OPTIMAL REGIONAL CONTROL OF
DISTRIBUTED PARAMETER
SYSTEMS

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Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1973

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ACKNOWLEDGEMENTS

I would like to express my appreciation for the help given by the following people:

Dr. Karl N. Reid, my major advisor and chairman of my thesis committee, who provided counsel and guidance during my graduate program.

Dr. Henry R. Sebesta, Dr. Charles M. Bacon, Dr. James E. Bose and Dr. Robert J. Mulholland who served on my thesis committee and offered helpful suggestions and criticisms.

I would like to acknowledge the financial support which I received in the form of graduate teaching and research assistantships from the School of Mechanical and Aerospace Engineering and from the Center for Systems Science (under NSF Grant GU-3160).

I would like to thank my parents for their encouragement and support.

Finally, I would like to thank Harriet for providing encouragement and for her effort in the preparation of this thesis.

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NOMENCLATURE

A	adjoint vector
a_i	adjoint variable
b_i	boundary condition coefficient
c	differential equation coefficient
F	state equation differential operator
g_i	polynomial control coefficient
h	Hamiltonian
J	performance index (scalar)
J*	modified performance index (scalar)
L	control location parameter vector
l_i	control location parameter
M^o	initial condition vector function
M^f	terminal condition vector function
N	boundary condition vector function
N^a	boundary condition vector function
N^b	boundary condition vector function
p	parameter for two-dimensional boundary
Q_1	state approximation vector
q_{1i}	state approximation variable
Q_2	adjoint approximation vector
q_{2i}	adjoint approximation variable
R	basis function vector
r_i	basis function variable

S	state vector
s	state variable
t	temporal independent variable
U	control vector
u	control variable
w	performance index function
w^a	performance index function
w^b	performance index function
X	spatial independent variable vector
x	spatial independent variable
y^o	performance index function
y^f	performance index function
z	performance index function
α	spline function parameter
β	spline function parameter
γ	gradient scaling parameter
ζ	independent variable
η	one-dimensional Hermite interpolation function
Θ	temporal control amplitude vector
θ	temporal control amplitude
κ	gradient constant
λ	Lagrangian interpolation function
μ	performance index coefficient
ν	performance index coefficient
ξ	independent variable
ρ	gradient variable
σ	gradient variable

ν	gradient scaling parameter
$\bar{\Phi}$	spatial control distribution vector
ϕ	spatial control distribution
ψ	two-dimensional Hermite interpolation function
Ω	two-dimensional spatial domain
Ω_b	two-dimensional spatial domain boundary

CHAPTER I

PROBLEM DEFINITION

Introduction

Although most of the research to date in the area of optimal control has been limited in application to systems described by ordinary differential equations, much interest in the optimal control of distributed parameter systems has been demonstrated in the literature. However, the gap between the theory and its applications remains large. Most methods of handling optimal control problems involving distributed parameter systems are limited to relatively simple problems because of the theoretical limitations of the methods or the impractical complexity of the application of the methods to more realistic problems. Problems involving one spatial dimension do not contain the possible boundary complexity of those of multiple spatial dimensions. Also, while controls may be theoretically completely distributed in space, realistic problems often contain controls which are constrained to regions which may be optimally located. These practically oriented problems form the basis of this research.

Statement of the Problem

This research involves an investigation of an optimal control problem for a class of distributed parameter systems and the development of a procedure for its solution. The class of systems under consideration is constrained to include only those which can be described by a set of coupled, parabolic partial differential equations. Both systems with one spatial independent variable and systems with two spatial independent variables together with the single temporal independent variable are included. An irregularly shaped spatial domain is allowed in the case of two spatial independent variables. System initial conditions and spatial boundary conditions are to be completely specified. The one or more spatially distributed controls are to be constrained to exist only in a finite number of spatial regions, the optimal locations and time-dependent amplitudes of which are to be determined.

Research Objectives

The fundamental research objective was the development of a procedure for determining the optimum open-loop control for a class of regionally controlled distributed parameter systems. This objective has been reached through the completion of a natural series of lesser objectives, the first of which was the conduction of a survey of the literature pertaining to the optimal control of distributed parameter systems. This survey contains information pertaining to the classification of optimal control problems involving distributed parameter systems, a discussion of formulations of the optimal control problem found in the literature, and background material directly related to the optimal regional control problem.

The second objective was the mathematical formulation of the optimal, open-loop, regional control problem for the class of distributed parameter systems described above. This problem was formulated as a constrained minimization problem. The employment of the Lagrange multipliers allowed the adjoining of the partial differential equation constraint and initial and boundary condition constraints to the integral performance index. Application of calculus of variations then resulted in a set of conditions necessary for optimality in the form of a boundary value problem.

The third objective was the development of a method for the solution of the boundary value problem. Various approximation techniques were considered and the Galerkin method was chosen for the reduction of the partial differential equations to sets of ordinary differential equations. Eigenfunctions, Hermite interpolation polynomials and fundamental splines were considered for functional expansions to be employed in conjunction with the Galerkin method. Application of a gradient approach in conjunction with the Galerkin approximation resulted in a computational algorithm for the boundary value problem solution.

The last objective was the application of the algorithm developed to example problems in one and two spatial independent variables. Example solutions were obtained for the open-loop control of the temperature distribution along a thin rod and on a thin, irregularly shaped plate.

The literature survey, mathematical formulation, boundary value problem solution, and applications comprise Chapter II through Chapter V. Chapter VI contains a discussion of the principal results of this research and recommendations for further study.

CHAPTER II

LITERATURE SURVEY

Background

In the past ten or twenty years optimal control theory for lumped parameter systems has been given much attention. Many texts as well as countless papers have been written on the topic. The theory upon which optimal control is based dates back to the end of the 17th century when Bernoulli posed the Brachistochrone problem. The extensive research conducted in the area of optimal control theory has lead to its application to continually expanding classes of problems.

Optimal control theory has been extended into the realm of distributed parameter systems as a result of interest in the control of systems described by partial differential equations. Although the problem had more than likely been mathematically considered earlier, Butkovskii and Lerner (14) presented the first work aimed at a practical, physical situation, the one-sided heating of a thin lamina moving through a furnace. Since 1960, interest has spread to more complex solid mechanics problems such as the stress constrained temperature control of the solid fuel rocket (7). The breadth of the field of possible application has even expanded to include the presently popular area of pollution control (31, 49). Before considering some of the contributions, some background information about distributed parameter systems might be of value.

The first question, "Why a distributed parameter model?" must be given some attention. The common approach, lumped parameter modeling, is a mathematical statement that the system of interest is adequately described in general, by a finite set of timewise continuous but spatially discrete functions (48). In practice, lumped parameter modeling is often sufficient and many times demanded by the physical situation; however, situations also exist where spatial discretizing imposes unrealistic constraints on the system, and thus detracts from the model accuracy and subsequently any optimization performed on that model. In this case, a discretized model must lead to a suboptimal control policy. Consequently, in what cases is the suboptimal control policy resulting from discretized modeling significantly different in comparison with the one which might result from optimization performed on a distributed parameter model? More accurately, when is the performance of the system significantly hampered by the employment of discretized modeling? Needless to say, a question of this magnitude is beyond the scope of this work, but is worth consideration. Undoubtedly, the complications introduced by the distributed parameter modeling over lumper parameter modeling must be justified, if only by the resulting insight into the physical situation. It should be noted that insight from lumped parameter optimal control has resulted in many approaches to suboptimally, but practically, controlling inherently lumped parameter systems.

Along the lines of suboptimal control resulting from discretizing another question must be posed. If optimization is to be performed computationally, at what point in the optimization procedure, modeling included, should the necessary discretizing take place? Athans (3) recommends the obvious, that the distributed parameter mathematical model be maintained as long as possible.

Another question of generality equal to those above involves the problem of performance indices. The optimization procedures so far developed for lumped parameter system models plague the engineer with the problem of selecting the proper performance index. This problem can only be compounded by the expansion from lumped parameter modeling to distributed parameter modeling. The possible choices of performance indices for distributed parameter systems becomes unimaginable with inclusion of equally many more varied physical situations to which the theory applies.

Above are only a few of the extremely general problems of interest to one considering the significant step from lumped parameter analysis to distributed parameter analysis. Following a brief review of a few of the physical situations to which distributed parameter optimization has been applied some more specific problems will be considered along with a classification of distributed parameter optimal control problems.

Classification of Optimal Control Problems

For the sake of organization of the distributed parameter system optimal control problem, it is of value to classify the types of optimal control problems under consideration. Although a listing of some examples cited by various authors could not be construed as a satisfactory method of classification, inclusion of such a discussion is appropriate. The most common example cited is the optimal temperature control problem where the plant is described by the nonhomogeneous heat conduction equation, a one-dimensional, linear, parabolic partial differential equation, (1, 4, 9, 11, 12, 13, 14, 17, 20, 23, 25, 29, 30, 36, 38, 39, 42, 44, 51, 53, 54). In the field of heat transfer, authors have considered various other examples, such as: the optimal control of coolant flow rate through a nuclear rocket to control the temperature gradient in fuel cells, the optimal control of a tubular reactor with radial diffusion, the optimal control of heat exchangers, and an optimal control problem involving an ablative shield on an aerodynamic re-entry vehicle (7, 16, 29, 31, 32, 33, 38). Other authors have cited examples involving the wave equation or the beam equation (8, 23, 29). The theory of the optimal control of distributed parameter systems has been applied to the optimal aeriation of a polluted river in order to control the biochemical oxygen demand (37, 49). The listing of interesting examples of applications of the theory could easily be extended.

Several authors have attempted the classification of optimal control problems involving plants described by the partial differential equations (8, 10, 38, 52, 53). Wang and Tung (53) and Wang (52) give a fairly comprehensive listing of classification possibilities. The most general of their classifications is concerned with the domain on which the system equation is defined. Fixed-domain systems are those having a specified spatial domain, while variable domain systems are those having domain boundaries which vary with time or certain variables defined on the domain. The state of the system defined on a variable domain must include additional variables which specify the instantaneous boundary motion.

A second common classification is according to the types of control variables involved, namely distributed control variables defined on the interior of the system domain, possibly at only specified points, and boundary control variables defined on the boundary or part of the system domain (52).

Classification of output transformation is another type of possibility suggested by Wang. These transformations may either be spatially dependent or spatially independent, i.e., a weighted spatial average.

Constaints provide a fourth classification approach. As in the lumped parameter case, constraints may be either equalities or inequalities. However, equality constraints may be spatial boundary conditions as well as initial conditions or temporal conditions. Inequality constraints might be bounded input amplitudes, bounded stated functions or bounded integral constraints.

Performance indices usually involve a spatial integral (terminal control) or spatial and time integrals. Closely associated with the performance indices is the basic objective of the problem, for example minimum energy or time optimal control.

Most lumped parameter optimal control work has delt with systems described by a set of coupled, first order ordinary differential equations. In most optimal control literature concerned with distributed parameter systems, the parallel approach of considering systems reducible to a set of partial differential equations of the first order in time is employed (4, 16, 26, 26, 31, 42, 43, 53, 54).

$$S_t(t,X) = F(S(t,X), S_X(t,X, \ldots, S_X k(t,X), U(t,X), t, X) \tag{1}$$

The above canonical form allows classification according to the number of state variables, the number of control variables, the number of independent spatial variables, and the highest order, possibly mixed, spatial partial differential operator.

By the above discussion of classifications of optimal control problems involving distributed parameter systems, the breadth of the field should be apparent. Some of the complications resulting from distributed parameter system modeling as opposed to lumped parameter system modeling can be seen in the above classifications while others may remain hidden without a general mathematical formulation of the optimal control problem.

An Optimal Control Problem Formulation

In the following section different formulations of the optimal control problem are discussed, various approaches to the derivation of necessary conditions are cited and several schemes for the solution of the resulting problem are considered. The objective of this discussion is to point out some of the more subtle problems associated with distributed parameter systems and their optimal control as well as to review some of the work presented in the literature.

Mathematical Statement of the Problem

The objective of the optimal control problem is to determine the control, unspecified initial conditions, unspecified terminal conditions, and unspecified boundary conditions such that, for the system described by a given set of partial differential equations, initial conditions, terminal conditions, boundary conditions and inequality constraints, the performance index is a minimum.

A discussion of literature involving the above optimal control problem begins with some comments concerning the partial differential equation set. As noted previously, most authors have utilized a form similar to the one presented above. Some authors have been somewhat more specific in their definition of a canonical form by considering linear equations of the form:

$$S_t(t,X) = L_X \cdot S(t,X) + U(t,X) \,, \tag{2}$$

where L_X is a linear, spatial, partial differential operator, (4, 5, 8). A third common approach is to begin with an integral equation which can be derived from the original partial differential equation (51).

$$S(t,X) = \int_0^t K(X,t,\tau)U(\tau)d\tau \tag{3}$$

Some problems arise in transforming higher order partial differential equations into the form of equation (1). Elliptic equations are not "well-posed" as initial value problems according to Brogan (5). He also notes that the obvious transformation for the one-dimensional wave equation leades to a set of two partial differential equations of second order in X which are not "well-posed".

Initial, terminal and boundary conditions give rise to a second area of problems. In most of the literature, the common problem of a completely specified set of initial conditions is considered. However, the general problem should include the possibility of the parallel of the initial condition manifold of the lumped parameter problem. This might be expressed in the form of a set of constraints on the initial state. Similarly, terminal conditions are usually unspecified; however, they might be totally specified or partially specified. The most interesting of the three specifications is boundary conditions. Completely specified boundary conditions are commonly considered. It is interesting to note that the unspecified boundary conditions may be viewed as boundary control functions to be specified by the optimization procedure. It is also interesting that there may exist problems in the specification of boundary conditions for a partial differential equation. The number of boundary conditions necessary for the solution of a particular partial differential equation depends on the type of equation being considered. In some instances, not all of the possible spatial derivatives may be independently specified on the boundary. Some references in which the boundary forcing function problem is considered are (8, 11, 16, 17, 25, 38, 44, 45, 54). Although most authors have considered only the unconstrained problem, a few have considered the problems of magnitude constraints on the distributed control function (1, 10, 11, 14, 45, 51). As in the case of lumped parameter optimization, constraints on the state variables are expected to cause additional difficulties (7, 13).

A variety of performance indices are utilized by authors in the literature. A spatial integral at the final time might be used for problems involving the deviation from a desired final state (1, 16, 29, 44, 51), while time integrals at a particular spatial point might be used for problems involving a time averaged deviation from a desired state at a fixed point in a spatial domain (9, 10, 11, 30). More general performance indices might be specified as a space-time integral (13, 16, 54), a space-time integral plus a space integral (26, 31, 42, 43), or a space-time integral plus a time integral (16, 25, 26). It should be noted that by suitable definition, different types may be converted to other types. The unfortunate usage of different forms of performance indices slightly alters the appearance of necessary conditions for optimality.

Formulation of Necessary Conditions

Most of the approaches to the formulation of necessary conditions follow three basic lines: calculus of variations, dynamic programming and functional analysis. Some authors employing the calculus of variations include Butovskii, Ergov and Lurie (13), Denn, Gray and Ferron (16), Hahn, Fan and Whang (26), Sage (43, 44), Sakawa (44, 45), Wiberg (54), and Brogan (8). Grahma and D'Souza (25) and Wang and Tung (53) employ dynamic programming to obtain necessary conditions. Axelband (5), Fleming (22), Lattes (34) and Lions (35) base their work on functional analysis.

The results of the application of the calculus of variations demonstrated in Chapter III compare with those found in the literature. Sage (42,43) considers the fixed initial and final time problem with specified initial and boundary conditions and no inequality constraints. Hahn (26) considers basically the same problem with the exception that boundary conditions are only partially specified. The result is a partial set of boundary conditions on the adjoint variables. Denn, Gray and Ferron (16) include a set of unknown functions of time in the boundary condition constraint functions. They employ spatial integrals of terminal condition constraint functions for a specialized application. Once the boundary value problem is formulated, the next step is to consider various methods of its solution.

Approaches to Boundary Value Problem Solution

Analytic solutions to optimal control problems involving the distributed parameter systems are difficult to obtain. Brogan (8) presents a Green's function approach and utilizes an "extended definition of the operator" to handle problems with boundary control functions. He presents analytical solutions for several common equations (the diffusion equation, the wave equation and the beam equation). The solutions inherently involve infinite series.

Various authors have contributed to the list of approximation techniques. Two basic approximation methods exist: discretization and eigenfunction truncation. In the first, the set of partial differential equations is spatially discretized and then handled with lumped parameter techniques. This method of reduction to a single independent variable by using finite differences is often called the method of lines (21). Butovskii (11), Sage (42, 43) and others discuss discretization methods. Wang and Tung (53) discuss some problems associated with discrete approximation, such as the controlability of approximate systems.

In their listing of forms of approximation, Wang and Tung (53) also mention spatial harmonic truncation, i.e., eignefunction truncation. Singh (47) discusses the eigenfunction method for conversion of partial differential equations to an infinite set of ordinary differential equations. This approach is basically the Green's function method which has not been taken to completion. A comparable approach, presented by Goodson (24) and Khatri and Goodson (30), is one in which transcendental terms found in the transfer functions of linear distributed parameter systems are approximated by infinite product expansions. However, it should be noted that in Khatri and Goodson's paper (30) only problems involving boundary control functions with performance indices involving time integrals are considered. In other words, the method is good for fairly accurate control at a particular spatial location.

Computational procedures for the solution of the optimal control problem have been presented in the literature. Denn, Gray and Ferron (29) and Hahn, Fan and Hwang (26) present gradient search techniques for distributed parameter systems. In both cases a suboptimal control is guessed and then iteratively improved. Sage (43) discusses a gradient approach and presents a method utilizing quasilinearization. He notes the two possibilities of linearization of the partial differential equation and linearization of a set of ordinary differential equations obtained by spatial discretizing. Brogan (8) presents several computational schemes for specialized problems.

The Regional Control Problem

Distributed parameter systems often involve inputs which can not be arbitrarily specified over the entire spatial domain. However, distributed inputs are commonly considered either spatially constant or spatially unconstrained. Athans (3) suggests that controls should be constrained to act at a finite number of locations, which might be optimally selected. He also notes that these controls should not be treated spatially as impulses. A somewhat specialized version of this problem is considered by Foster and Orner (23).

Foster and Orner formulate a linear regulator problem for a class of linear distributed parameter systems with two independent variables. The distributed inputs are constrained to a finite number of optimally located zones and the state of the system is observed at a finite number of optimal locations. They suggest two methods of reduction of the system partial differential equation to a finite order system of ordinary differential equations. The first method is a truncated eigenfunction expansion and the second is the Bubnov-Galerkin method. The solution of a matrix Riccati differential equation provides the feedback control law and a linear time-invariant dynamical observer is designed to provide an estimate of the state of the system.

The solution of an open-loop, regional control problem has not been found in literature. The remaining chapters are devoted to an approach for its solution.

CHAPTER III

MATHEMATICAL FORMULATION

The optimal control problem can be divided into two parts which are the mathematical formulation of the problem and the solution of the associated boundary value problem. The first step in the mathematical formulation of the optimal control problem is the transformation of a physically oriented statement of the problem to a mathematically oriented statement in the form of a constrained minimization problem. The constrained minimization problem is then converted to a set of conditions necessary for optimality by the application of Lagrange multipliers and the calculus of variations.

Constrained Minimization Problem

In this section, a general form of the open-loop, optimal, regional control problem with one spatial independent variable is formulated as a constrained minimization problem. Then the additional generality of two spatial independent variables is added.

Consider a system with a state vector $S(t,x)$ which is described by a set of coupled, parabolic, partial differential equations defined for all $t \epsilon (t_o, t_f)$ and $x \epsilon (x_a, x_b)$. That is:

$$S_t(t,x) = F(S(t,x), X_x(t,x), S_{xx}(t,x), U(t,x), t, x).$$

The system is constrained by a possibly partial set of initial and terminal conditions and a complete set of boundary conditions.

$$M^o(S(t,x),x) = 0 \quad ; \quad \forall\ x\epsilon(x_a,x_b), t = t_o$$
$$M^f(S(t,x),x) = 0 \quad ; \quad \forall\ x\epsilon(x_a,x_b), t = t_f$$
$$N^a(S(t,x).S_x(t,x)) = 0 \quad ; \quad \forall\ t\epsilon(t_o,t_f), x = x_a$$
$$N^b(S(t,x),S_x(t,x)) = 0 \quad ; \quad \forall\ t\epsilon(t_o,t_f), x = x_b$$

The control vector, $U(t,x)$, is constrained such that each element, $u_i(t,x)$, is the product of a vector of temporal functions and a vector of spatial distribution functions.

$$u_i(t,x) = \sum_j \theta_{ij}(t)\, \phi_{ij}(x,ij)$$

The spatial distribution function, $\phi_{ij}(x,l_{ij})$ are specified continuous functions of the parameters, l_{ij}, which determine the location or shape of the control and the temporal control functions, $\theta_{ij}(t)$, are unspecified but have continuous first derivitives.

The objective is to determine the temporal control functions, $\theta_{ij}(t)$, and the spatial distribution function parameters, l_{ij}, such that the state equation, initial conditions, terminal conditions, and boundary conditions are satisfied and such that a scalar performance index, J, is a minimum. The performance index may contain temporal integrals, spatial integrals and integrals over space and time.

$$J = \int_{t_o}^{t_f} w^a(S(t,x),S_x(t,x),t)\Big|_{x=x_a} + w^b(S(t,x),S_x(t,x),t)\Big|_{x=x_b} dt +$$

$$\int_{x_a}^{x_b} y^o(S(t,x),\ x)\Big|_{t=t_o} + y^f(S(t,x),x)\Big|_{t=t_f} dx +$$

$$\int_{t_o}^{t_f} \int_{x_a}^{x_b} z(S(t,x),S_x(t,x),S_{xx}(t,x),U(t,x),t,x)dxdt \quad .$$

A similar problem in two spatial dimensions can be formulated. Consider a two dimensional spatial domain, Ω , with a boundary Ω_b which is specified in terms of a linear parameter, p, representing distance along the boundary curve.

$$\Omega_b = \left\{ X = (x_1,x_2) \,\middle|\, x_1 = x_1(p), x_2 = x_2(p) \right\} .$$

The state equation and the initial, terminal, and boundary conditions can be written in the vector form.

$$S_t(t,X) = F(S(t,X),S_X(t,X),S_{XX}(t,X),U(t,X),t,X)$$

$$M^o(S(t,X),X) = 0 \qquad ; \qquad X\epsilon\Omega,\ t = t_o$$

$$M^f(S(t,X),S) = 0 \qquad ; \qquad X\epsilon\Omega,\ t = t_f$$

$$N(S(t,X),S_X(t,X)) = 0 \qquad ; \qquad t\epsilon(t_o,t_f), X\epsilon\Omega_b$$

For the two-dimensional location of each of the control spatial distributions, a vector set of parameters, L_{ij}, is required. The elements, $U_i(t,X)$, of the total control, $U(t,X)$, become:

$$u_i(t,X) = \sum_j \theta_{ij}(t)\phi_{ij}(X,L_{ij}).$$

The performance index for the problem in two spatial dimensions may contain integrals over the time interval and along the spatial boundary, integrals over the spatial domain and integrals over the time interval and the spatial domain.

$$J = \int_{t_o}^{t_f} \int_{\Omega_b} w(S(t,X),S_X(t,X),t,X)\Big|_{X\epsilon\Omega_b} dpdt \; +$$

$$\iint_{\Omega} y^{o}(S(t,X),X)\Big|_{t=t_o} + y^{f}(S(t,X),X)\Big|_{t=t_f} \, d\Omega +$$

$$\int_{t_o}^{t_f}\iint_{\Omega} z(X(t,X),S_X(t,X),S_{XX}(t,X),U(t,X),t,X)\, d\Omega dt$$

The constrained minimization problem remains to determine the temporal control functions, $\theta_{ij}(t)$, and the parameter vectors, L_{ij}, such that the state equation and initial, terminal, and boundary conditions are satisfied and such that the performance index is a minimum.

Additional generality might be added to the above constrained minimization problems, for example, by allowing an unspecified terminal time or inequality constraints. Once the constrained minimization problem is established, necessary conditions in the form of a boundary value problem can be derived.

Necessary Conditions

The derivation of conditions necessary for optimality begins with the adjoining of the state partial differential equation to the performance index with a Lagrange multiplier vector. The Hamiltonian is then defined and the first variation of the modified performance index is obtained. For the performance index to be a minimum, it is necessary that the first variation of the modified performance index be zero. This requirement yields the necessary conditions in the form of a boundary value problem.

Consider the constrained minimization problem in one spatial dimension. Adjoining the state partial differential equation to the performance index yields the modified performance index, J*.

$$J^* = J + \int_{t_o}^{t_f} \int_{x_a}^{x_b} A^T(t,x) \cdot$$

$$\left[F(S(t,x), S_x(t,x), S_{xx}(t,x), U(t,x), t, x) - S_t(t,x) \right] dxdt$$

The Hamiltonian, h, is then defined

$$h(S(t,x), S_x(t,x), S_{xx}(t,x), U(t,x), A(t,x)t,x) \triangleq$$

$$z(S(t,x), S_x(t,x), S_{xx}(t,x), U(t,x), t, x) +$$

$$A^T(t,x)\ F(S(t,x), S_x(t,x), S_{xx}(t,x), U(t,x), t, x)$$

and the first variation of the modified performance index is obtained.

$$\delta J^* = \int_{t_o}^{t_f} \left[\left(\frac{\partial w^a}{\partial S}\right)^T \delta S + \left(\frac{\partial w^a}{\partial S_x}\right)^T \delta S_x \Big|_{x=x_a} + \left(\frac{\partial w^b}{\partial S}\right)^T \delta S + \left(\frac{\partial w^b}{\partial S_x}\right)^T \delta S_x \Big|_{x=x_b} \right] dt +$$

$$\int_{x_a}^{x_b} \left[\left(\frac{\partial y^o}{\partial S}\right)^T \delta S \Big|_{t=0} + \left(\frac{\partial y^f}{\partial S}\right)^T \delta S \Big|_{t=t_f} \right] dx +$$

$$\int_{t_o}^{t_f} \int_{x_a}^{x_b} \left[\left(\frac{\partial h}{\partial S}\right)^T \delta S + \left(\frac{\partial h}{\partial S_x}\right)^T \delta S_x + \left(\frac{\partial h}{\partial S_{xx}}\right)^T \delta S_{xx} + \left(\frac{\partial h}{\partial U}\right)^T \delta U + \right.$$

$$\left. \left(\frac{\partial h}{\partial A}\right)^T \delta A - A^T \delta S_t - S_t^T \delta A \right] dxdt$$

Application of Green's Theorem is conjunction with the relationships

$$\delta S_t = \frac{\partial}{\partial t} (\delta S), \quad \delta S_x = \frac{\partial}{\partial x} (\delta S), \quad \delta S_{xx} = \frac{\partial^2}{\partial x^2} (\delta S)$$

and the control definition in terms of temporal functions and spatial function parameters yields a simplified form of δJ^*.

$$\delta J^* = \int_{t_o}^{t_f} \left\{ \left[\frac{\partial w^a}{\partial S} - \frac{\partial h}{\partial S_x} + \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial S_{xx}}\right) \right]^T \delta S + \left[\frac{\partial w^a}{\partial S_x} - \frac{\partial h}{\partial S_{xx}} \right]^T \delta S_x \Big|_{x=x_a} + \right.$$

$$\left[\frac{\partial w^b}{\partial S} + \frac{\partial h}{\partial S_x} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial S_{xx}}\right)\right]^T \delta S + \left[\frac{\partial w^b}{\partial S_x} + \left(\frac{\partial h}{\partial S_{xx}}\right)\right]^T \delta S_x \Bigg|_{x=b}\Bigg\} dt +$$

$$\int_{x_a}^{x_b}\left\{\left[\frac{\partial y^o}{\partial S} + A\right]^T \delta S\Bigg|_{t=t_o} + \left[\frac{\partial y^f}{\partial S} - A\right]^T \delta S\Bigg|_{t=t_f}\right\} dx +$$

$$\int_{t_o}^{t_f}\int_{x_a}^{x_b}\Bigg\{\left[\frac{\partial h}{\partial S} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial S_x}\right) + \frac{\partial^2}{\partial S_x^2}\left(\frac{\partial h}{\partial S_{xx}}\right) + A_t\right]^T \delta S +$$

$$\left[\frac{\partial h}{\partial A} - S_t\right]^T \delta A +$$

$$\sum_i\left[\frac{\partial h}{\partial u_i}\sum_j\left(\frac{\partial u_i}{\partial \theta_{ij}}\delta\theta_{ij} + \frac{\partial u_i}{\partial \phi_{ij}}\ \frac{\partial \phi_{ij}}{\partial l_{ij}}\ \delta l_{ij}\right)\right]\Bigg\} dxdt$$

The first variation of the initial, terminal and boundary condition constraint functions must be zero when evaluated at $t=t_o$, $t=t_f$, $x=x_a$, and $x=x_b$ respectively.

$$\delta M^o\Big|_{t=t_o} = \left[\frac{\partial}{\partial S}\left(M^o\right)^T\right]^T \delta S\Big|_{t=t_o} = 0$$

$$\delta M^f\Big|_{t=t_f} = \left[\frac{\partial}{\partial S}\left(M^f\right)^T\right]^T \delta S\Big|_{t=t_f} = 0$$

$$\delta N^a\Big|_{x=x_a} = \left[\frac{\partial}{\partial S}\left(N^a\right)^T\right]^T \delta S + \left[\frac{\partial}{\partial S_x}\left(N^a\right)^T\right]^T \delta S_x\Big|_{x=x_a} = 0$$

$$\delta N^b\Big|_{x=x_b} = \left[\frac{\partial}{\partial S}\left(N^b\right)^T\right]^T \delta S + \left[\frac{\partial}{\partial S_x}\left(N^b\right)^T\right]^T \delta S_x\Big|_{x=x_b} = 0$$

From the first variation of the modified performance index come the following relationships which in conjunction with the above relationships and the initial, terminal, and boundary conditions on the state equation form the boundary value problem:

$$S_t = \frac{\partial h}{\partial A}$$

$$A_t = -\frac{\partial h}{\partial S} + \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial S_x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial S_{xx}}\right)$$

$$\left[\frac{\partial y^o}{\partial S} + A\right]^T \delta S \Big|_{t=t_o} = 0$$

$$\left[\frac{\partial y^f}{\partial S} - A\right]^T \delta S \Big|_{t=t_f} = 0$$

$$\left[\frac{\partial w^a}{\partial S} - \frac{\partial h}{\partial S_x} + \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial S_{xx}}\right)\right]^T \delta S + \left[\frac{\partial w^a}{\partial S_x} - \frac{\partial h}{\partial S_{xx}}\right]^T \delta S_x \Big|_{x=x_a} = 0$$

$$\left[\frac{\partial w^b}{\partial S} + \frac{\partial h}{\partial S_x} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial S_{xx}}\right)\right]^T \delta S + \left[\frac{\partial w^b}{\partial S_x} + \frac{\partial h}{\partial S_{xx}}\right]^T \delta S_x \Big|_{x=x_b} = 0$$

The set of admissable controls are those of class C^1; therefore, the temporal control functions and spatial function parameters are required to satisfy the following relationships.

$$\int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}}\, dx = 0 \qquad ; \qquad t \in (t_o, t_f)$$

$$\int_{t_o}^{t_f}\int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}}\, dx dt = 0$$

A similar development of the necessary conditions is possible in the case of the problem in two spatial dimensions. The modified performance index becomes:

$$J^* = J + \int_{t_o}^{t_f}\iint_{\Omega} A^T(t,X)\cdot$$

$$\left[F(S(t,X),S_X(t,X),S_{XX}(t,X),U(t,X),t,X)-S_t(t,X)\right]\, d\Omega dt\ .$$

The Hamiltonian is defined in the same manner as for the problem in one spatial dimension

$$h(S(t,X),S_X(t,X),S_{XX}(t,X)U(t,X),A(t,X),t,X) =$$

$$z(S(t,X),S_X(t,X),S_{XX}(t,X)U(t,X),t,X) +$$

$$A^T(t,X)\, F(S(t,X),S_X(t,X),S_{XX}(t,X),U(t,X),t,X)$$

and the first variation of the modified performance index is obtained

$$\delta J^* = \int_{t_o}^{t_f}\int_{\Omega_b}\left[\left(\frac{\partial w}{\partial S}\right)^T \delta S + \left(\frac{\partial w}{\partial S_X}\right)^T \delta S_X\right]\Bigg|_{x\in\Omega_b} dp dt +$$

$$\iint_{\Omega}\left[\left(\frac{\partial y^o}{\partial S}\right)^T \delta S\Bigg|_{t=t_o} + \left(\frac{\partial y^f}{\partial S}\right)^T \delta S\Bigg|_{t=t_f}\right] d\Omega +$$

$$\int_{t_o}^{t_f}\iint\left[\left(\frac{\partial h}{\partial S}\right)^T \delta S + \left(\frac{\partial h}{\partial S_X}\right)^T \delta S_X + \left(\frac{\partial h}{\partial S_{XX}}\right)^T \delta S_X +\right.$$

$$\left.\left(\frac{\partial h}{\partial U}\right)^T \delta U + \left(\frac{\partial h}{\partial A}\right)^T \delta A \quad - \quad S_t^T\, \delta A - A^T \delta S_t\right] d\Omega dt\ .$$

The boundary line integral may be written in terms of the independent variables x_1 and x_2. Application of Green's Theorem and the Divergence Theorem yields a simplified form of the first variation.

$$J^* = \int_{t_o}^{t_f}\oint_{\Omega_b}\left\{\left[\left(\frac{\partial w}{\partial S}\right)^T \delta S + \left(\frac{\partial w}{\partial S_{x_1}}\right)^T \delta S_{x_1} + \left(\frac{\partial w}{\partial S_{x_2}}\right)^T \delta S_{x_2}\right]\frac{\partial p}{\partial x_1} \quad +\right.$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial S_{x_2x_2}}\right)^T \delta S - \left(\frac{\partial h}{\partial S_{x_2x_2}}\right)^T \delta S_{x_2} -$$

$$\left.\left(\frac{\partial h}{\partial S_{x_2}}\right)^T \delta S - \left(\frac{\partial h}{\partial S_{x_1x_2}}\right)^T \delta S_{x_1}\right\}\Bigg|_{(x_1,x_2)\in\Omega_b} dx_1\, dt +$$

$$\int_{t_o}^{t_f}\oint_{\Omega_b}\left\{\left[\left(\frac{\partial w}{\partial S}\right)^T \delta S + \left(\frac{\partial w}{\partial S_{x_1}}\right)^T \delta S_{x_1} + \left(\frac{\partial w}{\partial S_{x_2}}\right)^T \delta S_{x_2}\right]\frac{\partial p}{\partial x_2} -\right.$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial S_{x_1x_1}}\right)^T \delta S + \left(\frac{\partial h}{\partial S_{x_1x_1}}\right)^T \delta S_{x_1} +$$

$$\left.\left(\frac{\partial h}{\partial S_{x_1}}\right)^T \delta S - \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial S_{x_1x_2}}\right)^T \delta S\right\}\Bigg|_{(x_1,x_2)\in\Omega_b} dx_2 dt +$$

$$\iint_{\Omega}\left\{\left[\frac{\partial y^o}{\partial S} + A\right]^T \delta S\Big|_{t=t_o} + \left[\frac{\partial y^f}{\partial S} - A\right]^T \delta S\Big|_{t=t_f}\right\} dx_1 dx_2 +$$

$$\int_{t_o}^{t_f}\iint_{\Omega}\left\{\left[\frac{\partial h}{\partial S} - \frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial S_{x_1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial S_{x_2}}\right) + \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial h}{\partial S_{x_1x_1}}\right) +\right.\right.$$

$$\left.\frac{\partial^2}{\partial x_1 x_2}\left(\frac{\partial h}{\partial S_{x_1x_2}}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial h}{\partial S_{x_2x_2}}\right)\right]^T \delta S +$$

$$\left[\frac{\partial h}{\partial A} - S_t\right]^T \delta A +$$

$$\sum_i \frac{\partial h}{\partial u_i}\sum_j\left[\frac{\partial u_i}{\partial \theta_{ij}}\delta\theta_{ij} + \frac{\partial u_i}{\partial \phi_{ij}}\sum_k\left(\frac{\partial \phi_{ij}}{\partial l_{ijk}}\delta l_{ijk}\right)\right] dx_1 dx_2 dt$$

The first variation of the modified performance index, the initial, terminal, and boundary conditions and their zero first variations yield the following boundary value problem. The state and adjoint partial differential equations are:

$$S_t = \frac{\partial h}{\partial A}$$

$$A_t = -\frac{\partial h}{\partial S} + \frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial x_2}\right) - \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial h}{\partial S_{x_1x_1}}\right) -$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial h}{\partial S_{x_1x_2}}\right) - \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial h}{\partial S_{x_2x_2}}\right).$$

The initial and terminal conditions are specified by:

$$M^o \Big|_{t=t_o} = 0 \quad , \qquad M^f \Big|_{t=t_f} = 0$$

$$\left(\frac{\partial}{\partial S}(M^o)^T\right)^T \delta S \Big|_{t=t_o} = 0 \quad , \qquad \left(\frac{\partial}{\partial S}(M^f)^T\right)^T \delta S \Big|_{t=t_f} = 0$$

$$\left(\frac{\partial y^o}{\partial S} + A\right)^T \delta S \Big|_{t=t_o} = 0 \; , \qquad \left(\frac{\partial y^f}{\partial S} - A\right)^T \delta S \Big|_{t=t_f} = 0 \; .$$

The boundary conditions are specified by:

$$N \Big|_{(x_1,x_2)\in\Omega_b} = 0$$

$$\left(\frac{\partial}{\partial S}(N)^T\right)^T \delta S + \left(\frac{\partial}{\partial S_{x_1}}(N)^T\right)^T \delta S_{x_1} + \left(\frac{\partial}{\partial S_{x_2}}(N)^T\right)^T \delta S_{x_2} \Big|_{(x_1,x_2)\in\Omega_b} = 0$$

$$\left\{\left[\left(\frac{\partial w}{\partial S}\right)^T \delta S + \left(\frac{\partial w}{\partial S_{x_1}}\right)^T \delta S_{x_1} + \left(\frac{\partial w}{\partial S_{x_2}}\right)^T \delta S_{x_2}\right] \frac{\partial p}{\partial x_1} + \right.$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial S_{x_2 x_2}}\right)^T \delta S - \left(\frac{\partial h}{\partial S_{x_2 x_2}}\right)^T \delta S_{x_2} -$$

$$\left.\left(\frac{\partial h}{\partial S_{x_2}}\right)^T \delta S - \left(\frac{\partial h}{\partial S_{x_1 x_1}}\right)^T \delta S_{x_1}\right\}\Bigg|_{(x_1, x_1)\in\Omega_b} = 0$$

$$\left\{\left[\left(\frac{\partial w}{\partial S}\right)^T \delta S + \left(\frac{\partial w}{\partial S_{x_1}}\right)^T \delta S_{x_1} + \left(\frac{\partial w}{\partial S_{x_2}}\right)^T \delta S_{x_2}\right] \frac{\partial p}{\partial x_2} - \right.$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial S_{x_1 x_1}}\right)^T \delta S + \left(\frac{\partial h}{\partial S_{x_1 x_1}}\right)^T \delta S_{x_1} +$$

$$\left.\left(\frac{\partial h}{\partial S_{x_1}}\right)^T \delta S - \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial S_{x_1 x_2}}\right)^T \delta S\right\}\Bigg|_{(x_1, x_2)\in\Omega_b} = 0 .$$

The temporal control functions and spatial function parameters must satisfy the following relationships.

$$\iint_{\Omega} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}} \, dx_1 dx_2 = 0 \quad ; \quad \forall\, t \in (t_o, t_f)$$

$$\int_{t_o}^{t_f} \iint_{\Omega} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial \Gamma_{ijk}} \, dx_1 dx_2 dt = 0$$

Once the necessary conditions are established, some form of approximation must be made in order to solve the boundary value problem. The next chapter deals with approximation methods and a solution approach.

CHAPTER IV

BOUNDARY VALUE PROBLEM SOLUTION

Approximation Technique

The application of calculus of variations to the optimal regional control problem yields the original partial differential equation or equations governing the state of the distributed parameter system and the same number of adjoint partial differential equations together with initial, terminal and boundary conditions. Several approaches to the reduction of these partial differential equations to sets of ordinary differential equations are available. Among these are the method of lines, the collocation method, the subdomain method, the least squares approximation method, and Galerkin's method. All of these are functional expansion methods, with the exception of the first which is a finite difference method. Table I contains a brief description of the functional expansion methods.

Consider Galerkin's method for the reduction of partial differential equations to sets of ordinary differential equations. In the case of an equation in two independent variables, one of the independent variables may be eliminated by the assumption that the dependent variable can be expressed as the vector product of a specified set of basis functions of the one independent variable and a vector of unknown functions of the other independent variable. The requirement that the basis functions be each orthogonal to the error introduced in the equation by the substitution

TABLE I

METHODS FOR THE REDUCTION OF PARTIAL DIFFERENTIAL EQUATIONS TO SETS OF ORDINARY DIFFERENTIAL EQUATIONS

COLLOCATION	$e\Big\vert_{x_j} = 0 \qquad ; j = 1,2,\ldots,n$
SUBDOMAIN	$\int_{\Omega_j} e\,dx = 0 \qquad ; j = 1,2,\ldots,n$
LEAST SQUARES	$\frac{\partial}{\partial q_i}\int_{\Omega} e^2\,dx = 0 \; ; i = 1,2,\ldots,m$
GALERKIN	$\int_{\Omega} e\,r_i\,dx = 0 \qquad ; i = 1,2,\ldots,m$

Differential Equation: $s_t = f(s_{xx}, s_x, s, u)$

Functional Expansion: $s(t,x) = Q^T(t)R(x)$; $R(x)$ specified

Error of Approximation: $e = Q^T R_t - f(Q^T R_{xx}, Q^T R_x, Q^T R, u)$

of the vector product for the dependent variable be zero provides a number of basis functions equal to the number of new dependent variables. It is only required that the basis functions be chosen such that the boundary conditions which are eliminated in the reduction of the partial differential equations to ordinary differential equations are satisfied.

Eigenfunctions

Three types of functional expansions have been considered for application in conjunction with Galerkin's method for the reduction of partial differential equations to ordinary differential equations. The first type of basis functions considered is the traditional truncated eigenfunction expansion. This type has the basic advantage that it is an orthogonal set of functions, thus eliminating matrix inversion problems associated with nonorthogonal basis functions. While it is good for linear problems with regular boundaries, its application in the case of nonlinear problems is restricted to the approximation with the eigenfunctions of a similar linear equation. Complications also arise in the case of irregular boundaries.

Hermite Interpolation Polynomials

The second type of basis functions considered is the Hermite interpolation polynomials. This set of basis functions is similar to the Lagrangian interpolation functions. Lagrangian interpolation between two points is merely straight line interpolation, which is shown in Figure 1. From Figures 1 and 2, the value of $f(\zeta)$ at a point ζ_a between ζ_1 and ζ_2 may be represented as the sum of two functions

$$f(\zeta_a) = f(\zeta_1)\lambda_1(\zeta_a,\zeta_1,\zeta_2) + f(\zeta_2)\lambda_2(\zeta_a,\zeta_1,\zeta_2)$$

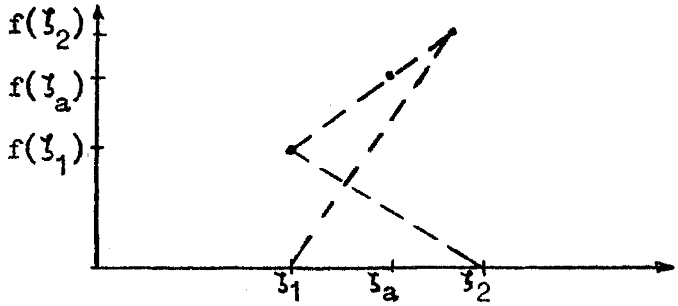


Figure 1. Two-Point Lagrangian Interpolation

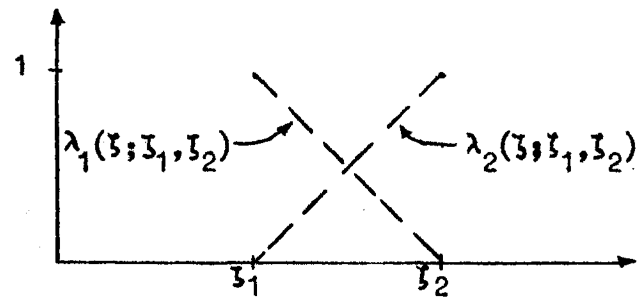


Figure 2. Interpolation Functions

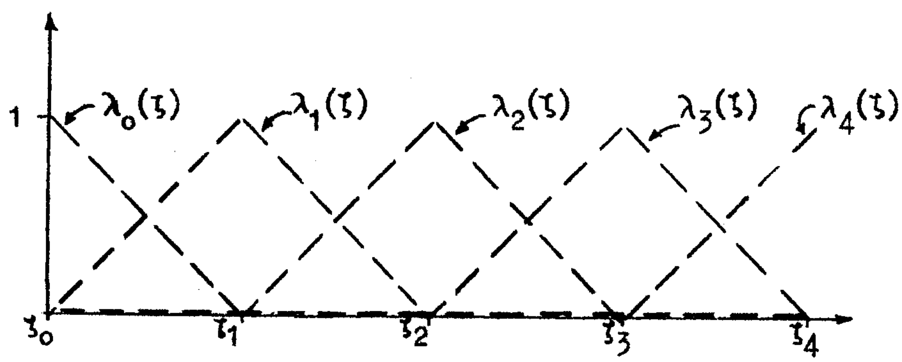


Figure 3. Lagrangian Interpolation Polynomials

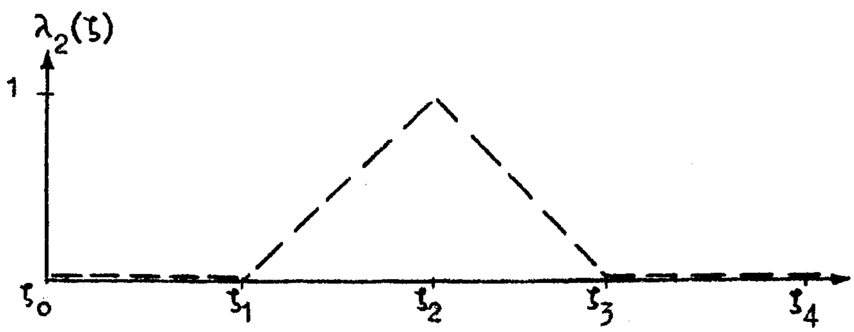


Figure 4. One Lagrangian Interpolation Polynomial

The combination of adjacent functions of this type results in the "hat" functions shown in Figures 3 and 4. Note that each "hat" function or Lagrange interpolation polynomial which is not centered at a boundary grid point, is nonzero over two subintervals and zero over all other subintervals. The Lagrange interpolation polynomials which are centered at the boundary grid points are nonzero over one subinterval and zero over all other subintervals. Therefore, all nonadjacent Lagrange interpolation polynomials are orthogonal over the total interval. They are piecewise polynomials. The Lagrange interpolation polynomials are elements of class C_p^1, i.e. continuous with piecewise continuous first derivatives and are elements of class C^1 within each of the subintervals. They obey the the relationship

$$\lambda_i(\zeta_j) = \delta_{ij}$$

at the grid points and are defined by

$$\lambda_i(\zeta) = \begin{cases} (\zeta - \zeta_{i-1})/(\zeta_i - \zeta_{i-1}) & ; \quad \in \left[\zeta_{i-1}, \zeta_i\right] \\ (\zeta_{i+1} - \zeta)/(\zeta_{i+1} - \zeta_i) & ; \quad \in \left(\zeta_i, \zeta_{i+1}\right] \\ 0 & ; \quad \zeta < \zeta_{i-1},\ \zeta > \zeta_{i+1} \end{cases} .$$

Hermite interpolation between two points of a discrete set involves knowledge of the first derivatives at the grid points as well as the values of the function at the grid points. Therefore, when Hermite interpolation between two points is employed, the interpolated value at a point between two points where the function values and its first derivatives are known is represented by the sum of four functions evaluated at that point, as shown in Figures 5 and 6.

$$f(\zeta_a) = f(\zeta_1)\eta^0(\zeta_a;\zeta_1,\zeta_2) + f(\zeta_2)\eta^0(\zeta_a;\zeta_1,\zeta_2) +$$
$$f\ (\zeta_1)\eta_2^1(\zeta_a;\zeta_1,\zeta_2) + f\ (\zeta_2)\eta_2^1(\zeta_a;\zeta_1,\zeta_2)$$

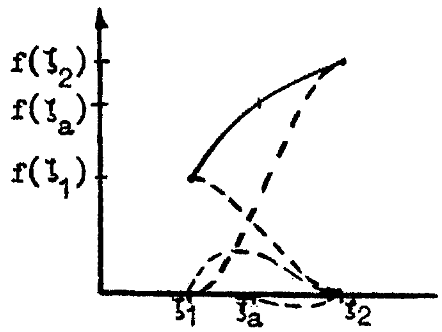


Figure 5. Two-Point Hermite Interpolation

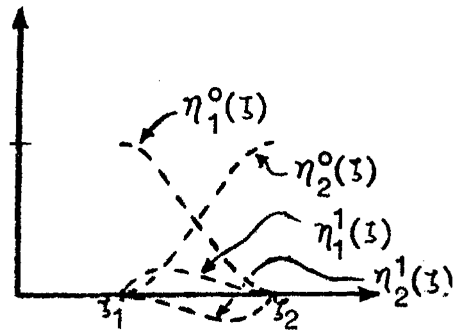


Figure 6. Interpolation Functions

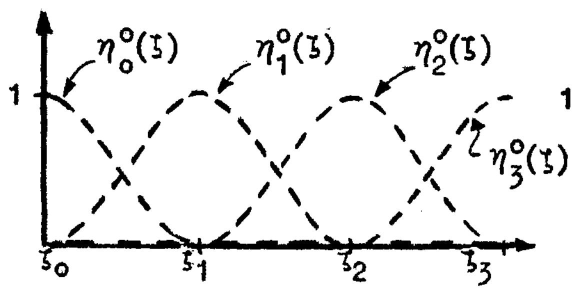


Figure 7. Hermite Interpolation Polynomials - Type Zero

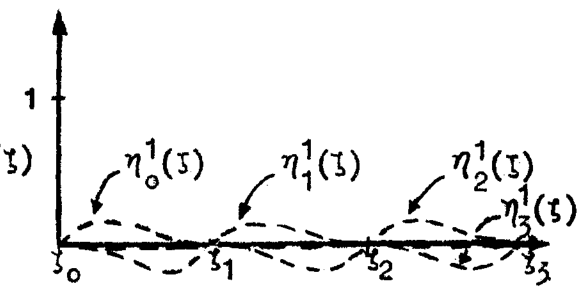


Figure 8. Hermite Interpolation Polynomials - Type One

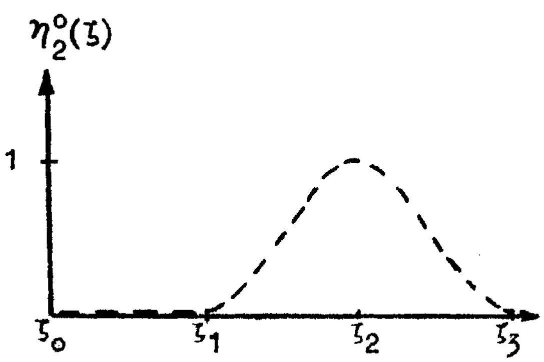


Figure 9. One Hermite Interpolation Polynomial - Type Zero

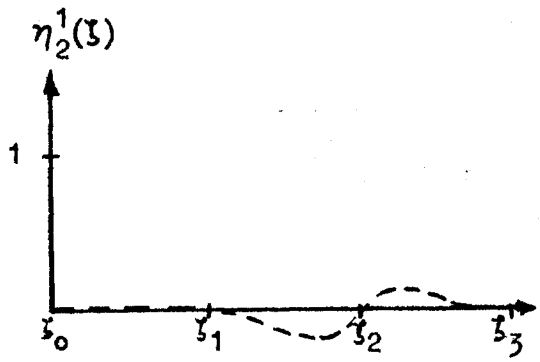


Figure 10. One Hermite Interpolation Polynomial - Type One

The combination of adjacent functions of this type results in a set of Hermite interpolation polynomials, half of which are smooth "hat" functions. The Hermite interpolation polynomials have additional continuity over the Lagrange interpolation polynomials. The Hermite interpolation polynomials are also piecewise polynomials, but they are elements of C_p^2, i.e. continuously differentiable with piecewise continuous second derivatives, and are elements of class C^2 within each of the subintervals. Like the Lagrange interpolation polynomials, the Hermite interpolation polynomials are zero except over the two subintervals between which the functions are centered. See Figures 7 through 10. The Hermite interpolation polynomials obey the following relationships at the grid points.

$$\eta_i^o(\zeta_j) = \delta_{ij} \quad ; \quad \eta_i^1(\zeta_j) = 0$$

$$\frac{d}{dx}\,\eta_i^o(\zeta_j) = 0 \quad ; \quad \frac{d}{dx}\,\eta_i^1(\zeta_j) = \delta_{ij}$$

They are defined as follows.

$$\eta_i^o(\zeta) = \begin{cases} -2\left(\dfrac{\zeta-(i-1)(\zeta_i-\zeta_{i-1})}{\zeta_i-\zeta_{i-1}}\right)^3 + 3\left(\dfrac{\zeta-(i-1)(\zeta_i-\zeta_{i-1})}{\zeta_i-\zeta_{i-1}}\right)^3 & ;\ \zeta\in[\zeta_{i-1},\zeta_i] \\[2ex] 2\left(\dfrac{\zeta-i(\zeta_{i+1}-\zeta_i)}{\zeta_{i+1}-\zeta_i}\right)^3 - 3\left(\dfrac{\zeta-i(\zeta_{i+1}-1)}{\zeta_{i+1}-\zeta_i}\right)^2 + 1 & ;\ \zeta\in[\zeta_i,\zeta_{i+1}] \\[2ex] 0 & ;\ \zeta<\zeta_{i-1},\ \zeta>\zeta_{i+1} \end{cases}$$

$$\eta_i^o(\zeta) = \begin{cases} \left(\dfrac{\zeta-(i-1)(\zeta_i-\zeta_{i-1})}{\zeta_i-\zeta_{i-1}}\right)^2 \cdot \left(\zeta - i\,(\zeta_i-\zeta_{i-1})\right) & ;\ \zeta\in\left[\zeta_{i-1},\zeta_i\right] \\ \left(\dfrac{(i+1)(\zeta_{i+1}-\zeta_i)-\zeta}{\zeta_{i+1}-\zeta_i}\right)^2 \cdot \left(\zeta - i\,(\zeta_{i+1}-\zeta_i)\right) & ;\ \zeta\in\left(\zeta_i,\zeta_{i+1}\right] \\ 0 & ;\ \zeta<\zeta_{i-1},\ \zeta>\zeta_{i+1} \end{cases}$$

The one-dimensional Hermite interpolation polynomials are cubic functions of the independent variable on a given subinterval with specific conditions met at the grid points. Similarly, the two-dimensional Hermite polynomials are bicubic functions of the two independent variables. on a given subdomain with specific conditions met along the grid edges and at the corners. They are elements of class C^2 on a given subdomain in each of the two dimensions and C_p^2 on the total two-dimensional domain in each of the two dimensions.

Two basic subdomain or grid block shapes, a rectangle and a right triangle, may be combined to form a somewhat general class of polygons. Arbitrary shapes as well as those polygons which can not be exactly formed by the connection of a series of grid intersections, i.e. by the combination of rectangles and right triangles, can be approximated this way.

Consider the two-dimensional Hermite interpolation polynomials associated with the rectangular grid block. While two Hermite interpolation polynomials are centered at each grid point in the one-dimensional case, four Hermite interpolation polynomials are centered at each grid intersection in the two-dimensional case. The interpolated value of a function at a point within a rectangle formed by four points at which the function value, its first derivative in both directions and the cross derivative are known, is given by the sum of sixteen weighted Hermite interpolation functions evaluated at the desired point.

$$f(\zeta_a,\xi_a) = \sum_{i=1}^{2} \sum_{j=1}^{2} \Big[f(\zeta_i,\xi_j)\psi_{i,j}^{0,0}(\zeta_a,\xi_a;\zeta_1,\zeta_2,\xi_1,\xi_2) + f_\zeta(\zeta_i,\xi_j)\psi_{i,j}^{1,0}(\zeta_a,\xi_a;\zeta_1,\zeta_2,\xi_1,\xi_2) + f_\xi(\zeta_i,\xi_j)\psi_{i,j}^{0,1}(\zeta_a,\xi_a;\zeta_1,\zeta_2,\xi_1,\xi_2) + f_{\zeta\xi}(\zeta_i,\xi_j)\psi_{i,j}^{1,1}(\zeta_a,\xi_a;\zeta_1,\zeta_2,\xi_1,\xi_2)\Big]$$

Each of the two-dimensional Hermite interpolation polynomials is the product of two one-dimensional Hermite interpolation polynomials.

$$\psi_{i,j}^{k,l}(\zeta,\xi) = \eta_i^k(\zeta)\,\eta_j^l(\xi)$$

The four two-dimensional Hermite interpolation polynomials centered at a grid intersection are shown in Figures 11 through 14. They are nonzero over at most the four adjacent rectangular grid blocks for which the grid intersection polynomial center is a common corner point. A two-dimensional Hermite interpolation polynomial is nonzero over less than four rectangular grid blocks only if the grid intersection about which it is centered lies on the system boundary. This is similar to the one-dimensional Hermite interpolation polynomial which is centered at one of the two end points of the system interval and, therefore, nonzero over only one subinterval.

The two-dimensional Hermite interpolation polynomials for rectangular grid blocks satisfy grid edge requirements as well as grid intersection requirements. While the one-dimensional Hermite interpolation polynomials are continuous and have a continuous slope at the points common to two subintervals, the two-dimensional Hermite interpolation polynomials for

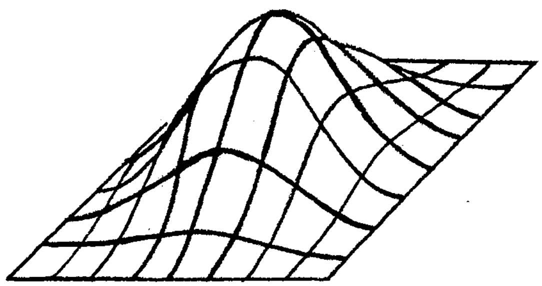


Figure 11. One Two-Dimensional Hermite Interpolation Polynomial Type - Zero-Zero

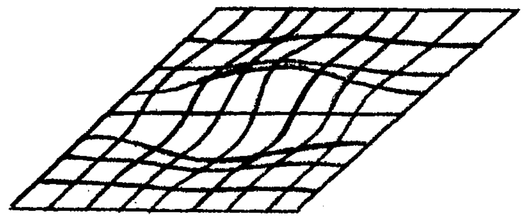


Figure 12. One Two-Dimensional Hermite Interpolation Polynomial Type - One-Zero

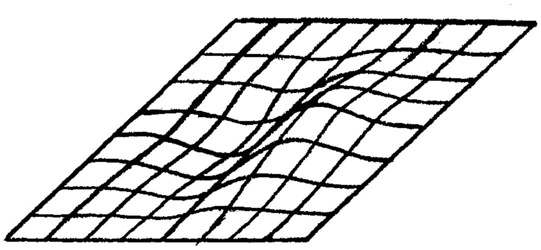


Figure 13. One Two-Dimensional Hermite Interpolation Polynomial Type - Zero-One

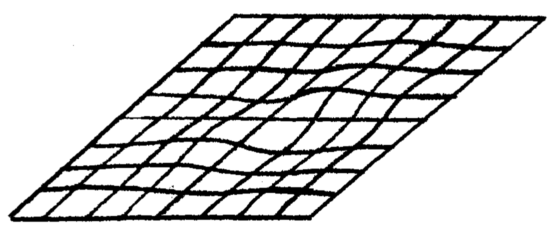


Figure 14. One Two-Dimensional Hermite Interpolation Polynomial Type - One-One

rectangular grid blocks are continuous and have continuous first derivatives in each direction and a continuous cross derivative along the grid block edges which are not sections of the boundary.

It was noted that a requirement of Galerkin's method for the reduction of partial differential equations to sets of ordinary differential equations is that the basis functions must be chosen such that all boundary conditions eliminated in the application of the method must be satisfied by the functional expansion. In the case involving one spatial independent variable, boundary conditions must be met at boundary points. For homogeneous Dirichlet or Neumann boundary conditions, substitution of the complete functional expansion into the boundary condition function and evaluation of the spatial functions at the boundary yields the requirement that one of the time varying coefficients of one of the Hermite interpolation polynomials must be zero. This simply eliminates one of the terms of the functional expansion. For the homogeneous boundary condition involving a function of the state and the normal derivative, the boundary condition provides a relationship between two of the time varying coefficients. The result is the replacement of two of the Hermite interpolation polynomials with a single modified Hermite interpolation polynomial which forces the satisfying of the boundary condition. Nonhomogeneous boundary conditions result in terms in the functional expansion which do not contain unspecified time varying coefficients.

Similar requirements are obtained for the application of Galerkin's method to the problem involving two spatial dimensions. However, in the two-dimensional case, boundary conditions may be functions of the distance along the boundary. Four two-dimensional Hermite interpolation polynomials are centered at each grid intersection. Therefore, along a rectangular grid edge which forms a section of the system boundary, eight of the two-dimensional Hermite interpolation polynomials in the functional expansion must be considered in the satisfying of boundary conditions. All other terms in the expansion will be zero and will have a zero normal derivative along the grid edge under consideration. For a homogeneous Dirichlet boundary condition, the four terms involving nonzero Hermite interpolation polynomials must have zero coefficients and be eliminated from the functional expansion. For a homogeneous Neumann boundary condition, the four terms involving Hermite interpolation polynomials with nonzero normal derivatives must have zero coefficients. For a homogeneous boundary condition, the eight Hermite interpolation polynomials are replaced by four modified Hermite interpolation polynomials and for a nonhomogeneous boundary condition, the functional expansion must contain a term without an unspecified time varying coefficient.

Rectangular grid blocks provide only a step approximation to oblique boundaries. Therefore, the right triangle is considered as a second grid block shape. While no set of bicubics satisfies all of the grid edge requirements, in particular those along the diagonal edge, C. A. Hall (27) describes a set of bicubics which allow the matching of the Dirichlet boundary condition along the diagonal edge. B. L. Hume (28) has modified this set in order to satisfy the Neumann boundary condition along the diagonal edge. These sets of bicubics are compatable with the set associated with the rectangular grid block; that is, the bicubics for the rectangular grid block match those for the right triangle grid block along grid edges. Note that only the two legs of the right triangle can also be edges of rectangular grid blocks. The piecewise definition of the two-dimensional Hermite interpolation polynomials centered at a given grid intersection depends on the types of grid blocks for which it provides a corner point. The two-dimensional Hermite interpolation polynomial remains an element of class C_p^2 even though it may be defined over various combinations of right triangles and rectangles.

Fundamental Splines

A third type of expansion is a series of fundamental splines. The fundamental splines are similar to the Hermite interpolation polynomials in that for the one-dimensional approximation they are cubic polynomials in each subinterval of a given total interval. However, the fundamental splines are elements of class C_p^3. The function value of the piecewise polynomial and its first and second derivatives are continuous at the points separating the subintervals. While the Hermite interpolation polynomials are nonzero over at most two subintervals and, therefore, demonstrate "banded orthogonality," the fundamental splines may be nonzero over all subintervals and are completely nonorthogonal.

A unique piecewise polynomial composed of cubics defined on each of n subintervals is specified by 4n conditions. Let $f_i(\zeta)$ be a cubic polynomial in the subinterval between the nodes at ζ_{i-1} and ζ_i with the set of nodes numbered from 0 to n. Continuity of the function value and its first and second derivatives yields 3(n-1) conditions.

$$f_i(\zeta_{i-1}) = f_{i-1}(\zeta_{i-1}) \qquad ; \qquad i = 1,2,\ldots,n-1$$

$$\frac{d}{d\zeta} f_i(\zeta_{i-1}) = \frac{d}{d\zeta} f_{i-1}(\zeta_{i-1}) \qquad ; \qquad i = 1,2,\ldots,n-1$$

$$\frac{d^2}{d\zeta^2} f_i(\zeta_{i-1}) = \frac{d^2}{d\zeta^2} f_{i-1}(\zeta_{i-1}) \qquad ; \qquad i = 1,2,\ldots,n-1$$

Specification of the value of the piecewise polynomial at each of the nodes yields n+1 conditions.

$$f_i(\zeta_i) = \alpha_i \qquad ; \qquad i = 0,1,\ldots,n$$

The last two conditions are obtained from the specification of the first derivative of the piecewise polynomial at the two end nodes.

$$\frac{d}{d\zeta} f_1(\zeta_o) = \beta_o$$

$$\frac{d}{d\zeta} f_n(\zeta_n) = \beta_n$$

The series of fundamental splines is a particular subset of the set of piecewise polynomials which satisfy the above 4n conditions. The series of fundamental splines is formed by choosing combinations of the n+3 constants α_i and β_j. A single fundamental spline results when one of the constants is chosen to be 1 and all others are chosen to be zero.

$$\left.\begin{array}{lll} \alpha_i = \delta_{ik} & ; & i = 0,1,\ldots,n \\ \beta_j = 0 & ; & j = 0,n \end{array}\right\} k = 0,1,\ldots,n$$

$$\left.\begin{array}{lll} \alpha_i = 0 & ; & i = 0,1,\ldots,n \\ \beta_j = \delta_{jk} & ; & j = 0,n \end{array}\right\} k = 0,n$$

One fundamental spline is centered at each interior node and two are centered at the end nodes.

The definition of two-dimensional fundamental splines is considerably more complex than that of one-dimensional fundamental splines. Continuity requirements must be satisfied along the grid edges as well as at grid intersections. For each grid block, the sixteen coefficients of the bicubic for each fundamental spline must be determined.

Comparison of Expansions

Both the Hermite interpolation polynomials and the fundamental splines have the disadvantage of being nonorthogonal, while the orthogonality of the eigenfunction allows reduced computation. The disadvantage of nonorthogonality appears to be inportant in one-dimensional expansions, but irregular boundaries provide sufficient justification for nonorthogonal expansions in multiple dimensions. In addition to the boundary advantages in multidimensional expansions, the Hermite interpolation polynomials and the fundamental splines allow varying accuracy over a region of interest by allowing irregular nodal spacing. However, this advantage is of questionable value in the case of the search for the optimum location of the region requiring the most accuracy. The Hermite interpolation polynomials have less continuity than the fundamental splines, but allow less initial computation.

Computational Algorithm

Optimization involves a search for one or more constants which cause the minimization of some performance index. There are basically two different directions to take in establishing an optimization procedure. One is to base the direction of change of the vector of constants to be optimized on the values of the performance index which result from changes in one or more of the constants. The other is to base that change on calculations in addition to that of the performance index at one point in the multidimensional space of the vector of constants. For example, the gradient direction for change of the vector of constants may be determined from samples or from gradient calculations at a single point.

Application of the calculus of variations with the Lagrange multiplier method provides a natural foundation for a gradient approach based on calculation of the gradient at a single point. The necessary conditions for optimality include the state equations, the adjoint equations, boundary conditions and conditions on the control. For the fixed time problem with specified initial conditions and balanced and specified spatial boundary conditions, the solution proceeds by the assumption of the control. All of the necessary conditions except those on the control are satisfied by forward integration of the state equations and backward integration of the adjoint equations. The variation of the performance index with constraints adjoined with respect to each of the control elements and evaluated along the state and adjoint trajectories provides a gradient direction for modification of the control.

Two slightly different approaches to the solution of the optimal control problem after application of Galerkin's method are possible. First, the time dependent control amplitude is discretized with an interval equal to the integration interval, creating a vector of constants equal in length to the number of time steps plus one. This vector of constants is searched for simultaneously with the constant locating the control. The second approach is to assume a polynomial series expansion for the time-dependent control amplitude and search for the coefficients of the series together with the constant locating the control.

Consider first the problem with one spatial independent variable. The gradient direction of the control modification is determined by examining the terms remaining in the first variation of the performance index after the extraction of the state and adjoint partial differential equations and the initial, terminal and boundary conditions.

$$\delta J^* = \int_{t_o}^{t_f} \int_{x_a}^{x_b} \sum_i \left[\frac{\partial h}{\partial u_i} \sum_j \left(\frac{\partial u_i}{\partial \theta_{ij}} \delta\theta_{ij} + \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}} \delta l_{ij} \right) \right] dxdt$$

Suppose that the time integral in the first term is discretized with the same time step as the state and adjoint equation integration. The parameters l_{ij} are constants and their variation can be taken out from under the two integrals, and the functions θ_{ij} before temporal discretization are functions of time only. The resulting approximation yields a form for the first variation of the modified performance index in terms of a set of parameters $\theta_{ij}(t_k)$ and l_{ij}.

$$\delta J^* = \sum_i \int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \sum_j \frac{\partial u_i}{\partial \theta_{ij}} \, dx \, \delta\theta_{ij} \, dt +$$

$$\sum_i \int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \sum_j \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}} \, dxdt \, \delta l_{ij}$$

$$= \sum_i \sum_j \sum_k \left[\Delta t \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}} \, dx \, \Bigg|_{t_k} \right] \delta\theta_{ij}(t_k) +$$

$$\sum_i \sum_j \int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}} \, dxdt \quad \delta l_{ij}$$

The change in the control should be in the direction in parameter space which maximizes the negative change in J^*.

$$\Delta\theta_{ij}(t_k) = -K \rho_{ijk} \Big/ \left[\sum_i \sum_j \sum_k \rho^2_{ijk} + \sum_i \sum_j \sigma^2_{ij} \right]^{\frac{1}{2}}$$

$$\Delta l_{ij} = -K \sigma_{ij} \Big/ \left[\sum_i \sum_j \sum_k \rho^2_{ijk} + \sum_i \sum_j \sigma^2_{ij} \right]^{\frac{1}{2}}$$

$$\rho_{ijk} \triangleq \Delta t \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}} \, dx \Big|_{t_k}$$

$$\sigma_{ij} \triangleq \int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}} \, dxdt$$

Suppose that the time dependent control amplitudes are to be expressed as polynomial series in time.

$$\theta_{ij}(t) = \sum_k g_{ijk} \, t^{k-1}$$

In this case the first variation of the modified performance index becomes a function of temporal parameters and spatial parameters.

$$\delta J^* = \sum_i \sum_j \sum_k \left[\int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}} \frac{\partial \theta_{ij}}{\partial g_{ijk}} \, dxdt \right] \delta g_{ijk} +$$

$$\sum_i \sum_j \left[\int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}} \, dxdt \right] \delta l_{ij}$$

The change in the control parameters should then be

$$\Delta g_{ijk} = -K \rho_{ijk} \Big/ \left[\sum_i \sum_j \sum_k \rho^2_{ijk} + \sum_i \sum_j \sigma^2_{ij} \right]^{\frac{1}{2}}$$

$$\Delta l_{ij} = -K \sigma_{ij} \Big/ \left[\sum_i \sum_j \sum_k \rho^2_{ijk} + \sum_i \sum_j \sigma^2_{ij} \right]^{\frac{1}{2}}$$

$$\rho_{ijk} \triangleq \int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}} \frac{\partial \theta_{ij}}{\partial g_{ij}} \, dxdt$$

$$\sigma_{ij} \triangleq \int_{t_o}^{t_f} \int_{x_a}^{x_b} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ij}} \, dxdt$$

The basic optimization approach is to integrate the state equations forward in time and check the performance index for a decrease. If a decrease in the performance index is obtained, integrate the adjoint equations backwards in time and calculate a new control based on the gradient described above. If a decrease in the performance index is not obtained, the past change in the control is reduced along the gradient and the step size constant, K, is reduced. This loop is maintained until a minimum control step size or a maximum number of iterations is reached.

For the problem involving two spatial independent variables, the spatial line integrals in the control changes become surface integrals. The first variation of the modified performance index after extraction of the state and adjoint equations and the initial, terminal and boundary conditions is a function of the variation in the temporal control functions and the spatial distribution parameters.

$$\delta J^* = \sum_i \sum_j \int_{t_o}^{t_f} \iint_{\Omega} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \theta_{ij}} \, dx_1 dx_2 \, \delta\theta_{ij} \, dt \; +$$

$$\sum_i \sum_j \sum_k \int_{t_o}^{t_f} \iint_{\Omega} \frac{\partial h}{\partial u_i} \frac{\partial u_i}{\partial \phi_{ij}} \frac{\partial \phi_{ij}}{\partial l_{ijk}} \, dx_1 dx_2 dt \, \delta l_{ijk}$$

If the temporal control is discretized, the gradient change in the discrete temporal control values and in the spatial distribution parameters becomes:

$$\Delta\theta_{ij}\,(t_m) = -K\rho_{ijm} \Big/ \left[\sum_i\sum_j\sum_m \rho^2_{ijm} + \sum_i\sum_j\sum_k \sigma^2_{ijk}\right]^{\frac{1}{2}}$$

$$\Delta l_{ijk} = -K\sigma_{ijk} \Big/ \left[\sum_i\sum_j\sum_m \rho^2_{ijm} + \sum_i\sum_j\sum_k \sigma^2_{ijk}\right]^{\frac{1}{2}}$$

$$\rho_{ijm} \triangleq \Delta t \iint_\Omega \frac{\partial h}{\partial u_i}\frac{\partial u_i}{\partial \theta_{ij}}\, dx_1 dx_2 \Bigg|_{t_m}$$

$$\sigma_{ijk} \triangleq \int_{t_o}^{t_f}\iint_\Omega \frac{\partial h}{\partial u_i}\frac{\partial u_i}{\partial \phi_{ij}}\frac{\partial \phi_{ij}}{\partial l_{ijk}}\, dx_1 dx_2 dt \;.$$

If the temporal control is approximated by a polynomial series in time, the gradient change in the temporal control parameters and in the spatial distribution parameters becomes:

$$g_{ijm} = -K\rho_{ijm} \Big/ \left[\sum_i\sum_j\sum_m \rho^2_{ijm} + \sum_i\sum_j\sum_k \sigma^2_{ijk}\right]^{\frac{1}{2}}$$

$$l_{ijk} = -K\sigma_{ijk} \Big/ \left[\sum_i\sum_j\sum_m \rho^2_{ijm} + \sum_i\sum_j\sum_k \sigma^2_{ijk}\right]^{\frac{1}{2}}$$

$$\rho_{ijm} \triangleq \int_{t_o}^{t_f}\iint_\Omega \frac{\partial h}{\partial u_i}\frac{\partial u_i}{\partial \theta_{ij}}\frac{\partial \theta_{ij}}{\partial g_{ijm}}\, dx_1 dx_2 dt$$

$$\sigma_{ijk} \triangleq \int_{t_o}^{t_f}\iint_\Omega \frac{\partial h}{\partial u_i}\frac{\partial u_i}{\partial \phi_{ij}}\frac{\partial \phi_{ij}}{\partial l_{ijk}}\, dx_1 dx_2 dt \;.$$

Scaling for Equal Sensitivity

Scaling is performed in order to correct for the problem of variations in the relative sensitivity of the performance index to changes in the variables to be optimized. The control variables are scaled such that the second derivatives of the performance index with respect to each of the control variables to be optimized are equal.

Consider a performance index which is a function of two variables which are free to be optimized and a linear transformation of the variables.

$$J = f(\zeta_1, \zeta_2)$$

$$\zeta_1 = \gamma_1 \xi_1 \; ; \; \zeta_2 = \gamma_2 \xi_2$$

The performance index and its first and second derivatives can be expressed as function of the new variables ξ_1 and ξ_2 and the scale factors γ_1 and γ_2.

$$J = f(\gamma_1 \xi_1, \gamma_2 \xi_2)$$

$$\frac{\partial J}{\partial \xi_1} = \frac{\partial f}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial \xi_1} = \gamma_1 \frac{\partial f}{\partial \zeta_1}$$

$$\frac{\partial J}{\partial \xi_2} = \frac{\partial f}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial \xi_2} = \gamma_2 \frac{\partial f}{\partial \zeta_2}$$

$$\frac{\partial^2 J}{\partial \xi_1^2} = \frac{\partial^2 f}{\partial \zeta_1^2} \left(\frac{\partial \zeta_1}{\partial \xi_1} \right)^2 = \gamma_1^2 \frac{\partial^2 f}{\partial \zeta_1^2}$$

$$\frac{\partial^2 J}{\partial \xi_2^2} = \frac{\partial^2 f}{\partial \zeta_2^2} \left(\frac{\partial \zeta_2}{\partial \xi_2} \right)^2 = \gamma_2^2 \frac{\partial^2 f}{\partial \zeta_2^2}$$

Setting the second derivatives equal yields one equation and two unknowns. Arbitrary specification of one of the scale factors allows determination of the other.

$$\gamma_1 = 1$$

$$\gamma_2 = \left[\frac{\partial^2 f}{\partial \zeta_1^2} \bigg/ \frac{\partial^2 f}{\partial \zeta_2^2} \right]^{\frac{1}{2}}$$

The scaled gradient changes in ζ_1 and ζ_2 are:

$$\Delta\zeta_1 = \gamma_1 \Delta\xi_1 = -\kappa\gamma_1^2 \ \frac{\partial f}{\partial \zeta_1} \Big/ \left[\gamma_1^2\left(\frac{\partial f}{\partial \zeta_1}\right)^2 + \gamma_2^2 \left(\frac{\partial f}{\partial \zeta_2}\right)^2\right]^{\frac{1}{2}}$$

$$\Delta\zeta_2 = \gamma_1 \Delta\xi_2 = -\kappa\gamma_1^2 \ \frac{\partial f}{\partial \zeta_2} \Big/ \left[\gamma_1^2\left(\frac{\partial f}{\partial \zeta_1}\right)^2 + \gamma_2^2 \left(\frac{\partial f}{\partial \zeta_2}\right)^2\right]^{\frac{1}{2}} .$$

Application of the above method of scaling to the optimal control problem in one spatial independent variable results in the following gradient changes in the control variables in the discretized version.

$$\Delta\theta_{ij}(t_k) = -\kappa \ \gamma_{ijk}^2 \ \rho_{ijk} \Big/ \left[\sum_i \sum_j \sum_k \gamma_{ijk}^2 \ \rho_{ijk}^2 + \sum_i \sum_j \nu_{ij}^2 \, \sigma_{ij}^2 \right]^{\frac{1}{2}}$$

$$\Delta l_{ij} = -\kappa\nu_{ij}^2 \, \sigma_{ij} \Big/ \left[\sum_i \sum_j \sum_k \gamma_{ijk}^2 \ \rho_{ijk}^2 + \sum_i \sum_j \nu_{ij}^2 \sigma_{ij}^2 \right]^{\frac{1}{2}}$$

$$\nu_{11}^2 = 1$$

$$\nu_{ij}^2 = \frac{\partial \sigma_{11}}{\partial l_{11}} \Big/ \frac{\partial \sigma_{ij}}{\partial l_{ij}}$$

$$\gamma_{ijk}^2 = \frac{\partial \sigma_{11}}{\partial l_{11}} \Big/ \frac{\partial \rho_{ijk}}{\partial \theta_{ij}(t_k)}$$

If the time dependent control amplitude is expressed as a polynomial function of time, the changes in the control parameters are scaled similarly. Scaling for problems in two spatial independent variables directly parallels that for problems in one spatial independent variable.

CHAPTER V

APPLICATIONS

Possible applications of the algorithm include the river and lake aeriation problems and the oil reservoir problem. In the aeriation problem described by Tarassov, Perlis and Davidson (49), the system is modeled by a pair of partial differential equations describing the biological oxygen demand and the dissolved oxygen level in the body of water. The objective is to determine an optimal aeriation policy for changing the BOD level. They consider three possible types of controls: a control which is free to vary in time and space, a control which is free to vary in time but constant in space, and a control which is free to vary in space but constant in time. The possibility of a regional optimal control as described in this research is not considered in their paper.

In the oil field reservoir problem, the objective is the proper placement of wells for water flooding of an oil field. Price and Varga (40) consider solutions of the diffusion-convection equation which is a simplified version of the higher order analog used to describe fluid flow in porous medium. Although this problem is extremely complex, the application of optimal regional control theory appears to be possible.

A third possible application would be in the area of thermosetting plastics. Accurate temperature profiles are required for the molds used for forming these plastic parts. A possible extension could include the location of the heaters in the three-dimensional molds and determination of the required time-dependent amplitudes of those heaters.

The approach to the solution of the optimal regional control problem discussed in the previous chapters is demonstrated by its application to systems which are described by the diffusion equation. The remainder of this chapter contains a discussion of example solutions obtained for problems involving diffusion systems with one and two spatial dimensions. In each case, the constrained minimization problem is formulated, necessary conditions are derived, the approximation technique is applied, and the gradient algorithm is used in obtaining numerical solutions.

A Two-dimensional System

A typical example of an optimal regional control problem is to determine the heat input to a thin rod of finite length such that the temperature distribution will approach a desired final temperature distribution in some optimum sense. The two-dimensional temperature distribution is a function of distance along the rod and time.

The state of the system, i.e., the temperature distribution which has been transformed such that the desired final temperature distribution is zero, must satisfy the diffusion equation with associated boundary and initial conditions.

$$s_t(t,x) = cs_{xx}(t,x) + u(t,x) \quad ; \quad \forall\, t\epsilon(t_o,t_f), x\epsilon(x_a,x_b)$$

$$s(t,x) = s_o(x) \quad ; \quad \forall\, x\, (x_a,x_b), t=t_o$$

$$b_{11}s(t,x) + b_{21}(t,x) = b_{31} \quad ; \quad \forall\, t\, (t_o,t_f), x=x_a$$

$$b_{12}s(t,x) + b_{22}(t,x) = b_{32} \quad ; \quad \forall\, t\, (t_o,t_f), x=x_b$$

The control, $u(t,x)$, is defined as the sum of a finite number of regional controls which are the products of specified spatial distributions, $\phi_i(\xi)$, about unspecified means, l_i, and unspecified time dependent amplitudes $\theta_i(t)$.

$$u(t,x) = \sum_i \theta_i(t)\phi_i(x-l_i)$$

To be determined are the locations of the means of the control regions and the time-dependent amplitudes of the control functions such that the state satisfies the state equation with the associated initial conditions and boundary conditions and such that the performance index, J, is a minimum.

$$J = \frac{1}{2}\int_{t_o}^{t_f}\int_{x_a}^{x_b}\left\{\eta\, s^2(t,x) + \nu\left[\sum_i \theta_i(t)\phi_i(x-l_i)\right]^2\right\}dxdt$$

Adjoining the state partial differential equation to the performance index with a two-dimensional Lagrange multiplier, $a(t,x)$, yields the modified performance index, J^*.

$$J^* = J + \int_{t_o}^{t_f}\int_{x_a}^{x_b} a(t,x)\left[s_{xx}(t,x) + \sum_i \theta_i(t)\phi_i(x-l_i) - s_t(t,x)\right] dxdt$$

The Hamiltonian is defined as follows.

$$h \triangleq \frac{1}{2}\eta s^2 + \frac{1}{2}\nu\left[\sum_i \theta_i(t)\phi_i(x-l_i)\right]^2 + a(t,x)\left[s_{xx}(t,x) + \sum_i \theta_i(t)\phi_i(x-l_i)\right]$$

Application of the calculus of variations yields the original state equation with initial and boundary conditions and the adjoint equation with terminal and boundary conditions.

$$a_t(t,x) = -ca_{xx}(t,x) - \mu s(t,x) \quad ; \quad \forall\, t\in(t_o,t_f), x\in(x_a,x_b)$$

$$a(t,x) = 0 \quad ; \quad \forall\, x\in(x_a,x_b), t=t_f$$

$$b_{11}a(t,x) + b_{21}a_x(t,x) = 0 \quad ; \quad \forall\, t\in(t_o,t_f), x=x_a$$

$$b_{21}a(t,x) + b_{22}a_x(t,x) = 0 \quad ; \quad \forall\, t\in(t_o,t_f), x=x_b$$

Remaining in the first variation of the modified performance index are the terms involving the first variations of $\theta_i(t)$ and l_i.

$$\delta J^* = \sum_i \int_{t_o}^{t_f} \int_{x_a}^{x_b} \left\{ \nu \sum_j \left[\theta_j(t)\phi_j(x-l_j)\right] + a(t,x) \right\} \phi_i(x-l_i)\,dx\,\delta\theta_i(t)\,dt + \sum_i \int_{t_o}^{t_f} \int_{x_a}^{x_b} \left\{ \nu \sum_j \left[\theta_j(t)\phi_j(x-l_j)\right] + a(t,x) \right\} \frac{\partial \phi_i(x-l_i)}{\partial l_i}\,dx\,\theta_i(t)\,dt\,\delta l_i$$

Since the control amplitudes, $\theta_i(t)$, and locations, l_i, are unconstrained, it is assumed that there exist optimum $\theta_i(t)$ and l_i such that the first variation of the modified performance index is zero. Therefore, the following conditions must hold.

$$\int_{x_a}^{x_b} \left\{ \nu \sum_j \left[\theta_j(t)\phi_j(x-l_j)\right] + a(t,x) \right\} \phi_i(x-l_i)\,dx = 0$$

$$\int_{t_o}^{t_f} \int_{x_a}^{x_b} \left\{ \nu \sum_j \left[\theta_j(t)\phi_j(x-l_j)\right] + a(t,x) \right\} \frac{\partial \phi_i(x-l_i)}{\partial l_i}\,dx\,\theta_i(t)\,dt = 0$$

Application of Galerkin's Method

Assume that the state of the system may be expressed as the vector product of a series of time dependent variables and a series of space dependent basis functions which when combined satisfy the boundary conditions. Since the differential equation and boundary conditions are self-adjoint, the same basis functions are used for the adjoint variable.

$$s(t,x) = Q_1^T(t)R(x)$$

$$a(t,x) = Q_1^T(t)R(x)$$

Each of the basis functions is required to be orthogonal to the error in the partial differential equations introduced by substitution of the approximating vector product.

$$\int_{x_a}^{x_b} R(x)\left[s_t(t,x) - cs_{xx}(t,x) - u(t,x)\right]dx = 0$$

$$\int_{x_a}^{x_b} R(x)\left[a_t(t,x) + ca_{xx}(t,x) + \mu s(t,x)\right]dx = 0$$

Integration by parts before substitution of the vector product yields simplified equation forms.

$$\int_{x_a}^{x_b} \left[R(x)\ s_t(t,x) + cR_x(x)s_x(t,x) - R(x)u(t,x)\right]dx - R(x)s_x(t,x)\Big|_{x_a}^{x_b} = 0$$

$$\int_{x_a}^{x_b} \left[R(x)\ a_t(t,x) - cR_x(x)a_x(t,x) + \mu R(x)s(t,x)\right]dx + R(x)a_x(t,x)\Big|_{x_a}^{x_b} = 0$$

For the special case of one adiabatic end ($b_{11} = b_{31} = 0$) and one isothermal end ($b_{22} = b_{32} = 0$), the terms evaluated at the boundaries are zero. Substitution of the vector product yields:

$$Q_{1t}(t) = - c \left[\int_{x_a}^{x_b} R(x) R^T(x) dx \right]^{-1} \left[\int_{x_a}^{x_b} R_x(x) R_x^T(x) dx \right] Q_1(t) +$$

$$\left[\int_{x_a}^{x_b} R_x(x) R_x^T(x) dt \right]^{-1} \sum_i \left[\int_{x_a}^{x_b} R(x) \phi_i (x - l_i) dx \, \theta_i(t) \right]$$

$$Q_{2t}(t) = c \left[\int_{x_a}^{x_b} R(x) R^T(x) dx \right]^{-1} \left[\int_{x_a}^{x_b} R_x(x) R_x^T(x) dx \right] Q_2(t) -$$

$$\mu Q_1(t)$$

The state initial condition and adjoint terminal condition are transformed by expansion in terms of the basis functions.

$$\int_{x_a}^{x_b} R(x) \left[s(t_o, x) - s_o(x) \right] dx = 0$$

$$\int_{x_a}^{x_b} R(x) \left[a(t_f, x) - 0 \right] dx = 0$$

Introduction of the vector product for $s(t,x)$ and $a(t,x)$ yields $Q_1(t_o)$

and $Q_2(t_f)$.

$$Q_1(t_o) = \left[\int_{x_a}^{x_b} R(x)R^T(x)dx\right]^{-1} \left[\int_{x_a}^{x_b} R(x)s_o(x)dx\right]$$

$$Q_2(t_f) = 0$$

The truncated eigenfunction expansion for the diffusion equation with one adiabatic end and one isothermal end at zero is:

$$R_i(x) = \cos\left[\frac{[1 + 2\ (i-1)]\pi}{2\ (x_b - x_a)} \quad (x - x_a)\right] \ ; \ i = 1,2,\ldots,n$$

In the case of the Hermite interpolation polynomial expansion, the proper elements of a complete expansion must be chosen to have zero coefficients in order that the boundary conditions are satisfied. Referring to previous definitions for the Hermite functions, the coefficients of $\eta_o^1(x)$ and $\eta_n^o(x)$ must be zero. For n + 1 nodal points numbered from 0 to n, the Hermite interpolation polynomial expansion includes the following terms.

$$R(x) = \left\langle r_1(x),\ r_2(x),\ldots,r_{2n}(x)\right\rangle$$

$$r_1(x) = \eta_o^o(x)$$

$$\left.\begin{aligned} r_{2i}(x) &= \eta_i^o(x) \\ r_{2i+1}(x) &= \eta_i^1(x) \end{aligned}\right\} \quad i = 1,2,\ldots,n-1$$

$$r_{2n}(x) = \eta_n^1(x)$$

The matrices of spatial integrals of $R(x)R^T(x)$ and $R_x(x)R_x^T(x)$ resulting from the application of Galerkin's method are single diagonal matrices in the case of the eigenfunction expansion due to the orthogonality of the elements. In the case of the Hermite interpolation polynomial expansion, these matrices are symmetric with zero elements in all but the seven center-most diagonals and, therefore, exhibit "banded orthogonality."

Gradient Control Modification

The gradient direction of the control modification is determined by examining the terms remaining in the first variation after extraction of the state and adjoint equations and initial, terminal and boundary conditions. If the temporal control amplitudes, $\theta_i(t)$, are discretized with the same time step as the state and adjoint equation integration interval, the iterative change in the control amplitudes and locations are given by the following relations. Note that these changes have been scaled for equal sensitivities.

$$\Delta\theta_i(t_j) = -\kappa\gamma_{ij}^2\rho_{ij} \Big/ \left[\sum_i\sum_j \gamma_{ij}^2\rho_{ij}^2 + \sum_i \nu_i^2\sigma_i^2\right]^{\frac{1}{2}}$$

$$\Delta l_i = -\kappa\nu_i^2\sigma_i \Big/ \left[\sum_i\sum_j \gamma_{ij}^2\rho_{ij}^2 + \sum_i \nu_i^2\sigma_i^2\right]^{\frac{1}{2}}$$

$$\rho_{ij} \triangleq \Delta t\left\{\nu\sum_k\left[\int_{x_a}^{x_b} \phi_k(x-l_k)\phi_i(x-l_i)\,dx\,\theta_k(t_j)\right] + \sum_m\left[\int_{x_a}^{x_b} r_m(x)\phi_i(x-l_i)dx\; q_{2m}(t_j)\right]\right\}$$

$$\sigma_i \triangleq \nu\sum_k\left[\int_{t_o}^{t_f}\theta_k(t)\theta_i(t)dt\int_{x_a}^{x_b}\phi_k(x-l_k)\,\frac{\partial\phi_i(x-l_i)}{\partial l_i}\,dx\right] +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\theta_i(t)dt \int_{x_a}^{x_b} r_m(x) \frac{\partial \phi_i(x-l_i)}{\partial l_i} dx \right]$$

$$\upsilon_1^2 \triangleq 1$$

$$\upsilon_i^2 \triangleq \left\{ \nu \left[\int_{t_o}^{t_f} \theta_1^2(t)\ dt \int_{x_a}^{x_b} \left(\frac{\partial \phi_1(x-l_1)}{\partial l_1} \right)^2 dx \right] + \right.$$

$$\nu \sum_k \left[\int_{t_o}^{t_f} \theta_k(t)\theta_i(t)dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial^2 \phi_1(x-l_1)}{\partial l_1^2} dx \right] +$$

$$\left. \sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\theta_1(t)dt \int_{x_a}^{x_b} r_m(x) \frac{\partial^2 \phi_1(x-l_1)}{\partial l_1^2} dx \right] \right\} \bigg/$$

$$\left\{ \nu \left[\int_{t_o}^{t_f} \theta_i^2(t)\ dt \int_{x_a}^{x_b} \left(\frac{\partial^2 \phi_i(x-l_i)}{\partial l_i^2} \right)^2 dx \right] + \right.$$

$$\nu \sum_k \left[\int_{t_o}^{t_f} \theta_k(t)\theta_i(t)dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial^2 \phi_i(x-l_i)}{\partial l_i^2} dx \right] +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\theta_i(t)dt \int_{x_q}^{x_b} r_m(x) \frac{\partial^2 \phi_i(x-l_i)}{\partial l_i^2} dx \right]$$

$$\gamma_{ij}^2 = \left\{ \nu \left[\int_{t_o}^{t_f} \theta_1^2(t)dt \int_{x_a}^{x_b} \left(\frac{\partial \phi_1(x-l_1)}{\partial l_1} \right)^2 dx \right] + \right.$$

$$\nu \sum_k \left[\int_{t_o}^{t_f} \theta_k(t)\theta_l(t)dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial^2 \phi_l(x-l_l)}{\partial l_l^2} dx \right] +$$

$$\left. \sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\theta_l(t)dt \int_{x_a}^{x_b} r_m(x) \frac{\partial^2 \phi_l(x-l_l)}{\partial l_l^2} dx \right] \right\} \Big/$$

$$\left\{ \Delta t\, \nu \left[\int_{x_a}^{x_b} \phi_i^2(x-l_i)dx \right] \right\}$$

If the temporal control amplitude is expressed as a polynomial in time,

$$\theta_i(t) = \sum_j g_{ij} t^{j-1} ,$$

the iterative changes in the control polynomial coefficients and locations are expressed in a similar manner.

$$\Delta g_{ij} = -\chi \gamma_{ij}^2 \rho_{ij} \Big/ \left[\sum_i \sum_j \gamma_{ij}^2 \rho_{ij}^2 + \sum_i \nu_i^2 \sigma_i^2 \right]^{\frac{1}{2}}$$

$$\Delta l_i = -\chi \nu_i^2 \sigma_i \Big/ \left[\sum_i \sum_j \gamma_{ij}^2 \rho_{ij}^2 + \sum_i \nu_i^2 \sigma_i^2 \right]^{\frac{1}{2}}$$

$$\rho_{ij} \triangleq \nu \sum_k \left[\int_{t_o}^{t_f} \theta_k(t) \frac{\partial \theta_i(t)}{\partial g_{ij}} dt \int_{x_a}^{x_b} \phi_k(x-l_k)\phi_i(x-l_i)\, dx \right] +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t) \frac{\partial \theta_i(t)}{\partial g_{ij}} dt \int_{x_a}^{x_b} r_m(x)\phi_i(x-l_i)dx \right]$$

$$\sigma_i \triangleq \nu \sum_k \left[\int_{t_o}^{t_f} \Theta_k(t)\Theta_i(t)dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial \phi_i(x-l_i)}{\partial l_i} dx \right] +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\Theta_i(t)dt \int_{x_a}^{x_b} r_m(x) \frac{\partial \phi_i(x-l_i)}{\partial l_i} dx \right]$$

$$\nu_1^2 \triangleq 1$$

$$\nu_i^2 \triangleq \left\{ \nu \left[\int_{t_o}^{t_f} \Theta_1^2(t)dt \int_{x_a}^{x_b} \left(\frac{\partial \phi_1(x-l_1)}{\partial l_1} \right)^2 dx \right] + \right.$$

$$\nu \sum_k \left[\int_{t_o}^{t_f} \Theta_k(t)\Theta_1(t)dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial^2 \phi_1(x-l_1)}{\partial l_1^2} dx \right] +$$

$$\left. \sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\Theta_1(t)dt \int_{x_a}^{x_b} r_m(x) \frac{\partial^2 \phi_1(x-l_1)}{\partial l_1^2} dx \right] \right\} \Big/$$

$$\left\{ \nu \left[\int_{t_o}^{t_f} \Theta_i^2(t)dt \int_{x_a}^{x_b} \left(\frac{\partial \phi_i(x-l_i)}{\partial l_i} \right)^2 dx \right] + \right.$$

$$\nu \sum_k \left[\int_{t_o}^{t_f} \Theta_k(t)\Theta_i(t)dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial^2 \phi_i(x-l_i)}{\partial l_i^2} dx \right] +$$

$$\left. \sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\Theta_i(t)dt \int_{x_a}^{x_b} r_m(x) \frac{\partial^2 \phi_i(x_i-l_i)}{\partial l_i^2} dx \right] \right\}$$

$$
\gamma_{ij}^{2} = \left\{ \nu \left[\int_{t_o}^{t_f} \theta_1^2(t)\,dt \int_{x_a}^{x_b} \left(\frac{\partial \phi_1 (x-l_1)}{\partial l_1} \right)^2 dx \right] + \nu \sum_k \left[\int_{t_o}^{t_f} \theta_k(t)\theta_1(t)\,dt \int_{x_a}^{x_b} \phi_k(x-l_k) \frac{\partial^2 \phi_1 (x-l_1)}{\partial l_1^2}\, dx \right] + \sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\theta_1(t)\,dt \int_{x_a}^{x_b} r_m(x) \frac{\partial^2 \phi_1 (x-l_1)}{\partial l_1^2}\, dx \right] \right\} \Bigg/ \left\{ \nu \left[\int_{t_o}^{t_f} \left(\frac{\partial \theta_i(t)}{\partial g_{ij}} \right)^2 dt \int_{x_a}^{x_b} \phi_i^2(x-l_i)\, dx \right] \right\}
$$

Computational Algorithm

The computational algorithm for the solution of the optimal regional control problem can be divided into four fundamental sections which are initialization, state equation integration, adjoint equation integration and control modification. The initialization section includes data input and the calculation of integrals which are not affected by changes in the control. These include the calculation of the matrix spatial integrals of $R(x)R^T(x)$, $R_x(x)R_x^T(x)$ and $R(x)s_o(x)$. The inverse of the integral of $R(x)R^T(x)$ and the initial condition on $Q_1(t)$ are also calculated.

The optimization loop contains the remaining three sections. In the state equation integration section, the matrix spatial integral of $R(x)\Phi^T(X-L)$ where $\Phi(X-L)$ is the control spatial distribution vector containing elements $\phi_j(x-l_j)$, is first calculated. Then the state equation is

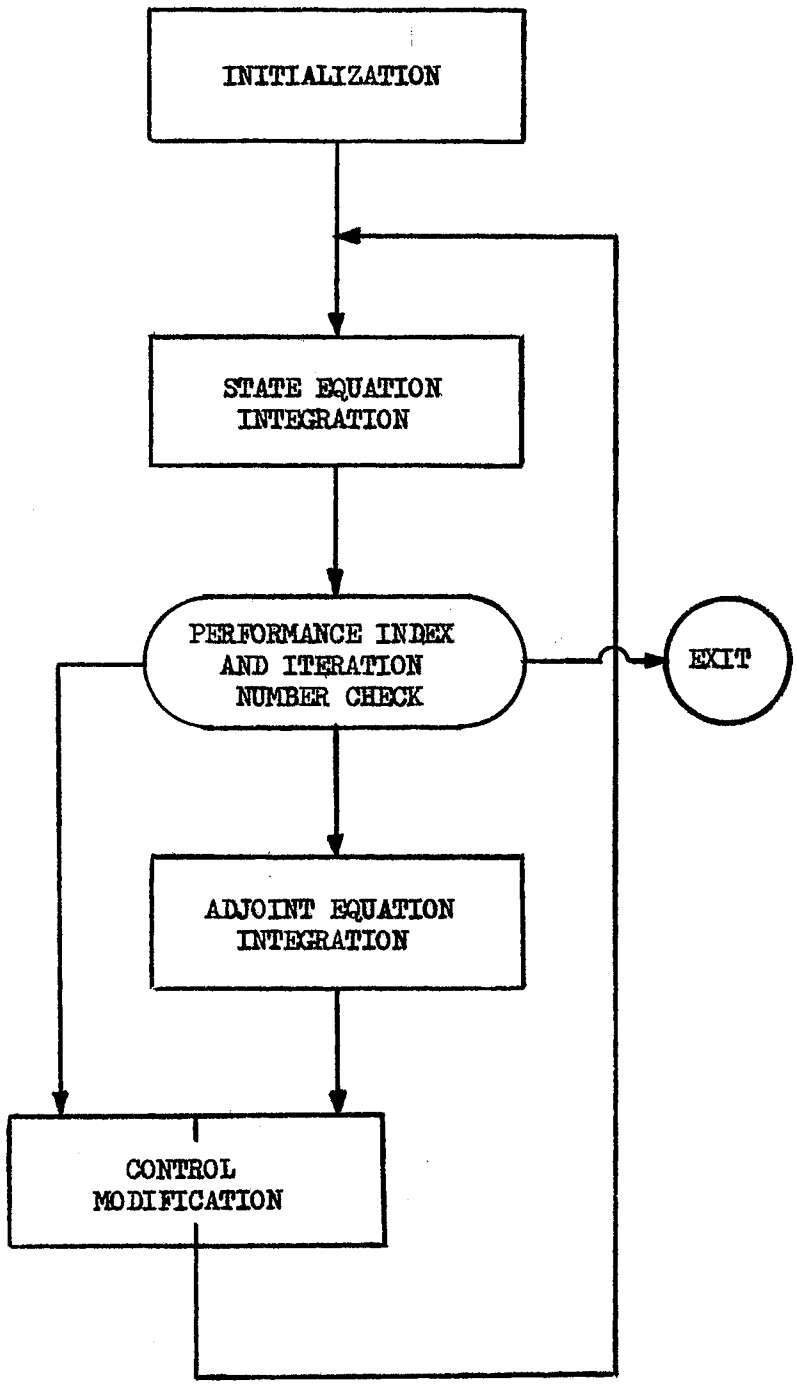


Figure 15. Fundamental Sections of Computational Algorithm

integrated forward in time. Calculation of the matrix integrals of $Q_1(t)\,Q_1^T(t)$, $\Phi(X-L)\Phi^T(X-L)$ and $\Theta(t)\Theta^T(t)$ allows calculation of the performance index.

Between the state equation integration and the adjoint integration section, the performance index is compared with the performance index of the past iteration except in the first iteration. If a decrease in the performance index is not obtained, the adjoint integration section is bypassed. If the iteration number is the preset maximum, the program is stopped.

The adjoint equation integration section contains only the background integration of the adjoint equation. The spatial integrals required for the coefficients in the adjoint equation have been previously calculated and the control in the adjoint equation is the system state, obtained in the state equation integration.

The control modification section contains a subsection for each of two types of control change calculations. If the new performance index is less than the past performance index indicating a control improvement, the gradient control changes are calculated. If the performance index change is not an improvement, the past control change is reduced along the past gradient and the gradient constant, K, is reduced. The optimization loop is then closed by the return to the state equation integration section.

Example Solutions

Example solutions have been obtained for the following state equation, initial condition, boundary conditions and performance index.

$$s_t(t,x) = s_{xx}(t,x) + \sum_i \theta_i(t)\phi_i(x-l_i) \quad ; \quad \forall t\in(0,.1), x\in(0,1)$$

$$s(o,x) = s_o(x) \quad ; \quad s_x(t,o) = 0 \quad ; \quad s(t,1) = 0$$

$$J = \frac{1}{2} \int_o^{.1} \int_o^1 \left\{ \mu s^2(t,x) + \left[\sum_i \theta_i(t)\phi_i(x - l_i) \right]^2 \right\} dxdt$$

The spatial control distributions are specified to be normal distributions and solutions have been obtained for standard deviations of .2 and .1,

$$\phi_i(x-l_i) = \exp\left[-\left(\frac{x-l_i}{\text{std. dev.}}\right)^2\right]$$

which appear in Figure 16a and b. Figure 17a and b show a comparison of the error resulting from the expansion of the spatial control distribution in terms of the first ten eigenfunctions for the above system and in terms of the Hermite interpolation piecewise polynomials associated with the spatial domain divided into five subdomains. The approximate value of ϕ is referred to as $\widetilde{\phi}$ and R represents the vector whose elements are those of the particular expansion.

$$\widetilde{\phi}(x-l) = R^T(x)\left[\int_o^1 R(x)R^T(x)dx\right]^{-1}\left[\int_o^1 R(x)\phi(x-l)dx\right]$$

It may be noted that the errors in the approximate values of the control utilizing the eigenfunction expansion and the Hermite interpolation polynomial expansion are nearly equivalent for a standard deviation of .1, while the eigenfunction expansion is significantly more accurate for a standard deviation of .2.

Example solutions have been obtained for problems involving one, two, and three control regions. Eigenfunction expansions and Hermite interpolation polynomial expansions are utilized. The temporal control functions are either dexcretized or constrained to be polynomial functions of time. The results are summarized in Table II. Appendix A contains plots of the

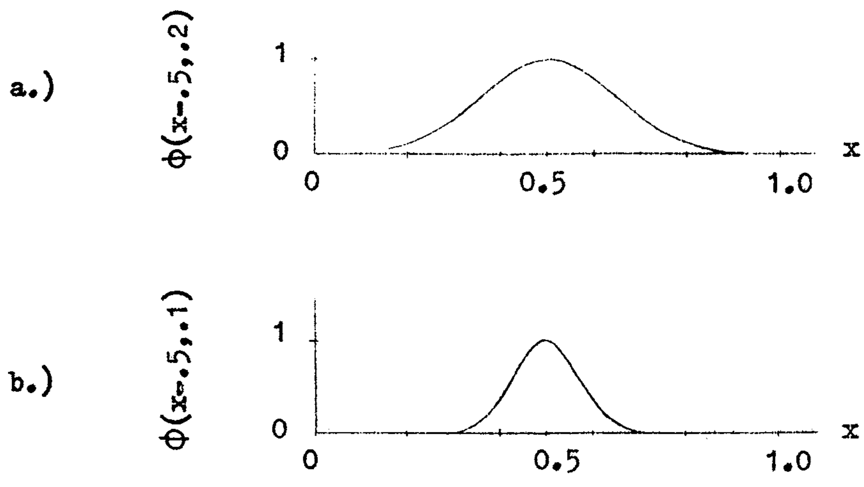


Figure 18. Spatial Control Functions

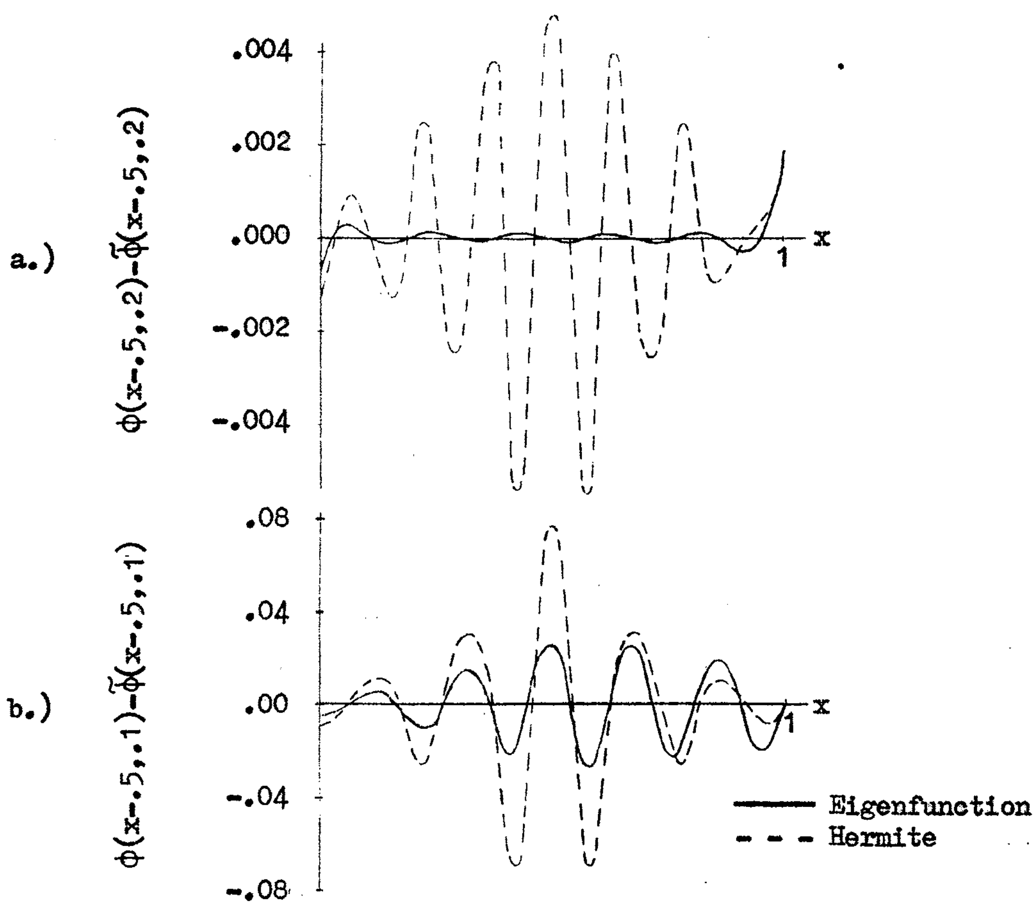


Figure 17. Error in Expansion for Spatial Control Function

TABLE II

EXAMPLE SOLUTIONS OF A TWO-DIMENSIONAL REGIONAL CONTROL PROBLEM

Number of Control Regions	Type of Spatial Expansion	Type of Temporal Control Approx.	Control Std. Dev.	Perf. Index Coeff.	Initial Guess for $\phi_i(t)$	Guessed Control Location l_1	Guessed Control Location l_2	Guessed Control Location l_3	Final Control Location l_1	Final Control Location l_2	Final Control Location l_3	Initial Performance Index	Final Performance Index	Figure Numbers: Append. A $\phi_1(t)$	Figure Numbers: Append. A l_1	Figure Numbers: Chapt. 5 State Surface	Figure Numbers: Chapt. 5 Control Surface
1	Eigen.	Poly.	.2	100	3-30t	.50			.236			1.171	1.095	43	45	18	19
	Hermite	Poly.	"	"	"	"			.236			1.152	1.079	43	45		
	Eigen.	Disc.	"	"	5-50t	"			.241			1.171	1.095	44	46		
	Eigen.	Disc.	"	"	5	.00			.243			1.203	1.095	44	46		
	Hermite	Disc.	"	"	5-50t	.50			.244			1.152	1.079	44	46		
	Hermite	Disc.	"	"	5	.00			.244			1.185	1.079	44	46		
2	Eigen.	Poly.	"	"	3-30t	.33	.66		.139	.527		1.137	1.051	47	49	20	21
	Hermite	Poly.	"	"	"	"	"		.134	.527		1.106	1.037	47	49		
	Eigen.	Disc.	"	"	"	"	"		.243	.242		1.124	1.095	48	50		
	Hermite	Disc	"	"	"	"	"		.238	.237		1.106	1.079	48	50		
2	Eigen.	Poly.	"	"	"	.10	.90		.144	.534		1.169	1.052	51	53		
	Hermite	Poly.	"	"	"	"	"		.149	.534		1.150	1.037	51	53		
	Eigen.	Disc.	"	"	"	"	"		.162	.460		1.169	1.059	52	54		
	Hermite	Disc.	"	"	"	"	"		.155	.402		1.150	1.052	52	54		
2	Eigen.	Poly.	.1	"	"	.33	.66		.112	.360		1.194	1.097	55	57	22	23
	Hermite	Poly.	"	"	"	"	"		.122	.375		1.174	1.080	55	57		
	Eigen.	Disc.	"	"	"	"	"		.186	.329		1.194	1.130	56	58		
	Hermite	Disc.	"	"	"	"	"		.180	.349		1.174	1.101	56	58		
2	Eigen.	Poly.	.2	1000	"	"	"		.420	.081		10.40	5.47	59	60		
3	Eigen.	Poly.	"	100	"	.25	.50	.75	.081	.359	.649	1.105	1.047	61	62	24	25

control amplitudes versus time and of the control location versus iteration number for each case.

The control amplitudes and control locations for the single control region problem resulting from the application of both the eigenfunction expansion and the Hermite interpolation polynomial expansion together with both a polynomial temporal control and a discretized temporal control demonstrate good correlation. A typical state surface and the corresponding control surface are shown in Figures 18 and 19. Control location convergence to an optimal location is obtained from different guessed locations. See Figures 43 through 46 in Appendix A.

For the two control region problem, example solutions are presented for different starting guesses, different control standard deviations and different performance index coefficients. A comparison of the solutions of the discretized versions of the problem for different starting locations indicates that the performance index hypersurface is not unimodal. The composite control for one of the starting locations is essentially the same as the optimal control of the single control problem, while the control regions resulting from the other starting guess are separated and yield a better performance index. The polynomial temporal control versions yield separated control regions for both starting locations. See figures 47 through 54 in Appendix A. A typical state surface and control surface are shown in Figures 20 and 21.

The resulting controls for the smaller standard deviation are closer together and closer to the adiabatic end of the rod. The state surface and control surface for two control regions with standard deviations of .1 are shown in Figures 22 and 23.

An increase in the coefficient of the state squared term in the

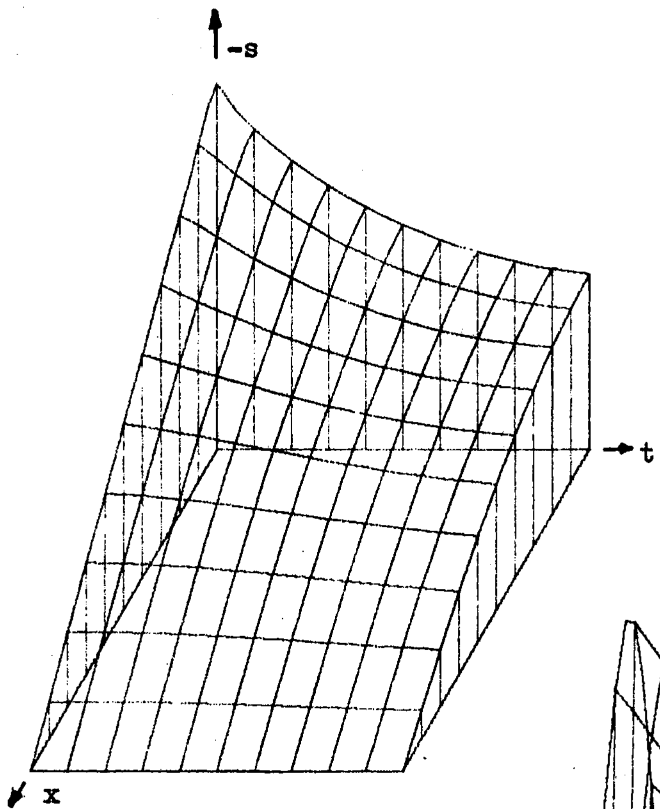


Figure 18. State Surface - One Control Region, Polynomial Temporal Control Function, Eigenfunction Expansion

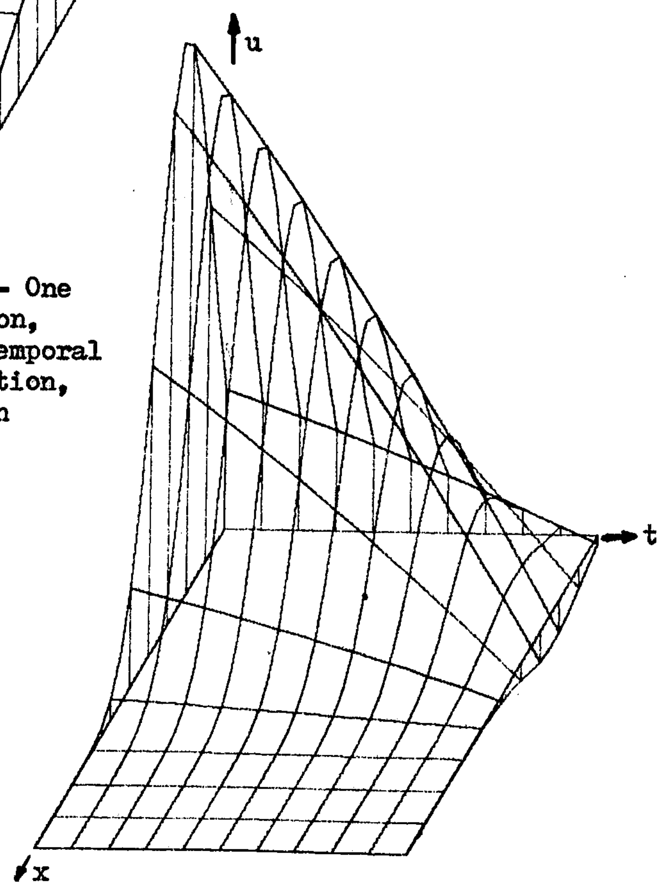


Figure 19. Control Surface - One Control Region, Polynomial Temporal Control Function, Eigenfunction Expansion

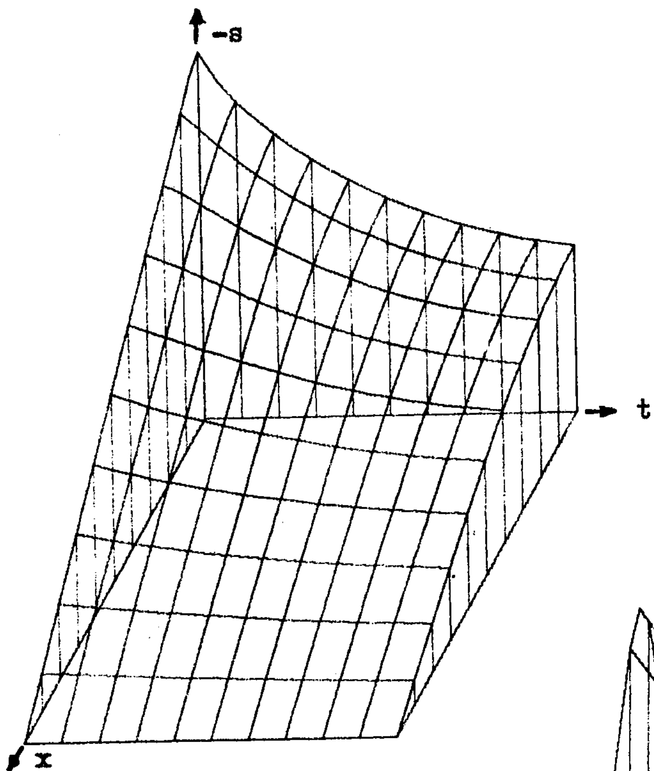


Figure 20. State Surface - Two Control Regions, Polynomial Temporal Control Function, Eigenfunction Expansion

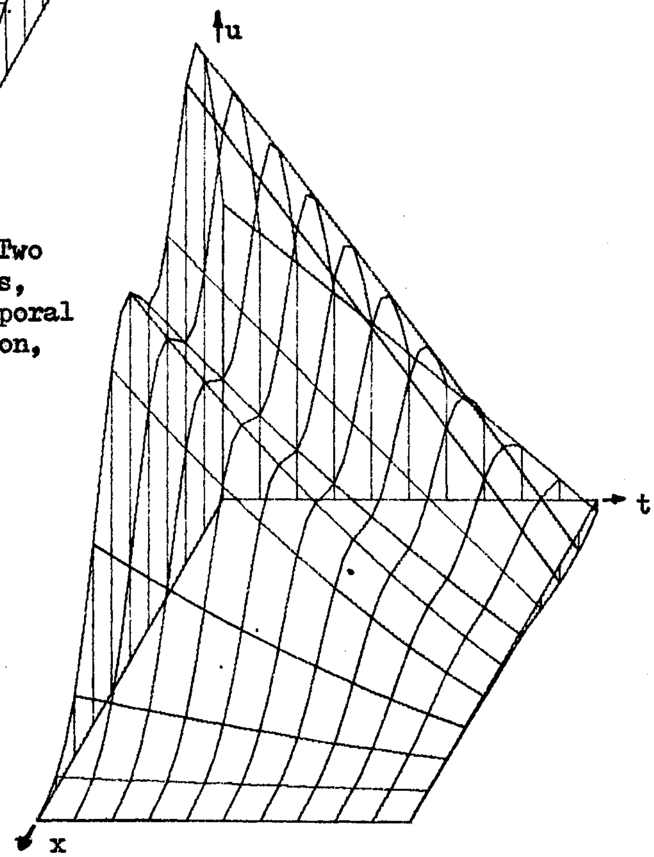


Figure 21. Control Surface - Two Control Regions, Polynomial Temporal Control Function, Eigenfunction Expansion

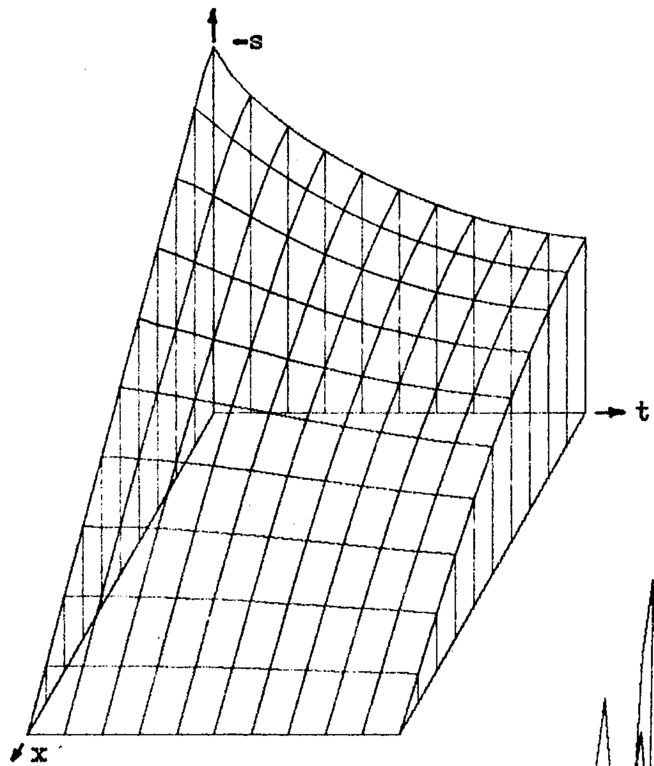


Figure 22. State Surface - Two Control Regions, Polynomial Temporal Control Function, Eigenfunction Expansion

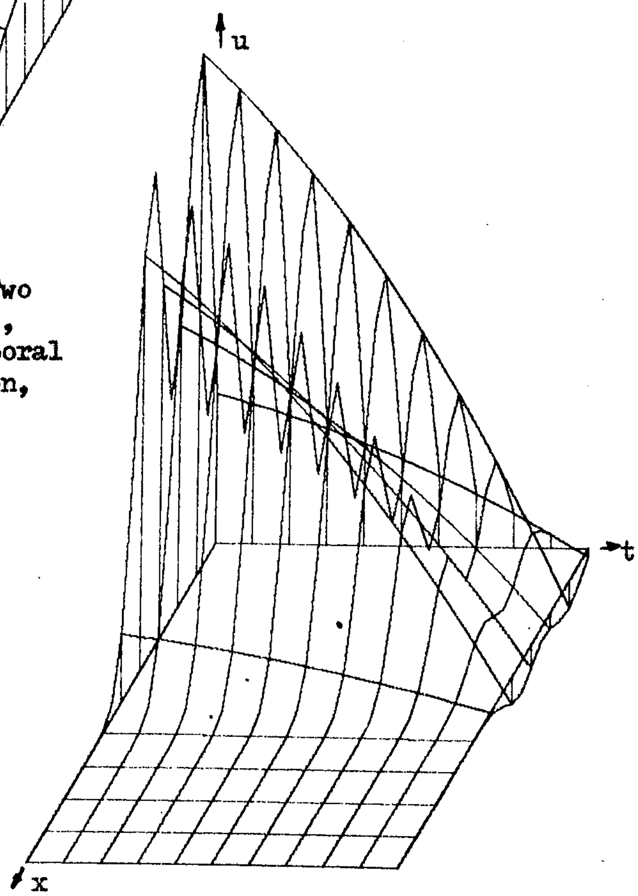


Figure 23. Control Surface - Two Control Regions, Polynomial Temporal Control Function, Eigenfunction Expansion

performance index yields a rise in the control amplitudes and a further separation of the control regions.

The addition of a third control region provides a smaller decrease in the performance index than the decrease obtained by the addition of the second control. The state surface and control surface are given for the three control problem in Figures 24 and 25.

The employment of the eigenfunction expansion is desirable for linear, two-dimensional problems due to the reduced computational time resulting from the orthogonality of its elements. In the case of nonlinear problems, the employment of the Hermite interpolation polynomial expansion may be justified.

For multiple control regions, the polynomial temporal control appears to exhibit better convergence than the discretized temporal control.

A Three-dimensional System

A typical example of a three-dimensional, optimal regional control problem results from the extension of the temperature control problem for a thin rod to that for a thin plate. The temperature distribution is now a function of time and the two spatial dimensions.

The state of the system must satisfy the diffusion equation within the spatial domain, Ω, and associated initial condition and boundary conditions along the spatial boundary Ω_b.

$$s_t(t,x_1,x_2) = c\left[s_{x_1x_1}(t,x_1,x_2) + s_{x_2x_2}(t,x_1,x_2)\right] + u(t,x_1,x_2) \;; \quad \forall t\in(t_o,t_f),(x_1,x_2)\in\Omega$$

$$s(t,x_1,x_2) = s_o(x_1,x_2) \;; \quad \forall(x_1,x_2)\in\Omega, t = t_o$$

$$b_1 s(t,x_1,x_2) + b_2 s_n(t,x_1,x_2) = b_3 \;; \quad \forall t\in(to,t_f),(x_1,x_2)\in\Omega_b$$

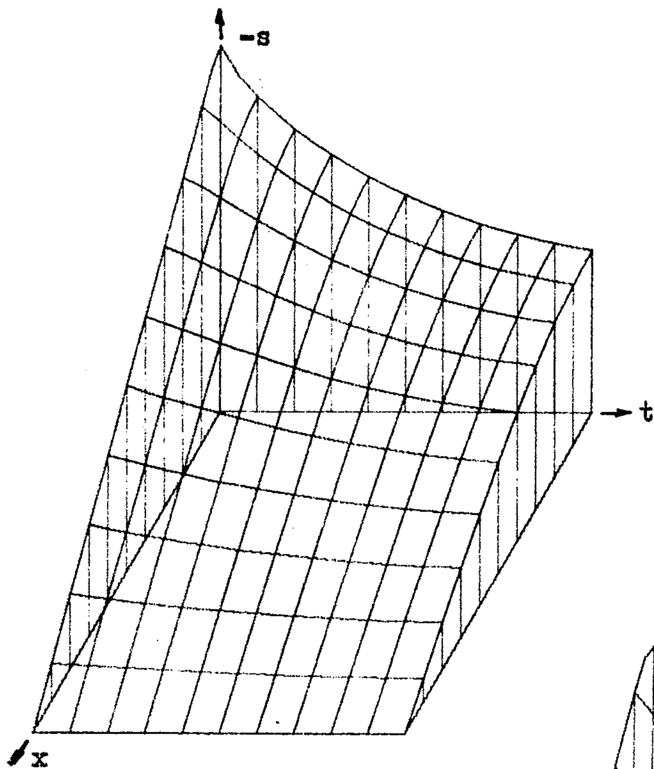


Figure 24. State Surface - Three Control Regions, Polynomial Temporal Control Function, Eigenfunction Expansion

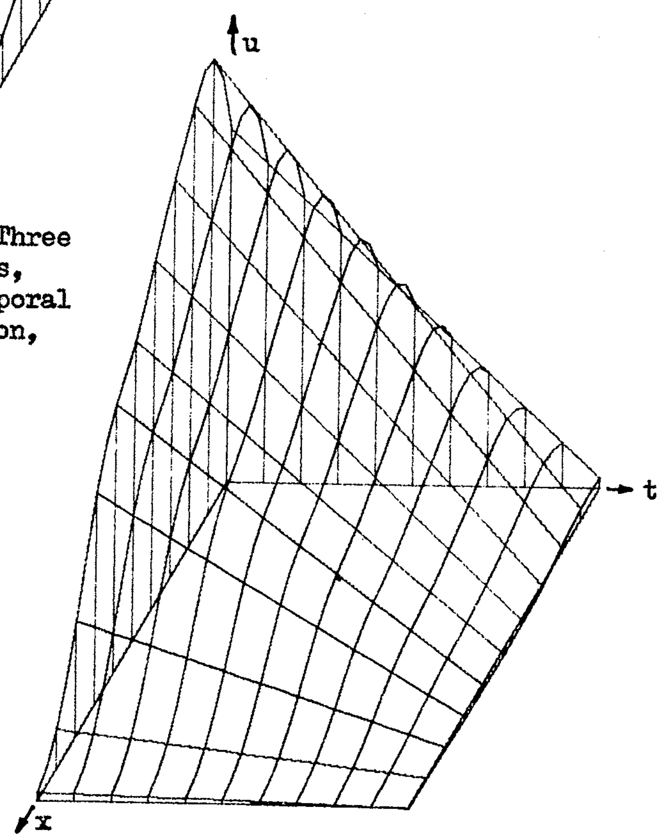


Figure 25. Control Surface - Three Control Regions, Polynomial Temporal Control Function, Eigenfunction Expansion

The control, $u(t,x_1,x_2)$, is defined as the sum of a finite number of regional controls similar to those in the problem with one spatial dimension.

$$u(t,x_1,x_2) = \sum_i \theta_i(t)\,\phi_i(x_1 - l_{i1}, x_2 - l_{i2})$$

To be determined are the control locations, which are specified by l_{i1} and l_{i2}, and time dependent amplitudes of the control functions such that the state satisfies the state equation with associated initial and boundary conditions and such that the performance index, J, is a minimum.

$$J = \frac{1}{2}\int_{t_o}^{t_f}\iint_{\Omega}\left\{\mu\, s^2(t,x_1x_2) + \left[\sum_i \theta_i(t)\phi_i(x_1 - l_{i1}, x_2 - l_{i2})\right]^2\right\}dx_1dx_2dt$$

The modified performance index is formed by adjoining the state equation to the performance index above with a three-dimensional lagrange multiplier, $a(t,x_1,x_2)$.

$$J^* = J + \int_{t_o}^{t_f}\iint_{\Omega} a(t,x_1,x_2)\Big[c(s_{x_1x_1} + s_{x_2x_2}) +$$

$$\sum_i \theta_i(t)\phi_i(x_1 - l_{i1}, x_2 - l_{i2}) - s_t(t,x_1,x_2)\Big]dx_1dx_2dt$$

The application of calculus of variations yields the original state equation with initial and boundary conditions and the adjoint equation with terminal and boundary conditions.

$$a_t(t,x_1,x_2) = -c(a_{x_1x_1} + a_{x_2x_2}) - \mu s(t_1x_1,x_2) \quad ;$$

$$\forall\, t\in(t_o,t_f),\ (x_1,x_2)\in\Omega$$

$$a(t,x_1,x_2) = 0 \qquad ; \quad \forall (x_1,x_2)\in\Omega \,, t = t_f$$

$$b_1 a(t,x_1,x_2) + b_2 a_n(t,x_1,x_2) = 0 \qquad ; \quad \forall\, t\in(t_o,t_f),(x_1,x_2)\in\Omega_b$$

The control amplitudes and location parameters must satisfy the following conditions.

$$\iint_{\Omega} \left\{ \nu \sum_j \left[\Theta_j(t)\phi_j(x_1 - l_{j1}, x_2 - l_{j2}) \right] + a(t,x) \right\} \phi_i(x_1 - l_{i1}, x_2 - l_{i2})\, dx = 0$$

$$\int_{t_o}^{t_f} \iint_{\Omega} \left\{ \nu \sum_j \left[\Theta_j(t)\phi_j(x_1 - l_{j1}, x_2 - l_{j2}) \right] + a(t,x) \right\} \frac{\partial \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{ik}}\, dx\, \Theta_i(t)\, dx = 0$$

Application of Galerkin's Method

The application of Galerkin's method to the three-dimensional problem follows directly its application to the two-dimensional problem considered previously. The state of the system is approximated by the vector product of a series of time dependent variables and a series of space dependent basis functions. The basis functions for the adjoint variable are the same as those for the state variable providing the state equation and boundary conditions are self-adjoint.

$$s(t,x_1,x_2) = Q_1^T(t)R(x_1,x_2)$$

$$a(t,x_1,x_2) = Q_2^T(t)R(x_1,x_2)$$

The orthogonality requirement of Galerkin's method yields relations for the new state and adjoint vectors, $Q_1(t)$ and $Q_2(t)$.

$$\iint_{\Omega} R(x_1,x_2)\left\{s_t(t,x_1,x_2) - c\left[s_{x_1x_1}(t,x_1,x_2) + s_{x_2x_2}(t,x_1,x_2)\right] - u(t,x_1,x_2)\right\} dx_1dx_2 = 0$$

$$\iint_{\Omega} R(x_1,x_2)\left\{a_t(t,x_1,x_2) + c\left[a_{x_1x_1}(t,x_1,x_2) + a_{x_2x_2}(t,x_1,x_2)\right] + \mu\, s(t,x_1,x_2)\right\} dx_1dx_2 = 0$$

The application of Green's theorm yields simplified forms of these vector equations.

$$\iint_{\Omega}\left\{R(x_1,x_2)s_t(t,x_1,x_2) + c\left[R_{x_1}(x_1,x_2)s_{x_1}(t,x_1,x_2) + R_{x_2}(x_1,x_2)s_{x_2}(t,x_1,x_2)\right] - R(x_1,x_2)u(t,x_1,x_2)\right\} dx_1dx_2 + \oint_{\Omega_b} R(x_1,x_2)s_{x_1}(t,x_1,x_2)dx_2 - \oint_{\Omega_b} R(x_1,x_2)s_{x_2}(t,x_1,x_2)\, dx_1 = 0$$

$$\iint_{\Omega}\left\{R(x_1,x_2)a_t(t,x_1,x_2) - c\left[R_{x_1}(x_1,x_2)a_{x_1}(t,x_1,x_2) + R_{x_2}(x_1,x_2)a_{x_2}(t,x_1,x_2)\right] + \mu R(x_1,x_2)s(t,x_1,x_2)\right\} dx_1dx_2 - \oint_{\Omega_b} R(x_1,x_2)a_{x_1}(t,x_1,x_2)dx_2 + \oint_{\Omega_b} R(x_1,x_2)Q_{x_2}(t,x_1,x_2)\, dx_1 = 0$$

For the special case of boundary segments being either adiabatic or homogeneous isothermal, the boundary terms in the above equations are zero or cancel each other. The new state and adjoint equations result from substitution of the vector product into each of these equations.

$$Q_{1t}(t) = -c\left[\iint_{\Omega} R(x_1,x_2)R^T(x_1,x_2)dx_1dx_2\right]^{-1} \cdot \left\{\left[\iint_{\Omega} R_{x_1}(x_1,x_2)R^T_{x_1}(x_1,x_2)dx_1dx_2\right] + \right.$$

$$
\left.\left[\iint_\Omega R_{x_2}(x_1,x_2)R_{x_2}^T(x_1,x_2)dx_1dx_2\right]\right\} Q_1(t) +
$$

$$
\left[\iint_\Omega R(x_1,x_2)R^T(x_1,x_2)dx_1dx_2\right]^{-1} \cdot
$$

$$
\sum_i \left\{\left[\iint_\Omega R(x_1,x_2)\phi_i(x_1 - l_{1i},x_2 - l_{2i})\ dx_1dx_2\right]\theta_i(t)\right\}
$$

$$
Q_{2t}(t) = c\left[\iint_\Omega R(x_1,x_2)R^T(x_1,x_2)dx_1dx_2\right]^{-1} \cdot
$$

$$
\left\{\left[\iint_\Omega R_{x_1}(x_1,x_2)R_{x_1}^T(x_1,x_2)dx_1dx_2\right] + \right.
$$

$$
\left.\left[\iint_\Omega R_{x_2}(x_1,x_2)R_{x_2}^T(x_1,x_2)dx_1dx_2\right]\right\} Q_2(t) \; -
$$

$$
\mu Q_1(t)
$$

The state initial condition and adjoint terminal condition become initial conditions on $Q_1(t)$ and terminal conditions on $Q_2(t)$ respectively.

$$
Q_1(t_o) = \left[R(x_1,x_2)R^T(x_1,x_2)dx_1dx_2\right]^{-1} \cdot \left[R(x_1,x_2)s_o(x_1,x_2)dx_1dx_2\right]
$$

$$
Q_2(t_f) = 0
$$

The two-dimensional Hermite interpolation polynomial expansion is chosen for the basis function set in order to accommodate irregular spatial boundaries. A possibly irreguarly spaced grid which covers the spatial domain must be established. Grid lines are straight and parallel or perpendicular. Each is continuous through the spatial domain. An edge of a given grid block is also an edge of, at most, one other grid block. Boundary grid blocks are rectangular or right-triangular. Four two-dimensional Hermite interpolation polynomials are centered at each interior grid intersection point. At grid intersection points which lie on the boundary, the proper two-dimensional Hermite interpolation polynomials are chosen to have zero coefficients such that the boundary conditions are satisfied. For example, consider a rectangular spatial domain with grid intersections numbered 0 to n and 0 to m in the two dimensions. A homogeneous isothermal boundary along the edge between the points (n,0) and (n,m) requires that the coeffiecients of $\psi^{oo}_{nj}(x_1,x_2)$ and $\psi^{o1}_{nj}(x_1,x_2)$ for j = 0,1,...,m be zero. An adiabatic boundary between the points (0,m) and (n,m) requires that the coefficients of $\psi^{o1}_{im}(x_1,x_2)$ and $\psi^{11}_{im}(x_1,x_2)$ for i = 0,1,...,n be zero.

The matrices of spatial integrals of $R(x_1,x_2)R^T(x_1,x_2)$, $R_{x_1}(x_1,x_2)\cdot R^T_{x_1}(x_1,x_2)$ and $R_{x_2}(x_1,x_2)R^T_{x_2}(x_1,x_2)$ are not single diagonal matrices due to the nonorthogonality of the two-dimensional Hermite interpolation polynomial expansion. The location of nonzero elements in the matrices is dependent on the order of the two-dimensional Hermite interpolation polynomials in the vector $R(x_1,x_2)$.

Gradient Control Modification

Examination of the terms remaining in the first variation after extraction of the state and adjoint equations and initial, terminal and boundary conditions yields the gradient direction of the control modification. If the control amplitudes $\theta_i(t)$ are expressed as polynomials in time,

$$\theta_i(t) = \sum_j g_{ij} t^{j-1},$$

then the scaled changes in the polynomial coefficients g_{ij} and the spatial distribution parameter l_{ik} are specified by:

$$\Delta g_{ij} = -K\gamma^2_{ij}\,\rho_{ij} \Big/ \left[\sum_i\sum_j \gamma^2_{ij}\rho^2_{ij} + \sum_i\sum_k \nu^2_{ik}\sigma^2_{ik}\right]^{\frac{1}{2}}$$

$$\Delta l_{ik} = -K\nu^2_{ik}\,\sigma_{ik} \Big/ \left[\sum_i\sum_j \gamma^2_{ij}\rho^2_{ij} + \sum_i\sum_k \nu^2_{ik}\sigma^2_{ik}\right]^{\frac{1}{2}}$$

$$\rho_{ij} \triangleq \nu \sum_k \left[\int_{t_o}^{t_f} \theta_k(t) \frac{\partial \theta_i(t)}{\partial g_{ij}} dt \cdot \right.$$

$$\left. \iint_\Omega \phi_k(x_1 - l_{k1}, x_2 - l_{k2}) \, \phi_i(x_1 - l_{i1}, x_2 - l_{i2}) \, dx_1 dx_2 \right] +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t) \frac{\partial \theta_i(t)}{\partial q_{ij}} \, dt \cdot \right.$$

$$\left. \iint_\Omega r_m(x_1, x_2) \, \phi_i(x_1 - l_{i1}, x_2 - l_{i2}) \, dx_1 dx_2 \right]$$

$$\sigma_{ik} \triangleq \nu \sum_j \left[\int_{t_o}^{t_f} \theta_j(t) \, \theta_i(t) dt \cdot \right.$$

$$\left. \iint_\Omega \phi_j(x_1 - l_{j1}, x_2 - l_{j2}) \frac{\partial \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{ik}} \, dx_1 dx_2 \right] +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t) \, \theta_i(t) dt \cdot \right.$$

$$\left. \iint_\Omega r_m(x_1, x_2) \frac{\partial \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{ik}} \, dx_1 dx_2 \right]$$

$$\nu_{11}^2 \triangleq 1$$

$$\nu_{ik}^2 \triangleq \left\{ \nu \left[\int_{t_o}^{t_f} \theta_1^2(t) \, dt \iint_\Omega \left(\frac{\partial \phi_1(x_1 - l_{11}, x_2 - l_{12})}{\partial l_{11}} \right)^2 dx_1 dx_2 \right] + \right.$$

$$\nu \sum_j \left[\int_{t_o}^{t_f} \theta_j(t) \, \theta_1(t) \, dt \cdot \right.$$

$$\iint_{\Omega} \phi_j(x_1 - l_{j1}, x_2 - l_{j2}) \frac{\partial^2 \phi_1(x_1 - l_{11}, x_2 - l_{12})}{\partial l_{11}^2} dx_1 dx_2 \Bigg] +$$

$$\sum_m \Bigg[\int_{t_o}^{t_f} q_{2m}(t) \theta_1(t) \, dt \cdot$$

$$\iint_{\Omega} r_m(x) \frac{\partial^2 \phi_1(x_1 - l_{11}, x_2 - l_{12})}{\partial l_{11}^2} dx \Bigg]\Bigg\} \Bigg/$$

$$\Bigg\{ \nu \Bigg[\int_{t_o}^{t_f} \theta_i^2(t) dt \iint_{\Omega} \left(\frac{\partial \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{ik}} \right)^2 dx_1 dx_2 \Bigg] +$$

$$\nu \sum_j \Bigg[\int_{t_o}^{t_f} \theta_j(t) \theta_i(t) dt \cdot$$

$$\iint_{\Omega} \phi_j(x_1 - l_{j1}, x_2 - l_{j2}) \frac{\partial^2 \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{ik}^2} dx_1 dx_2 \Bigg] +$$

$$\sum_m \Bigg[\int_{t_o}^{t_f} q_{2m}(t) \theta_i(t) dt \cdot$$

$$\iint_{\Omega} r_m(x_1, x_2) \frac{\partial^2 \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{ik}^2} dx_1 dx_2 \Bigg] \Bigg\}$$

$$\gamma_{ij}^2 = \Bigg\{ \nu \Bigg[\int_{t_o}^{t_f} \theta_1^2(t) \, dt \iint_{\Omega} \left(\frac{\partial \phi_1(x_1 - l_{11}, x_2 - l_{12})}{\partial l_{11}} \right)^2 dx_1 dx_2 \quad +$$

$$\nu \sum_k \int_{t_o}^{t_f} \theta_k(t) \theta_1(t) \, dt \cdot$$

$$\iint_{\Omega} \phi_k(x_1 - l_{k1}, x_2 - l_{k2}) \frac{\partial^2 \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{i1}^2} dx_1 dx_2 +$$

$$\sum_m \left[\int_{t_o}^{t_f} q_{2m}(t)\, \theta_i(t) dt \cdot \right.$$

$$\left. \left. \iint_{\Omega} r_m(x) \frac{\partial^2 \phi_i(x_1 - l_{i1}, x_2 - l_{i2})}{\partial l_{i1}^2} dx_1 dx_2 \right] \right\} \Bigg/$$

$$\left\{ \nu \left[\int_{t_o}^{t_f} \left(\frac{\partial \theta_i(t)}{g_{ij}} \right)^2 dt \iint_{\Omega} \phi_i^2(x_1 - l_{i1}, x_2 - l_{i2})\, dx_1 dx_2 \right] \right\}$$

Computational Algorithm

The basic algorithm for determining the optimum, open-loop regional control for the three-dimensional, parabolic, diffusion system is essentially an extension of that for the two-dimensional system discussed above. It includes an initialization section and an optimization loop in which the performance index is calculated and the gradient direction for the iterative modification of the control parameters is determined. The linearily of the problem being considered is utilized in that a transition matrix approach is followed for solving the state and adjoint sets of ordinary differential equations rather than numerical integration as in the case of the two-dimensional problem. While the initialization time is extended by the inclusion of the eigenvalue problem, the time per optimization iteration is shortened. Expression of the temporal control amplitudes as polynomials in time allows analytical precalculation of the convolution integral. In the initialization section grid block data, boundary conditions and control guesses are read into the program. The matrices of spatial integrals of RR^T, $R_{x_1}R_{x_1}^T$ and $R_{x_2}R_{x_2}^T$ are calculated. Then the inverse of the first of these and the initial condition on Q_1 are calculated. Also included in the initialization section are the calculation of the eigenvalues and eigenvector matrix associated with the coefficient matrix of the state vector in the state differential equation set and the inverse of the eigenvector matrix.

In the state integration section which is the first section of the optimization loop, the matrix of spatial integrals of $r_i\phi_j$ is first calculated. Then the state squared portion of the performance index is directly calculated. Calculation of the time dependent state is optional. After the control squared portion of the performance index is calculated, the total performance index is available to be checked with that of the previous iteration.

If an improvement in the performance index is obtained and the maximum number of iterations has not been reached, the adjoint integration section is entered. The matrices of temporal integrals of $q_{2i}(\partial\theta_j/\partial g_{ik})$ and $q_{2i}\theta_j$ which are required for the gradient control modification are directly calculated, without calculation of the adjoint vector, Q_2. The gradient direction for the control parameter modification is calculated.

If an improvement in the performance index is not obtained, the past control change is reduced along the past gradient and the gradient constant, K, is reduced. The optimization loop is then closed by the return to the state integration section.

Example Solutions

Example solutions have been obtained for the following state equation and performance index.

$$S_t(t,x_1,x_2) = S_{x_1x_1}(t,x_1,x_2) + S_{x_2x_2}(t,x_1,x_2) +$$
$$\sum_i \theta_i(t)\phi_i(x_1 - l_{i1},x_2 - l_{i2}) \quad ; \ \forall t\in(0,.1),(x_1,x_2)\in\Omega_b$$

$$J = \tfrac{1}{2}\int_0^{.1}\left\{\mu\, S^2(t,x_1,x_2) + \left[\sum_i \theta_i(t)\phi_i(x_1 - l_{i1}, - l_{i2})\right]^2\right\}dx_1dx_2$$

The temporal control functions, $\theta_i(t)$, are expressed as polynomial functions of time,

$$\theta_i(t) = \sum_j g_{ij}t^{j-1},$$

and the spatial distribution functions, $_i(x_1 - l_{i1},x_2 - l_{i2})$, are two-dimensional normal distributions,

$$\phi_i(x_1 - l_{i1},x_2 - l_{i2}) = \exp\left[-\left(\frac{x_1 - l_{i1}}{\text{std. dev.}}\right)^2 - \left(\frac{x_2 - l_{i2}}{\text{std. dev.}}\right)^2\right].$$

The solutions of the optimal regional control problem for four different spatial domain shapes with one and two regional controls are shown in Table III. First, a rectangular domain was considered for comparison with the rod problem in two dimensions, space and time. considered in the previous section. The resulting control locations for the three-dimensional problem match well those obtained in the nearly equivalent two-dimensional problem for both one and two regional controls. See Figures 26 through 29. Also, a study was conducted to determine if the control location found by by the program for the single regional control problem was an optimum. The pertinent control location parameter was held constant at several non-optimal values and the other parameters were left free to be optimized. The constrained optimal solutions obtained were found to be suboptimal when compared with the optimal solution. A plot of the constrained optimal

TABLE III

EXAMPLE SOLUTIONS OF A THREE-DIMENSIONAL REGIONAL CONTROL PROBLEM

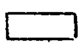
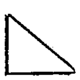
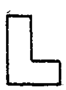
Spatial Domain Shape	Number of Control Regions	Initial Guess for $\Theta_i(t)$	Final $\Theta_1(t)$	Final $\Theta_2(t)$	Guessed Control Location for ϕ_1		Guessed Control Location for ϕ_2		Final Control Location for ϕ_1		Final Control Location for ϕ_2		Initial Performance Index	Final Performance Index	Control Surface
					l_{11}	l_{12}	l_{21}	l_{22}	l_{11}	l_{12}	l_{21}	l_{22}			
	1	5.-50.t	$6.51-54.7t+130.t^2$		.500	.1			.236	.100			.2307	.2161	28
	2	3.-30.t	$3.67-32.5t+55.7t^2$	$5.25-35.8t+209.t^2$	.667	.1	.333	.1	.523	.100	.138	.100	.2223	.2076	29
	1	5.-50.t	$3.85-50.3t+113.t^2$		.25	.25			.181	.181			.1608	.1555	33
	2	3.-30.t	$2.14-28.5t+81.6t^2$	$2.14-29.3t+81.5t^2$	.1	.8	.8	.1	.0897	.296	.295	.0896	.1759	.1544	34
	1	3.-30t	$5.56-28.5t+187.t^2$		.200	.200			.200	.725			.9664	.9106	37

TABLE III (Continued)

	2	3.-30.t	5.76-33.7t-237.t^2	6.38-33.9t-285.t^2	.200	.200	.200	.800	.208	.269	.194	.770	.9093	.8275	38
	1	3.-30.t	.686-41.2t-339.t^2		.100	.500			.184	.752			.6454	.5882	41
	2	3.-30.t	4.80-35.1t-176.t^2	6.09-40.5t-339.t^2	.200	.200	.200	.800	.145	.270	.192	.789	.6038	.5515	42

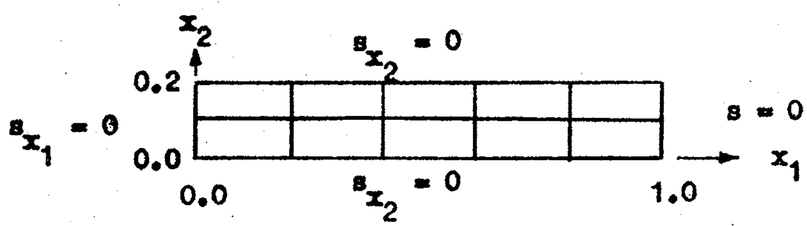


Figure 26. Rectangular Spatial Domain

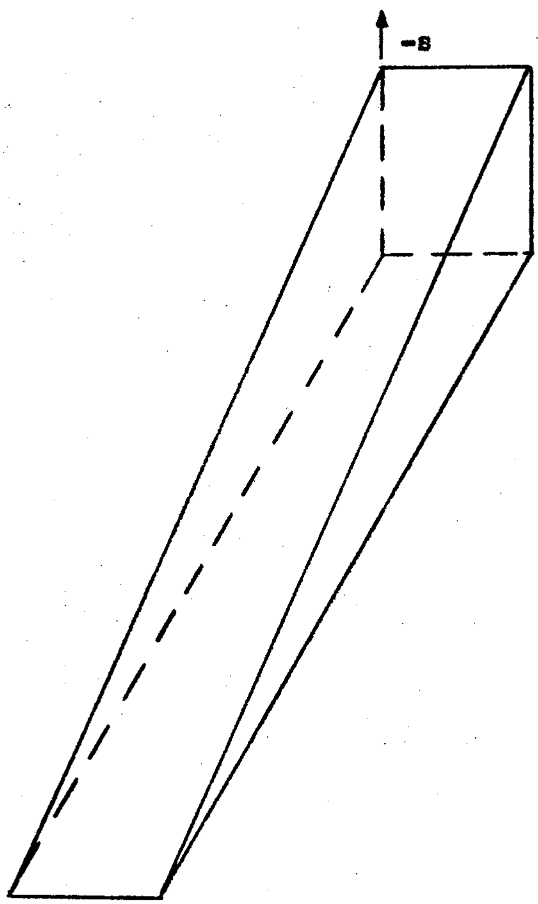


Figure 27. State Initial Condition for Rectangular Spatial Domain

a.) t=0

b.) t=.05

c.) t=.1

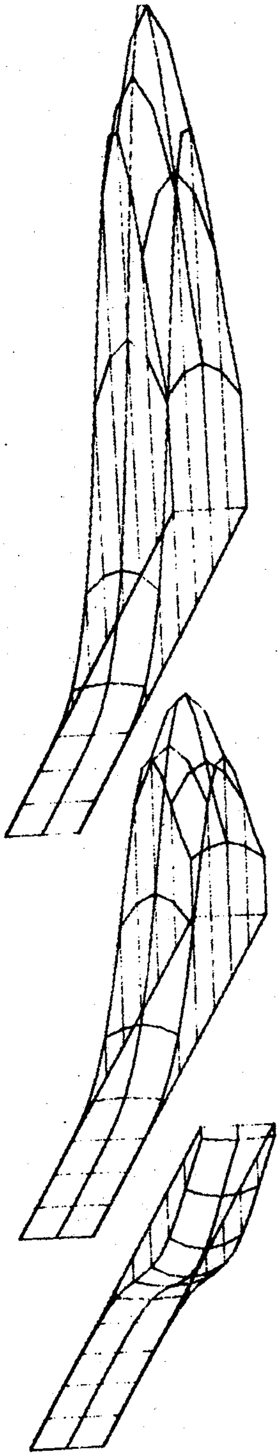


Figure 28. Control Surfaces for Rectangular Spatial Domain - One Control Region

a.) t=0

b.) t=.05

c.) t=.1

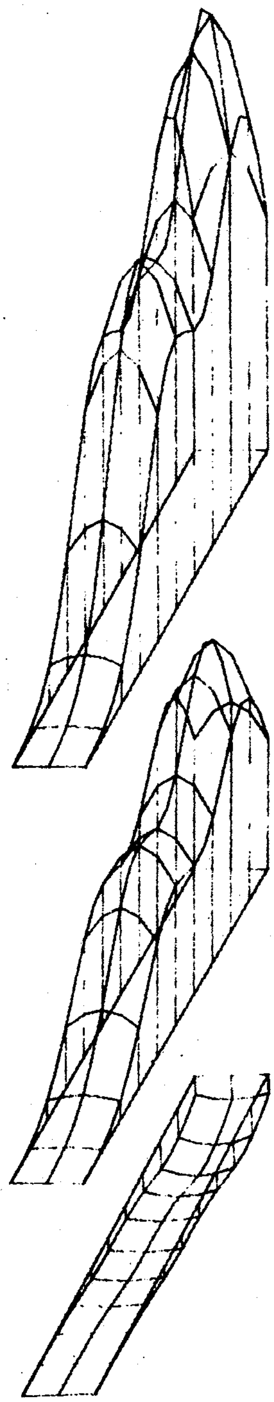


Figure 29. Control Surfaces for Rectangular Spatial Domain - Two Control Regions

performance index versus control location is given in Figure 30.

The other spatial domain shapes considered were a triangular shape, an L-shape and a U-shape. The spatial domains with boundary conditions and grids indicated, the state initial condition surfaces, and the composite control surfaces at the initial time, a midpoint time and the terminal time for both one and two regional controls are given for each of these three spatial domain shapes in Figures 31 through 42.

In the case of the triangular spatial domain, right triangular grid blocks used along the diagonal allow a more accurate matching of the spatial boundary. However, the use of the traingular grid blocks increases computation time due to the inability to calculate and multiply together a pair of line integrals. The initial condition on the state is assumed to be a planar surface which has a zero value along the diagonal boundary. The boundary conditions include adiabatic edges along the legs of the triangular domain and a homogeneous isothermal edge along the diagonal. In both the single and double control region problems the control locations moved from initial guesses near the isothermal diagonal edge to optimal locations nearer the adiabatic edges as expected. An improvement in the performance index is obtained by the addition of a second control over that of a single control.

In the cases of the L-shpaed and U-shaped spatial domains, rectangular grid blocks were employed. In both cases, one edge was specified to have a homogeneous isothermal boundary condition with all other edges adiabatic. In both cases, a single control tended to move toward regions partially bounded by adiabatic edges, as expected. The addition of second control regions provided improved performance indices. The control nearer the isothermal edge had lower time dependent amplitudes.

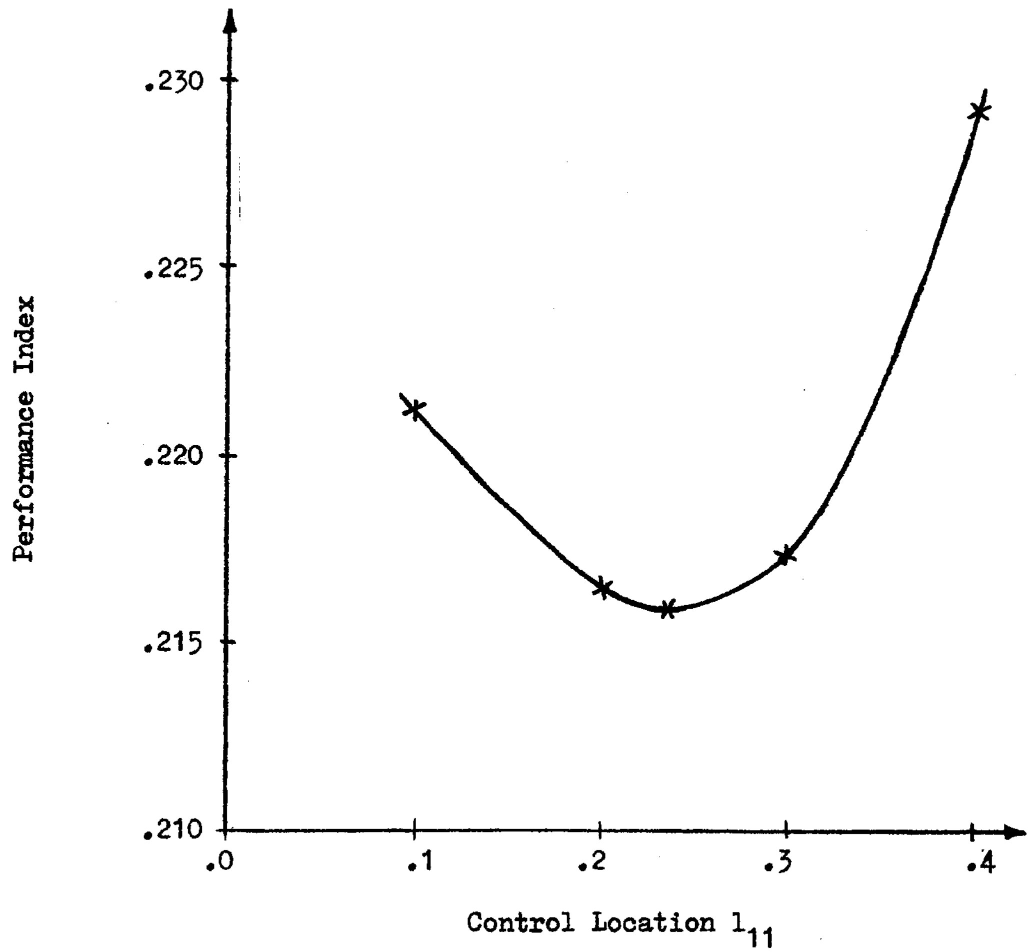


Figure 30. Performance Index Versus Control Location for Location Constrained Optimal Regional Control Problem

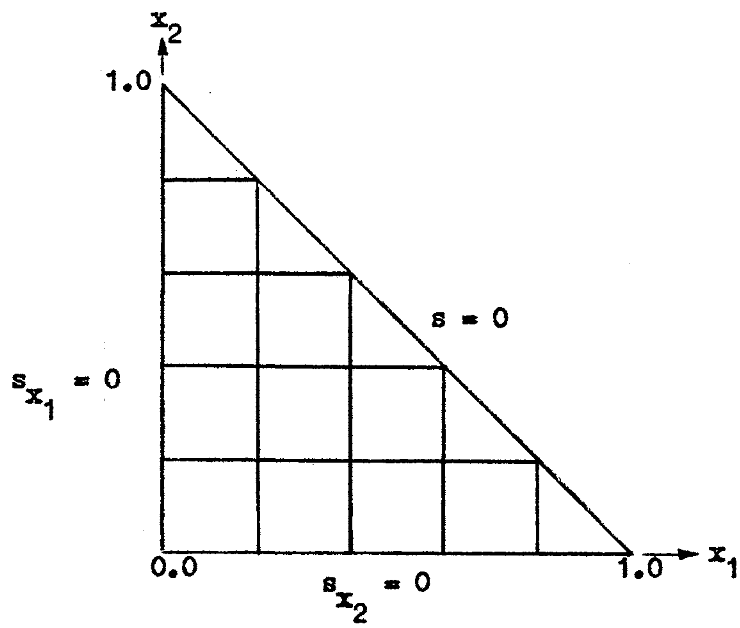


Figure 31. Triangular Spatial Domain

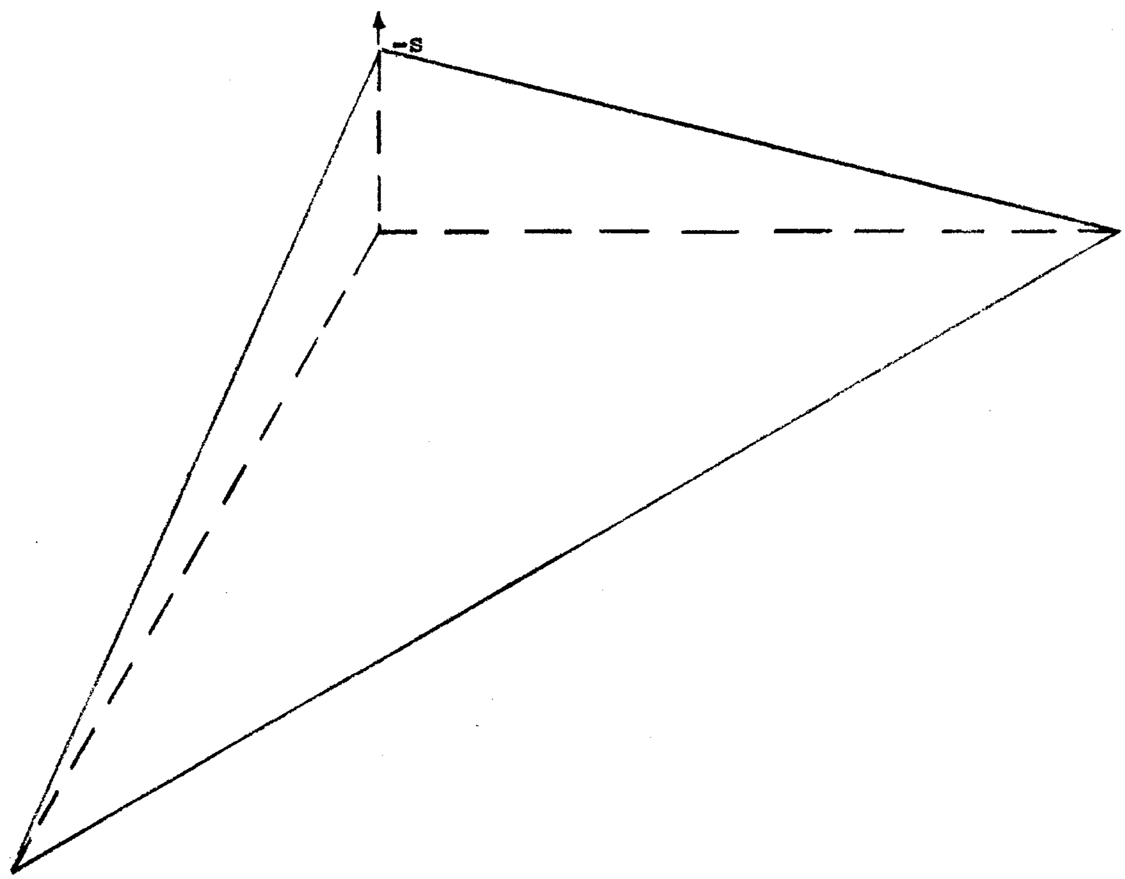


Figure 32. State Initial Condition for Triangular Spatial Domain

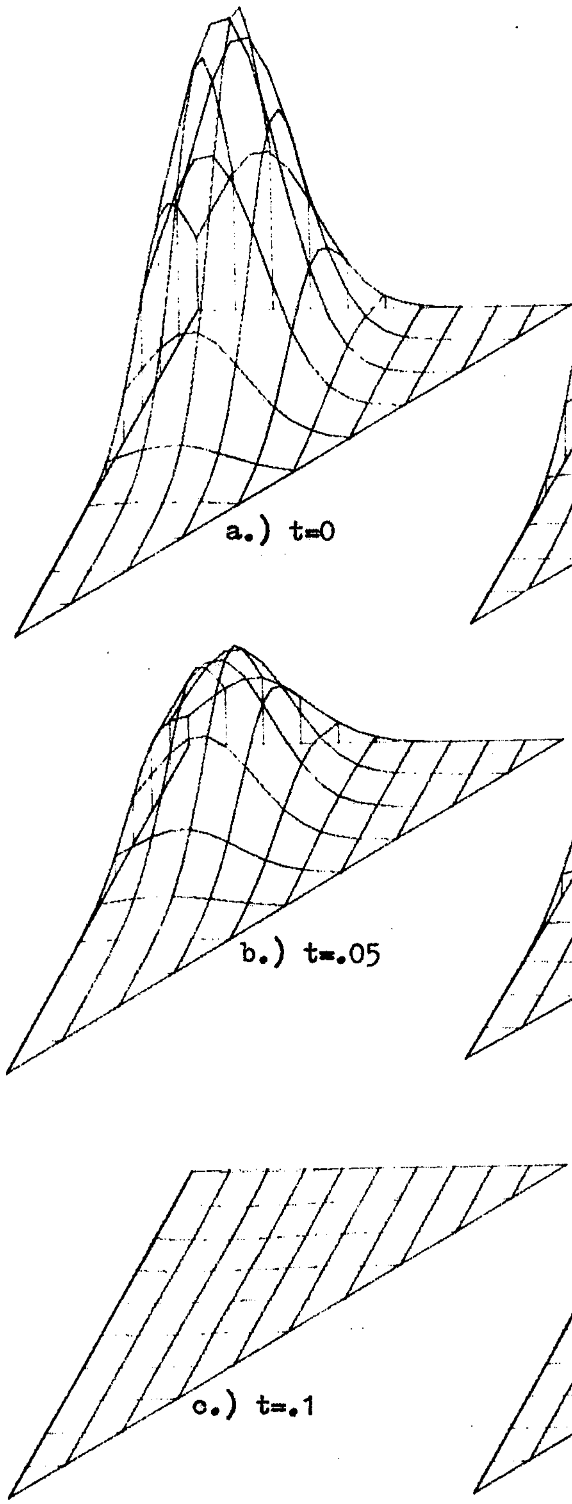


Figure 33. Control Surfaces for Triangular Spatial Domain - One Control Region

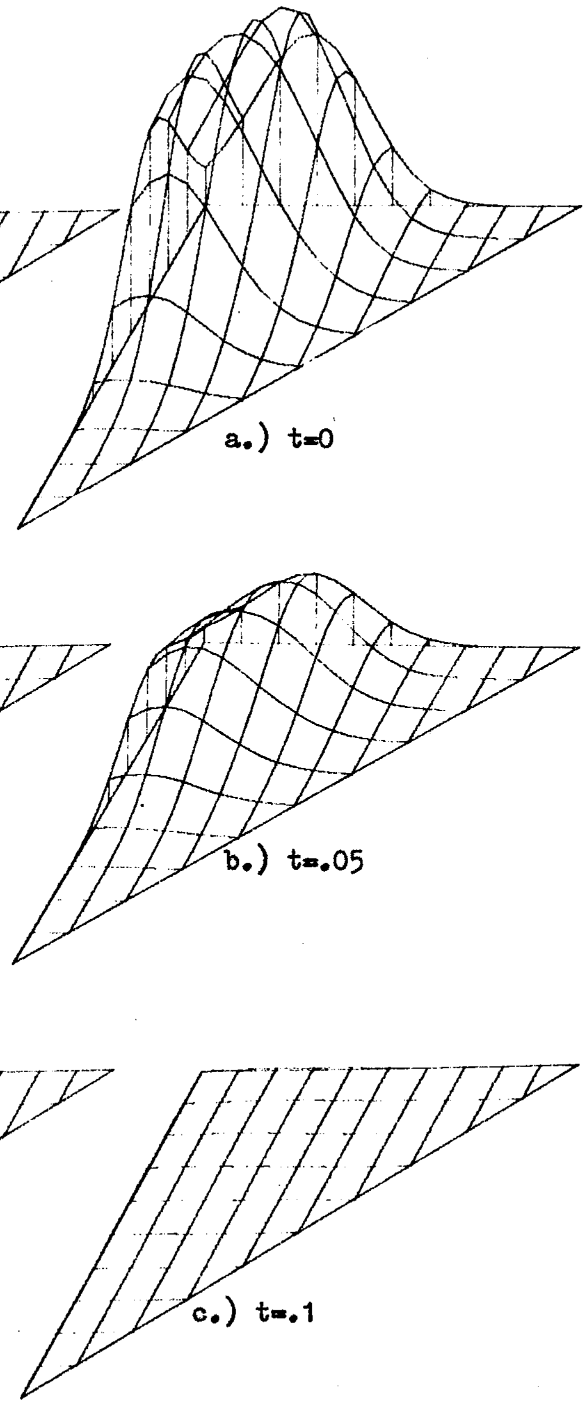


Figure 34. Control Surfaces for Triangular Spatial Domain - Two Control Regions

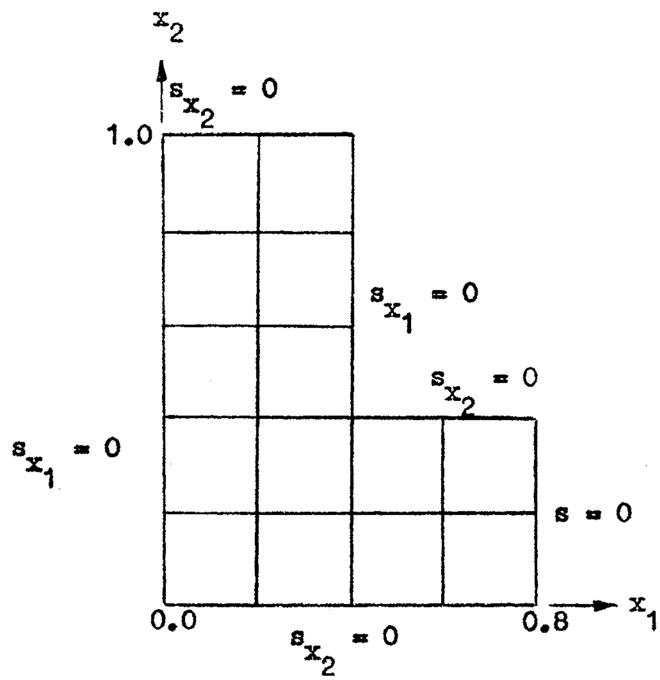


Figure 35. L - Shaped Spatial Domain

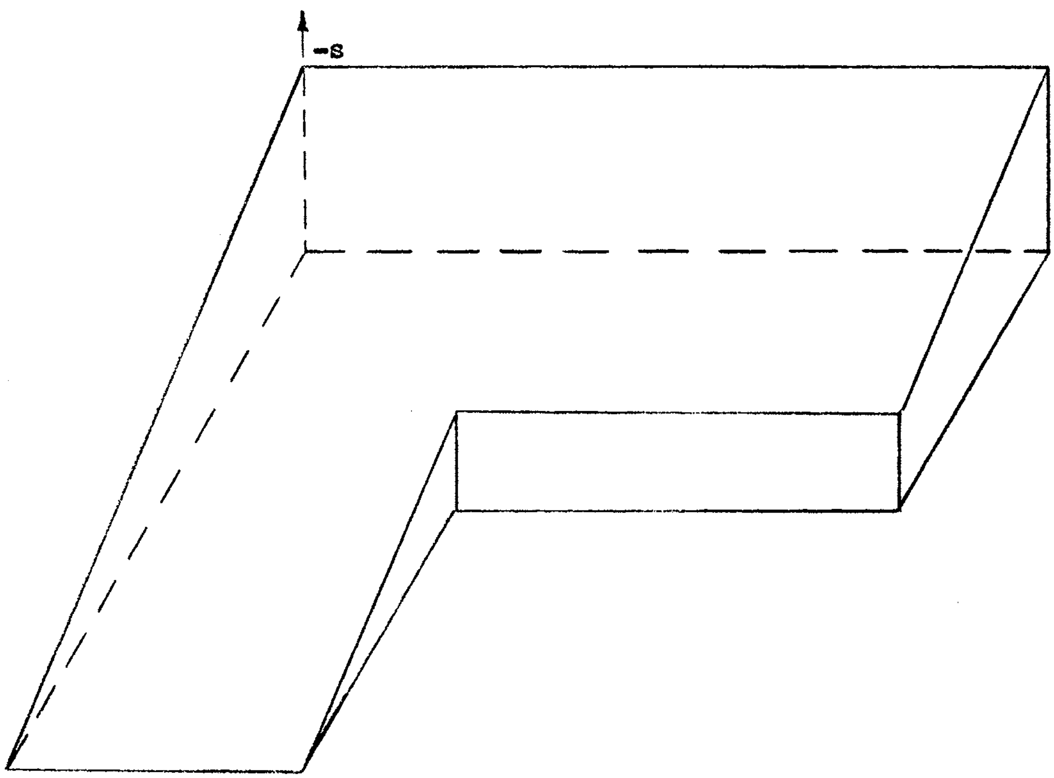


Figure 36. State Initial Condition for L - Shaped Spatial Domain

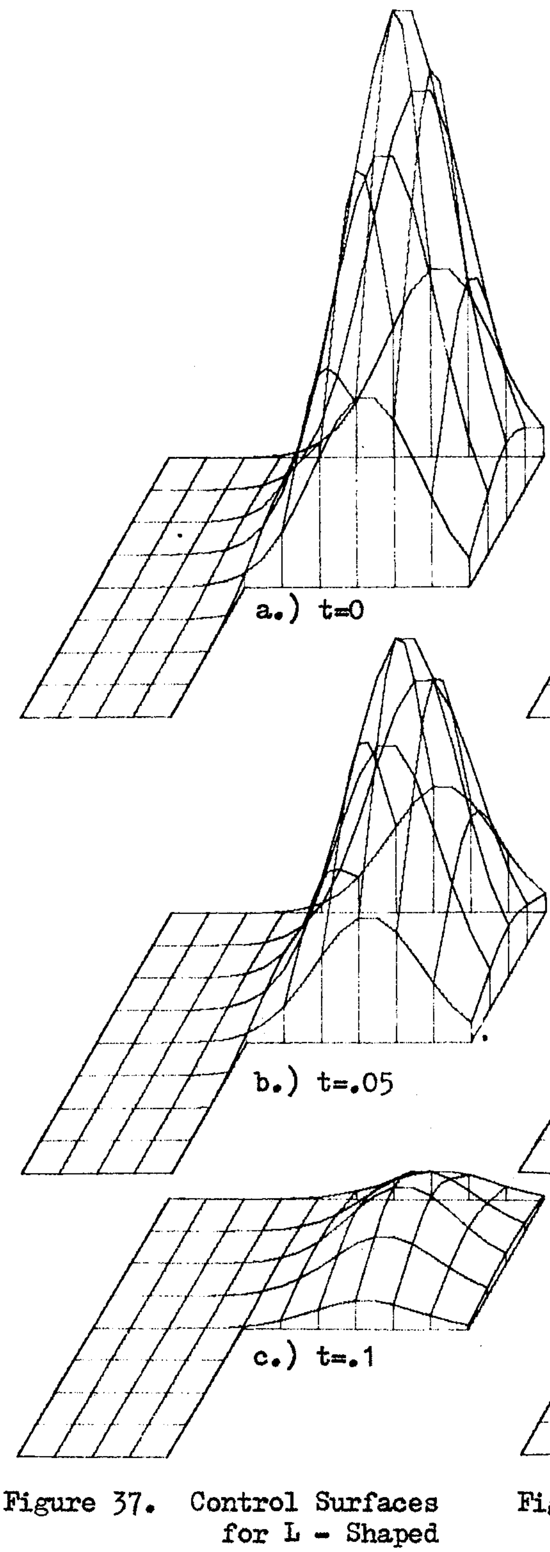


Figure 37. Control Surfaces for L - Shaped Spatial Domain - One Control Region

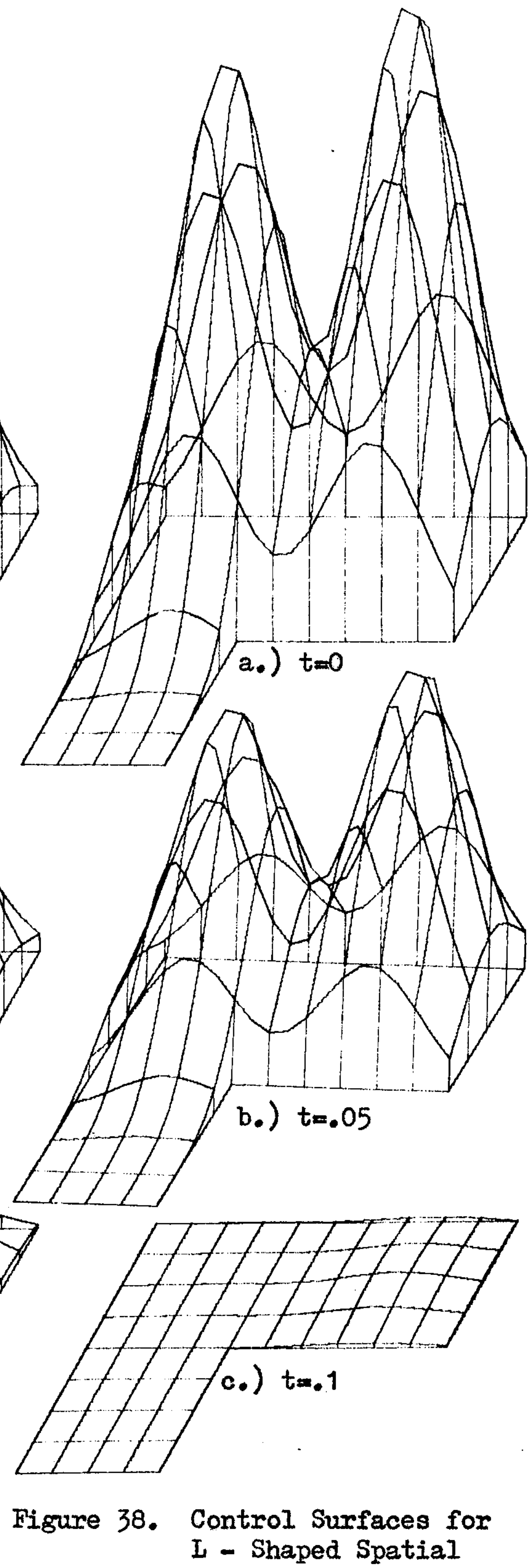


Figure 38. Control Surfaces for L - Shaped Spatial Domain - Two Control Regions

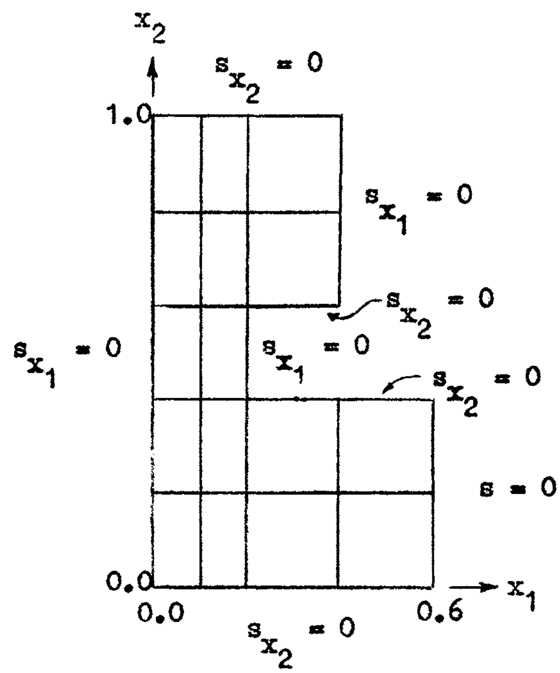


Figure 39. U - Shaped Spatial Domain

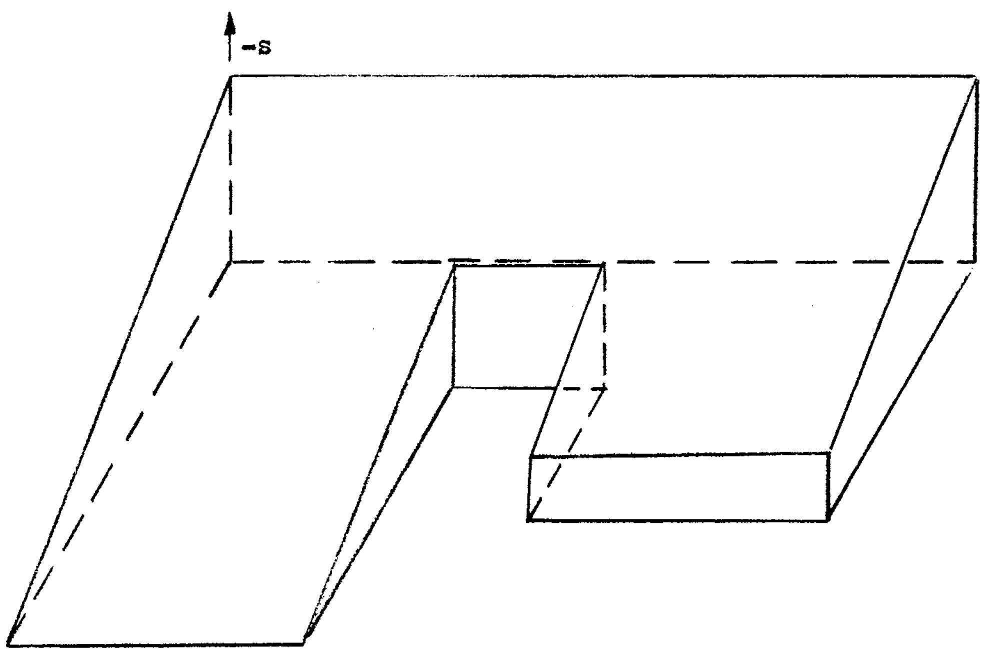


Figure 40. State Initial Condition for U - Shaped Spatial Domain

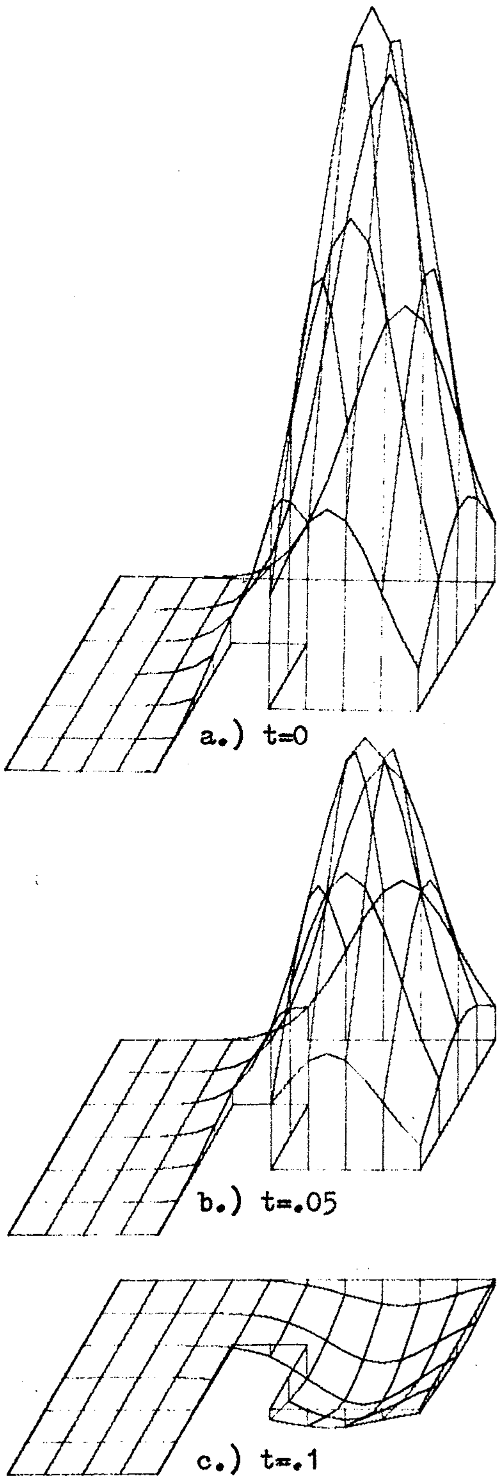


Figure 41. Control Surfaces for U - Shaped Spatial Domain - One Control Region

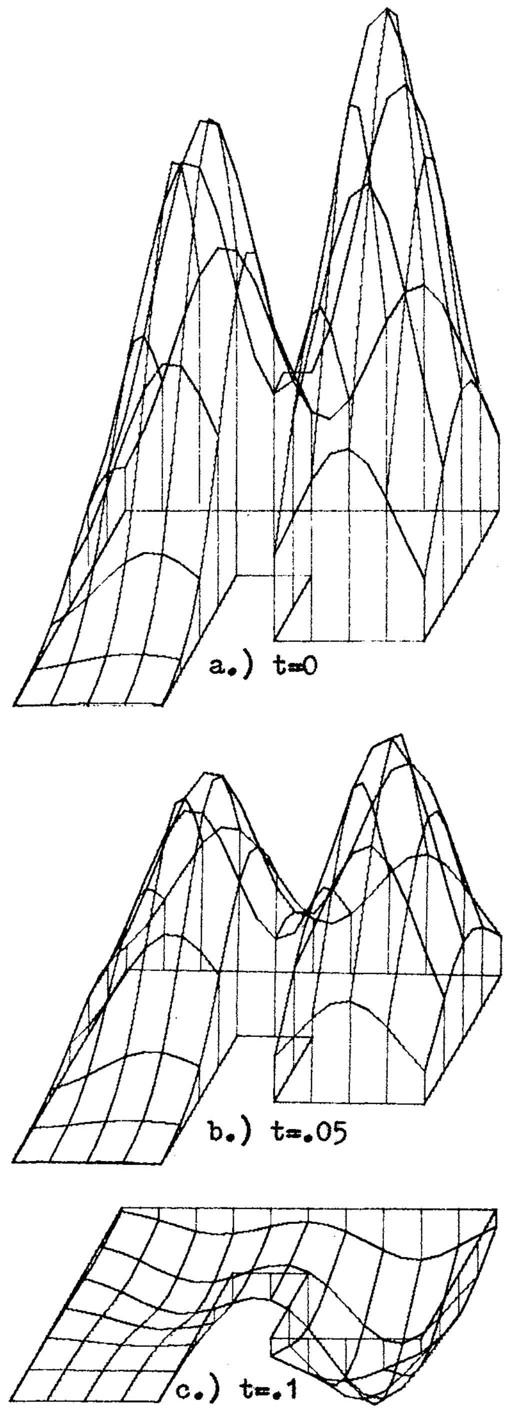


Figure 42. Control Surfaces for U - Shaped Spatial Domain - Two Control Regions

CHAPTER VI

SUMMARY

Principal Results

A procedure has been developed to determine the optimum, open-loop regional control for systems governed by sets of parabolic partial differential equations. The regional controls have been constrained to be products of temporal control amplitudes and spatial distribution functions. The temporal control amplitudes have been considered as either discretized functions or polynomial functions of time. The spatial distribution functions have been treated as specified functions with location parameters which are free to be optimized. Problems involving one-dimensional and irregular, two-dimensional spatial domains have been considered.

The procedure begins with a mathematical statement of the constrained minimization problem. The application of calculus of variations to the constrained minimization problem yields a set of conditions necessary for optimality in the form of a boundary value problem. The state and adjoint partial differential equations of the boundary value problem are reduced to ordinary differential equation sets by the application of Galerkin's method. Both one-dimensional Hermite interpolation polynomials and truncated eigenfunction expansions are used as basis function sets in the case of problems in two independent variables, i.e., one spatial independent variable and time. Two-dimensional Hermite interpolation polynomials are used in the case of problems in three independent variables, i.e., two spatial independent variables and time. An iterative algorithm, based on the gradient direction for the control modification is developed for the solution of the approximated boundary value problem.

The distributed nature of the system description is maintained throughout the development of the conditions necessary for optimality rather than an approximation technique being applied before the application of variational calculus. Galerkin's method provides a convenient approach for the reduction of the state and adjoint partial differential equations to sets of ordinary differential equations. For linear systems defined on one-dimensional spatial domains, the orthogonality of the elements of the truncated eigenfunction expansion makes this series more desirable than nonorthogonal series. However, for nonlinear systems, an eigenfunction expansion for a similar linear operator would have to be employed. This would make the application of one-dimensional Hermite interpolation polynomials more desirable. In the case of linear systems defined on regular, two-dimensional spatial domains, the truncated eigenfunction expansion appears to be most desirable. For irregular, two-dimensional spatial domains, the series of two-dimensional Hermite interpolation polynomials is applicable. System nonlinearity would also provide a justification for the use of a series of nonorthogonal basis functions. It should be noted that the proper choice of a series of basis functions depends on the system equations and boundaries to be approximated.

The computer program developed for the solution of the two-dimensional problem with one spatial independent variable and one temporal independent variable utilizes numerical integration for the solution of the state and adjoint ordinary differential equations sets, while a transition matrix approach is utilized in the program for three-dimensional systems. Although the transition matrix approach is limited to linear systems, a significant amount of time is saved by the direct calculation of terms in the performance index and in the gradient direction for iterative control modification. This approach also eliminates the problems associated with the numerical integration of different equation sets with widely varying eigenvalues.

Example solutions have been obtained for systems described by the diffusion equation defined on one-dimensional and irregular two-dimensional spatial domains. A quadratic performance index composed of the spatial and temporal integral of the system state squared plus the composite control squared has been minimized by the optimization of the time dependent amplitudes and the means of the normally distributed spatial functions of each of one, two or three regional controls. Although the unimodality of the performance index hypersurface is in question in the case of multiple regional controls, convergence is demonstrated in nearly all cases considered by the obtaining of essentially the same solutions from different starting control guesses.

Recommendations for Further Study

Further investigation of the problem of determining the optimal regional control of distributed parameter systems might be centered in any of three major areas. The first area might be the extension of this work to a larger class of systems. Although only three-dimensional systems defined on two-dimensional spatial domains are considered here, a direct extension to systems defined on three-dimensional spatial domains is possible. Also, multiple systems which are coupled through a common boundary might be considered. An example of this might be the temperature control of a mold which is in turn controlling the temperature of the material being formed. The input might be constrained to exist along a line to be determined rather than about a point in space as in this work. There are a vast number of possibilities for extending the class of systems under consideration.

The second area for further investigation is that of the choice of basis functions. The use of fundamental spline functions might be desirable because of the resulting reduction in the dimensionality of the approximate sets of ordinary differential equations. While four two-dimensional Hermite interpolation polynomials are centered at each grid intersection point, only one fundamental spline function would be centered at each grid intersection point. However, increased initialization time would be required and irregular spatial boundaries would cause some difficulty.

The third and most closely related area might include the representation of the control amplitude by a series of orthogonal functions rather than discrete values or polynomials. Also, a study might be conducted on the uniqueness of the solutions for problems with multiple control regions and the unimodality of the performance index hypersurface.

A SELECTED BIBLIOGRAPHY

(1) Andreyev, Yu. N. and A. G. Butkovski. "Optimal Control in Heating of Massive Bodies." Engineering Cybernetics, Volume 5, 1964, pp. 41-49.

(2) Arpaci, Vedat. Conduction Heat Transfer, Reading, Massachusetts: Addison-Wesley Publishing Company, 1966.

(3) Athans, Michael. "Some Remarks on the Control of Distributed Parameter Systems." Control of Distributed Parameter Systems, ed. Michael Athans, New York: ASME, 1969.

(4) Axelband, Elliot I. "An Approximation Technique for the Control of Linear Distrubuted Parameter Systems with Bounded Inputs." IEEE Transactions on Automatic Control, Volume AC-11, Number 1 January 1966, pp. 42-45.

(5) Axelband, Elliot I. "Optimal Control of Distributed Parameter Systems." Advances in Control Systems, Volume 7, ed. C. T. Leondes, New York: Academic Press, 1969.

(6) Birkhoff, Garrett and Carl R. de Boor. "Piecewise Polynomial Interpolation and Approximation." Approximation of Functions, ed. Henry L. Garabedian, New York: Elsevier Publishing Company, 1965.

(7) Briggs, D. L. and C. N. Shen. "Distributed Parameter Optimum Control of a Nuclear Rocket with Thermal Stress Constraints." Transactions of ASME, Journal of Basis Engineering, June 1967, pp. 300-306.

(8) Brogan, William L. "Optimal Control Theory Applied to Systems Described by Partial Differential Equations." Advances in Control Systems, Volume 6, ed. C. T. Leondes, New York: Academic Press, 1968.

(9) Butkovskii, A. G. "Optimum Processes in Systems with Distributed Parameters." Automation and Remote Control, Volume 22, Number 1, January 1961, pp. 13-21.

(10) Butovskii, A. G. "The Maximum Prinicple for Systems with Distributed Parameters." Automation and Remote Control, Volume 22, Number 10, October 1961, pp. 1156-1159.

(11) Butkovskii, A. G. "Some Approximate Methods for Solving Problems of Optimal Control of Distributed Parameter Systems."

Automation and Remote Control, Volume 22, Number 12, December 1961, pp. 1429-1438.

(12) Butkovskii, A. G. "The Method of Moments in the Theory of Optimal Control of Systems with Distributed Parameters." Automation and Remote Control, Volume 24, Number 9, September 1963, pp. 1106-1113.

(13) Butkovskii, A. G. and A. I. Egorvo and K. A. Lurie. "Optimal Control of Distributed Parameter Systems (A survey of Soviet Publications)." SIAM Journal of Control, Volume 6, Number 3, 1968, pp. 437-476.

(14) Butovskii, A. G. and A. Ya. Lerner. "The Optimal Control of Systems with Distributed Parameters." Automation and Remote Control, Volume 21, Number 6, June 1960, pp. 472-477.

(15) Crandall, Stephen H. Engineering Analysis, New York: McGraw-Hill Book Company, 1956.

(16) Denn, M. M. and R. D. Gray, Jr. and J. R. Ferron. "Optimization IN a Class of Distributed-Parameter Systems." I&EE Fundamentals, Volume 5, Number 1, February 1966, pp. 59-66.

(17) Denn, Mortin M. Optimization by Variational Methods, New York: McGraw-Hill Book Company, 1969.

(18) Douglas, Jim. "The Numerical Solution of a Compositional Model in Petroleum Reservoir Engineering." Numerical Solution of Field Problems in Continuum Physics, Volume 2, ed. Garrett Birkhoff and Richard S. Varga, Providence, Rhode Island: American Mathematical Society, 1970.

(19) Eveleigh, Virgil W. Adaptive Control and Optimization Techniques, New York: McGraw-Hill Book Company, 1967.

(20) Egorov, A. I. "Optimality Conditions of Systems Containing Components with Distributed Parameters." Mathematical Theory of Control, ed. A. V. Balkrishnan and Lucien W. Nuestadt, New York: Academic Press, 1967.

(21) Fairman, F. W. and D. W. Shen. "Parameter Identification for a Class of Distributed Systems." International Journal of Control, Volume 11, Number 6, 1970, pp. 929-940.

(22) Fleming, Wendel H. "Optimal Control of Diffusion Processes." Functional Analysis and Optimization, ed. E. R. Cainiello, New York: Academic Press, 1966.

(23) Foster, A. M. and P. A. Orner. "A Design Procedure for a Class of Distributed Parameter Systems." ASME Paper Number 70-WA/AUT-6.

(24) Goodson, R. E. "Distributed System Simulation Using Infinite Product Expansions." 1970 JACC, Session Paper 29-A, pp. 769-778.

(25) Graham, John W. and Frank D'Souza. "Optimal Control of Distributed Parameter Systems Subject to Quadratic Loss." 1969 JACC, pp. 703-709.

(26) Hahn, D. R. and L. T. Fann and C. L. Hwang. "Optimal Startup Control of a Jacketed Tubular Reactor." 1970 JACC, Session Paper 19-A, pp. 451-461.

(27) Hall, C. A. "Bicubic Interpolation over Triangles." Journal of Mathematics and Mechanics, Volume 6, Number 1, 1969.

(28) Hulme, B. L. "Picewise Bicubic Methods for Plate Bending Problems." Ph.D. Thesis, Harvard University, 1969.

(29) Kaiser, K. W. "Guidelines for Process Modeling in Optimal Control." 1969 JACC, pp. 937-941.

(30) Khatri, H. C. and R. E. Goodson. "Optimal Feedback Solutions for a Class of Distributed Systems." Transactions of ASME, Journal of Basic Engineering, June 1966, pp. 337-342.

(31) Koppel, Lowell B. and Yen-Ping Shih and Donald R. Coughanowr. "Optimal Feedback Control of a Class of Distributed-Parameter Systems with Space-Independent Controls." I&EC Fundamentals, Volume 7, Number 2, May 1968, pp. 286-295.

(32) Koppel, Lowell B. and Yen-Ping Shih. "Optimal Control of a Class of Distributed Parameter Systems with Distributed Controls." I&EC Fundamentals, Volume 7, Number 3, August 1968, pp. 414-422.

(33) Koppel, Lowell B. and Daniel T. Kamman and John L. Woolward. "Two-Point Practical Control of a Class of Distributed Processes." I&EC Fundamentals, Volume 9, Number 2, 1970, pp. 198-205.

(34) Lattes, Robert. "Non-well-set Problems and the Method of Quasi Reversibility." Functional Analysis and Optimization, ed. A. V. Balakrshnan and Lucien W. Neustadt, New York: Academic Press, 1967.

(35) Lions, J. L. "On Some Optimization Problems for Linear Parabolic Equations." Functional Analysis and Optimization, ed. A. V. Balakrishnan and Lucien W. Neustadt, New York: Academic Press, 1967.

(36) McCausland, I. "On Optimum Control of Temperature Distribution of a Solid." Journal of Electronics and Control, Volume 14, 1963, pp. 655-668.

(37) Perlis, H. J. and A. R. Cook. "An Improved Optimum Control Algorithm for a class of Water Pollution Problems." ISA-70 Silver Jubilee International Conference and Exhibit, October 1970, Paper number 805-70.

(38) Pike, H. E. "Case Study on the Control of a Slab Reheating Furnace." 1969 Case Studies in System Control, IEEE Group on Automatic Control, 1969.

(39) Porter, A. Modern Foundations of Systems Engineering, New York: The MacMillan Company, 1966.

(40) Price, Harvey S. and Richard S. Varga. "Error Bounds for Semidiscrete Galerkin Approximations of Parabolic Problems with Applications to Petroleum Reservoir Mechanics." Numerical Solution of Field Problems in Continuum Physics, Volume 2, ed. Garrett Birkhoff and Richard S. Varga, Providence, Rhode Island: American Mathematical Society, 1970.

(41) Rivlin, Theodore J. An Introduction to the Approximation of Functions, Waltham, Massachusetts: Blaisdell Publishing Company, 1969.

(42) Sage, Andrew P. Optimum Systems Control, Englewood Cliffs, New Jersey: Prentice-Hall, 1968.

(43) Sage, Andrew P. and Syama P. Chaudhuri. "Gradient and Quasi-Lineaization Computational Techniques for Distributed Parameter Systems." International Journal of Control, Volume 6, Number 1, 1967, pp. 81-98.

(44) Sakawa, Yoshiyuki. "Solution of an Optimal Control Problem in a Distributed-Parameter System." IEEE Transactions on Automatic Control, Volume AC-9, October 1964, pp. 402-426.

(45) Sakawa, Yoshiyuki. "Optimal Control of a Certain Type of Linear Distributed-Parameter Systems." IEEE Transactions on Automatic Control, Volume AC-11, Number 1, January 1966, pp. 35-41

(46) Scheonberg, I. J. "On Spline Functions." Inequalities, ed. Ovid Shisha, New York: Academic Press, 1968.

(47) Singh, R. N. P. "A Unified Approach to a State-space Model for Linear Distributed Systems." International Journal of Control, Volume 11, Number 3, 1970, pp. 471-478.

(48) Takahashi, Yasundo and Michael J. Rabins and David M. Auslander. Control and Dynamic Systems, Reading Massachusetts: Addison Wesley Publishing Company, 1970.

(49) Tarassov, Victor J. and Harlan J. Perlis and Burton Davidson. "Optimization of a Class of River Aeriation Problems by the Use of Multivariable Distributed Parameter Control Theory." Water Resources Research. Volume 5, Number 3, June 1969, pp. 563-573.

(50) Varga, Richard S. "Accurate Numerical Methods for Nonlinear Boundary Value Problems." Numerical Solution of Field Problems in Continuum Physics, Volume 2, ed. Garrett Birkhoff and Richard

S. Varga, Providence, Rhode Island: American Mathematical Society, 1970, pp. 152-167.

(51) Vidasagar M. "Optimal Control of a Distributed Parameter System With Quadratic Constraint." 1969 JACC, pp. 694-702.

(52) Wang, P. K. C. "Control of Distributed Parameter Systems." Advances in Control Systems, Volume 1, ed. C. T. Leondes, New York: Academic Press, 1964.

(53) Wang, P. K. C. and F. Tung. "Optimum Control of Distributed Parameter Systems." Transactions of ASME, Journal of Basic Engineering, March 1964, pp. 67-79.

(54) Wiberg, Donald M. "Feedback Control of Linear Distributed Systems." Transactions of ASME, Journal of Basic Engineering, June 1967, pp. 379-384.

APPENDIX A

PLOTS OF TWO-DIMENSIONAL OPTIMAL REGIONAL CONTROL PROBLEM SOLUTIONS

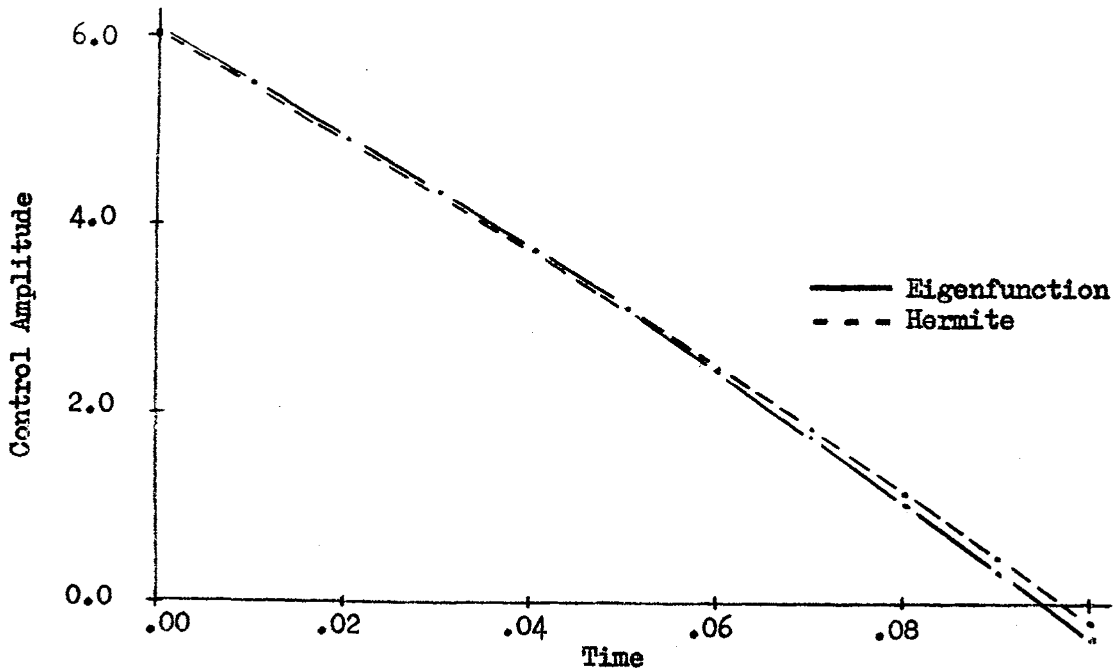


Figure 43. Control Amplitude Versus Time - One Control Region, Polynomial Temporal control function (std. dev. = .2, μ = 100)

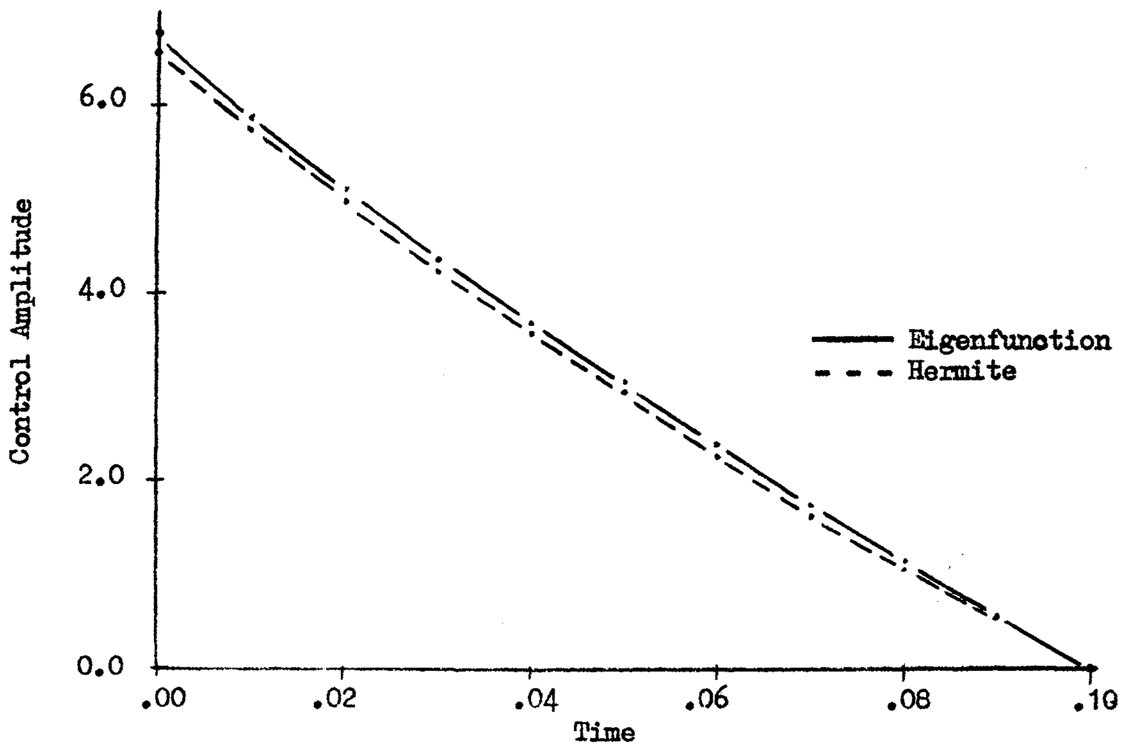


Figure 44. Control Amplitude Versus Time - One Control Region, Discretized Temporal Control Function (std. dev. = .2, μ = 100)

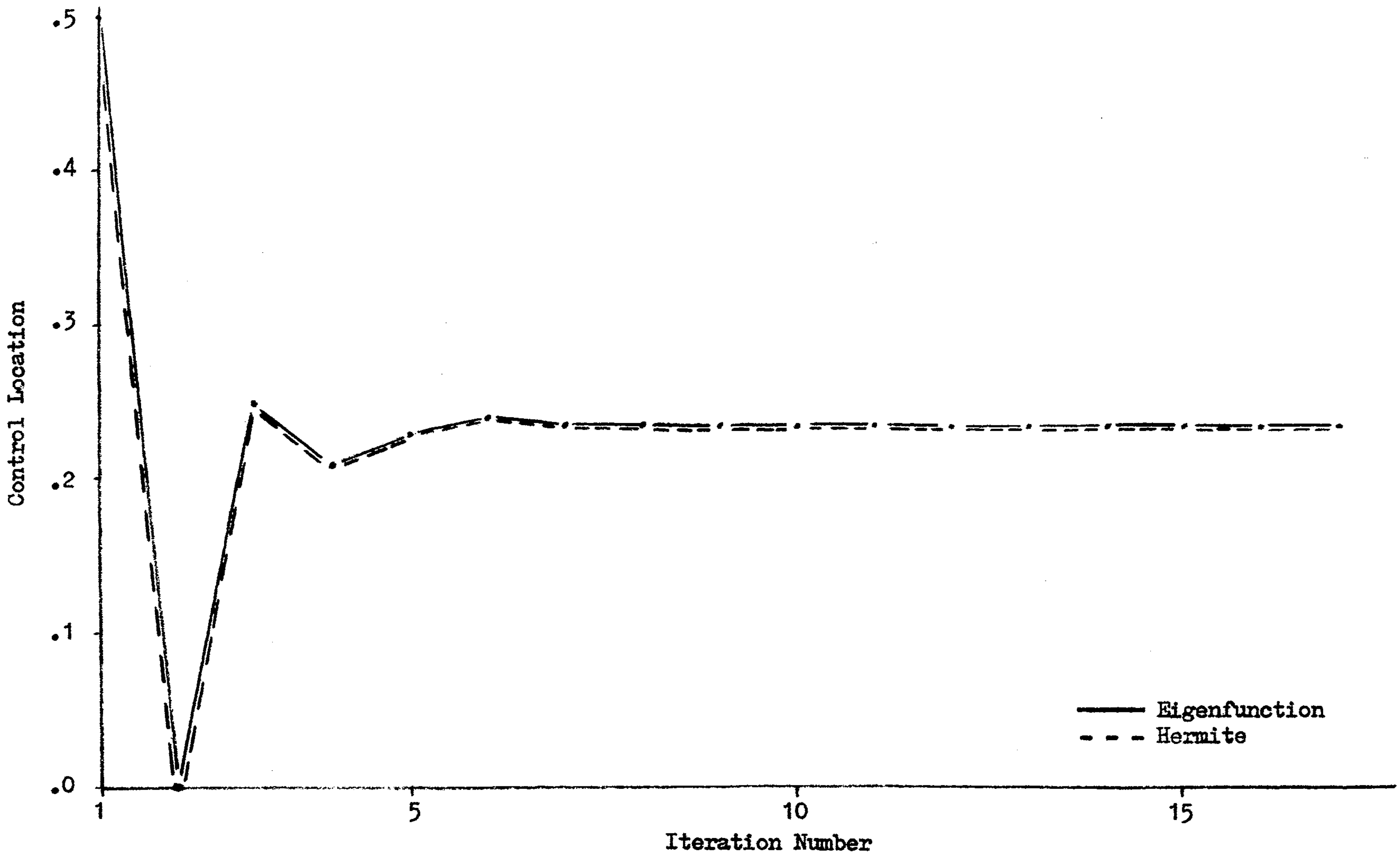


Figure 45. Control Location Versus Iteration Number - One Control Region, Polynomial Temporal Control (std. dev. = .2, ψ = 100)

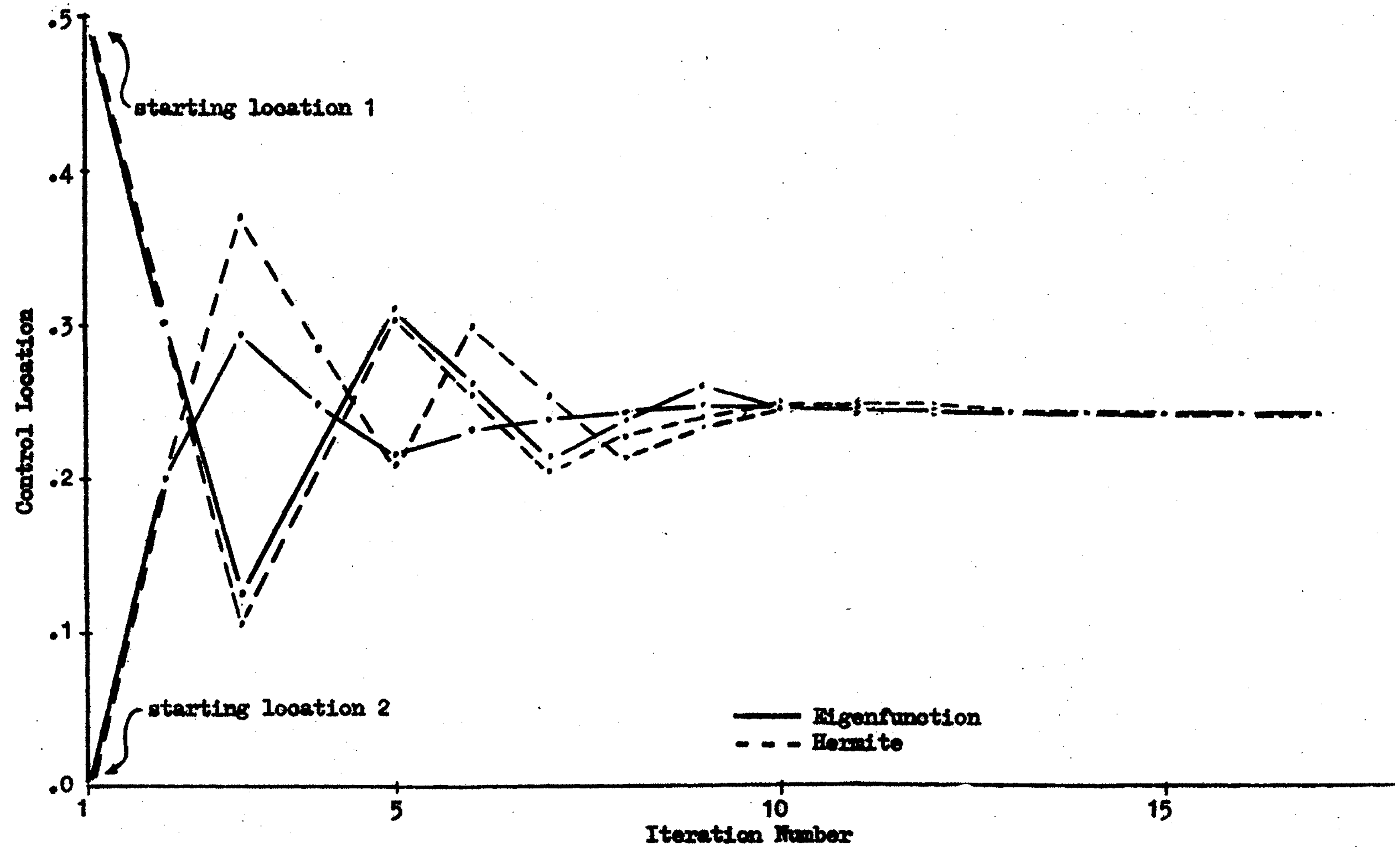


Figure 46. Control Location Versus Iteration Number - One Control Region, Discretized Temporal Control (std. dev. = .2, μ = 100)

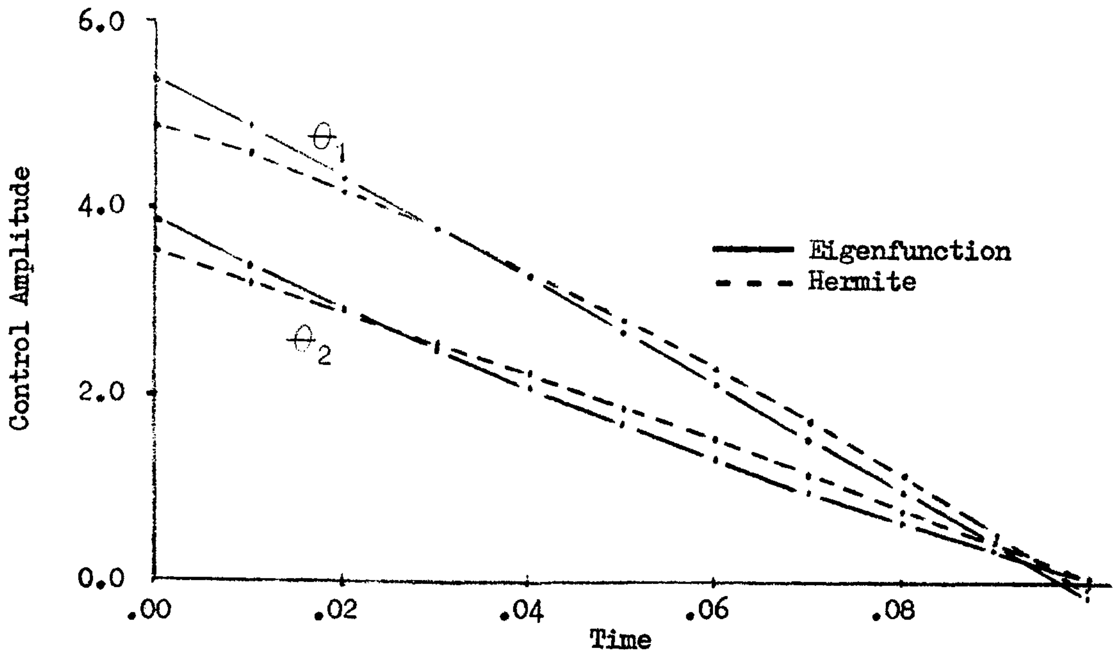


Figure 47. Control Amplitude Versus Time - Two Control Regions, Polynomial Temporal Control Functions (std. dev. = .2, = 100)

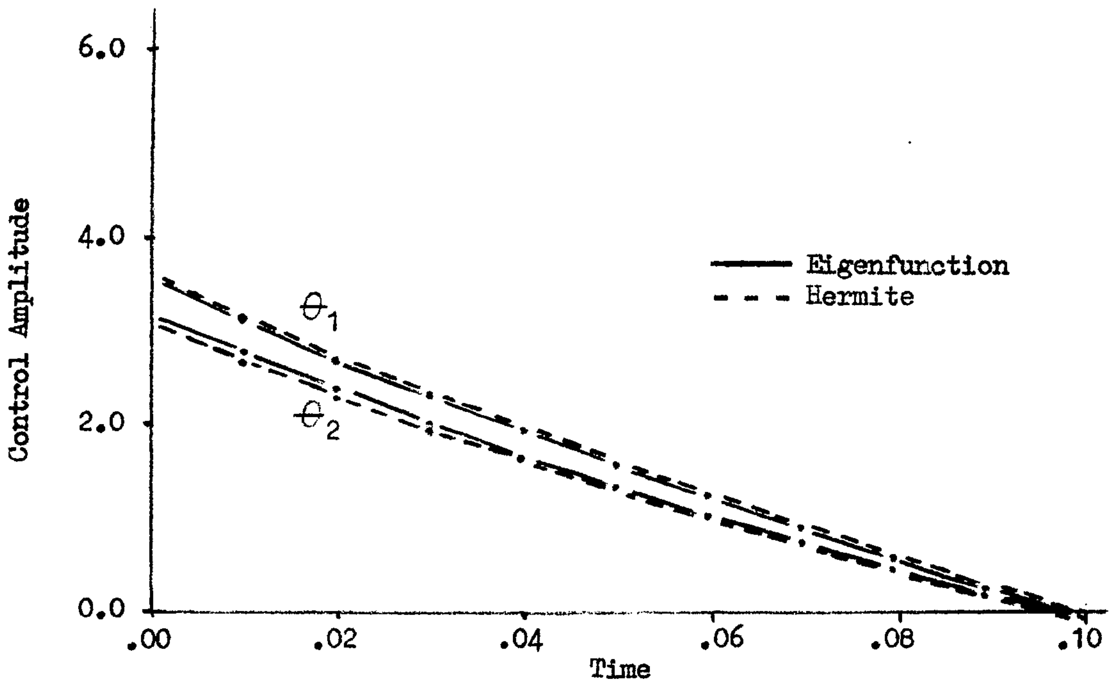


Figure 48. Control Amplitude Versus Time - Two Control Regions, Discretized Temporal Control Functions (std. dev. = .2, μ = 100)

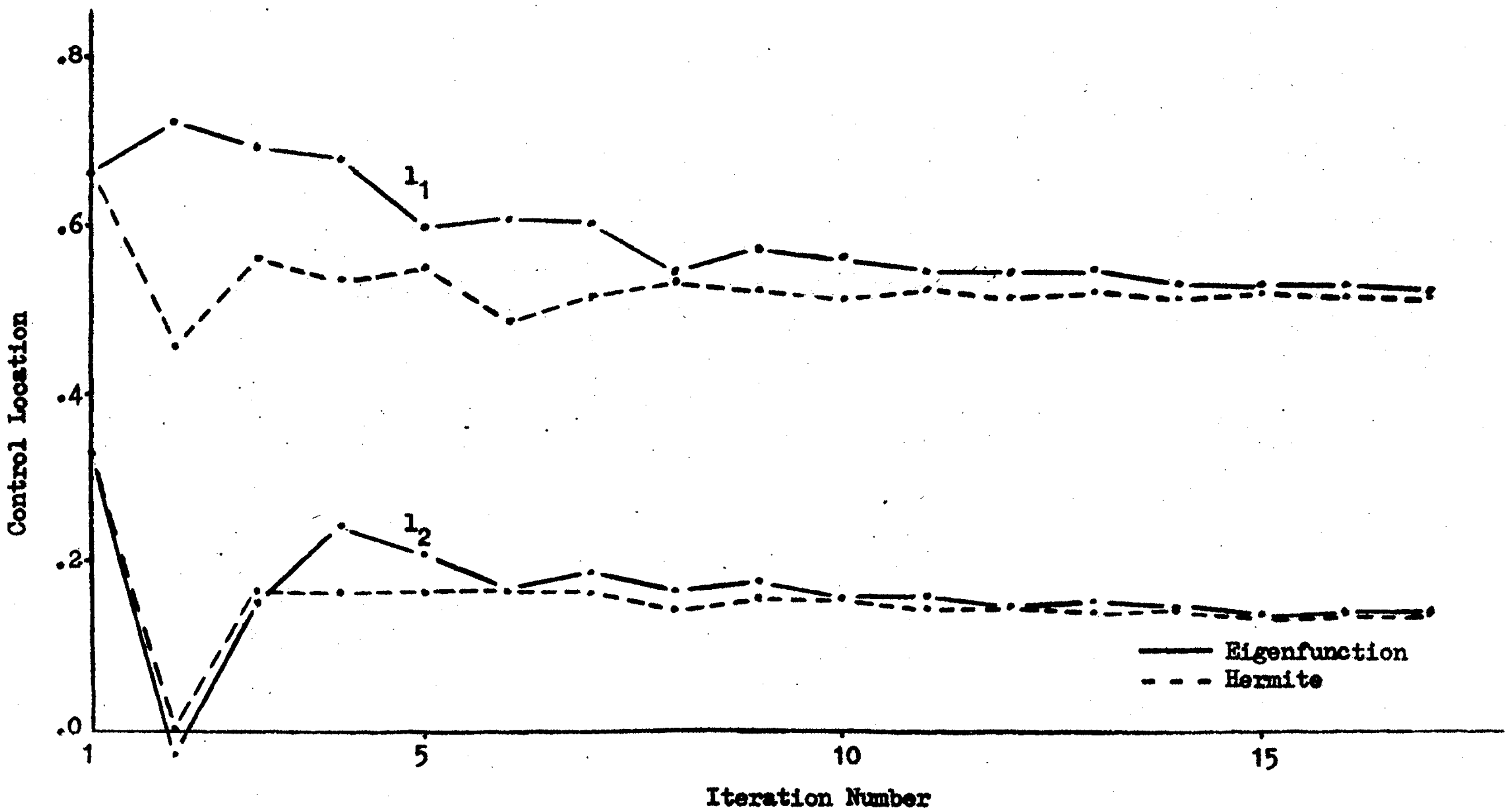


Figure 49. Control Location Versus Iteration Number - Two Control Regions, Polynomial Temporal Controls, (std. dev. = .2, μ = 100)

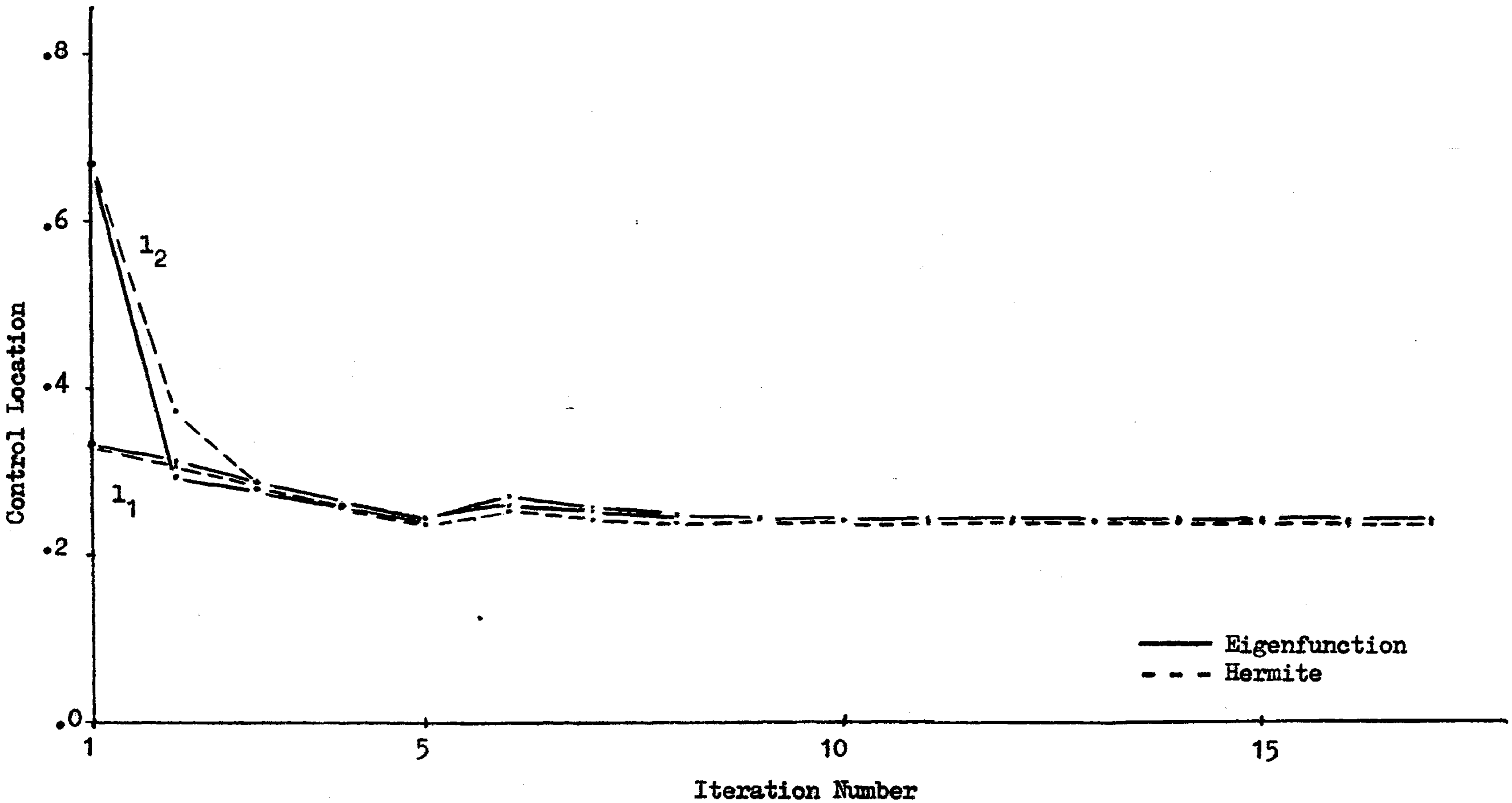


Figure 50. Control Location Versus Iteration Number - Two Control Regions, Discretized Temporal Controls (std. dev. = .2, μ = 100)

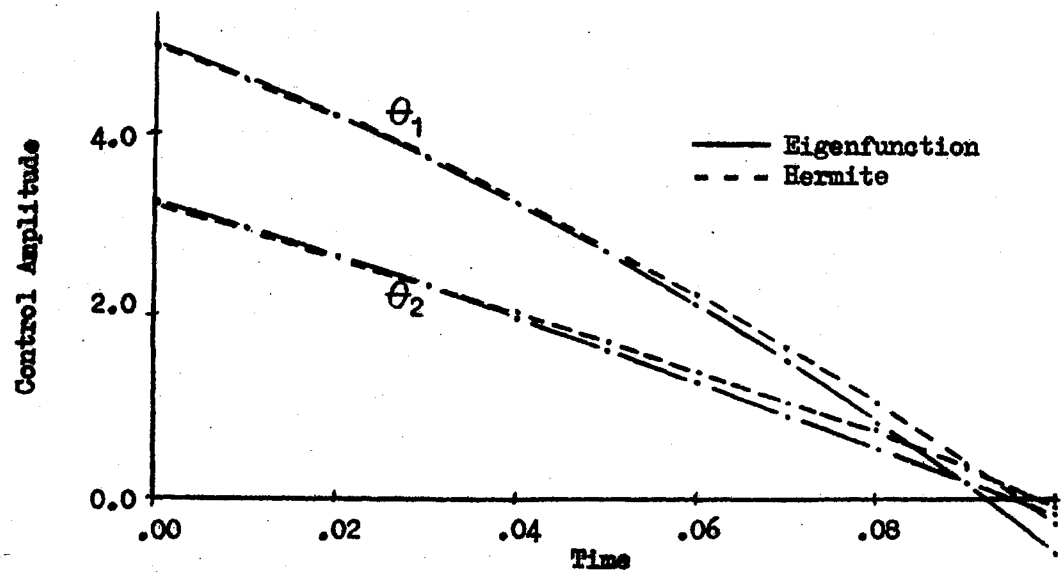


Figure 51. Control Amplitude Versus Time - Two Control Regions, Polynomial Temporal Control Functions (std. dev. = .2, μ = 100)

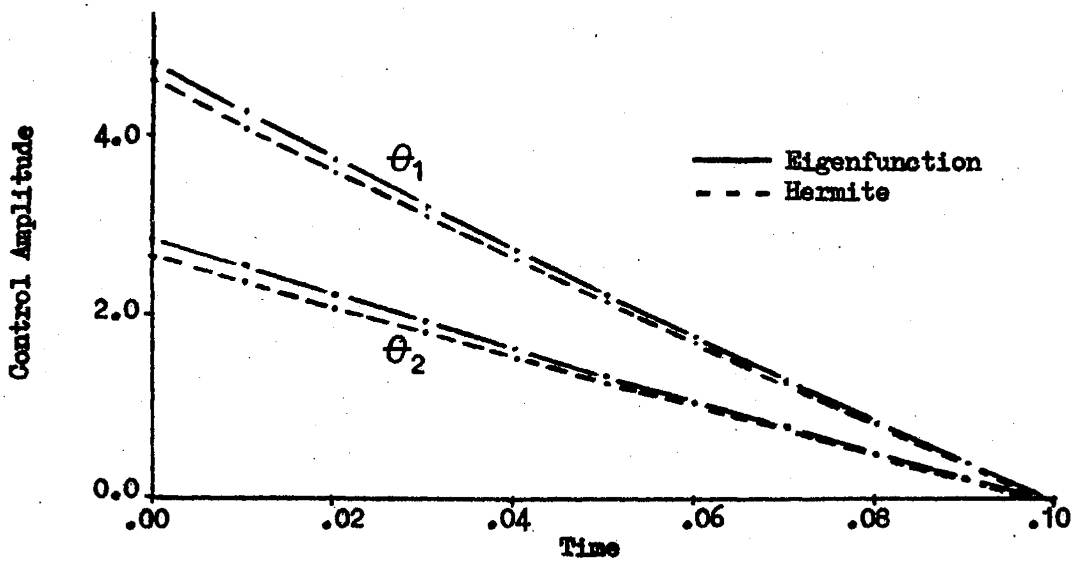


Figure 52. Control Amplitude Versus Time - Two Control Regions, Discretized Temporal Control Functions (std. dev. = .2, μ = 100)

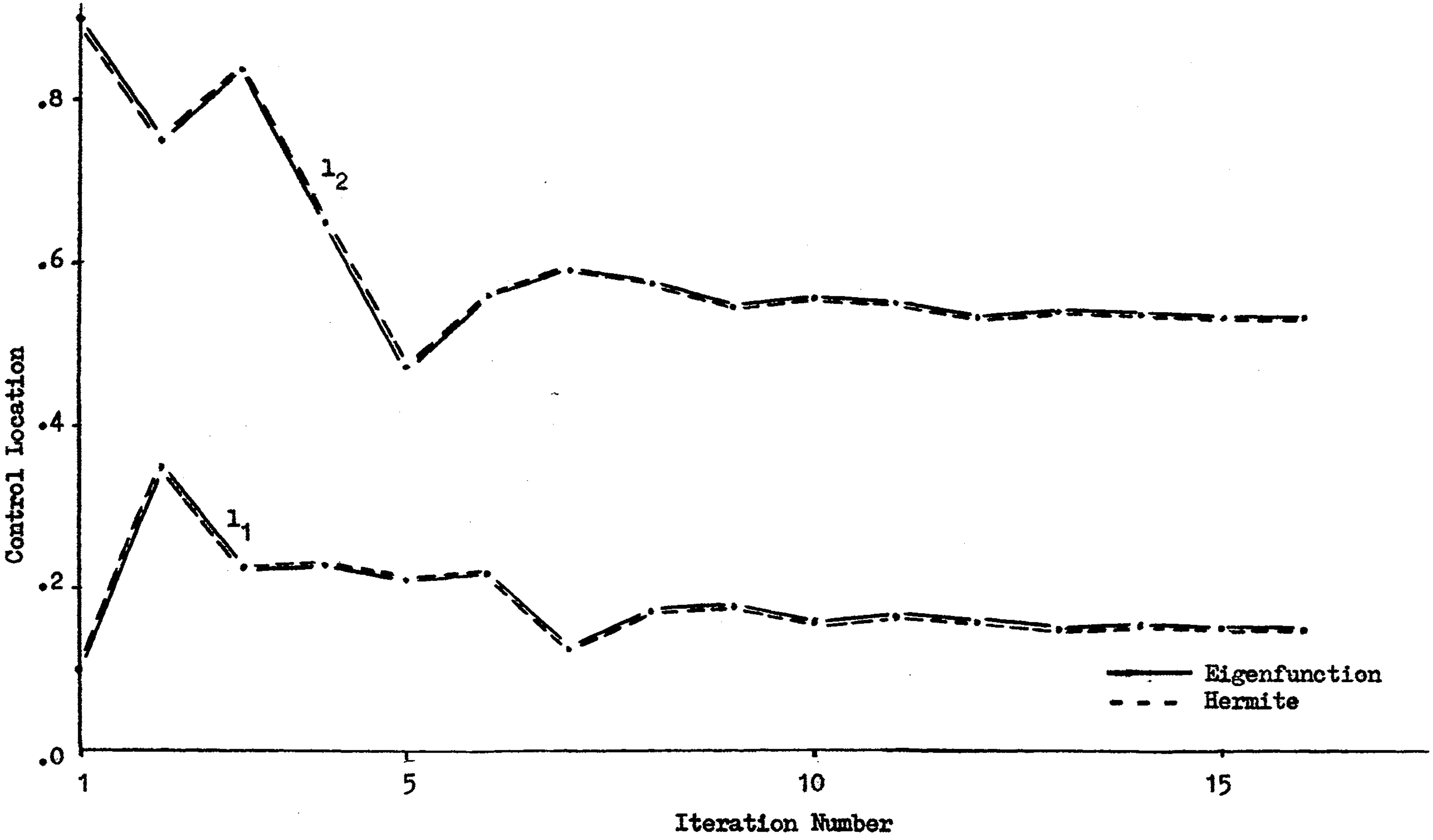


Figure 53. Control Location Versus Iteration Number - Two Control Regions, Polynomial Temporal Controls (std. dev. = .2, μ = 100)

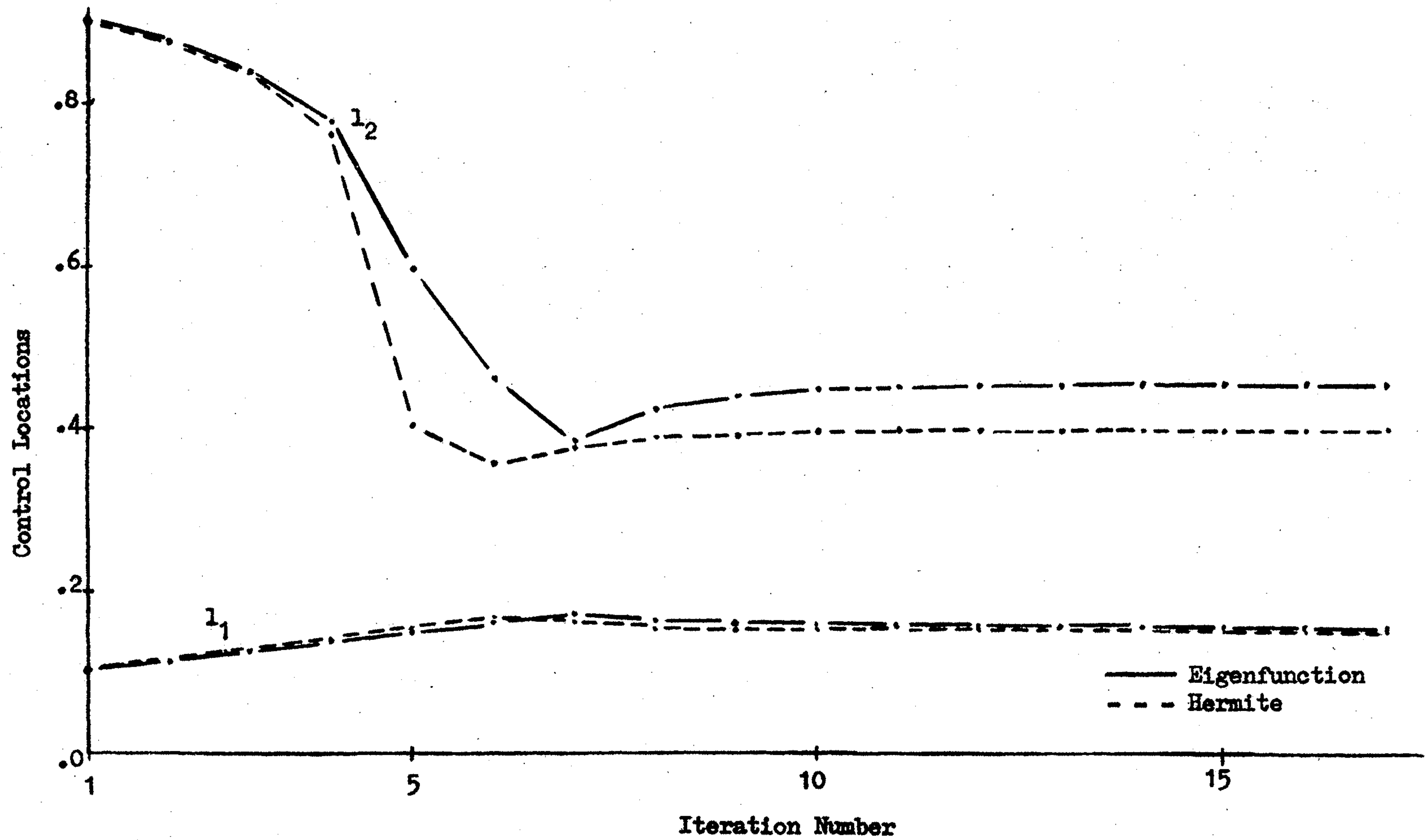


Figure 54. Control Location Versus Iteration Number - Two Control Regions, Discretized Temporal Controls (std. dev. = .2, γ = 100)

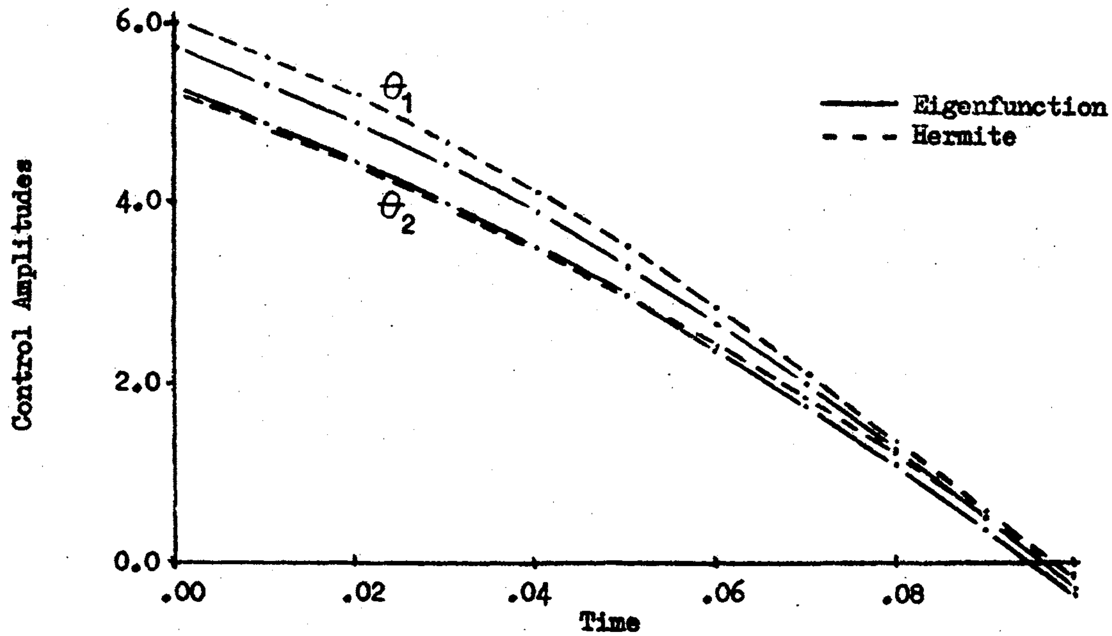


Figure 55. Control Amplitude Versus Time - Two Control Regions, Polynomial Temporal Control Functions (st . dev. = .1, $\mathcal{V}$ = 100)

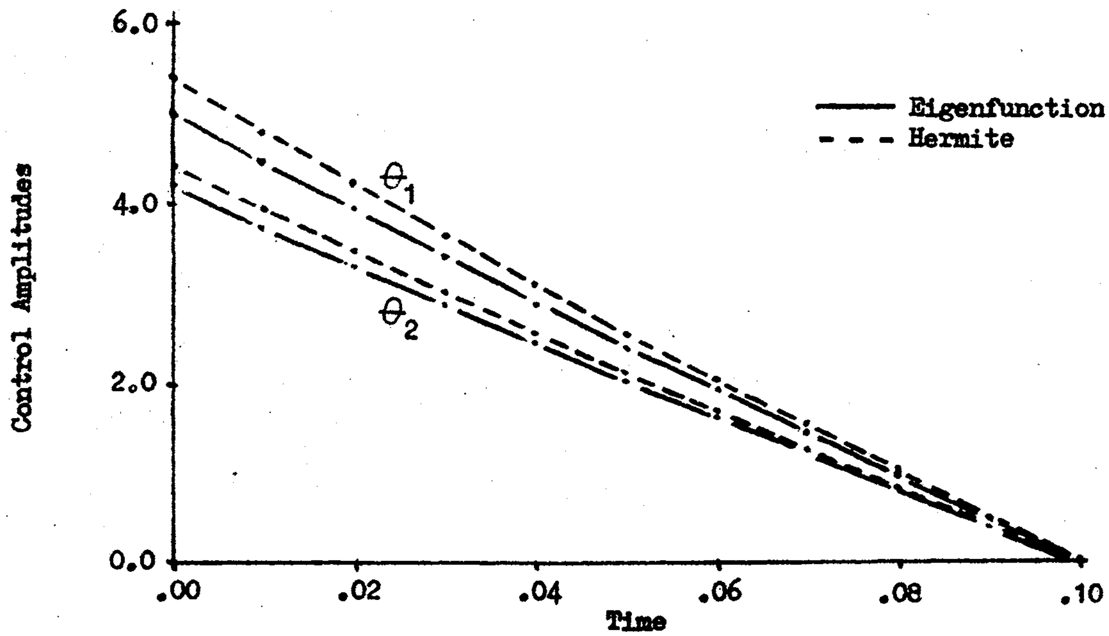


Figure 56. Control Amplitude Versus Time - Two Control Regions, Discretized Temporal Control Functions (std. dev. = .1, $\mathcal{V}$ = 100)

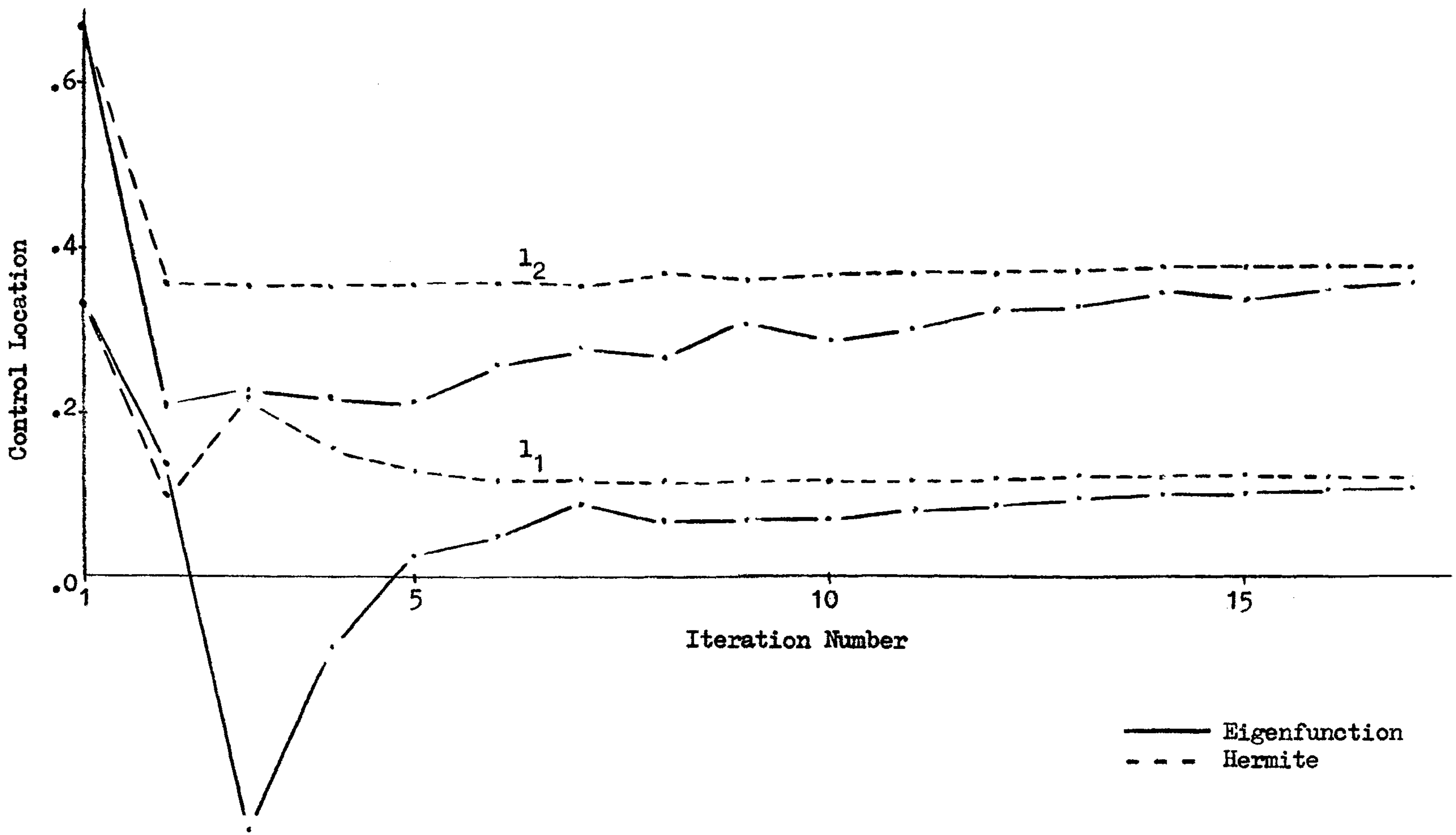


Figure 57. Control Location Versus Iteration Number - Two Control Regions, Polynomial Temporal Control (std. dev. = .1, μ = 100)

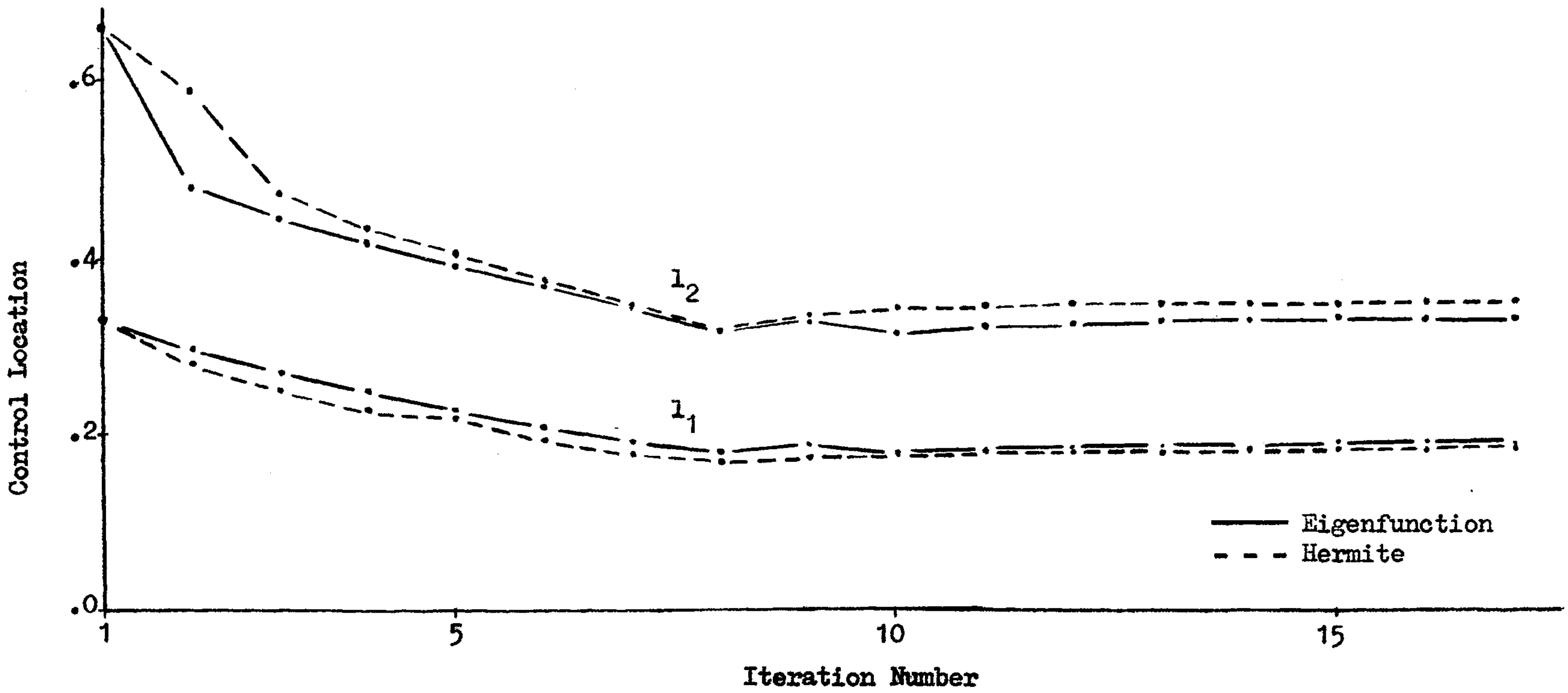


Figure 58. Control Location Versus Iteration Number - Two Control Regions, Discretized Temporal Control (std. dev. = .1, μ = 100)

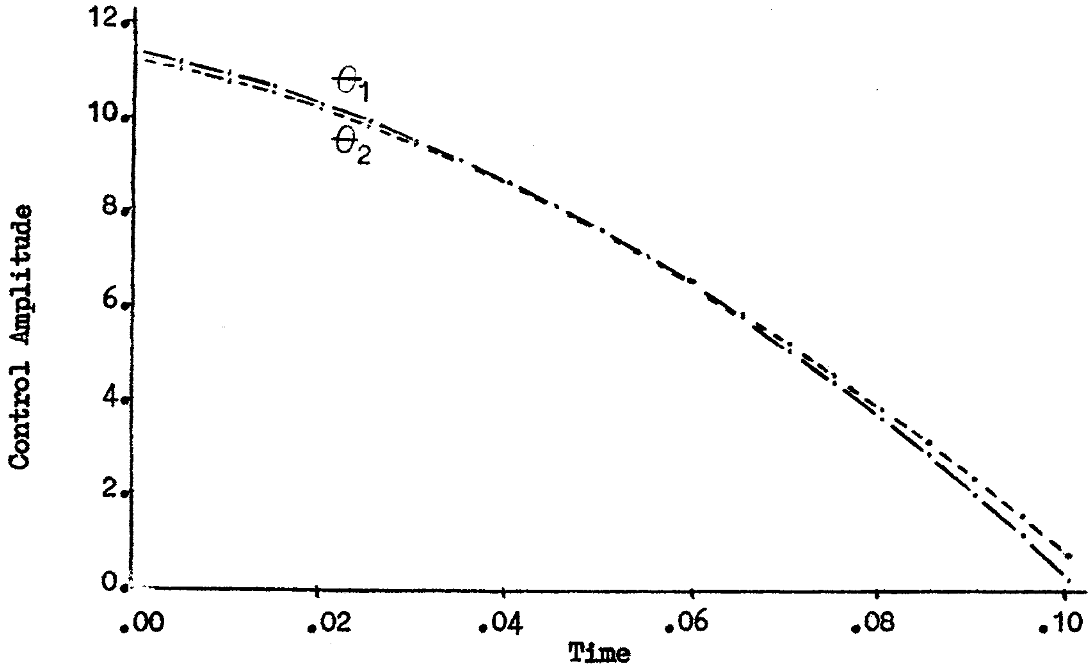


Figure 59. Control Amplitude Versus Time - Two Control Regions, Polynomial Temporal Control Functions, Eigenfunction Expansion (std. dev. = .2, μ = 1000)

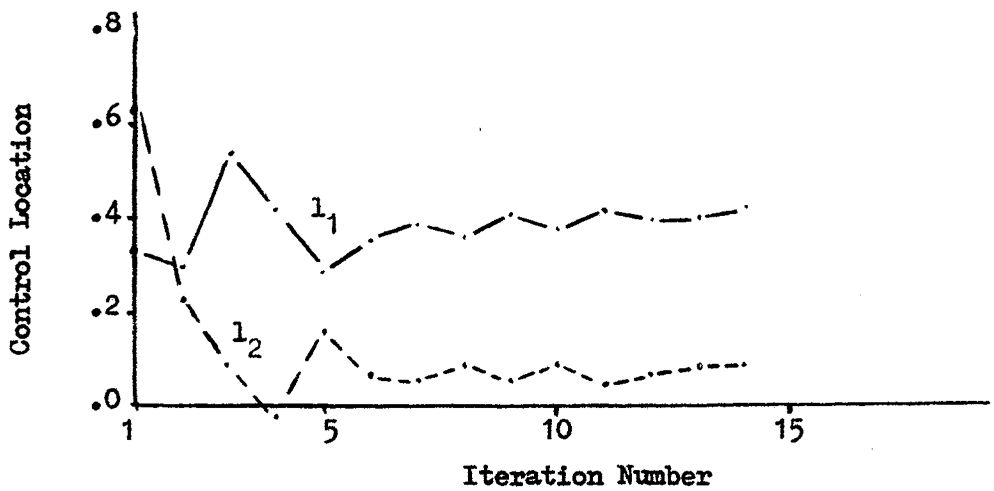


Figure 60. Control Location Versus Iteration Number - Two Control Regions, Polynomial Temporal Control Functions, Eigenfunction Expansion (std. dev. = .2, μ = 1000)

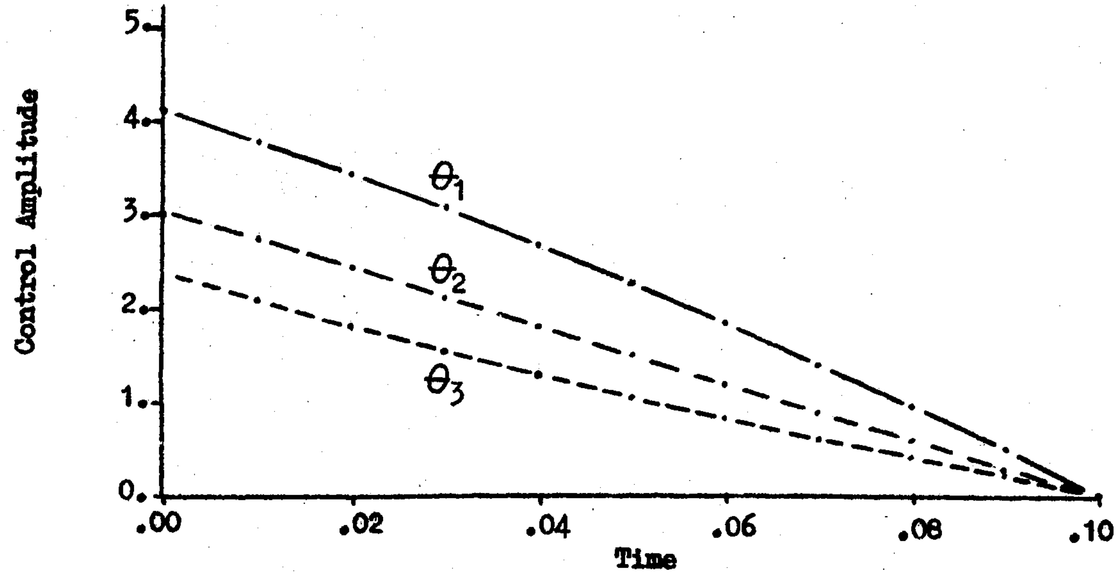


Figure 61. Control Amplitude Versus Time - Three Control Regions, Polynomial Temporal Control Functions, Eigenfunction Expansion (std. dev. = .2, = 100)

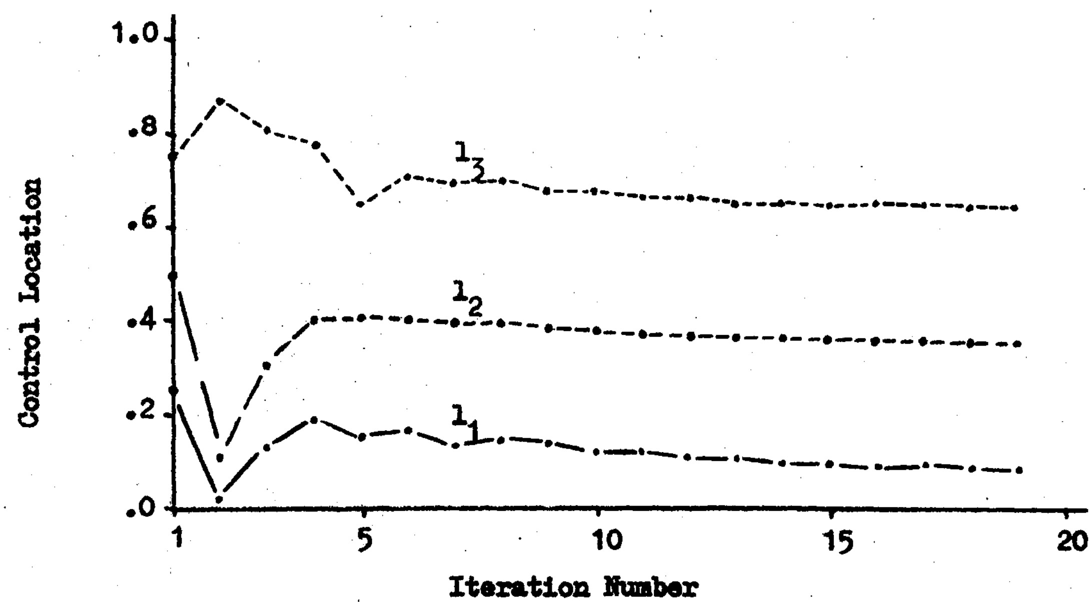


Figure 62. Control Location Versus Iteration Number - Three Control Regions, Polynomial Temporal Control Functions, Eigenfunction Expansion (std. dev. = .2, μ = 100)

APPENDIX B

A TRANSITION MATRIX APPROACH FOR THE THREE-DIMENSIONAL PROBLEM

APPENDIX B

A TRANSITION MATRIX APPROACH FOR THE THREE-DIMENSIONAL PROBLEM

In order to eliminate the numerical integration of the state equation set and the adjoint equation set, a transition matrix approach can be employed for linear problems. The n-dimensional state equation and initial condition for the optimal control problem in two spatial independent variables considered in Chapter V can be written as:

$$Q_{1t}(t) = -\left[IRR\right]^{-1}\left[IR_xR_x\right] Q_1(t) + \left[IRR\right]^{-1}\left[IR\Phi\right]\Theta(t)$$

$$Q_1(0) = \left[IRR\right]^{-1}\left[IRS_o\right]$$

where

$$\left[IRR\right] = \iint_{\Omega} RR^T \, dx_1 dx_2$$

$$\left[IR_xR_x\right] = -\iint_{\Omega} (R_{x_1}R_{x_1} + R_{x_2}R_{x_2}) \, dx_1 dx_2$$

$$\left[IR\Phi\right] = \iint_{\Omega} R\Phi^T dx_1 dx_2$$

$$\left[IRS_o\right] = \iint_{\Omega} R s_o \, dx_1 dx_2 \quad .$$

The solution of this ordinary differential equation set is a function of the eigenvalues, Λ, and the eigenvector matrix, M, of the coefficient matrix $-\left[IRR\right]^{-1}\left[IR_xR_x\right]$.

$$Q_1(t) = M\left[\exp(\Lambda t)\right]M^{-1} Q_1(0) + \int_0^t M\left[\exp(\Lambda(t-\tau))\right]M^{-1}\left[IRR\right]^{-1}\left[IR\Phi\right]\Theta(\tau)d\tau$$

If the temporal control amplitude vector, $\Theta(t)$, is expressed as a polynomial in time,

$$\Theta(t) = \sum_{j=1}^{m} G_j t^{j-1},$$

the state squared term in the performance index can be expressed as:

$$\int_0^{t_f} \iint_{\Omega} S^2 \, dx_1 dx_2 dt = \int_0^{t_f} \iint_{\Omega} Q_1^T RR^T Q_1 \, dx_1 dx_2 dt$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{m} \left\{ Y_i^T (M^{-1})^T \int_0^{t_f} \left[\alpha_i(\Lambda,t)\right]^T M^T \cdot \left[IRR\right] M \left[\alpha_j(\Lambda,t)\right] dt \, M^{-1} Y_j \right\},$$

where

$$Y_o = Q_1(0)$$

$$Y_i = \left[IRR\right]^{-1} \left[IR\Phi\right] G_i \qquad ; \qquad i = 1,2,\ldots,m$$

$$\left[\alpha_o(\Lambda,t)\right] = \left[\exp(\Lambda,t)\right]$$

$$\left[\alpha_i(\Lambda,t)\right] = \int_0^{t} \left[\exp(\Lambda(t-\tau))\right] \tau^{i-1} \, d\tau \; ; \; i = 1,2,\ldots,m.$$

The matrices, $\left[\alpha_i(\Lambda,t)\right]$, are diagonal. The time dependent portions of each term of the matrix integral can be analytically integrated.

$$\int_0^{t_f} \alpha_i(\lambda_k,t)\, \alpha_j(\lambda_l,t) \, dt \qquad ; \qquad \begin{array}{l} i,j = 0,1,\ldots,m \\ k,l = 1,2,\ldots,n \end{array}$$

If the eigenvalues, the eigenvector matrix, the state initial condition and the polynomial control coefficients are given, the state squared term in the performance index can be calculated without the numerical integration of the state equation.

Consider the two terms involving the adjoint vector,

$$\int_0^{t_f} Q_2(t)\left(\frac{\partial \theta_i}{g_{ij}}\right) dt$$

$$\int_0^{t_f} Q_2(t)\,\theta_i(t)\ dt,$$

which are required for the calculation of the gradient changes in the control parameters. The coefficient matrix of the adjoint vector in the adjoint equation,

$$Q_{2t}(t^*) = -\left[IRR\right]^{-1}\left[IR_x R_x\right]\ Q_2(t^*) + \mu\, Q_1(t^*),$$

is the same as that of the state equation if the independent variable time is transformed by:

$$t = t_f - t^* \ .$$

The terminal condition for the adjoint vector becomes an initial condition for backward integration.

$$Q_2(t = t_f) = Q_2(t^* = 0) = 0$$

The adjoint equation solution can be expressed in terms of the eigenvalues and eigenvectors matrix as:

$$Q_2(t^*) = \mu M \sum_{i=0}^{m}\left\{\left[\beta_i(\Lambda, t^*)\right]\ M^{-1}\ Y_i\right\},$$

where

$$\beta_i(\Lambda, t^*) = \int_0^{t^*}\left[\alpha_o(\Lambda, t^* - \tau)\right]\left[\alpha_i(\Lambda, t_f - \tau)\right]\ d\tau \ .$$

For a polynomial temporal control,

$$\frac{\partial \theta_i(t)}{\partial g_{ij}} = t^{j-1} \quad .$$

The two integrals of interest can now be expressed as:

$$\int_0^{t_f} Q_2(t^*)\ (t_f - t^*)^{j-1}\ dt^* =$$

$$\mu\, M \sum_{i=0}^{m} \left\{ \int_0^{t_f} \left[\beta_i(\Lambda, t^*)\right]\ (t_f - t^*)^{j-1}\ dt^*\ M^{-1} Y_i \right\}$$

and

$$\int_0^{t_f} Q_2(t^*) \theta_i(t^*)\ dt^* = \sum_{j=1}^{m} \left\{ g_{ij} \int_0^{t_f} Q_2(t^*)\ (t_f - t^*)^{j-1}\ dt^* \right\} .$$

Note that the integrals,

$$\int_0^{t_f} \left[\beta_i(\Lambda, t^*)\right]\ (t_f - t^*)^{j-1}\ dt^*,$$

can be analytically calculated.

APPENDIX C

DESCRIPTION OF COMPUTER PROGRAMS FOR OPTIMAL REGIONAL CONTROL PROBLEMS

TABLE IV

COMPUTATIONAL STEPS FOR OPTIMAL REGIONAL CONTROL PROBLEM

Step	Calculation	Subroutine 2-D Program	Subroutine 3-D Program
1	Enter data	DPSOPT	DPSOPT
2	$\int RR^T dx$	"	HHINT
3	$\int R_x R_x^T dx$	"	"
4	$\left[\int RR^T dx\right]^{-1}$	"	DPSOPT
5	$\int Rs_o dx$	"	HSINT
6	$\left[\int RR^T dx\right]^{-1} \cdot \left[\int R_x R_x dx\right]$	"	DPSOPT
7	$Q_1(0)$	"	AINTGR
8	$\int R\Phi^T dx$	"	HPINT
9	$\left[\int RR^T dx\right]^{-1}$	"	DPSOPT
10	State Equation Integration	RKIN24 DERFUN	AINTGR
11	$\int Q_1 Q_a^T dt$	DPSOPT	"
12	$\int \Theta \Theta^T dt$	"	DPSOPT
13	$\int \Phi \Phi^T dx$	"	PPINT
14	J	"	DPSOPT
15	Output	"	"
16	IF ITER = 1, go to 18	"	"
17	IF ITER = ITERMAX, STOP	"	"
18	IF J > OLD J, GO TO 32	"	"
19	Adjoint Equation Integration	RKIN24 DERFUN	AINTGR

TABLE IV (Continued)

20	$\int Q_2(\partial\Theta/\partial G)^T dt$	DPSOPT	AINTGR
21	$\int Q_2\Theta^T dt$	"	"
22	$\int\Phi(\partial\Phi/\partial L)^T dx$	"	PPINT
23	$\int\Phi(\partial^2\Phi/\partial L^2)dx$	"	"
24	$\int(\partial\Phi/\partial L)(\partial\Phi/\partial L)^T dx$	"	"
25	$\int R(\partial\Phi/\partial L)^T dx$	"	HPINT
26	$\int R(\partial^2\Phi/\partial L^2)dx$	"	"
27	$\int\Theta(\partial\Theta/\partial G)^T dt$	"	DPSOPT
28	ρ	"	"
29	σ	"	"
30	γ^2	"	"
31	ν^2	"	"
32	ΔL	"	"
33	ΔG (polynomial)	"	"
	$\Delta\Theta$ (discretized)	"	"
34	GO TO 8	"	"
35	$\Delta L = -\frac{1}{2}$ OLD ΔL	"	"
36	$\Delta G = -\frac{1}{2}$ OLD ΔG (polynomial)	"	"
	$\Delta\Theta = -\frac{1}{2}$ OLD $\Delta\Theta$(discretized)	"	"
37	$\kappa = \frac{1}{2}$ OLD κ	"	"
38	GO TO 8	"	"

Note: For program listings, Dr. Karl N. Reid may be contacted through the School of Mechanical and Aerospace Engineering, Oklahoma State University.

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