VARIANCE COMPONENTS IN TWO-WAY CLASSIFICATION MODELS

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CHAPTER I

INTRODUCTION

Estimation of variance components is one of the basic tools of research in several fields of scientific investigation. In any type of estimation, the properties of the estimators, such as whether or not the estimator is efficient, sufficient, consistent, unbiased, minimum variance, etc., should be known to the researcher so that he can ascertain which estimator is best suited to the needs of the particular problem he is considering. In practice, estimators which are unbiased and have minimum variance have proved useful to experimenters in many areas. Therefore, any investigation leading to minimum variance unbiased estimators would prove useful to those who do experimentation.

At present, if an experimental situation dictates the use of an incomplete block design, estimators have been proposed which have not been shown to possess the properties of being best (minimum variance) unbiased. In this thesis we shall be concerned with solving the problem of finding best unbiased estimators for the general two-way classification and for special types of incomplete block designs.

Any estimator which is to be best unbiased must be based on the observed values which are obtained in an experiment. A set of sufficient statistics has the property of containing all the information in the sample about the parameters of the model. Now, it would be very utilitarian if we could find a set of sufficient statistics which has the additional property of being minimal, that is, if ($s_1, s_2, \ldots, s_k$) is any set of sufficient statistics and ($s'_1, s'_2, \ldots, s'_m$) is a set of minimal sufficient statistics, then $k > m$. This latter concept has been set forth by Lehmann and Scheffe. [1]

The determination of a set of minimal sufficient statistics is not only useful when considered in the light of the discussion in the previous paragraph but such a determination is given stature when we consider a theorem proved by Rao and Blackwell [2] which states if T is a minimal sufficient statistic for θ and f(x) is an unbiased estimate of $g(\theta)$, then $h(T) = E[f(x) | T]$ is also an unbiased estimate of $g(\theta)$ based on T and such that $\sigma_f^2 > \sigma_h^2$ unless $f = h$. Thus we see if we are interested in determining minimum variance unbiased estimators of functions of the parameters, these estimators must be based on a set of minimal sufficient statistics.

The theorem does not enable us to determine which estimator is best if two or more unbiased estimators exist for a function $g(\theta)$ and each is based on a set of minimal sufficient statistics. If the density function from which the minimal set was obtained has the property of being complete, then an unbiased estimate of $g(\theta)$ based on the minimal sufficient statistics is unique and thus has minimum variance and the problem is solved. Unfortunately, none of the designs considered here possess density functions which are complete when an Eisenhart Model II is assumed. [3]

The problems of this thesis will be to consider the general two-way classification in order to determine bounds on the number of sufficient statistics in a minimal set and to determine what these statistics are in terms of functions of the observed random variables. Sets of minimal sufficient statistics will be found for the balanced incomplete block design and the group divisible, partially balanced incomplete block designs with two associate classes. The distribution of each statistic will be determined and stochastic independence of statistics in a set determined.

CHAPTER II

NOTATION AND LEMMAS

We shall present here the definitions of symbols which are frequently used in the body of the thesis. We divide them into two parts, those symbols which are scalars and those which are matrices.

(1) Scalars:

a. t is equal to the number of treatments in a design.

b. b is equal to the number of blocks in a design.

c. r is equal to the number of replicates of each treatment.

d. k is equal to the number of experimental units in each block.

e. BIB is an abbreviation for balanced incomplete block.

f. PBIB is an abbreviation for partially balanced incomplete block.

g. GD-PBIB is an abbreviation for group divisible, partially balanced incomplete block design. If GD is prefixed by S, SR or R it will denote the singular, semi-regular or regular group divisible, partially balanced incomplete block design respectively.

h. λ denotes in a BIB, the number of times two treatments occur together in all blocks.

i. λ_i $(i = 1, 2)$ denotes in a PBIB, the number of times two treatments which are i-th associates occur together in all blocks.

j. λ_j is the non-centrality parameter of the non-central chi-square distribution.

k. M is the total number of observations in a design.

l. n is the number of groups in a GD-PBIB design.

m. m is the number of treatments per group in a GD-PBIB design.

n. $v = k^{-1}(rk - r + \lambda_1) = k^{-1}[\lambda_2 t + m(\lambda_1 - \lambda_2)]$

o. $\mathcal{I}$ is an operation on a density function which, when properly defined, reduces the dimension of the space of the sufficient statistics.

p. E denotes mathematical expectation.

q. MVN is an abbreviation for multivariate normal.

(2) Matrices:

a. X is a design matrix of a two-way classification model.

b. X_1 is a partition of X corresponding to blocks.

c. X_2 is a partition of X corresponding to treatments.

d. Y is a vector of observable quantities.

e. J_q^s is an s x q matrix of all ones. j_1^n will be used to denote an n x 1 vector of ones.

f. $N = X_2'X_1$.

g. D is a diagonal matrix.

h. P is an orthogonal matrix. When partitioning a matrix, partitions will be denoted by the addition of a subscript. Further partitions of a partition will be denoted by the addition of an additional subscript. Thus $P = (P_1, P_2) = (P_{11}, P_{12}, P_{21}, P_{22}, P_{23})$.

i. Σ is a covariance matrix.

j. ϕ_w represents a w x w matrix of all zeroes.

k. $A = [X_2 - X_1(X_1X_1')^{-1}X_1'X_2]$.

l. I_w is the identity matrix of dimension w x w.

Additional symbols which occur less frequently will be defined as the discussion develops.

We shall now prove a few lemmas which will be needed for the proofs of the theorems in the ensuing chapters.

LEMMA 1. Let X denote the design matrix of a two-way classification model $Y = X\beta + e$ where the rank of X is $b + t - 1$ and where X is of the form $X = (j_1^M, X_1, X_2)$. Then there exists a set of $M - b - t + 1$ orthogonal rows P', such that $X_1'P = \phi$, $X_2'P = \phi$ and $j_M^1 P = \phi$.

Proof.

Consider the matrix product

$$XX' = (j_M^1, X_1, X_2) \begin{bmatrix} j_M^1 \\ X_1' \\ X_2' \end{bmatrix} = J_M^M + X_1X_1' + X_2X_2'$$

Since XX' is symmetric, there exists an orthogonal matrix Q such that $Q'XX'Q = D$ where D is a diagonal matrix. The number of non-zero elements on the diagonal of D is b+t-1 since X is rank b+t-1. Partition Q into $Q = (C, P)$ where C and P are of dimensions M x (b+t-1) and M x (M-b-t+1) respectively, and such that

$$Q'XX'Q = \begin{bmatrix} C' \\ P' \end{bmatrix} XX'(C, P) = \begin{bmatrix} D_1 & \phi \\ \phi & \phi \end{bmatrix}$$

where D_1 is (b+t-1) x (b+t-1). Therefore

$$P'J_M^M P + P'X_1X_1'P + P'X_2X_2'P = \phi$$

The matrices J_M^M, X_1X_1', and X_2X_2' are each positive semi-definite, each being the product of a matrix and its transpose. The matrices $P'J_M^M P$, $P'X_1X_1'P$ and $P'X_2X_2'P$ are also positive definite for the same reason. Since each diagonal element of each of these matrices is the sum of squares of real numbers and the sum of these sum of squares is zero, the diagonal elements of each of the three aforementioned matrices must be equal to zero. If any element off the diagonal is non-zero, there would be at least one of the principal minors which would be negative, a contradiction of the positive definiteness. We therefore conclude that each of the matrices must be equal to the null matrix.

It follows immediately that $j_M^1 P = \phi$, $X_1'P = \phi$ and $X_2'P = \phi$, which was to be shown.

<u>LEMMA 2. Let N be a t x b matrix of rank m. Let P be an orthogonal matrix such that P'NN'P = D where D is diagonal with the characteristic roots of NN' displayed on the diagonal. If $s \leq m$ of the characteristic roots are equal to d_o ($\neq 0$), then the matrix $d_o^{-1/2}P_o'N = C'$ (say) is a set of s orthogonal rows such that $C'N'NC = d_oI_s$ where P_o is such that $P_o'NN'P_o = d_oI_s$.</u>

Proof. By hypothesis, there are s characteristic roots of NN' equal to d_o. Therefore if we partition P into (P_o, P_1), we may write

$$(1) \qquad \begin{bmatrix} P_o' \\ P_1' \end{bmatrix} NN' (P_o, P_1) = D = \begin{bmatrix} d_o I_s & \phi \\ \phi & D_1 \end{bmatrix}$$

where D_1 is diagonal. Hence

$$P_o' NN' P_o = d_o I_s$$

or

$$(d_o^{-1/2} P_o' N)(N' P_o d_o^{-1/2}) = I_s .$$

Consider now

$$(d_o^{-1/2} P_o' N) N' N (N' P_o d_o^{-1/2}) = Z \text{ (say)}$$

Then we may write

$$Z = (d_o^{-1/2} P_o' N) N' (P_o P_o' + P_1 P_1') N (N' P_o d_o^{-1/2}).$$

From (1), $P_o' NN' P_1 = \phi$. Therefore

$$Z = d_o^{-1/2} (P_o' NN' P_o)(P_o' NN' P_o) d_o^{-1/2}$$

$$= d_o^{-1/2} (d_o I_s)(d_o I_s) d_o^{-1/2}$$

$$= d_o I_s$$

which was to be shown.

LEMMA 3. Let the matrix F be defined as follows:

$$\begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1n} \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ A_{n1} & A_{n2} & \cdot & \cdot & \cdot & A_{nn} \end{bmatrix}$$

where
$$A_{ij} = \begin{cases} (a_1 - b)I_m + bJ_m^m & \text{if } i = j. \\ (c - b)I_m + bJ_m^m & \text{if } i \neq j. \end{cases}$$

<u>Then the characteristic roots and multiplicities of F are as follows:</u>

<u>Roots</u>	<u>Multiplicities</u>
$a + (n - 1)c + n(m - 1)b$	1
$a + (n - 1)c - nb$	$m - 1$
$(a - c)$	$m(n - 1)$

Proof. To find the characteristic roots of F we must solve the determinantal equation $|F - \ell I| = 0$ for ℓ. Since a_1 occurs only on the diagonal of F, let $a_1 = a - \ell$. We then must find the value of the determinant of F defined in this manner.

Subtracting the last row from each of the other rows, we have

$$\begin{bmatrix} A_{11} - A_{n1} & \phi & \cdots & A_{1n} - A_{nn} \\ \phi & A_{22} - A_{n2} & \cdots & A_{2n} - A_{nn} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Now by adding each column to the last column, we have:

$$\begin{bmatrix} A_{11} - A_{n1} & \phi & \cdots & \phi \\ \phi & A_{22} - A_{n2} & \cdots & \phi \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} + \sum_i^{n-1} A_{ni} \end{bmatrix}$$

Now, $(A_{ii} - A_{ni}) = (a - c)I_m$ for $i = 1, 2, \ldots, n-1$ and $A_{nn} + \Sigma A_{ni} = [a_1 - nb + (n - 1)c]I_m + nbJ_m^m$. Therefore the determinant of F is equal to

$$(a_1 - c)^{m(n-1)}[a_1 + (n-1)c - nb]^{m-1}[a_1 + (n-1)c + n(m-1)b]$$

and since $a_1 = a - \ell$, by setting the above expression equal to zero and solving for ℓ, we have the result. [4]

LEMMA 4. If F is a symmetric matrix with characteristic roots $\ell_i \neq 0$ $(i = 1, 2, \ldots, r)$, then the characteristic roots of FF are ℓ_i^2.

Proof. Since F is symmetric, there exists an orthogonal matrix P such that $P'FP = D$ where D is diagonal with the characteristic roots of F on the main diagonal.

Consider now operating on FF with the matrix P of the foregoing paragraph. We then have

$$P'FFP = P'FIFP = (P'FP)(P'FP) = D\,D = D^2$$

Therefore P is the orthogonal matrix which also diagonalizes FF with the characteristic roots of FF on the diagonal. The diagonal matrix thus obtained is the square of the diagonal matrix obtained by operating on F with P. Hence the result follows.

LEMMA 5. If G is of the form:

$$\begin{bmatrix} c_1 I_{m-1} & \phi & c_5 I_{m-1} & \phi \\ \phi & c_2 I_{m(n-1)} & \phi & c_6 I_{m(n-1)} \\ c_5 I_{m-1} & \phi & c_3 I_{m-1} & \phi \\ \phi & c_6 I_{m(n-1)} & \phi & c_4 I_{m(n-1)} \end{bmatrix}$$

where the c_i are scalars, then G^{-1} is of the form:

$$\begin{bmatrix} c_3 d_1^{-1} I_{m-1} & \phi & -c_5 d_1^{-1} I_{m-1} & \phi \\ \phi & c_4 d_2^{-1} I_{m(n-1)} & \phi & -c_6 d_2^{-1} I_{m(n-1)} \\ -c_5 d_1^{-1} I_{m-1} & \phi & c_1 d_1^{-1} I_{m-1} & \phi \\ \phi & -c_6 d_2^{-1} I_{m(n-1)} & \phi & c_2 d_2^{-1} I_{m(n-1)} \end{bmatrix}$$

where

$$d_1 = c_1 c_3 - c_5^2 \quad \text{and} \quad d_2 = c_2 c_4 - c_6^2 .$$

Proof. By matrix multiplication we have $GG^{-1} = I$. Therefore by definition, the inverse of G is as given.

CHAPTER III

THE GENERAL TWO-WAY CLASSIFICATION

In this chapter we shall assume an Eisenhart Model II in the general two-way classification with unequal numbers in the sub-classes and obtain an upper bound on the number of statistics in a minimal set of sufficient statistics. We shall also show that the block totals, t-1 of the treatment totals and the intra-block error are a set of sufficient statistics for this design.

We shall assume an Eisenhart Model II of the form

$$Y = X\gamma + e$$

where the dimensions of the matrices are as follows:

Matrix	Dimension
Y	$M \times 1$
X	$M \times 1$
γ	$(b+t+1) \times 1$
X_o	$M \times 1$
X_1	$M \times b$
X_2	$M \times t$
e	$M \times 1$
β	$b \times 1$
τ	$t \times 1$

where

$$X = (X_o, X_1, X_2) \qquad\qquad \gamma' = (\mu, \beta', \tau').$$

The vectors e, β and τ are each distributed as the multivariate normal with the following properties:

(1) $E(e) = \phi$, $E(\beta) = \phi$, $E(\tau) = \phi$,

(2) $E(ee') = \sigma^2 I_M$, $E(\beta\beta') = \sigma_1^2 I_b$, $E(\tau\tau') = I_t\sigma_2^2$,

(3) $E(e\beta') = \phi$, $E(e\tau') = \phi$, $E(\beta\tau') = \phi$.

Since XX' is symmetric of rank b + t - 1, there exists an orthogonal matrix P such that

$$P'XX'P = \begin{bmatrix} W & \phi \\ \phi & \phi \end{bmatrix}$$

where W is diagonal of dimension (b+t-1) x (b+t-1). Partitioning X we have

$$P'XX'P = P'(X_o, X_1, X_2) \begin{bmatrix} X_o' \\ X_1' \\ X_2' \end{bmatrix} P = P'(X_1X_1' + X_2X_2' + X_oX_o')P$$

Partition P into (P_1, P_2) where P_1 and P_2 are of dimensions M x (b+t-1) and M x (M - b - t + 1) respectively.

Applying the result of Lemma 1, we have $P_2'X_o = \phi$, $P_2'X_1 = \phi$ and $P_2'X_2 = \phi$.

Consider now the distribution of the vector Y under the distributional assumptions we have made. Since Y is a linear combination of normally distributed variables, Y is distributed as the multivariate normal with

(1) mean $E(Y) = \bar{\mu}$ (say)

(2) covariance matrix $E(Y - \bar{\mu})(Y - \bar{\mu})' = \Sigma$ (say)

where

$$\bar{\mu} = \mu j_1^M$$

$$\Sigma = (X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I).$$

The joint distribution of the elements of Y is g(Y) where

$$g(Y) = (2\pi)^{-M/2}|\Sigma|^{-1/2} \exp -2^{-1}(Y - \mu)'\Sigma^{-1}(Y - \mu).$$

Consider now the operation I on $g(Y)$ to be

$$Ig(Y) = (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp 2^{-1}(Y - \mu)'PP'\Sigma PP'(Y - \mu),$$

where P is the orthogonal matrix described previously.

$$\text{Consider now } P'\Sigma P = P'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)P$$

$$= P'X_1X_1'P\sigma_1^2 + P'X_2X_2'P\sigma_2^2 + \sigma^2 I$$

By the argument previously considered we found that by partioning P into (P_1, P_2) we can find a set of rows P_2' such that $P_2'X_1 = \phi$ and $P_2'X_2 = \phi$.

We may therefore write

$$P'X_1X_1'P\sigma_1^2 = \begin{bmatrix} {}_1A_{11}\sigma_1^2 & \phi \\ \phi & \phi \end{bmatrix} \quad \text{and} \quad P'X_2X_2'P\sigma_2^2 = \begin{bmatrix} {}_2A_{11}\sigma_2^2 & \phi \\ \phi & \phi \end{bmatrix}$$

where ${}_kA_{11}$ $(k = 1, 2)$ is of dimension $(b+t-1) \times (b+t-1)$. Therefore

$$P'\Sigma P = \begin{bmatrix} T & \phi \\ \phi & \sigma^2 I \end{bmatrix}$$

where $T = {}_1A_{11}\sigma_1^2 + {}_2A_{11}\sigma_2^2$. We may then write

$$(P'\Sigma P)^{-1} = P'\Sigma^{-1}P = \begin{bmatrix} T^{-1} & \phi \\ \phi & \sigma^{-2}I_{M-b-t+1} \end{bmatrix}$$

$Ig(Y)$ then becomes

$$(2\pi)^{-M/2} |\Sigma|^{-1/2} \exp -2^{-1}[P'(Y - \mu X_o)]' \begin{bmatrix} T^{-1} & \phi \\ \phi & \sigma^{-2}I \end{bmatrix} [P'(Y - \mu X_o)]$$

$$= (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp -2^{-1} \begin{bmatrix} P_1'(Y - \mu X_o) \\ P_2'Y \end{bmatrix}' \begin{bmatrix} T^{-1} & \phi \\ \phi & \sigma^{-2}I \end{bmatrix} \begin{bmatrix} P_1'(Y - \mu X_o) \\ P_2'Y \end{bmatrix}$$

$$= (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp -2^{-1}[(P_1'Y - P_1'X_0\mu)'T^{-1}(P_1'Y - P'X_0\mu) + Y'P_2P_2'Y\sigma^{-2}]$$

Define now the b + t statistics $P_{1i}'Y$ (i = 1, 2, . . . , b+t-1) and $Y'P_2P_2'Y$ where P_{1i}' is the i-th row of P_1'. By definition these b + t statistics are a sufficient set of statistics for the parameters μ, σ^2, σ_1^2 and σ_2^2. From this discussion , we conclude that there are at most b + t sufficient statistics in a minimal set in the general two-way classification under the assumption of an Eisenhart Model II.[5]

We shall now show that the block and treatment totals (less one) and the intra-block error are a set of sufficient statistics for this design under Model II.

Consider now the matrix P and its partition (P_1, P_2). Let P_1 be of the form (P_{11}, P_{12}) such that

$$\begin{bmatrix} P_{11}' \\ P_{12}' \end{bmatrix} (P_{11}, P_{12}) = \begin{bmatrix} I_b & \phi \\ \phi & I_{t-1} \end{bmatrix}$$

Let $P_{11}' = D_1^{-1/2}X_1'$ where $D_1 = X_1'X_1$. Obviously $P_{11}'P_{11} = I_b$.

Consider now the matrix $A' = (X_2' - X_2'X_1D_1^{-1}X_1')$. Since $j_t^1 A' = \phi$, A' is of rank at most t - 1. We shall assume that the rank of A is exactly t - 1. Since $A'A$ is symmetric, there exists an orthogonal matrix Q such that $Q'A'AQ = D$ where D is diagonal with the characteristic roots of $A'A$ on the main diagonal. By assumption, the rank of A' is t - 1 and therefore there is one zero characteristic root of $A'A$. Since $j_t^1 A' = \phi$, let

$Q = (t^{-1/2}j_1^t, Q_1)$. Therefore

$$Q'A'AQ = \begin{bmatrix} 0 & \phi \\ \phi & D_2 \end{bmatrix}$$

where D_2 is diagonal with the non-zero characteristic roots of $A'A$ on the main diagonal. Now let $P'_{12} = D_2^{-1/2} Q'_1 A'$. Then

$$P'_{12}P_{12} = D_2^{-1/2} Q'_1 A'AQ_1 D_2^{-1/2} = D_2^{-1/2} D_2 D_2^{-1/2} = I_{t-1}$$

and

$$P'_{11}P_{12} = D_1^{-1/2} X'_1 A Q_1 D_2^{-1/2} = D_1^{-1/2}(X'_1X_2 - X'_1X_2)Q_1D_2^{-1/2} = \phi.$$

With P_{11} and P_{12} defined in this manner we see that P'_1 forms a set of $b + t - 1$ orthogonal rows.

Let us now examine $P'_1(Y - \mu j_1^M)$. We have then:

$$P'_1(Y - \mu j_1^M) = \begin{bmatrix} P'_{11} \\ P'_{12} \end{bmatrix} (Y - \mu j_1^M) = \begin{bmatrix} D_1^{-1/2} X'_1 (Y - j_1^M \mu) \\ D_2^{-1/2} Q'_1 A'(Y - j_1^M \mu) \end{bmatrix}$$

Define now $D_1^{-1/2}X'_1Y$ to be b statistics and $D_2^{-1/2}Q'_1 A'Y$ to be $t - 1$ statistics. Examining these two vectors, if we let $B = X'_1Y$ denote the vector of block totals and $V = X'_2Y$ denote the vector of treatment totals, we then have

$$D_1^{-1/2}X'_1Y = D_1^{-1/2} B$$

and

$$D_2^{-1/2}Q'_1A'Y = D_2^{-1/2}Q'(X'_2 Y - ND_1^{-1/2} X'_1Y) = D_2^{-1/2}Q'(V - ND_1^{-1}B)$$

In this form is is readily seen that these b+t-1 statistics are based on the block and treatment totals. It remains to be shown that $Y'P_2P'_2 Y$ is the intra-block error.

Consider now $Y'Y = Y'PP'Y = Y'(P_{11}P'_{11} + P_{12}P'_{12} + P_2P'_2)\,Y$. Substituting for P_{11} and P_{12} we have

$$Y'Y = Y'(X_1D_1^{-1}X_1' + AQ_1D_2^{-1}Q'_1A' + P_2P_2')\,Y$$

$Y'P_2P_2'Y$ is the intra-block error if it can be shown that $Y'X_1D_1^{-1}X'_1Y$ and $Y'AQ_1D_2^{-1}Q_1'A'Y$ are the blocks (ignoring treatments) and treatments (eliminating blocks) sums of squares, respectively.

To begin, consider the model

$$Y = \mu j_1^M + X_1\alpha + X_2\mathcal{T} + e$$

or

$$Y = X_1(\mu j_1^b + \alpha) + X_2\mathcal{T} + e$$

since $X_1 j_1^b = j_1^M$. Now define $(\mu j_1^b + \alpha) = \beta$ and we have

$$Y = X_1\beta + X_2\mathcal{T} + e.$$

The normal equations for this model are

$$(1) \quad X_1'X_1\hat{\beta} + X'_1X_2\hat{\mathcal{T}} = X'_1Y$$

$$(2) \quad X'_2X_1\hat{\beta} + X'_2X_2\hat{\mathcal{T}} = X'_2Y$$

Henceforth when we mention blocks (ignoring treatments), we will mean blocks and the mean (ignoring treatments).

In equation (1), if we ignore treatments, the sum of squares of blocks (ignoring treatments) is given by $\tilde{\beta}'X'_1Y$ where $\tilde{\beta}$ is a solution of the system

$$X'_1X_1\tilde{\beta} = X'_1Y.$$

Solving for $\tilde{\beta}$ we have $\tilde{\beta} = D_1^{-1}X'_1Y$. Thus $\tilde{\beta}X'_1Y = Y'X_1D_1^{-1}X_1'Y$ which is exactly $Y'P_{11}P'_{11}Y$. Hence $Y'P_{11}P'_{11}Y$ is the blocks (ignoring treatments) sum of squares.

Solving for $\hat{\beta}$ in equation (1) and substituting in (2) we obtain

$$(X_2'X_2 - X_2'X_1D_1^{-1}X_1'X_2)\hat{\tau} = (X_2' - X_2'X_1D_1^{-1}X_1')Y$$

or

$$(3) \qquad A'A\hat{\tau} = A'Y.$$

The treatments (eliminating blocks) sum of squares is given by $\tilde{\tau}'A'Y$ where $\tilde{\tau}$ is a solution to (3). Then from (3) we may write

$$Q'A'AQQ'\hat{\tau} = Q'A'Y$$

or

$$DQ'\hat{\tau} = Q'A'Y$$

or

$$\begin{bmatrix} 0 & \phi \\ \phi & D_2 \end{bmatrix} \begin{bmatrix} t^{-1/2}j_t^1 \\ Q_1' \end{bmatrix} \hat{\tau} = \begin{bmatrix} t^{-1/2}j_t^1 A'Y \\ Q_1'A'Y \end{bmatrix}$$

Then

$$(4) \qquad Q_1'\hat{\tau} = D_2^{-1}Q_1'A'Y$$

Since $QQ' = I_t$, $Q_1Q_1' = I_t - t^{-1}J_t^t$. Multiplying each side of (4) on the left by Q_1 and making the above substitution for Q_1Q_1', we have

$$(I_t - t^{-1}J_t^t)\hat{\tau} = Q_1D_2^{-1}Q_1'A'Y.$$

Since $(I_t - t^{-1}J_t^t)$ is not full rank, we add the restriction that $j_t^1\hat{\tau} = 0$.

We then have

$$(5) \qquad \begin{bmatrix} I_t - t^{-1}J_t^t & j_1^t \\ j_t^1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\tau} \\ z \end{bmatrix} = \begin{bmatrix} Q_1D_2^{-1}Q_1'A'Y \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} I_t - t^{-1}J_t^t & j_1^t \\ j_t^1 & 0 \end{bmatrix} \begin{bmatrix} I_t - t^{-1}J_t^t & t^{-1}j_1^t \\ t^{-1}j_t^1 & 0 \end{bmatrix} = \begin{bmatrix} I_t & \phi \\ \phi & 1 \end{bmatrix}$

(5) becomes

$$\begin{bmatrix} \tilde{\tau} \\ z \end{bmatrix} = \begin{bmatrix} I_t - t^{-1}J_t^t & t^{-1}j_1^t \\ t^{-1}j_t^1 & 0 \end{bmatrix} \begin{bmatrix} Q_1 D_2^{-1} Q_1' A'Y \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \tilde{\tau} \\ z \end{bmatrix} = \begin{bmatrix} (I_t - t^{-1}J_t^t)(Q_1 D_2^{-1} Q_1' A'Y) \\ 0 \end{bmatrix}.$$

It is then easily seen that $\tilde{\tau} A'Y = (Y'AQ_1D_2^{-1}Q_1')A'Y$, which is exactly $Y'P_{12}P_{12}'Y$. Therefore $Y'P_{12}P_{12}'Y$ is the treatments (eliminating blocks) sum of squares.

Since $Y'P_{11}P_{11}'Y$ and $Y'P_{12}P_{12}'Y$ are the presupposed quantities, we have that $Y'P_2P_2'Y$ is the intra-block error.

The results of this chapter may be summarized in the following theorem and corollary.

THEOREM. If an Eisenhart Model II is assumed in the general two-way classification with unequal numbers in the cells, then there are at most b + t statistics in a minimal set of sufficient statistics.

COROLLARY. If an Eisenhart Model II is assumed in the general two-way classification, then the intra-block error, the block totals and t - 1 of the treatment totals form a set of sufficient statistics.

CHAPTER IV

THE BALANCED INCOMPLETE BLOCK

In this chapter we will be concerned with finding a set of minimal sufficient statistics in the balanced incomplete block design when an Eisenhart Model II is assumed.

The balanced incomplete block (BIB) design is defined as a design in which t treatments are applied to $b \geq t$ blocks of $k < t$ experimental units. Each treatment appears exactly r times in the design with the treatments arranged such that any two treatments occur together in exactly λ blocks.

The model for the design may be written as a special case of the general two-way classification model. Specifically

$$y_{ijm} = \mu + \beta_i + \tau_j + e_{ijm}$$

where $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, t$; $m = 0, 1, \ldots, n_{ij}$; where n_{ij} is defined as follows:

$$n_{ij} = \begin{cases} 0 & \text{if treatment } j \text{ does not appear in block } i. \\ 1 & \text{if treatment } j \text{ appears in block } i. \end{cases}$$

If $n_{ij} = 0$, the observation y_{ijm} does not exist.

Under Model II, the following assumptions are made:

(1) β_i, τ_j and e_{ijm} are each distributed normally,

(2) $E(e_{ijm}) = 0$ for all i, j, m. $E(e_{ijm}e_{pqr}) = \begin{cases} \sigma^2 & \text{if } i=p, j=q, m=r. \\ 0 & \text{otherwise.} \end{cases}$

(3) $E(\beta_i) = 0$ for all i. $E(\beta_i\beta_p) = \begin{cases} \sigma_1^2 & \text{if } i = p. \\ 0 & \text{otherwise.} \end{cases}$

(4) $E(\tau_j) = 0$ for all j. $E(\tau_j \tau_s) = \begin{cases} \sigma_2^2 & \text{if } j = s, \\ 0 & \text{otherwise.} \end{cases}$

(5) $E(e_{ijm}\beta_s) = 0$ for all i, j, m, and s.

(6) $E(e_{ijm}\tau_p) = 0$ for all i, j, m, and p.

(7) $E(\beta_i \tau_j) = 0$ for all i and j.

(8) μ is a constant.

The following relationships hold in a BIB design:

(1) $\sum_i n_{ij} = r$, (2) $\sum_j n_{ij} = k$, (3) $\sum_i n_{ij}n_{ij'} = \lambda$ $(j \neq j')$

(4) $M = bk = tr$, (5) $\lambda t - \lambda = rk - r$. [6]

The matrix model which fulfills the conditions set forth above may be written as

$$Y = \mu j_1^M + X_1\beta + X_2\tau + e$$

where Y is the vector of M observations and we shall consider the elements ordered according to blocks, then treatments. X_1 and X_2 are M x b and M x t matrices respectively. β, τ, and e are vectors of b, t and M random variables respectively.

The distributional properties can be written in matrix form as follows:

(1) e is distributed as the MVN (ϕ , $\sigma^2 I_M$)

(2) β is distributed as the MVN (ϕ , $\sigma_1^2 I_b$)

(3) τ is distributed as the MVN (ϕ, $\sigma_2^2 I_t$)

(4) $E(e\beta') = \phi$, $E(e\tau')$, $E(\beta\tau') = \phi$.

The following relationships hold for the matrix model:

(1) $X_1'X_1 = kI_b$, (2) $X_2'X_2 = rI_t$, (3) $J_M^M X_1 = kJ_b^M$, (4) $J_b^M X_1' = J_M^M$,

(5) $J_M^M X_2 = r J_t^M$, (6) $J_t^M X_2' = J_M^M$, (7) $NN' = (r - \lambda) I_t + \lambda J_t^t$,

(8) $(X_2' - k^{-1}NX_1')X_2 = A'X_2 = \lambda k^{-1}(tI - J)$, (9) $(X_2' - k^{-1}NX_1')X_1 = \phi$.

We will now develop the operation $\mathcal{I}$ so that when the joint distribution of the elements of the vector Y has been operated on by $\mathcal{I}$, we obtain a set of sufficient statistics which is minimal.

The vector Y is distributed as the multivariate normal with mean $\bar{\mu}$ and covariance matrix Σ, where

$$\bar{\mu} = E(Y) = \mu j_1^M$$

and

$$\Sigma = E(Y - \bar{\mu})(Y - \bar{\mu})' = (X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)$$

The joint density of the elements of Y is then

$$g(Y) = (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp -2^{-1}(Y - \bar{\mu})' \Sigma^{-1}(Y - \bar{\mu}).$$

Consider now the operation $\mathcal{I}$ on g(Y) to be of the form

$$\mathcal{I} g(Y) = (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp -2^{-1}(Y - \bar{\mu})'PP'\Sigma^{-1}PP'(Y - \bar{\mu})$$

where P is an orthogonal M x M matrix to be defined. Let P be partitioned in the following manner: $P = (R_1, R_2, R_3, R_4)$ where the dimensions of R_i $(i = 1, 2, 3, 4)$ are M x 1, M x (b - 1), M x (t - 1) and M x (M - b - t + 1) respectively. We shall now define these four partitions of P so as to conform with the condition of orthogonality.

Let $R_1' = M^{-1/2} j_M^1$ and R_4 be constructed in the same manner as the matrix P of Lemma 1. We then have $R_1'R_1 = 1$ and $R_4'R_4 = I_{M-b-t+1}$.

Consider now the matrix $NN' = (r - \lambda)I + \lambda J$. The characteristic roots of NN' may be found by solving the determinantal equation

$$|NN' - \ell I| = 0$$

for ℓ. The characteristic roots of NN' are then $(r - \lambda)$ and $r + (t - 1)\lambda = rk$ of multiplicities $t - 1$ and 1 respectively. Let Q be an orthogonal $t \times t$ matrix which diagonalizes NN', that is

$$Q'NN'Q = \begin{bmatrix} rk & \phi \\ \phi & (r - \lambda) I_{t-1} \end{bmatrix}.$$

Partition Q into (P_1, P_3) where P_1 and P_3 are of dimension $t \times 1$ and $t \times (t - 1)$ respectively. Then

$$\begin{bmatrix} P_1' \\ P_3' \end{bmatrix} NN' (P_1, P_3) = \begin{bmatrix} rk & \phi \\ \phi & (r - \lambda) I_{t-1} \end{bmatrix} = D_1 \text{ (say)}$$

By Lemma 2, the orthogonal set of rows which diagonalizes $N'N$ and gives the non-zero characteristic roots of $N'N$ is $D_1^{-1/2}Q'N$. Thus

$$(D_1^{-1/2}Q'N)(N'N)(N'QD_1^{-1/2}) = D_1.$$

Since the rank of NN' is t, the rank of $N'N$ is also t. Since $N'N$ is $b \times b$ there will be $b - t$ zero characteristic roots of $N'N$. If by P_2 we denote the matrix which diagonalizes $N'N$, we may write

$$P_2'N'NP_2 = \begin{bmatrix} rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & (r - \lambda) I_{t-1} \end{bmatrix}$$

Partitioning P_2 into (P_{20}, P_{21}, P_{22}) we then have

$$P_2'N'NP_2 = \begin{bmatrix} P_{20}' \\ P_{21}' \\ P'_{22} \end{bmatrix} N'N\,(P'_{20},\ P'_{21},\ P'_{22}) = \begin{bmatrix} rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & (r-\lambda)\, I_{t-1} \end{bmatrix}$$

In subsequent discussions, by Lemma 2, we may make the substitution

$$P_{22}' = (r-\lambda)^{-1/2} P'_3 N.$$

Consider now the matrix $A' = (X_2' - k^{-1}N X_1')$. The orthogonal matrix which diagonalizes NN' will also diagonalize $A'A$, for

$$Q'(rI - k^{-1}NN')Q = rI - k^{-1}D_1$$

where

$$(rI - k^{-1}D_1) = \begin{bmatrix} 0 & \phi \\ \phi & k^{-1}\lambda t\, I_{t-1} \end{bmatrix}.$$

We now define the matrix P which we spoke of when the operation I was discussed. We will define P in the following manner:

$$P' = \begin{bmatrix} M^{-1/2} j_M^1 \\ k^{-1/2} P'_{21} X'_1 \\ k^{-1/2} P'_{22} X_1' \\ (k/\lambda t)^{1/2} P_3' A' \\ P_4' \end{bmatrix} = \begin{bmatrix} M^{-1/2} j_M^1 \\ k^{-1/2} P_{21}' X_1' \\ [k(r-\lambda)]^{-1/2} P'_3 N X_1' \\ (k/\lambda t)^{1/2} P_3' A' \\ P_4' \end{bmatrix}$$

where

$$R'_2 = \begin{bmatrix} k^{-1/2} P'_{21} X'_1 \\ k^{-1/2} P_{22}' X_1' \end{bmatrix} \quad \text{and} \quad R_3' = (k/\lambda t)^{1/2} P_3' A'.$$

and where we have let $P_4' = R'_4$ for consistency of notation. It can be verified that P is an orthogonal matrix.

With this definition of P let us examine the form of $P'\Sigma P$. In Appendix A it is shown that $P'\Sigma P$ assumes the form as shown in TABLE 1.

In order to find $P'\Sigma^{-1}P$, we note that $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$. We also note that if we have a matrix of the form

$$C = \begin{bmatrix} c_1 I_s & c_3 I_s \\ c_3 I_s & c_2 I_s \end{bmatrix} \quad \text{then } C^{-1} = (c_1 c_2 - c_3^2)^{-1} \begin{bmatrix} c_2 I_s & -c_3 I_s \\ -c_3 I_s & c_1 I_s \end{bmatrix}.$$

Using this fact, $P'\Sigma^{-1}P$ is as shown in TABLE II.

Secondly, let us examine the form of $P'(Y - \bar{\mu})$. We then have

$$P'(Y - \bar{\mu}) = \begin{bmatrix} M^{-1/2} j_M^{1}(Y - \mu j_1^M) \\ k^{-1/2} P'_{21} X'_1 (Y - \mu j_1^M) \\ k^{-1/2} P'_{22} X'_1 (Y - \mu j_1^M) \\ (k/\lambda t)^{1/2} P'_3 A'(Y - \mu j_1^M) \\ P'_4 (Y - \mu j_1^M) \end{bmatrix} = \begin{bmatrix} M^{-1/2}(y_{...} - \mu) \\ k^{-1/2} P'_{21} X'_1 Y \\ k^{-1/2} P'_{22} X'_1 Y \\ (k/\lambda t)^{1/2} P'_3 A' Y \\ P'_4 Y \end{bmatrix}$$

where $y_{...} = M^{-1} j_M^{1} Y$.

Letting $q = (Y - \bar{\mu})'PP'\Sigma^{-1}PP'(Y - \bar{\mu})$, we have

$$q = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1} M (y_{...} - \mu)^2 + [k(\sigma^2 + k\sigma_1^2)]^{-1} Y'X_1 P_{21} P'_{21} X'_1 Y$$

$$+ \quad k(\sigma^2 + k^{-1}\lambda t\sigma_2^2)\, d_1^{-1} Y'X_1 P_{22} P'_{22} X'_1 Y \quad + \quad \sigma^{-2} Y'P_4 P'_4 Y$$

$$+ \quad (k/\lambda t)[\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2]\, d_1^{-1} Y'AP_3 P'_3 A'Y$$

$$-2[k^{-2}\lambda t(r-\lambda)]^{1/2} d_1^{-1}\, Y'X_1 P_{22} P'_3 A'Y.$$

TABLE I

FORM OF $P' \Sigma P$

$$
\begin{bmatrix}
(\sigma^2 + k\sigma_1^2 + r\sigma_2^2) & \phi & \phi & \phi & \phi \\
\phi & (\sigma^2 + k\sigma_1^2)\, I_{b-t} & \phi & \phi & \phi \\
\phi & \phi & [\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda)\sigma_2^2]\, I_{t-1} & [k^{-2}\lambda t(r-\lambda)]^{1/2}\sigma_2^2 I_{t-1} & \phi \\
\phi & \phi & [k^{-2}\lambda t(r-\lambda)]^{1/2}\sigma_2^2 I_{t-1} & (\sigma^2 + k^{-1}\lambda t\sigma_2^2)\, I_{t-1} & \phi \\
\phi & \phi & \phi & \phi & \sigma^2 I_{M-b-t+1}
\end{bmatrix}
$$

TABLE II

FORM OF $P'\Sigma^{-1}P$

$$
\begin{bmatrix}
(\sigma^2+k\sigma_1^2+r\sigma_2^2)^{-1} & \phi & \phi & \phi & \phi \\
\phi & (\sigma^2+k\sigma_1^2)^{-1} I_{b-t} & \phi & \phi & \phi \\
\phi & \phi & d_1^{-1}(\sigma^2+k^{-1}\lambda t\sigma_2^2)\, I_{t-1} & -[k^{-2}\lambda t(r-\lambda)]^{1/2} d_1^{-1}\sigma_2^2 I_{t-1} & \phi \\
\phi & \phi & -[k^{-2}\lambda t(r-\lambda)]^{1/2} d_1^{-1}\sigma_2^2 I_{t-1} & [\sigma^2+k\sigma_1^2+k^{-1}(r-\lambda)\sigma_2^2]\, d_1^{-1} I_{t-1} & \phi \\
\phi & \phi & \phi & \phi & \sigma^{-2} I_{M-b-t+1}
\end{bmatrix}
$$

$$d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2$$

where $d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\,\sigma_1^2\sigma_2^2$.

Define now the six statistics s_i $(i = 1, 2, \ldots, 6)$ as follows:

$$s_1 = y_{\cdots}$$

$$s_2 = k^{-1}Y'X_1P_{21}P'_{21}X'_1Y \quad \text{if } b > t. \text{ Not defined if } b = t.$$

$$s_3 = k^{-1}Y'X'_1P_{22}P'_{22}X'_1Y$$

$$s_4 = k^{-1}(r - \lambda)^{1/2}Y'X_1P_{22}P'_3A'Y \quad \text{or} \quad k^{-1}Y'X_1N'P_3P'_3A'Y$$

$$s_5 = (k/\lambda t)Y'AP_3P'_3A'Y$$

$$s_6 = Y'P_4P'_4Y.$$

By definition, these six statistics form a set of sufficient statistics since we have factored $g(Y)$ into the form $\prod_i c_i\, h(s_i)$.

Lehmann and Scheffe' have given a scheme by which a set of sufficient statistics may be shown to be minimal sufficient. It consists of defining a function $K(Y, Y_0) = I\text{-}g(Y)/I\text{-}g(Y_0)$ and finding the condition under which $K(Y, Y_0)$ is independent of parameters. The symbol I- denotes an operation on $g(Y)$ which reduces the dimension of the space of sufficient statistics. In the case we are considering, we will define I to consist of operating on the exponent in $g(Y)$ with the matrix P as we have defined it. A set of sufficient statistics is minimal sufficient when $K(Y, Y_0)$ being independent of parameters, implies $s_i = s_{i0}$, where the s_i are the proposed set of minimal sufficient statistics and the s_{i0} are obtained from $I\text{-}g(Y_0)$ in the same manner as the s_i were obtained from $I\text{-}g(Y)$.

Proceeding with our problem, we have

$$K(Y, Y_o) = \exp -2^{-1}(q - q_o)$$

or

$$= \exp -2^{-1} \sum_i^6 f_i w_i$$

where the f_i are defined as follows:

$$f_1 = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}$$

$$f_2 = (\sigma^2 + k\sigma_1^2)^{-1}$$

$$f_3 = (\sigma^2 + k^{-1}\lambda t\sigma_2^2)d_1^{-1}$$

$$f_4 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2]\, d_1^{-1}$$

$$f_5 = -2\sigma_2^2\, d_1^{-1}$$

$$f_6 = \sigma^{-2}$$

and where $w_i = s_i - s_{io}$ $(i = 2, 3, \ldots, 6)$, $w_1 = M(s_1 - \mu)^2 - M(s_{1o} - \mu)^2$.

The function $K(Y, Y_o)$ will be independent of parameters only if the quantity $(q - q_o)$ is equal to a constant. Since none of the f_i involve a constant, we shall show that the only solution to the equation $\Sigma f_i w_i = 0$ is that the $w_i = 0$ for all i. In Appendix B it is shown that this is the case. Since $w_i = 0$ for all i, this implies $s_i = s_{io}$ $(i = 2, 3, \ldots, 6)$. For the case when i = 1, we have $w_1 = 0$ or $M(s_1 - \mu)^2 = M(s_{1o} - \mu)^2$. Since this is an identity in μ, let $\mu = 0$. This implies $s_1 = s_{1o}$. We therefore have $s_i = s_{io}$ $(i = 1, 2, \ldots, 6)$. Hence these six statistics form a minimal set of sufficient statistics if $b > t$ and a set of five statistics if $b = t$.

The expectations and distributions of the statistics are found in Appendix C. Pairwise independence of the statistics is investigated in

Appendix D.

We shall now examine each of the statistics in the minimal set to ascertain of what each consists in terms of block and treatment totals.

We now examine each statistic in turn.

(1) s_1. This statistic is simply the mean of all observations in the vector Y and is an unbiased estimate of the parameter μ.

(2) $s_3 = [k(r - \lambda)]^{-1} Y'X_1N'P_3P'_3NX'_1 Y$. The quantity NX'_1Y is a $t \times 1$ vector of elements T_j (say), where T_j is the total of all blocks containing treatment j.

For $P_3P'_3$ we may substitute $(I - t^{-1}J)$. Making this substitution, we have

$$\begin{aligned} s_3 &= [k(r - \lambda)]^{-1} Y'X_1N'(I - t^{-1}J)NX'_1Y \\ &= [k(r - \lambda)]^{-1}[Y'X_1N'NX'_1Y - t^{-1}Y'X_1N'JNX'_1Y] \\ &= [k(r - \lambda)]^{-1}[\Sigma T_j^2 - t^{-1}(kY...)^2] \\ &= [k(r - \lambda)]^{-1} \Sigma(T_j^2 - T.)^2 \end{aligned}$$

where $T. = t^{-1}\Sigma T_j$ and $Y... = j_M^1 Y$.

(3) $s_5 = (k/\lambda t)Y'AP_3P'_3A'Y$. Making the substitution $P_3P'_3 = (I - t^{-1}J)$, we obtain $s_5 = (k/\lambda t)Y'A(I - t^{-1}J)A'Y = (k/\lambda t)Y'AA'Y$.

Consider now $A'Y = (X'_2 - k^{-1}NX'_1)Y$. This quantity is a vector of what is conventionally called the Q_j's. We may then write $s_5 = (k/\lambda t)\Sigma Q_j^2$ where $Q_j = V_j - k^{-1}T_j$ with V_j denoting the j-th treatment total.

(4) $s_6 = Y'P_4P'_4 Y$. From the discussion in Chapter II, this statistic

is the intra-block error.

(5) $s_4 = k^{-1}Y'X_1N'P_3P_3'A'Y$. Substituting $(I - t^{-1}J)$ for P_3P_3' we have

$$s_4 = k^{-1}Y'X_1N'(I - t^{-1}J)A'Y = k^{-1}Y'X_1N'A'Y.$$

Since the j-th element of $Y'X_1N'$ is T_j and the j-th element of $A'Y$ is Q_j, this statistic may be written $k^{-1}\Sigma\, T_jQ_j$.

(6) In order to determine what s_2 is in terms of the block and treatment totals, consider

$$k^{-1}Y'X_1X_1'Y = k^{-1}Y'X_1(P_2P_2')X_1'Y = k^{-1}Y'X_1(P_{20}, P_{21}, P_{22})\begin{bmatrix} P_{20}' \\ P_{21}' \\ P_{22}' \end{bmatrix}X_1'Y.$$

We may now let $P_{20}P_{20}' = b^{-1}J_b^b$ since $b^{-1}j_b^{1}N'Nj_1^b = r^2tb^{-1} = rk$ which is a characteristic root of $N'N$ of multiplicity 1. We therefore write

$$k^{-1}Y'X_1X_1'Y - (bk)^{-1}Y'X_1JX_1'Y - k^{-1}Y'X_1P_{22}P_{22}'X_1'Y = k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

or writing this in terms of the block and treatment totals we have

$$k^{-1}\sum_i^b(B_i - B.)^2 - [k(r-\lambda)]^{-1}[\sum_j^t(T_j - T.)^2] = k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

where B_i is the i-th element of $X_1'Y$ and $B. = b^{-1}\Sigma\, B_i$.

The statistic s_2 may be obtained then by subtracting s_3 from the corrected sum of squares of blocks (ignoring treatments).

Summarizing the results of this chapter will be accomplished by means of the following theorem and corollaries.

<u>THEOREM 1. If an Eisenhart Model II is assumed in a balanced incomplete block design, then there are six statistics in a minimal set of sufficient statistics if $b > t$ and there are five statistics in a minimal set if $b = t$.</u>

COROLLARY 1.1. The explicit form of the statistics in a minimal set are as follows:

1. $s_1 = y_{\cdots}$.
2. $s_2 = k^{-1}Y'X_1P_{21}P'_{21}X'_1Y$ if $b>t$. s_2 is not defined if $b = t$.
3. $s_3 = k^{-1}Y'X_1P_{22}P'_{22}X'_1Y$ or $[k(r-\lambda)]^{-1}Y'X_1N'P_3P'_3NX'_1Y$.
4. $s_4 = [k^{-2}(r-\lambda)]^{1/2}Y'X_1P_{22}P'_3A'Y$ or $k^{-1}Y'X_1N'P_3P'_3A'Y$.
5. $s_5 = (k/\lambda t)^{-1}Y'AP_3P'_3A'Y$
6. $s_6 = Y'P_4P'_4Y$.

where $P'_{21}N'NP_{21} = \phi_{b-t}$ and $P'_3NN'P_3 = (r-\lambda)I_{t-1}$.

COROLLARY 1.2. The expectations of each of the statistics as defined in Corollary 1.1 are as follows:

1. $E(s_1) = \mu$
2. $E(s_2) = (b-t)(\sigma^2 + k\sigma_1^2)$ if $b>t$. Not defined if $b = t$.
3. $E(s_3) = (t-1)[\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda)\sigma_2^2]$
4. $E(s_4) = (t-1)k^{-2}(r-\lambda)\lambda t\sigma_2^2$
5. $E(s_5) = (t-1)(\sigma^2 + k^{-1}\lambda t\sigma_2^2)$
6. $E(s_6) = (M-b-t+1)\sigma^2$.

For the proof of this corollary see Appendix C.

COROLLARY 1.3. The distribution of each of the statistics of the minimal set as defined in Corollary 1.1 is as follows:

1. $s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)]$.
2. $s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(b-t)$ if $b>t$. Not defined if $b = t$.

3. $s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2]\ \chi^2(t - 1)$.

4. $s_5 \sim (\sigma^2 + k^{-1}\lambda t\sigma_2^2)\ \chi^2\ (t - 1)$.

5. $s_6 \sim \sigma^2\ \chi^2\ (M - b - t + 1)$

6. s_4 is distributed as a linear combination of independent chi-square variables, that is

$$s_4 \sim \Sigma\ p_i \chi^2(1)$$

where the p_i are the non-zero characteristic roots of $2^{-1}(A_4 + A'_4)\Sigma$ where $A_4 = k^{-1}X_1N'P_3P_3'A'$. [7]

The proof of this corollary appears in Appendix C.

COROLLARY 1.4. The statistics s_i $(i = 1, 2, \ldots, 6)$ are pairwise independent except for the pairs (s_3, s_4), (s_3, s_5) and (s_4, s_5).

The proof of this corollary appears in Appendix D.

COROLLARY 1.5. The six statistics as defined in Corollary 1.1 may be computed from the following Analysis of Variance Table (TABLE III).

TABLE III

ANALYSIS OF VARIANCE, BALANCED INCOMPLETE BLOCK

Source	Statistic
Mean	$My\ldots^2 = Ms_1^2$
Blocks (ignoring treatments)	$k^{-1}\Sigma(B_i - B.)^2$
Block-Treatment-Error Component	$[k(r-\lambda)]^{-1}\Sigma(T_j - T.)^2 = s_3$
Block-Error Component	By subtraction (s_2)
Treatment-Error Component	$(k/\lambda t)\ \Sigma\ Q_j^2 = s_5$
Intra-block Error	By subtraction (s_6)

with $s_4 = k^{-1}\ \Sigma\ T_jQ_j$.

CHAPTER V

GROUP DIVISIBLE, PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE CLASSES

In this chapter we shall consider PBIB designs and shall find sets of minimal sufficient statistics for each of the three types of group divisible designs. We begin the development by stating the definitions which will be needed as the discussion develops.

Definitions

An incomplete block design is said to be partially balanced with two associate classes if

(1) there are b blocks each with k experimental units,

(2) there are $t > k$ treatments, each of which satisfies the following:

(a) each treatment appears exactly r times in all blocks,

(b) each treatment has exactly n_i i-th associates,

(c) two treatments which are i-th associates occur in exactly λ_i blocks,

(3) any pair of treatments satisfy the following:

(a) the pair are either first or second associates,

(b) any pair of treatments which are i-th associates, the number of treatments common to the j-th associate of the first and the k-th associate of the second is p^i_{jk} and is independent of the pair of treatments.

From the above definitions, the following relationships hold:

(1) $bk = tr = M$, (2) $n_1 + n_2 = t - 1$, (3) $n_1\lambda_1 + n_2\lambda_2 = rk - r$.

A group divisible, partially balanced incomplete block design is defined as a design in which the treatments are arranged such that there are n groups of m treatments each, such that any two treatments of the same group occur in exactly λ_1 blocks, while any two treatments which are in different groups occur together in exactly λ_2 blocks.

For the group divisible designs, the following relationships hold:

(1) $t = mn$, (2) $n_1 = m - 1$, (3) $n_2 = m(n - 1)$, (4) $r \geqslant \lambda_1$

(5) $rk - \lambda_2 t \geqslant 0$, (6) $(m - 1)\lambda_1 + m(n - 1)\lambda_2 = r(k - 1)$.

The group divisible, partially balanced designs have been classified into three types by Bose, Clatworthy and Shrikhande [8]. They are

(1) Singular if $r = \lambda_1$,

(2) Semi-Regular if $rk - \lambda_2 t = 0$,

(3) Regular if $r > \lambda_1$ and $rk - \lambda_2 t > 0$.

General Considerations

We shall now examine each of the group divisible designs in order to determine a set of minimal sufficient statistics for each. We begin by discussing some of the general properties of all three types of designs.

We shall assume the same model as in the BIB design with the same distributional properties of the random variables. Explicitly, we have

$$Y = \mu j_1^M + X_1\beta + X_2\tau + e$$

with Y distributed as the multivariate normal, mean $\bar{\mu} = \mu j_1^M$ and covariance matrix $\Sigma = X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I$.

The joint distribution of the elements in the vector Y is

$$g(Y) = (2\pi)^{-M/2}|\Sigma|^{-1/2} \exp -2^{-1}(Y - \bar{\mu})'\Sigma^{-1}(Y - \bar{\mu})$$

In order to be able to define the operation $\mathcal{I}$ on g(Y) we shall first consider the matrix NN'. The jj'-th element of NN' is the number of times that treatment j occurs with treatment j' in all blocks. If we let $n_{ij} = 1$ if treatment j occurs in block i and equal 0 otherwise, the jj'-th element of NN' is equal to $\sum_i n_{ij}n_{ij'}$. For any GD-PBIB design

$$\sum_i n_{ij}n_{ij'} = \begin{cases} r & \text{if } j = j'. \\ \lambda_1 & \text{if } j \neq j' \text{ and } j \text{ and } j' \text{ are in the same group.} \\ \lambda_2 & \text{if } j \neq j' \text{ and } j \text{ and } j' \text{ are in different groups.} \end{cases}$$

Let the elements of the vector Y be ordered such that the matrix NN' assumes the form of the matrix F of Lemma 3. If we let $a = r$, $b = \lambda_2$ and $c = \lambda_1$, the characteristic roots of NN' are as displayed in TABLE IV.

TABLE IV

CHARACTERISTIC ROOTS OF NN' IN GD-PBIB DESIGNS

Multiplicities	Roots
1	rk
m - 1	$rk - \lambda_2 t$
m(n - 1)	$r - \lambda_1$

Imposing the restrictions on the roots for each of the three types of designs, we have the result as given in TABLE V.

TABLE V

CHARACTERISTIC ROOTS OF NN' FOR S, SR AND R GD-PBIB DESIGNS

Multiplicities	Roots	Roots	Roots
1	rk	rk	rk
$m - 1$	$rk - \lambda_2 t$	0	$rk - \lambda_2 t$
$m(n - 1)$	0	$r - \lambda_1$	$r - \lambda_1$

Since NN' is symmetric there exists an orthogonal matrix Q_3 such that $Q_3' NN' Q_3 = D_3$ where D_3 is diagonal with the characteristic roots of NN' displayed on the main diagonal. Partition Q_3 into (P_{30}, P_{31}, P_{32}) where P_{30}, P_{31}, and P_{32} are of dimension $t \times 1$, $t \times (m - 1)$ and $t \times m(n - 1)$ respectively. We then may write

$$\begin{bmatrix} rk & \phi & \phi \\ \phi & (rk - \lambda_2 t)I_{m-1} & \phi \\ \phi & \phi & \phi \end{bmatrix} \text{(S)}$$

$$\begin{bmatrix} P_{30}' \\ P_{31}' \\ P_{32}' \end{bmatrix} NN'(P_{30}, P_{31}, P_{32}) = \begin{bmatrix} rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & (r - \lambda_1)I_{m(n-1)} \end{bmatrix} \text{(SR)}$$

$$\begin{bmatrix} rk & \phi & \phi \\ \phi & (rk - \lambda_2 t)I_{m-1} & \phi \\ \phi & \phi & (r - \lambda_1)I_{m(n-1)} \end{bmatrix} \text{(R)}.$$

Since the non-zero characteristic roots of $N'N$ are equal to the non-zero characteristic roots of NN', there exists an orthogonal matrix Q_2 such that

$$Q'_2N'NQ_2 = \begin{bmatrix} rk & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & D_3 \end{bmatrix}$$

Partitioning Q_2 into (P_{20}, P_{21}, Q_{22}) where the dimensions of P_{20}, P_{21} and Q_{22} are b x 1, b x (b - t) and b x (t - 1) respectively, we may write:

$$\begin{bmatrix} P'_{20} \\ P'_{21} \\ Q'_{22} \end{bmatrix} N'N \ (P_{20}, P_{21}, Q_{22}) = \begin{bmatrix} rk & \phi & \phi \\ \phi & \phi_{b-t} & \phi \\ \phi & \phi & D_3 \end{bmatrix}$$

Consider the (t - 1) x b matrix of orthogonal rows Q'_{22}. Q_{22} is related to the matrix (P_{31}, P_{32}) of the previous discussion as given in Lemma 2. This relationship will be developed now. Partition Q_{22} into (P_{22}, P_{23}) where P_{22} and P_{23} are of dimension b x (m-1) and b x m(n-1) respectively. Then for

(1) S-GD-PBIB designs, $P'_{22} = (rk - \lambda_2 t)^{-1/2} P'_{31} N$,

(2) SR-GD-PBIB designs $P'_{23} = (r - \lambda_1)^{-1/2} P'_{32} N$

(3) R-GD-PBIB designs, the two relationships above hold.

The relationships of the matrices P_{20}, P_{21}, P_{22} and P_{23} to the various characteristic roots of each of the three types of GD-PBIB designs is as shown in TABLE VI.

In addition to the previous discussion, consider the matrix A'A. The orthogonal matrix which diagonalizes NN' also diagonalizes A'A, for

$$Q'_3A'AQ_3 = Q'_3(X'_2 - k^{-1}NX'_1)(X_2 - k^{-1}X_1N')Q_3 = Q'_3(rI - k^{-1}NN')Q_3$$

TABLE VI

RELATIONSHIPS BETWEEN MATRICES AND CHARACTERISTIC ROOTS IN GD-PBIB DESIGNS

$$
\begin{bmatrix} P'_{20} \\ P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} N'N(P_{20}, P_{21}, P_{22}, P_{23}) =
\begin{cases}
\begin{bmatrix} rk & \phi & \phi & \phi \\ \phi & \phi_{b-t} & \phi & \phi \\ \phi & \phi & (rk-\lambda_2 t)I_{m-1} & \phi \\ \phi & \phi & \phi & \phi_{m(n-1)} \end{bmatrix} & (S) \\[2ex]
\begin{bmatrix} rk & \phi & \phi & \phi \\ \phi & \phi_{b-t} & \phi & \phi \\ \phi & \phi & \phi_{m-1} & \phi \\ \phi & \phi & \phi & (r-\lambda_1)I_{m(n-1)} \end{bmatrix} & (SR) \\[2ex]
\begin{bmatrix} rk & \phi & \phi & \phi \\ \phi & \phi_{b-t} & \phi & \phi \\ \phi & \phi & (rk-\lambda_2 t)I_{m-1} & \phi \\ \phi & \phi & \phi & (r-\lambda_1)I_{m(n-1)} \end{bmatrix} & (R)
\end{cases}
$$

$= rI - k^{-1}D_3$. Since rI and $k^{-1}D_3$ are each diagonal, $rI - k^{-1}D_3$ is diagonal with the characteristic roots of A'A displayed on the main diagonal. The characteristic roots of A'A are as shown in TABLE VII.

TABLE VII

CHARACTERISTIC ROOTS OF A'A FOR GD-PBIB DESIGNS

Multiplicities	Roots
1	0
$m - 1$	$k^{-1}\lambda_2 t$
$m(n - 1)$	$k^{-1}[\lambda_2 t + n(\lambda_1 - \lambda_2)]$

Applying the restrictions for each of the three types of GD-PBIB designs, we display the characteristic roots of A'A for each of the three designs in TABLE VIII.

TABLE VIII

CHARACTERISTIC ROOTS OF A'A FOR S, SR, AND R GD-PBIB DESIGNS

Multiplicities	Roots (S)	Roots (SR)	Roots (R)
1	0	0	0
$m - 1$	$k^{-1}\lambda_2 t$	r	$k^{-1}\lambda_2 t$
$m(n - 1)$	r	v	v

Consider now an M x M orthogonal matrix P' defined in the following manner:

$$P' = \begin{bmatrix} R_1' \\ R_2' \\ C_3 R_3' \\ P_4' \end{bmatrix}$$

where we shall let $R_1' = M^{-1/2}j_M^1$, $R_2 = (k^{-1/2}P_{21}'X_1', k^{-1/2}P_{22}'X_1', k^{-1/2}P_{23}'X_1')$

and

$$C_3R_3' = \begin{cases} \begin{bmatrix} (k/\lambda_2 t)^{1/2}P_{31}'A' \\ r^{-1/2}P_{32}'A' \end{bmatrix} & \text{for S-GD-PBIB designs.} \\ \begin{bmatrix} r^{-1/2}P_{31}'A' \\ v^{-1/2}P_{32}'A' \end{bmatrix} & \text{for SR-GD-PBIB designs.} \\ \begin{bmatrix} (k/\lambda_2 t)^{1/2}P_{31}'A' \\ v^{-1/2}P_{32}'A' \end{bmatrix} & \text{for R-GD-PBIB designs.} \end{cases}$$

and P_4' be defined as the matrix P' of Lemma 1.

Consider the operation $\mathrm{I}g(Y)$ to be

$$\mathrm{I}\, g(Y) = (2\pi)^{-M/2}|\Sigma_*|^{-1/2}\exp{-2^{-1}(Y - \bar{\mu})'PP'\Sigma^{-1}PP'(Y - \bar{\mu})}$$

where P is as defined above.

In Appendix A, it is shown that $P'\Sigma P$ is of the form given in TABLE IX for each of the three types of group divisible designs.

In the next sections we shall consider each of the three types of group divisible designs separately using the results of this section.

<u>Singular, Group Divisible, Partially Balanced Incomplete Block Designs</u>

For this type of PBIB, in order to obtain a set of sufficient statistics we shall first examine the form of $P'\Sigma P$. The general form of $P'\Sigma P$ is as shown in TABLE IX.

In the light of the discussion in the previous section, we have the following relationships:

TABLE IX

GENERAL FORM OF $P'\Sigma P$ FOR GD-PBIB DESIGNS

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi \\ \phi & U_{22} & U_{23} & \phi \\ \phi & U_{32} & U_{33} & \phi \\ \phi & \phi & \phi & U_{44} \end{bmatrix}$$

where $U_{11} = \sigma^2 + k\sigma_1^2 + r\sigma_2^2$, $\quad U_{44} = \sigma^2 I_{M-b-t+1}$,

$$U_{22} = (\sigma^2 + k\sigma_1^2)I_{b-1} + \begin{bmatrix} P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} NN'(P_{21}, P_{22}, P_{23})k^{-1}\sigma_2^2$$

$$U_{23} = U'_{32} = k^{-3/2} \begin{bmatrix} P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} N'(rkI - NN')(P_{31}, P_{32})C_3\sigma_2^2$$

$$U_{33} = C_3 \begin{matrix} P'_{31} \\ P'_{32} \end{matrix} [k^{-1}(rkI - NN')](P_{31}, P_{32})C_3\sigma^2 + C_3 \begin{bmatrix} P'_{31} \\ P'_{32} \end{bmatrix} [k^{-1}(rkI - NN')]^2 (P_{31}, P_{32})C_3\sigma_2^2$$

and where C_3 is defined as follows:

$$C_3 = \begin{cases} \begin{bmatrix} [k/\lambda_2 t]^{1/2} I_{m-1} & \phi \\ \phi & r^{-1/2} I_{m(n-1)} \end{bmatrix} & \text{for S designs.} \\ \begin{bmatrix} r^{-1/2} I_{m-1} & \phi \\ \phi & v^{-1/2} I_{m(n-1)} \end{bmatrix} & \text{for SR designs.} \\ \begin{bmatrix} [k/\lambda_2 t]^{1/2} I_{m-1} & \phi \\ \phi & v^{-1/2} I_{m(n-1)} \end{bmatrix} & \text{for R designs.} \end{cases}$$

(1)

$$k^{-1}\begin{bmatrix} P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} N'N(P_{21}, P_{22}, P_{23}) = k^{-1}\begin{bmatrix} \phi_{b-t} & \phi & \phi \\ \phi & (rk-\lambda_2 t)I_{m-1} & \phi \\ \phi & \phi & \phi_{m(n-1)} \end{bmatrix}$$

(2)

$$k^{-3/2}\begin{bmatrix} P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} N'(rkI-NN')(P_{31}, P_{32})C_3 = k^{-3/2}\begin{bmatrix} P'_{21}N' \\ P'_{22}N' \\ P'_{23}N' \end{bmatrix}(rkI-NN')(P_{31}, P_{32})C_3$$

$$= k^{-3/2}\begin{bmatrix} \phi & \phi \\ P'_{22}N'(rkI - NN')P_{31}(k/\lambda_2 t)^{1/2} & P'_{22}N'(rkI - NN')P_{32}\, r^{-1/2} \\ \phi & \phi \end{bmatrix}$$

(3)

$$C_3\begin{bmatrix} P'_{31} \\ P'_{32} \end{bmatrix}[k^{-1}(rkI - NN')](P_{31}, P_{32})C_3 = \begin{bmatrix} I_{m-1} & \phi \\ \phi & I_{m(n-1)} \end{bmatrix}$$

(4)

$$C_3\begin{bmatrix} P'_{31} \\ P'_{32} \end{bmatrix}[k^{-1}(rkI-NN')][k^{-1}(rkI-NN')](P_{31}, P_{32})C_3 = \begin{bmatrix} (\lambda_2 t/k)I_{m-1} & \phi \\ \phi & rI_{m(n-1)} \end{bmatrix}$$

Examining the two non-null matrices in (2) above, we have

(a) $k^{-3/2}P'_{22}N'(rkI - NN')P_{31}(k/\lambda_2 t)^{1/2}$

$$= k^{-3/2}(rk - \lambda_2 t)^{-1/2}(k/\lambda_2 t)^{1/2}P'_{31}NN'(rkI - NN')P_{31}$$

$$= k^{-1}(rk - \lambda_2 t)^{-1/2}(\lambda_2 t)^{-1/2}[rk(rk - \lambda_2 t) - (rk - \lambda_2 t)^2]I_{m-1}$$

$$= k^{-1}(rk - \lambda_2 t)^{1/2}(\lambda_2 t)^{-1/2}(rk - r + \lambda_2 t)\, I_{m-1}$$

$$= k^{-1}(rk - \lambda_2 t)^{1/2}(\lambda_2 t)^{1/2}\, I_{m-1}$$

(b) $k^{-3/2}P'_{22}N'(rkI - NN')P_{32}r^{-1/2}$

$$= k^{-3/2}(rk - \lambda_2 t)^{-1/2}P'_{31}NN'(rkI - NN')P_{32}r^{-1/2}$$

$$= \phi$$

since $P'_{31}NN'P_{32} = \phi$ for this design.

Applying the results of (1), (2), (3) and (4) above to the general form of $P'\Sigma P$, we have the result as given in TABLE X .

TABLE X

FORM OF $P'\Sigma P$ FOR SINGULAR GD-PBIB DESIGNS

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & U_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & U_{33} & \phi & U_{35} & \phi & \phi \\ \phi & \phi & \phi & U_{44} & \phi & \phi & \phi \\ \phi & \phi & U_{53} & \phi & U_{55} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & U_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & U_{77} \end{bmatrix}$$

where

$$U_{11} = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2), \quad U_{22} = (\sigma^2 + k\sigma_1^2)I_{b-t}, \quad U_{44} = (\sigma^2 + k\sigma_1^2)I_{m(n-1)}$$

$$U_{33} = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk-\lambda_2 t)\sigma_2^2]I_{m-1}, \quad U_{55} = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)I_{m-1}$$

$$U_{66} = (\sigma^2 + r\sigma_2^2)I_{m(n-1)}, \qquad U_{77} = \sigma^2 I_{M-b-t+1},$$

$$U_{35} = U'_{53} = [k^{-2}\lambda_2 t(rk - \lambda_2 t)]^{1/2} I_{m-1}\sigma_2^2.$$

We must now determine the form of $P'\Sigma^{-1}P$. To accomplish this we note that $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$. The form of $P'\Sigma^{-1}P$ is given in TABLE XI.

TABLE XI

FORM OF $P'\Sigma^{-1}P$ FOR SINGULAR GD-PBIB DESIGNS

$$\begin{bmatrix} W_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & W_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & W_{33} & \phi & W_{35} & \phi & \phi \\ \phi & \phi & \phi & W_{44} & \phi & \phi & \phi \\ \phi & \phi & W_{53} & \phi & W_{55} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & W_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & W_{77} \end{bmatrix}$$

where

$$W_{11} = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}, \quad W_{22} = (\sigma^2 + k\sigma_1^2)^{-1} I_{b-t},$$

$$W_{33} = d_1^{-1}[\sigma^2 + (\lambda_2 t/k)\sigma_2^2]\, I_{m-1}, \quad W_{44} = (\sigma^2 + k\sigma_1^2)^{-1} I_{m(n-1)},$$

$$W_{55} = d_1^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(rk-\lambda_2 t)\sigma_2^2] I_{m-1}, \quad W_{66} = (\sigma^2 + r\sigma_2^2)^{-1} I_{m(n-1)},$$

$$W_{77} = \sigma^{-2} I_{M-b-t+1}, \quad W_{35} = W_{53}' = -[k^{-2}\lambda_2 t(rk-\lambda_2 t)]^{1/2} d_1^{-1} I_{m-1}\sigma_2^2,$$

with $d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda_2 t\sigma_1^2\sigma_2^2$.

Evaluating $P'(Y - \bar{\mu})$ we have

$$P'(Y - \bar{\mu}) = \begin{bmatrix} M^{1/2}(y_{\cdots} - \mu) \\ k^{-1/2} P_{21}' X_1' Y \\ k^{-1/2} P_{22}' X_1' Y \\ k^{-1/2} P_{23}' X_1' Y \\ (k/\lambda_2 t)^{1/2} P_{31}' A' Y \\ r^{-1/2} P_{32}' A' Y \\ P_4' Y \end{bmatrix}$$

Performing the multiplication $(Y - \mu)'PP'\Sigma^{-1}PP'(Y - \mu) = q$ (say) we have

$$q = M(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}(y_{...} - \mu)^2 + [k(\sigma^2 + k\sigma_1^2)]^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

$$[k(\sigma^2 + k\sigma_1^2)]^{-1}Y'X_1P_{23}P_{23}'X_1'Y + (kd_1)^{-1}(\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)Y'X_1P_{22}P_{22}'X_1'Y$$

$$[r(\sigma^2 + r\sigma_2^2)]^{-1}Y'AP_{32}P_{32}'A'Y - 2d_1^{-1}[k^{-2}(rk - \lambda_2 t)]^{1/2}Y'X_1P_{22}P_{31}'A'Y\sigma_2^2$$

$$+ [(\lambda_2 t/k)d_1]^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\, Y'AP_{31}P_{31}'A'Y + Y'P_4P_4'Y\sigma^{-2}$$

Let $(P_{21}P_{21}' + P_{23}P_{23}') = Q_{21}Q_{21}'$ and define the following seven statistics:

$s_1 = y_{...}$

$s_2 = k^{-1}Y'X_1Q_{21}Q_{21}'X_1'Y$ if $b > m$. Not defined if $b = m$.

$s_3 = k^{-1}Y'X_1P_{22}P_{22}'X_1'Y$

$s_4 = (k/\lambda_2 t)Y'AP_{31}P_{31}'A'Y$

$s_5 = r^{-1}Y'AP_{32}P_{32}'A'Y$

$s_6 = Y'P_4P_4'Y$

$s_7 = [k^{-2}(rk - \lambda_2 t)]^{1/2}\, Y'X_1P_{22}P_{31}'A'Y.$

By definition, these seven statistics are sufficient for the parameters μ, σ^2, σ_1^2 and σ_2^2 and we wish to show that these seven statistics form a minimal set of sufficient statistics.

First, we define the function $K(Y, Y_o) = g(Y)/g(Y_o)$ and find the condition under which K is independent of parameters. The function K in the case we are considering is of the form $\exp\text{-}2^{-1}(q - q_o)$. If we define $w_i = (s_i - s_{io})$ $(i = 2, 3, \ldots, 7)$ and $w_1 = M(s_1 - \mu)^2 - M(s_{1o} - \mu)^2$, then K may be written in the form $\exp\text{-}2^{-1}\Sigma f_i w_i$ where the f_i are functions of the parameters. Since the f_i involve no constant terms, K will be independent of parameters if $\Sigma f_i w_i = 0$. In Appendix B it is shown that the only solution to $\Sigma f_i w_i = 0$ is that $w_i = 0$. This implies that $s_i = s_{io}$ $(i = 2, 3, \ldots, 7)$. For w_1 we have $M(y_{\cdots} - \mu)^2 = M(y_{\cdots 0} - \mu)^2$. Since this is an identity in the parameter μ, we may choose $\mu = 0$. Then this implies $y_{\cdots} = y_{\cdots 0}$. Therefore $s_i = s_{io}$ $(i = 1, 2, \ldots, 7)$. When this condition holds, the s_i's form a set of minimal sufficient statistics.

We now summarize the results for the singular GD-PBIB designs by stating the following theorem and corollaries.

THEOREM 2. If an Eisenhart Model II is assumed in a singular, group divisible, partially balanced incomplete block design with two associate classes, then there are seven statistics in a minimal set of sufficient statistics if $b > m$ and six statistics if $b = m$.

COROLLARY 2.1. The explicit form of a set of minimal sufficient statistics for a S-GD-PBIB design is as follows:

$s_1 = y_{\cdots}$

$s_2 = k^{-1}Y'X_1Q_{21}Q_{21}'X_1'Y$ if $b > m$ and is not defined if $b = m$.

$s_3 = k^{-1}Y'X_1P_{22}P_{22}'X_1'Y$ or $[k(rk - \lambda_2 t)]^{-1}Y'X_1NP_{31}P_{31}'NX_1'Y$.

$s_4 = (k/\lambda_2 t)Y'AP_{31}P_{31}'A'Y$

$s_5 = r^{-1}Y'AP_{32}P_{32}'A'Y$

$s_6 = Y'P_4P_4'Y$

$s_7 = [k^{-2}(rk-\lambda_2 t)]^{1/2}Y'X_1P_{22}P_{31}'A'Y$ or $k^{-1}Y'X_1NP_{31}P_{31}'A'Y$,

where

$$\begin{bmatrix} P_{31}' \\ P_{32}' \end{bmatrix} NN'(P_{31}, P_{32}) = \begin{bmatrix} (rk - \lambda_2 t)I_{m-1} & \phi \\ \phi & \phi_{m(n-1)} \end{bmatrix}.$$

and $Q'_{21}N'NQ_{21} = \phi_{b-m}$.

COROLLARY 2.2. The distributions of the seven statistics as given in Corollary 2.1 are as follows:

$s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)]$

$s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(b - m)$ if $b > m$ and is not defined if $b = m$.

$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\chi^2(m - 1)$

$s_4 \sim (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)\chi^2(m - 1)$

$s_5 \sim (\sigma^2 + r\sigma_2^2)\chi^2[m(n - 1)]$

$s_6 \sim \sigma^2\chi^2(M - b - t + 1)$

$s_7 \sim \Sigma a_i\chi^2(1)$ where the a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A'_7)\Sigma$ where $A_7 = k^{-1}X_1NP_{31}P'_{31}A'$.

For proof of this corollary, see Appendix C.

COROLLARY 2.3. The statistics as defined in Corollary 2.1 are pairwise independent except for the pairs (s_3, s_4), (s_3, s_7) and (s_4, s_7).

For proof of this corollary see Appendix D.

COROLLARY 2.4. The expectations of the seven statistics as defined in Corollary 2.1 are as follows:

$E(s_1) = \mu$

$E(s_2) = (b - m)(\sigma^2 + k\sigma_1^2)$

$E(s_3) = (m - 1)[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]$

$E(s_4) = (m - 1)(\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)$

$E(s_5) = m(n - 1)(\sigma^2 + r\sigma_2^2)$

$E(s_6) = (M - b - t + 1)\sigma^2$

$E(s_7) = k^{-2}(m - 1)(rk - \lambda_2 t)\sigma_2^2$

Semi-Regular, GD-PBIB Designs

For this design we again examine the form of $P'\not\Sigma P$. The general form of $P'\not\Sigma P$ is as given in TABLE IX. In the light of the discussion previously, we have the following relationships:

(1)
$$k^{-1}\begin{bmatrix} P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} N'N\,(P_{21}, P_{22}, P_{23}) = k^{-1}\begin{bmatrix} \phi_{b-t} & \phi & \phi \\ \phi & \phi_{m-1} & \phi \\ \phi & \phi & (r-\lambda_1)I_{m(n-1)} \end{bmatrix}$$

(2)
$$C_3\begin{bmatrix} P'_{31} \\ P'_{32} \end{bmatrix}[k^{-1}(rkI - NN')](P_{31}, P_{32})C_3 = \begin{bmatrix} I_{m-1} & \phi \\ \phi & I_{m(n-1)} \end{bmatrix}$$

(3)
$$C_3\begin{bmatrix} P'_{31} \\ P'_{32} \end{bmatrix}[k^{-1}(rkI-NN')][k^{-1}(rkI-NN')](P_{31}, P_{32})C_3 = \begin{bmatrix} rI_{m-1} & \phi \\ \phi & vI_{m(n-1)} \end{bmatrix}$$

(4)
$$k^{-3/2}\begin{bmatrix} P'_{21} \\ P'_{22} \\ P'_{23} \end{bmatrix} N'(rkI - NN')(P_{31}, P_{32})C_3 = k^{-3/2}\begin{bmatrix} P'_{21}N' \\ P'_{22}N' \\ P'_{23}N' \end{bmatrix}(rkI-NN')(\sqrt{r}P_{31}, \sqrt{v}P_{32})$$

$$= \begin{bmatrix} \phi & \phi \\ \phi & \phi \\ k^{-3/2}P'_{23}N'(rkI - NN')P_{31}r^{-1/2} & k^{-3/2}P'_{23}N'(rkI - NN')P_{32}v^{-1/2} \end{bmatrix}.$$

Examining each of the non-null matrices in the last expression, we have

(a) $k^{-3/2}P'_{23}N'(rkI - NN')P_{31}r^{-1/2}$

$$= k^{-3/2}(r - \lambda_1)^{-1/2} P'_{32} NN'(rkI - NN')P_{31} r^{-1/2}$$

$$= \phi$$

since for this design $P'_{32}NN'P_{31} = P'_{32}NN'NN'P_{31} = \phi$.

(b) $k^{-3/2} P'_{23} N'(rkI - NN')P_{32} v^{-1/2}$

$$= k^{-3/2} v^{-1/2} (r - \lambda_1)^{-1/2} P'_{32} NN'(rkI - NN')P_{32}$$

$$= k^{-3/2} v^{-1/2} (r - \lambda_1)^{-1/2} (rkP'_{32}NN'P_{32} - P'_{32}NN'NN'P_{32})$$

$$= k^{-3/2} v^{-1/2} (r - \lambda_1)^{-1/2} [rk(r - \lambda_1) - (r - \lambda_1)^2] I_{m(n-1)}$$

$$= [k^{-1}v(r - \lambda_1)]^{1/2} I_{m(n-1)}$$

Applying the results of (1), (2), (3) and (4) above to the general form of $P'\Sigma P$ of TABLE IX, we have the result as given in TABLE XII.

TABLE XII

FORM OF $P'\Sigma P$ FOR SEMI-REGULAR GD-PBIB DESIGNS

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & U_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & U_{33} & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & U_{44} & \phi & U_{46} & \phi \\ \phi & \phi & \phi & \phi & U_{55} & \phi & \phi \\ \phi & \phi & \phi & U_{64} & \phi & U_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & U_{77} \end{bmatrix}$$

where

$U_{11} = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$, $U_{22} = (\sigma^2 + k\sigma_1^2)I_{b-t}$, $U_{33} = (\sigma^2 + k\sigma_1^2)I_{m-1}$

$U_{44} = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]I_{m(n-1)}$, $U_{55} = (\sigma^2 + r\sigma_2^2)I_{m-1}$,

$U_{66} = (\sigma^2 + v\sigma_2^2)I_{m(n-1)}$, $U_{77} = \sigma^2 I_{M-b-t+1}$, $U_{46} = [k^{-1}v(r-\lambda_1)]^{1/2} I_{m(n-1)}$

Noting that $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$, it is easily verified that the inverse of the matrix given in TABLE XII is as given in TABLE XIII.

TABLE XIII

FORM OF $P'\Sigma^{-1}P$ FOR SEMI-REGULAR GD-PBIB DESIGNS

$$\begin{bmatrix} W_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & W_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & W_{33} & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & W_{44} & \phi & W_{46} & \phi \\ \phi & \phi & \phi & \phi & W_{55} & \phi & \phi \\ \phi & \phi & \phi & W_{64} & \phi & W_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & W_{77} \end{bmatrix}$$

where

$$W_{11} = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}, \quad W_{22} = (\sigma^2 + k\sigma_1^2)^{-1} I_{b-t},$$

$$W_{33} = (\sigma^2 + k\sigma_1^2)^{-1} I_{m-1}, \quad W_{44} = (\sigma^2 + v\sigma_2^2)\, d_1^{-1} I_{m(n-1)}$$

$$W_{55} = (\sigma^2 + r\sigma_2^2)^{-1} I_{m-1}, \quad W_{66} = [\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda_1)\sigma_2^2] I_{m(n-1)}$$

$$W_{77} = \sigma^{-2} I_{M-b-t+1}, \quad W_{46} = W_{64}' = [k^{-1}(r - \lambda_1)v]^{1/2} d_1^{-1} \sigma_2^2 I_{m(n-1)}$$

and where $d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + kv\sigma^2\sigma_2^2$.

We must now ascertain the form of $P'(Y - \bar{\mu})$. We then have this quantity equal to the following:

$$\begin{bmatrix} M^{1/2}(y_{\cdots} - \mu) \\ k^{-1/2}P_{21}'X_1'Y \\ k^{-1/2}P_{22}'X_1'Y \\ k^{-1/2}P_{23}'X_1'Y \end{bmatrix}$$

$$\begin{bmatrix} r^{-1/2}P'_{31}A'Y \\ v^{-1/2}P'_{32}A'Y \\ P'_4Y \end{bmatrix}$$

Performing the multiplication we have for $(Y - \bar{\mu})'PP'\Sigma^{-1}PP'(Y - \bar{\mu})$

$$q = M(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}(y... - \mu)^2 + [kd_1]^{-1}(\sigma^2 + v\sigma_2^2)Y'X_1P_{23}P'_{23}X'_1Y$$

$$+ [k(\sigma^2 + k\sigma_1^2)]^{-1}Y'X_1(P_{21}P'_{21} + P_{22}P'_{22})X'_1Y + [r(\sigma^2 + r\sigma_2^2)]^{-1}Y'AP_{31}P'_{31}A'Y$$

$$+ (vd_1)^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda_1)\sigma_2^2]Y'AP_{32}P'_{32}A'Y + \sigma^{-2}Y'P_4P'_4Y$$

$$- 2(kd_1)^{-1}(r - \lambda_1)^{1/2}\sigma_2^2 Y'X_1P_{23}P'_{32}A'Y .$$

Let $(P_{21}P'_{21} + P_{22}P'_{22}) = Q_{21}Q'_{21}$. Define now the seven statistics

$$s_1 = y...$$
$$s_2 = k^{-1}Y'X_1Q_{21}Q'_{21}X'_1Y$$
$$s_3 = k^{-1}Y'X_1P_{23}P'_{23}X'_1Y$$
$$s_4 = r^{-1}Y'AP_{31}P'_{31}A'Y$$
$$s_5 = v^{-1}Y'AP_{32}P'_{32}A'Y$$
$$s_6 = Y'P_4P'_4Y$$
$$s_7 = [k^{-2}(r - \lambda_1)]^{1/2}Y'X_1P_{23}P'_{32}A'Y$$

By definition, these seven statistics form a set of sufficient statistics for this design. We wish to now show that these seven statistics form a set of minimal sufficient statistics. Again we define $K(Y, Y_o)$ and find the conditions under which this function is independent of parameters. K in the case we are considering is of the form $\exp -2^{-1}(q - q_o)$. If we define $w_i = (s_i - s_{io})$ $(i = 2, 3, \ldots, 7)$ and $w_1 = M(s_1 - \mu)^2 - M(s_{1o} - \mu)^2$

then we may write K in the form $\exp -2^{-1} \Sigma f_i w_i$ where the f_i are the coefficients of the w_i in K. K will be independent of parameters if $\Sigma f_i w_i = 0$. In Appendix B it is shown that the only solution to $\Sigma f_i w_i = 0$ is that $w_i = 0$ for all i. This in turn implies that $s_i = s_{io}$ $(i = 2, 3, \ldots, 7)$. For w_1 we have $M(s_1 - \mu)^2 = M(s_{1o} - \mu)^2$. Since this is an identity in the parameter μ, we may let $\mu = 0$. We then have $s_1 = s_{1o}$. Therefore, $s_i = s_{io}$ $(i = 1, 2, \ldots, 7)$ and since this is true, we have shown that these seven statistics form a minimal set of sufficient statistics.

The results of this section and the appendices pertaining thereto are summarized in the following theorem and corollaries.

THEOREM 3. In a semi-regular, group divisible, partially balanced incomplete block design with two associate classes, there are seven statistics in a minimal set of sufficient statistics if $b > t - m + 1$ and six statistics in a minimal set if $b = t - m + 1$.

COROLLARY 3.1. The explicit form of the statistics in a minimal set of sufficient statistics in a SR-GD-PBIB design as follows:

$s_1 = y_{\cdot\cdot\cdot}$

$s_2 = k^{-1} Y' X_1 Q_{21} Q'_{21} X'_1 Y$ if $b > t-m+1$. Not defined if $b = t-m+1$.

$s_3 = k^{-1} Y' X_1 P_{23} P'_{23} X'_1 Y$ or $[k(r-\lambda_1)]^{-1} Y' X_1 N' P_{32} P'_{32} N X'_1 Y$

$s_4 = r^{-1} Y' A P_{31} P'_{31} A' Y$

$s_5 = v^{-1} Y' A P_{32} P'_{32} A' Y$

$s_6 = Y' P_4 P'_4 Y$

$s_7 = k^{-1} Y' X_1 N' P_{32} P'_{32} A' Y$

where

$$\begin{bmatrix} P'_{31} \\ P'_{32} \end{bmatrix} NN'(P_{31}, P_{32}) = \begin{bmatrix} rI_{m-1} & \phi \\ \phi & vI_{m(n-1)} \end{bmatrix} \text{ and } Q'_{21}N'NQ_{21} = \phi_{b-t+m-1}.$$

COROLLARY 3.2. The distribution of each of the statistics as given in Corollary 3.1 is as follows:

$s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)]$

$s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(b - t + m - 1)$ if $b > t-m+1$ and is not defined if $b = t - m + 1$.

$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]\chi^2[m(n-1)]$

$s_4 \sim (\sigma^2 + r\sigma_2^2)\chi^2(m - 1)$

$s_5 \sim (\sigma^2 + v\sigma_2^2)\chi^2[m(n-1)]$

$s_6 \sim \sigma^2\chi^2(M - b - t + 1)$

$s_7 \sim \Sigma a_i\chi^2(1)$ where the a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A_7')\Sigma$ where $A_7 = [k(r - \lambda_1)]^{-1}X_1N'P_{32}P_{32}'A'$.

COROLLARY 3.3. The seven statistics as given in Corollary 3.1 are pair-wise independent except for the pairs (s_3, s_5), (s_3, s_7) and (s_5, s_7).

COROLLARY 3.4. The expectations of the seven statistics as given in Corollary 3.1 are as follows:

$E(s_1) = \mu$, $E(s_7) = k^{-1}v(r - \lambda_1)[m(n-1)]\sigma_2^2$

$E(s_2) = (b - t + m - 1)(\sigma^2 + k\sigma_1^2)$

$E(s_3) = [m(n-1)][\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]$

$E(s_4) = (m - 1)(\sigma^2 + r\sigma_2^2)$

$E(s_5) = [m(n-1)](\sigma^2 + v\sigma_2^2)$

$E(s_6) = (M - b - t + 1)\sigma^2$

Regular Group Divisible, Partially Balanced Incomplete Block Designs

In order to develop a set of sufficient statistics which we will test for minimality, we look first at $P'\Sigma P$ for this design. The general form of this matrix is as given in TABLE IX . The restrictions placed on P are as follows:

(1)

$$k^{-1}\begin{bmatrix}P'_{21}\\P'_{22}\\P'_{23}\end{bmatrix} N'N(P_{21}, P_{22}, P_{23}) = k^{-1}\begin{bmatrix}\phi_{b-t} & \phi & \phi\\ \phi & (rk-\lambda_2 t)I_{m-1} & \phi\\ \phi & \phi & (r-\lambda_1)I_{m(n-1)}\end{bmatrix}$$

(2)

$$C_3\begin{bmatrix}P'_{31}\\P'_{32}\end{bmatrix}[k^{-1}(rkI - NN')](P_{31}, P_{32})C_3 = \begin{bmatrix}I_{m-1} & \phi\\ \phi & I_{m(n-1)}\end{bmatrix}$$

(3)

$$C_3\begin{bmatrix}P'_{31}\\P'_{32}\end{bmatrix}[k^{-2}(rkI-NN')(rkI-NN')](P_{31}, P_{32})C_3 = \begin{bmatrix}(\lambda_2 t/k)I_{m-1} & \phi\\ \phi & vI_{m(n-1)}\end{bmatrix}$$

(4)

$$k^{-3/2}\begin{bmatrix}P'_{21}\\P'_{22}\\P'_{23}\end{bmatrix} N'(rkI - NN')(P_{31}, P_{32})C_3 = \begin{bmatrix}\phi & \phi\\ H_{21} & H_{22}\\ H_{31} & H_{32}\end{bmatrix}$$

where

(a)
$$\begin{aligned}H_{21} &= k^{-3/2}(k/\lambda_2 t)^{1/2}P'_{22}N'(rkI - NN')P_{31}\\ &= k^{-3/2}(rk-\lambda_2 t)^{-1/2}(k/\lambda_2 t)^{1/2}(rkP'_{31}NN'P_{31} - P'_{31}NN'NN'P_{31})\\ &= [k^{-2}(rk-\lambda_2 t)(\lambda_2 t)]^{1/2} I_{m-1}\end{aligned}$$

(b)
$$\begin{aligned}H_{22} &= k^{-3/2}P'_{22}N'(rkI - NN')P_{32}v^{-1/2}\\ &= [k^{-3}v^{-1}(rk-\lambda_2 t)^{-1}]^{1/2}P'_{31}NN'(rkI - NN')P_{32}\end{aligned}$$

$$= \phi \text{ since } P_{31}'NN'P_{32} = P_{31}'NN'NN'P_{32} = \phi.$$

$$\text{(c) } H_{31} = k^{-3/2}P_{23}'N'(rkI - NN')P_{31}(k/\lambda_2 t)^{1/2}$$

$$= k^{-3/2}(r - \lambda_1)^{-1/2}P_{32}'NN'(rkI - NN')P_{31}(k/\lambda_2 t)^{1/2}$$

$$= \phi \text{ for the same reason as in (b) above.}$$

$$\text{(d) } H_{32} = k^{-3/2}P_{23}'N'(rkI - NN')P_{32}v^{-1/2}$$

$$= k^{-3/2}(r - \lambda_1)^{-1/2}v^{-1/2}(rkP_{32}'NN'P_{32} - P_{32}'NN'NN'P_{32})$$

$$= k^{-3/2}(r - \lambda_1)^{-1/2}v^{-1/2}[rk(r - \lambda_1) - (r - \lambda_1)^2]I_{m(n-1)}$$

$$= k^{-1}(r - \lambda_1)^{1/2}(rk - r + \lambda_1)^{1/2} I_{m(n-1)}$$

Using the results of (1), (2), (3) and (4) above, $P'\Sigma P$ for this design becomes of the form as shown in TABLE XIV.

We now find the form of $(P'\Sigma P)^{-1} = P'\Sigma^{-1}P$ for this design. The form of this matrix is as shown in TABLE XV. Applying the result of Lemma 5 to the sub-matrix outlined by dotted lines in TABLE XIV will be useful in obtaining $P'\Sigma^{-1}P$ for this design.

We now find the form of $P'(Y - \bar{\mu})$ for this design. We then have

$$P'(Y - \bar{\mu}) = \begin{bmatrix} M^{1/2}(y_{\cdots} - \mu) \\ k^{-1/2}P_{21}'X_1'Y \\ k^{-1/2}P_{22}'X_1'Y \\ k^{-1/2}P_{23}'X_1'Y \\ (k/\lambda_2 t)^{1/2}P_{31}'A'Y \\ v^{-1/2}P_{32}'A'Y \\ P_4'Y \end{bmatrix}$$

TABLE XIV

FORM OF P'ΣP FOR REGULAR, GD-PBIB DESIGNS

$$\begin{bmatrix} U_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & U_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & U_{33} & \phi & U_{35} & \phi & \phi \\ \phi & \phi & \phi & U_{44} & \phi & U_{46} & \phi \\ \phi & \phi & U_{53} & \phi & U_{55} & \phi & \phi \\ \phi & \phi & \phi & U_{64} & \phi & U_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & U_{77} \end{bmatrix}$$

where

$$U_{11} = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$$

$$U_{22} = (\sigma^2 + k\sigma_1^2)\, I_{b-t}$$

$$U_{33} = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\, I_{m-1}$$

$$U_{44} = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]\, I_{m(n-1)}$$

$$U_{55} = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2) I_{m-1}$$

$$U_{66} = (\sigma^2 + v\sigma_2^2)\, I_{m(n-1)}$$

$$U_{77} = \sigma^2\, I_{M-b-t+1}$$

$$U_{35} = U_{53}' = [k^{-2}\lambda_2 t(rk - \lambda_2 t)]^{1/2}\, \sigma_2^2\, I_{m-1}$$

$$U_{46} = U_{64}' = [k^{-1}v(r - \lambda_1)]^{1/2}\, \sigma_2^2\, I_{m(n-1)}$$

TABLE XV

FORM OF $P \Sigma^{-1} P$ FOR REGULAR, GD - PBIB DESIGNS

$$\begin{bmatrix} W_{11} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & W_{22} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & W_{33} & \phi & W_{35} & \phi & \phi \\ \phi & \phi & \phi & W_{44} & \phi & W_{46} & \phi \\ \phi & \phi & W_{53} & \phi & W_{55} & \phi & \phi \\ \phi & \phi & \phi & W_{64} & \phi & W_{66} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & W_{77} \end{bmatrix}$$

where

$$W_{11} = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}$$

$$W_{22} = (\sigma^2 + k\sigma_1^2)\, I_{b-t}$$

$$W_{33} = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2) d_1^{-1}\, I_{m-1}$$

$$W_{44} = (\sigma^2 + v\sigma_2^2) d_2^{-1}\, I_{m(n-1)}$$

$$W_{55} = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2] d_1^{-1}\, I_{m-1}$$

$$W_{66} = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2] d_2^{-1}\, I_{m(n-1)}$$

$$W_{77} = \sigma^{-2}\, I_{M-b-t+1}$$

$$W_{35} = -[k^{-2}\lambda_2 t(rk - \lambda_2 t)]^{1/2} d_1^{-1} \sigma_2^2\, I_{m-1}$$

$$W_{46} = -[k^{-1} v(r - \lambda_1)]^{1/2} d_2^{-1} \sigma_2^2\, I_{m(n-1)}$$

with

$$d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda_2 t\sigma_1^2\sigma_2^2$$

$$d_2 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + kv\sigma_1^2\sigma_2^2$$

Performing the matrix multiplication, we now have

$$q = (Y - \bar{\mu})'PP'\Sigma PP'(Y - \bar{\mu}) = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}M(y_{\cdots} - \mu)^2$$

$$+ [k(\sigma^2 + k\sigma_1^2)]^{-1}Y'X_1P_{21}P_{21}'X_1'Y + (kd_1)^{-1}(\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)Y'X_1P_{22}P_{22}'X_1'Y$$

$$+ (\lambda_2 t d_1 k^{-1})^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\ Y'AP_{31}P_{31}'A'Y$$

$$+ [(r - \lambda_1)d_2]^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]\ Y'AP_{32}P_{32}'A'Y$$

$$+ \sigma^{-2}Y'P_4P_4'Y \quad -2(kd_1)^{-1}(rk - \lambda_2 t)^{1/2}Y'X_1P_{22}P_{31}'A'Y\ \sigma_2^2$$

$$- 2(kd_2)^{-1}(r - \lambda_1)^{1/2}Y'X_1P_{23}P_{32}'A'Y\ \sigma_2^2$$

We now define the nine statistics as follows:

$$s_1 = y_{\cdots}$$

$$s_2 = k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$$

$$s_3 = k^{-1}Y'X_1P_{22}P_{22}'X_1'Y$$

$$s_4 = k^{-1}Y'X_1P_{23}P_{23}'X_1'Y$$

$$s_5 = (k/\lambda_2 t)\ Y'AP_{31}P_{31}'A'Y$$

$$s_6 = v^{-1}Y'AP_{32}P_{32}'A'Y$$

$$s_7 = Y'P_4P_4'Y$$

$$s_8 = k^{-1}(rk - \lambda_2 t)^{1/2}Y'X_1P_{22}P_{31}'A'Y$$

$$s_9 = k^{-1}(r - \lambda_1)^{1/2}Y'X_1P_{23}P_{32}'A'Y.$$

By definition these nine statistics form a set of sufficient statistics for this design. We will now show that this is a set of minimal sufficient statistics.

Following the procedure in the three previous derivations, we define $K(Y, Y_o) = \frac{1}{4}g(Y) / \frac{1}{4}g(Y_o)$ and find the condition under which $K(Y, Y_o)$ is independent of parameters. We may write K in the form $\exp -2^{-1}(q - q_o)$ or $\exp -2^{-1} \Sigma f_i w_i$ where $w_i = (s_i - s_{io})$ $(i = 2, 3, \ldots, 9)$ with w_1 defined to be $M(y_{\cdots} - \mu)^2 - M(y_{\cdots o} - \mu)^2$ and with the f_i to be the coefficients of the w_i in the exponent of K. In Appendix B it is shown that the only condition under which $\Sigma f_i w_i = 0$ is that the $w_i = 0$. This implies $s_i = s_{io}$ $(i = 2, 3, \ldots, 9)$. For $w_1 = 0$, we have, by letting $\mu = 0$, $s_1 = s_{1o}$. Therefore, $s_i = s_{io}$ $(i = 1, 2, \ldots, 9)$. When this condition holds, the s_i form a set of minimal sufficient statistics. The results of this section and of the appendices pertaining thereto are summarized in the following theorem and corollaries.

THEOREM 4. Under the assumption of an Eisenhart Model II in a regular, group divisible, partially balanced incomplete block design with two associate classes, there are nine statistics in a minimal set of sufficient statistics if $b > t$ and eight statistics in a minimal set if $b = t$.

COROLLARY 4.1. A set of minimal sufficient statistics for a regular, group divisible, partially balanced incomplete block design is as follows:

$$s_1 = y_{\cdots}$$

$$s_2 = k^{-1} Y'X_1P_{21}P'_{21}X'_1Y$$

$$s_3 = k^{-1} Y'X_1P_{22}P'_{22}X'_1Y \text{ or } [k(rk - \lambda_2 t)]^{-1} Y'X_1N'P_{31}P'_{31}NX'_1Y$$

$$s_4 = k^{-1} Y'X_1P_{23}P'_{23}X'_1Y \text{ or } [k(r - \lambda_1)]^{-1} Y'X_1N'P_{32}P'_{32}NX'_1Y$$

$s_5 = k(\lambda_2 t)^{-1}\, Y'AP_{31}P'_{31}A'Y$

$s_6 = v^{-1} Y'AP_{32}P'_{32}A'Y$

$s_7 = Y'P_4P'_4Y$

$s_8 = k^{-1}(rk - \lambda_2 t)^{1/2} Y'X_1P_{22}P'_{31}A'Y$ or $k^{-1}Y'X_1N'P_{31}P'_{31}A'Y$

$s_9 = k^{-1}(r - \lambda)^{1/2} Y'X_1P_{23}P'_{32}A'Y$ or $k^{-1}Y'X_1N'P_{32}P'_{32}A'Y$.

COROLLARY 4.2. The distributions of the nine statistics as defined in Corollary 4.1 are as follows:

$s_1 \sim N[\,\mu\,,\ M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)\,]$

$s_2 \sim (\sigma^2 + k\sigma_1^2)\,\chi^2\,(b - t)$ if $b > t$. Not defined if $b = t$.

$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2\,]\,\chi^2(m - 1)$

$s_4 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2\,]\,\chi^2\,[m(n - 1)]$

$s_5 \sim (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)\,\chi^2\,(m - 1)$

$s_6 \sim (\sigma^2 + v\sigma_2^2\,)\,\chi^2[m(n - 1)]$

$s_7 \sim \sigma^2\chi^2\,(M - b - t + 1)$

$s_8 \sim \Sigma\, a_i\,\chi^2(1)$ where the a_i are the non-zero characteristic roots of $2^{-1}(A_8 + A'_8)\Sigma$ where $A_8 = k^{-1}X_1N'P_{31}P'_{31}A'$.

$s_9 \sim \Sigma\, b_i\chi^2(1)$ where the b_i are the non-zero characteristic roots of $2^{-1}(A_9 + A'_9\,)\,\Sigma$ where $A_9 = k^{-1}X_1N'P_{32}P'_{32}A'$. (Proved in Appendix C)

<u>COROLLARY 4.3. The nine statistics as defined in Corollary 4.1 are pairwise independent except for the pairs (s_3, s_8), (s_4, s_6), (s_4, s_9), (s_3, s_5) (s_5, s_8) and (s_6, s_9).</u> (Proved in Appendix D)

<u>COROLLARY 4.4. The expectations of the nine statistics as defined in Corollary 4.1 are as follows:</u>

$$E(s_1) = \mu$$

$$E(s_2) = (b - t)(\sigma^2 + k\sigma_1^2) \text{ if } b > t. \text{ Not defined if } b = t.$$

$$E(s_3) = (m - 1)[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]$$

$$E(s_4) = [m(n - 1)][\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]$$

$$E(s_5) = (m - 1)(\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)$$

$$E(s_6) = [m(n - 1)](\sigma^2 + v\sigma_2^2)$$

$$E(s_7) = (M - b - t + 1)\sigma^2$$

$$E(s_8) = k^{-2}(m - 1)\lambda_2 t(rk - \lambda_2 t)\sigma_2^2$$

$$E(s_9) = k^{-2}[m(n - 1)](r - \lambda_1)(rk - r + \lambda_1)\sigma_2^2$$

(Proved in Appendix C)

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APPENDIX A

EVALUATION OF $P'\Sigma P$

The Balanced Incomplete Block Design

With P as defined in Chapter IV, we will evaluate $P'\Sigma P$ for the balanced incomplete block design.

Letting $P'\Sigma P = (A_{ij});\ i, j = 1, 2, \ldots, 5,$ we then have evaluating A_{ij} for each i and j, the following:

(1) $A_{11} = M^{-1} j_M^1 \Sigma j_1^M = M^{-1} j_M^1 (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) j_1^M$

$= M^{-1}(bk^2\sigma_1^2 + tr^2\sigma_2^2 + M\sigma^2) = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$

(2) $A_{12} = k^{-1/2} M^{-1/2} j_M^1 \Sigma X_1 P_{21} = c_o j_M^1 (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) X_1 P_{21}$

$= c_o(\sigma^2 + k\sigma_1^2 + r\sigma_2^2) k j_b^1 P_{21} = \phi$

(3) $A_{13} = k^{-1/2} M^{-1/2} j_M^1 \Sigma X_1 P_{22} = c_o(\sigma^2 + k\sigma_1^2 + r\sigma_2^2) k\, j_b^1 P_{22} = \phi$

(4) $A_{14} = (k/\lambda t)^{1/2} M^{-1/2} j_M^1 \Sigma A P_3 = c_o(\sigma^2 + k\sigma_1^2 + r\sigma_2^2) j_M^1 A P_3 = \phi$

(5) $A_{15} = M^{-1/2} j_M^1 \Sigma P_4 = \phi$

(6) $A_{22} = k^{-1} P_{21}' X_1' \Sigma X_1 P_{21} = k^{-1} P_{21}' X_1' (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) X_1 P_{21}$

$= k^{-1} P_{21}' (k^2 \sigma_1^2 I_b + k\sigma^2 I_b + N'N\sigma_2^2) P_{21} = (\sigma^2 + k\sigma_1^2)\, I_{b-t}$

(7) $A_{23} = k^{-1} P_{21}' X_1' \Sigma X_1 P_{22} = k^{-1} P_{21}' X_1' (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) X_1 P_{22}$

$= k^{-1} P_{21}' (k^2 \sigma_1^2 I_b + k\sigma^2 I_b + N'N\sigma_2^2) P_{22} = \phi$

(8) $A_{24} = (\lambda t)^{-1/2} P'_{21} X'_1 \Sigma A P_3 = c_o P'_{21} X'_1 (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) A P_3$

$= c_o P'_{21} (k X'_1 \sigma_1^2 + N' X'_2 \sigma_2^2 + X'_1 \sigma^2) A P_3 = \phi$

(9) $A_{25} = k^{-1/2} P'_{21} X'_1 \Sigma P_4 = k^{-1/2} P'_{21} X'_1 (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) P_4 = \phi$

(10) $A_{33} = k^{-1} P'_{22} X'_1 \Sigma X_1 P_{22} = k^{-1} P'_{22} X'_1 (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) X_1 P_{22}$

$= k^{-1} P'_{22} (k^2 \sigma_1^2 I_b + N' N \sigma_2^2 + k \sigma^2 I_b) P_{22}$

Substituting $(r - \lambda)^{-1/2} P'_3 N$ for P'_{22} we have

$= (\sigma^2 + k\sigma_1^2) I_{t-1} + k^{-1} (r - \lambda)^{-1} P'_3 N N' N N' P_3$

$= [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2]\ I_{t-1}$

(11) $A_{34} = (\lambda t)^{-1/2} P'_{22} X'_1 \Sigma A P_3 = (\lambda t)^{-1/2} P'_{22} X'_1 (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) A P_3$

$= (\lambda t)^{-1/2} P'_{22} N' X'_2 A P_3\ \sigma_2^2$

Substituting for P'_{22} as in (10) we have

$= [\lambda t (r - \lambda)]^{-1/2} P'_3 N N' (rI - k^{-1} N N') P_3 \sigma_2^2$

$= [\lambda t (r - \lambda)]^{-1/2} [r(r - \lambda) - k^{-1}(r - \lambda)^2]\ \sigma_2^2\ I_{t-1}$

$= [k^{-2} \lambda t (r - \lambda)]^{1/2}\ \sigma_2^2\ I_{t-1}$

(12) $A_{35} = k^{-1/2} P'_{22} X'_1 \Sigma P_4 = \phi$

(13) $A_{44} = (k/\lambda t) P'_3 A' \Sigma A P_3 = (k/\lambda t) P'_3 A' (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) A P_3$

$= (k/\lambda t) P'_3 A' (X_2 X'_2 \sigma_2^2 + \sigma^2 I) A P_3$

$= (k/\lambda t) P'_3 (A' X_2 X'_2 A\ \sigma_2^2 + A' A \sigma^2) P_3 = (\sigma^2 + k^{-1} \lambda t \sigma_2^2)\ I_{t-1}$

(14) $A_{45} = (k/\lambda t)^{-1/2} P'_3 A' \Sigma P_4 = \phi$

(15) $A_{55} = P'_4 \Sigma P_4 = P'_4 (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) P_4 = \sigma^2 I_{M-b-t+1}$

The Group Divisible, Partially Balanced Incomplete Block Designs

$$\begin{bmatrix} M^{-1/2} j_M^1 \\ k^{-1/2} P_2' X_1' \\ C_3 P_3' A' \\ P_4' \end{bmatrix} [\, X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I \,] \begin{bmatrix} M^{-1/2} j_M^1 \\ k^{-1/2} P_2' X_1' \\ C_3 P_3' A' \\ P_4' \end{bmatrix}'$$

The product given above is the general form of the product $P'\Sigma P$ for the group divisible, partially balanced incomplete block designs. The result of performing the above multiplication will result in a matrix which has as elements, sixteen blocks of matrices. Letting A_{ij} denote the block in the i-th row and the j-th column and evaluating those blocks which are above the diagonal and including the diagonal elements we have:

(1) The matrices A_{12}, A_{13}, A_{14}, A_{24} and A_{34} are all equal to a null matrix.

(2) $A_{11} = M^{-1} j_M^1 \Sigma j_1^M = M^{-1} j_M^1 (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) j_1^M$

$= (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$

(3) $A_{22} = k^{-1} P_2' X_1' \Sigma X_1 P_2 = k^{-1} P_2' X_1' (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) X_1 P_2$

$= (\sigma^2 + k\sigma_1^2) I_{b-1} + k^{-1} P_2' N' N P_2 \sigma_2^2$

(4) $A_{23} = k^{-1/2} P_2' X_1' \Sigma A P_3 C_3 = k^{-1/2} P_2' X_1' (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) A P_3 C_3$

$= k^{-1/2} P_2' N' (rI - k^{-1} N N') P_3 C_3 \sigma_2^2$

(5) $A_{33} = C_3 P_3' A' \Sigma A P_3 C_3 = C_3 P_3' A' (X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I) A P_3 C_3$

$= C_3 P_3' A' (X_2 X_2' \sigma_2^2 + \sigma^2 I) A P_3 C_3$

$$= C_3P_3'[(X_2X_2' - k^{-1}NN')(X_2X_2' - k^{-1}NN')\sigma_2^2 + (X_2X_2' - k^{-1}NN')\sigma^2]P_3C_3$$

$$= C_3P_3'[(rI - k^{-1}NN')(rI - k^{-1}NN')\sigma_2^2 + (rI - k^{-1}NN')\sigma^2]P_3C_3$$

(6) $$A_{44} = P_4' \Sigma P_4 = P_4'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I_M)$$

$$= \sigma^2 I_{M-b-t+1}.$$

APPENDIX B

PROOF THAT $w_i = 0$ IN THE BIB AND GD-PBIB DESIGNS

The Balanced Incomplete Block Design

Now we shall show that the only solution to $\Sigma f_i w_i$ is $w_i = 0$ where the f_i are as defined below:

$$f_1 = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}, \quad f_2 = (\sigma^2 + k\sigma_1^2)^{-1}, \quad f_3 = (\sigma^2 + k^{-1}\lambda t\sigma_2^2)d_1^{-1},$$

$$f_4 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2]d_1^{-1}, \quad f_5 = \sigma_2^2 d_1^{-1}, \quad f_6 = \sigma^{-2},$$

where $d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2$ and where the -2 on the coefficient of f_5 has been omitted since it will have no bearing on this proof.

In order to show that this condition holds, we shall find the lowest common denominator of the six functions and subsequently the numerators of each of the functions. Then if we can select a 6 x 6 determinant of coefficients of like terms which is non-vanishing, we shall have shown that the only condition for which $\Sigma f_i w_i = 0$ is that the w_i be identically zero.

In order to simplify the algebra somewhat, we shall let

(1) $x_1 = \sigma^2, \quad x_2 = \sigma_1^2, \quad x_3 = \sigma_2^2,$

(2) $c_1 = k, \quad c_2 = r, \quad c_3 = k^{-1}\lambda t, \quad c_4 = k^{-1}(r - \lambda), \quad c_5 = \lambda t.$

The six functions then become:

$$f_1 = (x_1 + c_1x_2 + c_2x_3)^{-1}, \quad f_2 = (x_1 + c_1x_2)^{-1}, \quad f_6 = x_1^{-1},$$

$$f_3 = (x_1 + c_3x_3)(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3)^{-1},$$

$$f_4 = (x_1 + c_1x_2 + c_4x_3)(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3)^{-1},$$

$$f_5 = x_3(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3)^{-1}.$$

The lowest common denominator is:

$$x_1(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3).$$

with the numerators of the six functions as follows:

(1) f_1: $x_1(x_1 + c_1x_2)(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3)$

$$= x_1^4 + c_2x_1^3x_3 + 2c_1x_1^3x_2 + (c_5 + c_1c_2)x_1^2x_2x_3 + c_1^2x_1^2x_2^2 + c_1c_5x_1x_2^2x_3$$

(2) f_2: $(x_1 + c_1x_2 + c_2x_3)(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3)(x_1)$

$$= x_1^4 + 2c_2x_1^3x_3 + 2c_1x_1^3x_2 + (c_5 + 2c_1c_2)x_1^2x_2x_3 + c_1^2x_1^2x_2^2 + c_1c_5x_1x_2^2x_3$$
$$+ c_2^2x_1^2x_3^2 + c_2c_5x_1x_2x_3^2$$

(3) f_3: $x_1(x_1 + c_3x_3)(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)$

$$= x_1^4 + 2c_1x_1^3x_2 + (c_2 + c_3)x_1^3x_3 + c_1(c_2 + 2c_3)x_1^2x_2x_3 + c_2c_3x_1^2x_3^2$$
$$+ c_1^2x_1^2x_2^2 + c_1^2c_3x_1x_2^2x_3 + c_1c_2c_3x_1x_2x_3^2$$

(4) f_4: $x_1(x_1 + c_1x_2 + c_4x_3)(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)$

$$= x_1^4 + 3c_1x_1^3x_2 + (c_2 + c_4)x_1^3x_3 + 3c_1^2x_1^2x_2^2 + 2c_1(c_2 + c_4)x_1^2x_2x_3$$
$$+ c_2c_4x_1^2x_3^2 + c_1^3x_1x_2^3 + c_1^2(c_2 + c_4)x_1x_2^2x_3 + c_1c_2c_4x_1^2x_2x_3$$

(5) f_5: $x_1x_3(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)$

$$= x_1^3x_3 + 2c_1x_1^2x_2x_3 + c_2x_1^2x_3^2 + c_1^2x_1x_2^2x_3 + c_1c_2x_1x_2x_3^2$$

(6) f_6: $(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1^2 + c_2x_1x_3 + c_1x_1x_2 + c_5x_2x_3)$

$$= x_1^4 + 3c_1x_1^3x_2 + 2c_2x_1^3x_3 + 3c_1^2x_1^2x_2^2 + (4c_1c_2 + c_5)x_1^2x_2x_3 + c_2^2x_1^2x_3^2$$

$$+ 2c_1(c_1c_2 + c_5)x_1x_2^2x_3 + c_2(c_1c_2 + c_5)x_1x_2x_3^2 + c_1^3x_1x_2^3 + c_1^2c_5x_2^3x_3$$

$$+ c_1c_2c_5x_2^2x_3^2.$$

The results in the previous paragraph may be best represented by the following table:

Row	Term	f_1	f_2	f_3	f_4	f_5	f_6
1	x_1^4	1	1	1	1	0	1
2	$x_1^3x_3$	c_2	$2c_2$	c_2+c_3	c_2+c_4	1	$2c_2$
3	$x_1^3x_2$	$2c_1$	$2c_1$	$2c_1$	$3c_1$	0	$3c_1$
4	$x_1^2x_2x_3$	$c_5+c_1c_2$	$c_5+2c_1c_2$	$c_1(c_2+2c_3)$	$2c_1(c_2+c_4)$	$2c_1$	$4c_1c_2+c_5$
5	$x_1^2x_2^2$	c_1^2	c_1^2	c_1^2	$3c_1^2$	0	$3c_1^2$
6	$x_1x_2^2x_3$	c_1c_5	c_1c_5	$c_1^2c_3$	$c_1^2(c_2+c_4)$	c_1^2	$2c_1(c_1c_2+c_5)$
7	$x_1^2x_3^2$	0	c_2^2	c_2c_3	c_2c_4	c_2	c_2^2
8	$x_1x_2x_3^2$	0	c_2c_5	$c_1c_2c_3$	$c_1c_2c_4$	c_1c_2	$c_2(c_1c_2+c_5)$
9	$x_1x_2^3$	0	0	0	c_1^3	0	c_1^3
10	$x_2^3x_3$	0	0	0	0	0	$c_1^2c_5$
11	$x_2^2x_3^2$	0	0	0	0	0	$c_1c_2c_5$

We must now select six rows from the above array and show that the determinant does not vanish. It can be shown that the determinant formed by the rows 2, 3, 4, 6, 7 and 8 is equal to (ignoring the sign) $c_1^2 c_2^3 c_5^2 (c_1 c_2 - c_5)$ which in terms of the constants of the model is equal to $k^2 r^3 \lambda^2 t^2 (r - \lambda)$ which is non-zero for all but degenerate cases.

We therefore conclude that in order for $\Sigma f_i w_i = 0$, each of the w_i must be identically zero, which was to be shown.

Singular, Group Divisible, Partially Balanced Incomplete Block Designs

In this section of this appendix we shall prove that the only solution to the equation $\Sigma f_i w_i = 0$ is that w_i be identically equal to zero for the S-GD-PBIB designs.

The seven f_i are as follows:

$f_1 = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}$, $f_2 = (\sigma^2 + k\sigma_1^2)^{-1}$, $f_3 = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)d_1^{-1}$,

$f_4 = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]d_1^{-1}$, $f_5 = (\sigma^2 + r\sigma_2^2)^{-1}$, $f_6 = \sigma^{-2}$,

$f_7 = \sigma_2^2 d_1^{-1}$, where $d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda_2 t\sigma_1^2\sigma_2^2$ and where we have ignored the coefficient -2 of f_7 as it will not affect the result of this section.

In order that the algebra may be handled more easily in the ensuing discussion, we shall let

(1) $x_1 = \sigma^2$, $x_2 = \sigma_1^2$, $x_3 = \sigma_2^2$,

(2) $c_1 = k$, $c_2 = r$, $c_3 = k^{-1}\lambda_2 t$, $c_4 = k^{-1}(rk - \lambda_2 t)$, $c_5 = \lambda_2 t$.

The seven functions in this notation are:

$f_1 = (x_1 + c_1 x_2 + c_2 x_3)^{-1}$, $f_2 = (x_1 + c_1 x_2)^{-1}$, $f_5 = (x_1 + c_2 x_3)^{-1}$,

$f_3 = (x_1 + c_3x_3)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)^{-1}$, $f_6 = x_1^{-1}$,

$f_4 = (x_1 + c_1x_2 + c_4x_3)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)^{-1}$, $f_7 = x_3 d_1^{-1}$

The lowest common denominator of these seven functions is:

$$x_1(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)(x_1 + c_2x_3).$$

The numerators of the seven functions are then:

(1) f_1: $x_1(x_1 + c_1x_2)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)(x_1 + c_2x_3)$

$$= x_1^5 + 2c_1x_1^4x_2 + (c_5 + 3c_1c_2)x_1^3x_2x_3 + 2c_2x_1^4x_3 + c_1^2x_1^3x_2^2 + c_2^2x_1^3x_3^2$$

$$+ c_1(c_5 + c_1c_2)x_1^2x_2^2x_3 + c_2(c_5 + c_1c_2)x_1^2x_2x_3^2 + c_1c_2c_5x_1x_2^2x_3^2$$

(2) f_2: $x_1(x_1 + c_1x_2 + c_2x_3)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)(x_1 + c_2x_3)$

$$= x_1^5 + 2c_1x_1^4x_2 + (c_5 + 4c_1c_2)x_1^3x_2x_3 + 3c_2x_1^4x_3 + c_1^2x_1^3x_2^2 + 3c_2^2x_1^3x_3^2$$

$$+ c_1(c_5 + c_1c_2)x_1^2x_2^2x_3 + 2c_2(c_5 + c_1c_2)x_1^2x_2x_3^2 + c_1c_2c_5x_1x_2^2x_3^2$$

$$+ c_2^2c_5x_1x_2x_3^3 + c_2^3x_1^2x_3^3$$

(3) f_3: $x_1(x_1 + c_3x_3)(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1 + c_2x_3)$

$$= x_1^5 + 2c_1x_1^4x_2 + (2c_2 + c_3)x_1^4x_3 + c_1(3c_2 + 2c_3)x_1^3x_2x_3 + c_2(c_2 + 2c_3)x_1^3x_3^2$$

$$+ c_1^2x_1^3x_2^2 + c_1^2(c_2 + c_3)x_1^2x_2^2x_3 + c_1c_2(c_2 + 3c_3)x_1^2x_2x_3^2 + c_2^2c_3x_1^2x_3^3$$

$$+ c_1^2c_2c_3x_1x_2^2x_3^2 + c_1c_2^2c_3x_1x_2x_3^3$$

(4) f_4: $x_1(x_1 + c_1x_2 + c_4x_3)(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1 + c_2x_3)$

$$= x_1^5 + 3c_1x_1^4x_2 + (2c_2 + c_4)x_1^4x_3 + 3c_1^2x_1^3x_2^2 + c_1(5c_1 + 2c_4)x_1^3x_2x_3$$

$$+ c_2(c_2 + 2c_4)x_1^3x_3^2 + c_1^3x_1^2x_2^3 + c_1^2(4c_1 + c_4)x_1^2x_2^2x_3 + c_1c_2(2c_2 + 3c_4)x_1^2x_2x_3^2$$

$$+ c_2^2c_4x_1^2x_3^3 + c_1^3c_2x_1x_2^3x_3 + c_1^2c_2(c_2 + c_4)x_1x_2^2x_3^2 + c_1c_2^2c_4x_1x_2x_3^3$$

(5) f_5: $x_1(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)$

$$= x_1^5 + 3c_1x_1^4x_2 + 2c_2x_1^4x_3 + 3c_1^2x_1^3x_2^2 + (4c_1c_2 + c_5)x_1^3x_2x_3 + c_1^3x_1^2x_2^3$$

$$+ 2c_1(c_1c_2 + c_5)x_1^2x_2^2x_3 + c_2(c_1c_2 + c_5)x_1^2x_2x_3^2 + c_1^2c_5x_1x_2^3x_3$$

$$+ c_1c_2c_5x_1x_2^2x_3^2 + c_2^2x_1^3x_3^2$$

(6) f_6: $(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1 + c_2x_3)(x_1^2 + c_1x_1x_2 + c_5x_2x_3 + c_2x_1x_3)$

$$= x_1^5 + 3c_1x_1^4x_2 + 3c_1^2x_1^3x_2^2 + c_1^3x_1^2x_2^3 + 3c_2^2x_1^3x_3^2 + 3c_2x_1^4x_3 + c_2^3x_1^2x_3^3$$

$$+ (7c_1c_2 + c_5)x_1^3x_2x_3 + c_1(5c_1c_2 + 2c_5)x_1^2x_2^2x_3 + c_2(5c_1c_2 + 2c_5)x_1^2x_2x_3^2$$

$$+ c_1^2(c_1c_2 + c_5)x_1x_2^3x_3 + c_2^2(c_1c_2 + c_5)x_1x_2x_3^3 + c_1^2c_2c_5x_2^3x_3^2 + c_1c_2^2c_5x_2^2x_3^3$$

$$+ c_1c_2(2c_1c_2 + 3c_5)x_1x_2^2x_3^2$$

(7) f_7: $x_1x_3(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2)(x_1 + c_2x_3)$

$$= x_1^4x_3 + 2c_1x_1^3x_2x_3 + 2c_2x_1^3x_3^2 + 3c_1c_2x_1^2x_2x_3^2 + c_1^2x_1^2x_2^2x_3 + c_2^2x_1^2x_3^3$$

$$+ c_1^2c_2x_1x_2^2x_3^2 + c_1c_2^2x_1x_2x_3^3.$$

The results in (1) through (7) above may be best illustrated by use of the table on the following page.

If we consider the 7 x 7 determinant formed by the 7 rows, 4, 8, 10, 11, 12, 13, and 15, we find the absolute value of the determinant to be $3c_1^7 c_2^8 c_3 c_5 (c_1 c_2 - c_5)^2$. This expression in terms of the constants of the model for the singular designs is $3k^6 r^8 \lambda_2^2 t^2 (rk - \lambda_2 t)^2$. Since $rk - \lambda_2 t$ is greater than zero for singular designs, we conclude that the only solution to $\Sigma f_i w_i = 0$ is that the w_i be identically equal to zero, which was to be shown.

Semi-Regular, Group Divisible, PBIB Designs

In this section we shall prove that the only solution to $\Sigma f_i w_i = 0$ is that $w_i = 0$ $(i = 1, 2, \ldots, 7)$ for SR-GD-PBIB designs.

The seven functions (f_i) for this design are as follows:

$f_1 = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}$, $f_2 = (\sigma^2 + k\sigma_1^2)^{-1}$, $f_3 = (\sigma^2 + v\sigma_2^2)d_1^{-1}$,

$f_4 = (\sigma^2 + r\sigma_2^2)^{-1}$, $f_5 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]\, d_1^{-1}$, $f_6 = \sigma^{-2}$,

$f_7 = \sigma_2^2 d_1^{-1}$ where $d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + kv\sigma_1^2\sigma_2^2$ and where we have ignored the -2 coefficient of f_7 as it will not affect the result of this section.

By letting

(1) $\sigma^2 = x_1$, $\sigma_1^2 = x_2$, $\sigma_2^2 = x_3$,

(2) $k = c_1$, $r = c_2$, $v = c_3$, $k^{-1}(r - \lambda_1) = c_4$, $kv = c_5$,

the seven functions become:

Row	Term	f_1	f_2	f_3	f_4	f_5	f_6	f_7
1	x_1^5	1	1	1	1	1	1	0
2	$x_1^4x_2$	$2c_1$	$2c_1$	$2c_1$	$3c_1$	$3c_1$	$3c_1$	0
3	$x_1^3x_2x_3$	$(c_5+3c_1c_2)$	$(c_5+4c_1c_2)$	$c_1(3c_2+2c_3)$	$c_1(5c_1+2c_4)$	$(4c_1c_2+c_5)$	$(7c_1c_2+c_5)$	$2c_1$
4	$x_1^4x_3$	$2c_2$	$3c_2$	$(2c_2+c_3)$	$(2c_2+c_4)$	$2c_2$	$3c_2$	1
5	$x_1^3x_2^2$	c_1^2	c_1^2	c_1^2	$3c_1^2$	$3c_1^2$	$3c_1^2$	0
6	$x_1^2x_2^2x_3$	$c_1(c_5+c_1c_2)$	$c_1(c_5+c_1c_2)$	$c_1^2(c_1+c_3)$	$c_1^2(4c_1+c_4)$	$2c_1(c_1c_2+c_5)$	$c_1(5c_1c_2+2c_5)$	c_1^2
7	$x_1^2x_2x_3^2$	$c_2(c_5+c_1c_2)$	$2c_2(c_5+c_1c_2)$	$c_1c_2(c_2+3c_3)$	$c_1c_2(2c_2+3c_4)$	$c_2(c_1c_2+c_5)$	$c_2(5c_1c_2+2c_5)$	$3c_1c_2$
8	$x_1^3x_3^2$	c_2^2	$3c_2^2$	$c_2(c_2+2c_3)$	$c_2(c_2+2c_4)$	c_2^2	$3c_2^2$	$2c_2$
9	$x_1x_2^2x_3^2$	$c_1c_2c_5$	$c_1c_2c_5$	$c_1^2c_2c_3$	$c_1^2c_2(c_2+c_4)$	$c_1c_2c_5$	$c_1c_2(2c_1c_2+3c_5)$	$c_1^2c_2$
10	$x_1x_2x_3^3$	0	$c_2^2c_5$	$c_1c_2^2c_3$	$c_1c_2^2c_4$	0	$c_2^2(c_1c_2+c_5)$	$c_1c_2^2$
11	$x_1^2x_3^3$	0	c_2^3	$c_2^2c_3$	$c_2^2c_4$	0	c_2^3	c_2^2
12	$x_1^2x_2^3$	0	0	0	c_1^3	c_1^3	c_1^3	0
13	$x_1x_2^3x_3$	0	0	0	$c_1^3c_2$	$c_1^2c_5$	$c_1^2(c_1c_2+c_5)$	0
14	$x_2^3x_3^2$	0	0	0	0	0	$c_1^2c_2c_5$	0
15	$x_2^2x_3^3$	0	0	0	0	0	$c_1c_2^2c_5$	0

$f_1 = (x_1 + c_1x_2 + c_2x_3)^{-1}$, $f_2 = (x_1 + c_1x_2)^{-1}$, $f_3 = (x_1 + c_3x_3)d_1^{-1}$,

$f_4 = (x_1 + c_2x_3)^{-1}$, $f_5 = (x_1 + c_1x_2 + c_4x_3)d_1^{-1}$, $f_6 = x_1^{-1}$,

$f_7 = x_3d_1^{-1}$ with $d_1 = x_1^2 + c_1x_1x_2 + c_2x_1x_3 + c_5x_2x_3$.

In this form, the seven functions are the same as those in the foregoing section with the c_i defined differently. The absolute value of a 7 x 7 determinant was found to be $3c_1^7c_2^8c_3c_5(c_1c_2 - c_5)^2$. This becomes with the above definitions of the c_i, $3k^6r^8(rk - r + \lambda_1)(r - \lambda_1)^2$. This quantity is also non-zero for semi-regular designs, so we conclude that the only solution to $\Sigma f_iw_i = 0$ is that $w_i = 0$, which was to be shown.

Regular, Group Divisible, PBIB Designs

In this section we shall show that the only solution to $\Sigma f_iw_i = 0$ is that $w_i = 0$ $(i = 1, 2, \ldots, 9)$ for R-GD-PBIB designs.

The f_i are defined as follows:

$f_1 = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2)^{-1}$, $f_2 = (\sigma^2 + k\sigma_1^2)^{-1}$, $f_3 = (\sigma^2 + k^{-1}\lambda_2t\sigma_2^2)d_1^{-1}$,

$f_4 = (\sigma^2 + v\sigma_2^2)d_2^{-1}$, $f_5 = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2t)\sigma_2^2]d_1^{-1}$, $f_7 = \sigma^{-2}$,

$f_6 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]d_2^{-1}$, $f_8 = \sigma_2^2d_1^{-1}$, $f_9 = \sigma_2^2d_2^{-1}$, where

$d_1 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda_2t\sigma_1^2\sigma_2^2$ and $d_2 = \sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + kv\sigma_1^2\sigma_2^2$.

In order to simplify the algebra, we shall let:

(1) $x_1 = \sigma^2$, $x_2 = \sigma_1^2$, $x_3 = \sigma_2^2$,

(2) $c_1 = k$, $c_2 = r$, $c_3 = k^{-1}\lambda_2t$, $c_4 = v$, $c_5 = k^{-1}(rk - r + \lambda_1)$, $c_6 = \lambda_2t$

$c_7 = kv, \quad c_8 = k^{-1}(r - \lambda_1).$

The nine functions in this notation become:

$f_1 = (x_1 + c_1x_2 + c_2x_3)^{-1}, \quad f_2 = (x_1 + c_1x_2)^{-1}, \quad f_3 = (x_1 + c_3x_3)d_1^{-1},$

$f_4 = (x_1 + c_4x_3)d_2^{-1}, \quad f_5 = (x_1 + c_1x_2 + c_5x_3)d_1^{-1}, \quad f_7 = x_1^{-1},$

$f_6 = (x_1 + c_1x_2 + c_8x_3)d_2^{-1}, \quad f_8 = x_3d_1^{-1}, \quad f_9 = x_3d_2^{-1},$ where we have ignored the -2 coefficients of f_8 and f_9.

The lowest common denominator is $x_1d_1d_2(x_1 + c_1x_2)(x_1 + c_1x_2 + c_2x_3)$ with the numerators of the nine functions as follows:

(1) f_1: $d_1d_2x_1(x_1 + c_1x_2)$

$$= x_1^6 + 2c_2x_1^5x_3 + 3c_1x_1^5x_2 + (4c_1c_2 + c_6 + c_7)x_1^4x_2x_3 + c_2^2x_1^4x_3^2$$

$$+ c_2(c_1c_2 + c_6 + c_7)x_1^3x_2x_3^2 + 3c_1^2x_1^4x_2^2 + 2c_1(c_1c_2 + c_6 + c_7)x_1^3x_2^2x_3$$

$$+ [c_1c_2(c_6 + c_7) + c_6c_7]x_1^2x_2^2x_3^2 + c_1^3x_1^3x_2^3 + c_1^2(c_6 + c_7)x_1^2x_2^3x_3$$

$$+ c_1c_6c_7x_1x_2^3x_3^2$$

(2) f_2: $d_1d_2x_1(x_1 + c_1x_2 + c_2x_3)$

$$= x_1^6 + 3c_2x_1^5x_3 + 3c_1x_1^5x_2 + (6c_1c_2 + c_6 + c_7)x_1^4x_2x_3 + 3c_2^2x_1^4x_3^2$$

$$+ c_2(3c_1c_2 + 2c_6 + 2c_7)x_1^3x_2x_3^2 + 3c_1^2x_1^4x_2^2 + c_1(3c_1c_2 + 2c_6 + 2c_7)x_1^3x_2^2x_3$$

$$[c_6c_7 + 2c_1c_2(c_6 + c_7)]x_1^2x_2^2x_3^2 + c_1^2(c_6 + c_7)x_1^2x_2^3x_3 + c_1^3x_1^3x_2^3 + c_2^3x_1^3x_3^3$$

$$+ c_1c_6c_7x_1x_2^3x_3^2 + c_2^2(c_6 + c_7)x_1^2x_2x_3^3 + c_2c_6c_7x_1x_2^2x_3^3$$

(3) f_3: $x_1d_2(x_1 + c_1x_2)(x_1 + c_3x_3)(x_1 + c_3x_3)(x_1 + c_1x_2 + c_2x_3)$

$$= x_1^6 + (2c_2 + c_3)x_1^5x_3 + 3c_1x_1^5x_2 + (4c_1c_2 + 3c_1c_3 + c_7)x_1^4x_2x_3$$

$$+ c_2(c_2 + 2c_3)x_1^4x_3^2 + (4c_1c_2c_3 + c_1c_2^2 + c_2c_7 + c_3c_7)x_1^3x_2x_3^2 + 3c_1^2x_1^4x_2^2$$

$$+ c_1(3c_1c_3 + 2c_7 + 2c_1c_2)x_1^3x_2^2x_3 + c_1(2c_3c_7 + 2c_1c_2c_3 + c_2c_7)x_1^2x_2^2x_3^2$$

$$+ c_2^2c_3x_1^3x_3^3 + c_2c_3(c_1c_2 + c_7)x_1^2x_2x_3^3 + c_1^3x_1^3x_2^3 + c_1^2c_3c_7x_1x_2^3x_3^2$$

$$+ c_1^2(c_1c_3 + c_7)x_1^2x_2^3x_3 + c_1c_2c_3c_7x_1x_2^2x_3^3$$

(4) f_4: $d_1x_1(x_1 + c_1x_2)(x_1 + c_4x_3)(x_1 + c_1x_2 + c_2x_3)$

$$= x_1^6 + (2c_2 + c_4)x_1^5x_3 + 3c_1x_1^5x_2 + (4c_1c_2 + 3c_1c_4 + c_6)x_1^4x_2x_3$$

$$+ c_2(c_2 + 2c_4)x_1^4x_3^2 + (4c_1c_2c_4 + c_1c_2^2 + c_2c_6 + c_4c_6)x_1^3x_2x_3^2 + 3c_1^2x_1^4x_2^2$$

$$+ c_1(2c_6 + 2c_1c_2 + 3c_1c_4)x_1^3x_2^2x_3 + c_1(2c_4c_6 + 2c_1c_2c_4 + c_2c_6)x_1^2x_2^2x_3^2$$

$$+ c_2^2c_4x_1^3x_3^3 + c_2c_4(c_1c_2 + c_6)x_1^2x_2x_3^3 + c_1(c_1c_4 + c_6)x_1^2x_2^3x_3 + c_1^3x_1^3x_2^3$$

$$+ c_1^2c_4c_6x_1x_2^3x_3^2 + c_1c_2c_4c_6x_1x_2^2x_3^3$$

(5) f_5: $x_1d_2(x_1 + c_1x_2)(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2 + c_5x_3)$

$$= x_1^6 + 4c_1x_1^5x_2 + (2c_2 + c_5)x_1^5x_3 + 6c_1^2x_1^4x_2^2 + (6c_1c_2 + 3c_1c_5 + c_7)x_1^4x_2x_3$$

$$+ 4c_1^3x_1^3x_2^3 + 3c_1(2c_1c_2 + c_1c_5 + c_7)x_1^3x_2^2x_3 + c_2(c_2 + 2c_5)x_1^4x_3^2$$

$$+ \; c_1^2(2c_1c_2 + c_1c_5 + 3c_7)x_1^2x_2^3x_3 + (4c_1c_2c_5 + 2c_1c_2^2 + c_2c_7 + c_5c_7)x_1^3x_2x_3^2$$

$$+ \; c_1(c_1c_2^2 + 2c_1c_2c_5 + 2c_2c_7 + 2c_5c_7)x_1^2x_2^2x_3^2 + c_2^2c_5x_1^3x_3^3 + c_1^4x_1^2x_2^4$$

$$+ \; c_2c_5(c_1c_2 + c_7)x_1^2x_2x_3^3 + c_1^3c_7x_1x_2^4x_3 + c_1^2c_7(c_2 + c_5)\, x_1x_2^3x_3^2$$

$$+ \; c_1c_2c_5c_7x_1x_2^2x_3^3$$

(6) f_6: $x_1d_1(x_1 + c_1x_2)(x_1 + c_1x_2 + c_2x_3)(x_1 + c_1x_2 + c_8x_3)$

$$= x_1^6 + 4c_1x_1^5x_2 + (2c_2 + c_8)x_1^5x_3 + 6c_1^2x_1^4x_2^2 + (6c_1c_2 + 3c_1c_8 + c_6)x_1^4x_2x_3$$

$$+ \; 4c_1^3x_1^3x_2^3 + 3c_1(2c_1c_2 + c_1c_8 + c_6)x_1^3x_2^2x_3 + c_2(c_2 + 2c_8)x_1^4x_3^2$$

$$+ \; (4c_1c_2c_8 + 2c_1c_2 + c_2c_6 + c_6c_8)x_1^3x_2x_3^2 + c_1^2(2c_1c_2 + c_1c_8 + 3c_6)x_1^2x_2^3x_3$$

$$+ \; c_2^2c_8x_1^3x_3^3 + c_1^4x_1^2x_2^4 + c_1(c_1c_2^2 + 2c_1c_2c_8 + 2c_2c_6 + 2c_6c_8)x_1^2x_2^2x_3^2$$

$$+ \; c_1^3c_6x_1x_2^4x_3 + c_2c_8(c_1c_2 + c_6)x_1^2x_2x_3^3 + c_1^2c_6(c_2 + c_8)x_1x_2^3x_3^2$$

$$+ \; c_1c_2c_6c_8x_1x_2^2x_3^3$$

(7) f_7: $d_1d_2(x_1 + c_1x_2)(x_1 + c_1x_2 + c_2x_3)$

$$= x_1^6 + 3c_2x_1^5x_3 + 4c_1x_1^5x_2 + (9c_1c_2 + c_6 + c_7)x_1^4x_2x_3 + 3c_2x_1^4x_3^2$$

$$+ \; c_1^2c_6c_7x_2^4x_3^2 + 2c_2(3c_1c_2 + c_6 + c_7)x_1^3x_2x_3^2 + 6c_1^2x_1^4x_2^2 + 4c_1^3x_1^3x_2^3$$

$$+ \; c_1c_2c_6c_7x_2^3x_3^3 + (4c_1c_2c_6 + 4c_1c_2c_7 + 3c_1^2c_2^2 + c_6c_7)\, x_1^2x_2^2x_3^2$$

$$+ \; 3c_1(3c_1c_2 + c_6 + c_7)x_1^3x_2^2x_3 + 3c_1^2(c_1c_2 + c_6 + c_7)x_1^2x_2^3x_3 + c_2^3x_1^3x_3^3$$

$$+ 2c_1(c_6c_7 + c_1c_2c_6 + c_1c_2c_7)x_1x_2^3x_3^2 + c_1^3(c_6 + c_7)x_1x_2^4x_3 + c_1^4x_1^2x_2^4$$

$$+ c_2^2(c_1c_2 + c_6 + c_7)x_1^2x_2x_3^3 + c_2(c_6c_7 + c_1c_2c_6 + c_1c_2c_7)x_1x_2^2x_3^3$$

(8) f_8: $x_1x_3d_2(x_1 + c_1x_2)(x_1 + c_1x_2 + c_2x_3)$

$$= x_1^5x_3 + 3c_1x_1^4x_2x_3 + 2c_2x_1^4x_3^2 + 3c_1^2x_1^3x_2^2x_3 + (4c_1c_2 + c_7)x_1^3x_2x_3^2$$

$$+ c_2^2x_1^3x_3^3 + 2c_1(c_1c_2 + c_7)x_1^2x_2^2x_3^2 + c_2(c_1c_2 + c_7)x_1^2x_2x_3^3 + c_1^3x_1^2x_2^3x_3$$

$$+ c_1^2c_7x_1x_2^3x_3^2 + c_1c_2c_7x_1x_2^2x_3^3$$

(9) f_9: $x_1x_3d_1(x_1 + c_1x_2)(x_1 + c_1x_2 + c_2x_3)$

$$= x_1^5x_3 + 3c_1x_1^4x_2x_3 + 2c_2x_1^4x_3^2 + 3c_1^2x_1^3x_2^2x_3 + (4c_1c_2 + c_6)x_1^3x_2x_3^2$$

$$+ c_2^2x_1^3x_3^3 + 2c_1(c_1c_2 + c_6)x_1^2x_2^2x_3^2 + c_2(c_1c_2 + c_6)x_1^2x_2x_3^3$$

$$+ c_1^3x_1^2x_2^3x_3 + c_1^2c_6x_1x_2^3x_3^2 + c_1c_2c_6x_1x_2^2x_3^3.$$

By taking coefficients of like terms for each of the terms x_1^6, $x_1^5x_2$, $x_1^4x_2x_3$, $x_1^3x_2x_3^2$, $x_1^2x_2^3x_3$, $x_1x_2^3x_3^2$, $x_1^3x_3^3$, $x_1x_2^2x_3^3$, and $x_2^3x_3^3$ and forming a 9 x 9 determinant therefrom, it can be shown that the absolute value of this determinant is equal to $2c_1c_6c_7(c_6 - c_7)^3(c_2c_6 + c_2c_7 - c_1c_2^2 - c_5c_6)$ or in terms of the constants of the model, this is equal to the following,

$2k^2\lambda_2t(rk - r + \lambda_1)n^3(\lambda_1 - \lambda_2)^3[\lambda_2^2t^2 - rk(r - \lambda_1)]$ which is clearly not zero for R-GD- PBIB designs. We therefore conclude that the only solution to $\Sigma f_iw_i = 0$ is that $w_i = 0$, which was to be shown.

APPENDIX C

DISTRIBUTIONS AND EXPECTATIONS OF THE s_i IN THE BIB AND GD-PBIB DESIGNS

In this appendix, we shall find the distributions and expectations of each of the statistics in each of the minimal sets of sufficient statistics that we have found for the BIB and GD-PBIB designs.

We shall first state a theorem which will prove useful in the development of the proofs of this appendix.

Theorem. If Y is distributed as the multivariate normal, mean $\bar{\mu}$ and covariance matrix Σ, then $Y'AY$ is distributed as the non-central χ^2 with degrees of freedom k and non-centrality parameter λ if $A\Sigma$ is idempotent and where k is the rank of A and $\lambda = 2^{-1}\bar{\mu}'A\bar{\mu}$. [9]

The Balanced Incomplete Block Design

1. Distribution of $s_1 = y_{...}$,

Since $y_{...}$ is a linear combination of normal variables, $y_{...}$ is distributed normally, mean μ and variance $M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$ or

$$s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)].$$

2. Distribution and expectation of $s_2 = k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$.

a. Distribution of s_2.

Let $A_2 = k^{-1}X_1P_{21}P_{21}'X_1'$. Then $A_2A_2 = A_2$. In order to apply the theorem we must show that $A_2\Sigma A_2\Sigma = A_2\Sigma$ or equivalently that $A_2\Sigma A_2 = A_2$. Proceeding we have

$$A_2\Sigma A_2 = k^{-2}X_1P_{21}P_{21}'X_1'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1P_{21}P_{21}'X_1'$$

$$= k^{-2}X_1P_{21}P_{21}'[(\sigma^2 + k\sigma_1^2)I_b + N'N\sigma_2^2]P_{21}P_{21}'X_1'$$

$$= k^{-1}X_1P_{21}P_{21}'X_1'(\sigma^2 + k\sigma_1^2) \text{ since } P_{21}'N' = \phi.$$

$$= (\sigma^2 + k\sigma_1^2)\, A_2$$

Let $B_2 = (\sigma^2 + k\sigma_1^2)^{-1}A_2$. Then $Y'B_2Y \sim \chi'^2(k_2, \lambda_2)$ where

$$k_2 = \text{rank}\, B_2 = \text{rank}\, A_2 = \text{tr}\, A_2 = k^{-1}\text{tr}\, X_1P_{21}P_{21}'X_1' = \text{tr}\, P_{21}P_{21}' = (b-t)$$

and

$$\lambda_2 = \mu^2 j_M^1 X_1P_{21}P_{21}'X_1'j_1^M\, C(\sigma) = 0.$$

Therefore

$$s_2 \sim (\sigma^2 + k\sigma_1^2)\,\chi^2\,(b - t).$$

b. Expectation of s_2.

Since $s_2 \sim (\sigma^2 + k\sigma_1^2)\,\chi^2(b - t)$, and $E[\chi^2(p)] = p$, we have that

$$E(s_2) = (b - t)(\sigma^2 + k\sigma_1^2).$$

3. Distribution and expectation of $s_3 = k^{-1}Y'X_1P_{22}P_{22}'X_1'Y$.

a. Distribution of s_3.

Let $A_3 = k^{-1}X_1P_{22}P_{22}'X_1'$. Then $A_3A_3 = A_3$. Evaluating $A_3\Sigma A_3$ we have:

$$A_3 \Sigma A_3 = k^{-2} X_1 P_{22} P'_{22} X'_1 (X_1 X'_1 \sigma_1^2 + X_2 X'_2 \sigma_2^2 + \sigma^2 I) X_1 P_{22} P'_{22} X'_1$$

$$= k^{-2} X_1 P_{22} P'_{22} [k(\sigma^2 + k\sigma_1^2) + N'N\sigma_2^2] P_{22} P'_{22} X'_1$$

$$= k^{-1} X_1 P_{22} P'_{22} X'_1 [\sigma^2 + k\sigma_2^2 + k^{-1}(r - \lambda)\sigma_2^2]$$

$$= [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2] A_3.$$

Let $B_3 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2]^{-1} A_3$. Then

$$Y'B_3 Y \sim \chi^2(k_3, \lambda_3)$$

where

$$k_3 = \text{rank } B_3 = \text{rank } A_3 = \text{tr } A_3 = k^{-1} \text{tr } X_1 P_{22} P'_{22} X'_1 = \text{tr } I_{t-1} = t - 1,$$

and

$$\lambda_3 = \mu^2 j_M^{1} X_1 P_{22} P'_{22} X'_1 j_1^M C(\sigma) = 0.$$

Therefore

$$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2] \chi^2(t - 1).$$

b. Expectation of s_3.

Since s_3 is distributed as a central chi-square variate we have

$$E(s_3) = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\sigma_2^2](t - 1).$$

4. Distribution and expectation of $s_5 = k(\lambda t)^{-1} Y'AP_3P'_3A'Y$.

Let $A_5 = k(\lambda t)^{-1} AP_3P'_3A'$. Then $A_5A_5 = A_5$. We then have

$$A_5 \Sigma A_5 = k^2(\lambda t)^{-2} AP_3P'_3A'(X_1X'_1\sigma_1^2 + X_2X'_2\sigma_2^2 + \sigma^2 I)AP_3P'_3A'$$

$$= k^2(\lambda t)^{-2} AP_3P'_3[(rI - k^{-1}NN')^2\sigma_2^2 + (rI - k^{-1}NN')\sigma^2]P_3P'_3A'$$

$$= k(\lambda t)^{-1} AP_3(\sigma^2 + k^{-1}\lambda t\sigma_2^2)P'_3A' = (\sigma^2 + k^{-1}\lambda t\sigma_2^2)A_5.$$

Let $B_5 = (\sigma^2 + k^{-1}\lambda t\sigma_2^2)^{-1} A_5$. Then

$$Y'B_5Y \sim \chi'^2 (k_5, \lambda_5)$$

where

$$k_5 = \text{rank } B_5 = \text{rank } A_5 = \text{tr } A_5 = k(\lambda t)^{-1} \text{tr } AP_3P_3'A' = (t - 1)$$

and

$$\lambda_5 = \mu^2 j_M^1 AP_3P_3'A'j_1^M \, C(\sigma) = 0.$$

Therefore

$$s_5 \sim (\sigma^2 + k^{-1}\lambda t\sigma_2^2)\, \chi^2 (t - 1),$$

with

$$E(s_5) = (t - 1)(\sigma^2 + k^{-1}\lambda t\sigma_2^2).$$

5. Distribution and expectation of $s_6 = Y'P_4P_4'Y$.

Let $A_6 = P_4P_4'$. Then $A_6A_6 = A_6$. Therefore we have

$$A_6\Sigma A_6 = P_4P_4'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)P_4P_4' = \sigma^2 P_4P_4'$$

$$= \sigma^2 A_6$$

Let $B_6 = \sigma^{-2} A_6$. Then

$$Y'B_6Y \sim \chi'^2 (k_6, \lambda_6)$$

where

$$k_6 = \text{rank } B_6 = \text{rank } A_6 = \text{tr } P_4P_4' = \text{tr } P_4'P_4 = \text{tr } I_{M-b-t+1}$$

$$= M - b - t + 1$$

and

$$\lambda_6 = \mu^2 j_M^1 P_4P_4'j_1^M \, C(\sigma) = 0.$$

Therefore

$$s_6 \sim \sigma^2 \chi^2 (M - b - t + 1)$$

with

$$E(s_6) = (M - b - t + 1)\sigma^2$$

6. Distribution and expectation of $s_4 = k^{-1} Y'X_1N'P_3P_3'A'Y$.

a. Distribution of s_4.

Let $A_4 = k^{-1}X_1N'P_3P_3'A'$. Since A_4 is not symmetric, we may write $Y'A_4Y = 2^{-1}Y'(A_4 + A_4')Y$. Then since $4^{-1}(A_4+A_4')\Sigma(A_4+A_4')$ is not equal to $2^{-1}(A_4 + A_4')C(\sigma)$, s_4 is not distributed as a chi-square variate but as a linear combination of χ^2 variates, that is,

$$s_4 \sim \Sigma\, a_i\, \chi^2(1)$$

where the a_i are the non-zero characteristic roots of $2^{-1}(A_4 + A_4')$.

b. Expectation of s_4.

$$E(s_4) = E(k^{-1}Y'X_1N'P_3P_3'A'Y) = E\ \mathrm{tr}\ (k^{-1}Y'X_1N'P_3P_3'A'Y)$$

$$= k^{-1}\ \mathrm{tr}\ E(YY'X_1N'P_3P_3'A')$$

$$= k^{-1}\ \mathrm{trace}\ (X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1N'P_3P_3'A'$$

$$= k^{-1}\ \mathrm{tr}\ A'X_2X_2'X_1N'P_3P_3'\sigma_2^2 = k^{-1}\mathrm{tr}P_3'(rI - k^{-1}NN')NN'P_3\sigma_2^2$$

$$= k^{-1}\mathrm{tr}\ P_3'(rNN' - k^{-1}NN'NN')P_3\sigma_2^2$$

$$= k^{-1}\mathrm{tr}\ [r(r - \lambda) - k^{-1}(r - \lambda)^2]\ I_{t-1}\sigma_2^2$$

$$= k^{-2}(t - 1)(r - \lambda)\lambda t\ \sigma_2^2$$

Singular, Group Divisible, PBIB Designs

In this section of this appendix we will find the distributions and expectations of the statistics in a minimal set of sufficient statistics for the singular, group divisible, partially balanced incomplete block design.

1. Distribution of $s_1 = y\ldots$.

Since s_1 is a linear combination of normal variables, s_1 is normally distributed with mean $E(y\ldots) = \mu$, and variance $E(y\ldots - \mu)^2$ which is equal to $M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$. Symbolically then

$$s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)].$$

2. Distribution of $s_2 = k^{-1}Y'X_1Q_{21}Q_{21}'X_1'Y$.

Let $A_2 = k^{-1}X_1Q_{21}Q_{21}'X_1'$. Then $A_2A_2 = A_2$ and

$$A_2\Sigma A_2 = k^{-2}X_1Q_{21}Q_{21}'X_1'(X_1X_1'\sigma_2^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1Q_{21}Q_{21}'X_1'$$

$$= k^{-2}X_1Q_{21}Q_{21}'(k\sigma^2 + k^2\sigma_1^2)Q_{21}Q_{21}'X_1'$$

$$= (\sigma^2 + k\sigma_1^2)k^{-1}X_1Q_{21}Q_{21}'X_1' = (\sigma^2 + k\sigma_1^2)A_2.$$

Now let $B_2 = (\sigma^2 + k\sigma_1^2)^{-1}A_2$. We then have

$$Y'B_2Y \sim \chi'^2(k_2, \lambda_2),$$

where

$$k_2 = \text{rank } B_2 = \text{rank } A_2 = \text{tr } A_2 = k^{-1}\text{tr } X_1Q_{21}Q_{21}'X_1' = b - m,$$

and

$$\lambda_2 = \mu^2 j_M^1 X_1Q_{21}Q_{21}'X_1'j_1^M \, C(\sigma) = 0.$$

Therefore

$$s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(b - m)$$

if $b > m$ and is not defined if $b = m$.

3. Distribution of $s_3 = k^{-1}Y'X_1P_{22}P_{22}'X_1'Y$.

Let $A_3 = k^{-1}X_1P_{22}P_{22}'X_1'$. We then have

$$A_3\Sigma A_3 = k^{-2}X_1P_{22}P_{22}'X_1'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1P_{22}P_{22}'X_1'$$

$$= k^{-1}X_1P_{22}P_{22}'[(\sigma^2 + k\sigma_1^2)I_b + k^{-1}N'N\sigma_2^2]P_{22}P_{22}'X_1'$$

$$= k^{-1}X_1P_{22}[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]P_{22}'X_1'$$

$$= [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\ A_3.$$

Now let $B_3 = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]^{-1}\ A_3$. Then

$$Y'B_3Y \sim \chi'^2(k_3,\ \lambda_3)$$

where

$$k_3 = \text{rank } B_3 = \text{rank } A_3 = \text{tr } A_3 = k^{-1}X_1P_{22}P_{22}'X_1' = (m - 1)$$

and

$$\lambda_3 = \mu^2 j_M^1 X_1P_{22}P_{22}'X_1'j_1^M\ C(\sigma) = 0.$$

Therefore

$$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\chi^2(m - 1).$$

4. Distribution of $s_4 = k(\lambda_2 t)^{-1}Y'AP_{31}P_{31}'A'Y$.

Let $A_4 = k(\lambda_2 t)^{-1}AP_{31}P_{31}'A'$. Then $A_4A_4 = A_4$ and

$$A_4\Sigma A_4 = k^2(\lambda_2 t)^{-2}AP_{31}P_{31}'A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)AP_{31}P_{31}'A'$$

$$= k^2(\lambda_2 t)^{-2}[AP_{31}P_{31}'A'(X_2X_2')AP_{31}P_{31}'A'\sigma_2^2 + AP_{31}P_{31}'A'AP_{31}P_{31}'A\sigma^2]$$

$$= k^2(\lambda_2 t)^{-2}[AP_{31}P_{31}'(rI - k^{-1}NN')(rI - k^{-1}NN')P_{31}P_{31}'A'\ \sigma_2^2$$

$$+ AP_{31}P_{31}'(rI - k^{-1}NN')P_{31}P_{31}'A'\ \sigma^2]$$

$$= (\sigma^2 + k^{-1}\lambda_2 t\ \sigma_2^2)A_4$$

Now let $B_4 = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)^{-1}A_4$. Then

$$Y'B_4Y \sim \chi'^2(k_4, \lambda_4)$$

where

$$k_4 = \text{rank } B_4 = \text{rank } A_4 = \text{tr } A_4 = k(\lambda_2 t)^{-1}\text{tr } AP_{31}P_{31}'A' = (m - 1),$$

and

$$\lambda_4 = \mu^2 j_M^1 AP_{31}P_{31}'A' \, C(\sigma) = 0.$$

Therefore

$$s_4 \sim (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)\, \chi^2(m - 1).$$

5. Distribution of $s_5 = r^{-1}Y'AP_{32}P_{32}'A'Y$.

Let $A_5 = r^{-1}AP_{32}P_{32}'A'$. Then $A_5A_5 = A_5$ and

$$A_5\Sigma A_5 = r^{-2}[AP_{32}P_{32}'A'X_2X_2'AP_{32}P_{32}'A'\sigma_2^2 + AP_{32}P_{32}'A'AP_{32}P_{32}'A'\sigma^2]$$

$$= r^{-1}[AP_{32}P_{32}'A'\, r\sigma_2^2 + AP_{32}P_{32}'A'\,\sigma^2] = (\sigma^2 + r\sigma_2^2)A_5.$$

Now let $B_5 = (\sigma^2 + r\sigma_2^2)^{-1}A_5$. Then

$$Y'B_5Y \sim \chi'^2(k_5, \lambda_5)$$

where

$$k_5 = \text{rank } B_5 = \text{rank } A_5 = \text{tr } A_5 = r^{-1}\text{tr } AP_{32}P_{32}'A' = m(n - 1)$$

and

$$\lambda_5 = \mu^2 j_M^1 AP_{32}P_{32}'A' j_1^M C(\sigma) = 0.$$

Therefore

$$s_5 \sim (\sigma^2 + r\sigma_2^2)\, \chi^2\, [m(n - 1)].$$

6. Distribution of $s_6 = Y'P_4P_4'Y$.

First we have

$$P_4P_4'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)P_4P_4' = \sigma^2 P_4P_4'.$$

Therefore

$$\sigma^{-2} s_6 \sim \chi'^2(k_6, \lambda_6)$$

where

$$k_6 = \text{rank } P_4P_4' = \text{trace } P_4P_4' = \text{trace } P_4P_4' = M - b - t + 1$$

and

$$\lambda_6 = \mu^2 j_M^1 P_4 P_4' j_1^M C(\sigma) = 0.$$

Therefore

$$s_6 \sim \sigma^2 \chi^2(M - b - t + 1).$$

7. Distribution of $s_7 = [k^{-2}(rk - \lambda_2 t)]^{1/2} Y'X_1P_{22}P_{31}'A'Y$.

It is easily shown that if we let $P_{22}' = (rk - \lambda_2 t)^{-1/2} P_{31}'N$ and define $A_7 = k^{-1}X_1N'P_{31}P_{31}'A'$, since A_7 is not symmetric that $s_7 = Y'A_7Y = 2^{-1}Y'(A_7 + A_7')Y$. Also, $4^{-1}(A_7 + A_7')\Sigma(A_7 + A_7') \neq 2^{-1}(A_7 + A_7')$. Therefor s_7 is not distributed as a chi-square variate but as a linear combination of chi-square variates, that is,

$$s_7 \sim \Sigma a_i \chi^2(1)$$

where the a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A_7')\Sigma$.

8. Expectations of the s_i.

Since each of the s_i ($i = 2, 3, \ldots, 6$) is distributed as chi-square we have

$$E(s_2) = (\sigma^2 + k\sigma_1^2)(b - m)$$

$$E(s_3) = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2](m - 1)$$

$$E(s_4) = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)(m - 1)$$

$$E(s_5) = (\sigma^2 + r\sigma_2^2)[m(n-1)]$$

$$E(s_6) = (\sigma^2)(M - b - t + 1)$$

$$E(s_7) = Ek^{-1}Y'X_1N'P_{31}P'_{31}A'Y = E\ \mathrm{tr}\ YY'X_1N'P_{31}P'_{31}A'k^{-1}$$

$$= k^{-1}\mathrm{tr}\ (X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1N'P_{31}P'_{31}A'$$

$$= k^{-1}\mathrm{tr}\ A'X_2X_2'X_1N'P_{31}P'_{31}\sigma_2^2$$

$$= k^{-2}\mathrm{tr}\ (rkP'_{31}NN'P_{31} - P'_{31}NN'NN'P_{31})\sigma_2^2$$

$$= k^{-2}\mathrm{tr}\ [rk(rk - \lambda_2 t) - (rk - \lambda_2 t)^2]I_{m-1}\sigma_2^2$$

$$= k^{-2}(m-1)(rk - \lambda_2 t)(\lambda_2 t)\ \sigma_2^2$$

<u>Semi-regular, Group Divisible, PBIB Designs</u>

In this section of this appendix, we shall find the distributions and expectations of the statistics in the minimal set of sufficient statistics that were found for the semi-regular, group divisible, partially balanced incomplete block designs.

1. Distribution of $s_1 = y\ldots$.

It is easily verified that

$$s_1 \sim N[\mu,\ M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)].$$

2. Distribution of $s_2 = k^{-1}Y'X_1Q_{21}Q'_{21}X_1'Y$.

Let $A_2 = k^{-1}X_1Q_{21}Q'_{21}X_1'$. Then $A_2A_2 = A_2$ and

$$A_2\Sigma A_2 = k^{-2}X_1Q_{21}Q'_{21}X_1'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1Q_{21}Q'_{21}X_1'$$

$$= k^{-1}X_1Q_{21}Q_{21}'[(\sigma^2 + k\sigma_1^2)I_b + k^{-1}N'N\sigma_2^2]Q_{21}Q_{21}'X_1'$$

$$= k^{-1}(\sigma^2 + k\sigma_1^2)X_1Q_{21}Q_{21}'X_1' = (\sigma^2 + k\sigma_1^2)A_2.$$

Now let $B_2 = (\sigma^2 + k\sigma_1^2)^{-1}A_2$. Then $B_2\Sigma B_2\Sigma = B_2\Sigma$ and therefore

$$Y'B_2Y \sim \chi'^2(k_2, \lambda_2)$$

where

$$k_2 = \text{rank } B_2 = \text{rank } A_2 = \text{tr } A_2 = k^{-1}\text{tr } X_1Q_{21}Q_{21}'X_1' = b-t+m-1,$$

and

$$\lambda_2 = \mu^2 j_M^1 X_1Q_{21}Q_{21}'X_1'j_1^M C(\sigma) = 0.$$

Therefore

$$s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(b - t + m - 1).$$

3. Distribution of $s_3 = k^{-1}Y'X_1P_{23}P_{23}'X_1'Y$.

Let $A_3 = k^{-1}X_1P_{23}P_{23}'X_1'$. Then $A_3A_3 = A_3$ and

$$A_3\Sigma A_3 = k^{-2}X_1P_{23}P_{23}'X_1'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1P_{23}P_{23}'X_1'$$

$$= k^{-1}X_1P_{23}P_{23}[(\sigma^2 + k\sigma_1^2)I_b + k^{-1}N'N\sigma_2^2]P_{23}P_{23}'X_1'$$

$$= k^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]X_1P_{23}P_{23}'X_1'$$

$$= [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]A_3.$$

Now let $B_3 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]^{-1}A_3$. Then $B_3\Sigma B_3\Sigma = B_3\Sigma$

and we have

$$Y'B_3Y \sim \chi'^2(k_3, \lambda_3)$$

where

$$k_3 = \text{rank } B_3 = \text{rank } A_3 = \text{tr } A_3 = k^{-1}\text{tr } X_1P_{23}P_{23}'X_1' = m(n - 1),$$

and

$$\lambda_3 = \mu^2 j_M^1 X_1P_{23}P_{23}'X_1'j_1^M C(\sigma) = 0.$$

Therefore

$$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]\, \chi^2[m(n-1)].$$

4. Distribution of $s_4 = r^{-1}Y'AP_{31}P'_{31}A'Y$.

Let $A_4 = r^{-1}AP_{31}P'_{31}A'$. Then $A_4A_4 = A_4$ and

$$A_4\Sigma A_4 = r^{-2}AP_{31}P'_{31}A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)AP_{31}P'_{31}A'$$

$$= r^{-2}AP_{31}P'_{31}[(rI - k^{-1}NN')^2\sigma_2^2 + (rI - k^{-1}NN')\sigma^2]P_{31}P'_{31}A'$$

$$= (\sigma^2 + r\sigma_2^2)r^{-1}AP_{31}P'_{31}A' = (\sigma^2 + r\sigma_2^2)A_4.$$

Now let $B_4 = (\sigma^2 + r\sigma_2^2)^{-1}A_4$. Then $B_4\Sigma B_4 = B_4$, and therefore

$$Y'B_4Y \sim \chi'^2(k_4, \lambda_4)$$

where

$$k_4 = \text{rank } B_4 = \text{rank } A_4 = \text{tr } A_4 = r^{-1}\text{tr } AP_{31}P'_{31}A' = m - 1$$

and

$$\lambda_4 = \mu^2 j_1^{M} AP_{31}P'_{31}A' j_1^{M} C(\sigma) = 0.$$

Therefore

$$s_4 \sim (\sigma^2 + r\sigma_2^2)\, \chi^2(m-1)$$

5. Distribution of $s_5 = v^{-1}Y'AP_{32}P'_{32}A'Y$.

Let $A_5 = v^{-1}AP_{32}P'_{32}A'$. Then $A_5A_5 = A_5$ and

$$A_5\Sigma A_5 = v^{-2}AP_{32}P'_{32}A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)AP_{32}P'_{32}A'$$

$$= v^{-2}AP_{32}[P'_{32}(rI - k^{-1}NN')(rI - k^{-1}NN')P_{32}\sigma_2^2$$

$$+ P'_{32}(rI - k^{-1}NN')P_{32}\sigma^2]P'_{32}A'$$

$$= v^{-1}AP_{32}(\sigma^2 + v\sigma_2^2)P'_{32}A' = (\sigma^2 + v\sigma_2^2)A_5.$$

Now let $B_5 = (\sigma^2 + v\sigma_2^2)^{-1}A_5$. Then $B_5\Sigma B_5 = B_5$ and then

$$Y'B_5Y \sim \chi'^2(k_5, \lambda_5)$$

where

$$k_5 = \text{rank } B_5 = \text{rank } A_5 = \text{tr } A_5 = v^{-1}\text{tr } AP_{32}P'_{32}A' = m(n-1)$$

and

$$\lambda_5 = \mu^2 j_M^1 AP_{32}P'_{32}A' j_1^M C(\sigma) = 0.$$

Therefore

$$s_5 \sim (\sigma^2 + v\sigma_2^2)\chi^2[m(n-1)].$$

6. Distribution of $s_6 = Y'P_4P'_4Y$.

Following a proof identical to that of finding the distribution of s_6 in the previous section of this appendix, we have

$$s_6 \sim \sigma^2\chi^2(M - b - t + 1)$$

7. Distribution of $s_7 = k^{-1}Y'X_1N'P_{32}P'_{32}A'Y$.

Let $A_7 = k^{-1}X_1N'P_{32}P'_{32}A'$ and since A_7 is not symmetric let

$$s_7 = Y'A_7Y = 2^{-1}Y'(A_7 + A'_7)Y.$$

It is easily shown that $[2^{-1}(A_7 + A'_7)]\Sigma[2^{-1}(A_7 + A'_7)] \neq [2^{-1}(A_7 + A'_7)]$.

Therefore s_7 is not distributed as a chi-square variate. If this condition exists where $Y'BY$ is a quadratic form and $B\Sigma$ is not idempotent, then $Y'BY$ is distributed as a linear combination of independent χ^2 variates, that is, for the case we are considering,

$$s_7 \sim \Sigma a_i \chi^2(1)$$

where the a_i are the non-zero characteristic roots of $2^{-1}(A_7 + A_7')\Sigma$.

8. Expectations of the seven statistics.

Since the s_i are each distributed as χ^2 and $E(\chi^2(p)) = p$ we have

$$E(s_1) = \mu$$

$$E(s_2) = (b - t + m - 1)(\sigma^2 + k\sigma_1^2)$$

$$E(s_3) = [m(n - 1)][\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]$$

$$E(s_4) = (m - 1)(\sigma^2 + r\sigma_2^2)$$

$$E(s_5) = [m(n - 1)](\sigma^2 + v\sigma_2^2)$$

$$E(s_6) = (M - b - t + 1)\sigma^2$$

$$\begin{aligned}
E(s_7) &= k^{-1}E\,tr\,Y'X_1N'P_{32}P_{32}'A'Y = k^{-1}tr\,E\,YY'X_1N'P_{32}P_{32}'A' \\
&= k^{-1}tr\,(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1P_{32}P_{32}'A' \\
&= k^{-1}tr\,P_{32}'A'X_2X_2'X_1N'P_{32}\sigma_2^2 \\
&= k^{-1}P_{32}'(rI - k^{-1}NN')NN'P_{32}\sigma_2^2 \\
&= k^{-2}tr\,P_{32}'(rkNN' - NN'NN')P_{32}\sigma_2^2 \\
&= k^{-2}tr\,[rk(r - \lambda_1) - (r - \lambda_1)^2]I_{m(n-1)}\,\sigma_2^2 \\
&= m(n - 1)k^{-1}(r - \lambda_1)v\,\sigma_2^2
\end{aligned}$$

Regular, Group Divisible PBIB Designs

We shall in this section of this appendix find the distribution and expectation of each of the nine statistics in a minimal set of sufficient statistics for the regular, group divisible, partially balanced incomplete block designs.

1. Distribution of $s_1 = y\ldots$.

Parallel to the proof in the first section of this appendix, we have

$$s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)].$$

2. Distribution of $s_2 = k^{-1}Y'X_1P_{21}P_{21}'X_1'Y$.

Let $A_2 = k^{-1}X_1P_{21}P_{21}'X_1'$. Then $A_2A_2 = A_2$ and

$$A_2\Sigma A_2 = k^{-2}X_1P_{21}P_{21}'X_1'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2I)X_1P_{21}P_{21}'X_1'$$

$$= k^{-2}X_1P_{21}P_{21}'[k(\sigma^2 + k\sigma_1^2)I_b + N'N\sigma_2^2]P_{21}P_{21}'X_1'$$

$$= k^{-1}(\sigma^2 + k\sigma_1^2)X_1P_{21}P_{21}'X_1' = (\sigma^2 + k\sigma_1^2)A_2.$$

Let $B_2 = (\sigma^2 + k\sigma_1^2)^{-1}A_2$. Then $B_2\Sigma B_2 = B_2$ and

$$Y'B_2Y \sim \chi'^2(k_2, \lambda_2)$$

where

$$k_3 = \text{rank } B_2 = \text{rank } A_2 = \text{tr } A_2 = k^{-1}\text{tr } X_1P_{21}P_{21}'X_1' = b - t$$

and

$$\lambda_3 = \mu^2 j_M^1 X_1P_{21}P_{21}'X_1'j_1^M C(\sigma) = 0.$$

Therefore

$$s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(b - t).$$

3. Distribution of $s_3 = k^{-1}Y'X_1P_{22}P'_{22}X'_1Y$.

Let $A_3 = k^{-1}X_1P_{22}P'_{22}X'_1$. Then $A_3A_3 = A_3$ and

$$A_3\Sigma A_3 = k^{-2}X_1P_{22}P'_{22}X'_1(X_1X'_1\sigma_1^2 + X_2X'_2\sigma_1^2 + \sigma^2 I)X_1P_{22}P'_{22}X'_1$$

$$= k^{-1}X_1P_{22}P'_{22}[(\sigma^2 + k\sigma_1^2)I_b + k^{-1}N'N\sigma_2^2]P_{22}P'_{22}X'_1$$

$$= k^{-1}[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]X_1P_{22}P'_{22}X'_1$$

$$= [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]A_3.$$

Let $B_3 = [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]^{-1}A_3$. Then $B_3\Sigma B_3 = B_3$ and

$$Y'B_3Y \sim \chi'^2(k_3, \lambda_3)$$

where

$$k_3 = \text{rank } B_3 = \text{rank } A_3 = \text{tr } A_3 = k^{-1}\text{tr } X_1P_{22}P'_{22}X'_1 = m - 1$$

and

$$\lambda_3 = \mu^2 j_M^{\prime 1} X_1P_{22}P'_{22}X'_1 j_1^M C(\sigma) = 0.$$

Therefore

$$s_3 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]\chi^2(m - 1).$$

4. Distribution of $s_4 = k^{-1}Y'X_1P_{23}P'_{23}X'_1Y$.

Let $A_4 = k^{-1}X_1P_{23}P'_{23}X'_1$. Then $A_4A_4 = A_4$ and

$$A_4\Sigma A_4 = k^{-2}X_1P_{23}P'_{23}X'_1(X_1X'_1\sigma_1^2 + X_2X'_2\sigma_2^2 + \sigma^2 I)X_1P_{23}P'_{23}X'_1$$

$$= k^{-1}X_1P_{23}P'_{23}[(\sigma^2 + k\sigma_1^2)I_b + k^{-1}N'N\sigma_2^2]P_{23}P'_{23}X'_1$$

$$= [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]A_4.$$

Let $B_4 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]^{-1} A_4$. Then $B_4 \Sigma B_4 = B_4$ and

$$Y'B_4Y \sim \chi'^2(k_4, \lambda_4)$$

where

$$k_4 = \text{rank } B_4 = \text{rank } A_4 = \text{tr } A_4 = k^{-1}\text{tr } X_1P_{23}P_{23}'X_1' = m(n - 1)$$

and

$$\lambda_4 = \mu^2 j_M^1 X_1 P_{23} P_{23}' X_1' j_1^M C(\sigma) = 0.$$

Therefore

$$s_4 \sim [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2] \chi^2 [m(n - 1)].$$

5. Distribution of $s_5 = k(\lambda_2 t)^{-1} Y'AP_{31}P_{31}'A'Y$.

Let $A_5 = k(\lambda_2 t)^{-1} AP_{31}P_{31}'A'$. Then $A_5A_5 = A_5$ and

$$\begin{aligned}
A_5\Sigma A_5 &= k^2(\lambda_2 t)^{-2} AP_{31}P_{31}'A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)AP_{31}P_{31}'A' \\
&= k^2(\lambda_2 t)^{-2} AP_{31}P_{31}'A'(X_2X_2'\sigma_2^2 + \sigma^2 I)AP_{31}P_{31}'A' \\
&= k^2(\lambda_2 t)^{-2} AP_{31}P_{31}'[(rI - k^{-1}NN')^2\sigma_2^2 + (rI - k^{-1}NN')\sigma^2]P_{31}P_{31}'A' \\
&= k(\lambda_2 t)^{-1}(\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)AP_{31}P_{31}'A' = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)A_5.
\end{aligned}$$

Let $B_5 = (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)^{-1} A_5$. Then $B_5 \Sigma B_5 = B_5$ and therefore

$$Y'B_5Y \sim \chi'^2(k_5, \lambda_5)$$

where

$$k_5 = \text{rank } B_5 = \text{rank } A_5 = \text{tr } A_5 = k(\lambda_2 t)^{-1}\text{tr } AP_{31}P_{31}'A' = m - 1$$

and

$$\lambda_5 = \mu^2 j_M^1 AP_{31}P_{31}'A' j_1^M C(\sigma) = 0.$$

Therefore

$$s_5 \sim (\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)\,\chi^2(m - 1).$$

6. Distribution of $s_6 = v^{-1}Y'AP_{32}P_{32}'A'Y$.

Let $A_6 = v^{-1}AP_{32}P_{32}'A'$. Then $A_6A_6 = A_6$ and

$$A_6\Sigma A_6 = v^{-2}AP_{32}P_{32}'A'(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)AP_{32}P_{32}'A'$$

$$= v^{-1}AP_{32}(\sigma^2 + v\sigma_2^2)P_{32}'A' = (\sigma^2 + v\sigma_2^2)A_6.$$

Let $B_6 = (\sigma^2 + v\sigma_2^2)^{-1}A_6$. Then $B_6\Sigma B_6 = B_6$. Therefore

$$Y'B_6Y \sim \chi'^2(k_6, \lambda_6)$$

where

$$k_6 = \text{rank } B_6 = \text{rank } A_6 = \text{tr } A_6 = v^{-1}\text{tr } AP_{32}P_{32}'A' = m(n - 1)$$

and

$$\lambda_6 = \mu^2 j_M^1 AP_{32}P_{32}'A' j_1^M C(\sigma) = 0.$$

Therefore

$$s_6 \sim (\sigma^2 + v\sigma_2^2)\chi^2\,[m(n - 1)].$$

7. Distribution of $s_7 = Y'P_4P_4'Y$.

Parallel to the proof in the first section of this appendix for the distribution of s_6 we have

$$s_7 \sim \sigma^2\,\chi^2(M - b - t + 1).$$

8. Distribution of $s_8 = k^{-1}Y'X_1N'P_{31}P_{31}'A'Y$.

Let $A_8 = k^{-1}X_1N'P_{31}P_{31}'A'$. Then $s_8 = Y'[2^{-1}(A_8 + A_8')]Y$. It is easily shown that $4^{-1}(A_8 + A_8')\Sigma(A_8 + A_8') \neq 2^{-1}(A_8 + A_8')$. Therefore

s_8 is not distributed as a constant times a chi-square variate but as a linear combination of chi-square variates, that is,

$$s_8 \sim \Sigma\, a_i\, \chi^2(1)$$

where the a_i are the non-zero characteristic roots of $2^{-1}(A_8 + A_8')\Sigma$

9. Distribution of $s_9 = k^{-1} Y'X_1N'P_{32}P_{32}'A'Y$.

Let $A_9 = k^{-1}X_1N'P_{32}P_{32}'A'$. Then, following a discussion similar to that in 8 above, we have

$$s_9 \sim \Sigma\, b_i\, \chi^2(1)$$

where the b_i are the non-zero characteristic roots of $2^{-1}(A_9 + A_9')\Sigma$

10. Expectations of the statistics.

$$E(s_1) = \mu$$

$$E(s_2) = (b - t)(\sigma^2 + k\sigma_2^2)$$

$$E(s_3) = (m - 1)[\sigma^2 + k\sigma_1^2 + k^{-1}(rk - \lambda_2 t)\sigma_2^2]$$

$$E(s_4) = [m(n - 1)][\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda_1)\sigma_2^2]$$

$$E(s_5) = (m - 1)(\sigma^2 + k^{-1}\lambda_2 t\sigma_2^2)$$

$$E(s_6) = [m(n - 1)](\sigma^2 + v\sigma_2^2)$$

$$E(s_7) = (M - b - t + 1)\,\sigma^2$$

$$E(s_8) = E\ \mathrm{tr}\ k^{-1}Y'X_1N'P_{31}P_{31}'A'Y = E\ \mathrm{tr}\ k^{-1}YY'X_1N'P_{31}P_{31}'A'$$

$$= \mathrm{tr}\ k^{-1}(X_1X_1'\sigma_1^2 + X_2X_2'\sigma_2^2 + \sigma^2 I)X_1N'P_{31}P_{31}'A'$$

$$= \mathrm{tr}\ k^{-1}A'(X_2X_2'\sigma_2^2 + \sigma^2 I)X_1N'P_{31}P_{31}'$$

$$= \text{tr } k^{-1}P'_{31}(rI - k^{-1}NN')NN'P_{31}\sigma_2^2$$

$$= k^{-2}\text{tr } [rk(rk - \lambda_2 t) - (rk - \lambda_2 t)^2] I_{m-1}\sigma_2^2$$

$$= k^{-2}(m - 1)(\lambda_2 t)(rk - \lambda_2 t)\sigma_2^2$$

$$E(s_9) = E \text{ tr } k^{-1}Y'X_1N'P_{32}P'_{32}A'Y = E \text{ tr } k^{-1}YY'X_1N'P_{32}P'_{32}A'$$

$$= \text{tr } k^{-1}(X_1X'_1\sigma_1^2 + X_2X'_2\sigma_2^2 + \sigma^2 I)X_1N'P_{32}P'_{32}A'$$

$$= k^{-1}\text{tr } P'_{32}(rI - k^{-1}NN')NN'P_{32}\sigma_2^2$$

$$= k^{-2}\text{ tr } [rk(r - \lambda_1) - (r - \lambda_1)^2] I_{m(n-1)}$$

$$= k^{-2}m(n - 1)(r - \lambda_1)(rk - r + \lambda_1)\sigma_2^2$$

APPENDIX D

PAIRWISE INDEPENDENCE OF THE MINIMAL SUFFICIENT STATISTICS

In this appendix we will determine the pairwise independence of the statistics in each of the minimal sets which were found for each of the designs which were considered.

General Considerations

In order to determine pairwise independence, we shall first state a theorem on which the proofs in subsequent sections will be based.

Theorem. If the (M x 1) vector Z is distributed as the multivariate normal with mean μ and covariance matrix Σ and if $Z_1, Z_2, \ldots, Z_q$ are subvectors of Z such that $Z = (Z_1, Z_2, \ldots, Z_q)$, then a necessary and sufficient condition that the subvectors are jointly independent is that all the sub-matrices Σ_{ij} ($i \neq j$) be equal to the null matrix.

The Balanced Incomplete Block Design

In the balanced incomplete block design, we defined the vector Y and then transformed Y to Z by the relation $Z = P'Y$. Then

$$Z \sim MVN[\, P'\bar{\mu} \,,\, P'\Sigma P \,].$$

We then formed a partition of Z into $(Z_1, Z_2, Z_3, Z_4, Z_5)$. The form of $P'\Sigma P$ is as given in TABLE I and is the covariance matrix of Z.

By applying the theorem stated above, we have that the subvectors Z_1, Z_2, and Z_5 are mutually independent and are independent of Z_3 and Z_4, and that Z_3 and Z_4 are not independent. We now have the following relationships

$$s_1 = Z_1$$
$$s_2 = Z_2'Z_2$$
$$s_3 = Z_3'Z_3$$
$$s_5 = Z_4'Z_4$$
$$s_6 = Z_5'Z_5$$
$$s_4 = Z_3'Z_4.$$

Therefore we conclude that the statistics in the minimal set of sufficient statistics are pair-wise independent except for the pairs (s_3, s_4), (s_3, s_5) and (s_4, s_5).

The Singular, Group Divisible, PBIB Design

Following a procedure similar to that in the previous section and examining TABLE X, we have the result as stated in Corollary 2.3.

The Semi-regular, Group Divisible PBIB Design

Following a procedure similar to that of the first section and examining TABLE XII, we have the result as stated in Corollary 3.3.

The Regular, Group Divisible, PBIB Design

Again following the procedure of the first section and using TABLE XIV, we have the result as stated in Corollary 4.3.

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