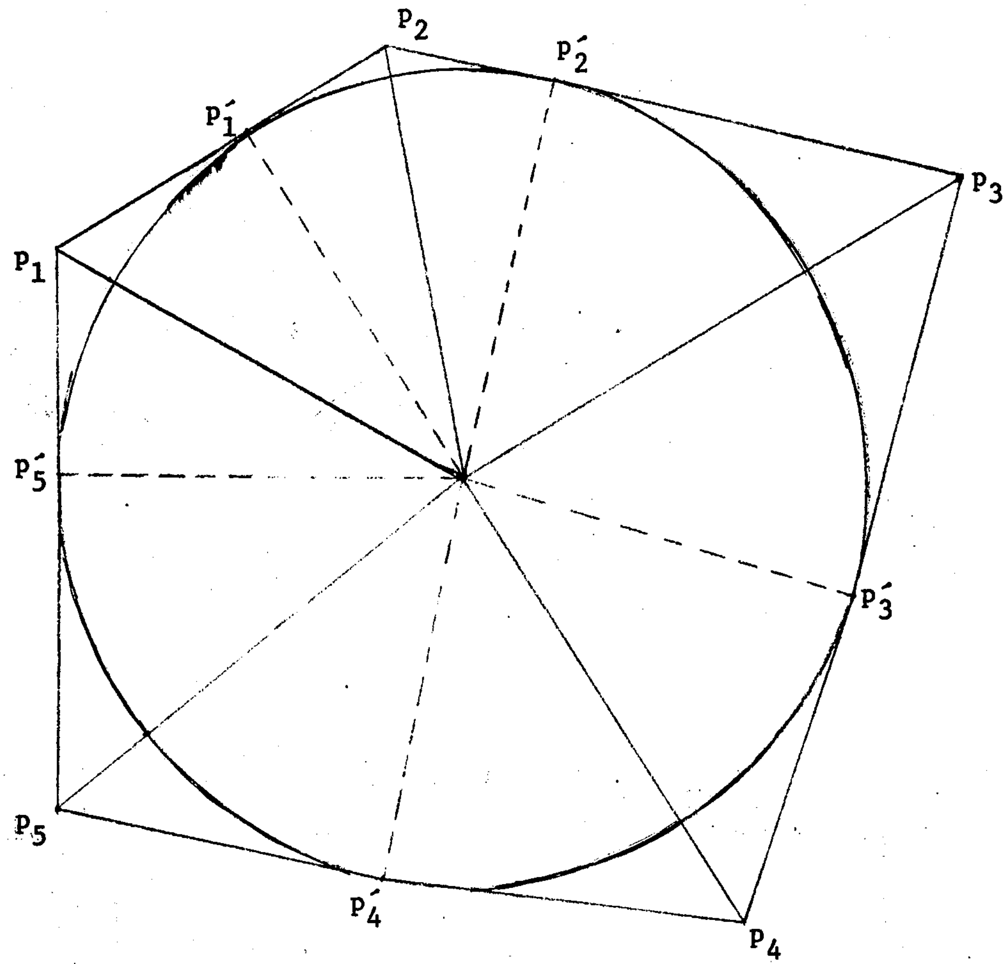


Figure 24. The Area of a Circumscribing n-gon

$$\ell_1\ell_2 \sin \alpha_2 = [(\ell_1 + \ell_2)^2 - (\ell_1 - \ell_2)^2] \sin \alpha_2/4$$

$$= [(\tfrac{1}{2}\ell)^2 - (\ell_1 - \ell_2)^2]\sin \alpha_2/4.$$

Hence, the area of P_4 is maximized when $\ell_1 - \ell_2 = 0$; i.e., when $\ell_1 = \ell_2$ so the parallelogram is a rhombus. ■

That the rhombus above inscribes a circle is also useful in the proof of the theorem below. The following argument leads to this conclusion.

In the rhombus above, $\alpha_1 = \alpha_3 = \pi - \alpha_2 = \pi - \alpha_4$, so that $\sin \alpha_1 = \sin \alpha_2 = \sin \alpha_3 = \sin \alpha_4$. Also, the distance between two opposite sides of a rhombus is $\ell_i \sin \alpha_i$, $1 \leq i \leq 4$. Since the ℓ_i are equal for $i = 1, 2, 3, 4$, then the distances between the pairs of opposite sides are the same for a rhombus. A circle centered at the intersection of the lines parallel to and midway between the pairs of opposite sides and with radius $\tfrac{1}{2}\ell_1 \sin \alpha_1$ is inscribed by the rhombus.

<u>Theorem</u> 3.1.2. Among all convex n-gons with given angles and a given perimeter, that one in which a circle can be inscribed has greatest area.

Proof: Induct on n, the number of sides of the polygon. Verification for the case $n = 3$: Any two triangles with corresponding angles equal and having the same perimeter must be congruent and, therefore, have the same area. Theorem 3.1.2 is trivially true for $n = 3$; therefore, since only one triangle with given angles and a given perimeter exists and it, like all traingles, inscribes a circle.

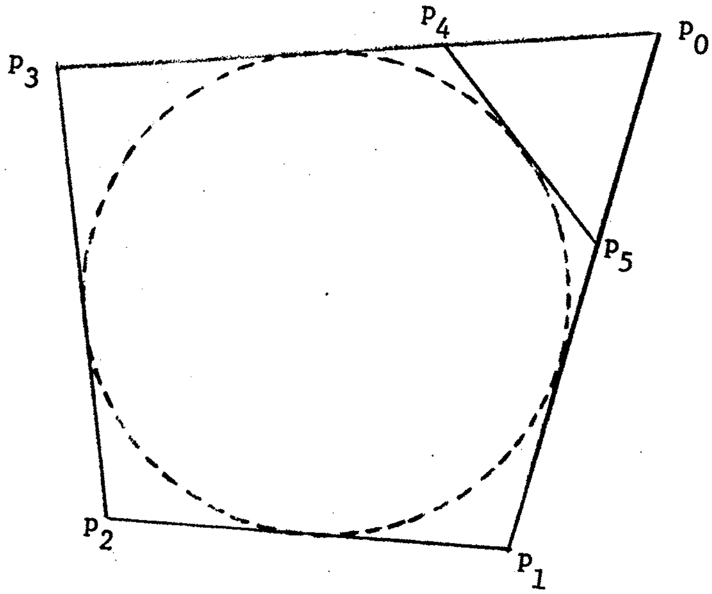
Now assume that the theorem holds for the case $n = k$. Then the theorem holds for the case when $n = k+1$: From Lemma 3.1.2-1 it will suffice to show that the (k+1)-sided polygon P_{k+1} with given angles and perimeter which is circumscribed about a circle is also that which maximizes the isoperimetric ratio $[m_2(P_{k+1})]/[m_1(P_{k+1})]^2$. Consider, therefore, two dissimilar but equiangular (k+1)-gons $P_{k+1} = p_1p_2 \cdots p_{k+1}$, circumscribed about a circle, and $P'_{k+1} = p'_1p'_2 \cdots p'_{k+1}$ (cf. Figure 25). Denote by α_i the interior angle of P_{k+1} at the vertex p_i (hence, also the interior angle of P'_{k+1} at p'_i). If no two consecutive angles sum to more than π radians, the following hold:

$$\begin{aligned}
\alpha_1 + \alpha_2 \qquad\qquad\qquad &\le \pi, \\
\alpha_2 + \alpha_3 \qquad\qquad &\le \pi, \\
\ddots \qquad &\vdots \\
\alpha_k + \alpha_{k+1} &\le \pi, \\
\alpha_1 \qquad\qquad\qquad + \alpha_{k+1} &\le \pi.
\end{aligned}$$

Hence, $2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{k+1} \le (k + 1)\pi$. But

$$2 \sum_{i=1}^{k+1} \alpha_i = 2[(k + 1 - 2)\pi] \le (k + 1)\pi,$$

so $2k - 2 \le k + 1$ and $k \le 3$ (or $k + 1 \le 4$). Therefore, if $k + 1 > 4$, there must exist at least one pair of consecutive angles from P_{k+1} and P'_{k+1} whose sum exceeds π radians. Without loss of generality, denote them by α_k and α_{k+1}. Replacing all the " $\le$ " symbols above by strict inequalities, it also follows that when



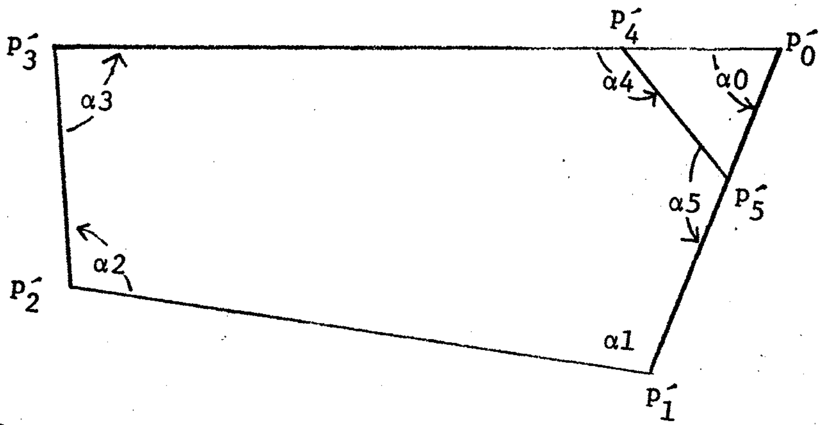


Figure 25. The Equilateral (k+1)-gons P_{k+1} and P'_{k+1}

$k + 1 = 4$, at least one pair of consecutive angles from P_{k+1} (and P'_{k+1}) must sum to at least π radians. If the sum exceeds π radians, denote the angles by α_k and α_{k+1} without loss of generality. If the sum is exactly π radians, then P_{k+1} (and P'_{k+1}) is a quadrilateral (a parallelogram) and the result follows from Lemma 3.1.2-3.

Now if $n \geq 5$, delete the sides $p_k p_{k+1}$ and $p'_k p'_{k+1}$ of P_{k+1} and P'_{k+1}, respectively, and prolong the adjacent sides to p_0 and p'_0, their respective intersections. Two k-gons $P_k = p_1 p_2 \cdots p_{k-1} p_0$ and $P'_k = p'_1 p'_2 \cdots p'_{k-1} p'_0$ with corresponding angles equal are thereby obtained, the first of which is circumscribed about a circle. Assume, without loss of generality, that P_{k+1} and P'_{k+1} were chosen so that the perimeters of P_k and P'_k have the common value of $2L$. By Lemma 3.1.2-2, the area of the k-gon P_k is rL, where r is the radius of the circle inscribed in this k-gon. Let αrL denote the area of P'_k. By the induction hypothesis, $\alpha \leq 1$ since P'_k cannot have greater area than P_k. Now denote the expression $|p_k p_0| + |p_0 p_{k+1}| - |p_{k+1} p_k|$ by 2ℓ and the expression $|p'_k p'_0| + |p'_0 p'_{k+1}| - |p'_{k+1} p'_k|$ by $2\kappa\ell$, where κ is the similarity ratio of triangles $p_k p_0 p_{k+1}$ and $p'_k p'_0 p'_{k+1}$. The area of triangle $p_k p_0 p_{k+1}$ is, therefore, given by $r\ell$, and the area of the triangle $p'_k p'_0 p'_{k+1}$, which is similar to triangle $p_k p_0 p_{k+1}$ with the ratio κ, must be $\kappa^2 r\ell$. Now if $a(S)$ is the area of the set S,

$$a(P_{k+1}) = a(P_k) - a(\Delta p_k p_0 p_{k+1}) = rL - r\ell,$$

and

$$|p_1'p_2'| + |p_2'p_3'| + \dots + |p_{k+1}p_1|$$

$$= |p_1p_2| + |p_2p_3| + \dots + |p_kp_0| + |p_0p_{k+1}|$$

$$- (|p_kp_0| + |p_0p_{k+1}| - |p_{k+1}p_k|)$$

$$= 2L - 2\ell.$$

Also,

$$a(P'_{k+1}) = a(P'_{k+1}) - a(\Delta p'_k p'_0 p'_{k+1}) = \alpha r\ell - \kappa^2 r\ell,$$

and

$$|p_1'p_2'| + |p_2'p_3'| + \dots + |p'_{k+1}p'_1|$$

$$= |p_1'p_2'| + |p_2'p_3'| + \dots + |p'_kp'_0| + |p'_0p'_{k+1}|$$

$$- (|p'_kp'_0| + |p'_0p'_{k+1}| - |p'_{k+1}p'_k|)$$

$$= 2L - 2\kappa\ell.$$

The following argument will show that the isoperimetric ratio associated with P_{k+1} exceeds the isoperimetric ratio associated with P'_{k+1}.

Now

$$\frac{m_2(P_{k+1})}{[m_1(P_{k+1})]^2} > \frac{m_2(P'_{k+1})}{[m_1(P'_{k+1})]^2},$$

is equivalent to

$$\frac{rL - r\ell}{(2L - 2\ell)^2} > \frac{\alpha rL - \kappa^2 r\ell}{(2L - 2\kappa\ell)^2},$$

which is equivalent to

$$\frac{1}{L - \ell} > \frac{\alpha L - \kappa^2 L}{(L - \kappa\ell)^2},$$

or, equivalently, $(L - \kappa\ell)^2 > (L - \ell)(\alpha L - \kappa^2\ell)$, since $L > \ell$. The last inequality holds if

$$\begin{aligned}(L - \kappa\ell)^2 - (L - \ell)(\alpha L - \kappa^2\ell) \\ &= (L - \kappa\ell)^2 - (L - \ell)(L - \kappa^2\ell) + (1 - \alpha)L(L - \ell) \\ &= L\ell(1 - \kappa)^2 + (1 - \alpha)(L - \ell)L > 0.\end{aligned}$$

Clearly from their definitions, $L - \ell > 0$, and from a previous inequality $1 - \alpha \geq 0$. If $1 - \alpha = 0$, then $\alpha = 1$ and P_k' is congruent to P_k. Also, $L\ell(1 - \kappa)^2 \geq 0$ and $L\ell(1 - \kappa)^2 = 0$ implies that $1 - \kappa = 0$, i.e., that $\kappa = 1$. Also if $\kappa = 1$ and $\alpha = 1$ simultaneously then triangle $p_k p_0 p_{k+1}$ is congruent to triangle $p_k' p_0' p_{k+1}'$ so P_{k+1} is congruent to P_{k+1}' -- a contradiction. Therefore, $\alpha \neq 1$ or $\kappa \neq 1$ so the inequality holds and the isoperimetric ratio associated with P_{k+1} exceeds that associated with P_{k+1}'. $\blacksquare$

<u>Optimal n-gons of Fixed Perimeter</u>

The Blaschke Convergence Theorem may be used to show that the collection C_n of all n-gons of a fixed perimeter ℓ contains a member of greatest possible area. The preceding theorems imply that a sequence of n-gons with nondecreasing area may be constructed using any member of C_n as the first term. The (k+1)st term is then alternately determined from the kth term by fixing the side lengths of the n-gon while inscribing it in a circle and fixing its angles while circumscribing it about a circle. The areas associated with the terms of this sequence are strictly increasing until a term is obtained which is both circumscribed about a circle and inscribed in a circle. Hence, the member of C_n of maximum possible area must be such an n-gon. The regular n-gon has both these properties but the discussion below will show that irregular n-gons also exist with both properties; hence, the preceding theorems are not alone sufficient to completely determine a member of C_n with the maximum possible area.

Let C_0 be the circle centered at 0 with radius r^* and let R_n be an n-gon inscribed in C_0. If p_1 is a vertex of R_n, construct a circle C_{r_1} centered at c (on the open segment $0p_1$) and with radius r_1 so that C_{r_1} fails to meet C_0 but meets every side of R_n. For $r \leq r_1$ denote by $\mathcal{s}(n,r)$ the sum of the lengths of the n successively connected arcs from C_0 clockwise from p_1 which are cut off by chords tangent to the circle C_r centered at c and with radius r (cf. Figure 26 for $n = 5$). Now, clearly, $\mathcal{s}(n,r_1) < 2\pi r^*$ and $\mathcal{s}(n,r) \to n\pi r^*$ as $r \to 0$. Since $\mathcal{s}$ is a continuous function of r, the intermediate value theorem guarantees the existence of an r_n' such that $\mathcal{s}(n,r_n') = 2\pi r^*$. Hence, there exists an irregular n-gon (since $c \neq 0$) which is inscribed in C_{r^*} and circumscribed about $C_{r_n'}$. A positive homothety of this n-gon will then yield an n-gon of the required perimeter ℓ.

The following theorem provides the additional constraint required.

<u>Theorem</u> 3.1.3. For each n-gon with unequal sides, an n-gon of the same perimeter exists which has greater area.

Proof: Let $P_n = p_1p_2 \cdots p_n$ be an n-gon with unequal sides. Assume, without loss of generality, that $|p_1p_2| \neq |p_2p_3|$. Let E be $\{p \in E_2\colon |p_1p| + |pp_3| = |p_1p_2| + |p_2p_3|\}$, i.e., the ellipse with foci at p_1 and p_3 and having $|p_1p_2| + |p_2p_3|$ as the length of its major axis (cf. Figure 27). Clearly E must contain p_2 but, since

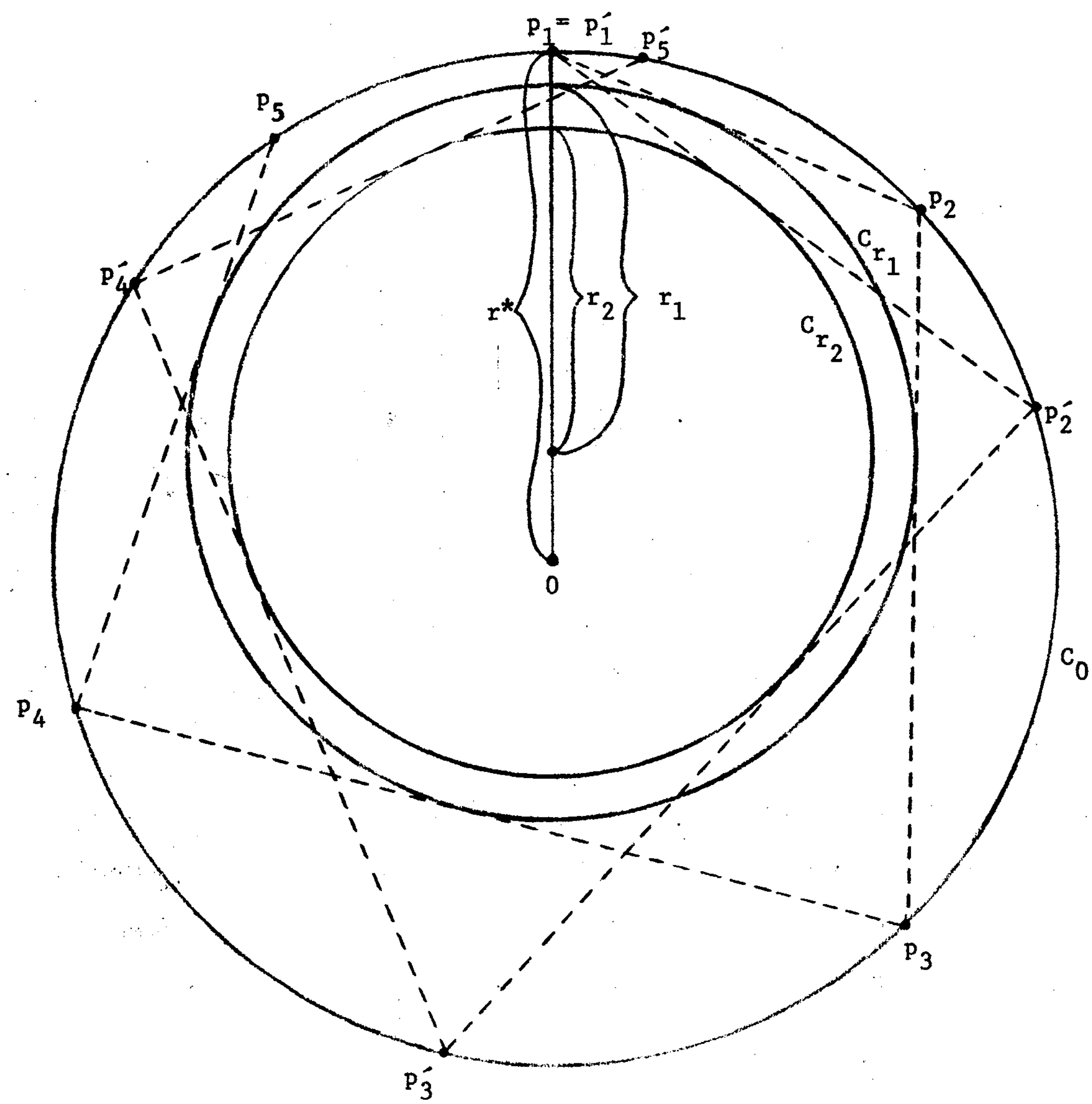


Figure 26. Existence of an Irregular 4-gon Circumscribed about a Circle and Inscribed in a Circle

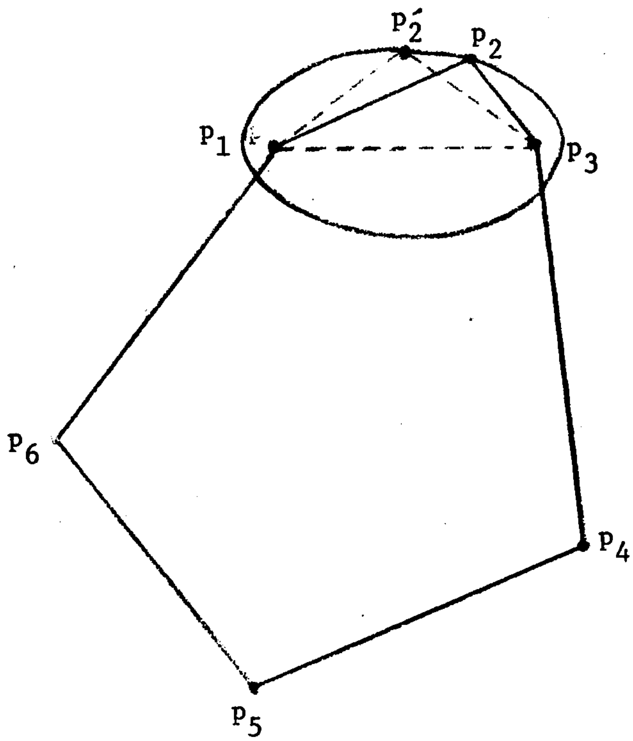


Figure 27. The n-gon P_n Has Greater Area than Does the n-gon P_n with Unequal Sides

$p_1p_2 \neq p_2p_3$, p_2 is not an endpoint of the minor axis of E. Let p_2' be the endpoint of the minor axis of E which lies on the arc of E containing p_2 and joining the endpoints of its major axis. Then the area of triangle $p_1p_2'p_3$ exceeds that of triangle $p_1p_2p_3$ since they have the same base and unequal heights. Hence, the area of $P_n' = p_1p_2'p_3 \cdots p_n$ is greater than that of P_n while their perimeters are equal, and P_n cannot be optimal. ■

Since the preceding theorem implies that an n-gon of maximum area must have equal sides, then if its perimeter is ℓ, each side must have length ℓ/n. From Theorem 3.1.1 (or Theorem 3.1.2) it is clear, therefore, that an n-gon of maximum area for fixed perimeter must also have equal angles; hence, it must be regular. Since the angles of an n-gon sum to $(n-2)\pi$, each one must be $(n-2)\pi/n$. Therefore, the sides and angles of the n-gon of maximum area are fixed, and it is unique. The following theorem has now been proved.

Theorem 3.1.4. The unique n-gon of maximum area for a fixed perimeter ℓ is the regular n-gon (whose sides are ℓ/n units in length and whose angles are $(n-2)\pi/n$ radians).

It is natural to inquire at this point about the existence of a polygon of greatest area for a fixed perimeter ℓ. The following theorem will imply that no such polygon exists, but rather that for each polygon of perimeter ℓ, a polygon of the same perimeter having greater area can be found.

<u>Theorem</u> 3.1.5. The regular n-gon R_n of perimeter ℓ bounds a region of greater area than does any other convex m-gon P_m of the same perimeter, where $m \leq n$.

Proof: The area $a(R_n)$ of the regular n-gon R_n of perimeter ℓ is

$$a(R_n) = \frac{\ell^2}{4n} \cot \frac{\pi}{n} .$$

It is a strictly increasing function of n for fixed ℓ. Also, from Theorem 3.1.4, the area $a(P_m)$ of an m-gon P_m of perimeter ℓ satisfies the inequality $a(P_m) \leq a(R_m)$, with equality only when $P_m = R_m$. Now $a(P_m) \leq a(R_m) \leq a(R_n)$ so $a(P_m) \leq a(R_n)$, with equality only in case $P_m = R_n$. $\blacksquare$

2. Reuleaux Polygons

A <u>set of constant width</u> is a convex body which is such that the breadth (distance between parallel lines of support) is constant. In the Euclidean plane a <u>Reuleaux polygon</u> is a set of constant width whose boundary consists of an odd number (greater than one) of circular arcs called sides. The center for a circular side of a Reuleaux polygon lies at its opposite vertex. A <u>Reuleaux n-gon</u> is a Reuleaux polygon with n sides. A Reuleaux n-gon is <u>regular</u> if its sides are of equal length.

From Barbier's theorem [17], all sets of the same constant width in the plane also have the same perimeter. In comparing the isoperimetric ratios of Reuleaux polygons, it will suffice, therefore, to compare the areas of all Reuleaux polygons of constant width one. The following theorem, which is proved in [6] and [13], is analogus to Theorem 3.1.4 for n-gons.

Theorem 3.2.1. Among all Reuleaux n-gons of constant width one, the regular Reuleaux polygon is the unique one of maximum area.

The next theorem also parallels a preceding theorem (Theorem 3.1.5) involving n-gons.

Theorem 3.2.2. Between two regular Reuleaux polygons of width one, that one which has more sides also has greater area.

Proof: The area $a(R_n)$ of a regular n-sided Reuleaux polygon R_n of constant width one is given by

$$a(R_n) = n[\frac{\pi}{n} - \tan \frac{\pi}{2n}] /2.$$

Then

$$\frac{d}{dn} [a(R_n)] = \frac{\pi}{4n} \sec^2 \frac{\pi}{2n} - \tfrac{1}{2}\tan \frac{\pi}{2n}$$

$$= \frac{\pi}{4n} \sec^2 \frac{\pi}{2n} \left[1 - \frac{\sin(\pi/n)}{(\pi/n)} \right] > 0.$$

Hence, $a(R_n)$ is an increasing function of n. ■

Thus the consideration of the isoperimetric problem for Reuleaux polygons yields results which are exactly analogus to the results obtained by its consideration for polygons.

3. The Honeycomb Isoperimetric Problem

As its name suggests, the basic problem considered in this section is closely associated with the architecture of the honeycombs found in nature. Pappus was the first to focus attention on the problem in

Book V of his Mathematical Collection in the fourth century A.D. He attributed to the honeybee a "deep geometric intuition" which allows it to construct a comb involving the most economical use of its materials, subject only to the constraints imposed by the uses of the comb. Biologists give assurance, however, that economy is among the least important considerations in the construction of the natural honeycomb. It is interesting, nonetheless, to accept Pappus' conjecture and to investigate what the bees do well and what they do not do so well in the construction of their comb. For this purpose, a precise mathematical definition of the honeycomb is now given.

A general n-dimensional honeycomb (n-comb) may be defined as a set of convex congruent polytopes called cells, filling the space between two parallel hyperplanes without overlapping or having interstices in such a way that:

(1) Each cell has a facet called the opening (or base), on one of the two hyperplanes bounding the comb and has no face (i.e. facet) in common with the other, and

(2) In the congruence of the cells, their bases correspond.

The width w of the comb is then defined to be the distance between the parallel bounding hyperplanes. The length of a comb is the number of cells contained in the comb. Two combs are said to be equal if each one can be mapped onto the other by a reflection or a translation or a composition of these two.

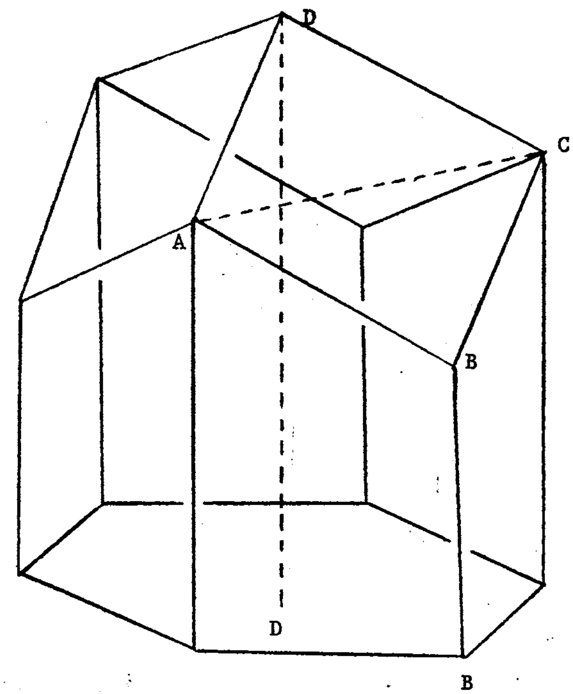
The honeycomb of the hive bees consists of cells having regular hexagonal bases and whose bottoms are closed by three equal rhombi forming diehedral angles of $2\pi/3$ radians with each other (cf. Figure 28(a)). The facets on the sides of the cell are perpendicular to its base. Two bee cells may, therefore, be joined at their bases to form an elongated rhombic dodecahedron (i.e., a polytope having twelve rhombic facets). Each cell in the honeycomb of the bees has a pentagonal face in common with each of six cells whose bases are coplanar with its own and a rhombic face in common with each of three cells whose bases lie on the other bounding hyperplane of the comb.

In order to facilitate an analysis of the natural honeycomb for economy of construction, the following isoperimetric problem is formulated.

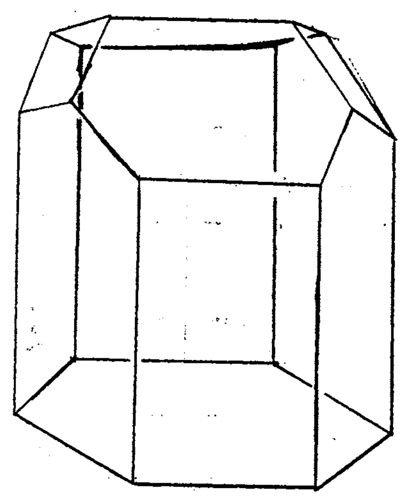
Honeycomb Isoperimetric Problem: Among the polyhedra of volume v generating an n-comb of width w, which has the minimum (n-1)-dimensional surface area (not counting the base)?

The (n-1)-dimensional area b of the opening at the base of a cell is determined by the specification of w and v in the statement of the problem. To see this note, that if a is the (n-1)-dimensional area of that portion of a bounding hyperplane used by the comb, then there exist $2a/b$ cells in the comb. Hence, the volume wa of the comb is also given by $2av/b$, and (since $wa = 2av/b$) it follows that $b = 2v/w$. Clearly then, the problem can be equivalently formulated by the specification of w and b, or v and b.

For the purpose of minimizing (n-1)-dimensional surface area, it seems reasonable to assume that each cell has a regular polygonal base. Let ℓ be the perimeter of that base and let r be the radius of the inscribed circle. The volume of a right prism bounded between two supporting hyperplanes of a comb and having its base congruent to that



(a) A Bee Cell



(b) Half of a Truncated Octahedron

Figure 28. The Comb of the Hive Bees

of a cell from the comb is (from Lemma 3.1.5-1) equal to $r\ell w/2$. Since any cell of a comb can be dissected and pieced together with a second cell so that the two cells exactly fill the prism, the volume v of a cell is $r\ell w/4$ and hence $\ell = 4v/(wr)$. If A is the total surface area of a cell, then A is greater than $r\ell/2$ (the area of the base). Now, since $\ell = 4v/(wr)$, A is greater than $r(4v/(wr))/2 = 2v/2$ and $A \to \infty$ as $w \to 0$. It is also evident that A is greater than or equal to $\ell w/2$ (half the combined area of the sides of the prism). Since $\ell = 4v/(wr)$, then A is greater than or equal to $(4v/(wr))w/2 = 2v/r$ and as $w \to \infty$, $r \to 0$ so $A \to \infty$.

Since for fixed v the (n-1)-dimensional surface area A_{n-1} of a cell increases as w becomes very small or very large, there is an absolute best cell for each value of v.

Pappus' conjecture concerning the structure of a bee cell was primarily due to his analysis of its base. The regular hexagonal shape of the base of a bee cell is, among all convex space fillers, that which minimizes perimeter for a fixed area. To see that this is true among regular polygonal space fillers, note that the interior angles at the vertices of a regular n-gon must be divisors of 2π if the regular n-gon is to be a space filler. The interior angles at the vertices of a regular n-gon measure $n\pi/(n-2)$ radians, hence only the choices $n = 3$, 4, or 6 lead to values of θ which are exact divisors of 2π. Theorem 3.1.5 now implies that from a finite collection of regular n-gons with equal area, that one has least perimeter which has more sides; i.e., the regular hexagon. Since the cells are relatively deep, their surface areas consist mostly of cell walls rather than the rhombi which close their bottom. Clearly, therefore, the hive bees do much to minimize cell area when they choose a regular hexagonal base.

If, for some reason, the bees must maintain a cell bottom consisting of three congruent rhombi, they have one degree of freedom (cf. Figure 28(a)). When the planes containing the rhombi (such as that containing ABCD in Figure 28(a)) are rotated about the horizontal diagonals (such as diagonal AC of the rhombus ABCD in Figure 28(a)), new rhombi are formed which yield the same values for v and w in their associated comb. The rhombi thus obtained have two new vertices which lie on the lines perpendicular to the base which contain the vertices they replace (lines DD' and BB' in Figure 28(a)). The values of w and v are maintained since the horizontal diagonal of each rhombus is $w/2$ units from the planes containing the cell bases and the amount of volume lost under one side of the horizontal diagonal is equal to that gained under the other. Again the bees make an excellent choice since the minimum surface area occurs when the diehedral angles formed by the rhombi measure $2\pi/3$ radians as in a bee cell.

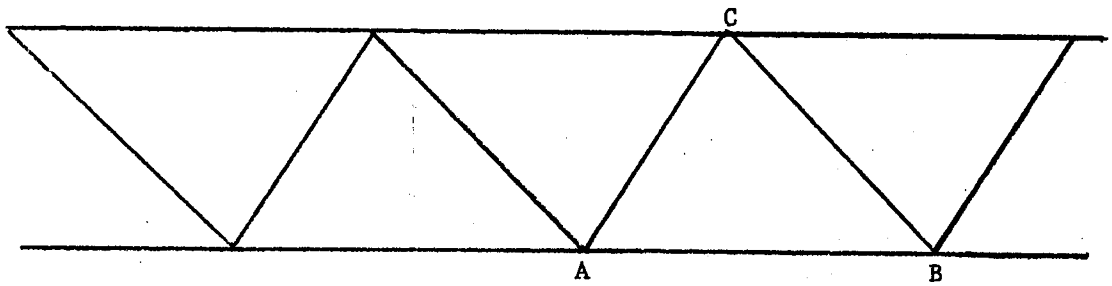
The above considerations do not justify the conclusion, however, that the bee cell is a solution to the honeycomb isoperimetric problem when $n = 3$. The various parts of the cell were analyzed <u>separately</u> and the cell bottom was severly restricted as to form. The solution to the problem is, in fact, not yet known. It is known, however, that the cell constructed by the hive bees is not the solution for any choice of the parameters v and w.

Two-dimensional Honeycombs

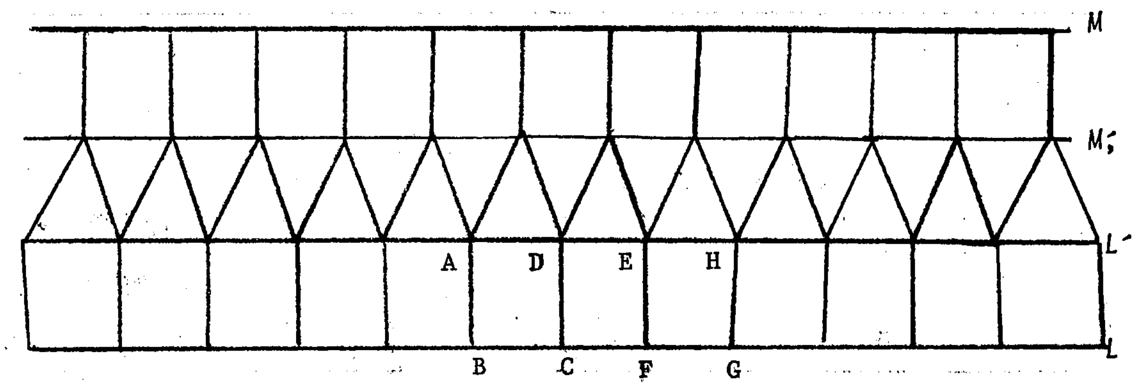
The basic types of two-dimensional honeycombs are relatively few so they can be enumerated and then compared for economy of construction. The solution will thus be obtained for the two-dimensional honeycomb isoperimetric problem.

A reduced comb is a comb, each of whose cells have points in common with each of its faces (bounding hyperplanes). The enumerability of the basic types of combs is due to the fact that the cells of a reduced comb are triangles. To see this, note from property (1) of a cell that each cell of a reduced comb has exactly one point C on one of the faces of the comb and its base AB on the other (cf. Figure 29(a)). That portion of the cell wall which joins A to C is also a portion of the wall of an adjacent cell. Since both the cells are convex, it follows that AC must simultaneously be convex and concave. Hence, AC is a straight line. A similar result holds for BC; hence, the cell is a triangle.

Let L and M be the hyperplanes (lines) bounding a nonreduced two-dimensional comb. Let L' be the member of F, the family of lines parallel to L and having a common point with each cell having its base on M, which is closest to L and define M' similarly (cf. Figure 29(b)). Since the cells lying along L are congruent in a base preserving manner, each can be made coincident with any other by a translation or a reflection with respect to a line perpendicular to L. Clearly, the same is true of the parts of the cells lying between L and L'. If DCFE is a translate of ABCD then ABCD must be a parallelogram. If DCFE is a reflection (and not a translation) of



(a) The Cells of a Reduced Comb are Triangles



(b) A Cell of a Nonreduced Comb is the Union of a Triangle and a Parallelogram

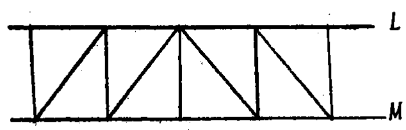
Figure 29. The Cells of a Two Dimensional Honeycomb

ABCD then angle CFE is equal to angle ABC and angles BCD and DCF have a common measure of $\pi/2$ radians (cf. Figure 29(b)). Also if DCFE is not a translate of ABCD, then angle ABC is not equal to $\pi/2$ radians. Hence, if EFGH is that portion of a cell adjacent to DCFE as shown in Figure 29(b), it cannot be congruent to ABCD with corresponding bases. This contradiction implies that ABCD must be a parallelogram. A similar argument implies that the partial cells between M and M' must also be parallelograms.

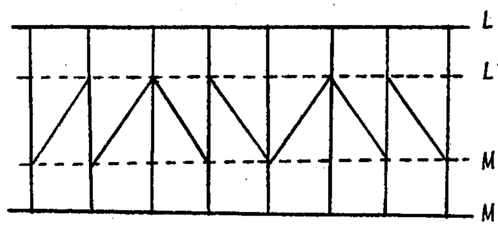
Theorem 3.3.1. A cell of a two-dimensional comb is either a triangle, a parallelogram, or a pentagon which is formed by adjoining a triangle and a parallelogram along a common side such that the cell base is opposite the common side.

Proof: If L' and M' coincide, the cells are parallelograms as shown above. If L' and M' do not coincide, that part of the comb between L' and M' is a reduced comb; hence, its cells must be triangles as shown above. If $L' = L$ and $M' = M$, then the comb consists of triangular cells. Otherwise, the cells are pentagons (cf. Figure 29(b)). Each pentagon is the union of a triangle and a parallelogram where each cell's base is the side of the parallelogram opposite the triangle. ■

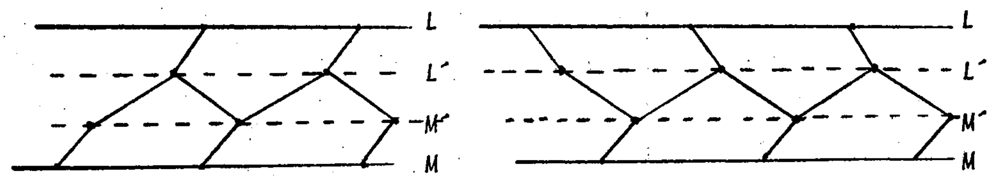
Any convex polygon of one of the types described in Theorem 3.3.1 generates at least one comb. In the case of triangular cells, the comb is uniquely determined except when the cell is a right triangle having a leg as its base. In that case, there are finitely many combs of fixed finite length (cf. Figure 30(a)). Similarly the comb is



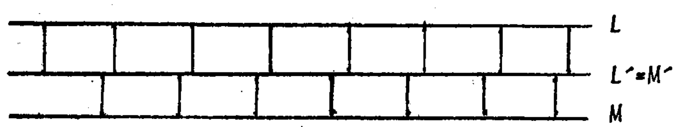
(a) Right Triangular Cell



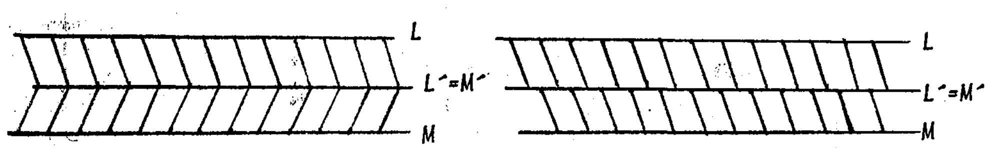
(b) Cell Is Union of Rectangle and Right Triangle



(c) Two Honeycombs Determined by Cells Which Are the Union of Isosceles Triangle and Parallelogram



(d) Continuous Number of Combs Determined By Rectangular Cells



(e) Two Honeycombs Determined by Parallelogram Cells

Figure 30. Types of Two-dimensional Honeycombs

uniquely determined when the cell is the union of a triangle and a parallelogram except when (i) the cell is formed by a right triangle and a rectangle adjoining to its legs or (ii) an isosceles traingle and a nonrectangular parallelogram adjoining to the base (cf. Figure 30(b) and (e), respectively). In case (i), there are again finitely many combs of a fixed finite length, and exactly two combs for a fixed finite length in case (ii). There is an infinite set of combs determined by a rectangular cell, and two infinite sets of combs if the cell is a nonrectangular parallelogram (cf. Figure 30(d) and (e), respectively).

The following theorem gives a solution to the two-dimensional honeycomb isoperimetric problem.

Theorem 3.3.2. Among the cells of given area which generate a comb of given width w, that cell having the least perimeter is either a pentagon composed of a rectangle and an isosceles triangle having an angle equal to $2\pi/3$ radians at its apex, or an isosceles triangle having an angle greater than or equal to $2\pi/3$ radians at its apex.

Proof: In view of the relation $w = 2a/b$, the constraints on the cells under consideration in the above theorem may be expressed in terms of the area a of the cell and the length b of its base. It is also clear that for the purpose of minimizing perimeter the pentagonal cells under consideration may be restricted to cells composed of a rectangle and an isosceles triangle (cf. Figure 31). If the length of a "vertical" side of the cell is $(w/2) - x$, where $0 \leq x \leq w/2$, then the total perimeter p of the cell is

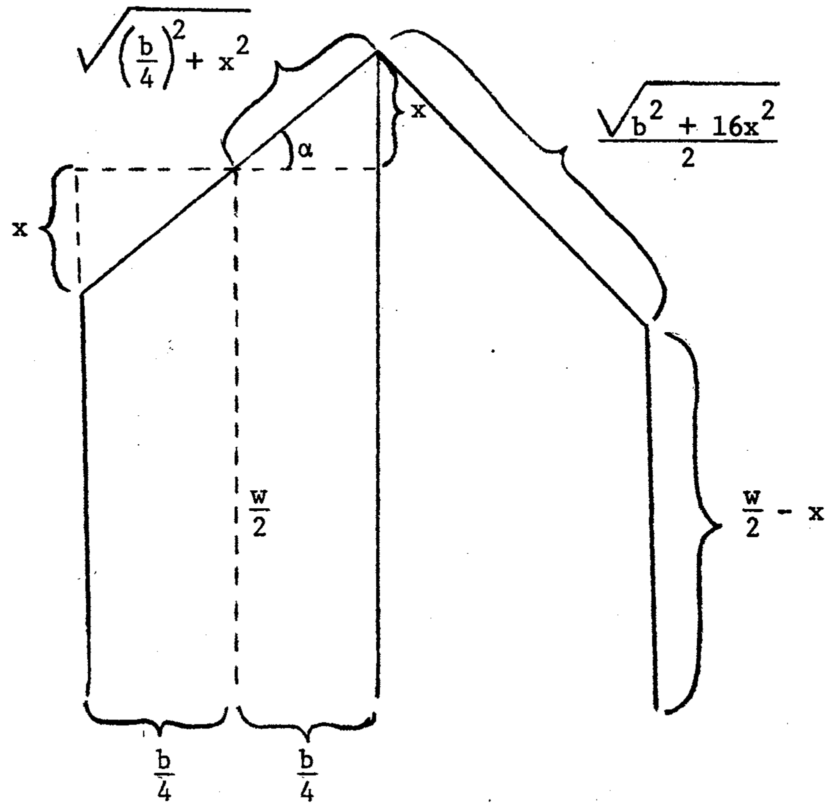


Figure 31. The Cell of Minimum Perimeter for a Fixed Area a and Width w

$$p = w - 2x + \sqrt{b^2 + 16x^2} .$$

A necessary condition for a minimum perimeter is

$$\frac{dp}{dx} = -2 + \frac{16x}{\sqrt{b^2 + 16x^2}} = 0,$$

as

$$2x = \sqrt{(b/4)^2 + x^2} .$$

Thus

$$\sin \alpha = \frac{1}{2} = \frac{x}{(b/4)^2 + x^2} .$$

Hence, the three angles at the bottom of the cell must measure $2\pi/3$ radians.

The above analysis (for pentagons) holds only when $0 < x < w/2$. Since the above equality also implies

$$x = \frac{1}{2} \frac{b}{\sqrt{12}} ,$$

the analysis is valid only in case $b < w\sqrt{12}$. If $b \geq w\sqrt{12}$ (i.e., the trainagle case), the best cell is the isosceles triangle determined by b and w. When $b = w\sqrt{12}$, the vertex angle is $2\pi/3$ radians and when $b > w\sqrt{12}$, the vertex angle is greater than $2\pi/3$ radians. $\blacksquare$

It is interesting to inquire which cell is the "absolute best" from the standpoint of minimization of perimeter for a fixed area (or comb width or base length). The results concerning the polygonal isoperimetric problem and an argument similar to that employed in Dido's problem show that the solution is a pentagon which is half a regular hexagon. Amazingly, half a regular hexagon remains the optimal shape of a cell when properties (1) and (2) of a cell are relaxed.

An Improved Bee Cell

The three-dimensional honeycombs are not so easy to enumerate and compare. For this reason, no comb has yet been shown to minimize surface area for cells of fixed volume v. It is possible, however, to describe a cell of volume v which has a smaller surface area than the one which is actually constructed by the bees, thereby showing that the bee cell is not the solution. This improved bee cell can be formed by halving a truncated octahedron by a plane orthogonal to one of its hexagonal zones of faces (cf. Figure 28(b)). This cell could very well be the solution to the problem in three-dimensional space.

CHAPTER IV

GENERALIZED ISOPERIMETRIC RATIOS

In this chapter, attention is focused on the problem of maximizing "generalized" isoperimetric ratios on the collection $\mathcal{P}_n$ of n-dimensional polytopes P. The isoperimetric ratio is generalized from the "classical" ratio

$$I(P) = \frac{\sqrt[n]{V_n(P)}}{\sqrt[n-1]{S_{n-1}(P)}}$$

to the more general ratio for $1 \leq i,j \leq n$

$$I_{ij}(P) = \frac{\sqrt[i]{m_i(P)}}{\sqrt[j]{m_j(P)}} \quad ,$$

where $m_k(P)$ denotes the sum of the measures (Lebesque) of the various k-faces of P if $1 \leq k \leq n$. Specific inquiry is made into the problems of maximizing the applicable generalized ratios on the collection of all right prisms and the collection of all tetrahedrons as subsets of $\mathcal{P}_3$. These results are then used as a basis for certain conjectures concerning the types of polytopes in $\mathcal{P}_3$ which maximize the generalized isoperimetric ratios. Finally, there is interest in determining which of the generalized isoperimetric ratios on $\mathcal{P}_n$ are bounded and on which polytopes (if any) the maxima are attained.

1. Definitions and Examples

A subset of Euclidean space is called a polytope if and only if it is the convex hull of a finite set of points or, equivalently, is the intersection of a finite number of closed halfspaces and is bounded. An n-polytope is simply a polytope that is n-dimensional. For example, a solid polygon, such as a triangle or a rectangle, is a 2-polytope while a disc is a two dimensional convex set which is not a polytope. Similarly, a tetrahedron, cube, or prism is a 3-polytope while a sphere is a three dimensional convex set which is not a polytope. For $0 \leq k < n$, a k-face of p is a k-polytope which is the intersection of P with a supporting hyperplane; the entire polytope P is its only n-face. For $0 \leq k \leq n$, $m_k(P)$ will denote the sum of the k-measures (Lebesque) of the various k-faces of P. Hence $m_0(P)$ is the number of vertices of P, $m_1(P)$ is the sum of the lengths of the edges of P, $m_2(P)$ is the sum of the areas of the 2-faces of P, ..., $m_n(P)$ is the n-dimensional volume of P. For example, if T is an equilateral triangle and S is a square (each of side length s) then $m_0(T) = 3$, $m_1(T) = 3s$, and $m_2(T) = \sqrt{3}s^2/4$, while $m_0(S) = 4$, $m_1(S) = 4s$, and $m_2(S) = s^2$. Similarly, if T_3 is a regular tetrahedron and P_T is a right equilateral triangular prism (each edge of both polytopes having length s), then $m_0(T_3) = 4$, $m_1(T_3) = 6s$, $m_2(T_3) = \sqrt{3}\, s^2$, $m_3(T_3) = s^3\sqrt{2}/12$, while $m_0(P_T) = 6$, $m_1(P_T) = (\sqrt{3} + b)s^2/2$, and $m_3(P_T) = s^2\sqrt{3}/4$ (cf. Figures 32 and 33).

Let I_n denote the set of all pairs (i,j) of distinct integers between 1 and n. Each $(i,j) \in I_n$ is associated with a generalized (i,j)-isoperimetric ratio applicable to each $P \in \mathcal{P}_n$.

The following generalization of Equation 1.1.2 is now possible:

$$I_{ij}(\lambda P) = \frac{\sqrt[i]{m_i(\lambda P)}}{\sqrt[j]{m_j(\lambda P)}} = \frac{\sqrt[i]{\lambda^i m_i(P)}}{\sqrt[j]{\lambda^j m_j(P)}}$$

$$= \frac{|\lambda| \sqrt[i]{m_i(P)}}{|\lambda| \sqrt[j]{m_j(P)}} = I_{ij}(P),\ \lambda \neq 0. \qquad (4.1.1)$$

Hence the generalized (i,j)-isoperimetric ratio also is dependent on the shape rather than the size of a polytope.

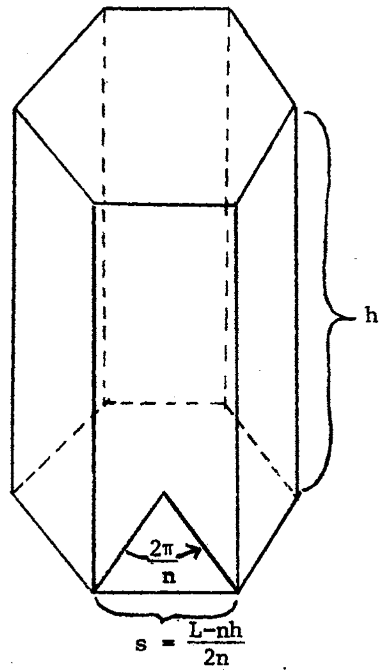
2. Maximization of Ratios on the Prisms in E_3

The ratios which are of greatest interest in E_3 are those associated with the ordered pairs (1,2),(1,3),(2,1),(2,3),(3,1) and (3,2). If P_h is the square right prism of side length s and height h, then $m_1(P_h) = 8s+4h$, $m_2(P_h) = 2s^2+rsh$, and $m_3(P_h) = s^2h$ (cf. Figure 32). Hence

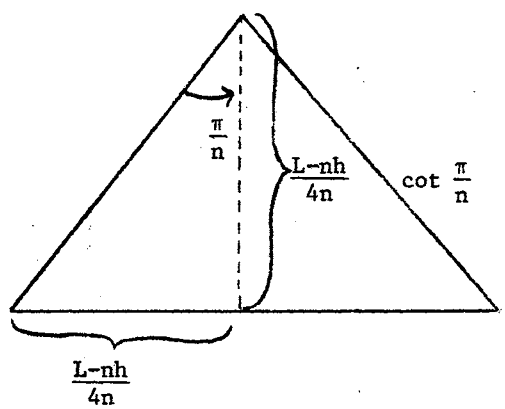
$$I_{12}(P_h) = \frac{m_1(P_h)}{\sqrt{m_2(P_h)}} = \frac{8s + 4h}{\sqrt{2s^2 + 4sh}} = \frac{6s}{\sqrt{2s^2 + 4sh}} + \sqrt{2s + 4h}$$

so $I_{12}(P_h) \to \infty$ as $h \to \infty$,

$$I_{13}(P_h) = \frac{m_1(P_h)}{\sqrt[3]{m_3(P_h)}} = \frac{8s + 4h}{\sqrt[3]{s^2h}} = 8\left(\frac{s}{h}\right)^{(1/3)} + 4\left(\frac{h}{s}\right)^{(2/3)}$$



(a) A Regular Right n-sided Prism of Height h and Side Length s, such that Total Edge Length Is L



(b) A Triangle Portion of the Base of the Prism

Figure 32. A Right Prism with a Regular n-gon as Base

so $I_{13}(P_h) \to \infty$ as $h \to \infty$, and

$$I_{23}(P_h) = \frac{\sqrt{m_2(P_h)}}{\sqrt[3]{m_3(P_h)}} = \sqrt[6]{\frac{8s^2}{h^2} + \frac{48s}{h} + 96 + 64\frac{h}{s}}$$

so $I_{23}(P_h) \to \infty$ as $h \to \infty$. The following theorems will show that upper bounds exist for the remaining ratios, however.

Theorem 4.2.1. (Kömhoff) If $\mathcal{P}_3$ is the class of prisms in E_3, then $I_{21}(P) \leq (2(12 - \sqrt{3}))^{-(1/2)}$ for $P \in \mathcal{P}_3$ with equality if and only if P is an equilateral triangular right prism whose height h is $(6 - \sqrt{3})/9$ times the perimeter of its base $p(B)$.

Proof: (i). In view of Equation 4.1.1 it may be assumed without loss of generality that $m_1(P) = L$.

(ii). Among the prisms of total edge length L with fixed base B, the right prism has maximum surface area. To see this, note that Theorem 3.1.1 implies that a rectangle encloses greater area than does any other quadrilateral with sides of the same length and the same order of succession. Only the right prism has all its sides rectangular.

(iii). Theorem 3.1.4 implies, therefore, that among the prisms of height h, total edge length L, and with $2n$ vertices, the right prism with a regular n-gon as base maximizes surface area A.

(iv). The right prism with a regular n-gon as base, fixed edge length L, and height h has area given by

$$A(h) = h \;\frac{L-nh}{2} \;+ \left(\frac{L-nh}{2}\right)\left(\frac{L-nh}{4n} \cot \frac{\pi}{n}\right)$$

$$= \frac{Lh}{2} - \frac{nh^2}{2} + \left(\frac{L^2}{8n} - \frac{Lh}{4} + \frac{nh^2}{8}\right) \cot \frac{\pi}{n}$$

(cf. Figure 32).

Since $A(h)$ is a polynomial in h, it is differentiable everywhere so $A'(h) = 0$ at relative maxima and minima. Now

$$A'(h) = \frac{L}{2} - nh - \frac{L}{4} \cot \frac{\pi}{n} + \frac{nh}{4} \cot \frac{\pi}{n} = 0$$

if and only if

$$h = \frac{L}{n} \frac{\cot(\pi/n) - 2}{\cot(\pi/n) - 4} \quad .$$

Since $A(h)$ is continuous and differentiable on the interval $[0, L/n]$ of feasible values for h on which $A(h) \geq A(0) = A(L/n) = 0$, no minima of $A(h)$ may occur on $(0, L/n)$ without the simultaneous occurence of a maxima at some other point of $(0, L/n)$. Hence the maximum value of $A(h)$ on $[0, L/n]$ must occur when

$$h = \frac{L}{n} \frac{\cot(\pi/n) - 2}{\cot(\pi/n) - 4} \quad .$$

(v). The right prism with total edge length L, a regular n-gon as base, and height

$$h = \frac{L}{n} \frac{\cot(\pi/n) - 2}{\cot(\pi/n) - 4}$$

has total surface area given as a function of n by

$$A(n) = \frac{L-nh}{2}\left(h + \frac{L-nh}{4n}\cot\frac{\pi}{n}\right)$$

$$= \frac{L - n\left(\frac{L}{n}\frac{\cot(\pi/n) - 2}{\cot(\pi/n) - 4}\right)}{2}\left(\frac{L}{n}\frac{\cot(\pi/n) - 2}{\cot(\pi/n) - 4} + \frac{L - n\left(\frac{L}{n}\frac{\cot(\pi/n) - 2}{\cot(\pi/n) - 4}\right)}{4n}\cot(\pi/n)\right)$$

$$= \frac{L^2}{2n(4-\cot(\pi/n))} .$$

The function $A(n)$ is continuous and differentiable on the interval $[3, \pi/(\operatorname{arccot} 4)]$ containing all the feasible values for n. On this interval

$$A'(n) = \frac{-L^2}{2n^2}\left(\frac{1}{4-\cot(\pi/n)} + \frac{\pi\csc^2(\pi/n)}{n(4-\cot(\pi/n))^2}\right) < 0,$$

so $A(n)$ is decreasing on this interval since $4 - \cot(\pi/n) > 0$ and hence must attain its minimum at the left endpoint; i.e., $n = 3$.

Hence the isoperimetric ratio is maximized at the equilateral traingular right prism P_T whose height is given by

$$h = \frac{L}{3}\ \frac{1 - 2\sqrt{3}}{1 - 4\sqrt{3}} .$$

Thus the maximum value is

$$I_{21}(P_T) = \frac{\sqrt{m_2(P_T)}}{m_1(P_T)} = \frac{\sqrt{L^2/6(4-\cot(\pi/3))}}{L} = \frac{1}{\sqrt{2(12-\sqrt{3})}} .$$

Since

$$h = \frac{L}{3}\ \frac{1 - 2\sqrt{3}}{1 - 4\sqrt{3}} = \frac{L}{3}\ \frac{6 - \sqrt{3}}{12 - \sqrt{3}} ,$$

it follows that $(36-3\sqrt{3})h = (6-\sqrt{3})\ L$ which implies that

$$h = \frac{6 - \sqrt{3}}{9}\left(\frac{L - 3h}{2}\right) .$$

But $p(B) = (L - 3h)/2$ so $h = (6 - \sqrt{3})p(B)/9$ as required. ■

The following theorem shows that the (3,1) ratio is also maximized on an equilateral triangular right prism. However, the height of the prism is different from that of the above prism.

Theorem 4.2.2. If P_3 is the class of prisms in E_3, then $I_{31}(P) \leq 2^{-(2/3)}3^{-(11/6)}$ with equality only when P is an equilateral triangular right prism all of whose edges have the same length.

Proof: (i). In view of Equation 4.2.1, it may again be assumed without loss of generality that $m_1(P) = L$ or that $m_3(P) = V$.

(ii). Among those prisms of a fixed base B and a fixed height h (hence also of fixed volume), the right prism has minimum total edge length L. This follows from the fact that the shortest segment with endpoints in each of two parallel planes is perpendicular to both planes. Only the right prism has its edges perpendicular to its base.

(iii). Among those right prisms of fixed height h, total edge length L and with $2n$ vertices, that prism whose base is a regular n-gon has the greatest volume. This follows as an easy consequence of Theorem 3.1.4.

(iv). The right prism with a regular n-gon as base and fixed edge length L has volume given as a function of height h by

$$V(h) = h\left(\frac{L-nh}{4}\right)\left(\frac{L-nh}{4n}\right)\cot\left(\frac{\pi}{n}\right)$$
$$= (L^2h - 2Lnh^2 + n^2h^3)\,\frac{\cot\left(\frac{\pi}{n}\right)}{16n}$$

(cf. Figure 32).

Again $V(h)$ is a polynomial in h so it is differentiable everywhere and $V'(h) = 0$ at relative maxima and minima. Then

$$V'(h) = \frac{1}{16n}\,(L^2 - 4Lnh + 3n^2h^2)\,\cot\left(\frac{\pi}{n}\right) = 0$$

if and only if

$$h = \frac{4Ln \pm \sqrt{(4Ln)^2 - 4(3n^2)L^2}}{2(3n^2)} = \frac{L}{n} \text{ or } \frac{L}{3n} \quad .$$

Also

$$V''(h) = \frac{1}{16n}\,(-4Ln + 6n^2h)\,\cot\left(\frac{\pi}{n}\right)$$

so

$$V''\left(\frac{L}{n}\right) = \frac{L}{8}\cot\left(\frac{\pi}{n}\right) > 0$$

and hence $V(h)$ has a relative minimum at $h = \frac{L}{n}$. Furthermore,

$$V''\left(\frac{L}{3n}\right) = -\frac{1}{8}\,L\,\cot\left(\frac{\pi}{n}\right) < 0$$

and $V(h)$ has a relative maximum at $h = \frac{L}{3n}$.

$$V(0) = V\left(\frac{L}{n}\right) = 0,\ V\left(\frac{L}{3n}\right) > 0, \text{ and } 0 < \frac{L}{3n} < \frac{L}{n},$$

$V(h)$ must achieve its maximum at $h = L/(3n)$.

(v). A right prism with fixed edge length L, regular n-gon as base, and height $h = L/(3n)$ has volume

$$V(n) = h\left(\frac{L-nh}{4}\right)\left(\frac{L-nh}{4n}\right)\cot\left(\frac{\pi}{n}\right)$$

$$= \frac{L}{3n}\ \frac{L-Ln/(3n)}{4}\ \frac{L-Ln/(3n)}{4n}\cot\left(\frac{\pi}{n}\right)$$

$$= \frac{L^3}{108\ n^2}\cot\frac{\pi}{n}$$

$$= \frac{L^3}{108\pi^2}\left(\frac{\pi}{n}\right)^2\cot\frac{\pi}{n},$$

for $n = 3,4,5,\ldots$

Let $f(x) = X^2 \cot x$, then $f'(x) = 2x \cot x - x^2\csc^2 x$. So for $x \in (0, \pi/2)$, $f'(x) > 0$ if and only if $2 \cos x > x \csc x$; i.3., if and only if $\sin x \cos x > x/2$. But $\tan x > x$ for $x \in (0, \pi/2)$, so $f'(x) > 0$ if $\sin x \cos x > (\tan x)/2$ that is, if $\cos x > 2^{-(1/2)}$. Now $\cos x > 2^{-(1/2)}$ when $x \in (0, \pi/4)$ so $f'(x) > 0$ and f is increasing on $(0, \pi/4)$.

If $g(n) = \pi/n$, then $g(n)$ is decreasing for $n = 4,5,6,\ldots$. Since $V(n) = L^3 f(g(n))/(108\pi^2)$, then $V(n)$ is decreasing for $n = 4,5,6,\ldots$. Also

$$V(3) = \frac{L^3}{108}\ \frac{1}{9\sqrt{3}} > \frac{L^3}{108}\,\frac{1}{16} = V(4)$$

so $V(n)$ is decreasing on its domain. Therefore, $V(n)$ (hence also $I_{31}(P)$) is maximized when $n = 3$. The (3,1) ratio for this "completely equilateral" triangular prism is

$$I_{31}(P_{T'}) = \frac{\sqrt[3]{m_3(P_{T'})}}{m_1(P_{T'})} = \frac{\sqrt[3]{L^3/(972\sqrt{3})}}{L} = 2^{-(2/3)}3^{-(11/6)}.$$

The following theorem shows that the least upper bound of $I_{32}(P)$ is attained by no prism in E_3.

Theorem 4.2.3. Let $\mathcal{P}_3$ denote the collection of prisms in E_3. Then $\sqrt[6]{(54\pi)^{-1}}$ is the least upper bound for the set $\{I_{32}(P):\ P \in \mathcal{P}_3\}$ but it is attained by no $P \in \mathcal{P}_3$.

Proof: (i). As before, we may assume without loss of generality that $m_3(P) = B$.

(ii). Among those prisms of a given height h and base B (hence also of constant volume) the right prism minimizes surface area. To see this, note that among the rectangles whose opposite sides are parallel segments of fixed length lying in two fixed parallel planes, that one has minimum area which lies in a plane perpendicular to the given ones. Since the right prism is the only one all of whose sides lie in planes perpendicular to its bases, the result follows.

(iii). Theorem 3.1.4 implies that among the right prisms of a fixed height h and having as base an n-gon of fixed area A (hence also having fixed volume V), that one has minimum base perimeter (hence minimum surface area) which has a regular n-gon for its base.

(iv). Theorem 3.1.5 implies that between two right prisms having fixed height h and a regular n-gon for its base of fixed area A

(therefore also having a fixed volume V) is the one with least surface area has the greater number of sides. Hence surface area is minimized as the base approaches the shape of a circle.

(v). A right circular cylinder with height h and volume V is the limit (Hansdorff metric) of the sequence $\{P_n\}$ of right prisms P_n of height h, volume V and with a regular n-gon as base. Since surface area is continuous, the surface area of the cylinder (hence also the limit of the sequence of (3,2)-isoperimetric ratios of $\{P_n\}$ is approximated to any required degree of accuracy by that of each term of the sequence $\{A(P_n)\}$ optimal height of these limit circular cylinders. The right circular cylinder C_h of radius r, volume V and height h satisfies the following

$$V = \ r^2 h$$

and

$$m_2(C_h) = 2\ r^2 + 2\ rh$$

$$= 2\ r^2 + 2V/r\ .$$

Let $f(r) = 2\ r^2 + 2V(1/r)$. Then $f\ (r) = 4\ r - (2V/r)$ if and only if

$$r^3 = \frac{V}{2} = \frac{r^2 h}{2}\ .$$

That is, if $h = 2r$. Also $f''(r) = 4\pi + 4V/r^3$ so $f''(h/2) = 4 + 32V/h^3 > 0$ and $h = 2r$ is a relative minimum for $m_2(C_h)$. Furthermore, f has an absolute minimum at $r = h/2$ since $f(0) = f(\infty) = \infty$. If C^* is a right cylinder whose height is the diameter, $2r$, of its base, then $m_2(C^*) = 6\pi r^2$ and $m_3(C^*) = 2\pi r^3$.

Therefore,

$$I_{32}(C^*) = \frac{\sqrt[3]{m_3(C^*)}}{\sqrt{m_2(C^*)}} = \frac{\sqrt[3]{2\pi r^3}}{\sqrt{6\pi r^2}} = \sqrt[6]{\frac{1}{54\pi}} ,$$

but

$$I_{32}(P) < \sqrt[6]{(54\pi)^{-1}}$$

for every $P \in \mathcal{P}_3$ as required. ■

Note that C^* is a right cylinder circumscribed about a sphere.

3. Maximization of Ratios on the Tetrahedrons in E_3

The ratios which will be investigated in this section are again those associated with the ordered pairs (1,2),(1,3),(2,1),(2,3),(3,1) and (3,2). Let T_h be the tetrahedron whose base has sides of equal length s and whose vertex V lies h units above the centroid C of the base. Then

$$m_1(T_h) = 3s + 3\sqrt{s^2/3 + h^2} ,$$

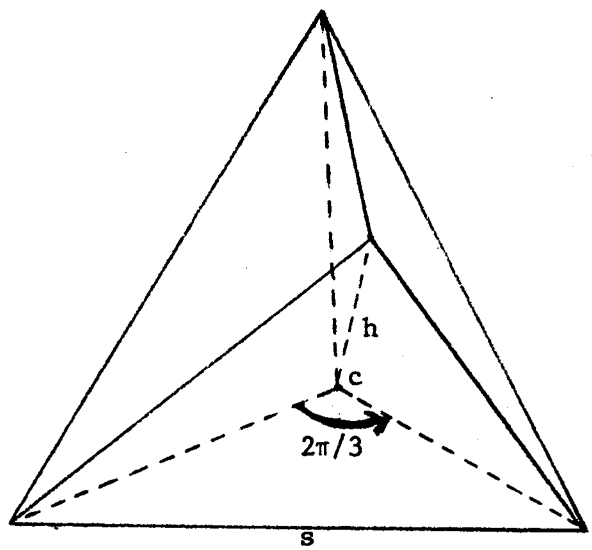
$$m_2(T_h) = \frac{3}{2} s \sqrt{h_2 + \frac{s^2}{12}} + \frac{\sqrt{3}s^2}{4} ,$$

$$3s\sqrt{h^2 + (s^2/12)}/2 + \sqrt{3}s^2/4$$

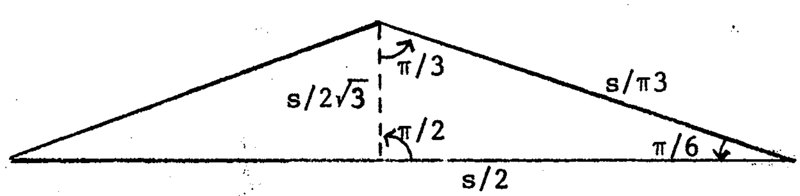
and

$$m_3(T_h) = \sqrt{3}/12s^2h$$

(cf. Figure 33).



(a) A Tetrahedron with Congruent Sides, Height h, and an Equilateral Base of Side Length s



(b) A Portion of the Base of the Tetrahedron

Figure 33. A Tetrahedron with an Equilateral Base and Congruent Sides

Note that $m_1(T_h) > h$ and $3sh/2 < m_2(T_h) < 3s(h+s)/2$. Hence

$$I_{12}(T_h) = \frac{m_1(T_h)}{m_2(T_h)} > \frac{h}{\sqrt{3s(h+s)/2}} = \sqrt{2(h+s)/3s} - \sqrt{\frac{2s}{3}}\Big/\sqrt{h+s}$$

so $I_{12}(T_h) \to \infty$ as $h \to \infty$.

$$I_{13}(T_h) = \frac{m_1(T_h)}{\sqrt[3]{m_3(T_h)}} > \frac{h}{\sqrt{3s(h+s)/2}} = \sqrt[3]{\frac{12}{s^2\sqrt{3}}}\; h^{(2/3)}$$

so $I_{13}(T_h) \to \infty$ as $h \to \infty$, and

$$I_{23}(T_h) = \frac{\sqrt{m_2(T_h)}}{\sqrt[3]{m_3(T_h)}} > \frac{\sqrt{(3/2)sh}}{\sqrt[3]{(\sqrt{3}/12)s^2h}} = \left(\frac{162}{s}\,h\right)^{(1/6)}$$

so $I_{23}(T_h) \to \infty$ as $h \to \infty$.

The following theorems will show that upper bounds do exist for the remaining ratios, however.

<u>Theorem</u> 4.3.1. If $\mathcal{T}$ is the class of tetrahedrons in E_3, then $I_{21}(T) \le (12\sqrt{3})^{-\frac{1}{2}}$ for $T \in \mathcal{T}$, with equality if and only if T is regular.

Kömhoff [8] proved a theorem which implies that $I_{21}(t) \le (12\sqrt{3})^{-\frac{1}{2}}$ for all polytopes T with triangular faces, with equality if and only if T is a regular tetrahedron. In particular, this inequality must hold on the subcollection of polytopes $\mathcal{T}$.

The next theorem shows that the regular tetrahedron which maximizes the (2,1)-ratio on $\mathcal{T}$ also maximizes the (3,1)-ratio on $\mathcal{T}$. Hence for a given total edge length, the regular tetrahedron maximizes both surface area and volume on $\mathcal{T}$.

<u>Theorem</u> 4.3.2. If $\mathcal{T}$ is the class of tetrahedrons in E_3, then $I_{31}(T) \leq \left(6 \sqrt[3]{6} \sqrt[2]{2}\right)^{-1}$ for $T \in \mathcal{T}$, with equality if and only if T is regular.

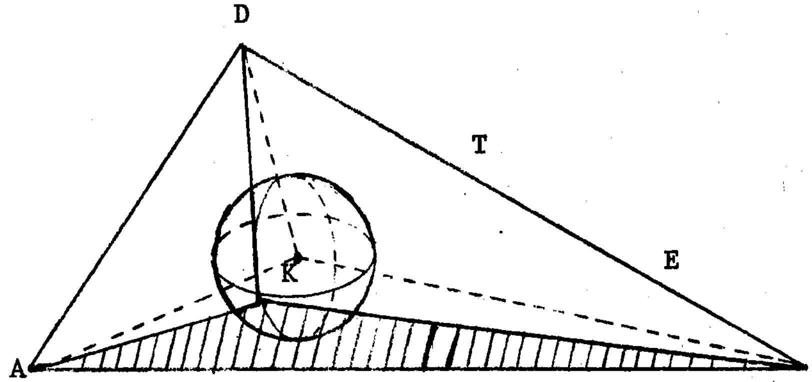
Z. A. Melzak [10] proved an equivalent theorem by a vector argument.

The remaining theorem shows that the regular tetrahedron also maximizes the (3,2)-ratio on $\mathcal{T}$. Hence it is optimal for each applicable ratio on $\mathcal{T}$ and among those tetrahedrons of a given volume, the regular tetrahedron minimizes both edge length and surface area.

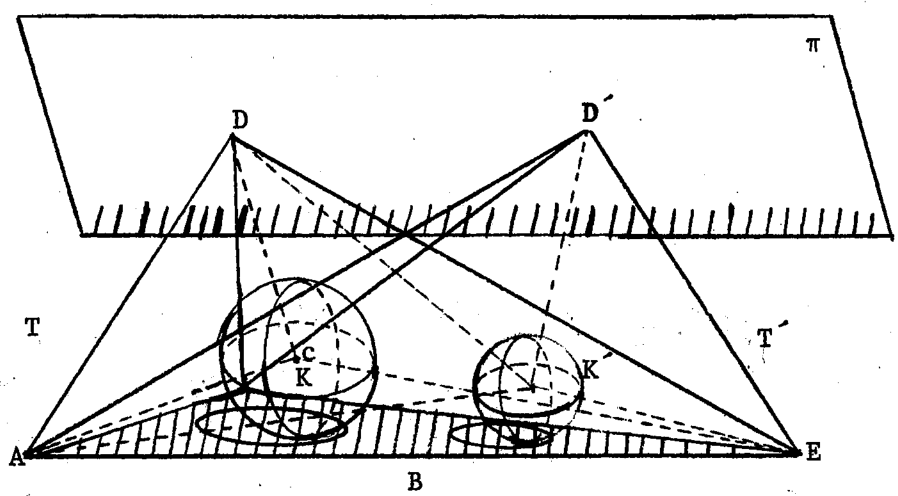
<u>Theorem</u> 4.3.3. If $\mathcal{T}$ is the class of tetrahedrons in E_3, then $I_{32}(T) \leq (\sqrt{2} \sqrt[4]{3})^{-1}$ for $T \in \mathcal{T}$, with equality if and only if T is regular.

Proof: (i). Among the tetrahedrons of a given base B and which a fixed height h (therefore also of a fixed volume), that one has minimum surface area whose vertex opposite B lies on the lines perpendicular to B at the center of its inscribed circle.

To see this, note first that a tetrahedron may be divided into four tetrahedrons whose bases are the four faces of the original tetrahedron and whose common vertex is the center K of the inscribed sphere (cf. Figure 34(a)). Hence the volume of a tetrahedron is (1/3)Ar; where A is the surface area of T and r is the radius of



(a) Tetrahedron AECD Is Divided into Four Tetrahedra ADCK, AECK, AEDK and ECDK



(b) Maximizing the Radius of the Inscribed Circle

Figure 34. Minimizing Surface Area on the Collection of Tetrahedrons with Fixed Base and Height

its inscribed sphere. Note also that the volume of a tetrahedron is (1/3)ah where a is the area of B the base. Hence volume is constant on the collection $\mathcal{T}_{B,h}$ of tetrahedrons with a fixed base B and whose vertices lie in a plane h parallel to and at a distance h from the plane containing B. The conclusion now follows from the fact that among the tetrahedrons in $\mathcal{T}_{B,h}$, that one whose vertex lies on the line perpendicular to B at the center of its inscribed circle maximizes the radius of the inscribed sphere and hence minimizes A since (1/3)Ar = a constant (cf. Figure 34(b)).

(ii). Among the tetrahedrons of fixed base area a and height h (therefore also of fixed volume) and whose vertices lie on the lines perpendicular to their bases at the centers of their inscribed circles, that one minimizes surface area whose base is equilateral. To see this note that if p is the perimeter of base B and r is the radius of its inscribed circle, then base area $a = rp/2$ (cf. Figure 33(a)). Also the total surface area of T is given by $A_T = a + p\sqrt{r^2 + h^2}/2 = A(r,p)$ (cf. Figure 33(b)). Hence $A_T = A(r) = a + a(1 + \sqrt{r^2 + h^2}/4)$. Then $A'(r) = -ah^2/r^2\sqrt{r^2 + h^2} < 0$, so A(r) is a decreasing function of r which will be minimized when r is maximized. Since $r = 2a/p$ and a is fixed, r will be maximized when p is minimized and Theorem 3.1.4 implies that p is minimized when B is equilateral.

(iii). Among the tetrahedrons of a given volume, the regular tetrahedron alone minimizes surface area.

This follows from the preceding steps in the following manner: A convergent (Hausdorff metric) sequence of tetrahedrons has a tetrahedron as limit. An argument similar to that employed in the proof of the Existence Theorem of Chapter I implies therefore that a tetrahedron of minimum surface area exists among those of a given volume. Steps (i) and (ii) together imply that this minimizing tetrahedron has an equilateral triangular base and its fourth vertex lies on a line perpendicular to this base at the center of its inscribed circle. Hence the three sides of this tetrahedron's base must be congruent. Now regarding one of the other faces as the base, the preceding argument implies that all sides of the tetrahedron are congruent so that the tetrahedron is regular.

If R is a regular tetrahedron of side length s and height h, then $h = s\sqrt{b}/3$. Hence $m_3(R) = s^2h\sqrt{3}/4 = s^3\sqrt{2}/4$ and $m_2(R) = 3s/2.$ $\sqrt{h^2 + s^2/12} + (\sqrt{3}/4)s^2 = \sqrt{3}\ s^2$ and

$$I_{32}(R) = \frac{\sqrt[3]{(\sqrt{2}/4)s^3}}{\sqrt{\sqrt{3}\ s^2}} = \frac{1}{\sqrt{2}\sqrt{3}} . \quad \blacksquare$$

4. Maximization of Ratios on the Polytopes in E_n

Let

$$B(n,i,j) = \sup\{I_{ij}(P) : P \text{ is an } n\text{-polytope}\}.$$

It follows from Sections 2 and 3, for example, that $B(3,1,2) = B(3,1,3) = B(3,2,3) = \infty$. Lower bounds were also established for $B(3,2,1)$, $B(3,3,1)$, and $B(3,3,2)$ in those sections.

From Section 2,

$$B(3,2,1) \geq \frac{1}{\sqrt{2(12 - \sqrt{3})}} \approx .2207.$$

Kömhoff [9] improved slightly on this bound when he calculated the (2,1)-ratios associated with a sequence of "shell-polytopes" to show that

$$B(3,2,1) \geq \frac{1}{\sqrt{2(\sqrt{3} + 8\pi/3)}} \approx .2224.$$

Oliver Aberth [1] obtained an upper bound of $(\sqrt{6}\pi)^{-1} \approx .2303$ for this ratio. Hence the value of $B(3,2,1)$ must be between .2224 and .2303. It is interesting to note that the optimal polygons described in Theorems 3.2.1 and 3.3.1 have the minimum possible number of faces consistent with the contraints of those theorems. This feature seems to override in desirability the characteristic suggested by Theorem 2.4.5 of having each facet a many sided regular polygon.

In Section 2 it was shown that $B(3,3,1) \geq 2^{-(1/2)}3^{-(11/6)}$. Melzak [12] conjectures that $B(3,3,1) = 2^{-(2/3)}3^{-(11/6)}$, where this value is assumed only when P is a right prism whose base is an equilateral triangle with sides equal in length to the height of the prism.

Since the area and volume functions are continuous (Hausdorff metric), Theorem 1.5.5 implies that $B(3,3,2) = \left(\sqrt[3]{6}\,\sqrt[6]{\pi}\right)^{-1} = I(S)$, where S is a ball. Also from Theorem 1.5.5, $I_{32}(P) \leq \left(\sqrt[3]{6}\,\sqrt[6]{\pi}\right)^{-1}$ for each polytope P in E_3. From the continuity of area and volume, therefore, those polytopes have greater (3,2)-ratios which better approximate a ball. The following conjecture seems reasonable: Let $\mathcal{T}_p$ be the collection of polytopes P_T in E_3 which are the intersections of translates of the halfspaces whose boundaries contain the sides of a fixed polytope P and whose intersection is P. Those members of $\mathcal{T}_p$ which are circumscribed about a sphere maximize the (3,2)-isoperimetric ratio. This conjecture is analogus to Theorem 3.1.2 and is consistent with the result obtained in Theorem 4.2.3. It also implies for example, that a cube maximizes the (3,2)-isoperimetric ratio on the collection of right rectangular prisms.

More generally, it is interesting to investigate the question of which ordered triples (n,i,j) lead to finite values for $B(n,i,j)$. Since for $1 \leq i, j,k \leq n$ and $P \in E_n$, $I_{ij}(P) = I_{ik}(P) \cdot I_{kj}(P)$ it follows that $B(n,i,j) \leq B(n,i,k) \cdot B(n,k,j)$. Therefore $B(n,i,j)$ must be finite when $B(n,i,k)$ and $B(n,k,j)$ are finite. The finiteness of the (3,2)-isoperimetric ratio associated with the solution to the classical isoperimetric problem implies that $B(n,n,n-1)$ is finite for each value of $n \geq 2$. Eggleston, Grünbaum and Klee [5] extended this result by showing that $B(n,i,j)$ is finite when $i=n$, $i=n-1 > j$, or i is a multiple of j. Klee [7] showed that $B(n,i,j) = \infty$ when $i < j \leq n$ and conjectured in all other cases $B(n,i,j)$ is finite as is certainly the case when $n \leq 4$ (cf. Table 1). Even when $B(n,i,j)$ is known to be finite, its exact value has been determined only for the cases corresponding to the classical isoperimetric problem.

Although $B(n,n,n-1)$ is finite, it is not attained by any polytope. Grunbaum conjectures that all other finite bounds are attained on some polytope.

TABLE I

VALUES OF B(n,i,j) WHERE f, μ, AND ∞ INDICATE FINITE, UNKNOWN, AND INFINITE, RESPECTIVELY

j \ i	1	2	3	4	5	6	7	8	...	n-1	n
n	∞	∞	∞	∞	∞	∞	∞	∞		∞	1
n-1	∞	∞	∞	∞	∞	∞	∞	∞		1	f
...											
8	∞	∞	∞	∞	∞	∞	∞	1		f	f
7	∞	∞	∞	∞	∞	∞	1	μ		f	f
6	∞	∞	∞	∞	∞	1	μ	μ		f	f
5	∞	∞	∞	∞	1	μ	μ	μ		f	f
4	∞	∞	∞	1	μ	μ	μ	f		f	f
3	∞	∞	1	μ	μ	f	μ	μ		f	f
2	∞	1	μ	f	μ	f	μ	f		f	f
1	1	f	f	f	f	f	f	f		f	f

TABLE II

VALUES OF B(3,i,j) = x

j \ i	1	2	3
3	∞	∞	1
2	∞	1	$x = .4547$
1	1	$.2224 \leq x \leq .2302$	$x \geq .0841$

BIBLIOGRAPHY

1. Oliver Aberth, An isoperimetric inequality for polyhedra and its application to an extremal problem, Proceedings of the London Mathematical Society, (3) 13 (1963) 322-336.

2. Russell V. Benson, Euclidean Geometry and Convexity, McGraw Hill Book Company, New York. 1966.

3. T. Bonnesen and W. Fenchel, Theorie Der Konvexen Korper, Chelsea Publishing Company, New York, 1948.

4. R. Courant and H. Robbins, Plateau's problem, The World of Mathematics (Newman-editor) Vol. 2, Simon and Schuster, New York (1956) 901-909.

5. H. G. Eggleston, B. Grunbaum and V. Klee, Some semi-continuity theorems for convex polytopes and cell complexes, Comment. Math. Helv., 39 (1964) 165-188.

6. William J. Firey, Isoperimetric ratios of Reuleaux polygons, Pacific Journal of Mathematics, 19 (1960) 823-829.

7. Victor Klee, Which isoperimetric ratios are bounded? The American Mathematical Monthly, 77 (1970) 288-289.

8. Magelone Kömhoff, An isoperimetric inequality for convex polyhedra with triangular faces, Canadian Mathematical Bulletin, 8 (1966) 667-669.

9. Magelone Kömhoff, On a 3-dimensional isoperimetric problem, Canadian Mathematical Bulletin, 13 (1970) 447-449.

10. Z. A. Melzak, An isoperimetric inequality for tetrahedra, Canadian Mathematical Bulletin, 8 (1966) 667-669.

11. Z. A. Melzak, Numerical evaluation of an isoperimetric constant, Mathematical Comp., 22 (1968) 188-190.

12. Z. A. Melzak, Problems connected with convexity, Canadian Mathematical Bulletin, 8 (1965) 565-573.

13. G. T. Sallee, Maximal Areas of Reuleaux polygons, Canadian Mathematical Bulletin, 13 (1970) 175-179.

14. L. Fejes Tŏth, What the bees know and what they don't know, Bulletin of the American Mathematical Society, 70 (1964) 468-481

15. L. Fejes Tŏth and M. N. Bleicher, Two-dimensional honeycombs, The American Mathematical Monthly, 72 (1965) 969-973.

16. Frederick A. Valentine, Convex Sets, McGraw Hill Book Company, New York, 1964.

17. I. M. Yaglom and V. G. Boltyanski, Convex Figures, Holt, Rinehart and Winston, New York, 1961.

VITA

James Vernon Balch, Jr.

Candidate for the Degree of

Doctor of Education

Thesis: ISOPERIMETRIC PROBLEMS WITH SIDE CONDITIONS INVOLVING CONVEX BODIES IN E_n

Biographical:

Personal Data: Born in Batesville, Arkansas, September 9, 1942, the son of Mr. and Mrs. James V. Balch, Sr.

Education: Graduated from Batesville High School, Batesville, Arkansas in May, 1960; received the Bachelor of Arts degree from Arkansas College in Batesville, Arkansas in 1964 with a major in mathematics; received the Master of Science degree in mathematics from the Oklahoma State University in Stillwater, Oklahoma in July, 1969; completed requirements for Doctor of Education degree in higher education with emphasis in mathematics from the Oklahoma State University in December, 1973.

Professional Experience: Graduate Teaching Assistant, Department of Mathematics, Oklahoma State University, 1964-1971; Instructor, Arkansas College, 1965-1966; Instructor, Department of Mathematics, Oklahoma State University, 1971-1972; Assistant Professor of Mathematics, Milligan College, 1972-1974.