MINIMAL SUFFICIENT STATISTICS
FOR EISENHART'S MODEL III

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PREFACE

In 1950 there appeared in the Statistical journal Sankhya a number of results concerning minimal sufficient statistics. These results were in the form of very general mathematical theorems. Since 1950 these theorems have been applied to several statistical models, but very little has been said about the variance component models with some parameters fixed and others random. In this thesis minimal sufficient statistics for this class of models have been investigated.

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CHAPTER I

INTRODUCTION

Statistics is a science which deals with data. An important phase of this science is the condensation of the data without loss of information. Suppose the objective of an experiment is to estimate the mean of a certain population. In order to estimate the mean, a random sample of size n is drawn. The n dimensional sample then provides information concerning the value of the population mean, but the average of the sample measurements provides an equivalent amount of information concerning the population mean. The average sample value is called a statistic and in this case we have a condensation of information from an n dimensional vector to a scalar. Statistics which condense without loss of information are termed sufficient statistics.

The original observations trivially always represent a sufficient statistic. One would prefer to work with a condensation and generally a condensation of small dimension. A sufficient statistic is called minimal if its dimension is less than or equal to the dimension of any other sufficient statistic.

A minimal sufficient statistic for a specific statistical design is not unique. We shall later define a property called completeness which the probability density function of some statistics possess. Sufficient statistics with complete density functions are called sufficient complete statistics. When a sufficient complete statistic exists, then every estimable function of the parameters possesses an unbiased estimate with uniformly smallest variance and this estimate is the unique unbiased estimate based on the sufficient complete statistic.

Statement of the Problem

The objective of this thesis is to exhibit minimal sufficient statistics for a class of statistical designs which fall in the category of Eisenhart's Model III (1). The exact definition of Eisenhart's Model III will be given later. The entire solution for the one-way classification of data and the two-way classification of data appears in this thesis. In addition, solutions for special sets of assumptions are given for the n-way classification situation.

Notation and Definitions

The entire thesis has been written in terms of matrix and vector notation. Eisenhart's Model III is a special case of the general linear hypothesis model which in matrix notation takes the form $Y = X\beta + e$.

Here Y is an (n x 1) vector of observations, X is an (n x b) matrix of known constants, β is a (b x 1) vector of unknown parameters and e is an (n x 1) vector of random errors. Eisenhart's restrictions for Model III are that certain of the parameters are fixed unknown constants, while the remaining parameters are distributed normally. e is also assumed to be distributed normally.

Previous Work in this Field

The basic theorems used in the solution of the problems encountered in this thesis are found in Lehmann and Scheffe's (2) 1950 paper. These theorems are in very general mathematical terms. Papers by F. Graybill and the author (3) and F. Graybill and D. Weeks (4) discuss minimal sufficient statistics for Eisenhart's Model II and Eisenhart's Model III.

CHAPTER II

GENERAL PROCEDURE

In Chapter III the one-way classification will be examined; in Chapter IV the two-way classification will be discussed; and in Chapter V certain aspects of the n-way classification are developed. Rather than discuss in each chapter those aspects of the theory which are similar, we instead treat them in this chapter.

Criterion for Sufficiency

Let $f(Y; \theta)$ be the joint probability density function of the vector of observations. Neyman's Criterion for S to be a sufficient statistic is that $f(Y; \theta)$ can be written $f(Y; \theta) = G(Y) \cdot H(S; \theta)$ where $G(Y)$ is independent of θ. Throughout this paper $f(Y; \theta)$ will be the multivariate normal density function; hence, an equivalent criterion is to write the quadratic form for $f(Y; \theta)$ as

$$Q(Y; \theta) = Q_1(Y) + Q_2(S; \theta)$$

Criterion for Completeness

A family of density functions such as $f(Y; \theta)$ is said to be complete if $\int h(Y)f(Y;\theta)dY \equiv 0$ implies $h(Y) \equiv 0$ almost everywhere. We shall use this definition as the criterion for determining whether a sufficient statistic is complete.

Criterion for Minimality

To show that a sufficient statistic S is minimal, we shall use a result stated and proved by Lehmann and Scheffe (2). The ratio

$$K(Y, Y_o) = \frac{f(Y;\theta)}{f(Y_o;\theta)} = \frac{f_1(S;\theta)}{f_1(S_o;\theta)}$$

is examined where Y_o is some point other than Y in the n dimensional sample space. The condition for S to be minimal is that $K(Y, Y_o)$ be independent of θ if and only if $S = S_o$. For multivariate normal density functions this procedure is equivalent to investigating whether or not the difference of the quadratic forms $Q(S) - Q(S_o)$ is independent of θ if and only if $S = S_o$.

Procedure for Condensing the Information

Consider now the quadratic form Q of the joint probability density function of the normal variables in the vector Y.

$$Q = (Y-EY)'V^{-1}(Y-EY)$$

where E is the operator denoting the expected value of Y and the covariance matrix V is by definition

$$V = E(Y-EY)(Y-EY)'$$

The procedure used in the following chapters for rewriting Q is as follows. There exists for Model III in the one-way classification, the two-way classification and in certain n-way situations an orthogonal matrix P of known constants such that PVP' is diagonal. We refer to the matrix P as an orthogonal matrix which will diagonalize V. The quadratic form can be written

$$Q = (Y-EY)'P'PV^{-1}P'P(Y-EY)$$

$$Q = (PY-EPY)'(PVP')^{-1}(PY-EPY) \ .$$

The theory from this point digresses depending upon the model involved. The objective in each case is to choose P so that Q can be written in a form which will exhibit a minimal sufficient statistic.

CHAPTER III

THE ONE WAY CLASSIFICATION VARIANCE COMPONENT MODEL

Consider the model $Y = \mu j + X\tau + e$ where j is a vector of ones, X is a matrix of zeros and ones, μ is a scalar parameter, and τ is a vector of t treatment parameters ordered according to the number of observations for the treatment. τ is assumed to be independent of e and distributed normally with mean ϕ and covariance matrix $\sigma_\tau^2 I$. e is assumed to be distributed normally with mean ϕ and covariance matrix $\sigma^2 I$. Let m_i be the number of treatments having n_i observations and let a equal the number of distinct n_i's.

In a variance component model as described above we have the following relationships.

X which is an (n x t) matrix is of rank $t < n$.

$$j_n' X = (n_1 j_{m_1}', \ldots, n_i j_{m_i}', \ldots, n_a j_{m_a}')$$

$$j_t' X' = j_n'$$

$$X'X = D_t$$

where D_t is a diagonal matrix of the form

$$\begin{bmatrix} n_1 I_{m_1} & \cdots & \phi & \cdots & \phi \\ \vdots & & \vdots & & \vdots \\ \phi & \cdots & n_i I_{m_i} & \cdots & \phi \\ \vdots & & \vdots & & \vdots \\ \phi & \cdots & \phi & \cdots & n_a I_{m_a} \end{bmatrix}$$

A Sufficient Statistic for the One-Way Classification Model

We use these relationships to define a partitioning of an orthogonal matrix P defined in Chapter II. Since $X'XX'X = D_t^2$ it follows that $D_t^{-1/2}X'XX'XD_t^{-1/2} = D_t$, and $D_t^{-1/2}X'XD_t^{-1/2} = I$. $P_1 = D_t^{-1/2}X'$ thus consists of t orthogonal rows which diagonalize XX'.

Consider next the matrix $W = (j, X)$ and choose P_2 to be any set of n-t orthogonal rows such that $P_2WW'P_2' = \phi$. There exists such a matrix because W is $(n \times t + 1)$ and has rank t. $P_2(jj' + XX')P_2' = P_2jj'P_2' + P_2XX'P_2' = \phi$. Since jj' and XX' are both positive semi-definite, $P_2jj'P_2' = \phi$ and $P_2XX'P_2' = \phi$. It then follows that $P_2j = \phi$ and $P_2X = \phi$. $P_2P_1' = P_2XD_t^{-1/2} = \phi$, hence $P' = (P_1', P_2')$ is an orthogonal $(n \times n)$ matrix.

Consider now the quadratic form for this model.

$$Q = (PY-EPY)'(PVP')^{-1}(PY-EPY)$$

$$EPY = P\mu j_n = \begin{bmatrix} \mu P_1 j_n \\ \mu P_2 j_n \end{bmatrix} = \begin{bmatrix} \mu D_t^{-1/2} X' j_n \\ \phi \end{bmatrix} = \begin{bmatrix} \mu D_t^{-1/2} D_t j_t \\ \phi \end{bmatrix} = \begin{bmatrix} \mu D_t^{1/2} j_t \\ \phi \end{bmatrix}$$

$$V = E(Y-\mu j)(Y-\mu j)' = E(X\tau + e)(X\tau + e)'$$

$$V = E(X\tau\tau'X') + 2E(e\tau'X') + E(ee') = \sigma_\tau^2 XX' + \sigma^2 I$$

$$PXX'P' = \begin{bmatrix} P_1XX'P_1' & P_1XX'P_2' \\ P_2XX'P_1' & P_2XX'P_2' \end{bmatrix} = \begin{bmatrix} D_t & \phi \\ \phi & \phi \end{bmatrix}$$

$$PVP' = \sigma_\tau^2 PXX'P' + \sigma^2 I$$

$$PVP' = \begin{bmatrix} \sigma_\tau^2 D_t + \sigma^2 I & \phi \\ \phi & \sigma^2 I \end{bmatrix}$$

$$Q = \begin{bmatrix} D_t^{-1/2}X'Y - \mu D_t^{1/2} j_t \\ P_2Y \end{bmatrix}' \begin{bmatrix} (\sigma_\tau^2 D_t + \sigma^2 I)^{-1} & \phi \\ \phi & \sigma^{-2} I \end{bmatrix} \begin{bmatrix} D_t^{-1/2}X'Y - \mu D_t^{1/2} j_t \\ P_2Y \end{bmatrix}$$

If we now substitute for D_t the quadratic form Q can be written

$$\begin{bmatrix} n_1^{-1/2}(T_1 - n_1\mu j_{m_1}) \\ \cdots\cdots\cdots \\ n_i^{-1/2}(T_i - n_i\mu j_{m_i}) \\ \cdots\cdots\cdots \\ n_a^{-1/2}(T_a - n_a\mu j_{m_a}) \\ P_2Y \end{bmatrix}' \begin{bmatrix} (n_1\sigma_\tau^2 + \sigma^2)^{-1} I_{m_1} & & & & \\ & \ddots & & & \\ & & (n_i\sigma_\tau^2 + \sigma^2)^{-1} I_{m_i} & & \\ & & & \ddots & \\ & & & & (n_a\sigma_\tau^2 + \sigma^2)^{-1} I_{m_a} \\ & & & & & \sigma^{-2} I \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

where T_i is the vector of the treatment totals for those treatments having n_i $(i = 1, \ldots, a)$ observations.

$$Q = \sum_{i=1}^{a} (n_i\sigma_\tau^2 + \sigma^2)^{-1} n_i^{-1} (T_i' T_i - 2n_i\mu j_{m_i}' T_i + m_i n_i^2 \mu^2) + \sigma^{-2} Y'P_2'P_2Y$$

This form of Q exhibits a sufficient statistic of dimension $2a + 1 - s$ where s is the number of m_i equal to one. In the case where all $m_i \geq 2$ for all i, the $2a + 1$ components are

$$T_i' T_i \quad (i = 1, \ldots a)$$

$$j_{m_i}' T_i \quad (i = 1, \ldots a)$$

$$Y'P_2'P_2Y$$

Computation of the Sufficient Statistic

The above statistic is readily computed, including the component $Y'P_2'P_2Y$ which we shall now show to be a function of the treatment totals and the total sum of squares.

$$Y'Y = Y'P'PY = Y'P_1'P_1Y + Y'P_2'P_2Y$$

$$Y'P_2'P_2Y = Y'Y - Y'XD_t^{-1}X'Y$$

$$Y'P_2'P_2Y = Y'Y - \sum_{i=1}^{a} n_i^{-1} T_i' T_i$$

The above results are now summerized in the following:

Theorem 3.1. The sum of the treatment totals for those treatments having n_i $(i = 1, \ldots a)$ observations, the sum of squares of the treatment totals for those treatments having n_i $(i = 1, \ldots, a)$ observations, and the total sum of squares form a sufficient statistic of $2a + 1$ components.

Minimal Sufficient Statistics for One-Way Models

Theorem 3.2. Let m_i, n_i, and a be defined as above for the one-way classification variance component model, then a minimal sufficient statistic has dimension $2a + 1 - s$ where s is the number of $m_i = 1$. One minimal sufficient statistic has as its components

T_i when $m_i = 1$

$j'_{m_i} T_i$ when $m_i \geq 2$

$T_i' T_i$ when $m_i \geq 2$

$Y'Y$

Consider first the case where $s = 0$. We shall show that when the number of treatments having n_i observations is greater than two for all i, then a minimal sufficient statistic is a statistic having as its components the sum and the sum of squares of the treatment totals, for those totals having n_i $(i = 1, \ldots, a)$ observations, and the total sum of squares.

Proof: Following the procedure of Chapter II we form the difference

$$Q-Q_0 = g_0[Y'P_2'P_2Y - Y_0'P_2'P_2Y_0] + \sum_{i=1}^{a} g_i[T_i'T_i - T_{i0}'T_{i0} - 2n_i\mu(j_{m_i}'T_i - j_{m_i}'T_{i0})]$$

where $g_0 = \sigma^{-2}$ and $g_i = (n_i\sigma_\tau^2 + \sigma^2)^{-1}n_i^{-1}$ $(i = 1, \ldots a)$. When

$$S = (T_1'T_1, j'T_1, T_2'T_2, j'T_2, \ldots T_a'T_a, j'T_a, Y'P_2'P_2Y)$$

is equal to S_0 with corresponding components $T_{i0}'T_{i0}$ etc. we have $Q - Q_0 = 0$. Next we set $Q - Q_0$ identically equal to zero in the parameters μ, σ_τ^2, and σ^2. The form of $Q - Q_0$ and the linear independence of the g_i's, (see appendix for proof) implies $Y'P_2'P_2Y = Y_0'P_2'P_2Y_0$, $T_i'T_i = T_{i0}'T_{i0}$ $(i = 1, \ldots a)$ and $j_{m_i}'T_i = j_{m_i}'T_{i0}$ $(i = 1, \ldots a)$. These are the conditions for the above set to be a minimal sufficient statistic. This concludes the proof of Theorem 3.2 for the case where $m_i \geq 2$ for all i. Observe that when $m_i = 1$, we have $T_i'T_i = T_i^2$ which is a function of the statistic $j_1'T_i = T_i$. Hence when $m_i = 1$ the component $T_i'T_i$ is deleted and the dimension of the sufficient statistic is reduced by one each time an $m_i = 1$.

It is interesting to note that when a = 1, XX' commutes with jj' and our model becomes a special case of the n-way problem discussed in Chapter V. In the case where a = 1, $n_1 = 1$, we have no estimate of σ_τ^2.

An Example

Consider the following data:

Treatment	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
Observations	2.3	3.8	3.5	2.9	2.9	4.1
	2.7	3.6	4.1	2.7	3.8	4.5
	3.2	3.8	3.1			3.8
	2.5	3.1	3.3			
Totals	10.7	14.3	14.0	5.6	6.7	12.4

For this example the parameters and total vectors are $n_1 = 4$, $m_1 = 3$, $n_2 = 2$, $m_2 = 2$, $n_3 = 3$, $m_3 = 1$, $a = 3$, $T_1' = (10.7, 14.3, 14.0)$, $T_2' = (5.6, 6.7)$, $T_3' = 12.4$. According to Theorem 3.2 the following are the values of the components of a minimal sufficient statistic for this example:

$$T_1'T_1 = 514.98$$

$$j'T_1 = 39.0$$

$$T_2'T_2 = 76.25$$

$$j'T_2 = 12.3$$

$$T_3 = 12.4$$

$$Y'Y = 220.13$$

Distribution of the Minimal Sufficient Statistic

In this section we shall discuss the distribution of $Y'P_2'P_2Y$ and the distribution of the components of the minimal sufficient statistic exhibited in Theorem 3.2.

Consider the partition of X into

$$X = (X_1, \ldots X_i, \ldots X_a)$$

where the column dimension of X_i is the number m_i. In terms of X_i and Y the vector of treatment totals T_i can be written $T_i = X_i'Y$. Now since Y is distributed $N(\mu j, \sigma_\tau^2 XX' + \sigma^2 I)$ it then follows that T_i is distributed $N[\mu X_i'j, X_i'(\sigma_\tau^2 XX' + \sigma^2 I)X_i]$. Upon simplifying this result we conclude that T_i is distributed $N[n_i \mu j, n_i(n_i\sigma_\tau^2 + \sigma^2)I]$. $j'T_i$ is then distributed $N[n_i m_i \mu, n_i m_i (n_i \sigma_\tau^2 + \sigma^2)]$.

In order to determine the distribution of $T_i'T_i$, we write $T_i'T_i$ as $Y'X_iX_i'Y$ and apply the following theorem:

If Y is distributed as a normal vector with mean μj and covariance V, then a necessary and sufficient condition that $Y'AY$ be distributed as $\chi'^2(p, \lambda)$ is that VA be idempotent. Here p is the rank of A and $\lambda = 2^{-1}\mu'A\mu$. (5)

(a) Let

$$(\sigma_\tau^2 XX' + \sigma^2 I) \cdot \frac{X_iX_i'}{n_i(n_i\sigma_\tau^2 + \sigma^2)}$$

play the role of VA in the above theorem. This quantity reduces to $n_i^{-1}X_iX_i'$ which is idempotent.

(b) the rank of X_iX_i' is m_i.

(c)

$$\lambda = \frac{2^{-1}\mu^2 j'X_iX_i'j}{n_i(n_i\sigma_\tau^2 + \sigma^2)} = 2^{-1}n_im_i(n_i\sigma_\tau^2 + \sigma^2)^{-1}\mu^2$$

The above statements imply that

$$\frac{Y'X_iX_i'Y}{n_i(n_i\sigma_\tau^2 + \sigma^2)}$$

is distributed as $\chi'^2[m_i,\ 2^{-1}n_im_i(n_i\sigma_\tau^2 + \sigma^2)^{-1}\mu^2]$. $T_i'T_i$ is then distributed as $n_i(n_i\sigma_\tau^2 + \sigma^2)$ times a noncentral chi square variate with m_i degrees of freedom and non centrality parameter

$$\lambda = 2^{-1}n_im_i\ (n_i\sigma_\tau^2 + \sigma^2)^{-1}\ \mu^2 .$$

$T_i'T_i$ and $T_j'T_j$; $(i \neq j)$ are independent because T_i and T_j have a covariance matrix equal to ϕ. The same conclusion holds concerning the independence of $j'T_k$ and $j'T_u$; $(k \neq u)$. If we delete $Y'Y$ from the components of the minimal sufficient statistic the remaining components are a mutually independent set.

We now wish to investigate the distribution of $Y'P_2'P_2Y$.

(a) $[\sigma^{-2}P_2'P_2(\sigma_\tau^2 XX' + \sigma^2 I)][\sigma^{-2}P_2'P_2(\sigma_\tau^2 XX' + \sigma^2 I)]$

$$= \sigma^{-4}\left\{\sigma_\tau^2\ P_2'P_2XX'P_2'P_2 + \sigma^2 P_2'P_2P_2'P_2\right\}\left\{\sigma_\tau^2 XX' + \sigma^2 I\right\}$$

$$= \sigma^{-2}P_2'P_2(\sigma_\tau^2 XX' + \sigma^2 I)$$

Thus $\sigma^{-2}P_2'P_2(\sigma_\tau^2 XX' + \sigma^2 I)$ is idempotent.

(b) The rank of $P_2'P_2$ is $n - t$.

(c) $\lambda = 2^{-1}\mu^2 j'P_2'P_2 j\sigma^{-2} = 0.$

The above statements imply that $Y'P_2'P_2Y$ is distributed as $\chi^2(n-t)$. In order to verify that $Y'P_2'P_2Y$ is independent of $T_i'T_i$ we again write $T_i'T_i$ as $Y'X_iX_i'Y$. $P_2'P_2(\sigma_\tau^2 XX' + \sigma^2 I)X_iX_i' = \phi$ is the condition for the independence of these quadratic forms. This is indeed satisfied for $P_2X = \phi$ and $P_2X_i = \phi$.

We have previously seen that $Y'Y$ can be written

$$Y'Y = \sum_{i=1}^{a} n_i^{-1} T_i'T_i + Y'P_2'P_2Y$$

$Y'Y$ is therefore distributed as a linear combination of independent chi square variables, one of which is a central chi square variable.

CHAPTER IV

THE TWO-WAY CLASSIFICATION MODEL III

Consider the model $Y = X\tau + Z\beta + e$ where τ is $(t \times 1)$ and is a vector of fixed estimable treatment parameters independent of β and e; and β is a vector of b normally distributed block components independent of e and ordered according to the number of plots k_i for the block. β is assumed to have mean ϕ and covariance matrix $\sigma_\beta^2 I$. Suppose that X and Z which are $(n \times t)$ and $(n \times b)$ matrices are each of full rank. e is assumed to be distributed normally with mean ϕ and covariance $\sigma^2 I$. Let m_i be the number of blocks having k_i plots and let a equal the number of distinct k_i's.

In the model described above we have the following relationship. $Z'Z = D_b$ where D_b is a diagonal matrix of the form

$$\begin{bmatrix} k_1 I_{m_1} & \phi & \phi \\ & \ddots & \\ \phi & k_i I_{m_i} & \phi \\ & \ddots & \\ \phi & \phi & k_a I_{m_a} \end{bmatrix}$$

The Partition of P

We now define a partitioning of P into $P' = [P_1', P_2', P_3']$. Since $Z'Z = D_b$, then $Z'ZZ'Z = D_b^2$. $D_b^{-1/2} Z'$ will then diagonalize ZZ' and since $D_b^{-1/2} Z'ZD^{-1/2} = I$, $D_b^{-1/2} Z'$ is a set of b orthogonal rows. Let $P_1 = D_b^{-1/2} Z'$.

Consider now the symmetric matrix $X'HX$ where H denotes the idempotent matrix $I - Z(Z'Z)^{-1}Z'$. The matrix $X'HX$ appears in connection with the normal equations of the model $Y = X\tau + Z\beta + e$.

$$X'X\hat{\tau} + X'Z\hat{\beta} = X'Y$$

$$Z'X\hat{\tau} + Z'Z\hat{\beta} = Z'Y$$

Solving for $\hat{\beta}$ in the second set of equations we have

$$\hat{\beta} = (Z'Z)^{-1} Z'Y - (Z'Z)^{-1} Z'X\hat{\tau} .$$

Substituting in the first set of equations we have

$$X'X\hat{\tau} + X'Z(Z'Z)^{-1} Z'Y - X'Z(Z'Z)^{-1} Z'X\hat{\tau} = X'Y$$

or

$$[X'X - X'Z(Z'Z)^{-1}Z'X]\hat{\tau} = [X' - X'Z(Z'Z)^{-1}Z']Y$$

In terms of the matrix H this is

$$X'HX\hat{\tau} = X'HY$$

$X'HX$ has rank at most equal to $(t - 1)$ since $j'X'HX = \phi$. That is, the sum of the rows and the sum of the columns of $X'HX$ each add to zero. Since the treatment differences are estimable, $X'HX$ has rank exactly equal to $(t - 1)$.

There exists a $(t \times t)$ orthogonal matrix $\overline{U}^*$ which will diagonalize the matrix $X'HX$. Furthermore, we can choose the first row of $\overline{U}^*$ to be j', whereupon we can write

$$\overline{U}^* X'HX\overline{U}^{*\prime} = \begin{bmatrix} 0 & \phi \\ \phi & D \end{bmatrix}$$

Let $\overline{U}$ be the matrix $\overline{U}^*$ with the row j' deleted. Then $UX'HXU' = I$ where $U = D^{-1/2}\overline{U}$. Let $P_2 = UX'H$. Since $Z'H = Z' - Z'Z(Z'Z)^{-1}Z' = \phi$, we then have $P_1P_2' = \phi$. Thus

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

is a set of $b + t - 1$ orthogonal rows.

Let $W = (X, Z)$ and let P_3 be any set of $n - t - b + 1$ orthogonal rows which are such that $P_3WW'P_3' = \phi$. There exist such orthogonal rows because W has rank $(t + b - 1)$, wherein it follows that WW' has $(n-t-b+1)$ characteristic roots equal to zero.

$$P_3WW'P_3' = P_3(XX' + ZZ')P_3' = P_3XX'P_3' + P_3ZZ'P_3' = \phi$$

Since XX' and ZZ' are positive semi-definite, we have

$$P_3XX'P_3' = \phi$$

and

$$P_3ZZ'P_3' = \phi .$$

Then $P_3X = \phi$ and $P_3Z = \phi$. Consequently $P_3P_1' = P_3ZD_b^{-1/2} = \phi$, and

$$P_3P_2' = P_3HXU' = P_3XU' - P_3Z(Z'Z)^{-1}ZXU' = \phi$$

Thus $P' = (P_1', P_2', P_3')$ with P_1, P_2, and P_3 defined as above, is an orthogonal (n x n) matrix.

Sufficient Statistics for the Two-Way Model

Consider now the quadratic form for this model.

$$Q = (PY - EPY)'(PVP')^{-1}(PY - EPY)$$

$$EPY = PXT = \begin{pmatrix} P_1X\tau \\ P_2X\tau \\ P_3X\tau \end{pmatrix} = \begin{pmatrix} D_b^{-1/2}Z'X\tau \\ UX'HX\tau \\ \phi \end{pmatrix}$$

$$V = E(Y - X\tau)(Y - X\tau)' = E(Z\beta + e)(Z\beta + e)'$$

$$V = E(Z\beta\beta'Z') + 2E(e\beta'Z') + E(ee') = \sigma_\beta^2 ZZ' + \sigma^2 I$$

$$PVP' = \sigma_\beta^2 PZZ'P' + \sigma^2 I$$

$$PZZ'P' = \begin{bmatrix} P_1ZZ'P_1' & P_1ZZ'P_2' & P_1ZZ'P_3' \\ P_2ZZ'P_1' & P_2ZZ'P_2' & P_2ZZ'P_3' \\ P_3ZZ'P_1' & P_3ZZ'P_2' & P_3ZZ'P_3' \end{bmatrix}$$

$$P_1ZZ'P_1' = D_b$$

$$P_2ZZ'P_2' = UX'HZZ'HXU' = \phi$$

$$P_3ZZ'P_3 = \phi$$

$$P_1ZZ'P_2' = P_1ZZ'HXU' = \phi$$

$$P_1ZZ'P_3' = \phi$$

$$P_2ZZ'P_3' = \phi$$

The quadratic form can now be written

$$Q = \begin{bmatrix} D_b^{-1/2}Z'Y - D_b^{-1/2}Z'X\tau \\ P_2Y - UX'HX\tau \\ P_3Y \end{bmatrix}' \begin{bmatrix} (\sigma_\beta^2 D_b + \sigma^2 I)^{-1} & & \\ & \sigma^{-2}I & \\ & & \sigma^{-2}I \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix}$$

Let the vector of block totals $Z'Y$ be denoted by B; let the vector of treatment totals X'Y be denoted by T; and let B_i be the vector of block totals for those blocks having k_i (i = 1, . . . a) plots. Let N = Z'X and let (N_1', . . . N_i', . . . N_a') be the partition of N' corresponding to the partition of B' into (B_1', . . . B_i', . . . B_a'). Q can then be written

$$Q = \sum_{i=1}^{a} k_i^{-1}(k_i\sigma_\beta^2 + \sigma^2)^{-1}(B_i - N_i\tau)'(B_i - N_i\tau) +$$

$$\sigma^{-2}(P_2Y-P_2X\tau)'(P_2Y-P_2X\tau) + \sigma^{-2}Y'P_3'P_3Y$$

The total sum of squares Y'Y can be written

$$Y'Y = Y'P'PY = Y'(P_1'P_1 + P_2'P_2 + P_3'P_3)Y.$$

Then

$$Y'P_3'P_3Y = Y'Y - Y'P_1'P_1Y - Y'P_2'P_2Y$$

$$Y'P_3'P_3Y = Y'Y - Y'ZD_b^{-1}Z'Y - Y'HXU'UX'HY$$

$Y'P_3'P_3Y$ is the intrablock error if Y'HXU'UX'HY can be shown to be the reduction due to treatments adjusted for blocks. In our notation the reduction due to treatments adjusted $R(\tau / \beta) = \hat{\tau}$'X'HY where $\hat{\tau}$ is a solution to the system of equations $X'HX\hat{\tau} = X'HY$. Augment this system in the following manner:

$$\begin{bmatrix} X'HX & j \\ j' & 0 \end{bmatrix} \begin{bmatrix} \hat{\tau} \\ g \end{bmatrix} = \begin{bmatrix} X'HY \\ 0 \end{bmatrix}$$

The matrix

$$\begin{bmatrix} X'HX & j \\ j' & 0 \end{bmatrix}$$

has an inverse which we shall now show to be

$$\begin{bmatrix} U'U & t^{-1}j \\ t^{-1}j' & 0 \end{bmatrix}$$

$$t^{-1}j'j = 1$$

$$j'U'U = \phi$$

$$t^{-1}j'X'HX = \phi$$

There remains to show that $U'UX'HX + t^{-1}J = I$. We shall show this in the following manner. Since $UX'HXU' = I$, then $U'UX'HXU' = U'$. Multiplying by DU we have $U'UX'HXU'DU = U'DU$. Now $\overline{U}^{*\prime}\overline{U}^{*} = I$, hence

$$(t^{-1/2}j,\ \overline{U}') \begin{bmatrix} t^{-1/2}j' \\ \overline{U} \end{bmatrix} = t^{-1}jj' + \overline{U}'\overline{U} = I$$

$$t^{-1}J + U'D^{1/2}D^{1/2}U = I$$

$$U'DU = I - t^{-1}J$$

Substituting in the above equation we have

$$U'UX'HX - U'UX'HX\,t^{-1}J = I - t^{-1}J.$$

$HXj = \phi$, hence we have $U'UX'HX + t^{-1}J = I$. We may now write

$$\begin{pmatrix} \hat{\tau} \\ g \end{pmatrix} = \begin{bmatrix} U'U & t^{-1}j \\ t^{-1}j' & 0 \end{bmatrix} \begin{pmatrix} X'HY \\ 0 \end{pmatrix} = \begin{pmatrix} U'UX'HY \\ 0 \end{pmatrix}$$

$\hat{\tau} = U'UX'HY$ is then a solution of the system $X'HX\hat{\tau} = X'HY$ and $R(\tau/\beta) = Y'HXU'UX'HY$. $Y'P_3'P_3Y$ is consequently the intra-block error and $Y'P_3'P_3Y = Y'Y - R(\beta) - R(\tau/\beta)$.

The following theorem is a summary of the results of the preceding discussion.

Theorem 4.1. The vector of block totals B, the vector of treatment totals T and the total sum of squares Y'Y form a sufficient statistic of (b + t + 1) components.

Proof: If we let

$$g_i = k_i^{-1}(k_i\sigma_\beta^2 + \sigma^2)^{-1}$$

then the quadratic form Q can be written

$$Q = \sum_{i=1}^{a} g_i(B_i - N_i\tau)'(B_i - N_i\tau) + \sigma^{-2}(P_2Y - P_2X\tau)'(P_2Y - P_2X\tau) + \sigma^{-2}Y'P_3'P_3Y.$$

$$P_2Y = UX'HY = UX'Y - UX'Z(Z'Z)^{-1}Z'Y$$

$$P_2Y = UT - UX'ZD_b^{-1}B .$$

Substituting for P_2Y and $Y'P_3'P_3Y$, Q can be written

$$Q = \sum_{i=1}^{a} g_i(B_i - N_i\tau)'(B_i - N_i\tau) + \sigma^{-2}[(UT - UX'ZD_b^{-1}B) - (UX'X\tau + UX'ZD_b^{-1}Z'X\tau)]'[(UT - UX'ZD_b^{-1}B) - (UX'X\tau + UX'ZD_b^{-1}Z'X\tau)]$$
$$+ \sigma^{-2}[(Y'Y - R(\beta) - R(\tau/\beta)] .$$

This form of Q exhibits the vectors B and T and the sum of squares Y'Y as a sufficient statistic of (b + t + 1) components.

Minimal Sufficient Statistics

We now direct our attention to finding minimal sufficient statistics.

Theorem 4.2. Let n_i be the rank of N_i. When $n_i = m_i$ for all i (i = 1, . . . a) the dimension of a minimal statistic is b + t. The block totals B, (t - 1) of the treatment totals and Y'Y form a minimal sufficient statistic when $n_i = m_i$ for all i (i = 1, . . . a).

Proof:

$$Q = \sum_{i=1}^{a} g_i(B_i - N_i\tau)'(B_i - N_i\tau) +$$
$$\sigma^{-2}[(P_2Y - P_2X\tau)'(P_2Y - P_2X\tau) + Y'P_3'P_3Y]$$

This form of Q exhibits a sufficient statistic of b + t components. These components are the b block totals, the (t - 1) components of the vector P_2Y and the scalar $Y'P_3'P_3Y$. To show that this statistic is minimal we apply the procedure of Chapter II.

$$Q - Q_0 = \sum_{i=1}^{a} g_i[(B_i'B_i - B_{i0}'B_{i0}) - 2(B_i - B_{i0})'N_i\tau] + \sigma^{-2}[(Y'P_2' \cdot P_2Y - Y_0'P_2' \cdot P_2Y_0) - 2(P_2Y - P_2Y_0)'P_2X\tau + (Y'P_3'P_3Y - Y_0'P_3'P_3Y_0)] \equiv 0 .$$

According to Lemma 1, found in the appendix, the set $\{g_i\}$ is a set of linearly independent functions of the parameters involved. The linear independence of the g_i's implies that

$$(B_i'B_i - B_{i0}'B_{i0}) - 2(B_i - B_{i0})'N_i\tau \equiv 0$$

and

$$(Y'P_2' \cdot P_2Y - Y_0'P_2' \cdot P_2Y_0) - 2(P_2Y - P_2Y_0)'P_2X\tau + Y'P_3'P_3Y - Y_0'P_3'P_3Y_0 \equiv 0$$

The independence of the m_i rows of N_i implies $N_i\tau$ is a vector of m_i linearly independent functions of the parameters $\tau_1 \ldots \tau_t$. This in turn implies $B_i - B_{i0} = \phi$ or $B = B_{i0}$. In like manner we have $P_2Y = P_2Y_0$ which implies $Y'P_2' \cdot P_2Y - Y_0'P_2' \cdot P_2Y_0 = 0$. Finally then $Y'P_3'P_3Y$

must equal $Y_0'P_3'P_3Y_0$. Together these relationships are Lehmann and Scheffe's condition that the t + b components exhibited are a minimal sufficient statistic.

Since $j_b'B = j_t'T$, the block totals B, (t - 1) of the treatment totals and Y'Y form a sufficient statistic. Since this statistic has dimension t + b, it is a minimal sufficient statistic.

We now extend this theorem to the case where $n_i < m_i$. (Notice that this is the only alternative case. The rank of m_i rows can never be greater than m_i).

Theorem 4.3. Let S be the set $\{i \mid n_i \le m_i\}$ and let $\bar{S}$ be the complement of S. In this case the dimension of a minimal sufficient statistic is

$$\sum_{i \in S} (n_i + 1) + \sum_{i \in \bar{S}} n_i + t .$$

The total sum of squares Y'Y, (t - 1) treatment totals, the block totals for those blocks where $n_i = m_i$, n_i linearly independent functions of the block totals for each set of blocks where $n_i < m_i$ and the sum of squares $B_i'B_i$ for each set of blocks where $n_i < m_i$ form a minimal sufficient statistic for this case.

Proof: We have from the proof of Theorem 4.2

$$P_2Y = P_2Y_0$$

$$Y'P_3'P_3Y = Y_0'P_3'P_3Y_0$$

and

$$(B_i'B_i - B_{i0}'B_{i0}) - 2(B_i - B_{i0})'N_i\tau \equiv 0$$

Consider first n_i where $i \in S$. Since the rank of N_i, equal to n_i, is less than the dimension of N_i, this identity does not imply that all of the block totals are present in a given minimal statistic.

Let N_i be partitioned into $N_i' = (N_{i1}', N_{i2}')$ where N_{i1} has rank n_i and dimension $(n_i \times t)$ and N_{i2} has dimension $[(m_i - n_i) \times t]$. (For sake of notation consider N_{i1} to be the first n_i rows of N_i.) If we now partition B_i into $B_i' = (B_{i1}', B_{i2}')$ with dimensions corresponding to those of N_{i1} and N_{i2} then we can write $B_i'N_i\tau = B_{i1}'N_{i1}\tau + B_{i2}'N_{i2}\tau$. Since the rows of N_{i2} are linear combinations of the rows of N_{i1}, there exists a matrix G_i such that $N_{i2} = G_iN_{i1}$. Thus $B_i'N_i\tau = B_{i1}'N_{i1}\tau + B_{i2}'G_iN_{i1}\tau = (B_{i1} + G_i'B_{i2})'N_{i1}\tau$ where N_{i1} is of full rank. The linear independence of the g_i by Lemma 1 implies that

$$(B_i'B_i - B_{i0}'B_{i0}) - 2[(B_{i1} + G_i'B_{i2})' - (B_{i10} + G_i'B_{i20})']N_{i1}\tau \equiv 0$$

which in turn implies that $B_i'B_i = B_{i0}'B_{i0}$ and $B_{i1} + G_i'B_{i2} = B_{i10} + G_i'B_{i20}$. This is a set of $n_i + 1$ components for a given $i \in S$. Summing over $i \in S$ we obtain

$$\sum_{i \in S} (n_i + 1)$$

components of this type. For n_i where $i \in \bar{S}$ we proved in Theorem 4.2 that $B_i = B_{i0}$. There are $\sum_{i \in \bar{S}} n_i$ statistics of this type. Lehmann and Scheffe's condition is thus satisfied for the

$$\sum_{i \in S} (n_i + 1) + \sum_{i \in \bar{S}} n_i + t$$

above components.

We have proved the first statement of Theorem 4.3. There remains to show that the following

$$\sum_{i \in S} (n_i + 1) + \sum_{i \in \bar{S}} n_i + t$$

components are sufficient and thus minimal sufficient:

$Y'Y$

$(t - 1)$ treatment totals

B_i where $i \in \bar{S}$

$B_i'B_i$ and $B_{i1} + G_i'B_{i2}$ where $i \in S$

The quadratic form Q written as

$$Q = \sum_{i \in S} g_i [B_i - N_i\tau]'[B_i - N_i\tau] + \sigma^{-2}[P_2Y - P_2X\tau]'[P_2Y - P_2X\tau]$$

$$+ \sigma^{-2}Y'P_3'P_3Y + \sum_{i \in S} g_i [B_i'B_i - 2(B_{i1} + G_i'B_{i2})'N_{i1}\tau + \tau'N_i'N_i\tau]$$

exhibits P_2Y and $Y'P_3'P_3Y$ as components of a minimal statistic. Consider P_2Y. $P_2Y = UT - UN'D_b^{-1}B$. The vector $N'D_b^{-1}B$ can be partitioned into subvectors of dimension m_i corresponding to the partitioning of B into $B' = (B_1', \ldots B_a')$. The general term of this partitioning is $k_i^{-1} N_i'B_i$ for $i \in \overline{S}$. From the above discussion we see that P_2Y can be written in terms of the block totals B_i for $i \in \overline{S}$, $(B_{i1} + G_i'B_{i2})$ for $i \in S$, and the treatment totals T. We shall now show that one of the treatment totals is unnecessary. Since $N_{i2} = G_iN_{i1}$, we have $N_{i2}j = G_iN_{i1}j$. Then

$$k_ij_{m_i-n_i} = k_iG_ij_{n_i},$$

$$j_{m_i-n_i} = G_ij_{n_i},$$

or

$$j_{n_i}'G_i' = j_{m_i-n_i}'.$$

Now

$$\sum_{i\in S} j_{n_i}'(B_{i1} + G_i'B_{i2}) + \sum_{i\in\overline{S}} j_{m_i}'B_i = \sum_{i\in S}(j_{n_i}'B_{i1} + j_{m_i-n_i}'B_{i2}) + \sum_{i\in\overline{S}} j_{m_i}'B_i$$
$$= \sum_{i=1}^{a} j_{m_i}'B_i = j'B = j'T$$

We can thus conclude that P_2Y is a function of t - 1 treatment totals and the above functions of block totals.

$$Y'P_3'P_3Y = Y'Y - Y'ZD_b^{-1}Z'Y - Y'HXU'UX'HY$$

$$Y'ZD_b^{-1}Z'Y = \sum_{i=1}^{a} k_i^{-1} B_i'B_i$$

$$Y'HX = Y'X - Y'ZD_b^{-1}N = T' - B'D_b^{-1}N$$

We have already seen that $B'D_b^{-1}N$ is a function of $B_{i1} + G_i'B_{i2}$ when $i \in S$ and B_i when $i \in \overline{S}$. We can here conclude that $Y'P_3'P_3Y$ is a function of $Y'Y$ and the above mentioned components. This completes the proof of Theorem 4.3 for we have shown that the stated statistic is a function of a minimal sufficient statistic each statistic having the same dimension.

Example for the Two-Way Classification

Data

	Treatments	τ_1	τ_2	τ_3
	β_1^*	2.1	4.2	---
		3.5	3.9	---
	β_2^*	1.8	5.3	3.8
		---	---	3.0
Block	β_3^*	1.5	5.5	3.2
		2.6	4.8	---
		2.2	---	---
	β_4^*	2.0	4.8	4.1
		---	---	3.2
	β_5^*	2.8	---	3.8
		3.2	---	4.2
	β_6^*	1.8	4.0	3.5
		2.0	---	---

We observe first that blocks numbered 1, 2, 4, 5, 6, all have 4 plots per block and block numbered 3 has 6 plots. This calls for a renumbering of the blocks. But before we renumber observe that in the matrix N* several rows are linear combinations of other rows.

$$N^* = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

Rows 1, 2, 5, are linearly independent, hence in the renumbering we place the blocks corresponding to these numbers first.

Data Reordered According to Plots per Block and Independence of Rows

	τ_1	τ_2	τ_3
β_1	2.1	4.2	---
	3.5	3.9	---
β_2	1.8	5.3	3.8
	---	---	3.0
β_3	2.8	---	3.8
	3.2	---	4.2
β_4	2.0	4.8	4.1
	---	---	3.2
β_5	1.8	4.0	3.5
	2.0	---	---
β_6	1.5	5.5	3.2
	2.6	4.8	---
	2.2	---	---

According to Theorem 4.1 the block totals $B' = (13.7, 13.9, 14.0, 14.1, 11.3, 19.8)$, the treatment totals $T' = (25.5, 32.5, 28.8)$, and the total sum of squares $Y'Y = 321.4$ are the values of the components of a sufficient statistic for this example.

$$N = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

We have $m_1 = 5$ and $m_2 = 1$. The partition of N is as follows.

$$N_1 = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \qquad N_2 = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

The rank of N_1 is $n_1 = 3$. We partition N_1 into

$$N_{11} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} \qquad N_{12} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

N_{12} can be written in terms of N_{11} through the matrix equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

Let

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Now Theorem 4.2 tells us that the dimension of a minimal statistic for this data is $n_1 + 1 + n_2 + t = 8$. The values of the components of a minimal statistic for this example are

$$T' = (25.5,\ 32.5),$$

$$B_2 = 19.8,$$

$$(B_{11} + G'B_{12})' = (19.35,\ 28.0,\ 19.65),$$

$$B_1'B_1 = 903.4$$

and

$$Y'Y = 321.4$$

Distribution of the Minimal Sufficient Statistic

In this section we shall discuss the distribution of $Y'P_i'P_iY$, $(i = 1, 2, 3)$ and the distribution of the components of the minimal sufficient statistic exhibited in Theorem 4. 3.

Since Y is distributed $N[\mu j, \sigma_\beta^2 ZZ' + \sigma^2 I]$, we can immediately write the distribution of the vector of treatment totals $T = X'Y$ as $N[\mu X'j, X'(\sigma_\beta^2 ZZ' + \sigma^2 I)X]$, or equivalently $N[\mu X'j, (\sigma_\beta^2 N'N + \sigma^2 D_t)]$.

The vector of block totals $B = Z'Y$ is distributed $N[\mu Z'j, Z'(\sigma_\beta^2 ZZ' + \sigma^2 I)Z]$, or equivalently $N[\mu Z'j, (\sigma_\beta^2 D_b^2 + \sigma^2 D_b)]$. We now partition Z into $Z = (Z_1, \ldots, Z_i, \ldots Z_a)$ such that $B_i = Z_i'Y$ is the vector of block totals for those blocks having k_i plots per block. B_i is distributed $N[\mu Z_i'j, Z_i'(\sigma_\beta^2 ZZ' + \sigma^2 I)Z_i]$, or equivalently $N[k_i\mu j_{m_i}, (k_i^2\sigma_\beta^2 + k_i\sigma^2)I_{m_i}]$. The distribution of the components B_{i1} and $(B_{i1} + G'B_{i2})$ are immediate consequences of the above statement.

B_{i1} is distributed $N[k_i\mu j_{n_i}, (k_i^2\sigma_\beta^2 + k_i\sigma^2)I_{n_i}]$, $G_i'B_{i2}$ is distributed $N[k_i\mu G_i'j, (k_i^2\sigma_\beta^2 + k_i\sigma^2)G_i'G_i]$, and $B_{i1} + G_i'B_{i2}$ is distributed $N[k_i\mu(I + G_i')j, (k_i^2\sigma_\beta^2 + k_i\sigma^2)(I + G_i'G_i)]$.

Let us now investigate the distribution of $B_i'B_i$. Observe first that $(k_i^2\sigma_\beta^2 + k_i\sigma^2)^{-1}I$ multiplied by the variance of B_i is the identity matrix of dimension m_i. This is the necessary and sufficient condition for $(k_i^2\sigma_\beta^2 + k_i\sigma^2)^{-1}B_i'B_i$ to be distributed as $\chi'^2(m_i, \lambda)$ where

$$\lambda = \frac{2^{-1}\mu^2 k_i^2 j'j}{k_i^2\sigma_\beta^2 + k_i\sigma^2} = \frac{2^{-1}k_i m_i \mu^2}{k_i\sigma_\beta^2 + \sigma^2}.$$

$B_i'B_i$ is then distributed as $k_i(k_i\sigma_\beta^2 + \sigma^2)\chi'^2[m_i, 2^{-1}k_i m_i\mu^2(k_i\sigma_\beta^2 + \sigma^2)^{-1}]$ $Z_iZ_i'(\sigma_\beta^2 ZZ' + \sigma^2 I)Z_jZ_j' = \phi$ for $i \neq j$. This is sufficient to imply the independence of $B_i'B_i$ and $B_j'B_j$ $(i \neq j)$.

We shall now turn our attention to a discussion of the distribution of $Y'P_i'P_iY$; $(i = 1, 2, 3)$. Since $P_1 = D_b^{-1/2}Z'$ the quadratic form $Y'P_1'P_1Y$ can be written $Y'ZD_b^{-1}Z'Y$. If we now partition Z as above $Y'P_1'P_1Y$ can be written

$$Y'P_1'P_1Y = \sum_{i=1}^{a} k_i^{-1}Y'Z_iZ_i'Y = \sum_{i=1}^{a} k_i^{-1}B_i'B_i.$$

$Y'P_1'P_1Y$ is then distributed as a linear combination of independent non central chi square variables.

Consider the quantities $\sigma^{-2}Y'P_i'P_iY$ $(i = 2, 3)$; and the following relationships.

(a) $\sigma^{-2}P_i'P_i(\sigma_\beta^2 ZZ' + \sigma^2 I)$ reduces to $P_i'P_i$;

(b) $P_2'P_2$ is idempotent of rank $t - 1$;

$P_3'P_3$ is idempotent of rank $n - t - b + 1$;

(c) $\lambda_i = 2^{-1}\mu^2 j'P_i'P_i j\sigma^{-2} = 0$, $(i = 2, 3)$.

The above relationships imply that $Y'P_2'P_2Y$ is distributed as $\sigma^2\chi^2(t-1)$ and $Y'P_3'P_3Y$ is distributed as $\sigma^2\chi^2(n - t - b + 1)$. Since $P_i'P_i(\sigma_\beta^2 ZZ' + \sigma^2 I)P_j'P_j = \phi$; $(i \neq j; i, j = 1, 2, 3)$, the quadratic forms $Y'P_i'P_iY$ are an independent set.

We have previously seen that Y'Y can be written

$$Y'Y = \sum_{i=1}^{3} Y'P_i'P_iY = \sum_{i=1}^{a} k_i^{-1}B_i'B_i + Y'P_2'P_2Y + Y'P_3'P_3Y$$

Y'Y is therefore distributed as a linear combination of independent chi square variables two of which are central chi square variables.

Balanced and Partially Balanced Designs

Balanced and partially balanced incomplete block designs are of course special cases of the two way classification problem just considered. In this section we shall exhibit, in a more familiar notation, a minimal sufficient statistic for these designs. As before let n equal the rank of the matrix Z'X in the model $Y = X\tau + Z\beta + e$. Componentwise the model for the balanced and partially balanced designs is

$$y_{ijk} = \tau_i + \beta_j + e_{ijk}$$

(i = 1, . . . t) τ_i a fixed constant

(j = 1, . . . b) β_j distributed $N(0, \sigma_\beta^2)$ and independently

$k = n_{ij} = 0$ if treatment i does not appear in block j

$k = n_{ij} = 1$ if treatment i appears in block j.

In Table I which follows the block totals the treatment totals are defined as

$$y_{\cdot j \cdot} = \sum_{ik} y_{ijk}$$

$$y_{i\cdot\cdot} = \sum_{jk} y_{ijk}$$

Table I

Minimal Sufficient Statistics for Two-Way Classification Designs

Design	Dimension	A Minimal Sufficient Statistic
Balanced Complete Block	t+2	$\sum_{ijk} y_{ijk}^2$, $\sum_j y_{\cdot j \cdot}^2$, $y_{i\cdot\cdot}$ $(i=1,\ldots,t)$
Balanced Incomplete Block with b > t	2t+1	$\sum_{ijk} y_{ijk}^2$, $\sum_j y_{\cdot j \cdot}^2$, $y_{i\cdot\cdot}$ $(i=1,\ldots,t-1)$, $\sum_j n_{ij} y_{\cdot j \cdot}$ $(i=1,\ldots,t)$
Balanced Incomplete Block with b = t	2t	$\sum_{ijk} y_{ijk}^2$, $y_{\cdot j \cdot}$ $(j=1,\ldots,b)$, $y_{i\cdot\cdot}$ $(i=1,\ldots t-1)$
Partially Balanced Incomplete Block with n < b	n+t+1	$\sum_{ijk} y_{ijk}^2$, $y_{i\cdot\cdot}$ $(i=1,\ldots,t-1)$, $\sum_j y_{\cdot j \cdot}^2$ and n linearly independent functions of block totals.
Partially Balanced Incomplete Block with n = b	b+t	$\sum_{ijk} y_{ijk}^2$, $y_{\cdot j \cdot}$ $(j=1,\ldots,b)$, $y_{i\cdot\cdot}$ $(i=1,\ldots,t-1)$

We have seen that the dimension of a minimal sufficient statistic in a partially balanced incomplete block design is a function of the rank n of the matrix Z'X where Z and X are matrices in the model $Y = X\tau + Z\beta + e$. Of special interest are the partially balanced group divisible designs with two associate classes (6). In these designs the observations can be divided into c groups of d each such that any two treatments of the same group are first associates while two treatments from different groups are second associates. Table II, which follows, gives the value of the rank of Z'X in terms of the number of treatments t and the parameter d (7).

Table II

The Rank of Z'X For Partially Balanced Group Divisible Incomplete Block Designs with Two Associate Classes

Design	Rank
Singular	d
Semi-regular	t - d + 1
Regular	t

CHAPTER V

A CLASS OF N-WAY CLASSIFICATION MODELS

Consider the model

$$Y = \sum_{j=1}^{v} X_j \tau_j + \sum_{k=1}^{h} Z_k \beta_k$$

where X_j and Z_k are matrices of known constants; τ_j is a vector of t_j parameters; and β_k is a vector of b_k parameters. Throughout this chapter the following assumptions are made.

(a) τ_j (i = 1, . . . , v) are vectors of fixed functionally independent unknown parameters.

(b) β_k (k = 1, . . . , h) are vectors distributed normally with mean ϕ and covariance $\sigma_k^2 I$. All the components of the vectors β_k (k = 1, . . . , h) are stochastically independent.

(c) σ_k^2 (k = 1, . . . , h) are functionally independent and each σ_k^2 is independent of τ_j (j = 1, . . . , v).

(d) All pairs of matrices from the set $\left\{X_1X_1' \ldots X_vX_v', Z_1Z_1' \ldots Z_hZ_h'\right\}$ commute.

(e) For some k, say k_0, we have $Z_{k_0} = I$.

(f) The matrices X_jX_j' (j = 1, . . . , v), Z_kZ_k' (k = 1, . . . , h) are linearly independent.

Definition: The matrices A_i $(i = 1, \ldots, k)$ are said to be linearly independent if for any set of real constants α_i $(i = 1, \ldots, k)$

$$\sum_{i=1}^{k} \alpha_i A_i = \phi$$

implies $\alpha_i = 0$.

Partitioning P

There exists an orthogonal matrix P which has the property that $PX_jX_j'P' = D_j$ $(j = 1, \ldots, v)$ and $PZ_kZ_k'P' = E_k$ $(k = 1, \ldots, h)$ where D_j and E_k are diagonal (5). Let the rank of X be denoted by q where $X = (X_1, X_2, \ldots, X_v)$, then

$$XX' = \sum_{j=1}^{v} X_jX_j'$$

$$PXX'P' = \sum_{j=1}^{v} D_j.$$

Since the rank of $PXX'P'$ is also q, exactly q of the diagonal elements of $PXX'P'$ are nonzero. Let the rows of P be arranged such that the q nonzero characteristic roots of XX' are the first q characteristic roots on the diagonal of $PXX'P'$. Thus $P_uXX'P_u' \neq 0$ for $u = 1, \ldots, q$ and $P_uXX'P_u' = 0$ for $u = q+1, \ldots, n$. This, however, implies $P_uX \neq 0$ for $u = 1, \ldots, q$ and $P_uX = 0$ for $u = q+1, \ldots, n$. Let $P' = (R', S')$ where R is the $(q \times n)$ matrix of the first q rows of P and S is the $[(n-q) \times n]$ matrix of the last $n - q$ rows of P.

Let $\tau' = (\tau_1', \ldots, \tau_v')$. $RX\tau$ is then a (q x 1) vector of linearly independent estimable functions of the parameters of τ. Furthermore, the vector $RX\tau$ is composed of linear combinations of the parameters which span the space of all linearly independent estimable functions of the parameters of τ. $SX\tau$ on the other hand is ϕ.

Let V denote the covariance matrix of Y. It readily follows by applying the definition of covariance to the model considered that

$$V = \sum_{k=1}^{h} \sigma_k^2 Z_k Z_k'.$$

Now

$$PVP' = \sum_{k=1}^{h} \sigma_k^2 E_k$$

which is a diagonal matrix with the characteristic roots of V on the diagonal.

$$PV^{-1}P' = \begin{pmatrix} R \\ S \end{pmatrix} V^{-1}[R', S'] = \begin{pmatrix} RV^{-1}R' & RV^{-1}S' \\ SV^{-1}R' & SV^{-1}S' \end{pmatrix}$$

$RV^{-1}S'$ and $SV^{-1}R'$ equal ϕ. $RV^{-1}R'$ and $SV^{-1}S'$ are diagonal matrices with diagonal elements equal to reciprocals of the characteristic roots of V.

$$Q = (Y - X\tau)'V^{-1}(Y - X\tau)$$

$$Q = (PY - PX\tau)'PV^{-1}P'(PY - PX\tau)$$

$$Q = \begin{bmatrix} RY - RX\tau \\ SY \end{bmatrix}' \begin{bmatrix} RV^{-1}R' & \phi \\ \phi & SV^{-1}S' \end{bmatrix} \begin{bmatrix} RY - RX\tau \\ SY \end{bmatrix}$$

Let s be the number of distinct diagonal elements of $SV^{-1}S'$. Denote them by d_i^{-1} $(i = 1, \ldots, s)$ where the d_i are s of the distinct characteristic roots of V. Arrange the rows of S so that the like characteristic roots are grouped together on the diagonal of SVS'. Then partition S' into $(S_1', S_2', \ldots, S_s')$ such that $S_iV^{-1}S_i' = d_i^{-1}I$ where the dimension of I is the multiplicity of d_i in the set of diagonal elements of SVS'.

Sufficient Statistics for the N-Way Model

If we now denote $R_uX\tau$ by θ_u, the quadratic form Q can be written

$$Q = \sum_{u=1}^{q} g_u^{-1}(R_uY - \theta_u)^2 + \sum_{i=1}^{s} d_i^{-1} Y'S_i'S_iY$$

where g_u $(u = 1, \ldots, q)$ are the q diagonal elements of RVR'. This form exhibits a sufficient statistic of q + s components namely R_uY $(u = 1, \ldots, q)$ and $Y'S_i'S_iY$ $(i = 1, \ldots, s)$.

Distribution of the Sufficient Statistic

R_uY is distributed as a univariate normal variable with mean θ_u and variance $g_u = R_uVR_u'$, and

$$\frac{Y'S_i'S_iY}{d_i}$$

is distributed as a central chi square variable with degrees of freedom n_i equal to the multiplicity of d_i in the set of diagonal elements of SVS'. The q + s components of the sufficient statistic are independent.

Minimal Sufficient Statistics

Theorem 5.1. The sufficient statistic with q + s components R_uY ($u = 1, \ldots, q$) and $Y'S_i'S_iY$ ($i = 1, \ldots, s$) is a minimal sufficient statistic.

Proof: We form the difference of quadratic forms

$$Q - Q_0 = \sum_{u=1}^{q} g_u^{-1}[(R_uY - \theta_u)^2 - (R_uY_0 - \theta_u)^2] +$$

$$\sum_{i=1}^{s} d_i^{-1}[Y'S_i'S_iY - Y_0'S_i'S_iY_0]$$

$Q - Q_0$ equals zero when $R_uY = R_uY_0$ ($u = 1, \ldots, q$) and $Y'S_i'S_iY = Y_0'S_i'S_iY_0$ ($i = 1, \ldots, s$). Now set $Q - Q_0$ identically equal to zero in the g_u^{-1}, the θ_u, and the d_i^{-1}. The θ_u, the d_i^{-1}, and the distinct g_u^{-1} form a linearly independent set of parameters. If d_i is not equal to some g_u, we immediately have $Y'S_i'S_iY = Y_0'S_i'S_iY_0$. If g_u equals some d_i, say d_k, then the above identity implies

$$\sum_u^* [(R_uY - \theta_u)^2 - (R_uY_0 - \theta_u)^2] + Y'S_k'S_kY - Y_0'S_k'S_kY_0 \equiv 0$$

Here $\sum_u^*$ indicates the sum over all u where $g_u = d_k$. Expanding we have

$$\sum_u^* [(R_u Y)^2 - (R_u Y_0)^2 - 2\theta_u (R_u Y - R_u Y_0)] + Y' S_k' S_k Y - Y_0' S_k' S_k Y_0 \equiv 0$$

In this form it can be seen that $R_u Y = R_u Y_0$ from which it follows that $Y' S_k' S_k Y = Y_0' S_k' S_k Y_0$, (for all k where d_k equals some g_u). The fact that $R_u Y = R_u Y_0$ when g_u is not equal to some d_i follows as a special case of the above situation. We thus have $R_u Y = R_u Y_0$ (u = 1, . . . , q) and $Y' S_i' S_i Y = Y_0' S_i' S_i Y_0$ (i = 1, . . . , s) for all cases. This is Lehmann and Scheffe's condition for a sufficient statistic to be a minimal sufficient statistic.

A Complete Sufficient Statistic

Theorem 5. 2. *When the covariance matrix V has h distinct characteristic roots, then the sufficient statistic $R_u Y$ (u = 1, . . . , q) and $Y' S_i' S_i Y$ (i = 1, . . . , s) is a complete sufficient statistic of dimension q + s.*

Proof: Write the quadratic form Q as follows.

$$Q = \sum_{u=1}^{q} g_u^{-1} (R_u Y - \theta_u)^2 + \sum_{u=q+1}^{q+s} d_{u-q}^{-1} Y' S_{u-q}' S_{u-q} Y$$

Let $Z_u = R_u Y$ (u = 1, . . . , q) $Z_u = Y' S_{u-q}' S_{u-q} Y$ (u = q, 1, . . . , s). Let $P_{\theta, \sigma^2}(Z_1, \ldots, Z_{q+s})$ be the joint probability density function of Z_u (u = 1, . . . , q+s). Since these statistics are stochastically independent, we have

$$P_{\theta,\sigma^2}(Z_1 \ldots Z_{q+s}) = C \prod_{u=q+1}^{q+s} Z_u^{\frac{n_u - 2}{2}} e^{-\frac{Z_u}{2d_{u-q}}} e^{-\frac{1}{2}\sum_{u=1}^{q} g_u^{-1}(Z_u - \theta_u)^2}$$

Suppose there exists a function $f(Z_1 \ldots Z_{q+s})$ such that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} f(Z_1 \ldots Z_{q+s}) \prod_{u=q+1}^{q+s} Z_u^{\frac{n_u-2}{2}} e^{-\frac{Z_u}{2d_{u-q}}} e^{-\frac{1}{2}\sum_{u=1}^{q} g_u^{-1}(Z_u - \theta_u)^2} \equiv 0$$

According to Fubini's Theorem this can be written

$$\int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-\frac{1}{2}\sum_{u=1}^{q} g_u^{-1}(Z_u^2 - 2\theta_u Z_u)} \phi(Z_1 \ldots Z_q) dZ_1 \ldots dZ_q \equiv 0$$

where

$$\phi(Z_1 \ldots Z_q) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(Z_1 \ldots Z_{q+s}) \prod_{u=q+1}^{q+s} Z_u^{\frac{n_u-2}{2}} e^{-\frac{Z_u}{2d_{u-q}}} dZ_{q+1} \ldots dZ_{q+s}$$

$$\int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{\sum_{u=1}^{q} \frac{\theta_u Z_u}{g_u}} \left[e^{-\frac{1}{2}\sum_{u=1}^{q} \frac{Z_u^2}{g_u}} \phi(Z_1 \ldots Z_q) \right] dZ_1 \ldots dZ_q \equiv 0 .$$

Since the parameters in the set $\{\theta_u\}$ $(u = 1, \ldots, q)$ were obtained by orthogonal transforming a set of functionally independent parameters the θ_u are themselves functionally independent. Each θ_u is functionally independent of the characteristic roots of **V** hence we can apply the uniqueness theorem of the unilateral laplace transform to assert that the above identity implies

$$e^{-\frac{1}{2}\sum_{u=1}^{q}\frac{Z_u^2}{g_u}}\ \phi(Z_1 \ldots Z_q) \equiv 0.$$

$$e^{-\frac{1}{2}\sum_{u=1}^{q}\frac{Z_u^2}{g_u}}$$

is not equal to zero hence $\phi(Z_1 \ldots Z_q) \equiv 0$.

The elements of the set $\left\{d_{u-q}\right\}$ $(u = q+1, \ldots, q+s)$ are functionally independent according to Lemma 2 found in the appendix. The uniqueness theorem of the bilateral laplace transform is now used to assert that $\phi(Z_1 \ldots Z_q) \equiv 0$ implies

$$f(Z_1 \ldots Z_{q+s}) \prod_{u=q+1}^{q+s} Z_u^{\frac{n_u-2}{2}} \equiv 0.$$

Except on a set with probability measure zero

$$\prod_{u=q+1}^{q+s} Z_u^{\frac{n_u - 2}{2}} \neq 0$$

hence $f(Z_1 \ldots Z_{q+s}) \equiv 0$ except on a set with probability measure zero.

Thus the statistic with components $R_u Y$ ($u = 1, \ldots, q$) and $Y' S_i' S_i Y$ ($i = 1, \ldots, s$) is a complete sufficient statistic when the number of distinct characteristic roots of V is equal to h.

CHAPTER VI

SUMMARY

Through an orthogonal transformation of the vector of observations the data was so transformed that estimates of the same function of parameters could be combined. The combined transformed data was then examined for the following properties: sufficiency, completeness, and minimal dimension. The objective was to exhibit minimal sufficient statistics for a class of statistical designs which fall in the category of Eisenhart's Model III.

In the variance component model for the one-way classification of data we have the following theorem:

The sum and the sum of squares of the treatment totals for those treatments having n_i $(i = 1, \ldots, a)$ observations, augmented by the total sum of squares form a sufficient statistic.

Conditions are given for determining which of these components form a minimal sufficient statistic. In Eisenhart's Model III for two-way classification of data we have the following theorem:

The block totals, the treatment totals, and the total sum of squares form a sufficient statistic.

Conditions are given for condensing these components into a minimal sufficient statistic. Minimal sufficient statistics and complete sufficient statistics are given for special sets of assumptions in the n-way classification situation.

Suggestions for Future Study

In many cases the quantities used in the conventional analysis of variance method of computing estimates of parameters are not the quantities which appear as components of the minimal sufficient statistics exhibited in this thesis. For maximum use to be made of the results of this thesis, a new computing technique must be devised or the existing analysis of variance method must be written in terms of the components of a minimal sufficient statistic.

In many models there remains the problem of how best to combine estimates of the same function of parameters.

The n-way classification problem is only partially solved in this thesis. Questions related to interaction still remain unsolved in the n-way classification problem.

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APPENDIX

Lemma 1. If the distinct positive quantities d_u, $(u = 1, \ldots, k)$ are of the form $d_u = b_u + a$ where $a \neq 0$ and a is functionally independent of each b_u, then the quantities d_u^{-1} $(u = 1, \ldots, k)$ are linearly independent.

Proof: Consider the set of constants c_u $(u = 1, \ldots, k)$, such that

$$\sum_{u=1}^{k} \frac{c_u}{d_u} = 0.$$

It follows then that

$$\sum_{u=1}^{k} (c_u \prod_{v \neq u} d_v) = 0 \quad \text{or equivalently} \quad \sum_{u=1}^{k} [c_u \prod_{v \neq u} (b_v + a)] = 0.$$

Expanding and collecting coefficients of powers of a we have

$$a^{k-1}(\sum_u c_u) = 0$$

$$a^{k-2}[\sum_u c_u(\sum_{v \neq u} b_v)] = 0$$

$$\cdots\cdots\cdots\cdots$$

$$c_u \prod_{v \neq u} b_v = 0$$

The above system of k equations can be written as $AC = \phi$ where $C' = (c_1, \ldots, c_k)$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \sum_{v \neq 1} b_v & \sum_{v \neq u} b_v & \sum_{v \neq k} b_k \\ \sum_{\substack{v \neq 1 \\ v_1 < v_2}} b_{v_1} b_{v_2} & \sum_{\substack{v \neq u \\ v_1 < v_2}} b_{v_1} b_{v_2} & \sum_{\substack{v \neq k \\ v_1 < v_2}} b_{v_1} b_{v_2} \\ \cdots & \cdots & \cdots \\ b_2 b_3 \cdots b_k & b_1 \cdot \cdot \underset{v \neq u}{b_v} \cdot b_k & b_1 b_2 \cdots b_{k-1} \end{bmatrix}$$

We will now prove by induction that $|A| \neq 0$. If $k = 2$

$$|A| = \begin{vmatrix} 1 & 1 \\ b_2 & b_1 \end{vmatrix} = (b_1 - b_2).$$

If $k = 3$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ b_2 + b_3 & b_1 + b_3 & b_1 + b_2 \\ b_2 b_3 & b_1 b_3 & b_1 b_2 \end{vmatrix} = (b_1 - b_2)(b_1 - b_3)(b_2 - b_3)$$

Assuming for $k = m$ that

$$|A| = \prod_{u < j}^{m} (b_u - b_j)$$

we consider

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ \sum_{v \neq 1} b_v & \sum_{v \neq u} b_v & \sum_{v \neq m+1} b_v \\ \sum_{\substack{v \neq 1 \\ v_1 < v_2}} b_{v_1} b_{v_2} & \sum_{\substack{v \neq u \\ v_1 < v_2}} b_{v_1} b_{v_2} & \sum_{\substack{v \neq m+1 \\ v_1 < v_2}} b_{v_1} b_{v_2} \\ \cdots & \cdots & \cdots \\ b_2 \cdots b_{m+1} & b_1 \cdot \underset{v \neq u}{b_v} \cdot b_{m+1} & b_1 b_2 \cdots b_m \end{vmatrix}$$

Subtracting column m + 1 from the remaining columns and expanding by the first row of the resulting matrix we obtain

$$|A| = \prod_{u=1}^{m} (b_u - b_{m+1}) \prod_{u<j}^{m} (b_u - b_j) = \prod_{u<j}^{m+1} (b_u - b_j) .$$

Since the d_u are distinct, the b_u are also distinct and $|A| \neq 0$. This implies $C = \phi$ which asserts that the quantities d_u^{-1} $(u = 1, \ldots, k)$, are linearly independent.

<u>Lemma 2.</u> <u>Let the covariance matrix for a vector Y of observations be of the form</u>

$$V = \sum_{k=1}^{h} \sigma_k^2 Z_k Z_k'$$

<u>where σ_k^2 are functionally independent and the matrices $Z_k Z_k'$ are linearly independent. Let P be a matrix of orthogonal rows such</u>

that $PZ_kZ_k'P'$ $(k = 1, \ldots, h)$ and PVP' are diagonal matrices. Let s equal the number of distinct diagonal elements of PVP'. If $s = h$, then the distinct characteristic roots d_k $(k = 1, \ldots, h)$ are functionally independent.

Proof: From the form of V we have

$$PVP' = \sum_{k=1}^{h} \sigma_k^2 PZ_kZ_k'P' .$$

Let D^* and D_k^* be the vectors of the diagonal elements of the matrices PVP' and $PZ_kZ_k'P'$, respectively. D^* can then be written

$$D^* = \sum_{k=1}^{h} \sigma_k^2 D_k^* = (D_1^*, \ldots, D_h^*) \not\supseteq$$

where $\not\supseteq' = (\sigma_1^2, \ldots, \sigma_h^2)$. Since the Z_kZ_k' are linearly independent matrices, the D_k^* are linearly independent vectors which implies the matrix $(D_1^*, \ldots, D_h^*)$ has rank h. This together with the fact that $\not\supseteq$ has h functionally independent elements implies D^* has h functionally independent elements. These clearly are the h distinct characteristic roots $d_1, \ldots, d_h$.

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