ALGEBRAIC CARRY-OVER IN TWO DIMENSIONAL SYSTEMS

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Submitted to the Faculty of the Graduate School
of the Oklahoma State University of
Agriculture and Applied Science
in partial fulfillment of
the requirements for
the degree of
DOCTOR OF PHILOSOPHY
August, 1959

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Thesis Approved:

Thesis Adviser

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PREFACE

The analysis of the general second order problem defined by the Laplace or the Poisson equation, by means of summing infinite geometric series, is presented in this dissertation. The corresponding finite-difference equations are formulated in four different coordinate systems: rectangular, skew, polar, and triangular. For each network, it is proved that these equations can be solved in algebraic form by a carry-over procedure which yields a finite value for the unknown function at each point of the net. Each algebraic value is the sum of one or more infinite series each term of which is an infinite series.

This research has grown out of lectures given by Professor Jan Tuma in December, 1956, on the method of numerical carry-over in plate structures, and from initial investigations into an algebraic approach made by him during the summer and fall of 1956. The idea of solving finite-difference networks by summing infinite series was originated by Professor Tuma as a result of his four years' prior research in solving the analogical problem of continuous frames. In a letter to Dr. Clark A. Dunn, Director of the Division of Engineering Research, in January, 1957, he formally acknowledged the possibility and desirability of extending the general philosophy of geometric series to problems in continuous elastic systems. The following areas were indicated:

1. Calculation of critical loads for columns of constant and variable cross-sections.
2. Calculation of critical loads for beam-columns of constant and variable cross-sections
3. Analysis of beams on elastic foundations
4. Analysis of torsion of simple and continuous beams of variable cross-sections
5. Analysis of plates and grids
6. Analysis of shells and space lattices.

It was immediately evident that an even more widespread use of the infinite series approach was possible due to the extensive class of engineering problems having similar mathematical representations.

The decision to formulate the general problems of the Poisson and Laplace equations in algebraic difference form and to extend the original investigations of Professor Tuma in 1956 to various coordinate systems was made by the writer in the summer of 1958. The solutions to the skew, polar, and triangular systems investigated were obtained during the fall of 1958 and the early months of 1959.

A special application of the basic research in carry-over that Professor Tuma has conducted for the past three years and in which the writer has assisted since 1957 was a project with the McDonnell Aircraft Corporation in the analysis of simply supported rectangular plates. This project was begun in September, 1957, and the final report, by Professors Tuma and French and the writer, was completed in November, 1958. Volume I of this report illustrates certain of the basic concepts of algebraic carry-over, and demonstrates the algebraic solution of the basic plate equation in finite-difference form (25).

In completing this, the final phase of his graduate work, the writer wishes to express his appreciation to the following individuals and organizations:

To the Continental Oil Company and the School of Civil Engineering for awarding the fellowship which made his first two years of graduate study financially possible;

To the Division of Engineering Research, Dr. Clark A. Dunn, Director, for sponsoring this research during the past academic year;

To Professor Roger L. Flanders, Professor James V. Parcher, Dr. L. Wayne Johnson, and Dr. R. B. Deal, for their helpful advice and encouragement throughout the writer's graduate program;

To Professor Jan J. Tuma for his guidance, understanding, and friendship, and, most of all, for providing the writer with the opportunity and the inspiration to pursue a life of university teaching and research;

To his wife, Roberta, for her thoughtful consideration, constant faith, and gentle encouragement during the four years of his graduate work.

In addition, gratitude is due Mrs. Robby F. Tollison and Mrs. Willie P. Bernardi, who typed the final manuscript, and Mr. Edward R. Sturm, who prepared some special drawings.

July 1, 1959
Stillwater, Oklahoma

Kerry S. Havner

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NOMENCLATURE

a, b, c . . . . Carry-Over Factors

$a_{i+1, i}, a_{i-1, i}, b_i$ . . . . Carry-Over Factors in Polar Coordinates

i, j, k, l . . . . Network Points

n . . . . Number of Radial Lines in Polar Coordinates

r, θ . . . . Polar Coordinates

$t = \frac{\Delta x}{\Delta y} = \frac{\Delta u}{\Delta v}$ . . . . Network Interval Ratios

u, v, w . . . . Triangular Coordinates

x, y . . . . Rectangular and Skew Coordinates

A_m, B_m, C_m, D_m, E_m . . . . Direct Final Carry-Over Factors

$A_{mn}, B_{mn}, C_{mn}, D_{mn}$ . . . . Direct Final Carry-Over Factors

(B) . . . . Symbol for Basic Series

(C) . . . . Symbol for Central Carry-Over Series

(CO) . . . . Symbol for Carried-Over Value

D . . . . Domain of Definition for the Poisson or Laplace Equation

(E) . . . . Symbol for External Series

F . . . . Given Function in the Domain

G . . . . Given Function on the Boundary

(I) . . . . Symbol for Internal Series

O . . . . Origin of Coordinate System

Q . . . . Unknown Function in the Domain

Q_{ij}^{kl} . . . . Function Value at ij Due to Starting Value at kl

S	Closed Curve Bounding the Domain
(S)	Symbol for Circulatory Series
(Sk)	Symbol for Skew Series
S_{mn}, X_{mn}	Denominators of Basic Series
U, V, W, X, Y, Z	Coordinate Axes
U_m, V_m	Denominators of Carry-Over Series
K_{mn}, Y_{mn}, Z_{mn}	Denominators of Carry-Over Series
α, β	Angles Between Coordinate Axes
$\alpha_n, \beta_n, \gamma_n, \delta_n$	Terms of Carry-Over Series
λ, λ_i	Starting Values
$\Delta r, \Delta \theta$	Network Intervals in Polar Coordinates
$\Delta u, \Delta v, \Delta w$	Network Intervals in Triangular Coordinates
$\Delta x, \Delta y$	Network Intervals in Rectangular and Skew Coordinates
∇^2	Laplace Partial Differential Operator

CHAPTER I

INTRODUCTION

1-1 Historical Study. The method of finite differences has been in use for more than fifty years in the solution of the differential equations of engineering physics. Equations of finite differences were originally introduced by Brook Taylor (1) in the eighteenth century. The first application of these equations to elasticity was made by Runge (2) in solving torsional problems in beams (1908). An approximate solution of the finite difference equations was obtained by Richardson (3) in 1910 using a numerical iteration process. A more rapidly convergent iteration procedure was given in 1918 by Liebmann.(4).

Marcus (5) made an extensive application of finite-differences to the analysis of thin plates, and introduced the membrane analogy for the plate by replacing the fourth order partial differential (biharmonic) equation by two second order (harmonic) equations. The work of Marcus was publicized in the United States in two papers by Wise. (6, 7). Hencky (8) applied the method of finite-differences to the large deflection theory of plates.

The convergency and rate of convergency of the Liebmann iteration process were discussed by Wolf (9) and Courant (10). Shortley and Weller (11) developed a highly mathematical improved rate of convergency involving the error function. Frankel (12) developed an "extrapolated Liebmann method" and discussed the improved rate of convergency over the Liebmann and Richardson procedures. Young(13) generalized Frankel's method to a "successive over relaxation method" applicable to the general linear, elliptic partial differential equation. Frocht (14) used block iteration of certain "key values" on the finite difference net. French (15) improved the convergency by solving a simple geometric series at each cycle of the iteration.

The relaxation procedure for the solution of finite difference equations was developed by Sir Richard Southwell (16) during the period 1935-40. This procedure was applied to torsion problems by Southwell and Christopherson (17) and to the analysis of extension and flexure of thin plates by Southwell and Fox (18), and Southwell (19, 20). Temple (21) gave an analytical proof of the convergence of the relaxation method, using the principle of minimum energy, and extended the general approach to include all linear systems by his "method of steepest descents". The use of higher order difference equations and a coarser net was proposed by Fox (22) and discussed by Southwell (23) and Christopherson (24).

The method of solving finite difference equations by summing infinite, geometric series was developed by Tuma, Havner, and French (25). This philosophy had its beginning in 1932 with the solution of beam and frame problems by Cross (26, 27), who conceived the basic series, and is founded upon the concepts of carry-over and circulatory series developed by Tuma (28) in 1950. The algebraic series were applied to the analysis of continuous beams by Tuma and Anderson (29), to continuous frames by Tuma (30) and Tuma, Havner, and Hedges (31), and to grid systems by Cellis (32). A simpler procedure of algebraic carry-over for beams was developed by Tuma (33). Methods of using combined algebraic and numerical relaxation procedures were investigated by Yoshimura (34), Yoshimura and Marakami (35), and Pauw (36, 37).

1-2 The Equations of Poisson and Laplace. A number of important problems of engineering physics have their mathematical formulation in either Poisson's or Laplace's equation. Considering a domain D bounded by a closed curve S (Fig. 1), the Poisson equation has the form

$$\nabla^2 Q = -F \qquad (1)$$

where F is a given function in the domain and Q is zero on the boundary. Examples of this equation include the bending of simply supported plates under lateral loads, the deflection of a uniformly stretched membrane, and the torsion of non-circular sections.

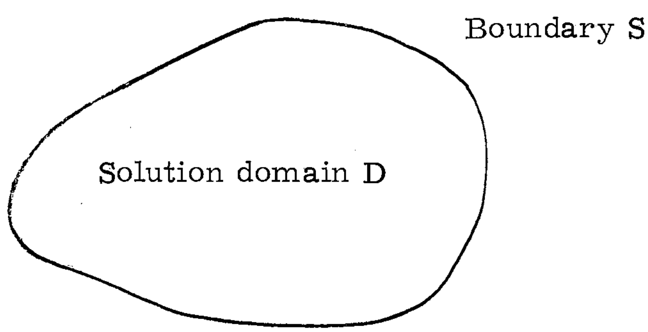


Fig. 1
Domain of Definition for
Laplace's or Poisson's Equation

The Laplace equation over the domain D is

$$\nabla^2 Q = 0 \qquad (2)$$

with Q equal to a given function G on the boundary. Temperature distribution in the steady state, the bending of simply supported plates under moments distributed along the edges, and the plane stress problem are examples of Laplace's equation.

1-3 Finite Differences and Algebraic Carry-Over. The solution of either the Poisson or the Laplace equation by the method of finite differences has the advantage of mathematical simplicity in that the partial differential equation considered is replaced by a system of linear algebraic equations. The method has the fundamental disadvantage of becoming rather cumbersome when a large number of unknowns are involved. In addition, two approximations are introduced in applying the method of finite differences. The first approximation is that of the net itself; the second occurs in solving the linear difference equations by the numerical method chosen: either an iteration or a relaxation technique.

To eliminate this second approximation and achieve a feasible solution for the finite difference equations in general algebraic form, an entirely different approach is necessary. Visualizing the problem of solving the network as one of determining the flow of function values from a specified starting point as that point begins to affect those surrounding it, the idea of the algebraic carry-over of these values may immediately be conceived.

Performing this carry-over procedure simultaneously over the entire network, power series are formed whose sums cannot be determined, and no insight into the functional mechanics of the process is possible. If, however, the network is properly divided into component parts, it is found that the solution of each isolated point set is achieved by summing simple geometric series. Interrelating these isolated parts by a gradual relaxing back and forth between them, the higher order geometric series of carry-over are formed, and the final solution of the network is accomplished.

The principles of algebraic carry-over, and their application to the solution of Poisson's equation, were demonstrated for rectangular coordinate systems in reference 25, and are restated here (Chapter II) in order to form a basis for later comparisons. The extension of these principles to the algebraic solution of the Poisson and Laplace equations in skew, polar, and triangular coordinate systems is the purpose of this dissertation. Only simple geometric shapes are considered, and the solutions are accomplished by means of infinite, convergent, geometric series. A starting value for the function Q in the domain leads to the solution of Poisson's equation, a starting value for Q on the boundary to the solution of Laplace's equation.

CHAPTER II

RECTANGULAR SYSTEMS

2-1 Linear Finite - Difference Equations. In rectangular coordinates the Poisson equation has the form (38)

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = - F(x,y) \ . \qquad (3)$$

The corresponding finite difference equation written for point ij of the finite difference net is (Fig. 2) (38)

$$\frac{Q_{i-1,j} - 2Q_{ij} + Q_{i+1,j}}{\Delta x^2} + \frac{Q_{i,j-1} - 2Q_{ij} + Q_{i,j+1}}{\Delta y^2} = - F_{ij} \ .$$

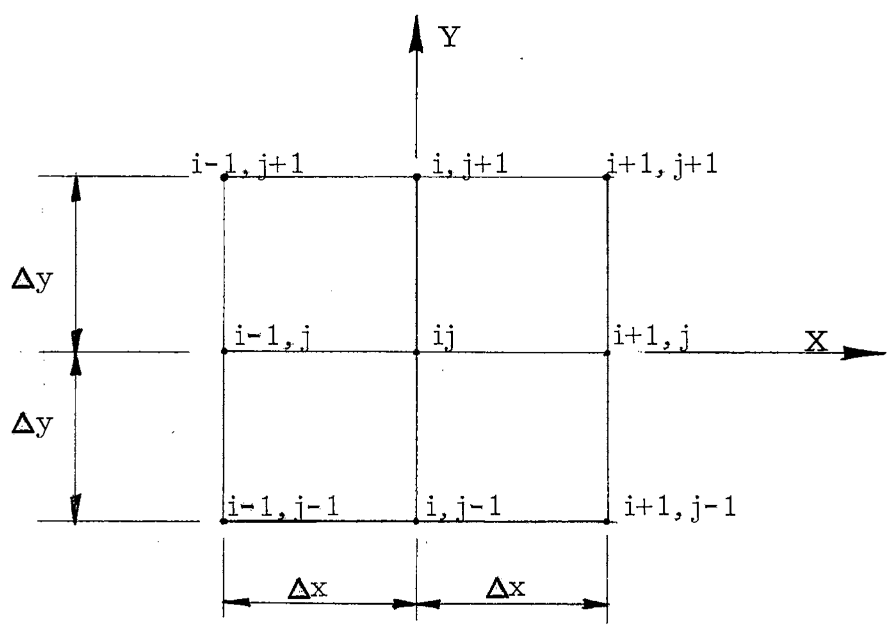


Fig. 2
Finite Difference Net in Rectangular Coordinates

Introducing the notation

$$a = \frac{1}{2(1+t^2)} \qquad b = \frac{t^2}{2(1+t^2)}$$

$$\lambda = \frac{t}{2(1+t^2)} \qquad t = \frac{\Delta x}{\Delta y} \tag{4}$$

this equation may be written

$$Q_{ij} = \left\{ \begin{array}{l} a(Q_{i-1,j} + Q_{i+1,j}) \\ \\ b(Q_{i,j-1} + Q_{i,j+1}) \end{array} \right\} + Q_{ij}^{*} \tag{5}$$

where

$$Q_{ij}^{*} = \lambda F_{ij} \Delta x \Delta y \tag{6}$$

is the starting value for Q_{ij}, assuming the Q's at the four adjacent points to be zero.

It is evident from Eq. (5) that a and b are carry-over factors on the finite difference net in the X- and Y- directions, respectively. These carry-over factors represent the influences of the Q values at the adjacent points i-1, j, i+1, j, i, j-1, and i, j+1 on the value at point ij.

2-2 The Basic Series. The analysis of a two dimensional, linear, nine point set (Fig. 3) by algebraic carry-over yields final results for function values which are equal to the algebraic sums of infinite geometric series.

Considering a starting value λ at point 13, it is apparent from the figure that a value carried from that point to any one of the adjacent points (8, 12, 14, 18) will return to 13 multiplied by the square of the corresponding carry-over factor. A simple geometric series is thus developed which is called the basic series. The results of algebraic carry-over on this point set are:

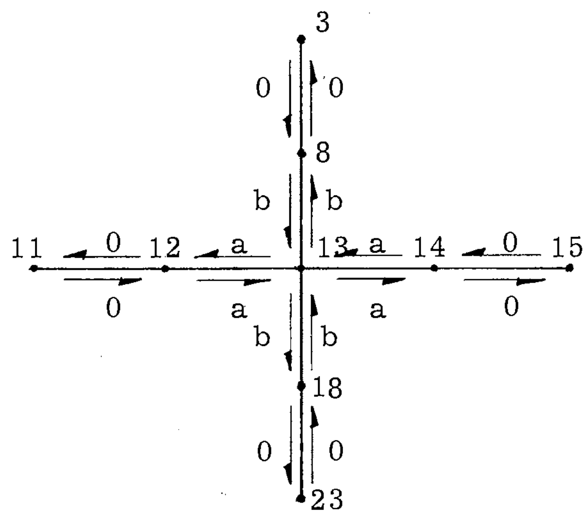


Fig. 3
Nine Point Set - Basic Series

$$Q_{13}^{13(B)} = \lambda \left[1 + (2a^2 + 2b^2) + (2a^2 + 2b^2)^2 + \ldots \right]$$

$$= \lambda \frac{1}{X_{22}}$$

$$Q_8^{13(B)} = \lambda \frac{b}{X_{22}} = Q_{18}^{13(B)}$$

$$Q_{12}^{13(B)} = \lambda \frac{a}{X_{22}} = Q_{14}^{13(B)}$$

where

$$X_{22} = 1 - 2a^2 - 2b^2 \quad .$$

A diagrammatic representation of the final values is shown in Fig. 4. From these results it is evident that:

(a) The constant $\frac{1}{X_{22}}$ is the over-relaxation factor for the basic series

(b) The final function value at the center is equal to the starting value multiplied by the over-relaxation factor

(c) The final function value at any other point is equal to the final central value multiplied by the corresponding carry-over factor.

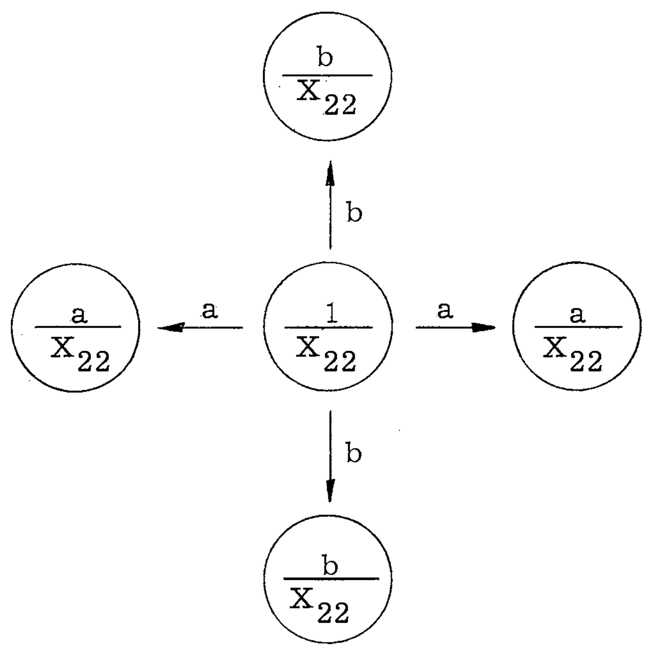


Fig. 4
Final Results - Basic Series

For a starting value λ at any other point the function coefficients may be obtained by the method of involution:

(a) The starting value is carried-over to the central point 13, multiplied by the corresponding carry-over factor

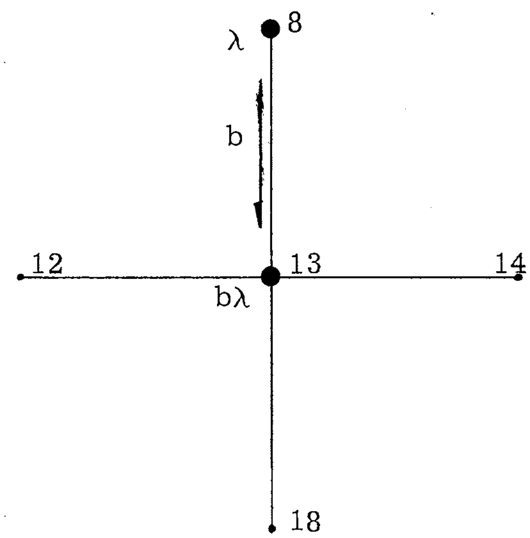


Fig. 5a
Involuted Starting Value at Point 13

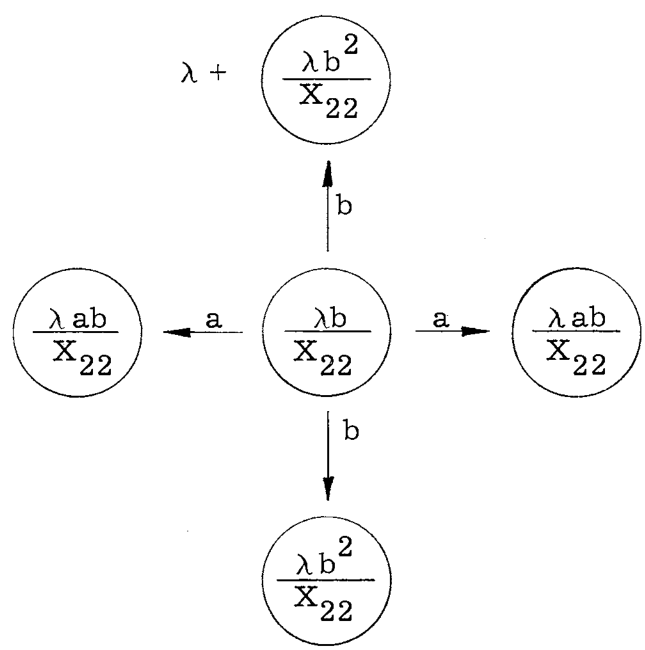


Fig. 5b
Final Results - Starting Value at 8

(b) The carried-over value becomes a new starting value, forming the basic series

(c) The final results are obtained by superimposing the initial starting value upon the results of the basic series, using the over-relaxation factor and direct carry-over.

Thus for a starting value at point 8 the involution and the final results are as shown in Fig.'s 5a, 5b.

2-3 The Circulatory Series. Considering an eight point closed ring (Fig. 6) and applying the algebraic carry-over method to the computation of function coefficients, each final value may be represented as the sum of four infinite geometric series.

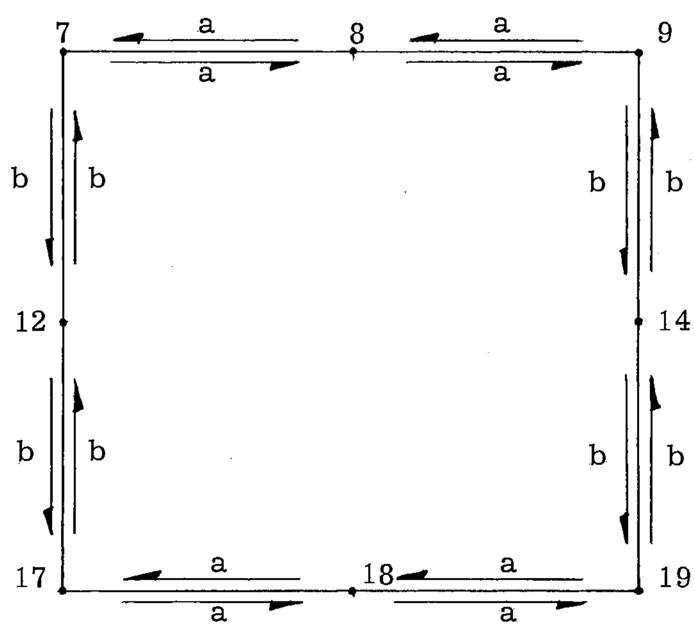


Fig. 6
Eight Point Ring - Circulatory Series

For a starting value λ at point 7, the algebraic analysis of the system can be simplified by the methods of resolution and superposition:

(a) The initial system is resolved into four basic cases as shown in Fig. 7, taking advantage of symmetry and antisymmetry

(b) The algebraic results from the individual cases are superimposed to give the final values for function coefficients on the closed ring.

Case I From the symmetry of this system, the algebraic procedure can be further reduced by using modified carry-over factors as shown in Fig. 8. This reduced point set is obtained from the fact that function values at opposite points (ie. 7 and 9) must be equal after each cycle of carry-over.

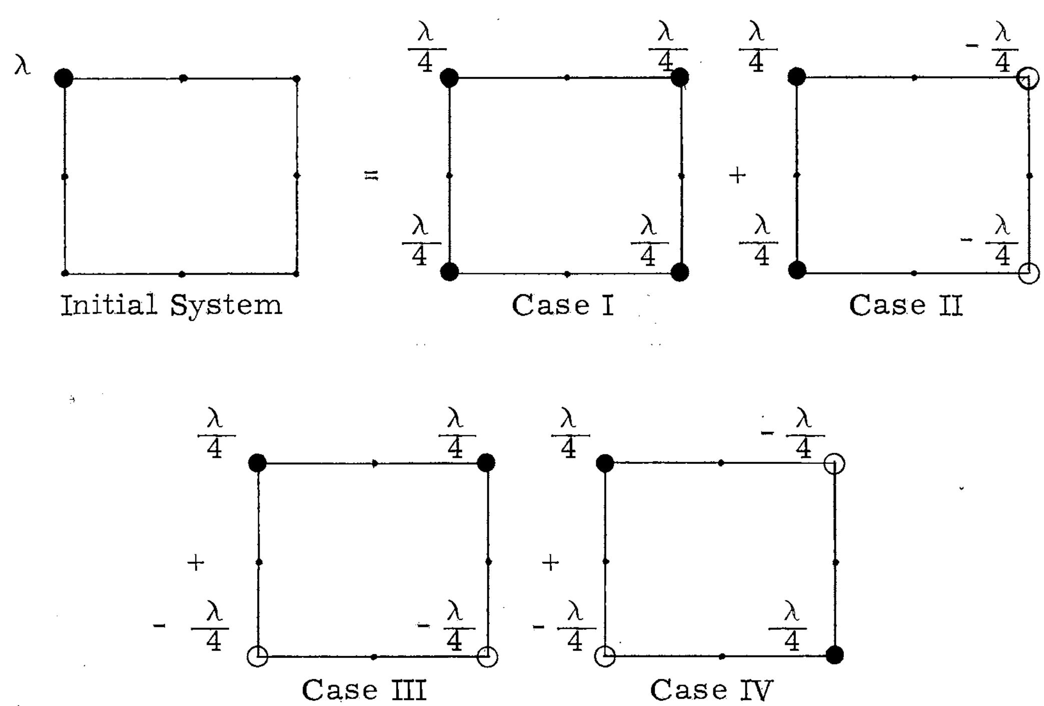


Fig. 7

Resolution of Circulatory System with Starting Value at Point 7 into Four Basic Cases

Performing algebraic carry-over on the modified three point set the results are:

$$Q_7^{7(S)I} = \frac{1}{X_{22}} \frac{\lambda}{4} \quad \Bigg| \quad Q_8^{7(S)I} = \frac{2a}{X_{22}} \frac{\lambda}{4} \quad \Bigg| \quad Q_{12}^{7(S)I} = \frac{2b}{X_{22}} \frac{\lambda}{4} .$$

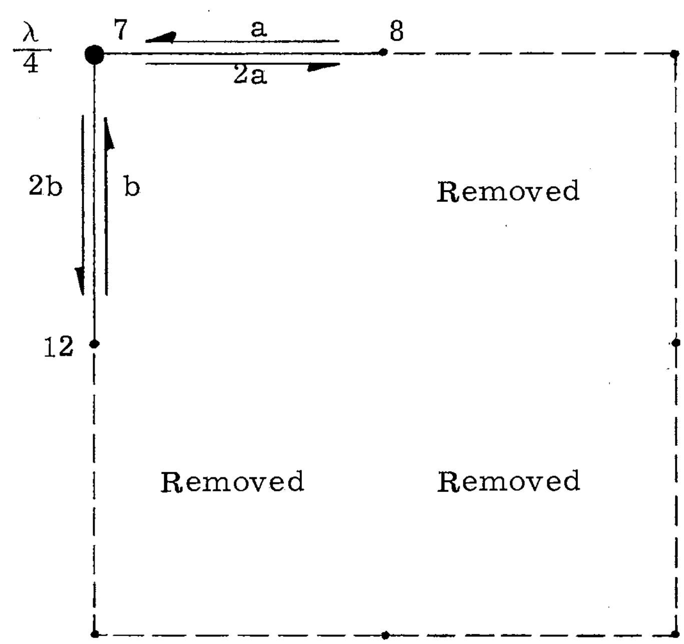


Fig. 8
Modified Circulatory System
Case I

Case II. This system, symmetrical with respect to the central X-axis and antisymmetrical with respect to the Y, can be reduced to two independent systems (Fig. 9) and algebraic carry-over performed.

The results are

$$Q_7^{7(S)II} = \frac{\lambda}{4} \frac{1}{X_{02}} \qquad Q_9^{7(S)II} = -\frac{\lambda}{4} \frac{1}{X_{02}}$$

$$Q_{12}^{7(S)II} = \frac{\lambda}{4} \frac{2b}{X_{02}} \qquad Q_{14}^{7(S)II} = -\frac{\lambda}{4} \frac{2b}{X_{02}}$$

where

$$X_{02} = 1 - 2b^2 \ .$$

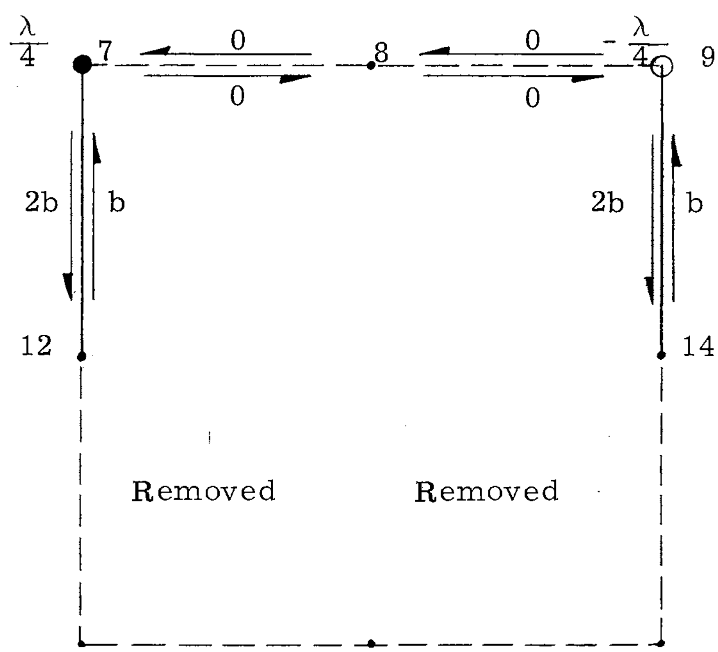


Fig. 9

Modified Circulatory System
Case II

Case III. The reduction of this system is shown in Fig. 10. The values of function coefficients after carry-over are

$$Q_7^{7(S)III} = \frac{\lambda}{4} \frac{1}{X_{20}} \qquad Q_{17}^{7(S)III} = -\frac{\lambda}{4} \frac{1}{X_{20}}$$

$$Q_8^{7(S)III} = \frac{\lambda}{4} \frac{2a}{X_{20}} \qquad Q_{18}^{7(S)III} = -\frac{\lambda}{4} \frac{2a}{X_{20}}$$

where

$$X_{20} = 1 - 2a^2 \quad .$$

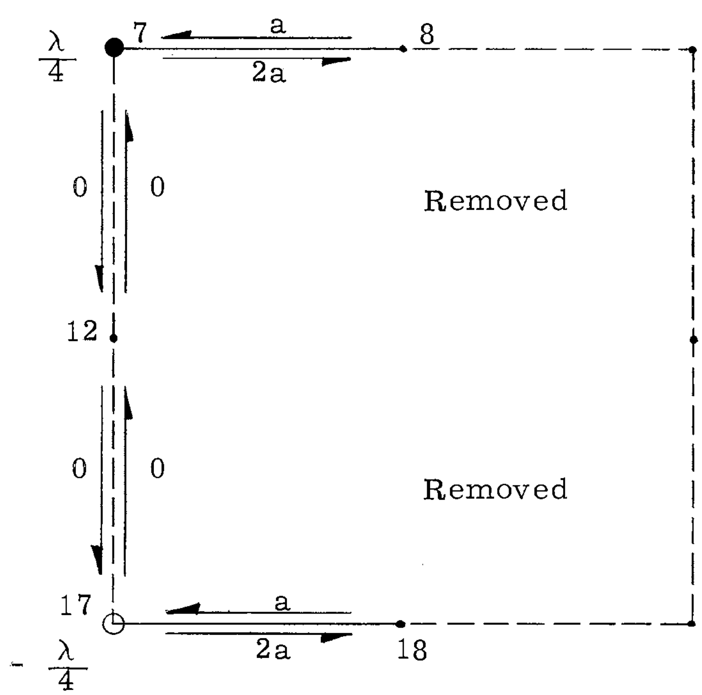


Fig. 10
Modified Circulatory System
Case III

Case IV. No algebraic carry-over procedure is possible, and the starting values represent final results (Fig. 11).

From the superposition of Cases I through IV, it is evident that the single cell series which forms in a geometrically symmetrical closed ring may be resolved into simple geometric series. The results in more complex multicell rings may also be interpreted as basic series in certain cases. (25). In general, however, these are higher order, or carry-over, series.

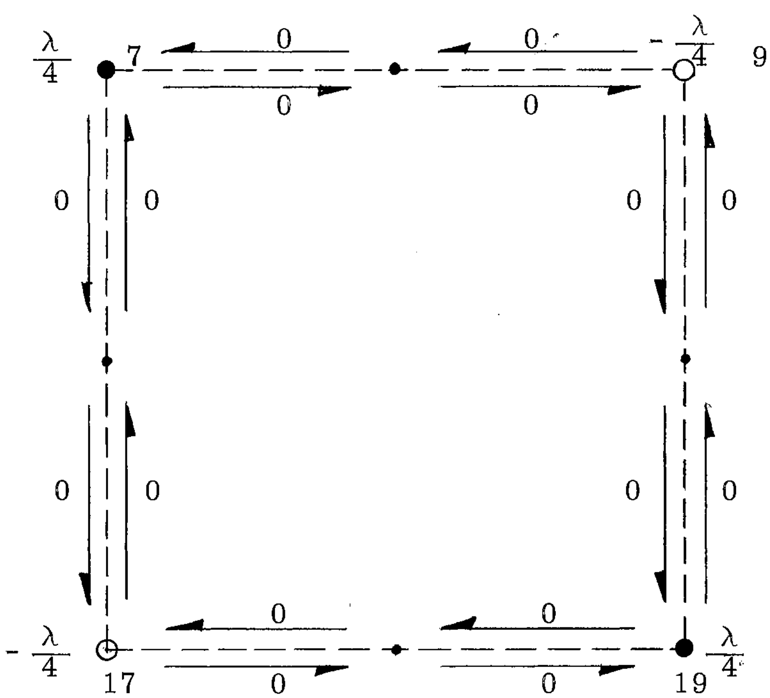


Fig. 11
Modified Circulatory System
Case IV

2-4 The Carry-Over Series. If now the algebraic carry-over method is applied to the analysis of a two dimensional twenty-five point set (Fig. 12), each final result is found to be the finite algebraic sum of an infinite geometric series each term of which is an infinite series.

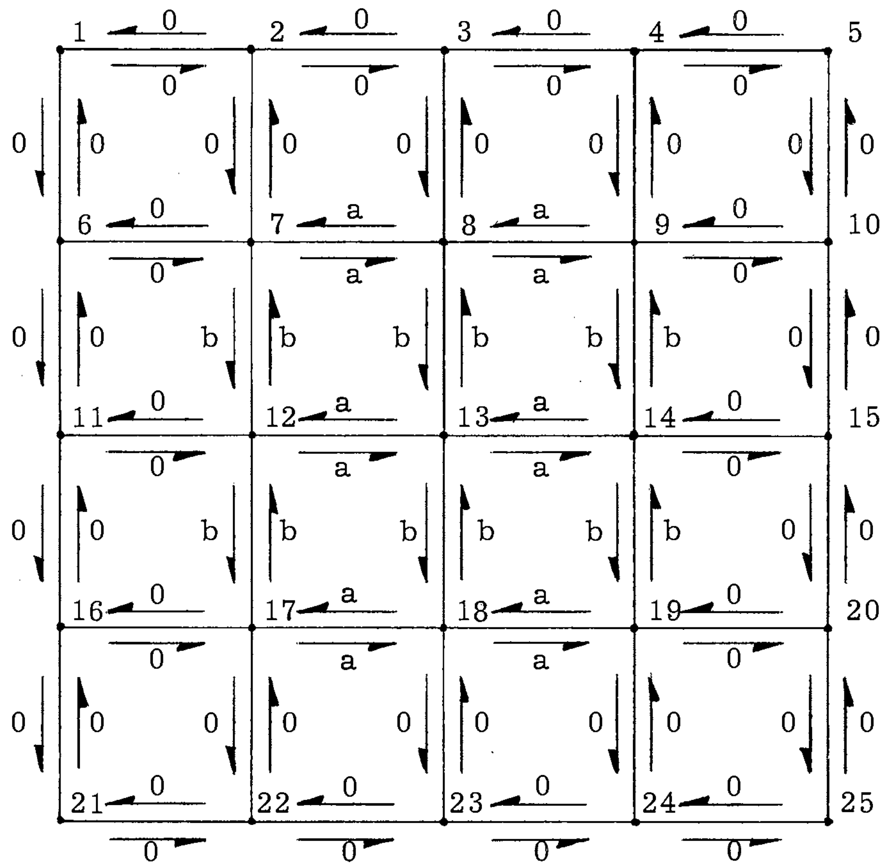


Fig. 12
Twenty-Five Point Set

It is evident from Fig. 12 that a starting value λ carried-over from point 13 to the adjacent points (8, 12, 14, 18) will return to 13 as well as begin to circulate through the closed ring 7, 8, 9, 14, 19, 18, 17 and 12. If the procedure is carried-out algebraically, complex power series are generated whose sums it is practically impossible to determine.

In order to eliminate this difficulty a new concept must be introduced, the principle of the suppressed or zero point:

> A portion of the network can be isolated and analyzed independently by surrounding it with points for which the function values are temporarily assumed to be zero.

This procedure is known as suppressing the point or introducing a zero at the point. The inverse procedure is called releasing the point or removing the zero at the point. After the isolated point set is solved, values are carried to the released points and the algebraic carry-over procedure continues throughout the network.

Thus in Fig. 12, a basic series forming on the five point set 8, 12, 13, 14, and 18 can be isolated (Fig. 13a) by introducing zeros at the corner points 7, 9, 17, and 19. The function values corresponding to the basic series are (Art. 2-2):

$$Q_8^{13(B)} = \lambda \frac{b}{X_{22}} = Q_{18}^{13(B)} \qquad Q_{13}^{13(B)} = \lambda \frac{1}{X_{22}}$$

$$Q_{12}^{13(B)} = \lambda \frac{a}{X_{22}} = Q_{14}^{13(B)} .$$

Removing the zeros at the corners and simultaneously suppressing the central point 13, an eight point closed ring is isolated (Fig. 13b) with the carried over value

$$\alpha_0 = \lambda \frac{2ab}{X_{22}}$$

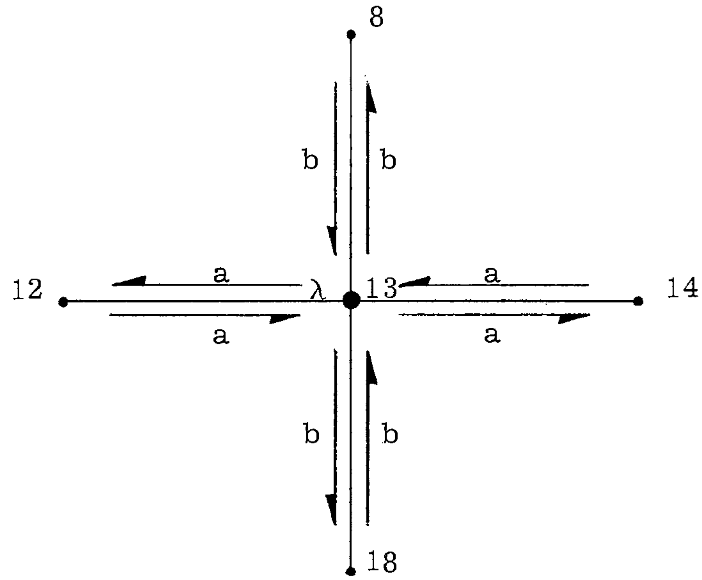


Fig. 13a
Isolated Five Point Set - Basic Series

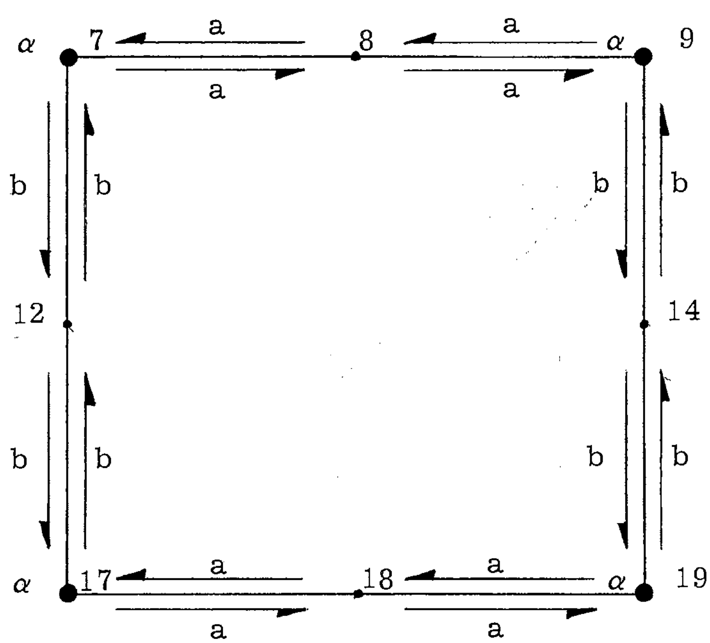


Fig. 13b
Isolated Eight Point Ring - Circulatory Series

at points 7, 9, 17, and 19. This isolated system is identical with Case I of the circulatory series (Art. 2-3). The function values are

$$Q_7^{13(S)} = \alpha_0 \frac{1}{X_{22}} = Q_9^{13(S)} = Q_{17}^{13(S)} = Q_{19}^{13(S)}$$

$$Q_8^{13(S)} = \alpha_0 \frac{2a}{X_{22}} = Q_{18}^{13(S)} \qquad \Big| \qquad Q_{12}^{13(S)} = \alpha_0 \frac{2b}{X_{22}} = Q_{14}^{13(S)} .$$

The analysis of the twenty-five point set shown in Fig. 12 becomes, therefore, a matter of determining the carry-over series which form between these two isolated systems. This is accomplished by releasing point 13 and finding the value carried-back to the center, thus completing one full cycle of carry-over on the network:

$$Q_{13}^{13(CO)} = \frac{8ab}{X_{22}} \alpha_0 = \frac{16\, a^2 b^2}{X^2_{22}} \lambda = \beta_1 \quad .$$

The ratio of the returned value β_1 to the starting value $\beta_0 = \lambda$ is the common ratio of the carry-over series developed by repeating this procedure infinite times.

These carry-over series α and β are infinite geometric series all terms of which are infinite series. Their sums are

$$\sum_0^\infty \alpha_n = \alpha_0 + \alpha_1 + \ldots = \frac{2ab}{X_{22}\, Y_{22}} \lambda$$

$$\sum_0^\infty \beta_n = \beta_0 + \beta_1 + \ldots = \frac{1}{Y_{22}} \lambda$$

where

$$Y_{22} = 1 - \frac{16\, a^2 b^2}{X^2_{22}} \quad .$$

Superimposing the α and β series, the final values for function coefficients on the twenty-five point set become

$$Q_7^{13} = \frac{1}{X_{22}} \sum_0^\infty \alpha_n = \frac{C_{22}}{Z_{22}} \lambda = Q_9^{13} = Q_{17}^{13} = Q_{19}^{13}$$

$$Q_8^{13} = \frac{2a}{X_{22}} \sum_0^\infty \alpha_n + \frac{b}{X_{22}} \sum_0^\infty \beta_n = \frac{B_{22}}{Z_{22}} \lambda = Q_{18}^{13}$$

$$Q_{12}^{13} = \frac{2b}{X_{22}} \sum_0^\infty \alpha_n + \frac{a}{X_{22}} \sum_0^\infty \beta_n = \frac{A_{22}}{Z_{22}} \lambda = Q_{14}^{13} \qquad (7)$$

$$Q_{13}^{13} = \frac{1}{X_{22}} \sum_0^\infty \beta_n = \frac{1}{Z_{22}} \lambda \quad .$$

The new equivalents used in these equations are:

$$A_{22} = a(1 + \frac{4b^2}{X_{22}}) \qquad C_{22} = \frac{2ab}{X_{22}}$$

$$B_{22} = b(1 + \frac{4a^2}{X_{22}}) \qquad Z_{22} = X_{22}(1 - \frac{16a^2b^2}{X^2_{22}}).$$

These constants may be interpreted from the diagrammatic representation of the final values (Fig. 14) and conclusions drawn similar to those made for the basic series in Art. 2-2. Thus

(a) The constant $\frac{1}{Z_{22}}$ is the over-relaxation factor for the system

(b) The final function value at the center is equal to the starting value multiplied by the over-relaxation factor

(c) The final function value at any other point is equal to the final central value multiplied by the corresponding direct final carry-over factor A_{22}, B_{22}, or C_{22} .

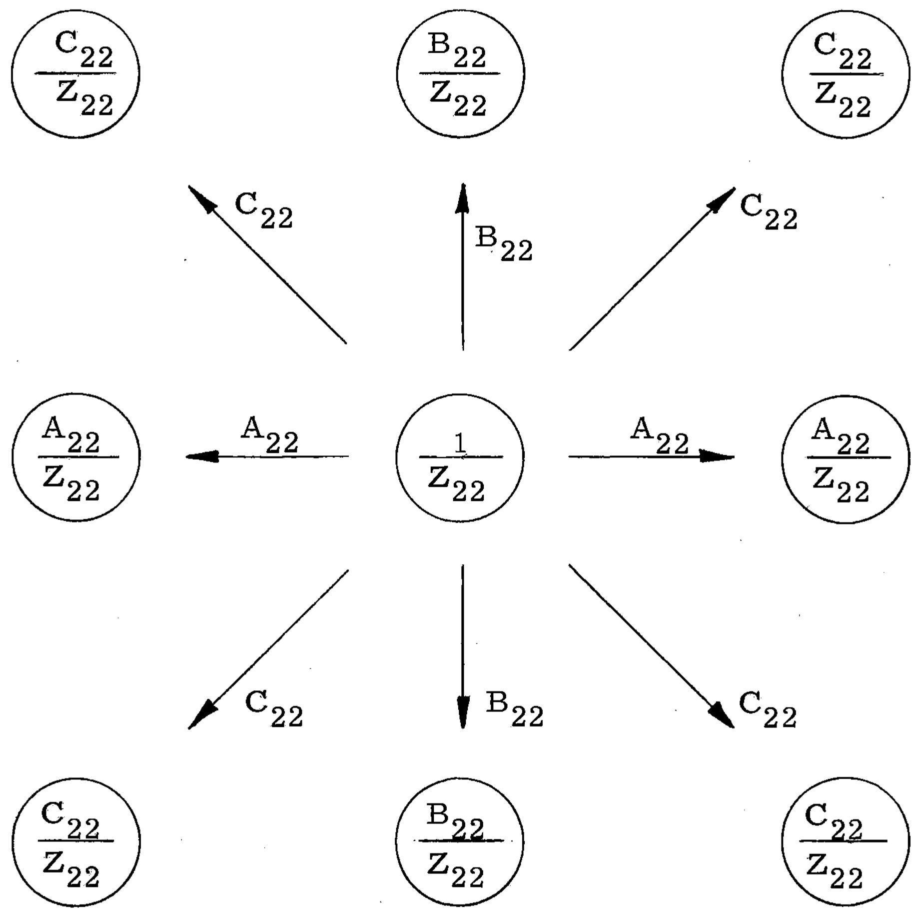


Fig. 14
Final Results - Carry-Over Series

<u>2-5 Resolution, Superposition, and Involution.</u> The methods of involution, resolution, and superposition, demonstrated in Art's. 2-2 and 2-3, may be stated as three basic principles applicable to the analysis of finite-difference networks by algebraic carry-over.

The principle of resolution states that:

> The analysis of any geometrically symmetrical two dimensional system with unsymmetrical starting values can be simplified by resolving the system

> into four or more basic cases each of which contains axes of symmetry and or antisymmetry.

The principle of superposition, the inverse of resolution, states that:

> The final results for function values on the network due to the initial system of starting values are equal to the algebraic sums of the results corresponding to each of the resolved systems.

The principle of involution, illustrated in Art. 2-2, states that:

> The function values on the finite-difference net due to a starting value λ at some point kl are equal to the algebraic sums of the function values due to starting values λ at points adjacent to kl multiplied by the corresponding carry-over factors.

The use of these methods in completing the analysis of the twenty-five point set (Fig. 12) for a starting value at any point is now demonstrated.

For a starting value λ at point 7, the system is resolved into four basic cases as shown in Fig. 15. The results for these cases are obtained by superimposing the values from the circulatory and the central carry-over series.

Case I. Temporarily suppressing the central point 13, an eight point closed ring is isolated (Fig. 16). This isolated system is identical with Case I of the circulatory series (Art. 2-3), and the results are:

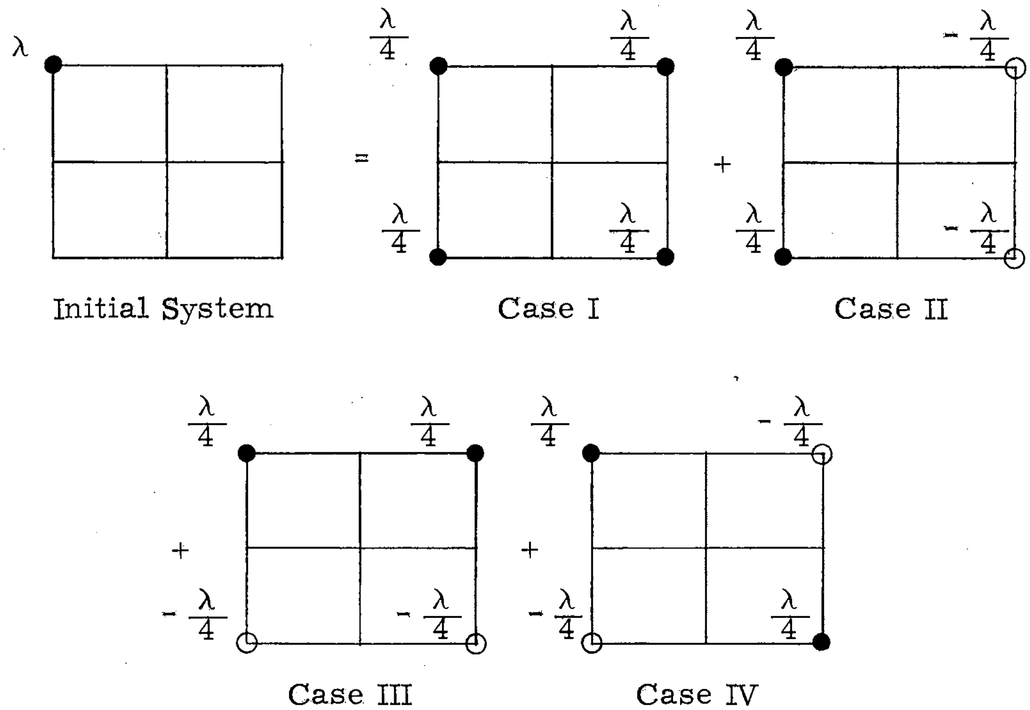


Fig. 15

Resolution of Twenty-Five Point Set with Starting Value at Point 7 into Symmetrical and Antisymmetrical Cases

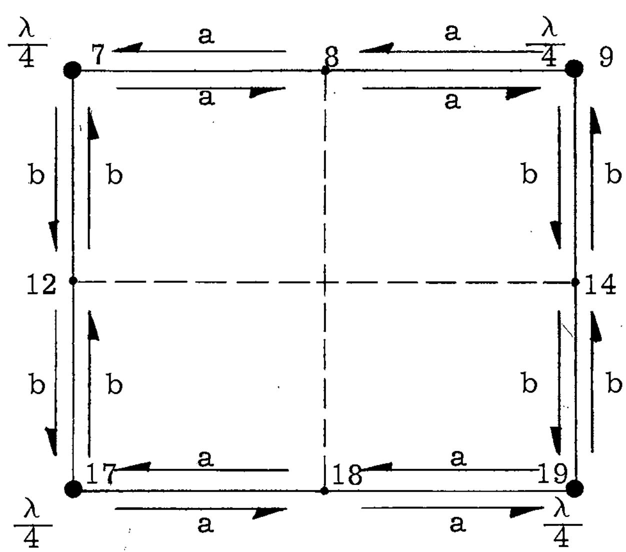


Fig. 16

Isolated Eight Point Ring - Case I

$$Q_7^{13(S)I} = \frac{1}{X_{22}} \frac{\lambda}{4} = Q_9^{13(S)I} = Q_{17}^{13(S)I} = Q_{19}^{13(S)I}$$

$$Q_8^{13(S)I} = \frac{2a}{X_{22}} \frac{\lambda}{4} = Q_{18}^{13(S)I} \qquad \Bigg| \qquad Q_{12}^{13(S)I} = \frac{2b}{X_{22}} \frac{\lambda}{4} = Q_{14}^{13(S)I} .$$

Releasing point 13, the carried-over value

$$C_{22} \lambda = \frac{8ab}{X_{22}} \frac{\lambda}{4}$$

becomes a new starting value which forms the central carry-over series. From Eq's. (7), Art. 2-4, the results due to this starting value are:

$$Q_7^{7(C)I} = \frac{C_{22}^2}{Z_{22}} \lambda = Q_9^{7(C)I} = Q_{17}^{7(C)I} = Q_{19}^{7(C)I}$$

$$Q_8^{7(C)I} = \frac{B_{22} C_{22}}{Z_{22}} \lambda = Q_{18}^{7(C)I}$$

$$Q_{12}^{7(C)I} = \frac{A_{22} C_{22}}{Z_{22}} \lambda = Q_{14}^{7(C)I}$$

$$Q_{13}^{7(C)I} = \frac{C_{22}}{Z_{22}} \lambda .$$

The final values are obtained by superimposing these results with those of the circulatory series.

Case II. This system is antisymmetrical with respect to the central Y-axis (Fig. 17). Thus there is no carry-over to the central point 13 and this case is identical with Case II of the circulatory series (Fig. 9). From Art. 2-3 the values for function coefficients are

$Q_7^{7(II)} = \frac{\lambda}{4}\frac{1}{X_{02}}$	$Q_{12}^{7(II)} = \frac{\lambda}{4}\frac{2b}{X_{02}}$	$Q_{17}^{7(II)} = \frac{\lambda}{4}\frac{1}{X_{02}}$
$Q_9^{7(II)} = -\frac{\lambda}{4}\frac{1}{X_{02}}$	$Q_{14}^{7(II)} = -\frac{\lambda}{4}\frac{2b}{X_{02}}$	$Q_{19}^{7(II)} = -\frac{\lambda}{4}\frac{1}{X_{02}}$.

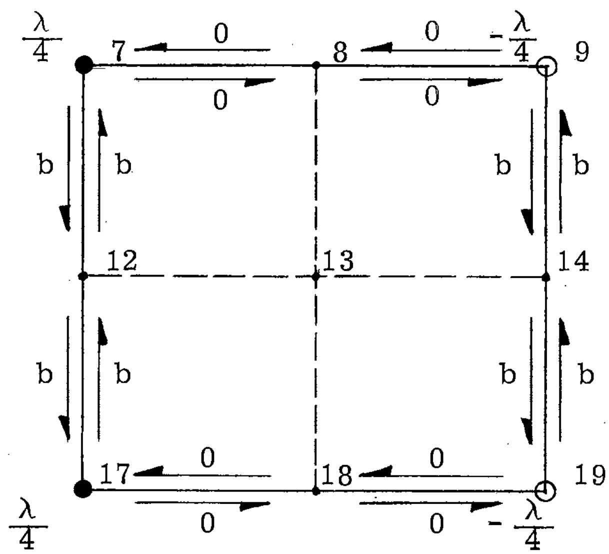


Fig. 17
Case II

<u>Case III.</u> This system (Fig. 18) is identical with Case III of the circulatory series (Fig. 10), described in Art. 2-3. The results are:

$Q_7^{7(III)} = \frac{\lambda}{4}\frac{1}{X_{20}}$	$Q_8^{7(III)} = \frac{\lambda}{4}\frac{2a}{X_{20}}$	$Q_9^{7(III)} = \frac{\lambda}{4}\frac{1}{X_{20}}$
$Q_{17}^{7(III)} = -\frac{\lambda}{4}\frac{1}{X_{20}}$	$Q_{18}^{7(III)} = -\frac{\lambda}{4}\frac{2a}{X_{20}}$	$Q_{19}^{7(III)} = -\frac{\lambda}{4}\frac{1}{X_{20}}$.

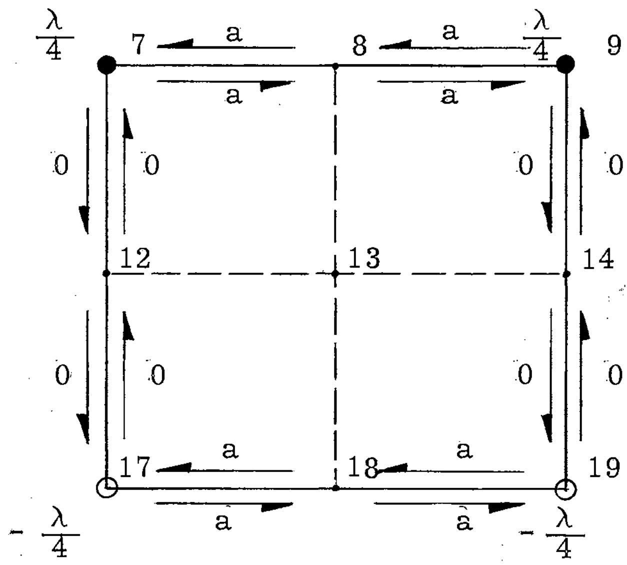


Fig. 18
Case III

Case IV. No algebraic carry-over is possible, the system being identical with Case IV of the circulatory series (Fig. 11), and the starting values represent final results (Fig. 19).

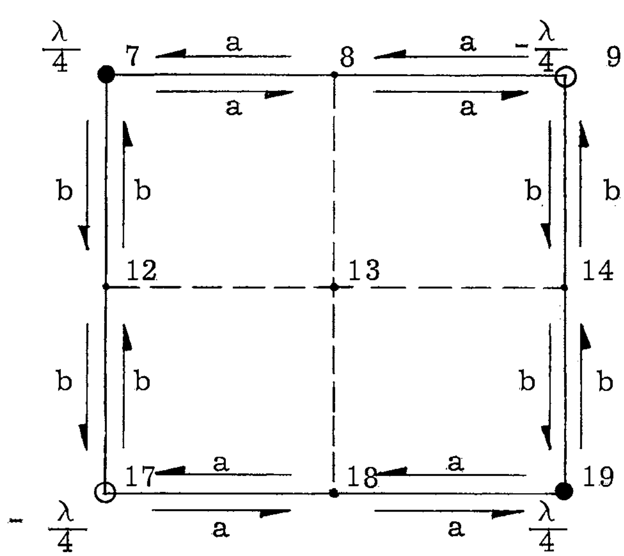


Fig. 19
Case IV

The final values for the function coefficients, starting value at point 7, are obtained by superimposing results from Cases I, II, III, and IV:

$$Q_{ij}^{7} = Q_{ij}^{I} + Q_{ij}^{II} + Q_{ij}^{III} + Q_{ij}^{IV}$$

where ij is any pivotal point of the network.

The function coefficients corresponding to a starting value λ at point 9, 17, or 19 (Fig. 12) may be obtained by a similar procedure, or they may be obtained directly from the results for starting value at 7 by cyclosymmetry. In terms of Cases I, II, III, and IV, the equations for the coefficients are:

$$Q_{ij}^{9} = Q_{ij}^{I} - Q_{ij}^{II} + Q_{ij}^{III} - Q_{ij}^{IV}$$

$$Q_{ij}^{17} = Q_{ij}^{I} + Q_{ij}^{II} - Q_{ij}^{III} - Q_{ij}^{IV}$$

$$Q_{ij}^{19} = Q_{ij}^{I} - Q_{ij}^{II} - Q_{ij}^{III} + Q_{ij}^{IV} \quad .$$

For a starting value at one of the other points the final results may be obtained by involution. Considering a starting value λ at point 8, this value is carried - over to 7, 9, and 13, thus introducing involuted starting values at these points (Fig. 20). Each of these involuted values develops series which have already been defined and determined. Superimposing these series the final results are:

$$Q_{8}^{8} = \lambda + a\lambda\, Q_{8}^{7} + a\lambda\, Q_{8}^{9} + b\lambda\, Q_{8}^{13}$$

and for any other point

$$Q_{ij}^{8} = a\lambda\, Q_{ij}^{7} + a\lambda\, Q_{ij}^{9} + b\lambda\, Q_{ij}^{13} \quad .$$

Similar equations may be written for a starting value at 12, 14, or 18.

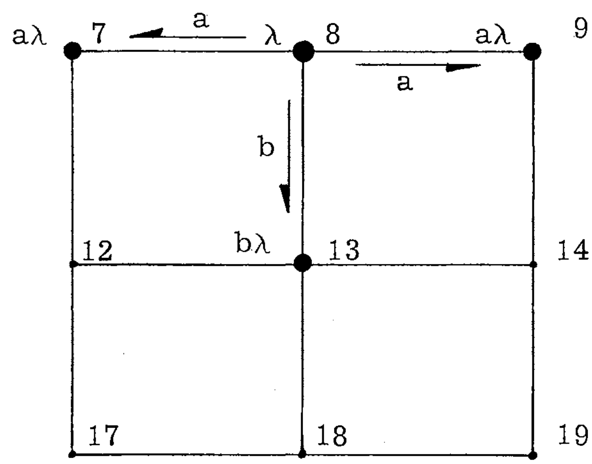


Fig. 20
Involuted Starting Values

2-6 The Laplace Equation. In rectangular coordinates the Laplace equation has the form

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = 0 \tag{8}$$

with Q equal to a given function G(x, y) on the boundary (38) .

The corresponding finite-difference equation written for an interior point ij of the net is (Fig. 2)

$$Q_{ij} = \left\{ \begin{array}{l} a\,(\,Q_{i-1,j} + Q_{i+1,j}\,) \\ \\ b\,(\,Q_{i,j-1} + Q_{i,j+1}\,) \end{array} \right\} . \tag{9}$$

At a boundary point kl, the function Q takes on the value of the known function G :

$$Q_{kl} = G_{kl} \quad . \tag{10}$$

Comparing Eq's (9) and (10) with Eq. (5), Art. 2-1, it is evident that there is a basic difference in the concept of algebraic carry-over as applied to the Laplace and the Poisson equations. In the latter case, carry-over begins from a network point with a specified starting value and proceeds to the surrounding points, the algebraic carry-over method being concerned with the determination of the resulting geometric series. Thus the solution of Poisson's equation gives final values for function coefficients which vary from a maximum at or near the point of starting value λ to zero at the boundary.

In the case of Laplace's equation, however, there are no starting values at interior points of the finite-difference net (Eq. 9), and the flow of function values takes place in an inverse manner. The boundary values are the starting values, and the algebraic carry-over proceeds from the boundary into the interior domain. Thus the nature of the carry-over solution of the two problems is fundamentally different, the investigation of the Poisson equation corresponding to an outward flow of values, and the investigation of the Laplace equation corresponding to an inward flow.

Although there is a difference in concept, it is possible to relate solutions of these equations because of the similarity in the carry-over procedure. This can be accomplished as follows. Writing the finite-difference equation for a point ij adjacent to the boundary (Fig. 21), and noting that Q takes on the value of the given function G at the boundary point i-1, j, the equation becomes

$$Q_{ij} = \left\{ \begin{array}{c} a(Q_{i+1,j}) \\ \\ b(Q_{i,j-1} + Q_{i,j+1}) \end{array} \right\} + a\, G_{i-1,j} \; .$$

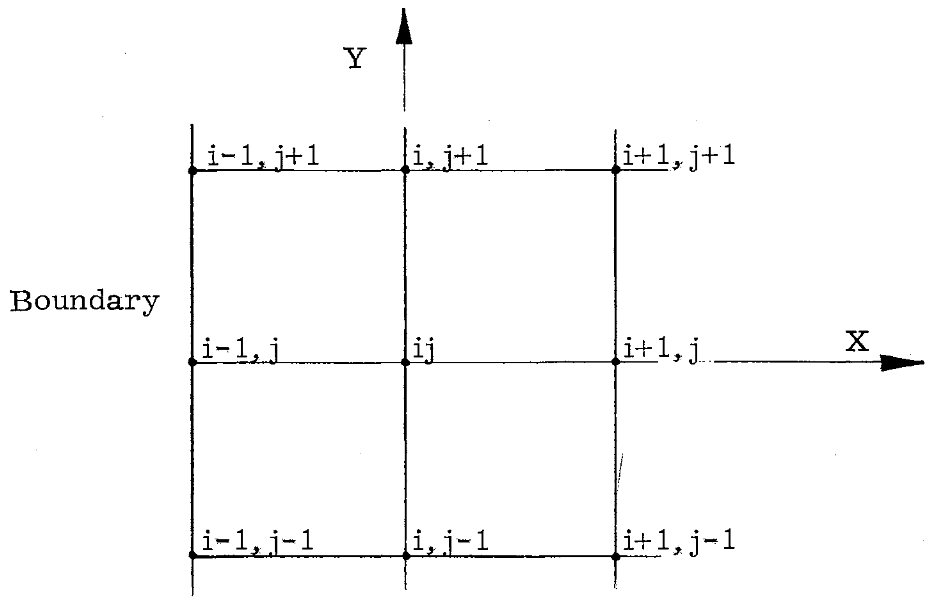


Fig. 21
Finite Difference Net Adjacent to Boundary

Comparing this equation with the Poisson equation in finite-difference form (Eq. 5), it is readily seen that the carried-over value $aG_{i-1,j}$ above corresponds with the value Q_{ij}^{*}. Thus $aG_{i-1,j}$ may be considered a new starting value at point ij and the algebraic carry-over procedure performed as before.

In this way the solution of the Laplace equation is replaced by the solution of the Poisson equation, and the following conclusion is made:

> Final results for function coefficients due to a starting value λ at some boundary point of the network are equal to the final results due to a starting value λ at the adjacent interior point, multiplied by the corresponding carry-over factor.

CHAPTER III

SKEW SYSTEMS

3-1 Linear Finite - Difference Equations. In skew coordinates the Poisson equation has the form (38)

$$\frac{1}{\mathrm{Sin}^2\,\alpha}\,\frac{\partial^2 Q}{\partial x^2} - \frac{2\,\mathrm{Cos}\,\alpha}{\mathrm{Sin}^2\,\alpha}\,\frac{\partial^2 Q}{\partial x\,\partial y} + \frac{1}{\mathrm{Sin}^2\,\alpha}\,\frac{\partial^2 Q}{\partial y^2} = -F(x,y). \quad (11)$$

The corresponding finite difference equation written for point ij of the finite difference net is (Fig. 22) (38)

$$\frac{Q_{i-1,j} - 2Q_{ij} + Q_{i+1,j}}{\Delta x^2} - \frac{Q_{i+1,j+1} - Q_{i-1,j+1} - Q_{i+1,j-1} + Q_{i-1,j-1}}{2\Delta x\Delta y}\,\mathrm{Cos}\,\alpha$$

$$+ \frac{Q_{i,j-1} - 2Q_{ij} + Q_{i,j+1}}{\Delta y^2} = -F_{ij}\,\mathrm{Sin}^2\,\alpha \quad .$$

Introducing the notation

$$a = \frac{1}{2(1+t^2)} \qquad b = \frac{t^2}{2(1+t^2)}$$

$$c = \frac{t\,\mathrm{Cos}\,\alpha}{4(1+t^2)} \qquad \lambda = \frac{t\,\mathrm{Sin}\,\alpha}{2(1+t^2)} \qquad (12)$$

$$t = \frac{\Delta x}{\Delta y}$$

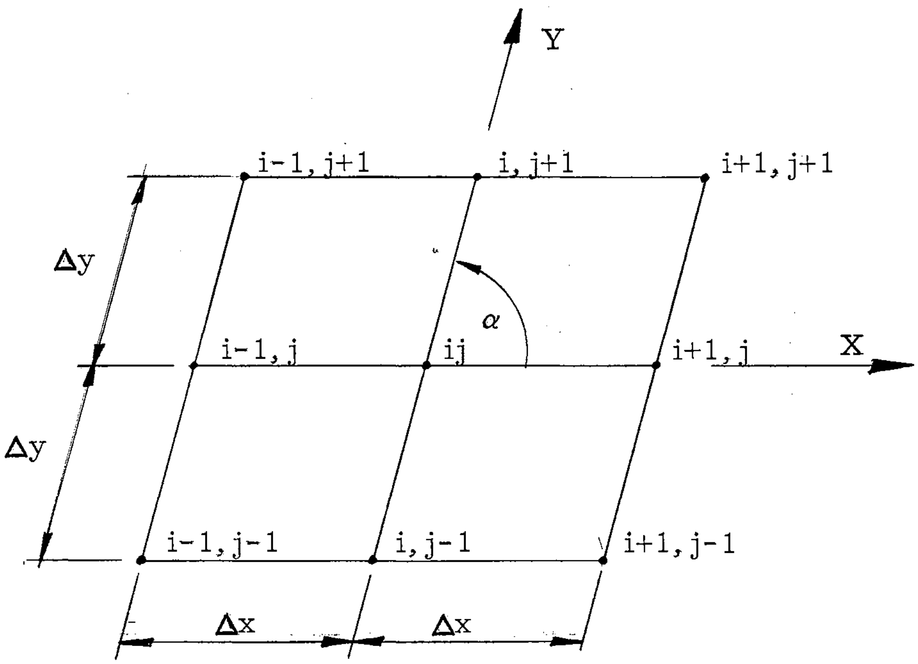


Fig. 22
Finite Difference Net in Skew Coordinates

this equation may be written

$$Q_{ij} = \left\{ \begin{matrix} a(Q_{i-1,j} + Q_{i+1,j}) + b(Q_{i,j-1} + Q_{i,j+1}) \\ \\ c(Q_{i-1,j+1} + Q_{i+1,j-1}) - c(Q_{i-1,j-1} + Q_{i+1,j+1}) \end{matrix} \right\} + Q^*_{ij} \qquad (13)$$

where

$$Q^*_{ij} = \lambda F_{ij} \Delta x \Delta y \ \text{Sin}\ \alpha \qquad (14)$$

is the starting value for Q_{ij}, assuming the Q's at the eight adjacent points to be zero. For $\alpha = \frac{\pi}{2}$, Eq's. (12), (13), and (14) reduce to Eq's. (4), (5), and (6), respectively.

It is evident from Eq. (13) that a, b, c, and -c are carry-over factors on the finite difference net. These carry-over factors represent the influences of the function values at the adjacent points i-1, j-1, i, j-1, i+1, j-1, i-1, j, i+1, j, i-1, j+1, i, j+1, and i+1, j+1 on the value at point ij.

3-2 The Basic Series. Considering a two dimensional nine point skew set with starting value λ at point 13 (Fig. 23a) and using algebraic carry-over to determine the function values, each final result is the algebraic sum of an infinite geometric series.

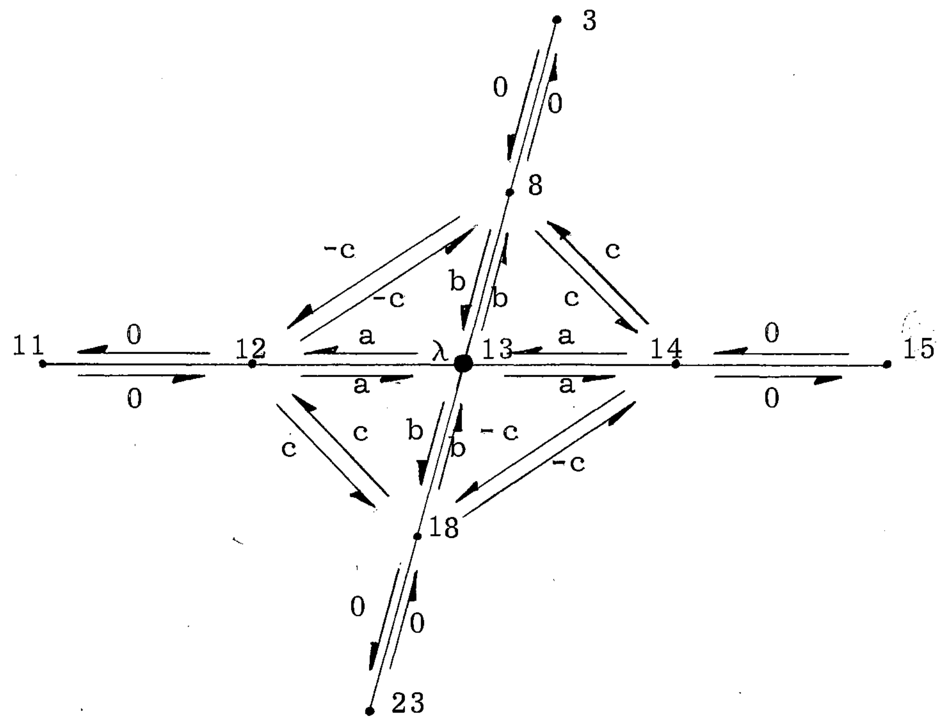


Fig. 23a
Nine Point Set - Basic Series

From the figure it may be seen that any value carried from the central point 13 to points 8, 12, 14 and 18 first flows through these points before returning to 13. From the nature of the carry-over factors, however, carried over values on the skew ring (Fig. 23b) sum to zero at every point. Thus no series develops on this isolated skew point set (8, 12, 18, 14), and the value which returns to point 13 is simply the starting value multiplied by $2a^2 + 2b^2$.

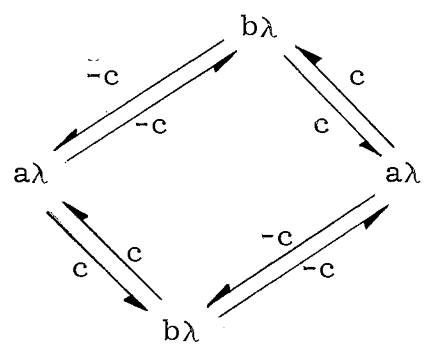


Fig. 23b
Isolated Skew Point Set

Continuing this procedure, an infinite geometric series is formed which corresponds with the basic series of the rectangular point set (Art. 2-2). The final function values are

$$Q_{13}^{13(B)} = \lambda \left[1 + (2a^2 + 2b^2) + (2a^2 + 2b^2)^2 + \ldots \right] = \lambda \frac{1}{X_{22}}$$

$$Q_8^{13(B)} = \lambda \frac{b}{X_{22}} = Q_{18}^{13(B)} \qquad \Big| \qquad Q_{12}^{13(B)} = \lambda \frac{a}{X_{22}} = Q_{14}^{13(B)}$$

where $$X_{22} = 1 - 2a^2 - 2b^2 \quad .$$

These results are identical with the solution of the basic series in Art. 2-2. A diagrammatic representation can again be made (Fig. 24) and similar conclusions stated:

(a) The constant $\frac{1}{X_{22}}$ is the over-relaxation factor for the basic series of the skew network.

(b) The final function value at the center is equal to the starting value multiplied by the over-relaxation factor

(c) The final function value at any other point is equal to the final central value multiplied by the corresponding carry-over factor.

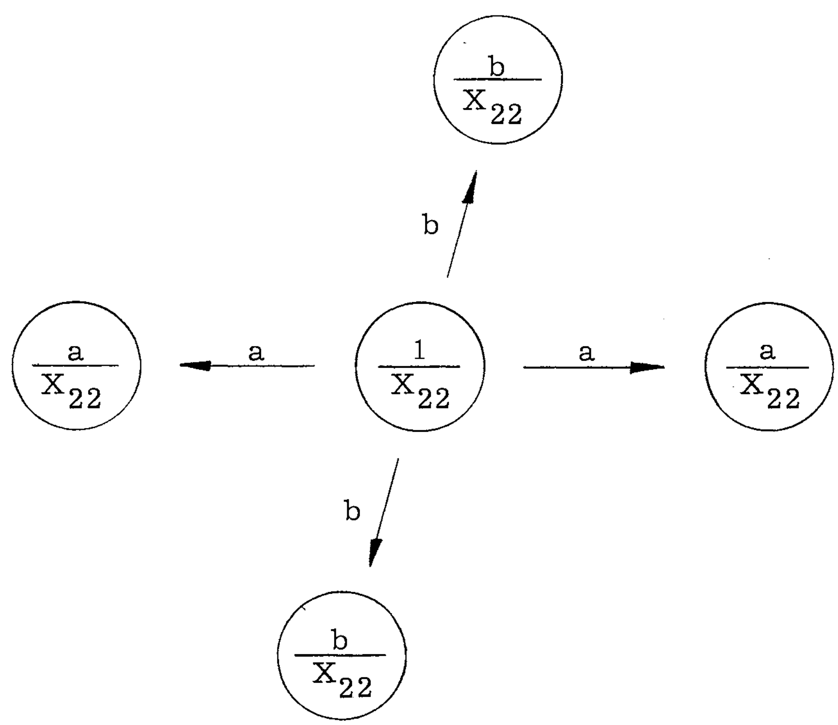


Fig. 24
Final Results - Basic Series

<u>3-3 The Skew Series.</u> For a starting value λ at point 8 of the nine point set (Fig. 23a), values are carried-over to points 12 and 14 as well as to the central point 13 (Fig. 25a). Thus the final values of function coefficients can not be found from the basic series by simple involution, as explained in Art. 2-2 for rectangular point sets, and the series due to the unsymmetrical starting values at points 12 and 14 must be determined.

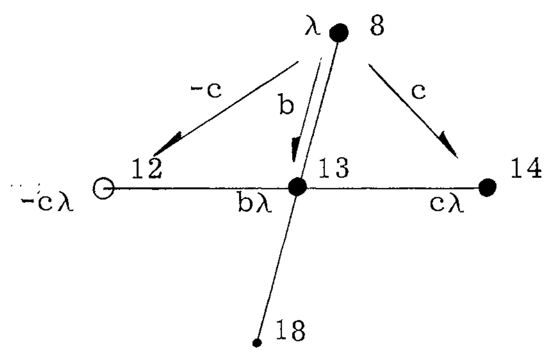


Fig. 25a
Involuted Starting Values

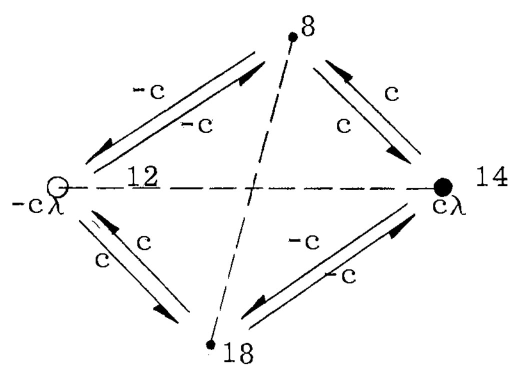


Fig. 25b
Isolated Four Point Set - Skew Series

From the antisymmetry of the starting values at these points (12, 14) with respect to point 13, the carry-over to 13 is equal to zero and the series forms on the skew point set shown in Fig. 25b. Considering the first cycle of carry-over on this skew set, it is seen that $c^2\lambda$ is carried to point 8 from both points 12 and 14, and $-c^2\lambda$ is carried to 18 from both these points. Thus the value returned to either point 12 or point 14 is equal to the corresponding starting value multiplied by $4c^2$. Continuing this procedure a simple infinite geometric series is developed which is called the skew series. The results for function values due to this series are:

$$Q_{14}^{8(Sk)} = c\lambda \left[1 + (4c^2) + (4c^2)^2 + \dots \right] = c\lambda \frac{1}{S_{22}}$$

$$Q_{12}^{8(Sk)} = - c\lambda \frac{1}{S_{22}} \qquad\Bigg|\qquad Q_8^{8(Sk)} = c\lambda \frac{2c}{S_{22}} = -Q_{18}^{8(Sk)}$$

where $$S_{22} = 1 - 4c^2 .$$

The final values for function coefficients, starting value λ at point 8, are obtained by superimposing these results with those from the basic series due to the involuted starting value $b\lambda$ at point 13. Thus

$$Q_8^{8} = \lambda + bQ_8^{13} + Q_8^{(Sk)}$$

and

$$Q_{ij}^{8} = bQ_{ij}^{13} + Q_{ij}^{(Sk)}$$

for any other point ij.

Similar results may be obtained for a starting value at 12, 14, or 18. For the particular angle α (90^{o}) at which the skew point set becomes orthogonal, the skew series vanishes and the results reduce to those of the rectangular point set.

3-4 The Circulatory Series. Applying the algebraic carry-over procedure to the analysis of an eight point closed ring (Fig. 26), each final function value is found to be the sum of four infinite geometric series.

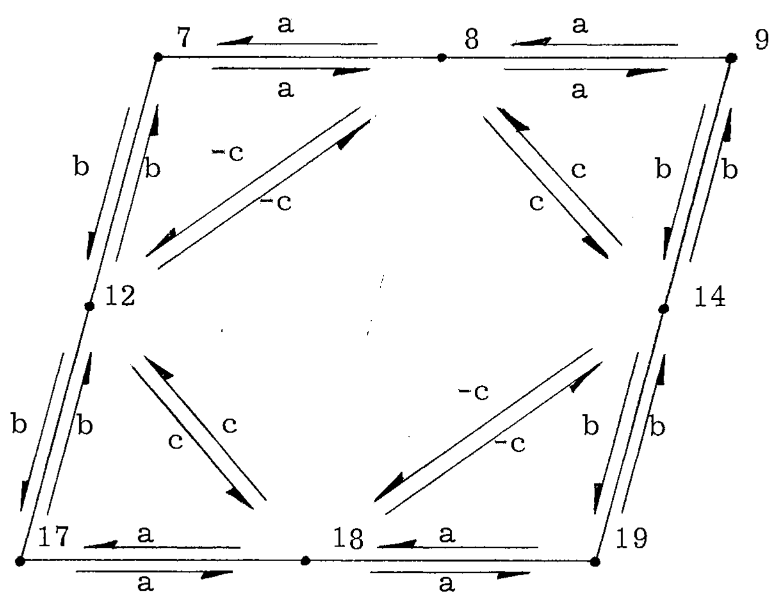


Fig. 26
Eight Point Ring - Circulatory Series

For a starting value λ at point 7, the algebraic analysis of the system can be simplified in a manner similar to that of the rectangular network (Art. 2-3) by using the methods of resolution and superposition:

(a) The initial system is resolved into four basic cases as shown in Fig. 27, taking advantage of skew symmetry and antisymmetry

(b) The algebraic results from the individual cases are superimposed to give the final values for function coefficients on the skew ring.

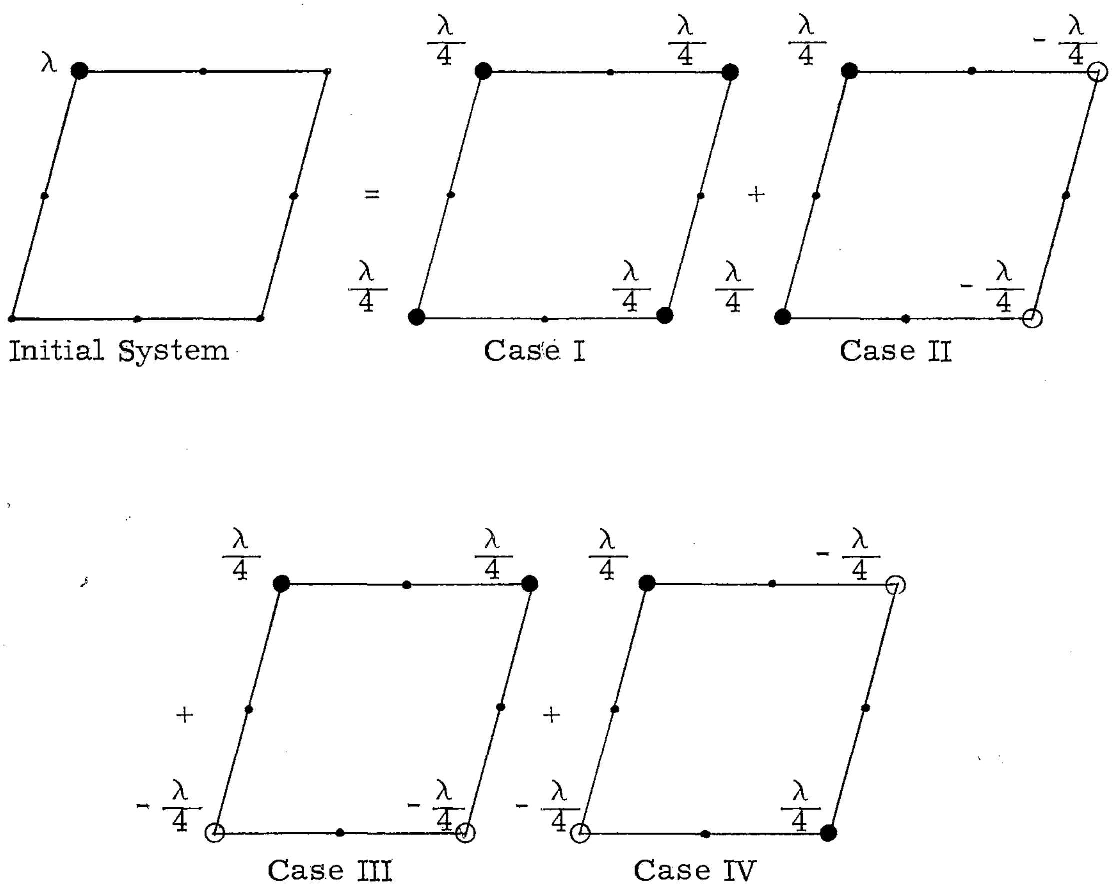


Fig. 27

Resolution of Circulatory System with Starting Value at Point 7 into Four Basic Cases

<u>Case I.</u> The algebraic carry-over procedure for this system can be simplified by using modified carry-over factors as was done in Art. 2-3 for the rectangular set.

From the skew symmetry of the starting values (Fig. 27) and the antisymmetrical nature of the carry-over factors between points 8, 12, 18, and 14 (Fig. 26), it is evident that no series develops on this skew point set. Thus only the simple circulatory series is formed, and the reduced point set of Fig. 28 may be used to determine the function values. Performing algebraic carry-over the results are:

$$Q_7^{7(S)I} = \frac{1}{X_{22}} \frac{\lambda}{4} \quad \Big| \quad Q_8^{7(S)I} = \frac{2a}{X_{22}} \frac{\lambda}{4} \quad \Big| \quad Q_{12}^{7(S)I} = \frac{2b}{X_{22}} \frac{\lambda}{4}.$$

Thus the final values for Case I of the circulatory series are identical in rectangular and skew systems.

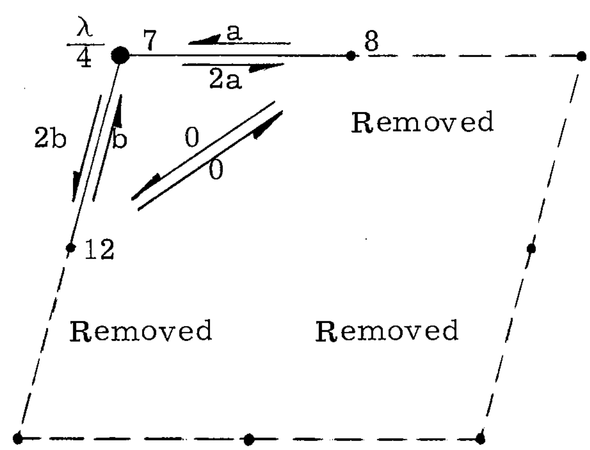


Fig. 28
Modified Circulatory System
Case I

Case II and Case III. For these systems algebraic carry-over becomes more complex, and a higher order series is formed on the closed ring. This series is called the external series and is discussed in the next article (3-5).

Case IV. For this antisymmetrical set of starting values, the first cycle of carried-over values sums to zero at points 8, 12, 14 and 18. Thus no algebraic carry-over procedure is possible (Fig. 29), and the starting values represent final results.

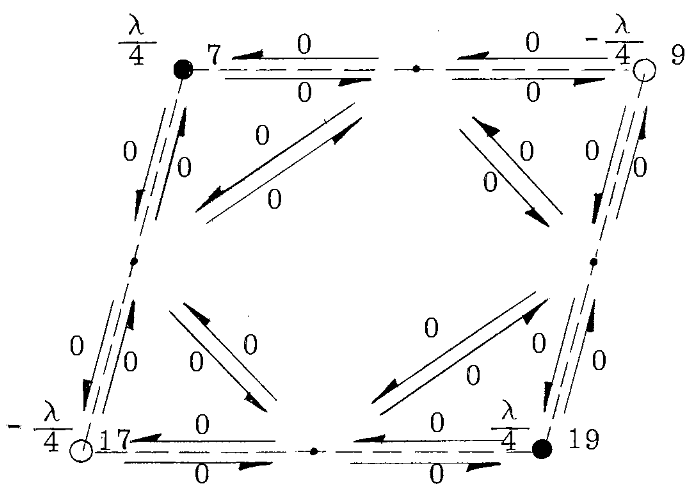


Fig. 29
Modified Circulatory System
Case IV

3-5 The External Series. For the skew- symmetrical set of starting values (Case I, Art. 3-4) it was shown that the skew series on points 8, 12, 18 and 14 vanishes (Fig. 28) and only the circulatory series of the corresponding rectangular point set remains. In the instance of starting value patterns as given in Cases II and III of Fig. 27, however, the skew series does not vanish and a higher order series forms on the circumferential ring which interrelates the skew series with the circulatory series. The development of this series by algebraic carry-over follows.

Case II. If the analysis of the system shown in Fig. 30 is begun by carrying-over the starting values at the corners to points 8, 12, 14, and 18, solving the resulting skew series on this set, and carrying back to the corners, it is found that the returned values are not equal multiples of the corresponding starting values. Continuing this procedure, complex power series are generated whose sums cannot be determined.

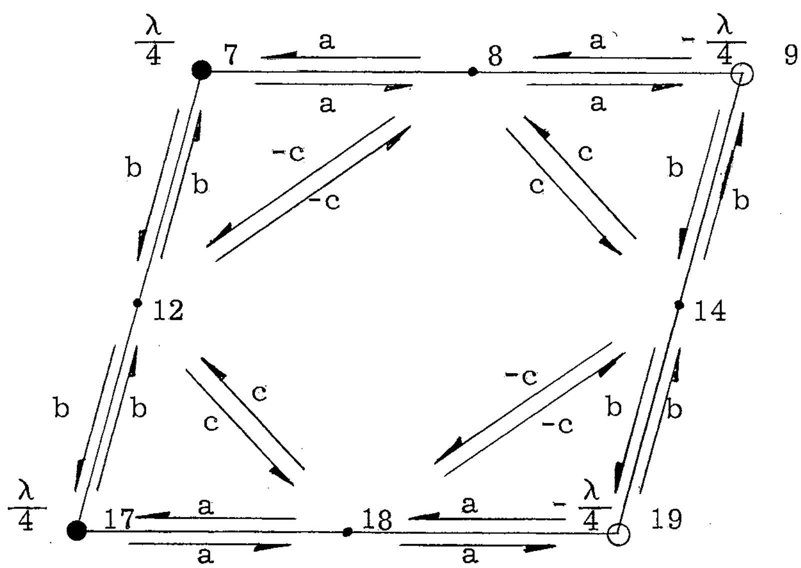


Fig. 30
External Series
Case II

To avoid this problem, linear point sets 7, 12, 17; 9, 14, 19 and 7, 8, 9; 17, 18, 19 are alternately isolated and solved by again using the concept of the suppressed or zero point. Interrelating the resulting linear series through the skew carry-over factors $\pm$ c, each final result may be represented as the sum of an infinite, geometric series each term of which is an infinite series.

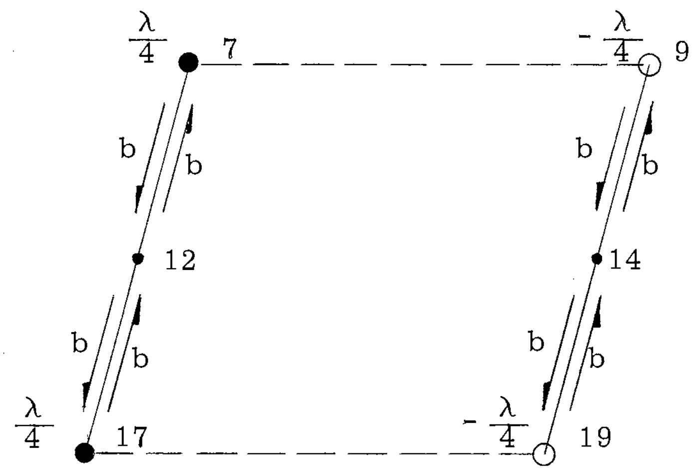


Fig. 31
One Dimensional Series-Starting
Values at 7, 9, 17, 19

Introducing zero points at 8 and 18, two independent one dimensional point sets are obtained (Fig. 31). Performing algebraic carry-over the results are:

$$Q_7^{7(S1)II} = \frac{\lambda}{4}\frac{1}{X_{02}} \qquad Q_{12}^{7(S1)II} = \frac{\lambda}{4}\frac{2b}{X_{02}} \qquad Q_{17}^{7(S1)II} = \frac{\lambda}{4}\frac{1}{X_{02}}$$

$$Q_9^{7(S1)II} = -\frac{\lambda}{4}\frac{1}{X_{02}} \qquad Q_{14}^{7(S1)II} = -\frac{\lambda}{4}\frac{2b}{X_{02}} \qquad Q_{19}^{7(S1)II} = -\frac{\lambda}{4}\frac{1}{X_{02}}$$

where

$$X_{02} = 1 - 2b^2 .$$

At this stage the function values correspond with those from Case II in rectangular coordinates. The new external series which is now demonstrated is due to the skew carry-over factors.

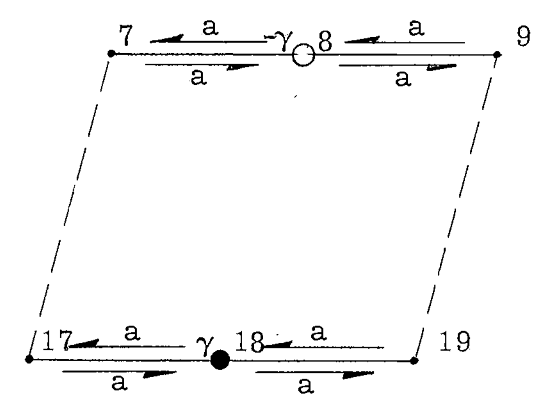


Fig. 32a
One Dimensional Series
in X-Direction

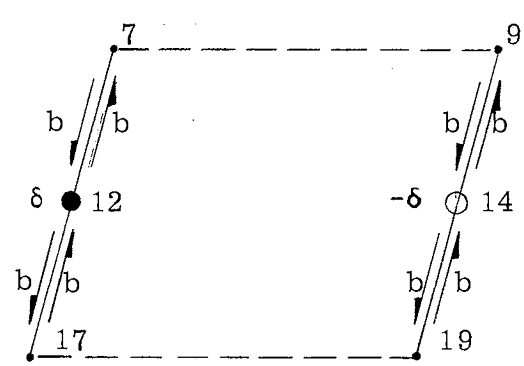


Fig. 32b
One Dimensional Series
in Y-Direction

Removing the zeros at points 8 and 18 and simultaneously suppressing points 12 and 14, the carried-over values from the corners sum to zero at 8 and 18 due to the antisymmetry of these values (Fig. 30). The antisymmetrical skew carry-over factors, however, introduce values at these points (Fig. 32a) which are carried from 12 and 14. These carried-over values

$$Q_8^{7(CO)II} = -\frac{4bc}{X_{02}} \frac{\lambda}{4} = -\gamma_0$$

$$Q_{18}^{7(CO)II} = \frac{4bc}{X_{02}} \frac{\lambda}{4} = \gamma_0$$

develop one dimensional series on the point sets 7, 8, 9 and 17, 18, 19, respectively. The results of algebraic carry-over are (Fig. 32a):

$Q_7^{7(S2)II} = -\gamma_0 \frac{a}{X_{20}}$	$Q_8^{7(S2)II} = -\gamma_0 \frac{1}{X_{20}}$	$Q_9^{7(S2)II} = -\gamma_0 \frac{a}{X_{20}}$
$Q_{17}^{7(S2)II} = \gamma_0 \frac{a}{X_{20}}$	$Q_{18}^{7(S2)II} = \gamma_0 \frac{1}{X_{20}}$	$Q_{19}^{7(S2)II} = \gamma_0 \frac{a}{X_{20}}$

where

$$X_{20} = 1 - 2a^2 .$$

The values returned to points 12 and 14, found by releasing these points and reintroducing zeros at points 8 and 18, are

$$Q_{12}^{7(CO)II} = \frac{2c}{X_{20}} \gamma_0 = \frac{4c^2}{X_{20}X_{02}} (2b) \frac{\lambda}{4} = \delta_1$$

$$Q_{14}^{7(CO)II} = -\frac{2c}{X_{20}} \gamma_0 = -\delta_1 .$$

These values develop series on the linear point sets 7, 12, 17 and 9, 14, 19 (Fig. 32b) whose sums are :

$$Q_7^{7(S3)II} = \delta_1 \frac{b}{X_{02}} \quad\Bigg|\quad Q_{12}^{7(S3)II} = \delta_1 \frac{1}{X_{02}} \quad\Bigg|\quad Q_{17}^{7(S3)II} = \delta_1 \frac{b}{X_{02}}$$

$$Q_9^{7(S3)II} = -\delta_1 \frac{b}{X_{02}} \quad\Bigg|\quad Q_{14}^{7(S3)II} = -\delta_1 \frac{1}{X_{02}} \quad\Bigg|\quad Q_{19}^{7(S3)II} = -\delta_1 \frac{b}{X_{02}} \quad .$$

One full cycle of carry-over between the two sets of isolated systems is completed by determining the values carried back to points 8 and 18 when these points are again released. Thus

$$Q_8^{7(CO)II} = -\frac{2c}{X_{02}}\delta_1 = -\frac{4c^2}{X_{20}X_{02}}\gamma_0 = -\gamma_1$$

$$Q_{18}^{7(CO)II} = \frac{2c}{X_{02}}\delta_1 = \frac{4c^2}{X_{20}X_{02}}\gamma_0 = \gamma_1 \quad .$$

Repeating this procedure infinite times, new carry-over series γ and δ are formed having the common ratio

$$\frac{\gamma_1}{\gamma_0} = \frac{4c^2}{X_{20}X_{02}} \quad .$$

These series are called the external carry-over series, and their sums are:

$$\sum_0^{\infty} \gamma_n = \gamma_0 + \gamma_1 + \ldots = \frac{4bc}{X_{02}K_{22}} \frac{\lambda}{4}$$

$$\sum_0^{\infty} \delta_n = \delta_0 + \delta_1 + \ldots = \frac{2b}{K_{22}} \frac{\lambda}{4}$$

where

$$K_{22} = 1 - \frac{4c^2}{X_{20}X_{02}} .$$

Superimposing these series the final values, Case II, are:

$$Q_7^{7(E)II} = \frac{\lambda}{4} + \frac{b}{X_{02}} \sum_0^\infty \delta_n - \frac{a}{X_{20}} \sum_0^\infty \gamma_n$$

$$= \frac{X_{20} - 4c(ab + c)}{X_{20}\, X_{02}\, K_{22}} \frac{\lambda}{4} = - Q_{19}^{7(E)II}$$

$$Q_8^{7(E)II} = - \frac{1}{X_{20}} \sum_0^\infty \gamma_n = - \frac{4bc}{X_{20}X_{02}K_{22}} \frac{\lambda}{4} = - Q_{18}^{7(E)II}$$

$$Q_9^{7(E)II} = - \frac{\lambda}{4} - \frac{b}{X_{02}} \sum_0^\infty \delta_n - \frac{a}{X_{20}} \sum_0^\infty \gamma_n$$

$$= - \frac{X_{20} + 4c(ab - c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4} = - Q_{17}^{7(E)II}$$

$$Q_{12}^{7(E)II} = \frac{1}{X_{02}} \sum_0^\infty \delta_n = \frac{2b}{X_{02}K_{22}} \frac{\lambda}{4} = - Q_{14}^{7(E)II} .$$

Case III. Applying the algebraic carry-over method to the analysis of the system shown in Fig. 33, the external carry-over series is again developed, each final value being the finite sum of an infinite number of infinite series.

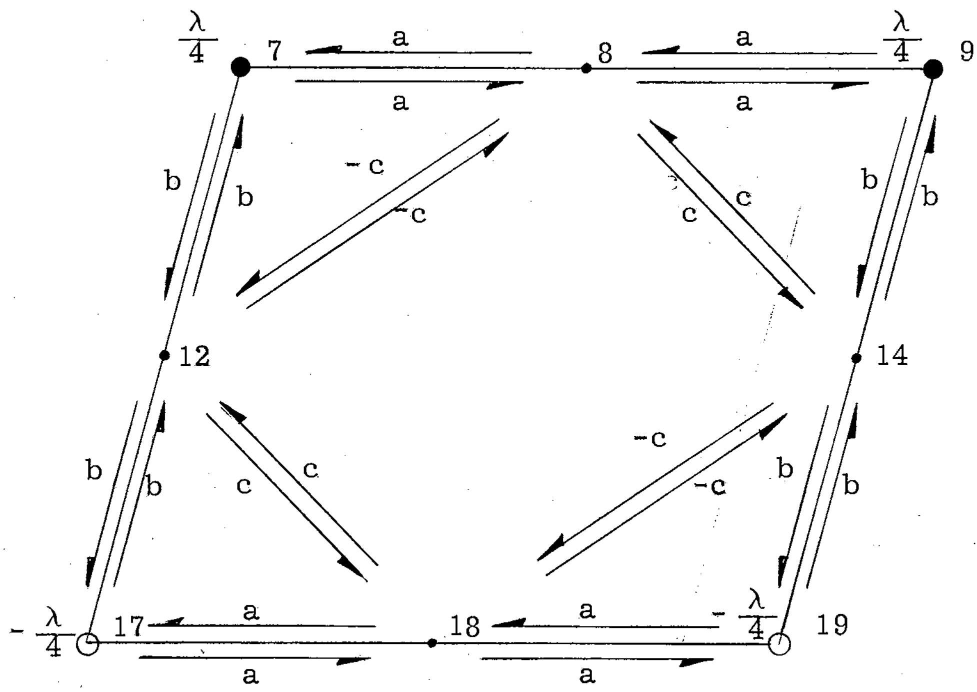


Fig. 33
External Series
Case III

The same procedure is followed as in Case II, with the exception that the point sets of Fig's. 32 are somewhat simplified by using modified carry-over factors. These reduced systems, with the modified skew carry-over factors between point sets indicated, are shown in Fig's. 34. Thus, introducing a zero at point 12, the two point set of Fig. 34a is isolated. Performing algebraic carry-over, with a starting value $\frac{\lambda}{4}$ at point 7, the results are:

$$Q_7^{7(S1)III} = \frac{\lambda}{4} \frac{1}{X_{20}} \qquad\Bigg|\qquad Q_8^{7(S1)III} = \frac{\lambda}{4} \frac{2a}{X_{20}} \quad .$$

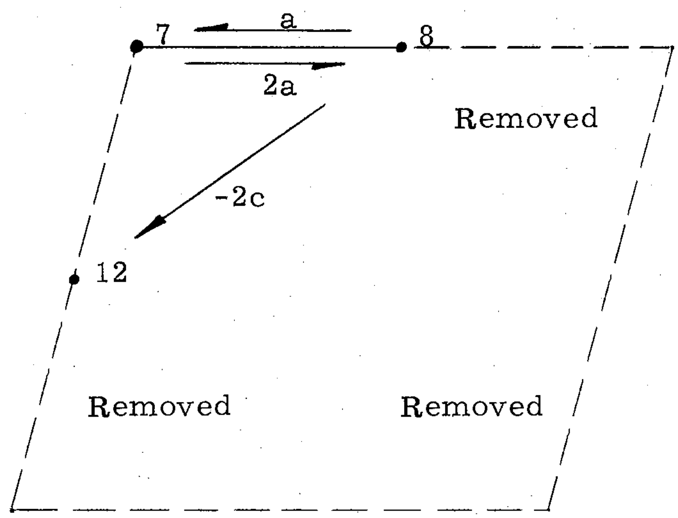


Fig. 34a

One Dimensional Series in X-Direction
Modified Point Set

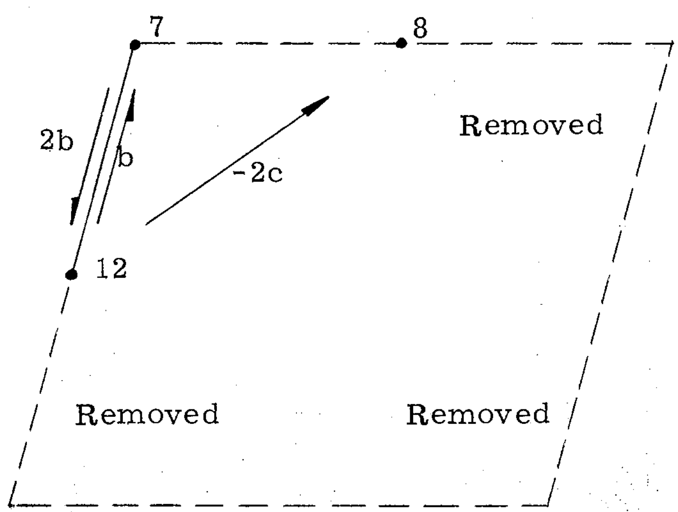


Fig. 34b

One Dimensional Series in Y-Direction
Modified Point Set

Releasing point 12 and simultaneously suppressing point 8, the carried over value

$$Q_{12}^{7(CO)III} = - \frac{4ac}{X_{20}} \frac{\lambda}{4} = - \delta_0$$

forms series on the two point set 7, 12 (Fig. 34b) whose sums are

$$Q_7^{7(S2)III} = - \delta_0 \frac{b}{X_{02}} \quad \Big| \quad Q_{12}^{7(S2)III} = - \delta_0 \frac{1}{X_{02}} .$$

Replacing the zero point at 12 and removing that at 8, the returned value is

$$Q_8^{7(CO)III} = \frac{2c}{X_{02}} \delta_0 = \frac{4c^2}{X_{20}X_{02}} (2a) \frac{\lambda}{4} = \gamma_1$$

and the results of algebraic carry-over on the reduced point set (Fig. 34a) are

$$Q_7^{7(S3)III} = \gamma_1 \frac{a}{X_{20}} \quad \Big| \quad Q_8^{7(S3)III} = \gamma_1 \frac{1}{X_{20}} .$$

Finally, releasing point 12 the carried over value becomes

$$Q_{12}^{7(CO)III} = - \frac{2c}{X_{20}} \gamma_1 = - \frac{4c^2}{X_{20}X_{02}} \delta_0 = - \delta_1 .$$

The common ratio of the carry-over series relating these two isolated point sets is again seen to be

$$\frac{\delta_1}{\delta_0} = \frac{4c^2}{X_{20}X_{02}} .$$

Continuing this procedure the external carry-over series γ and δ are formed having the sums

$$\sum_0^\infty \gamma_n = \gamma_0 + \gamma_1 + \dots = \frac{2a}{K_{22}} \frac{\lambda}{4}$$

$$\sum_0^\infty \delta_n = \delta_0 + \delta_1 + \dots = \frac{4ac}{X_{20}K_{22}} \frac{\lambda}{4} .$$

Superimposing these carry-over series the results of Case III become

$$Q_7^{7(E)III} = \frac{\lambda}{4} + \frac{a}{X_{20}} \sum_0^\infty \gamma_n - \frac{b}{X_{02}} \sum_0^\infty \delta_n$$

$$= \frac{X_{02} - 4c(ab + c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4} = - Q_{19}^{7(III)}$$

$$Q_8^{7(E)III} = \frac{1}{X_{20}} \sum_0^\infty \gamma_n = \frac{2a}{X_{20}K_{22}} \frac{\lambda}{4} = - Q_{18}^{7(III)}$$

$$Q_9^{7(E)III} = \frac{\lambda}{4} + \frac{a}{X_{20}} \sum_0^\infty \gamma_n + \frac{b}{X_{02}} \sum_0^\infty \delta_n$$

$$= \frac{X_{02} + 4c(ab - c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4} = - Q_{17}^{7(III)}$$

$$Q_{12}^{7(E)III} = - \frac{1}{X_{02}} \sum_0^\infty \delta_n = - \frac{4ac}{X_{20}X_{02}K_{22}} \frac{\lambda}{4} = - Q_{14}^{7(III)} .$$

The final function values on the eight point closed ring (Fig. 26), starting value λ at point 7, are obtained by superimposing the results from Cases I, II, III, and IV :

$$Q_{ij}^{7(S)} = Q_{ij}^{I(S)} + Q_{ij}^{II(E)} + Q_{ij}^{III(E)} + Q_{ij}^{IV(S)} \quad .$$

From the analysis of these cases it may be concluded that:

(a) The single cell series which forms in a skew closed ring can be resolved into four goemetric series

(b) Two of these series are simple geometric series and yield results for function values identical to those from the corresponding orthogonal ring (Cases I and IV)

(c) The remaining series are infinite geometric series all terms of which are infinite series (external carry-over series). These higher order series vanish when the skew ring becomes orthogonal, leaving simple geometric series identical with those of the corresponding rectangular sets (Cases II and III).

3-6 The Carry-Over Series. Considering now a two dimensional twenty-five point skew set (Fig. 35), and applying the algebraic carry-over method to the determination of function coefficients, each final value is the sum of an infinite geometric series each term of which is an infinite series.

As in the case of the twenty-five point rectangular set (Art. 2-4), a starting value λ carried-over from point 13 (Fig. 35) to the adjacent points (7, 8, 9, 12, 14, 17, 18, 19) will return to 13 as well as circulate through the closed ring 7, 8, 9, 14, 19, 18, 17, and 12. In order to separate the various series formed by this infinite carry-over process, the method of suppressed points is again used.

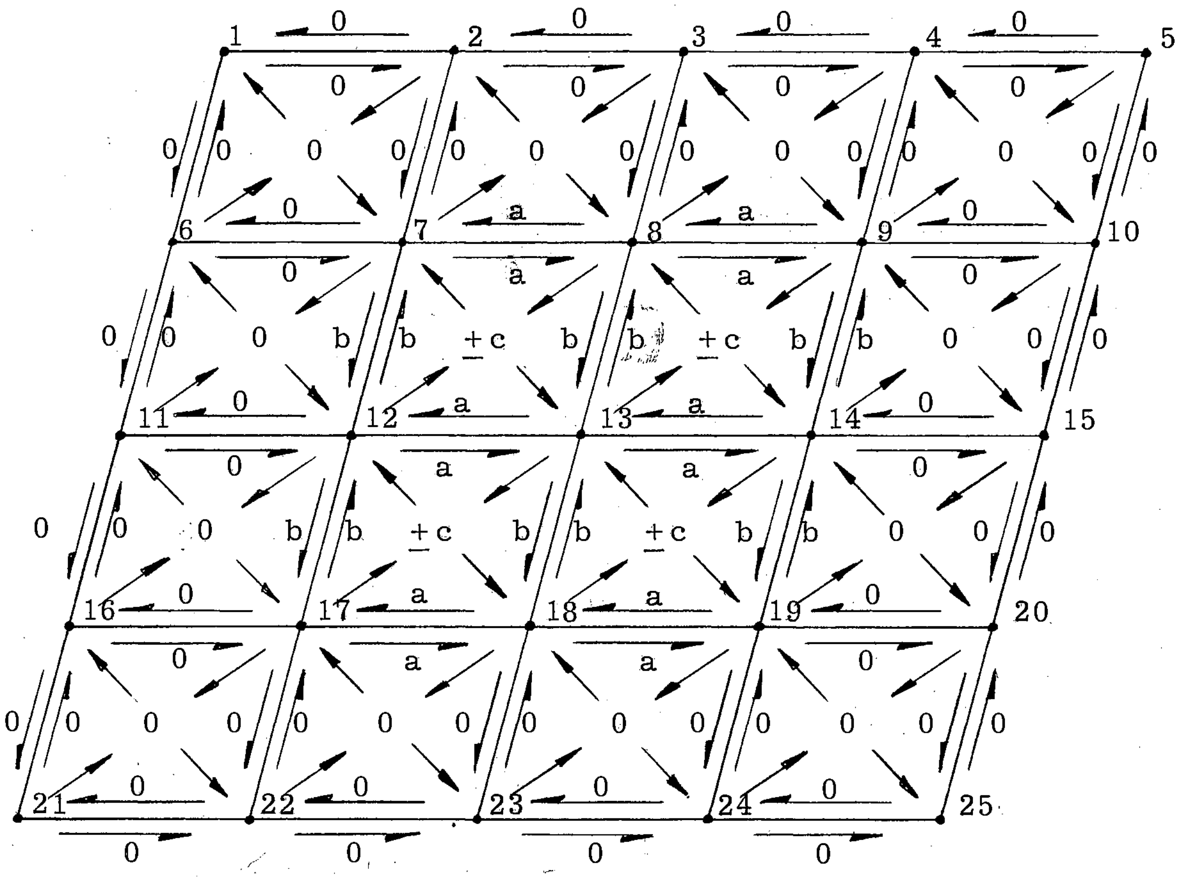


Fig. 35
Twenty - Five Point Skew Set

Thus the basic series on the five point set 8, 12, 13, 14, 18 can be isolated (Fig. 36a) by introducing zero points at 7, 9, 17 and 19. The function coefficients corresponding to the basic series are (Art. 3-2)

$$Q_8^{13(B)} = \lambda \frac{b}{X_{22}} = Q_{18}^{13(B)} \qquad Q_{13}^{13(B)} = \lambda \frac{1}{X_{22}}$$

$$Q_{12}^{13(B)} = \lambda \frac{a}{X_{22}} = Q_{14}^{13(B)} \qquad .$$

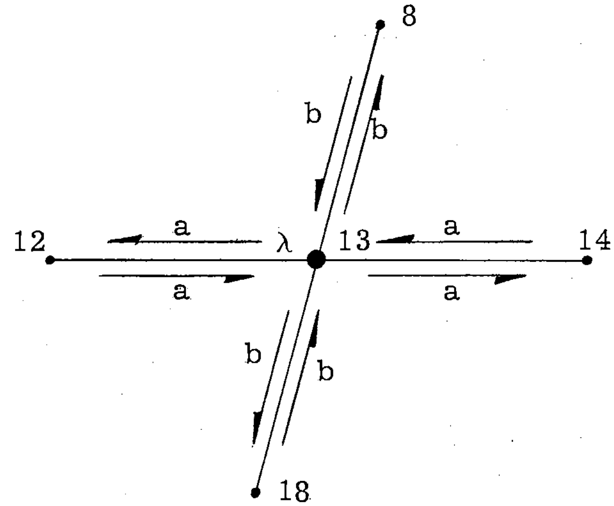


Fig. 36a
Isolated Five Point Set - Basic Series

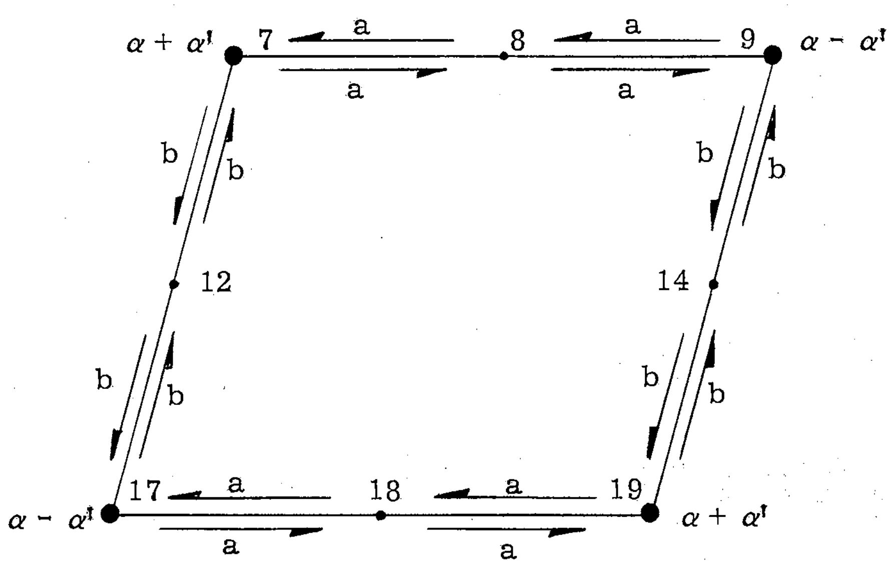


Fig. 36b
Isolated Eight Point Ring - Circulatory Series

Releasing the corners and simultaneously suppressing point 13, an eight point skew ring is isolated (Fig. 36b) with the carried-over values

$$Q_7^{13(CO)} = \frac{2ab + c}{X_{22}} \lambda = Q_{19}^{13(CO)} = \alpha_0 + \alpha'_0$$

$$Q_9^{13(CO)} = \frac{2ab - c}{X_{22}} \lambda = Q_{17}^{13(CO)} = \alpha_0 - \alpha'_0$$

where

$$\alpha_0 = \frac{2ab}{X_{22}} \lambda \qquad \Big| \qquad \alpha'_0 = \frac{c}{X_{22}} \lambda \quad .$$

This isolated system may be resolved into two basic cases as shown in Fig. 37, the first corresponding to Case I of the circulatory series and the second to Case IV (Art. 3-4). The function values from Case I are

$$Q_7^{13(S)I} = \alpha_0 \frac{1}{X_{22}} = Q_9^{13(S)I} = Q_{17}^{13(S)I} = Q_{19}^{13(S)I}$$

$$Q_8^{13(S)I} = \alpha_0 \frac{2a}{X_{22}} = Q_{18}^{13(S)I} \qquad \Big| \qquad Q_{12}^{13(S)I} = \alpha_0 \frac{2b}{X_{22}} = Q_{14}^{13(S)I} \; .$$

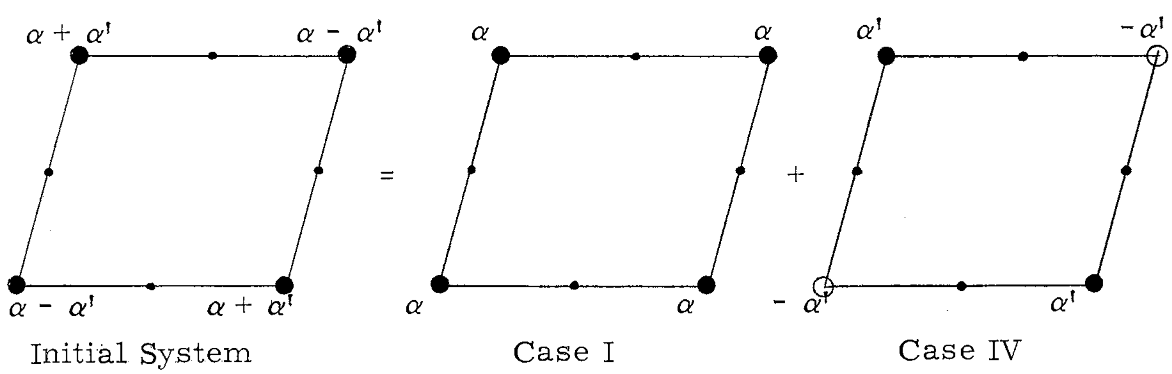


Fig. 37
Resolution of Isolated Circulatory System

From Case IV, no algebraic carry-over is necessary and the starting values represent final results.

The first cycle of carry-over is completed by releasing point 13 and finding the returned value:

$$Q_{13}^{13(CO)} = \frac{8ab}{X_{22}} \alpha_0 + 4c\alpha'_0 = \left(\frac{16a^2b^2}{X^2_{22}} + \frac{4c^2}{X_{22}} \right) \lambda = \beta_1 \quad .$$

Thus the common ratio of the carry-over series between the basic and the circulatory systems is

$$\frac{\beta_1}{\beta_0} = \frac{\beta_1}{\lambda} = \frac{16\,a^2b^2}{X^2_{22}} + \frac{4c^2}{X_{22}} \quad .$$

Repeating this procedure infinite times, the carried-over values α, α', and β form infinite geometric series whose sums are

$$\sum_0^\infty \alpha_n = \alpha_0 + \alpha_1 + \dots = \frac{2ab}{X_{22}\,Y_{22}} \lambda$$

$$\sum_0^\infty \alpha'_n = \alpha'_0 + \alpha'_1 + \dots = \frac{c}{X_{22}\,Y_{22}} \lambda$$

$$\sum_0^\infty \beta_n = \beta_0 + \beta_1 + \dots = \frac{1}{Y_{22}} \lambda$$

where

$$Y_{22} = 1 - \frac{16a^2b^2}{X^2_{22}} - \frac{4c^2}{X_{22}} \quad .$$

Superimposing these carry-over series, the final values for function coefficients are

$$Q_7^{13} = \frac{1}{X_{22}} \sum_0^\infty \alpha_n + \sum_0^\infty \alpha'_n = \frac{C_{22} + c}{Z_{22}} \lambda = Q_{19}^{13}$$

$$Q_8^{13} = \frac{2a}{X_{22}} \sum_0^\infty \alpha_n + \frac{b}{X_{22}} \sum_0^\infty \beta_n = \frac{B_{22}}{Z_{22}} \lambda = Q_{18}^{13}$$

$$Q_{12}^{13} = \frac{2b}{X_{22}} \sum_0^\infty \alpha_n + \frac{a}{X_{22}} \sum_0^\infty \beta_n = \frac{A_{22}}{Z_{22}} \lambda = Q_{14}^{13} \qquad (15)$$

$$Q_9^{13} = \frac{1}{X_{22}} \sum_0^\infty \alpha_n - \sum_0^\infty \alpha'_n = \frac{C_{22} - c}{Z_{22}} \lambda = Q_{17}^{13}$$

$$Q_{13}^{13} = \frac{1}{X_{22}} \sum_0^\infty \beta_n = \frac{1}{Z_{22}} \lambda \quad .$$

The new equivalents used in these equations are:

$$A_{22} = a(1 + \frac{4b^2}{X_{22}}) \qquad C_{22} = \frac{2ab}{X_{22}}$$

$$B_{22} = b(1 + \frac{4a^2}{X_{22}}) \qquad Z_{22} = X_{22}(1 - \frac{16a^2b^2}{X^2_{22}}) - 4c^2 \quad .$$

These constants may be interpreted from the diagrammatic presentation of final results in Fig. 38. Thus

(a) The constant $\frac{1}{Z_{22}}$ is the over-relaxation factor for the skew system

(b) The final function value at the center is equal to the start-value multiplied by the over-relaxation factor

(c) The final function value at any other point is equal to the final central value multiplied by the corresponding direct final carry-over factor $A_{22}, B_{22}, C_{22} + c$, or $C_{22} - c$.

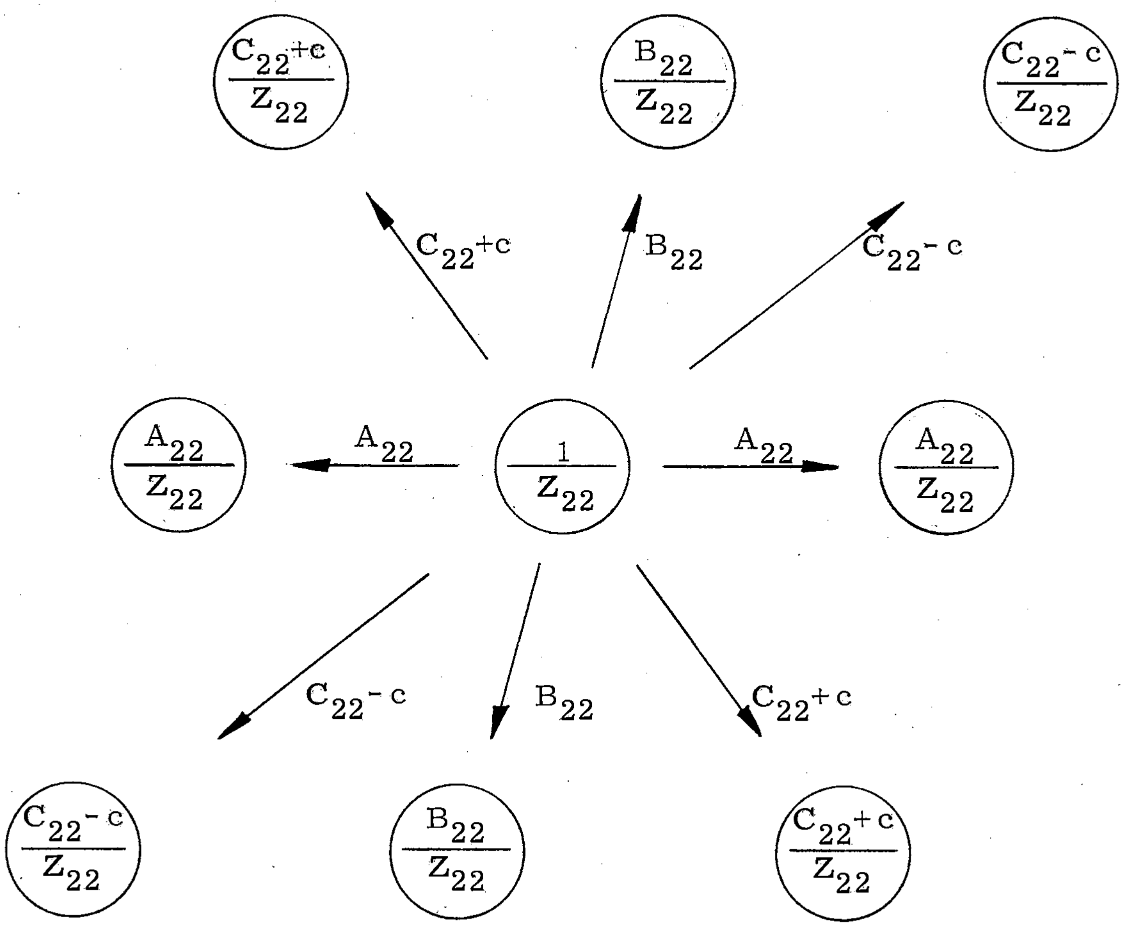


Fig. 38
Final Results - Carry-Over Series

These function values are similar to those of the twenty-five point rectangular set given in Fig. 14, and become identical for $\alpha = 90^{\circ}$ ($c = 0$) .

3-7 Resolution, Superposition, and Involution. The principles of resolution, superposition, and involution, first discussed in Chapter II and demonstrated for skew point sets in Articles 3-3 and 3-4, may be used to complete the analysis of the twenty-five point skew set (Fig. 35) as follows.

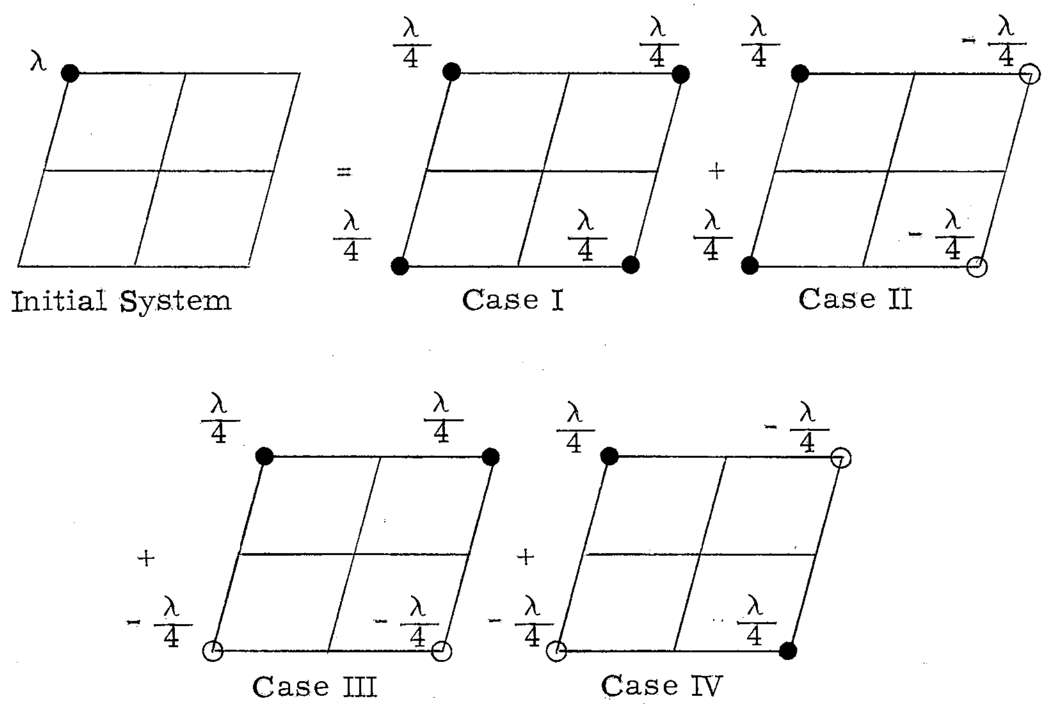


Fig. 39

Resolution of Twenty-Five Point Skew Set with Starting Value at Point 7 into Basic Cases

Considering first a starting value λ at point 7, the system is resolved into four basic cases as shown in Fig. 39. The results for these cases are obtained by superimposing the values from the circulatory and the central carry-over series.

Case I. Temporarily suppressing the central point 13, an eight point skew ring is isolated (Fig. 40) which is identical with Case I of the circulatory series (Art. 3-4). The results are

$$Q_7^{13(S)I} = \frac{1}{X_{22}} \frac{\lambda}{4} = Q_9^{13(S)I} = Q_{17}^{13(S)I} = Q_{19}^{13(S)I}$$

$$Q_8^{13(S)I} = \frac{2a}{X_{22}} \frac{\lambda}{4} = Q_{18}^{13(S)I} \quad \Bigg| \quad Q_{12}^{13(S)I} = \frac{2b}{X_{22}} \frac{\lambda}{4} = Q_{14}^{13(S)I} .$$

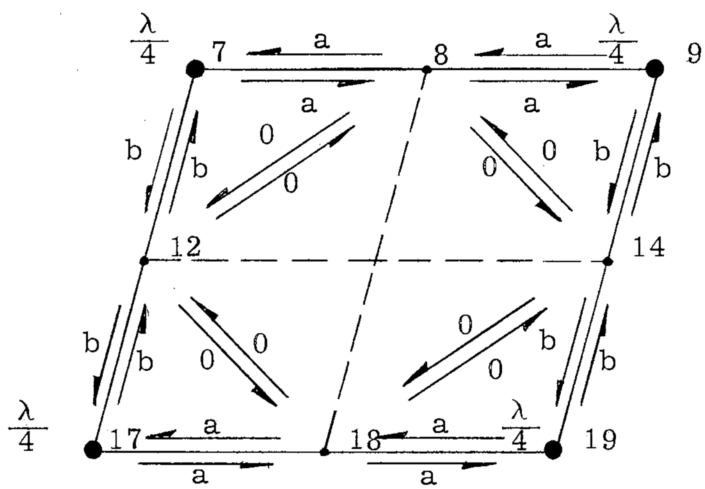


Fig. 40
Isolated Eight Point Ring
Case I

Removing the zero point at 13, the carried-over value

$$C_{22} \lambda = \frac{8ab}{X_{22}} \frac{\lambda}{4}$$

becomes a new starting value which forms the central carry-over series discussed in Art. 3-6.

Superimposing the results of the central series (Eq's. 15) with those of the circulatory series above, the final values for function coefficients, Case I, are :

$$Q_7^{7(I)} = Q_{19}^{7(I)} = \frac{1}{X_{22}} \frac{\lambda}{4} + \frac{C_{22} + c}{Z_{22}} C_{22} \lambda$$

$$Q_9^{7(I)} = Q_{17}^{7(I)} = \frac{1}{X_{22}} \frac{\lambda}{4} + \frac{C_{22} - c}{Z_{22}} C_{22} \lambda$$

$$Q_8^{7(I)} = Q_{18}^{7(I)} = \frac{2a}{X_{22}} \frac{\lambda}{4} + \frac{B_{22}}{Z_{22}} C_{22} \lambda$$

$$Q_{12}^{7(I)} = Q_{14}^{7(I)} = \frac{2b}{X_{22}} \frac{\lambda}{4} + \frac{A_{22}}{Z_{22}} C_{22} \lambda$$

$$Q_{13}^{7(I)} = \frac{1}{Z_{22}} C_{22} \lambda \quad .$$

Case II. The starting values of this system are antisymmetrical with respect to the central point 13 (Fig. 41). Thus the carry-over to that point is equal to zero and this case becomes identical with Case II of the circulatory system, the external series (Art. 3-5, Fig. 30). The values for function coefficients are therefore

$$Q_7^{7(II)} = - Q_{19}^{7(II)} = \frac{X_{20} - 4c(ab + c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4}$$

$$Q_9^{7(II)} = - Q_{17}^{7(II)} = - \frac{X_{20} + 4c(ab - c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4}$$

$$Q_8^{7(II)} = - Q_{18}^{7(II)} = - \frac{4bc}{X_{20}X_{02}K_{22}} \frac{\lambda}{4}$$

$$Q_{12}^{7(II)} = - Q_{14}^{7(II)} = \frac{2b}{X_{02}K_{22}} \frac{\lambda}{4} \quad .$$

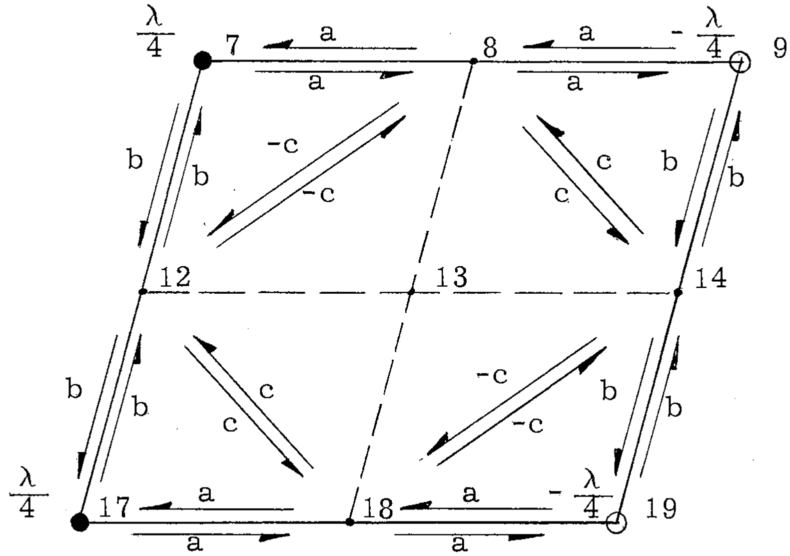


Fig. 41
Case II

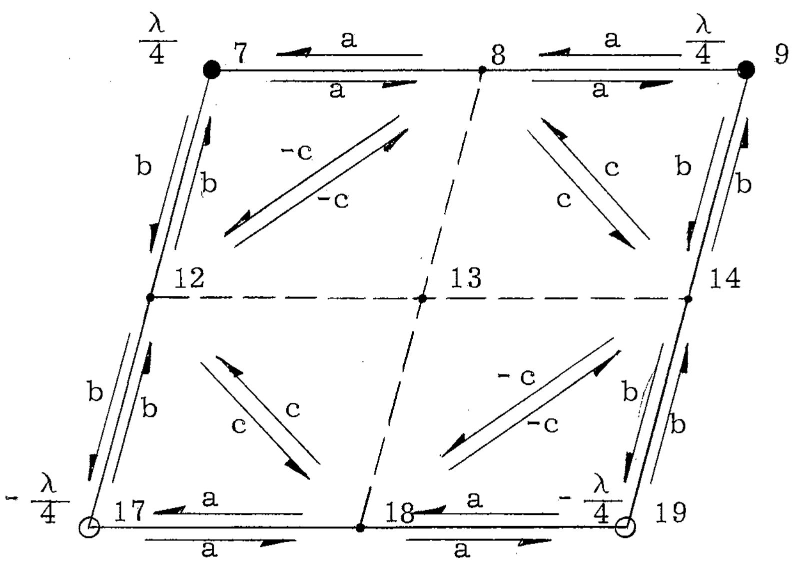


Fig. 42
Case III

Case III. This system is also antisymmetrical with respect to the central point 13 (Fig. 42). The carry-over to that point is again zero, and this case is seen to be identical with the external series formed by Case III of the circulatory system (Art. 3-5, Fig. 33). The results are

$$Q_7^{7(III)} = - Q_{19}^{7(III)} = \frac{X_{02} - 4c(ab + c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4}$$

$$Q_9^{7(III)} = - Q_{17}^{7(III)} = \frac{X_{02} + 4c(ab - c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{4}$$

$$Q_8^{7(III)} = - Q_{18}^{7(III)} = \frac{2a}{X_{20}K_{22}} \frac{\lambda}{4}$$

$$Q_{12}^{7(III)} = - Q_{14}^{7(III)} = - \frac{4ac}{X_{20}X_{02}K_{22}} \frac{\lambda}{4} \quad .$$

Case IV. Temporarily suppressing the central point 13, an eight point closed ring is isolated (Fig. 43) which is identical to Case IV of the circulatory series (Art. 3-4). Thus no algebraic carry-over procedure is possible and the starting values become final results for this isolated system.

Releasing point 13, the carried over value is

$$c\lambda = 4c \frac{\lambda}{4} \quad .$$

The final function values, Case IV, are obtained by superimposing the initial values at the corners and the results of the central series (Eq's. 15) due to this new starting value at 13. Thus

$$Q_7^{7(IV)} = Q_{19}^{7(IV)} = \frac{\lambda}{4} + \frac{C_{22} + c}{Z_{22}} c\lambda$$

$$Q_9^{7(IV)} = Q_{17}^{7(IV)} = -\frac{\lambda}{4} + \frac{C_{22} - c}{Z_{22}} c\lambda$$

$$Q_8^{7(IV)} = Q_{18}^{7(IV)} = \frac{B_{22}}{Z_{22}} c\lambda$$

$$Q_{12}^{7(IV)} = Q_{14}^{7(IV)} = \frac{A_{22}}{Z_{22}} c\lambda$$

$$Q_{13}^{7(IV)} = \frac{1}{Z_{22}} c\lambda \quad .$$

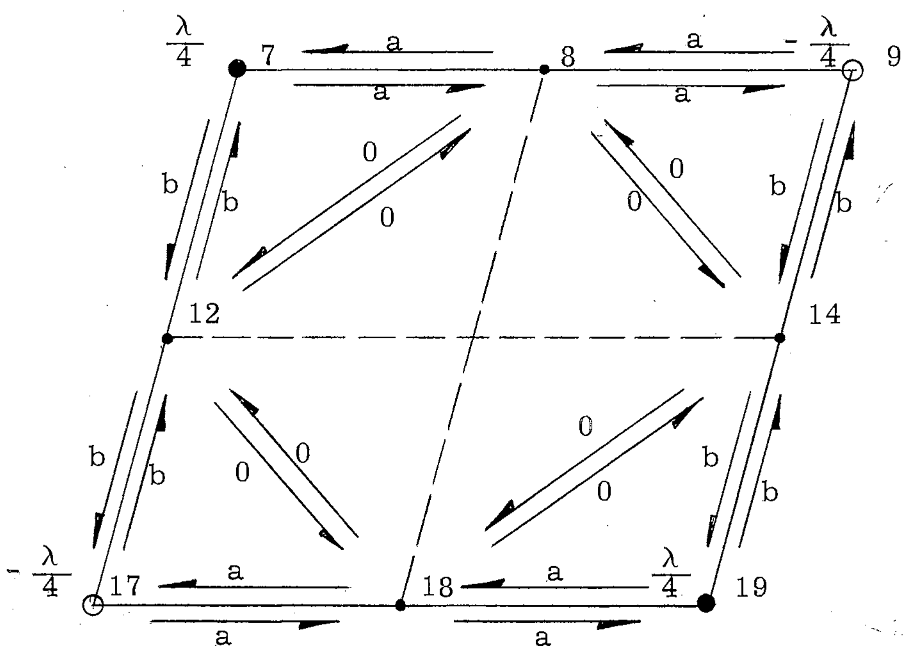


Fig. 43

Isolated Eight Point Ring
Case IV

Superposition of Cases I, II, III, and IV yields the final results, starting value λ at point 7:

$$Q_{ij}^{7} = Q_{ij}^{I} + Q_{ij}^{II} + Q_{ij}^{III} + Q_{ij}^{IV} .$$

From cyclosymmetry, the equations of function coefficients corre-sponding to a starting value at points 9, 17, or 19 are :

$$Q_{ij}^{9} = Q_{ij}^{I} - Q_{ij}^{II} + Q_{ij}^{III} - Q_{ij}^{IV} .$$

$$Q_{ij}^{17} = Q_{ij}^{I} + Q_{ij}^{II} - Q_{ij}^{III} - Q_{ij}^{IV} .$$

$$Q_{ij}^{19} = Q_{ij}^{I} - Q_{ij}^{II} - Q_{ij}^{III} + Q_{ij}^{IV} .$$

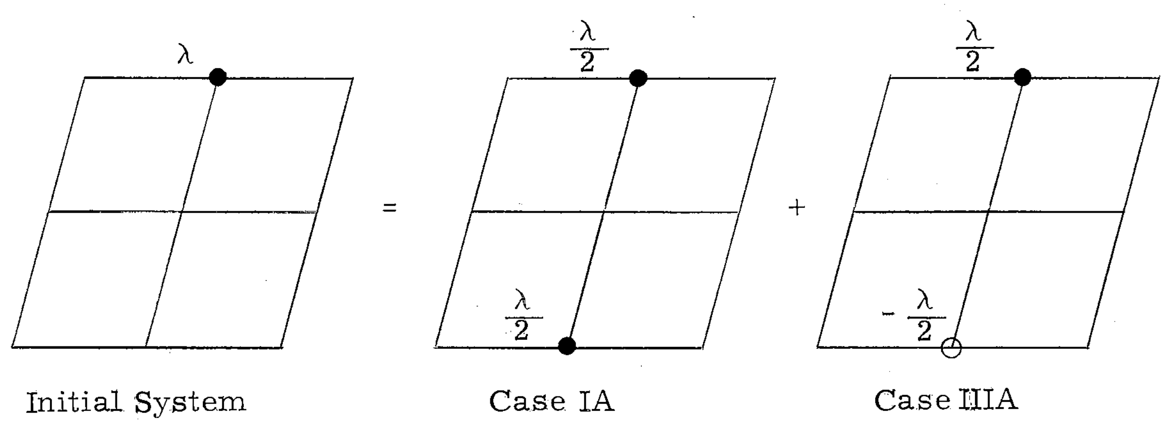


Fig. 44
Resolution of Twenty-Five Point Set with Starting Value at Point 8 into Basic Cases

For a starting value λ at point 8, the twenty-five point skew set is resolved into two basic cases as shown in Fig. 44 and the results superimposed.

Case IA. This system is a modification of Case I, and the final results may be obtained by involution. The values at 8 and 18 are carried-over to 7, 9, 17, 19, and 13, thus introducing involuted starting values at these points (Fig. 45) which develop series previously defined and determined. Superimposing these series the function values are:

$$Q_8^{8(IA)} = \frac{\lambda}{2} + 2a\, Q_8^{I} + b\, Q_8^{13} = Q_{18}^{8(IA)}$$

and for any other point ij

$$Q_{ij}^{8(IA)} = 2a\, Q_{ij}^{I} + b\, Q_{ij}^{13} \quad .$$

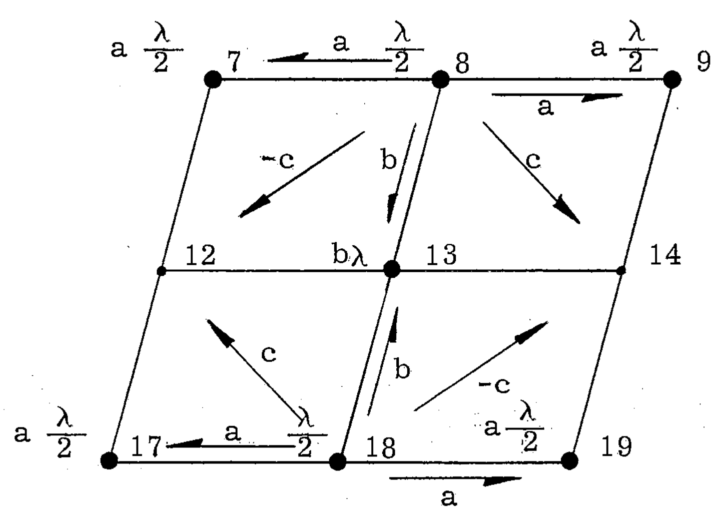


Fig. 45
Involuted Starting Values
Case IA

Case IIIA. This system of starting values, a modification of Case III, develops the external carry-over series on the modified point sets of Fig's. 46a, b. Proceeding as in Article 3-5, the final results are

$$Q_7^{8(IIIA)} = \frac{a}{X_{20}K_{22}} \frac{\lambda}{2} - \frac{2bc}{X_{20}X_{02}K_{22}} \frac{\lambda}{2}$$

$$= \frac{a - 2b\,(ab + c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{2} = - Q_{19}^{8(IIIA)}$$

$$Q_8^{8(IIIA)} = \frac{1}{X_{20}K_{22}} \frac{\lambda}{2} = - Q_{18}^{8(IIIA)}$$

$$Q_9^{8(IIIA)} = \frac{a}{X_{20}K_{22}} \frac{\lambda}{2} + \frac{2bc}{X_{20}X_{02}K_{22}} \frac{\lambda}{2}$$

$$= \frac{a - 2b\,(ab - c)}{X_{20}X_{02}K_{22}} \frac{\lambda}{2} = - Q_{17}^{8(IIIA)}$$

$$Q_{12}^{8(IIIA)} = - \frac{2c}{X_{20}X_{02}K_{22}} \frac{\lambda}{2} = - Q_{14}^{8(IIIA)} \quad .$$

Superimposing these values and those from Case IA, the final results, starting value at 8, are

$$Q_{ij}^{8} = Q_{ij}^{IA} + Q_{ij}^{IIIA} \quad .$$

By cyclosymmetry, the final results for starting value at point 18 are

$$Q_{ij}^{18} = Q_{ij}^{IA} - Q_{ij}^{IIIA} \quad .$$

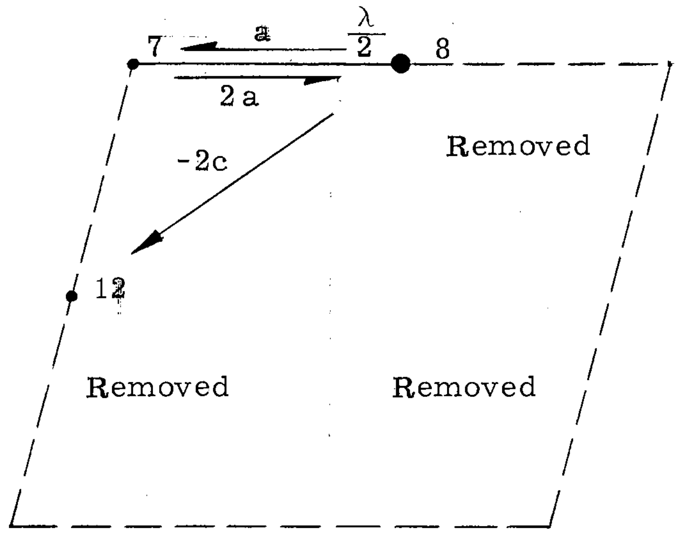


Fig. 46a

One Dimensional Series in X-Direction
Modified Point Set - Case IIIA

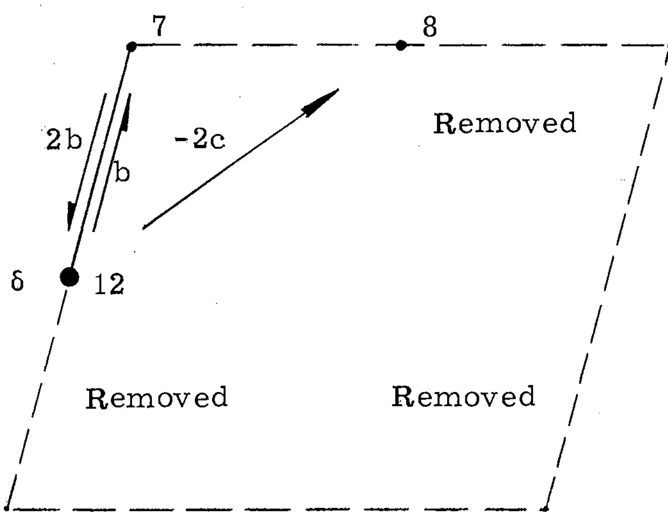


Fig. 46b

One Dimensional Series in Y-Direction
Modified Point Set - Case IIIA

The function coefficients due to a starting value at point 12 or 14 may be determined by involution. Thus considering a starting value λ at 12 (Fig. 47) the results are:

$$Q_{12}^{12} = \frac{\lambda}{2} + b\,Q_{12}^{7} + a\,Q_{12}^{13} - c\,Q_{12}^{8} + c\,Q_{12}^{18}$$

and

$$Q_{ij}^{12} = b\,Q_{ij}^{7} + a\,Q_{ij}^{13} - c\,Q_{ij}^{8} + c\,Q_{ij}^{18}$$

for any other point of the net.

Similar equations can be written for a starting value at 14.

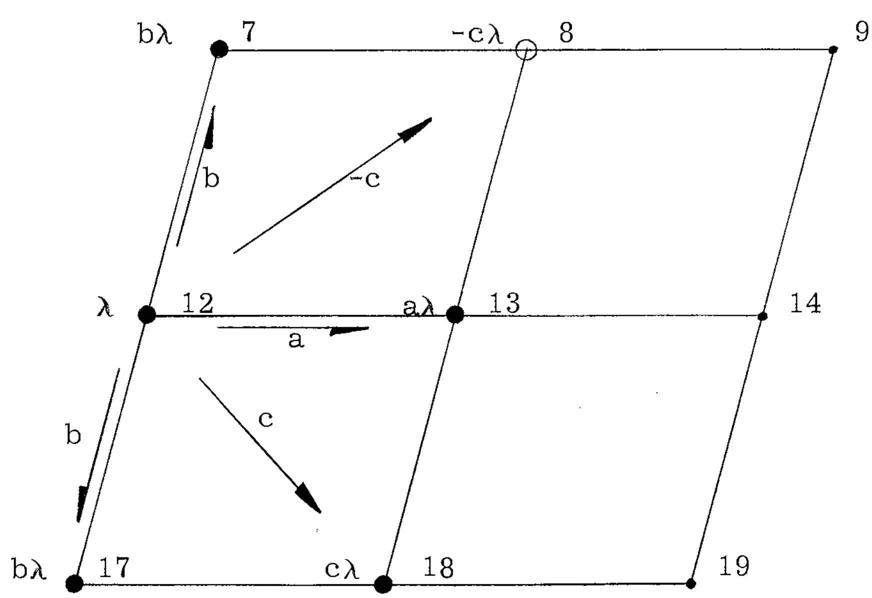


Fig. 47
Involuted Starting Values

3-8 The Laplace Equation. In skew coordinates the Laplace equation has the form

$$\frac{\partial^2 Q}{\partial x^2} - 2 \cos \alpha \frac{\partial^2 Q}{\partial x \partial y} + \frac{\partial^2 Q}{\partial y^2} = 0 \tag{16}$$

with Q equal to a given function G(x, y) on the boundary (38).

The corresponding difference equation for an interior point ij of the finite difference net is (Fig. 22)

$$Q_{ij} = \left\{ \begin{array}{l} a(Q_{i-1,j} + Q_{i+1,j}) + b(Q_{i,j-1} + Q_{i,j+1}) \\ \\ c(Q_{i-1,j+1} + Q_{i+1,j-1}) - c(Q_{i-1,j-1} + Q_{i+1,j+1}) \end{array} \right\} . \tag{17}$$

At a boundary point kl, the function Q takes on the value of the given function G:

$$Q_{kl} = G_{kl} . \tag{18}$$

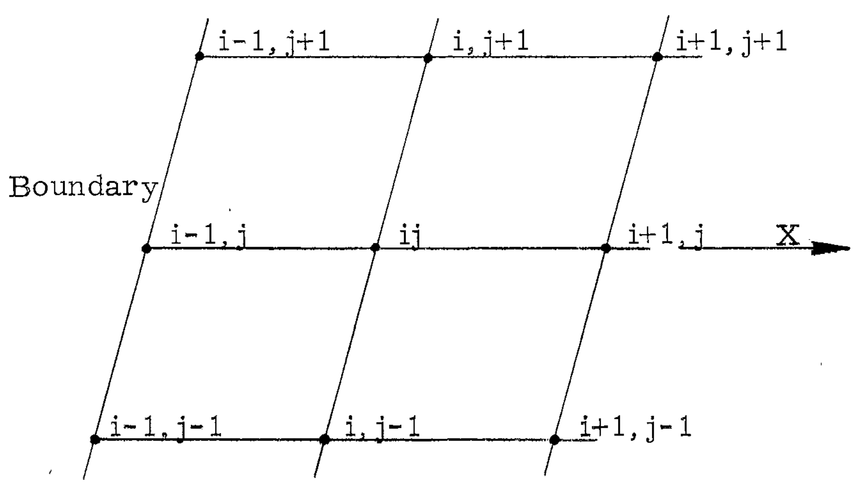


Fig. 48
Finite Difference Net Adjacent to the Boundary

The carry-over solution of the Laplace equation in finite-difference form is again achieved by relating it to the solution of the Poisson equation, as explained in Art. 2-6. Writing the finite-difference equation for point ij adjacent to the boundary (Fig. 48)

$$Q_{ij} = \left\{ \begin{array}{l} a(Q_{i+1,j}) + b(Q_{i,j-1} + Q_{i,j+1}) \\ \\ c(Q_{i+1,j-1}) - c(Q_{i+1,j+1}) \end{array} \right\} + \left\{ \begin{array}{c} a\,G_{i-1,j} \\ \\ c\,G_{i-1,j+1} - c\,G_{i-1,j-1} \end{array} \right\}$$

and comparing this equation with the Poisson equation in finite difference form (Eq. 13), it is evident that the sum of the carried over values above (terms in the second brackett) corresponds with the value Q_{ij}^*. Thus the sum may be considered as a starting value at point ij and the algebraic carry-over procedure performed as before.

The Laplace equation in skew coordinates is therefore solved by algebraic carry-over in the same way as is the equation in rectangular coordinates. Final results for function coefficients due to a starting value λ at a boundary point of the network are equal to the sum of final results due to starting values λ at the adjacent interior points, multiplied by the corresponding carry-over factors.

CHAPTER IV

POLAR SYSTEMS

<u>4-1 Linear Finite - Difference Equations.</u> In polar coordinates the Poisson equation has the form (38)

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Q}{\partial \theta^2} = - F(r,\theta) \quad . \tag{19}$$

The corresponding finite difference equation written for point ij of the finite difference net is (Fig. 49) (38)

$$\frac{Q_{i+1,j} - 2Q_{ij} + Q_{i-1,j}}{\Delta r^2} + \frac{Q_{i+1,j} - Q_{i-1,j}}{2r_i \, \Delta r} + \frac{Q_{i,j+1} - 2Q_{ij} + Q_{i,j-1}}{r_i^{\,2} \, \Delta\theta^2} = -F_{ij}.$$

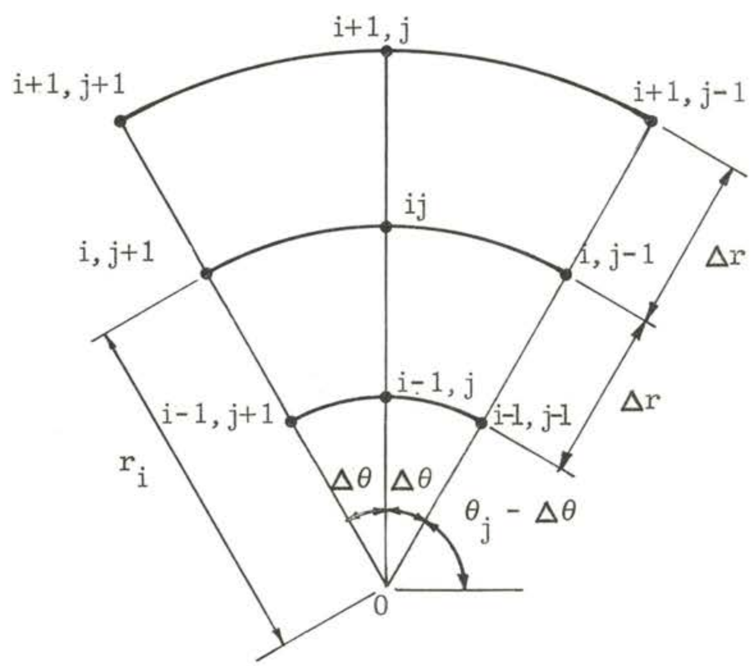


Fig. 49
Finite Difference Net in Polar Coordinates

Introducing the notation

$$a_{i+1,i} = \frac{i(2i+1)\,\Delta\theta^2}{4i^2\,\Delta\theta^2 + 2} \qquad a_{i-1,i} = \frac{i(2i-1)\,\Delta\theta^2}{4i^2\,\Delta\theta^2 + 2}$$

$$b_i = \frac{1}{4i^2\,\Delta\theta^2 + 2} \qquad \lambda_i = \frac{4i\,\Delta\theta}{6i^2\,\Delta\theta^2 + 3} \tag{20}$$

$$i = \frac{r_i}{\Delta r}$$

this equation may be written

$$Q_{ij} = \left\{ \begin{array}{c} a_{i+1,i}\,Q_{i+1,j} + a_{i-1,i}\,Q_{i-1,j} \\ b_i\,(Q_{i,j+1} + Q_{i,j-1}) \end{array} \right\} + Q_{ij}^{*} \tag{21}$$

where

$$Q_{ij}^{*} = \lambda_i F_{ij}\,\frac{3}{4}\,r_i\,\Delta r\,\Delta\theta \tag{22}$$

is the starting value for Q_{ij}, assuming the Q's at the four adjacent points to be zero.

It is evident from Eq. (21) that $a_{i+1,i}$, $a_{i-1,i}$, and b_i are carry-over factors on the finite difference net. The carry-over factors $a_{i+1,i}$ and $a_{i-1,i}$ represent the respective influences of the adjacent values in the i+1 st and i-1 st circumferential rings on the value Q_{ij} in the ith ring. The carry-over factor b_i represents the influences of the adjacent values in the ith ring upon this value.

To determine the carry-over factor a_{10} from the first ring into the origin it is necessary to apply l' Hospital's rule (39), the second and third terms of Eq. (19) being indeterminate when r = 0. Thus

$$(\nabla^2 Q)_{r=0} = \lim_{r \to 0} \left(\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r} \frac{\partial Q}{\partial r} + \frac{1}{r^2} \frac{\partial^2 Q}{\partial \theta^2} \right)$$

which becomes, upon differentiating the last two terms:

$$(\nabla^2 Q)_{r=0} = 2 \left(\frac{\partial^2 Q}{\partial r^2} \right)_{r=0} + \frac{1}{2} \left(\frac{\partial^4 Q}{\partial r^2 \partial \theta^2} \right)_{r=0} .$$

The corresponding finite difference equation written for the origin in the direction j is (Fig. 50)

$$\frac{2(Q_{1,j} - 2Q_0 + Q_{1,j'})}{\Delta r^2} + \frac{Q_{1,j+1} - 2Q_{1,j} + Q_{1,j-1} + Q_{1,j'+1} - 2Q_{1,j'} + Q_{1,j'-1}}{2\Delta r^2 \, \Delta\theta^2}$$

$$= - F_0$$

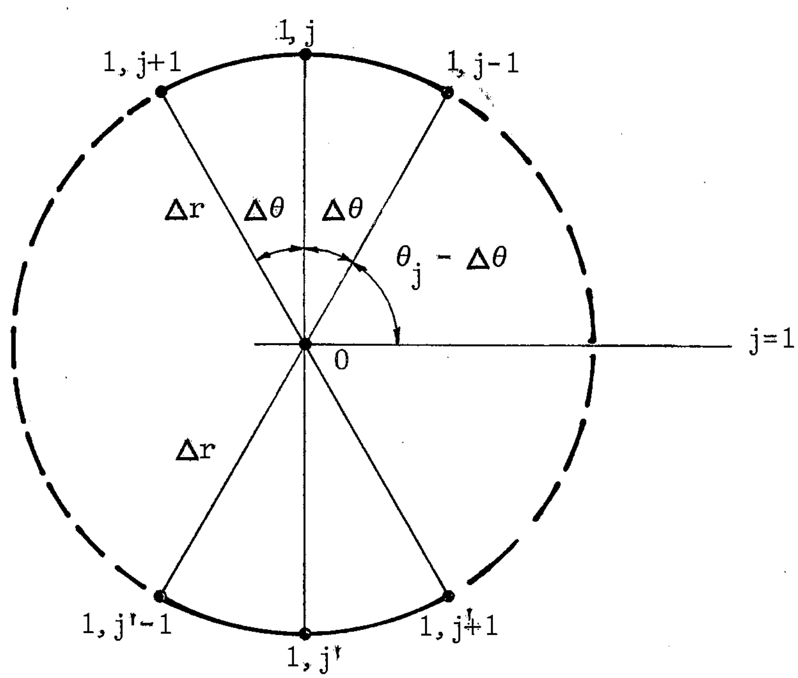


Fig. 50
Origin of the Polar Coordinate Network

where

$$j' = j + \frac{1}{2} n$$

and $n = \frac{2\pi}{\Delta\theta}$ is the total number of radial lines j .

Rearranging terms this equation may be written

$$Q_0 = \frac{2\Delta\theta^2 - 1}{4\Delta\theta^2} (Q_{1,j} + Q_{1,j'})$$

$$+ \frac{1}{8\Delta\theta^2} (Q_{1,j+1} + Q_{1,j-1} + Q_{1,j'+1} + Q_{1,j'-1}) + \frac{1}{4} F_0 \Delta r^2 \quad .$$

Taking directional derivatives along all radial lines and summing, the resulting expression is

$$nQ_0 = \sum_{j=1}^{n} Q_{1,j} + nF_0 \frac{\Delta r^2}{4} \quad .$$

Thus the value of Q at the origin is given by the equation

$$Q_0 = \frac{1}{n} \sum_{j=1}^{n} Q_{1,j} + F_0 \frac{\Delta r^2}{4} \quad . \tag{23}$$

and the carry-over factor into the origin by the equation

$$a_{10} = \frac{1}{n} \quad . \tag{24}$$

Equation (23) compares with Marcus' expression (40) derived by statics using the physical analogy of a laterally loaded network of uniformly tensioned strings.

For the special case of an axially symmetrical function F(r), the Poisson equation (Eq. 19) reduces to the ordinary differential equation (38)

$$\frac{d^2Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} = - F(r) \quad . \tag{25}$$

The corresponding finite difference equation (Eq. 21) may be written

$$Q_i = a_{i+1,i}\, Q_{i+1} + a_{i-1,i}\, Q_{i-1} + F_i \frac{\Delta r^2}{2} \tag{26}$$

in which the carry-over factors are given by the expressions

$$a_{i+1,i} = \frac{1}{2} + \frac{1}{4i} \qquad\Bigg|\qquad a_{i-1,i} = \frac{1}{2} - \frac{1}{4i} \quad . \tag{27}$$

At the origin Eq. (23) becomes

$$Q_0 = Q_1 + F_0 \frac{\Delta r^2}{4} \tag{28}$$

from which $a_{10} = 1$.

Eq. (26) is identical in form to the three moment equation for a continuous beam on rigid supports (33). Thus final values of function coefficients due to unit axially symmetrical starting values may be obtained from the algebraic carry-over solution of this analogous problem, which has been accomplished in other papers and is not considered here (33, 41).

4-2 The Axial Symmetric Basic Series. Applying the algebraic carry-over method to the analysis of an axially symmetric point set composed of one ring and n radial lines (Fig. 51), each final value is found to be an infinite, geometric series. The determination of this series is facilitated by modifying the system to a two point linear set as shown in Fig. 52. The modification is accomplished as follows.

From the equality of function values on the circumferential ring, the finite difference equation written for point 1 becomes (Fig. 51)

$$Q_1 = a_{01} Q_0 + b_1 Q_1 + b_1 Q_1$$

from which

$$Q_1 = \frac{a_{01} Q_0}{1 - 2b_1} .$$

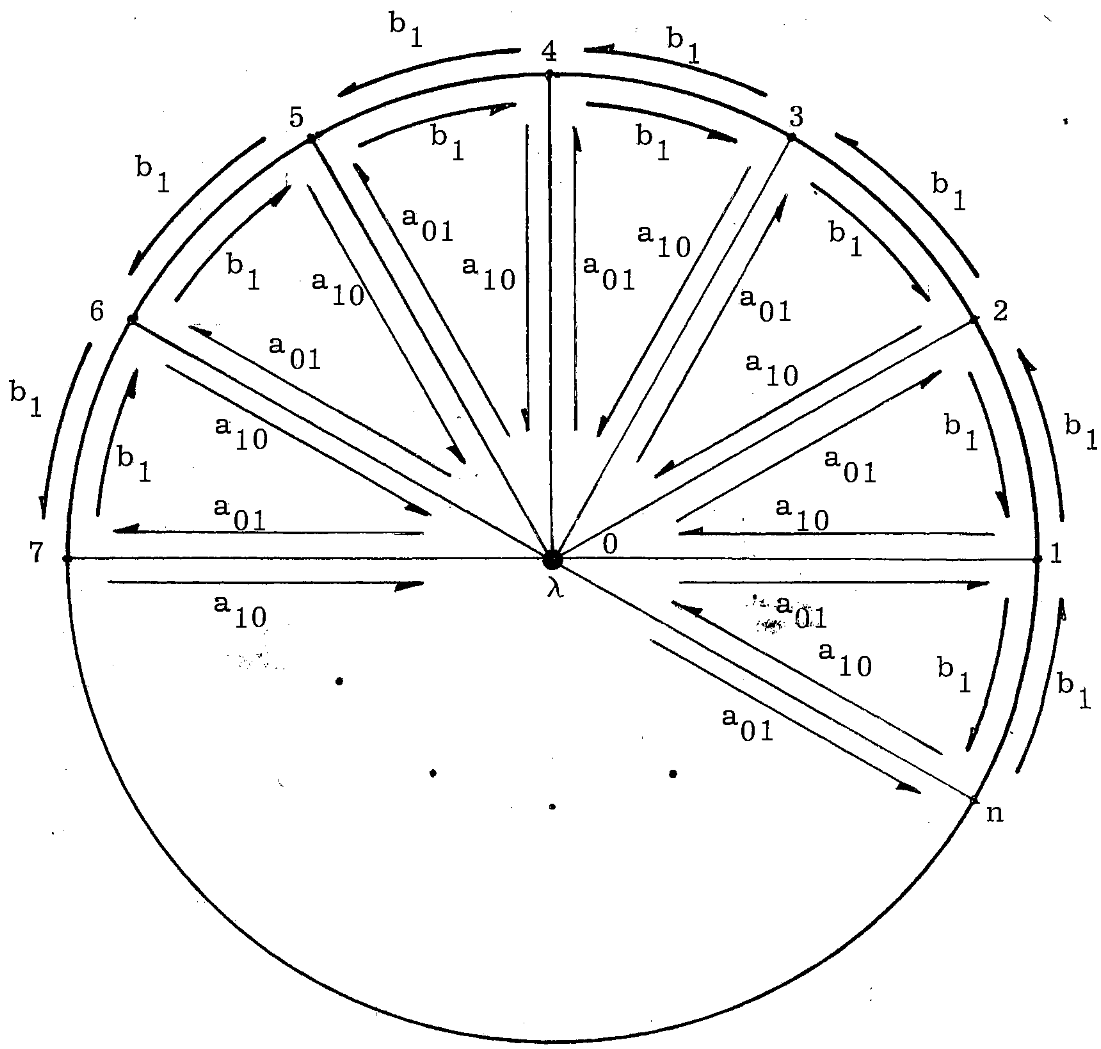


Fig. 51
Axially Symmetric Point Set - Basic Series

The constant $\frac{1}{1 - 2b_1}$ may be interpreted as the over-relaxation factor for the axial symmetric circulatory series forming on the n point ring. Thus the function value at a circumferential point is equal to the value $a_{01}Q_0$ carried-over from the origin multiplied by the over-relaxation factor.

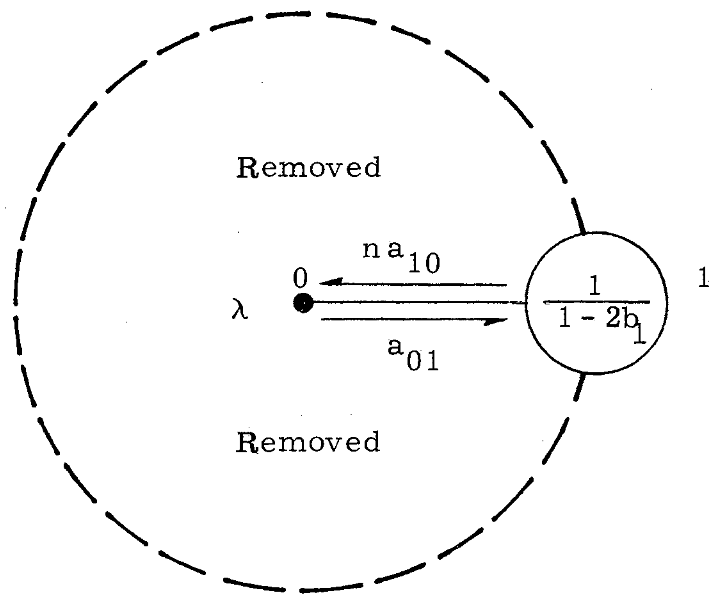


Fig. 52
Reduced Point Set - Axial Symmetric
Basic Series

The modified carry-over factor $n a_{10}$ from point 1 into the origin may be obtained by writing the finite - difference equation for that point, or may be deduced from axial symmetry.

Using these modified constants, it is evident from Fig. 52 that a value carried from point 0 to the circumferential point will return to 0 multiplied by the product of the carry-over factors and the over-relaxation factor. The infinite geometric series formed by continuing this process is called the axial symmetric basic series. The final results for function values are

$$Q_0^{\,0(B)} = \lambda \left[1 + \left(\frac{na_{10}a_{01}}{1 - 2b_1} \right)^1 + \left(\frac{na_{10}a_{01}}{1 - 2b_1} \right)^2 + \dots \right]$$

$$= \lambda \frac{1}{X_1}$$

$$Q_1^{\,0(B)} = \lambda \frac{a_{01}}{(1 - 2b_1)X_1} = Q_2^{\,0} = \dots = Q_n^{\,0}$$

where

$$X_1 = 1 - \frac{na_{10}a_{01}}{1-2b_1} = 1 - \frac{a_{01}}{1 - 2b_1}$$

in terms of Eq. (24).

The final values are given in Fig. 53. Similar conclusions can be made as before (Art. 2-2):

(a) The constant $\frac{1}{X_1}$ is the over-relaxation factor for the axial symmetric basic series

(b) The final function value at the origin is equal to the starting value multiplied by the over-relaxation factor

(c) The constant value $\frac{a_{01}}{1 - 2b_1}$, incorporating the over-relaxation factor of the axially symmetric circulatory series, may be interpreted as a direct final carry-over factor a'_{01}

(d) The final function value at each circumferential point is equal to the final central value multiplied by the direct carry-over factor

(e) The algebraic results are independent of the number of radial lines chosen (from the identity $n\, a_{10} = 1$).

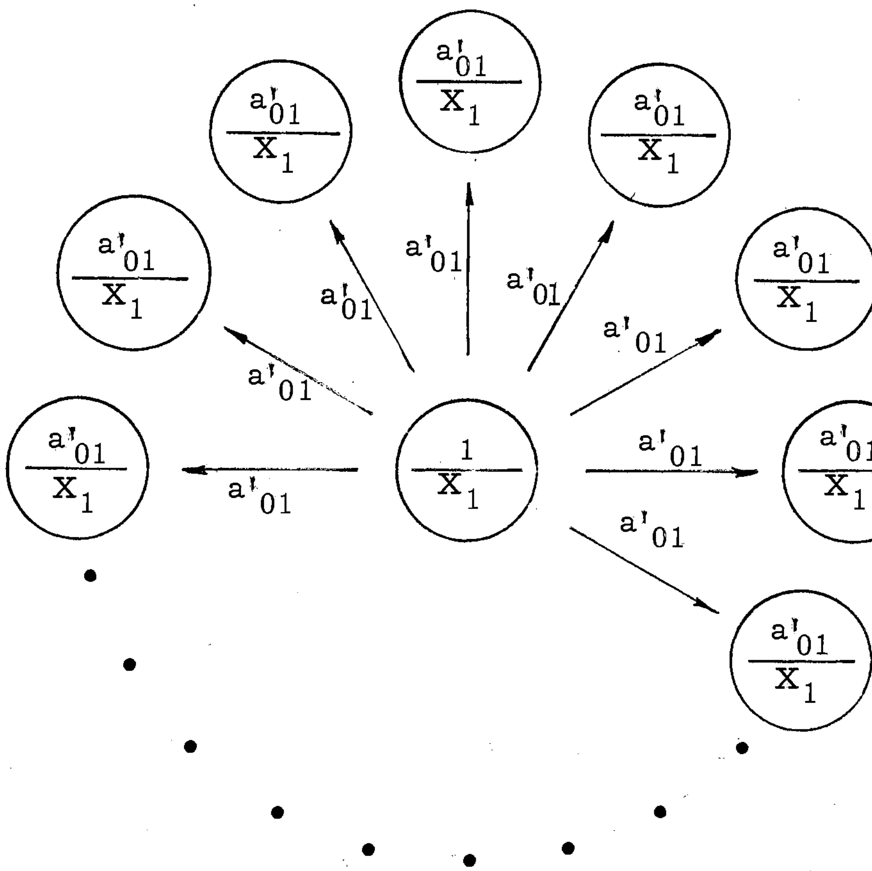


Fig. 53
Final Results - Axial Symmetric Basic Series

4-3 The Single Ring Circulatory Series. For non-axially symmetrical patterns of starting values, it is necessary to investigate the flow of function values in single or multiple rings. The circulatory series which form in single rings may frequently be interpreted as basic series; in multiple rings they are usually higher order series.

Considering an eight point single ring (Fig. 54), the algebraic carry-over method yields final results which may be represented as the sums of three simple geometric series. For a starting value λ at point 1, the system is resolved into basic cases (Fig. 55), as discussed for the rectangular set in Art. 2-3, and the results superimposed.

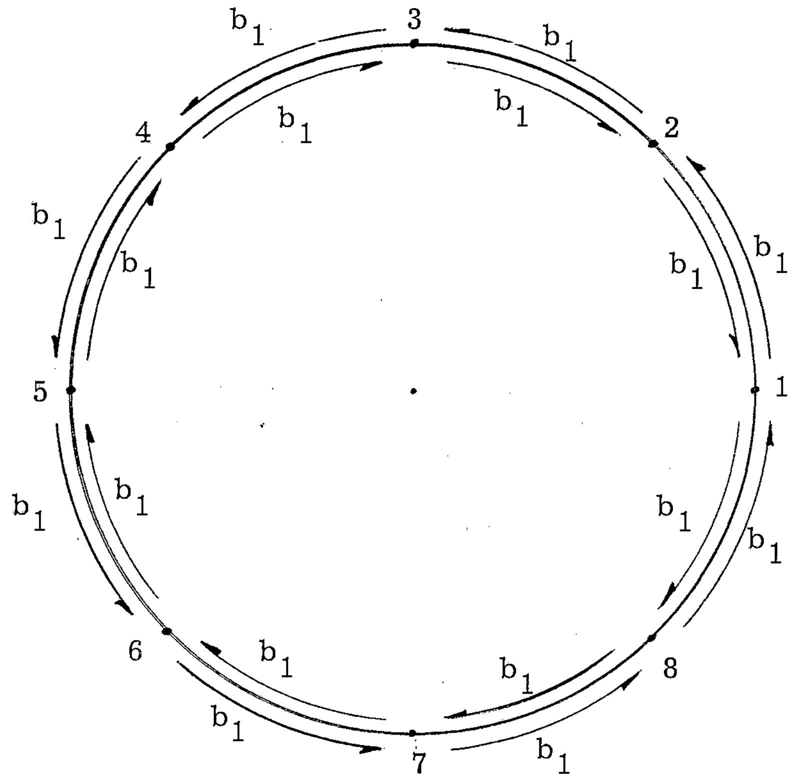


Fig. 54
Eight Point Ring - Circulatory Series

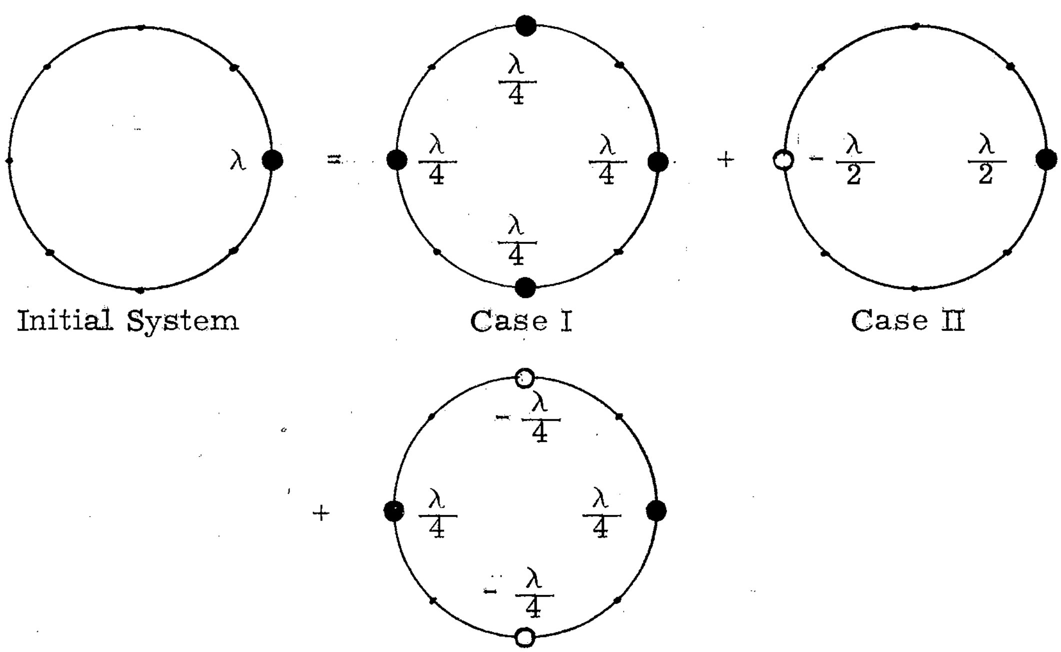


Fig. 55
Resolution of Circulatory System with Starting Value at Point 1 into Three Basic Cases

Case I. Observing the symmetry of this system with respect to all diameters of the ring, modified carry-over factors can be introduced and the point set reduced to that shown in Fig. 56. Performing algebraic carry-over the results are:

$$Q_1^{1(S)I} = \frac{1}{X_{04}} \frac{\lambda}{4} \qquad\Big|\qquad Q_2^{1(S)I} = \frac{2b_1}{X_{04}} \frac{\lambda}{4}$$

where

$$X_{04} = 1 - 4b_1^2 .$$

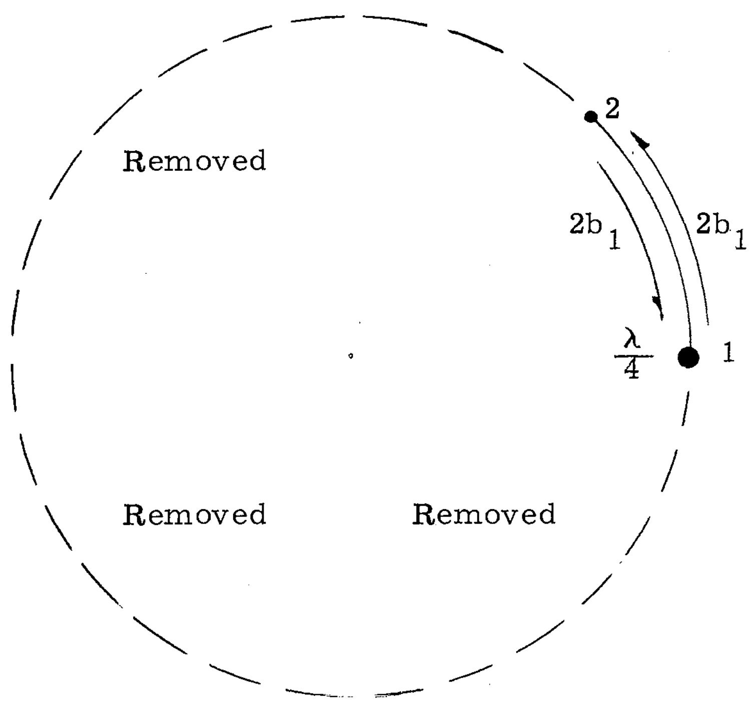


Fig. 56
Modified Circulatory System
Case I

Case II. This system, symmetrical with respect to the horizontal diameter and antisymmetrical with respect to the vertical, may be resolved into two independent systems (Fig. 57). The results of algebraic carry-over are

$$Q_1^{1(S)II} = \frac{1}{X_{02}} \frac{\lambda}{2} \qquad Q_5^{1(S)II} = -\frac{1}{X_{02}} \frac{\lambda}{2}$$

$$Q_2^{1(S)II} = \frac{b_1}{X_{02}} \frac{\lambda}{2} \qquad Q_4^{1(S)II} = -\frac{b_1}{X_{02}} \frac{\lambda}{2}$$

where

$$X_{02} = 1 - 2b_1^2 .$$

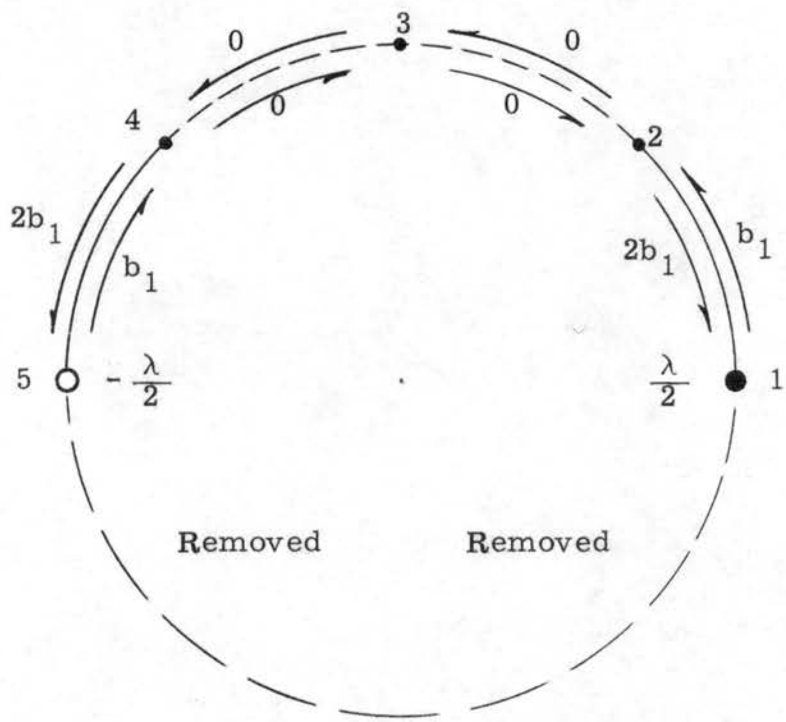


Fig. 57
Modified Circulatory System
Case II

Case III. For this system no algebraic carry-over procedure is possible, and the starting values represent final results (Fig. 58).

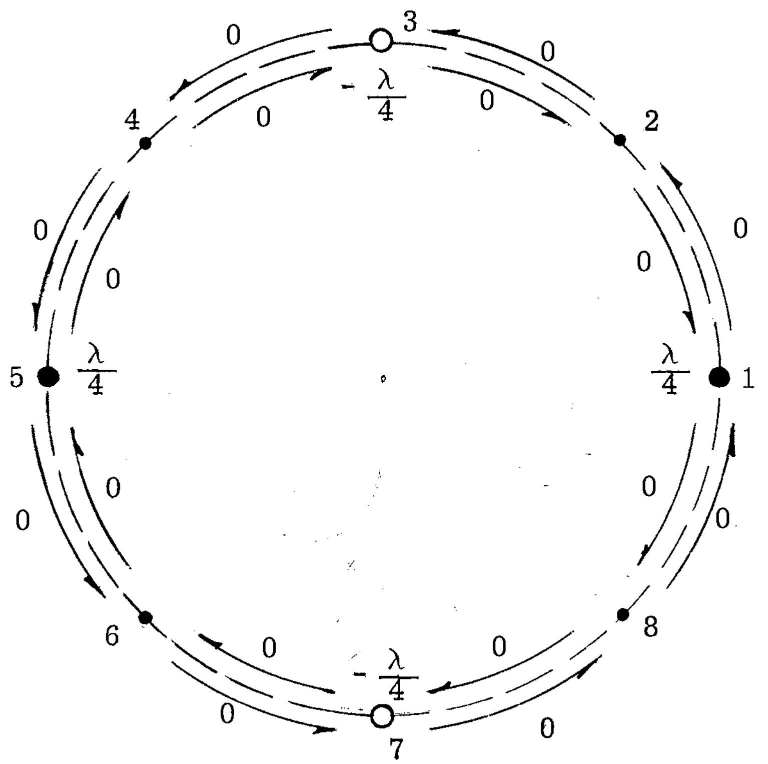


Fig. 58
Modified Circulatory System
Case III

The final values for function coefficients are obtained by superimposing Cases I, II, and III. It is evident from this superposition that the series forming in an axially summetrical ring, starting value at any point, can be resolved into simple goemetric series.

4-4 The Axial Symmetric Carry-Over Series. If algebraic carry-over is used to determine the function coefficients on a two ring axially symmetric point set, starting value λ at the origin (Fig. 59), each final value is the finite sum of an infinite geometric series of geometric series.

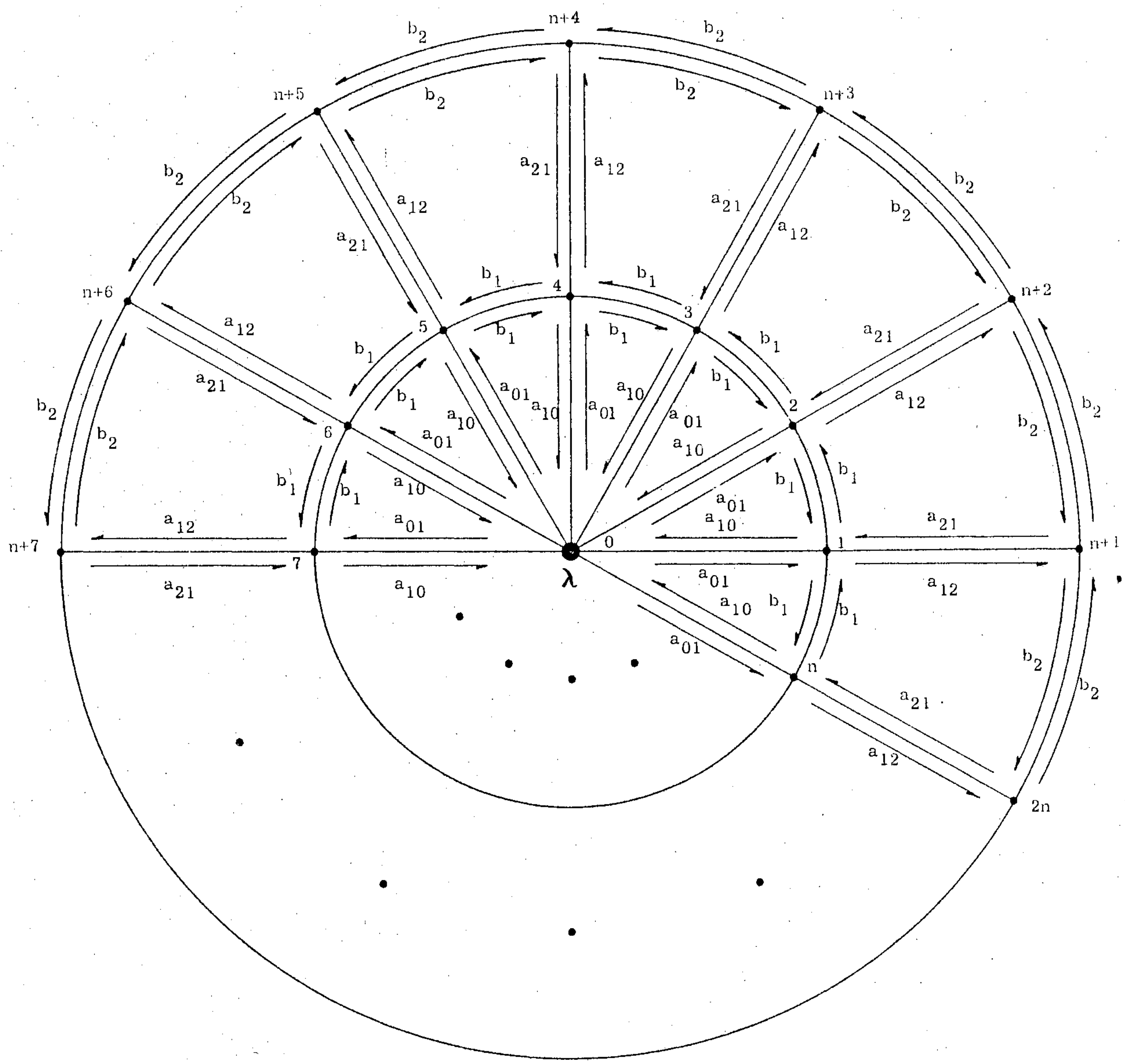


Fig. 59

Axially Symmetric Point Set - Carry-Over Series

As was explained for the single ring set in Art. 4-2, this system may be reduced to the linear point set of Fig. 60. The constants $\frac{1}{1-2b_1}$ and $\frac{1}{1-2b_2}$ are the over-relaxation factors for the axially symmetric circulatory series forming on the first and second n point rings.

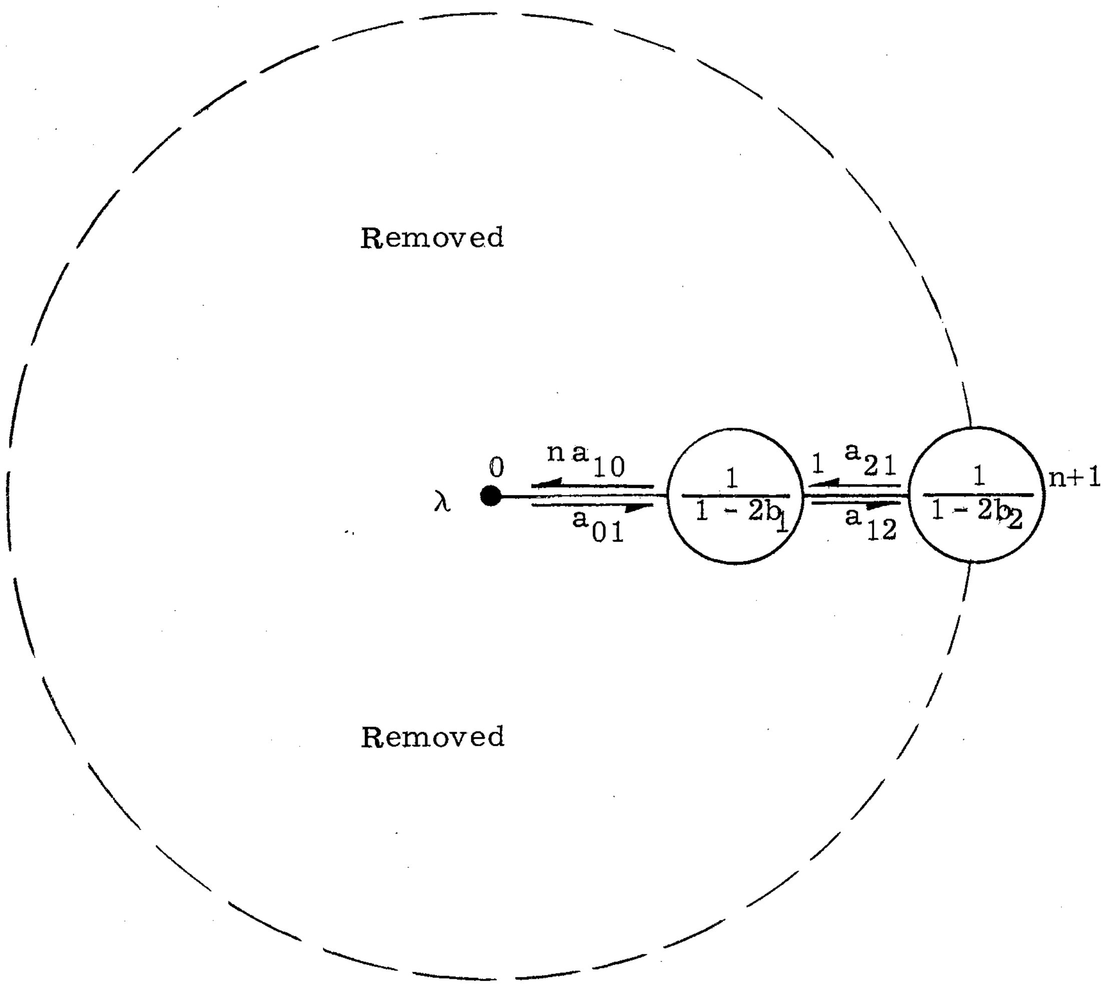


Fig. 60
Reduced Point Set - Axial Symmetric
Carry-Over Series

Because only three points are involved in the carry-over procedure, it is possible to represent the final results by basic series. Considering, however, that series which form in the circumferential rings are hidden by the over-relaxation factors at 1 and n+1, the one dimensional series on the three point set is more logically interpreted as a carry-over series. Again the concept of the zero point is used. Thus, carrying-over from 0 to 1, then temporarily suppressing point 0, the resulting series on the two point set yields the values

$$Q_1^{0(B1)} = \lambda \frac{a_{01}}{1 - 2b_1} \left[1 + \left(\frac{a_{12}\, a_{21}}{(1 - 2b_1)(1 - 2b_2)} \right)^1 + \left(\frac{a_{12}\, a_{21}}{(1 - 2b_1)(1 - 2b_2)} \right)^2 + \ldots \right]$$

$$= \lambda \frac{a_{01}}{(1 - 2b_1)\, X'_2} = \lambda \frac{a'_{01}}{X'_2}$$

$$Q_{n+1}^{0(B1)} = \lambda \frac{a_{01}\, a_{12}}{(1 - 2b_1)(1 - 2b_2)\, X'_2} = \lambda \frac{a'_{01}\, a'_{12}}{X'_2}$$

where

$$a'_{01} = \frac{a_{01}}{1 - 2b_1} \quad \Bigg| \quad a'_{21} = \frac{a_{21}}{1 - 2b_1} \quad \Bigg| \quad a'_{12} = \frac{a_{12}}{1 - 2b_2}$$

and

$$X'_2 = 1 - \frac{a_{12}\, a_{21}}{(1 - 2b_2)(1 - 2b_1)} = 1 - a'_{12} a'_{21} \ .$$

The quantities a'_{01}, a'_{21}, a'_{12}, which incorporate the over-relaxation factors $\frac{1}{1 - 2b_1}$ and $\frac{1}{1 - 2b_2}$, may be considered modified carry-over factors.

Removing the zero point at the origin the returned value is

$$Q_0^{0(CO)} = \lambda \frac{n\, a_{10}\, a_{01}}{(1 - 2b_1)\, X'_2} \ .$$

The quantity

$$\frac{n\, a_{10}\, a_{01}}{(1 - 2b_1)\, X'_2} = \frac{a'_{01}}{X'_2}$$

is therefore the common ratio of the axial symmetric carry-over series developed by continuing this procedure infinite times. The final results are

$$Q_0^{\ 0} = \lambda \left[1 + \left(\frac{a'_{01}}{X'_2} \right)^1 + \left(\frac{a'_{01}}{X'_2} \right)^2 + \ldots \right] = \lambda \frac{1}{Y_2}$$

$$Q_1^{\ 0} = \lambda \frac{a'_{01}}{X'_2 Y_2} = Q_2^{\ 0} = \ldots = Q_n^{\ 0} \qquad (29)$$

$$Q_{n+1}^{\ 0} = \lambda \frac{a'_{01} a'_{12}}{X'_2 Y_2} = Q_{n+2}^{\ 0} = \ldots. = Q_{2n}^{\ 0}$$

where

$$Y_2 = 1 - \frac{n\, a_{10} a_{01}}{(1 - 2b_1)\, X'_2} = 1 - \frac{a_{01}}{(1 - 2b_1)\, X'_2} .$$

Introducing the equivalents

$$A_{01} = \frac{a_{01}}{(1 - 2b_1)\, X'_2} \qquad \Bigg| \qquad A_{02} = \frac{a_{01}\, a_{12}}{(1 - 2b_1)(1 - 2b_2)\, X'_2}$$

the final values can be represented diagrammatically (Fig. 61), and similar conclusions made as for the single ring basic series (Art. 4-2). Thus

(a) The constant $\frac{1}{Y2}$ is the over-relaxation factor for the axial symmetric carry-over series

(b) The final function value at the origin is equal to the starting value multiplied by the over-relaxation factor

(c) The final function value at any circumferential point is equal to the final central value multiplied by the corresponding direct final carry-over factor A_{01} or A_{02}

(d) The algebraic results are independent of the number of radial lines chosen.

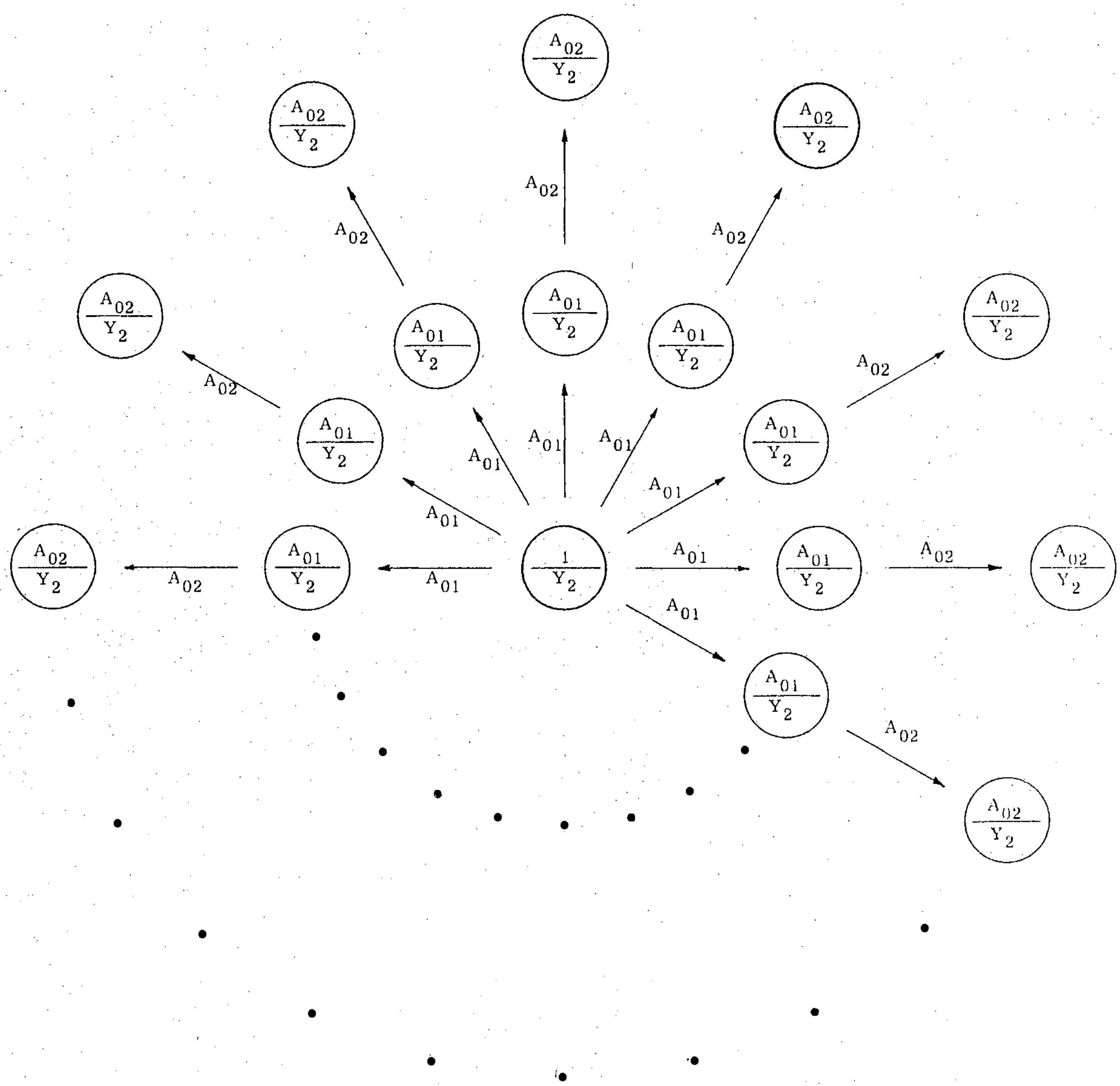


Fig. 61
Final Results - Axial Symmetric Carry-Over Series

4-5 The Double Ring Circulatory Series. Considering now the analysis of a sixteen point double ring (Fig. 62) by algebraic carry-over, it is again convenient to resolve an unsymmetrical set of starting values into basic cases as was done in Art. 4-3 for single rings. This resolution for a starting value λ at point 1 is given in Fig. 63.

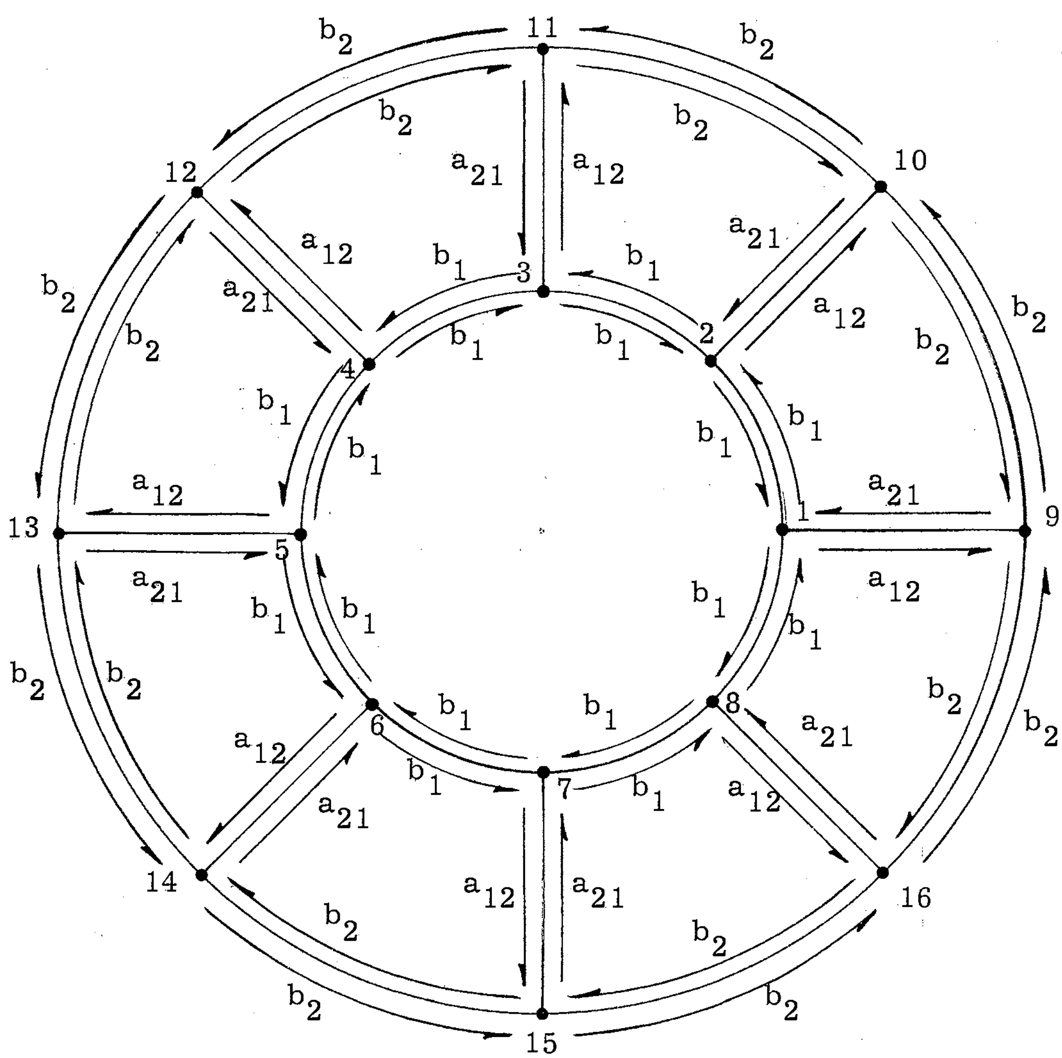


Fig. 62
Sixteen Point Double Ring - Circulatory Series

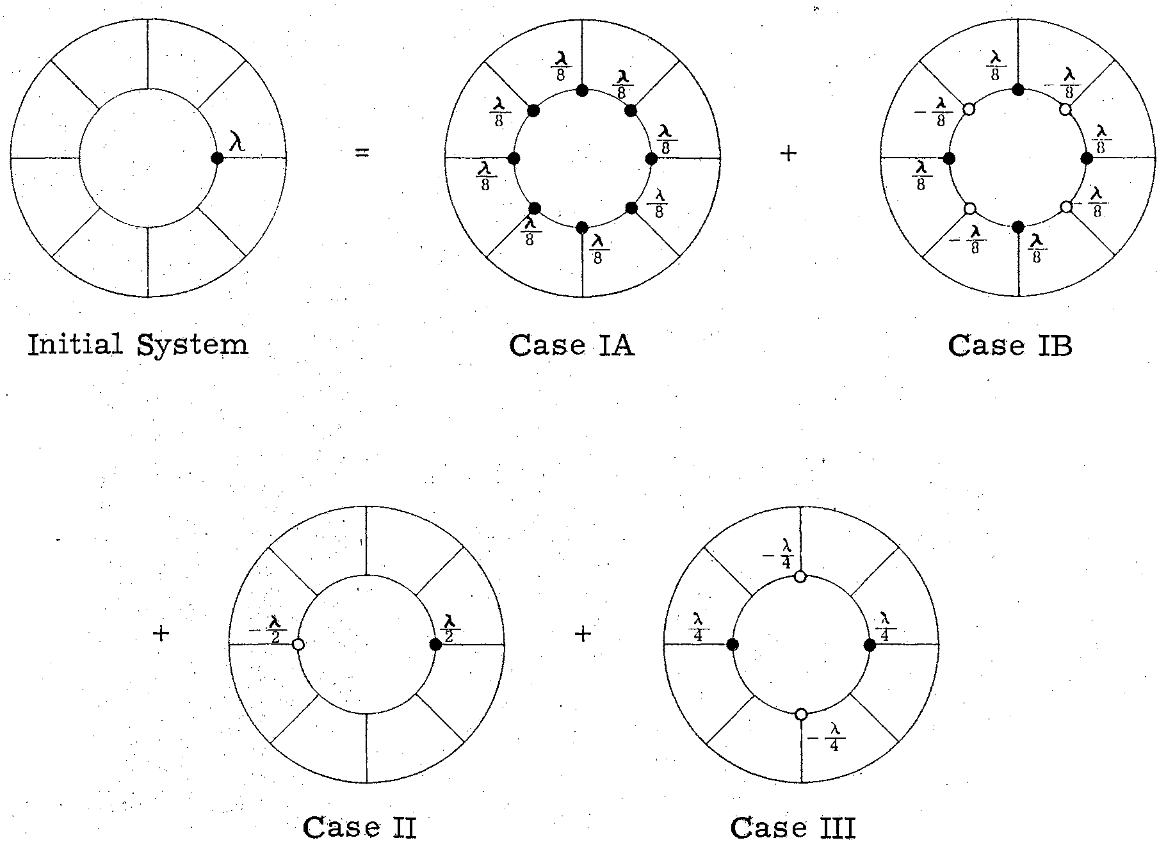


Fig. 63

Resolution of Double Ring Circulatory System with Starting Value at Point 1 into Basic Cases

Case IA. From the axial symmetry of this system (Fig. 63), it is evident that a reduction to the modified point set of Fig. 64 is possible. Performing algebraic carry-over on this reduced set the results are:

$$Q_1^{1(S)IA} = \frac{1}{(1 - 2b_1) X'_2} \frac{\lambda}{8}$$

$$Q_9^{1(S)IA} = \frac{a_{12}}{(1 - 2b_1)(1 - 2b_2) X'_2} \cdot \frac{\lambda}{8} \quad .$$

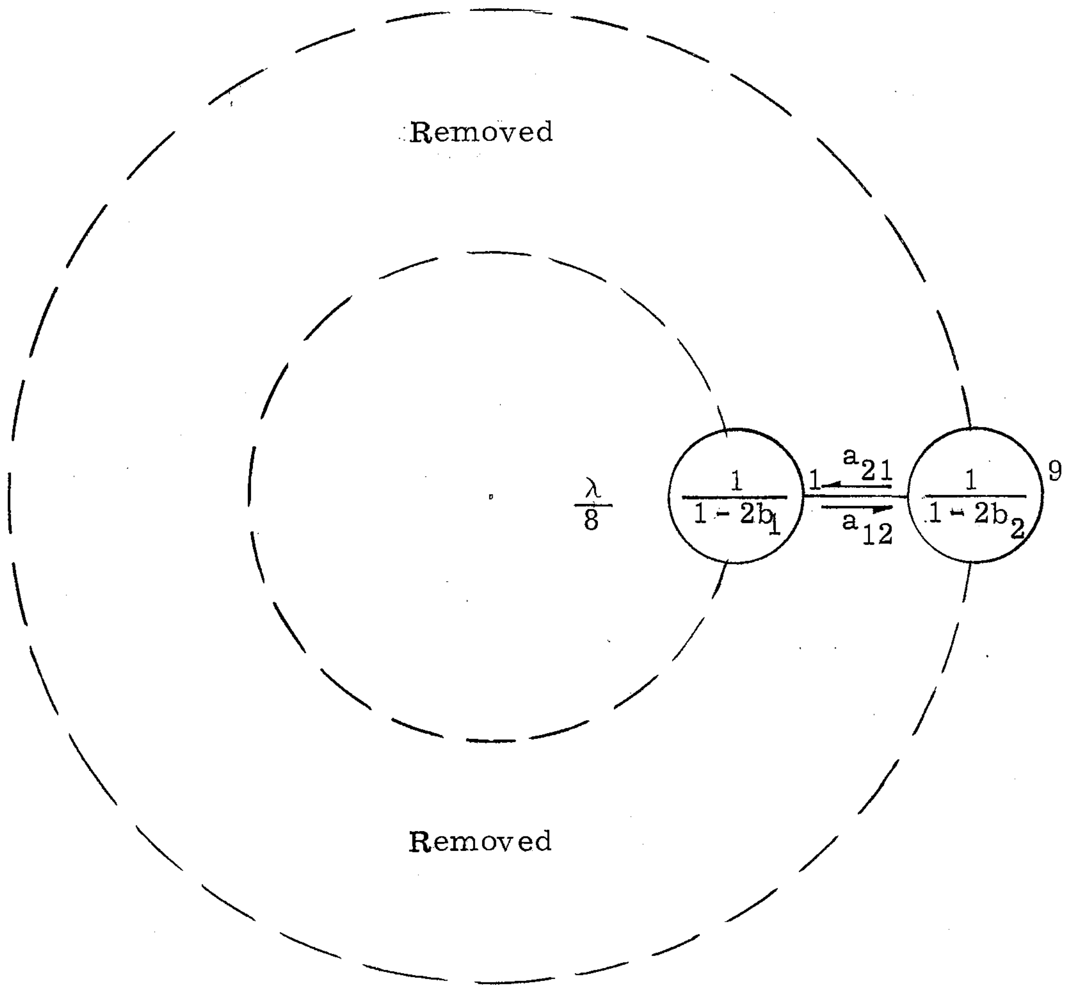


Fig. 64
Modified Double Ring Circulatory System
Case IA

Case IB. This system of starting values is symmetrical with respect to all diameters of the double ring, but it is not axially symmetrical (Fig. 63). The over-relaxation factors at points 1 and 9 of the reduced system are obtained by writing the finite-difference equations at these points.

Thus (Fig. 63)

$$Q_1 = b_1(-Q_1) + b_1(-Q_1) + a_{21}Q_9 + Q_1^*$$

$$Q_9 = b_2(-Q_9) + b_2(-Q_9) + a_{12}Q_1$$

from which

$$Q_1 = \frac{a_{21}Q_9}{1 + 2b_1} + \frac{Q^*_1}{1 + 2b_1}$$

$$Q_9 = \frac{a_{12}Q_1}{1 + 2b_2} \quad .$$

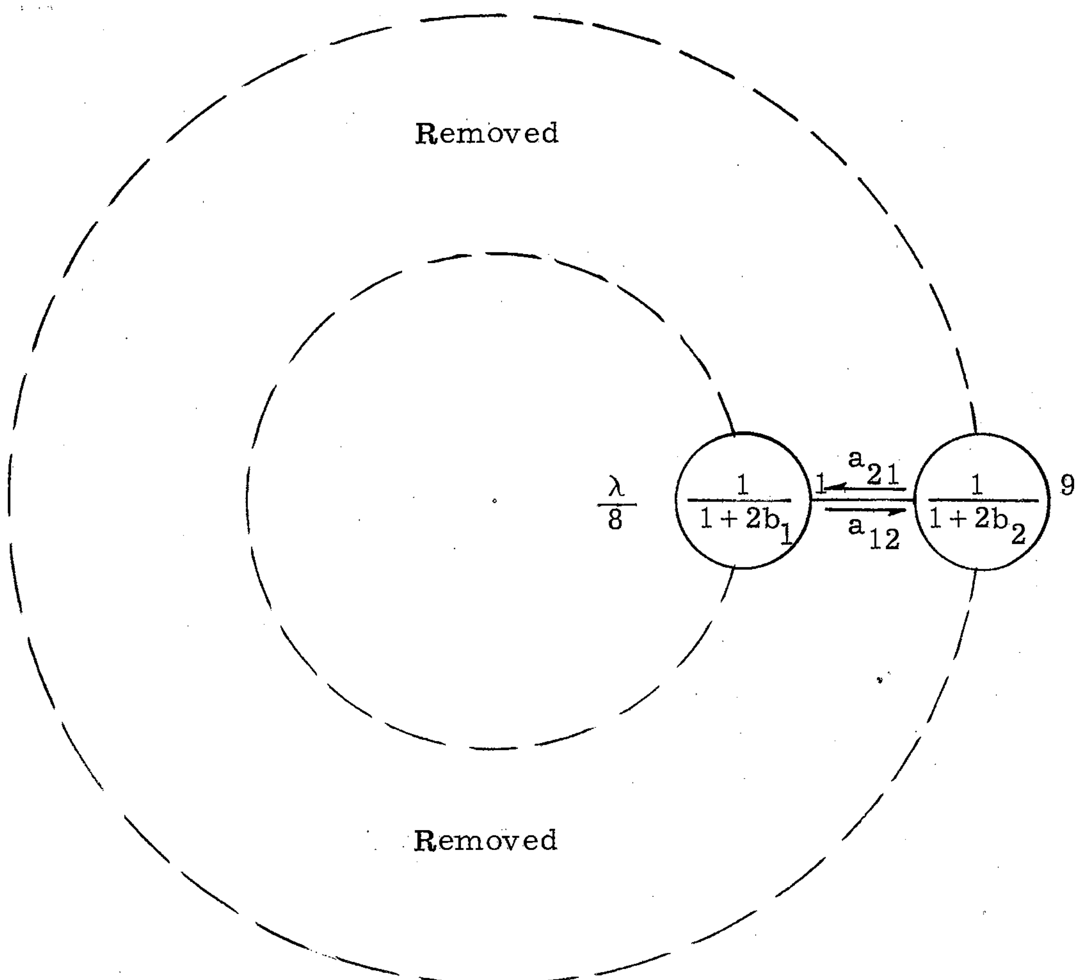


Fig. 65

Modified Double Ring Circulatory System
Case IB

The over-relaxation factors corresponding to the series forming on the first and second rings are therefore $\frac{1}{1 + 2b_1}$ and $\frac{1}{1 + 2b_2}$. Using these factors the system is reduced to the linear set of Fig. 65. Performing algebraic carry-over the results are

$$Q_1^{1(S)IB} = \frac{1}{(1 + 2b_1)X''_2} \frac{\lambda}{8} \quad \Big| \quad Q_9^{1(S)IB} = \frac{a_{12}}{(1 + 2b_1)(1 + 2b_2)X''_2} \frac{\lambda}{8}$$

where

$$X''_2 = 1 - \frac{a_{12}a_{21}}{(1+2b_2)(1+2b_1)} .$$

<u>Case II.</u> This system, symmetrical with respect to the horizontal diameter and antisymmetrical with respect to the vertical, is resolved into two independent systems as shown in Fig. 66. The solution of each of these systems is the solution of a geometrically unsymmetrical four point ring.

From the nature of the carry-over procedure, the flow of function values in these rings generates carry-over series. In order to determine these series, the method of alternately suppressed points is again adopted. Introducing a zero point at 10 and applying algebraic carry-over to the isolated three point set 1, 2, 9, the function values are

$$Q_1^{1(B1)II} = \frac{1}{X_{21}} \frac{\lambda}{2} \quad \Big| \quad Q_2^{1(B1)II} = \frac{b_1}{X_{21}} \frac{\lambda}{2} \quad \Big| \quad Q_9^{1(B1)II} = \frac{a_{12}}{X_{21}} \frac{\lambda}{2}$$

where

$$X_{21} = 1 - a_{12}a_{21} - 2b_1^2 .$$

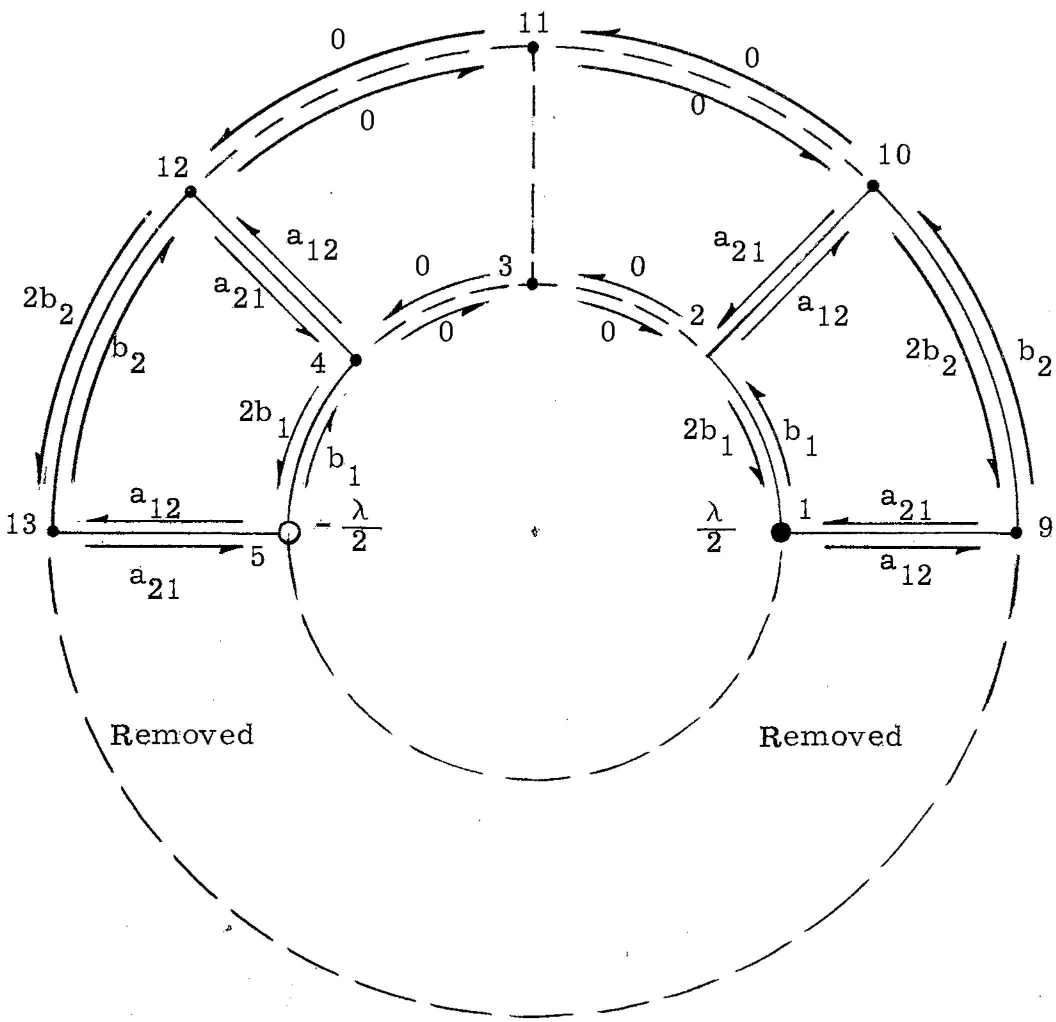


Fig. 66
Modified Double Ring Circulatory System
Case II

Releasing point 10 and suppressing point 1, the carried-over value

$$Q_{10}^{1(CO)} = \frac{a_{12}(b_1 + b_2)}{X_{21}} \quad \frac{\lambda}{2} = \alpha_0$$

forms series on the isolated set 2, 10, 9. The results of algebraic carry-over are

$$Q_{10}^{1(B2)II} = \frac{1}{X_{22}} \alpha_0 \quad \Bigg| \quad Q_2^{1(B2)II} = \frac{a_{21}}{X_{22}} \alpha_0 \quad \Bigg| \quad Q_9^{1(B2)II} = \frac{2b_2}{X_{22}} \alpha_0$$

where

$$X_{22} = 1 - a_{12}a_{21} - 2b_2^2 \quad .$$

The common ratio of the infinite geometric carry-over series forming between these two isolated point sets is determined by releasing point 1 and finding the returned value

$$Q_1^{1(CO)} = \frac{2\,a_{21}(b_1 + b_2)}{X_{22}} \alpha_0 = \frac{2\,a_{12}a_{21}(b_1 + b_2)^2}{X_{21}X_{22}} \frac{\lambda}{2} = \beta_1$$

which becomes a new starting value at 1. Dividing β_1 by the initial starting value $\beta_0 = \frac{\lambda}{2}$, the common ratio is

$$\frac{2\,a_{12}a_{21}\,(b_1 + b_2)^2}{X_{21}X_{22}} \quad .$$

The carry-over series α and β are infinite geometric series of series whose sums are

$$\sum_0^\infty \alpha_n = \alpha_0 + \alpha_1 + \ldots = \frac{a_{12}(b_1 + b_2)}{X_{21}Y_{22}} \frac{\lambda}{2}$$

$$\sum_0^\infty \beta_n = \beta_0 + \beta_1 + \ldots = \frac{1}{Y_{22}} \frac{\lambda}{2}$$

where

$$Y_{22} = 1 - \frac{2\,a_{12}a_{21}(b_1 + b_2)^2}{X_{21}X_{22}} \quad .$$

Superimposing these series the final values for function coefficients, Case II, become

$$Q_1^{1(II)} = \frac{1}{X_{21}} \sum_0^\infty \beta_n = \frac{1}{Z_{21}} \frac{\lambda}{2} = -Q_5^{1(II)}$$

$$Q_2^{1(II)} = \frac{b_1}{X_{21}} \sum_0^\infty \beta_n + \frac{a_{21}}{X_{22}} \sum_0^\infty \alpha_n = \frac{B_{21}}{Z_{21}} \frac{\lambda}{2} = -Q_4^{1(II)}$$

$$Q_9^{1(II)} = \frac{a_{12}}{X_{21}} \sum_0^\infty \beta_n + \frac{2b_2}{X_{22}} \sum_0^\infty \alpha_n = \frac{A_{21}}{Z_{21}} \frac{\lambda}{2} = -Q_{13}^{1(II)}$$

$$Q_{10}^{1(II)} = \frac{1}{X_{22}} \sum_0^\infty \alpha_n = \frac{C_{21}}{Z_{21}} \frac{\lambda}{2} = -Q_{12}^{1(II)} \quad .$$

The new constants introduced above are

$$A_{21} = a_{12}\left[1 + \frac{2b_2(b_1 + b_2)}{X_{22}}\right] \qquad C_{21} = \frac{a_{12}(b_1 + b_2)}{X_{22}}$$

$$B_{21} = b_1\left[1 + \frac{a_{12}a_{21}\left(1 + \frac{b_2}{b_1}\right)}{X_{22}}\right] \qquad Z_{21} = X_{21}\,Y_{22} \quad .$$

Interpreting these constants as direct final carry-over factors A_{21}, B_{21}, C_{21} and over-relaxation factor $\frac{1}{Z_{21}}$ for the unsymmetrical four point ring (Fig. 67), and comparing these results with those corresponding to the carry-over series in rectangular coordinates (Art. 2-4), it is evident that the final values are similar in form. When the circular panel (1, 2, 9, 10) is transformed into the corresponding rectangular panel, the final values are found to be the solution of a twenty point set symmetrical with respect to the X-axis. Thus a circular panel can be considered a more general form of a rectangular panel.

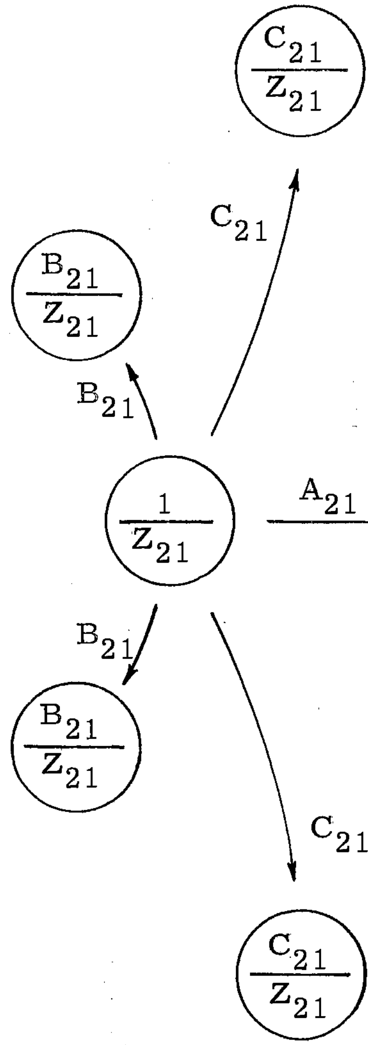


Fig. 67
Final Results - Circular Panel Carry-Over Series

Case III. From the antisymmetry of this system with respect to alternate diameters, all modified carry-over factors in the circumferential direction are equal to zero (Fig. 68). The system thus reduces to four isolated two point sets. Performing algebraic carry-over the results are

$$Q_1^{1(S)III} = \frac{1}{X_2}\,\frac{\lambda}{4} = Q_5^{1(S)III} = -Q_3^{1(S)III} = -Q_7^{1(S)III}$$

$$Q_9^{1(S)III} = \frac{a_{12}}{X_2}\frac{\lambda}{4} = Q_{13}^{1(S)III} = -Q_{11}^{1(S)III} = -Q_{15}^{1(S)III}$$

where

$$X_2 = 1 - a_{12}a_{21} \quad .$$

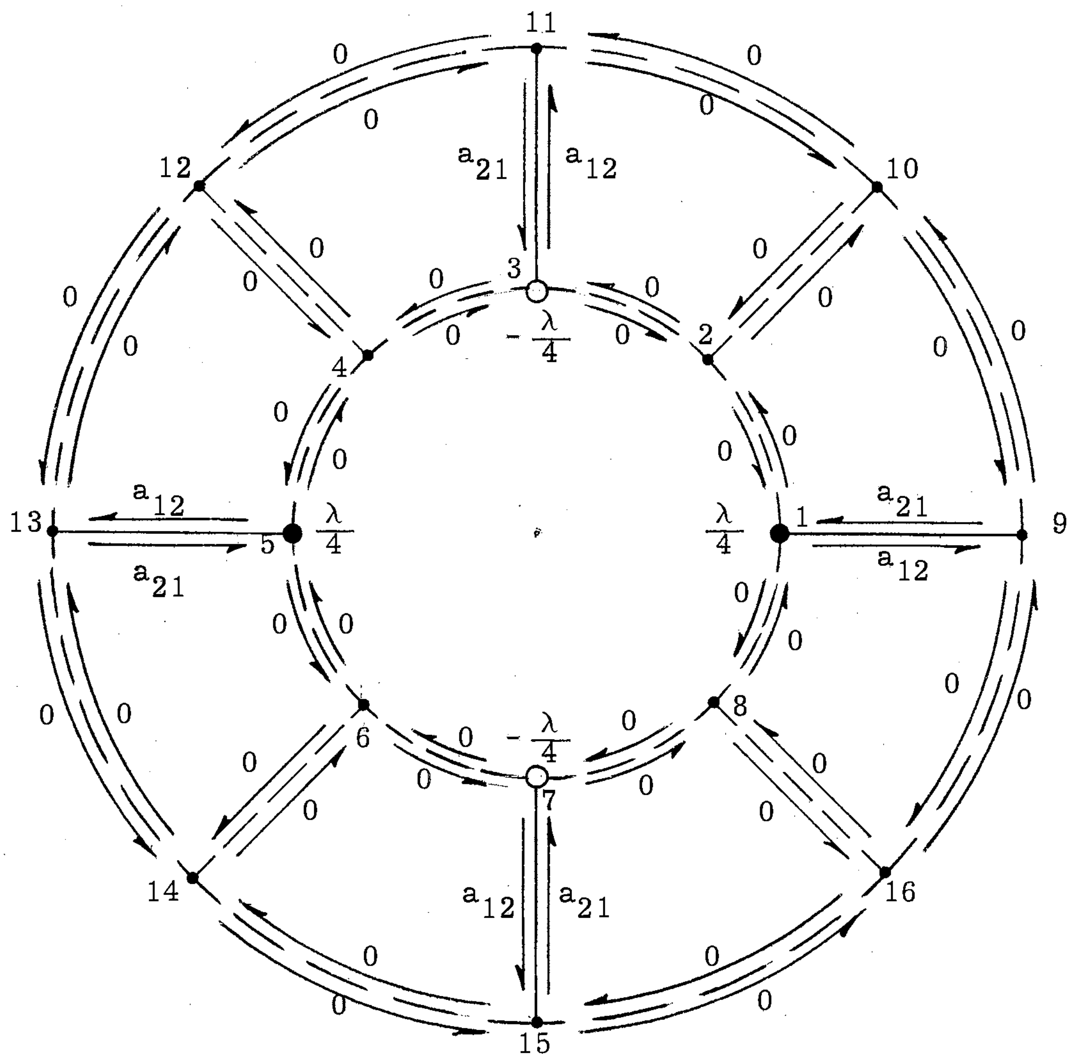


Fig. 68
Modified Double Ring Circulatory System
Case III

The final values for function coefficients on the sixteen point double ring, starting value λ at point 1, are obtained by superimposing Cases IA, IB, II and III. Thus the solution of multiple ring circulatory systems involves both simple geometric and carry-over series.

4-6 Resolution and Superposition. If now the algebraic carry-over method is applied to the analysis of a geometrically axial symmetric twenty-five point set (Fig. 69), the final results due to a starting value λ at any point can be found by again using the principles of resolution and superposition defined in Art. 2-5.

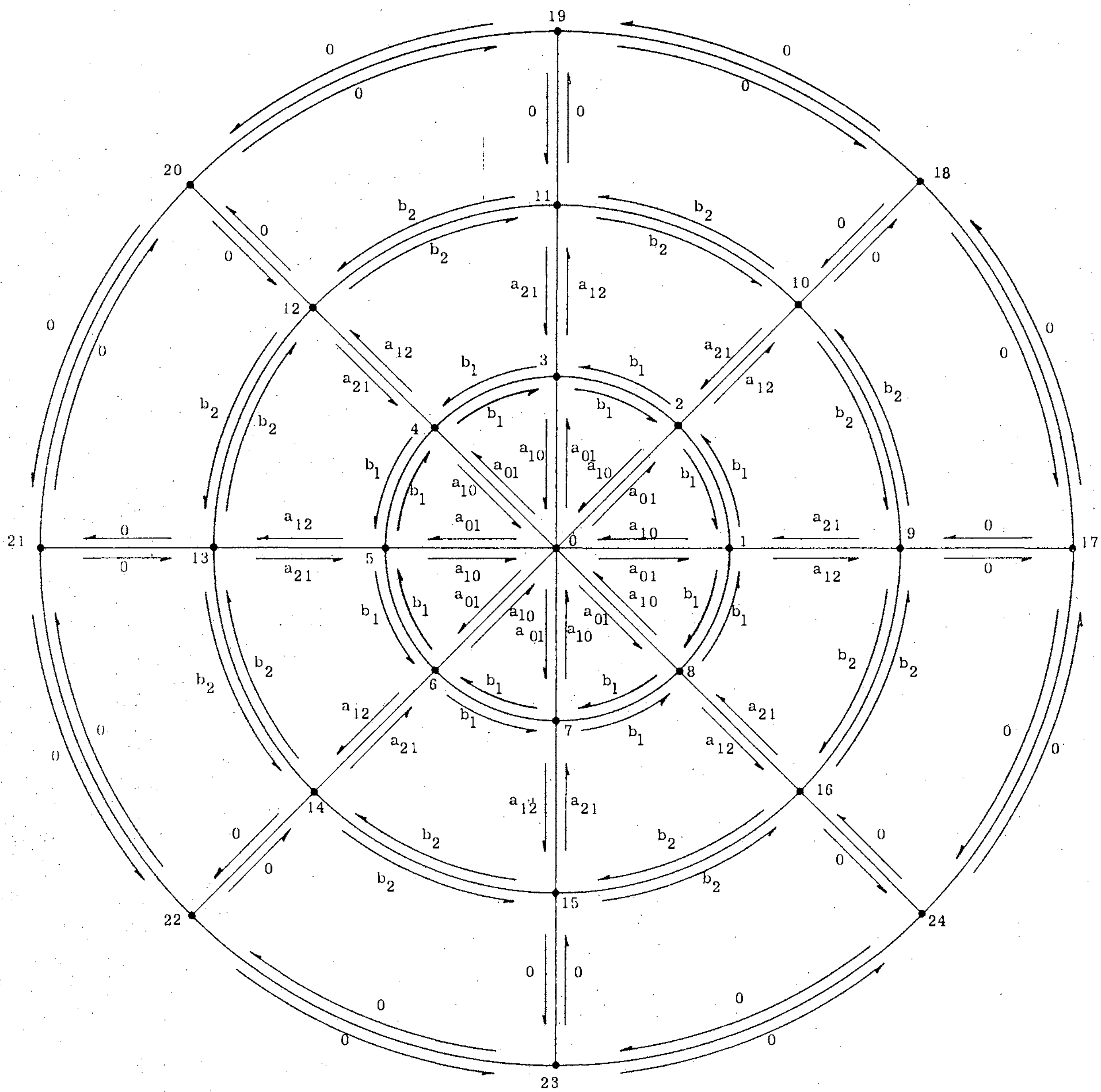


Fig. 69
Twenty-Five Point Set

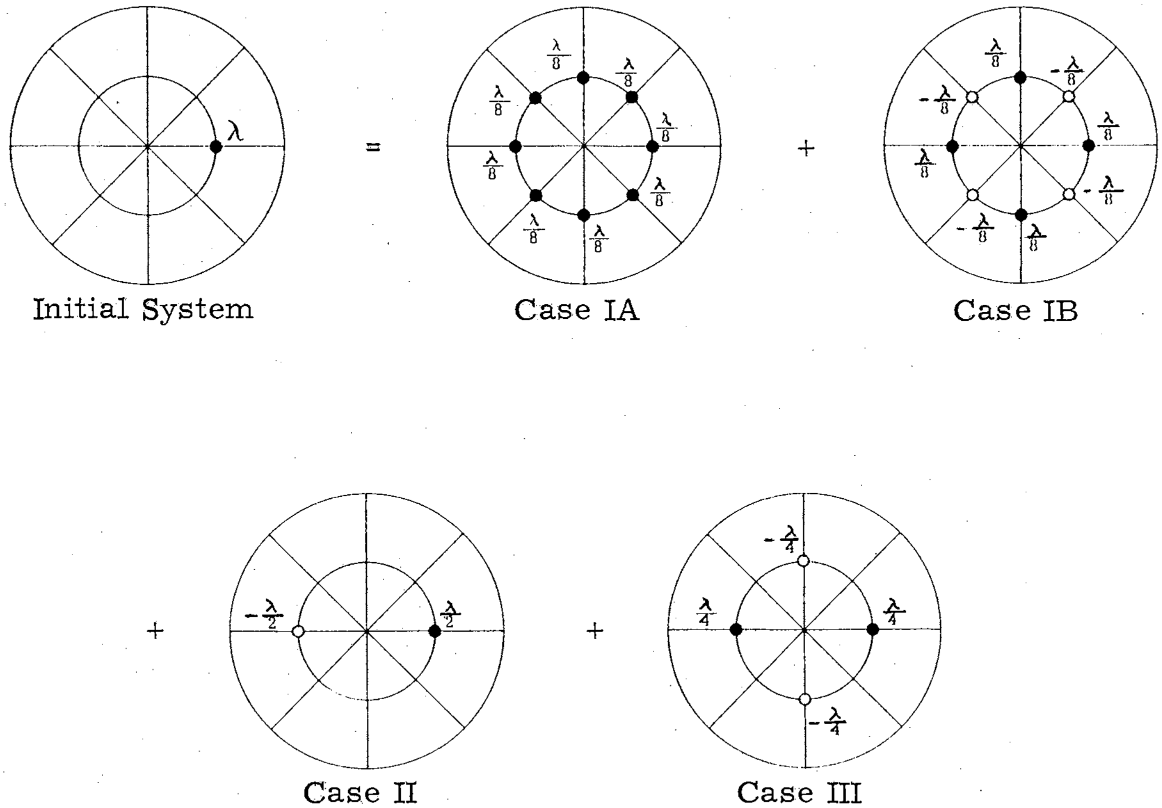


Fig. 70
Resolution of Twenty-Five Point Set with Starting Value at Point 1 into Basic Cases

Considering first a starting value at the origin, the problem is simply that of the axially symmetrical carry-over series discussed in Art. 4-4. The final function values are given in that article and need not be restated here.

For a starting value at point 1, the final results are obtained by superimposing the double ring circulatory series and the axial symmetric carry-over series. Resolving the system as shown in Fig. 70, it is evident that Cases IB, II, and III have no carry-over into the origin. These systems therefore reduce to the corresponding Cases IB, II, and III of the double ring circulatory system (Art. 4-5). Case IA, however, develops the axial symmetric carry-over series. Modifying

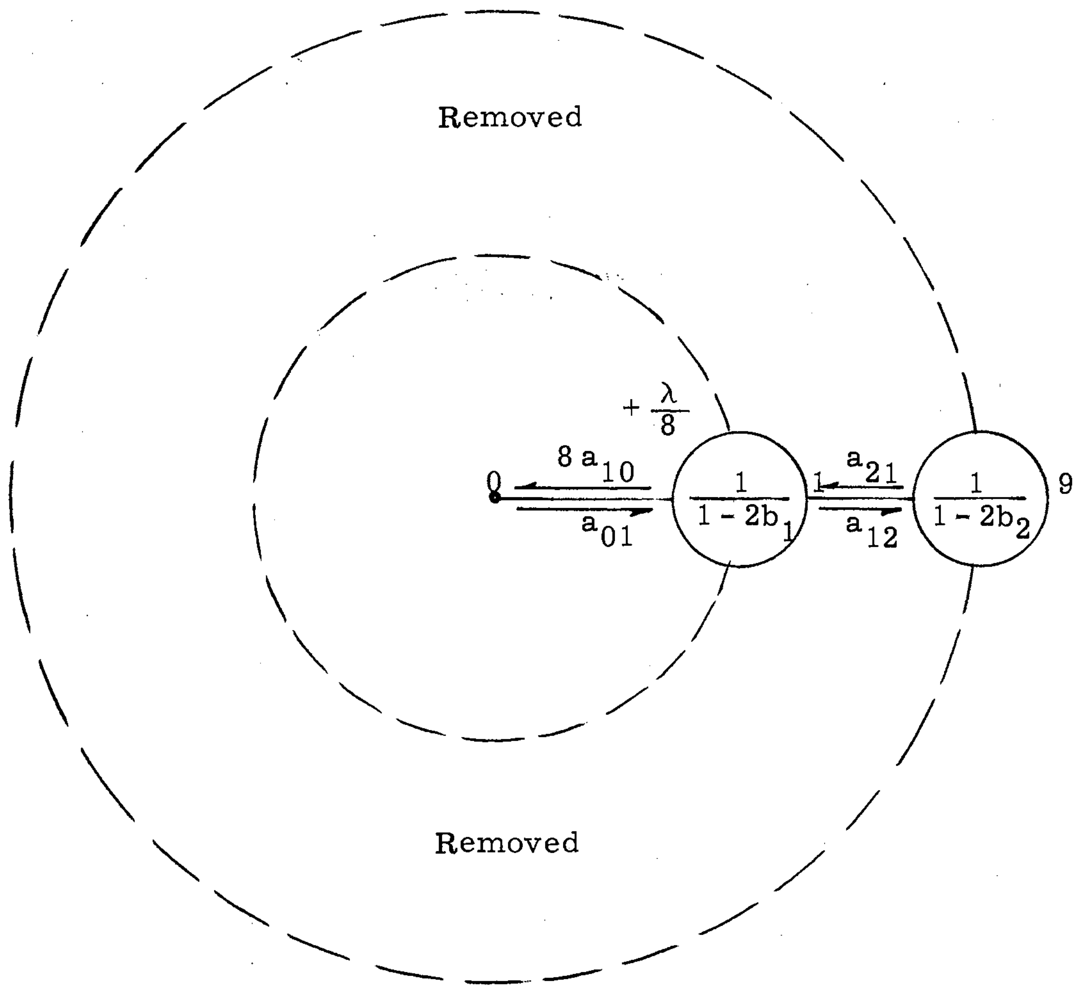


Fig. 71
Modified Point Set
Case IA

this system to the linear point set of Fig. 71 and temporarily introducing a zero point at the origin 0, the series forming on the two point set 1, 9 is identically Case IA of the double ring circulatory series, Art. 4-5:

$$Q_1^{1(S)IA} = \frac{1}{(1-2b_1)X'_2}\,\frac{\lambda}{8} \quad \Big| \quad Q_9^{1(S)IA} = \frac{a_{12}}{(1-2b_1)(1-2b_2)X'_2}\,\frac{\lambda}{8}\,.$$

Releasing point 0, the carried over value

$$Q_0^{1(CO)} = \frac{8\ a_{10}}{(1 - 2b_1)\ X'_2}\ \frac{\lambda}{8} = \frac{1}{(1 - 2b_1)\ X'_2}\ \frac{\lambda}{8} = \frac{A_{01}}{a_{01}}\ \frac{\lambda}{8}$$

becomes a new starting value which forms the axial symmetric carry-over series. From Eq's 29, Art. 4-4, the function coefficients due to this starting value are

$$Q_0^{1(C)IA} = \frac{A_{01}}{a_{01}\,Y_2}\ \frac{\lambda}{8}$$

$$Q_1^{1(C)IA} = \frac{A_{01}^2}{a_{01}\,Y_2}\ \frac{\lambda}{8} = Q_2^{1(C)IA} = \ldots = Q_8^{1(C)IA}$$

$$Q_9^{1(C)IA} = \frac{A_{01}\ A_{02}}{a_{01}\ Y_2}\ \frac{\lambda}{8} = Q_{10}^{1(C)IA} = \ldots = Q_{16}^{1(C)IA} .$$

The final results are equal to the superposition of these values with the results from the double ring circulatory system.

Finally, considering a starting value λ at point 9, the system is resolved into four basic cases as shown in Fig. 72 and the results superimposed.

Case IA1. This system is a modification of Case IA. Suppressing point 0, the results of algebraic carry-over on the reduced point set 1, 9 are (Fig. 73)

$$Q_1^{9(S)IA1} = \frac{a_{21}}{(1 - 2b_1)(1 - 2b_2)X_2'}\ \frac{\lambda}{8} \quad \Bigg| \quad Q_9^{9(S)IA1} = \frac{1}{(1 - 2b_2)X_2'}\ \frac{\lambda}{8} .$$

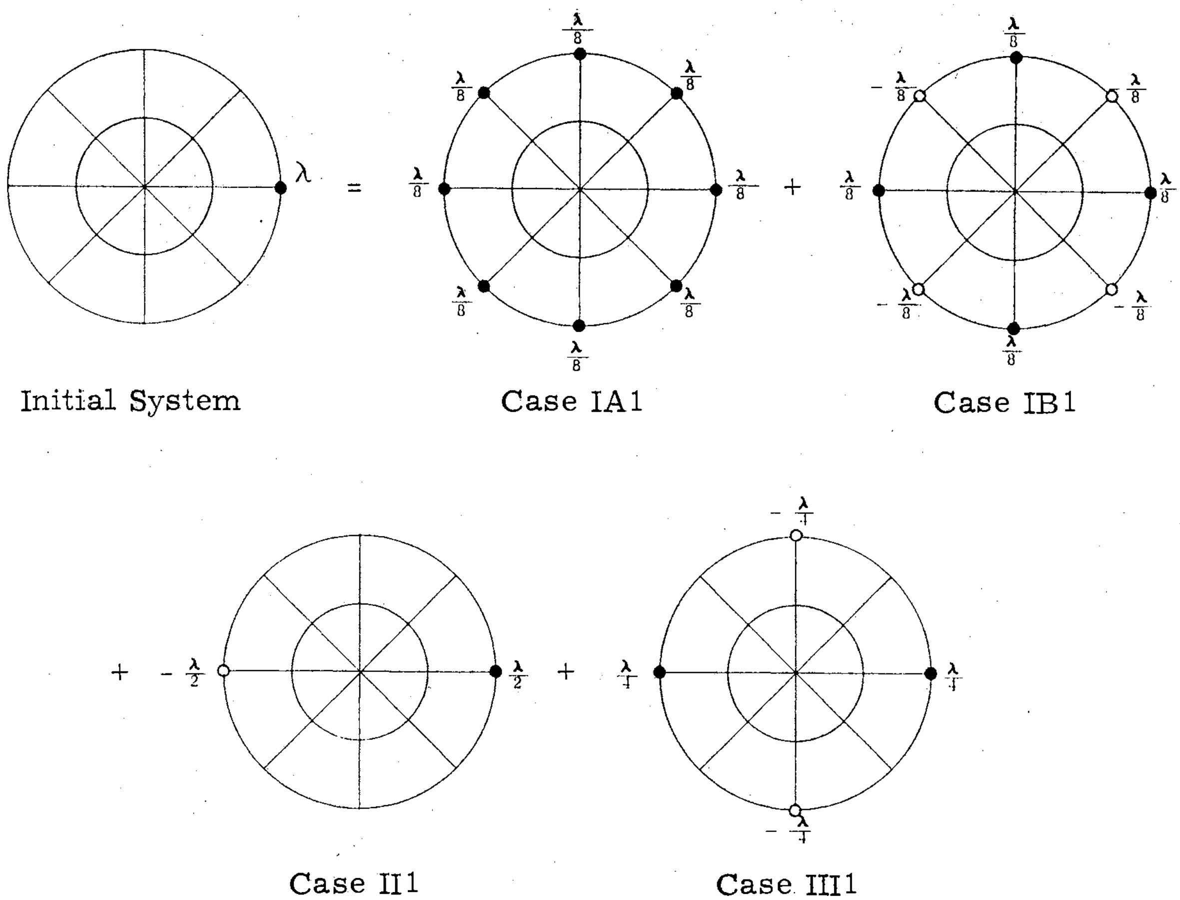


Fig. 72
Resolution of Twenty-Five Point Set with Starting Value at Point 9 into Basic Cases

Releasing point 0 the carried-over value

$$Q_0^{9(CO)} = \frac{8\, a_{10}\, a_{21}}{(1 - 2b_1)(1 - 2b_2)X'_2} \; \frac{\lambda}{8} = \frac{a_{21}\, A_{01}}{(1 - 2b_2)a_{01}} \; \frac{\lambda}{8}$$

forms the central carry-over series. The corresponding function values become (Eq's 29) :

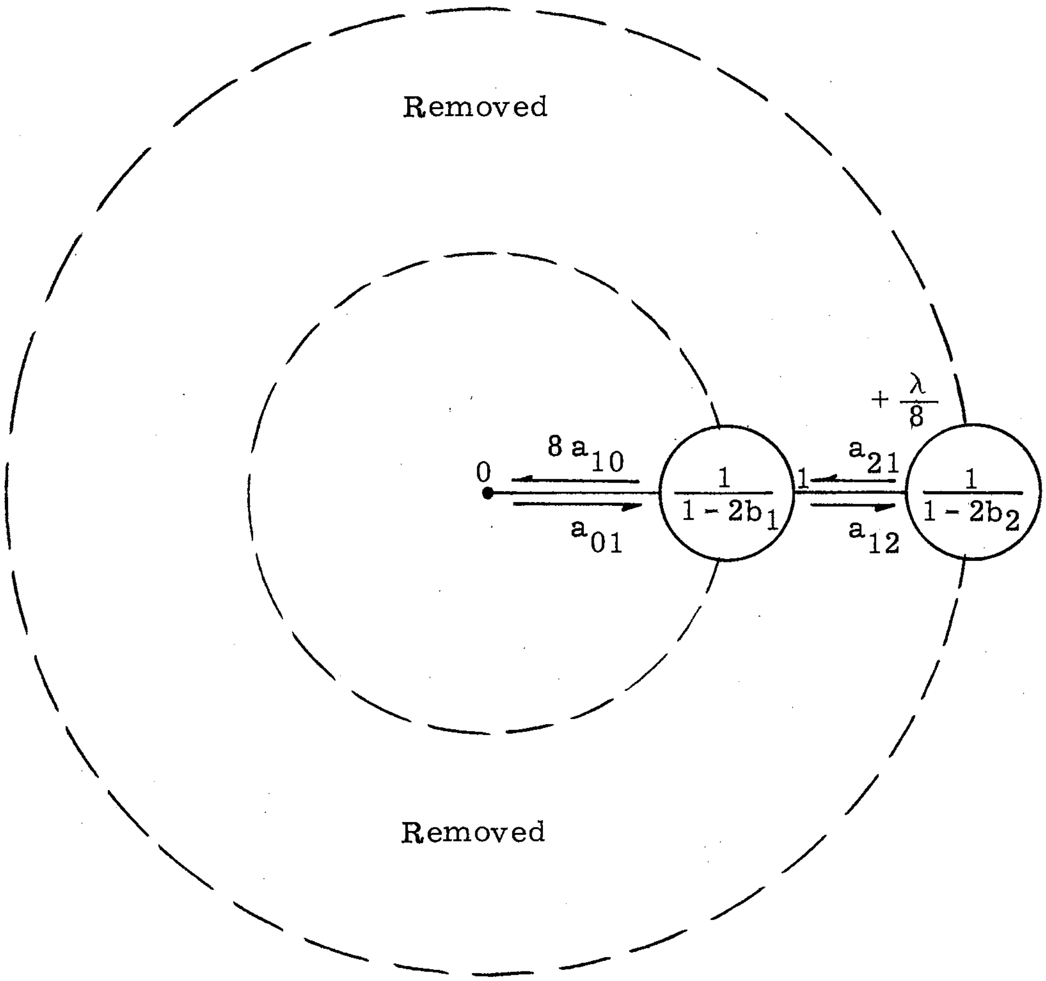


Fig. 73
Modified Point Set - Case IA1

$$Q_0^{1(C)IA1} = \frac{a_{21}A_{01}}{(1-2b_2)a_{01}Y_2}\,\frac{\lambda}{8}$$

$$Q_1^{1(C)IA1} = \frac{a_{21}A_{01}^2}{(1-2b_2)a_{01}Y_2}\,\frac{\lambda}{8} = Q_2^{1(C)IA1} = \ldots = Q_8^{1(C)IA1}$$

$$Q_9^{1(C)IA1} = \frac{a_{21}A_{01}A_{02}}{(1-2b_2)a_{01}Y_2}\,\frac{\lambda}{8} = Q_{10}^{1(C)IA1} = \ldots = Q_{16}^{1(C)IA1}\,.$$

The final results, Case IA1, are obtained by superimposing these values and those of the axially symmetric circulatory series above.

Case IB1. As in Case IB, there is no carry-over into the origin, and the system can be simplified to that of Fig. 74. Performing algebraic carry-over on this linear set the final values are

$$Q_1^{9(S)IB1} = \frac{a_{21}}{(1 + 2b_1)(1 + 2b_2)X''_2} \frac{\lambda}{8} \quad \Big| \quad Q_9^{9(S)IB1} = \frac{1}{(1 + 2b_2)X''_2} \frac{\lambda}{8} .$$

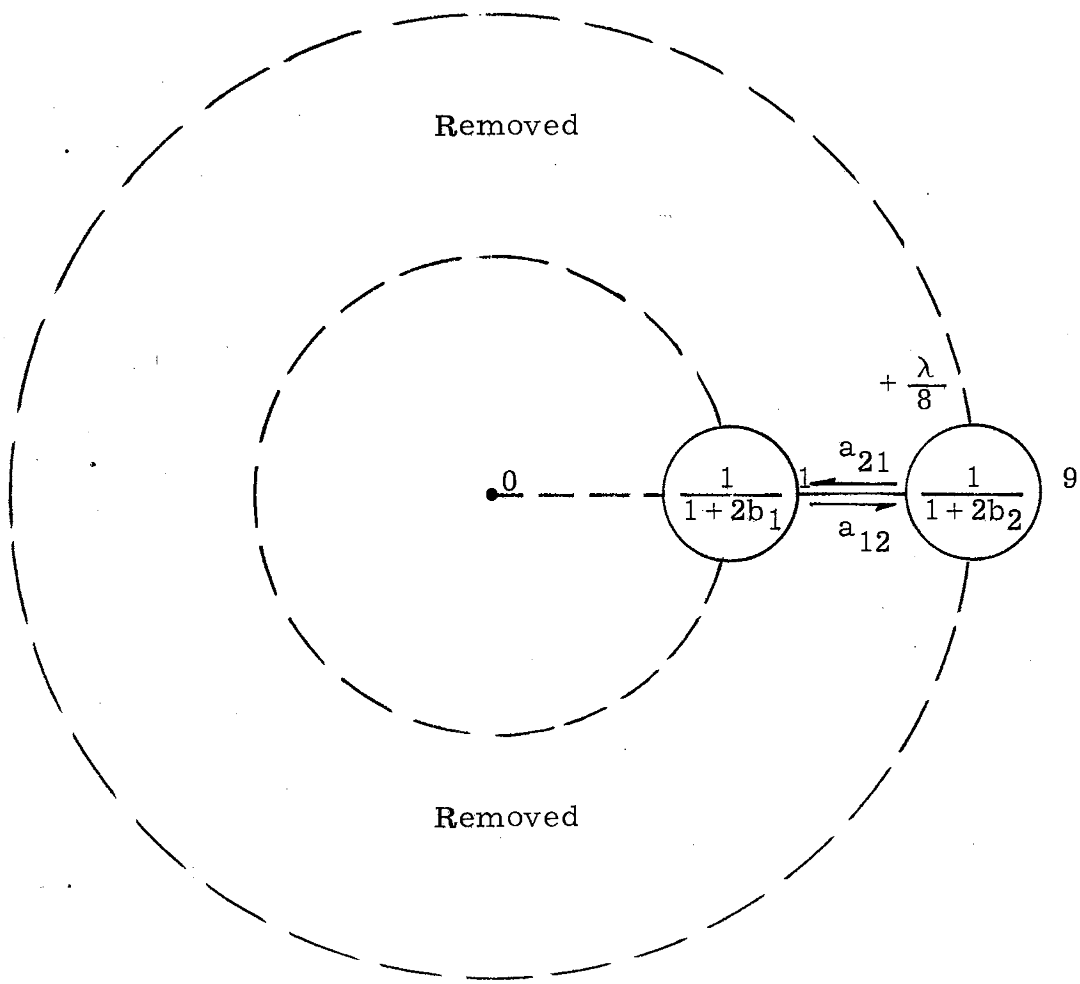


Fig. 74
Modified Point Set
Case IB1

Case II1. This system, a modification of Case II, again forms the circular panel carry-over series with zero value at the origin. Considering the reduced point set of Fig. 75 and temporarily suppressing point 2, the results of algebraic carry-over on points 1, 9, 10 are:

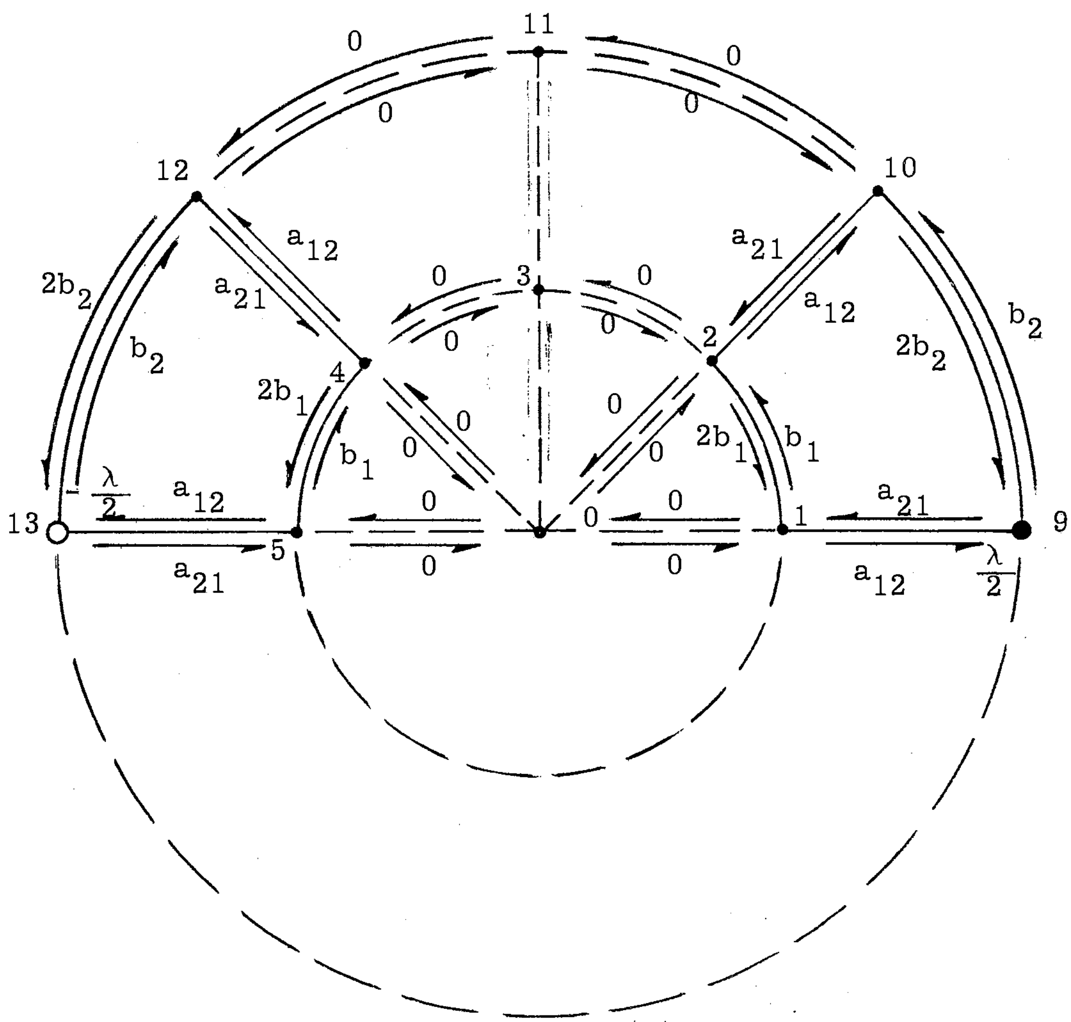


Fig. 75
Modified Point Set
Case II1

$$Q_9^{9(B1)II1} = \frac{1}{X_{22}} \frac{\lambda}{2} \quad \Bigg| \quad Q_1^{9(B1)II1} = \frac{a_{21}}{X_{22}} \frac{\lambda}{2} \quad \Bigg| \quad Q_{10}^{9(B1)II1} = \frac{b_2}{X_{22}} \frac{\lambda}{2} \quad .$$

Releasing point 2 and simultaneously suppressing point 9, the carried-over value is

$$Q_2^{9(CO)} = \frac{a_{21}(b_1 + b_2)}{X_{22}} \frac{\lambda}{2} = \alpha_0 \quad .$$

This value forms series on the three point set 2, 1, 10, the results of which are

$$Q_2^{9(B2)II1} = \frac{1}{X_{21}} \alpha_0 \quad \Bigg| \quad Q_1^{9(B2)II1} = \frac{2b_1}{X_{21}} \alpha_0 \quad \Bigg| \quad Q_{10}^{9(B2)II1} = \frac{a_{12}}{X_{21}} \alpha_0 \quad .$$

Removing the zero point at 9 the returned value is

$$Q_9^{9(CO)} = \frac{2\,a_{12}(b_1 + b_2)}{X_{21}} \alpha_0 = \frac{2\,a_{12}a_{21}(b_1 + b_2)^2}{X_{21}X_{22}} \frac{\lambda}{2} = \beta_1$$

and the common ratio is again (Case II, Art. 4-5)

$$\frac{2\,a_{12}a_{21}(b_1 + b_2)^2}{X_{21}X_{22}} \quad .$$

Repeating this procedure infinite times, the carried-over values α and β form infinite geometric carry-over series whose sums are:

$$\sum_0^{\infty} \alpha_n = \alpha_0 + \alpha_1 + \ldots = \frac{a_{21}(b_1 + b_2)}{X_{22}Y_{22}} \frac{\lambda}{2}$$

$$\sum_0^{\infty} \beta_n = \beta_0 + \beta_1 + \ldots = \frac{1}{Y_{22}} \frac{\lambda}{2} \quad .$$

Superimposing the α and β series, the final values, Case II1, become

$$Q_2^{9(II1)} = \frac{1}{X_{21}} \sum_0^\infty \alpha_n = \frac{C_{22}}{Z_{22}} \frac{\lambda}{2} = -Q_4^{9(II1)}$$

$$Q_1^{9(II1)} = \frac{2b_1}{X_{21}} \sum_0^\infty \alpha_n + \frac{a_{21}}{X_{22}} \sum_0^\infty \beta_n = \frac{A_{22}}{Z_{22}} \frac{\lambda}{2} = -Q_5^{9(II1)}$$

$$Q_{10}^{9(II1)} = \frac{a_{12}}{X_{21}} \sum_0^\infty \alpha_n + \frac{b_2}{X_{22}} \sum_0^\infty \beta_n = \frac{B_{22}}{Z_{22}} \frac{\lambda}{2} = -Q_{12}^{9(II1)}$$

$$Q_9^{9(II1)} = \frac{1}{X_{22}} \sum_0^\infty \beta_n = \frac{1}{Z_{22}} \frac{\lambda}{2} = -Q_{13}^{9(II1)} \quad .$$

The new equivalents used in these equations are

$$A_{22} = a_{21}\left[1 + \frac{2b_1(b_1 + b_2)}{X_{21}}\right] \qquad C_{22} = \frac{a_{21}(b_1 + b_2)}{X_{21}}$$

$$B_{22} = b_2\left[1 + \frac{a_{12}a_{21}(1 + \frac{b_1}{b_2})}{X_{21}}\right] \qquad Z_{22} = X_{22}\,Y_{22} \quad .$$

These constants can be interpreted as over-relaxation and direct carry-over factors, and conclusions drawn similar to those in Case II, Art. 4-5.

<u>Case III1.</u> This system is a modification of Case III (Fig. 68). Performing algebraic carry-over on the isolated two point sets (Fig. 76), the results are

$$Q_9^{9(III1)} = \frac{1}{X_2} \frac{\lambda}{4} = Q_{13}^{9(III1)} = -Q_{11}^{9(III1)} = -Q_{15}^{9(III1)}$$

$$Q_1^{9(III1)} = \frac{a_{21}}{X_2} \frac{\lambda}{4} = Q_5^{9(III1)} = -Q_3^{9(III1)} = -Q_7^{9(III1)} \quad .$$

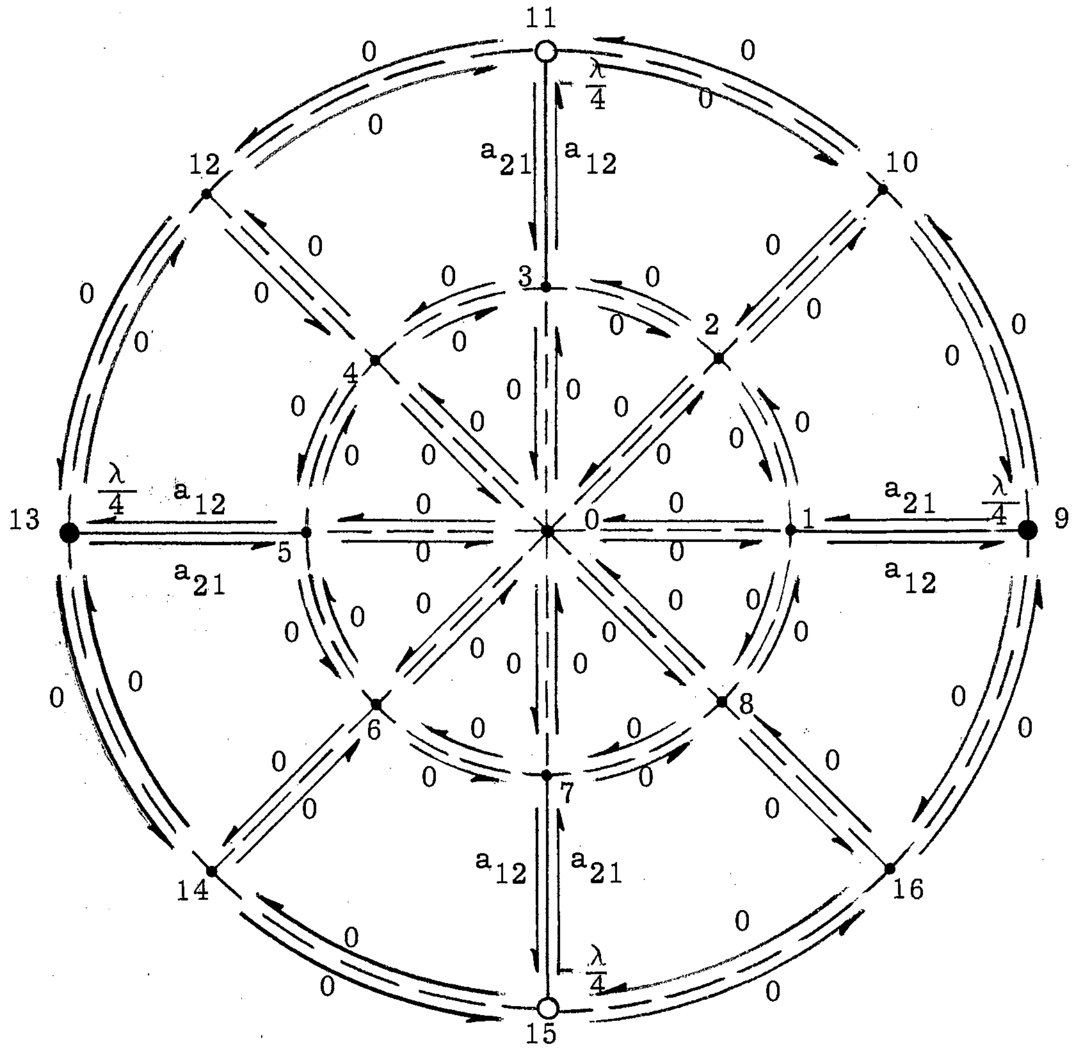


Fig. 76
Modified Point Set
Case III1

The final function coefficients, starting value λ at point 9, are obtained by superimposing the results from Cases IA1, IB1, II1, and III1. For a starting value at any other point on the network, the final values are identical with these results (or those corresponding to starting value λ at point 1) after a simple rotation of the pivotal point numbers.

4-7 The Laplace Equation. In polar coordinates the Laplace equation has the form

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\,\frac{\partial Q}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 Q}{\partial \theta^2} = 0 \qquad (30)$$

with Q equal to a given function $G(r, \theta)$ on the boundary (38).

The corresponding finite - difference equation written for an interior point ij of the network is (Fig. 49)

$$Q_{ij} = \left\{ \begin{matrix} a_{i+1,i}\, Q_{i+1,j} + a_{i-1,i}\, Q_{i-1,j} \\ \\ b_{i}(Q_{i,j+1} + Q_{i,j-1}) \end{matrix} \right\} \qquad (31)$$

and at a boundary point kl

$$Q_{kl} = G_{kl} \quad . \qquad (32)$$

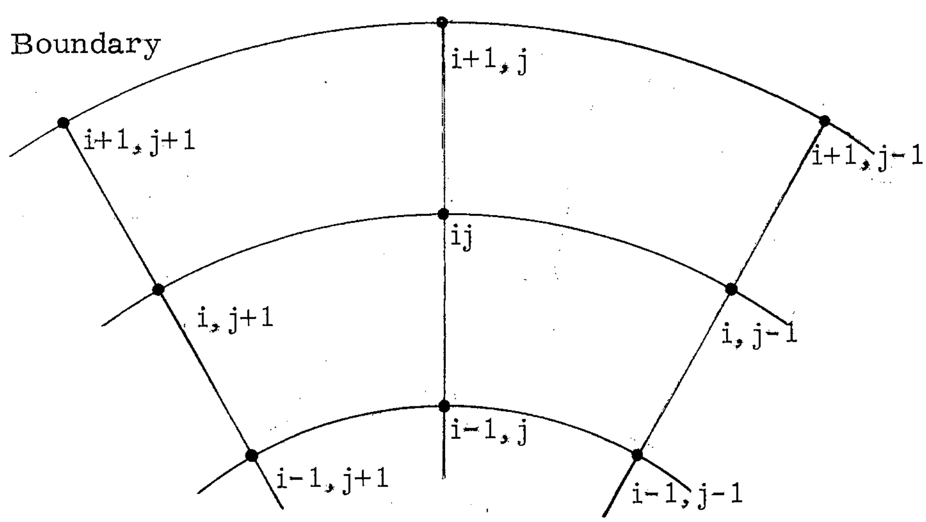


Fig. 77
Finite Difference Net Adjacent to Boundary

The boundary values are the starting values for the carry-over process, as explained in Art. 2-6. The finite-difference solution is again obtained from a corresponding solution of the Poisson equation by considering the difference equation written for a point ij adjacent to the boundary. Thus (Fig. 77)

$$Q_{ij} = \left\{ \begin{matrix} a_{i-1,i}\,(Q_{i-1,j}) \\ \\ b_i\,(Q_{i,j+1} + Q_{i,j-1}) \end{matrix} \right\} + a_{i+1,i}\, G_{i+1,j} \quad .$$

The carried-over value $a_{i+1,i}\, G_{i+1,j}$, which corresponds with the value Q_{ij}^{*} of the Poisson equation (Eq. 21), is a new starting value at point ij, and the algebraic carry-over is performed as before.

The final function coefficients corresponding to a starting value λ at a boundary point of the finite-difference net are therefore equal to the final results due to a starting value λ at the adjacent interior point, multiplied by the carry-over factor from the outer ring.

CHAPTER V

TRIANGULAR SYSTEMS

5-1 Linear Finite - Difference Equations. In triangular coordinates the Poisson equation has the form (38)

$$\frac{\partial^2 Q}{\partial u^2} \text{Sin } 2(\beta - \alpha) - \frac{\partial^2 Q}{\partial v^2} \text{Sin } 2\beta + \frac{\partial^2 Q}{\partial w^2} \text{Sin } 2\alpha$$

$$= - 2 \text{ Sin } \alpha \text{ Sin } \beta \text{ Sin } (\beta - \alpha) \text{ F } (u, v, w) . \qquad (33)$$

For the symmetrical case ($\beta = \pi - \alpha$), the corresponding finite-difference equation written for point ij of the network is (Fig. 78)

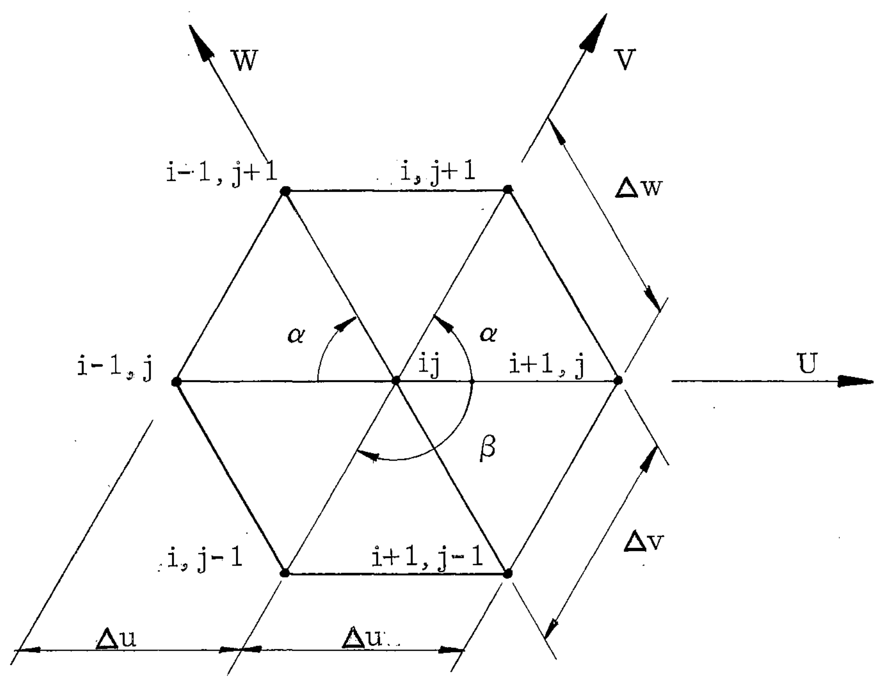


Fig. 78
Finite Difference Net in Triangular Coordinates

$$\frac{Q_{i+1,j} - 2Q_{ij} + Q_{i-1,j}}{\Delta u^2} 2 \sin 2\alpha (2\sin^2 \alpha - 1) + \frac{Q_{i,j+1} - 2Q_{ij} + Q_{i,j-1}}{\Delta v^2} \sin 2\alpha$$

$$+ \frac{Q_{i-1,j+1} - 2Q_{ij} + Q_{i+1,j-1}}{\Delta w^2} = -2 \sin^2 \alpha \sin 2\alpha F_{ij} .$$

Introducing the notation

$$a = \frac{2 \sin^2 \alpha - 1}{2(t^2 + 2 \sin^2 \alpha - 1)} \qquad b = \frac{t^2}{4(t^2 + 2 \sin^2 \alpha - 1)}$$

$$\lambda = \frac{t \sin^2 \alpha}{2(t^2 + 2 \sin^2 \alpha - 1)} \qquad t = \frac{\Delta u}{\Delta v} = \frac{\Delta u}{\Delta w} \tag{34}$$

this equation becomes

$$Q_{ij} = \left\{ \begin{array}{c} a\,(Q_{i+1,j} + Q_{i-1,j}) \\ \\ b(Q_{i,j+1} + Q_{i,j-1} + Q_{i+1,j-1} + Q_{i-1,j+1}) \end{array} \right\} + Q^*_{ij} \tag{35}$$

where

$$Q^*_{ij} = \lambda F_{ij} \Delta u \Delta v \tag{36}$$

is the starting value for Q_{ij}, assuming the Q's at the six adjacent points to be zero.

The quantities a and b are carry-over factors on the finite-difference net in the horizontal and diagonal direction, respectively. They represent the influences of the Q values at the points $i+1, j$, $i-1, j$, $i, j+1$, $i, j-1$, $i+1, j-1$, and $i-1, j+1$ on the value at point ij.

<u>5-2 The Three Point Basic Series.</u> Isolating a three point set with starting value λ at point 5 (Fig. 79a) and applying algebraic carry-over to the computation of function coefficients, each final result is the sum of an infinite geometric series.

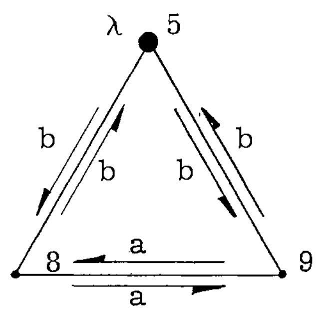


Fig. 79a
Three Point Set - Basic Series

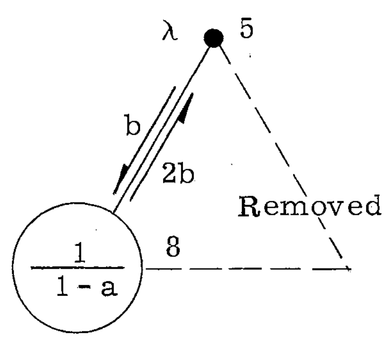


Fig. 79b
Reduced Point Set - Basic Series

From the figure it is evident that the flow of values takes place symmetrically with respect to point 5. The point set may therefore be reduced to that of Fig. 79b by introducing the modified carry-over factor 2b and the over-relaxation factor $\frac{1}{1-a}$. The over-relaxation factor corresponds with the series forming between the symmetric points 8 and 9.

Performing carry-over the results are

$$Q_5^{5(B)} = \lambda \left[1 + \left(\frac{2b^2}{1-a}\right)^1 + \left(\frac{2b^2}{1-a}\right)^2 + \ldots \right] = \frac{1}{X'_{02}} \lambda$$

$$Q_8^{5(B)} = \frac{b}{(1-a)X'_{02}} \lambda = Q_9^{5(B)} = \frac{b'}{X'_{02}} \lambda$$

where

$$X'_{02} = 1 - \frac{2b^2}{1-a} = 1 - 2bb' \quad .$$

The quantity

$$b' = \frac{b}{1-a}$$

may be considered a modified carry-over factor.

From the diagrammatic representation of final values (Fig. 80) it is evident that:

(a) The constant $\frac{1}{X'_{02}}$ is the over-relaxation factor for the basic series on the symmetrical three point set

(b) The final function value at the apex is equal to the starting value multiplied by the over-relaxation factor.

(c) The final function value at each of the other points is equal to the final value at the apex multiplied by the modified carry-over factor b' .

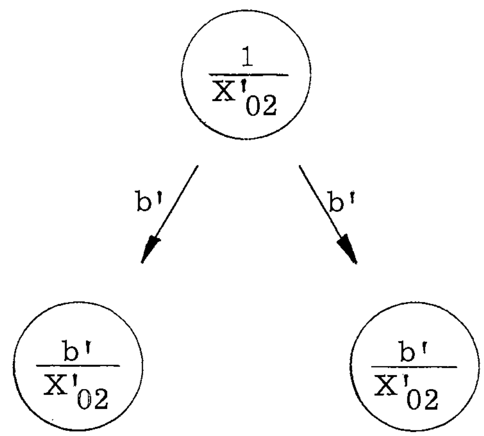


Fig. 80
Final Results - Three Point Basic Series

5-3 The Hexagonal Circulatory Series. A six point closed ring of hexagonal shape is considered (Fig. 81), and the function values are determined by algebraic carry-over . For a starting value λ at point 8, the carry-over procedure is simplified by resolving the initial system into four basic cases (Fig. 82) and superimposing the results. In this way each final value is found to be the algebraic sum of four simple geometric series.

Case I: This system is symmetrical with respect to both the X- and Y- axes, and can be reduced to the point set of Fig. 83 by using modified carry-over factors and the over-relaxation factor $\frac{1}{1-a}$. Performing algebraic carry-over the results are.

$$Q_8^{8(S)I} = \frac{1}{(1-a)X'_{02}} \frac{\lambda}{4} \quad \Bigg| \quad Q_{12}^{8(S)I} = \frac{2b}{(1-a)X'_{02}} \frac{\lambda}{4} .$$

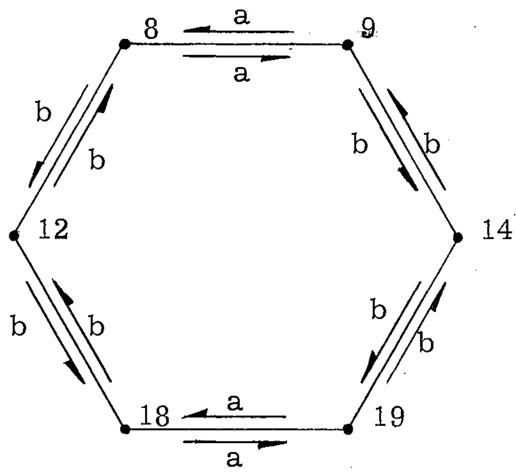


Fig. 81

Six Point Ring - Circulatory Series

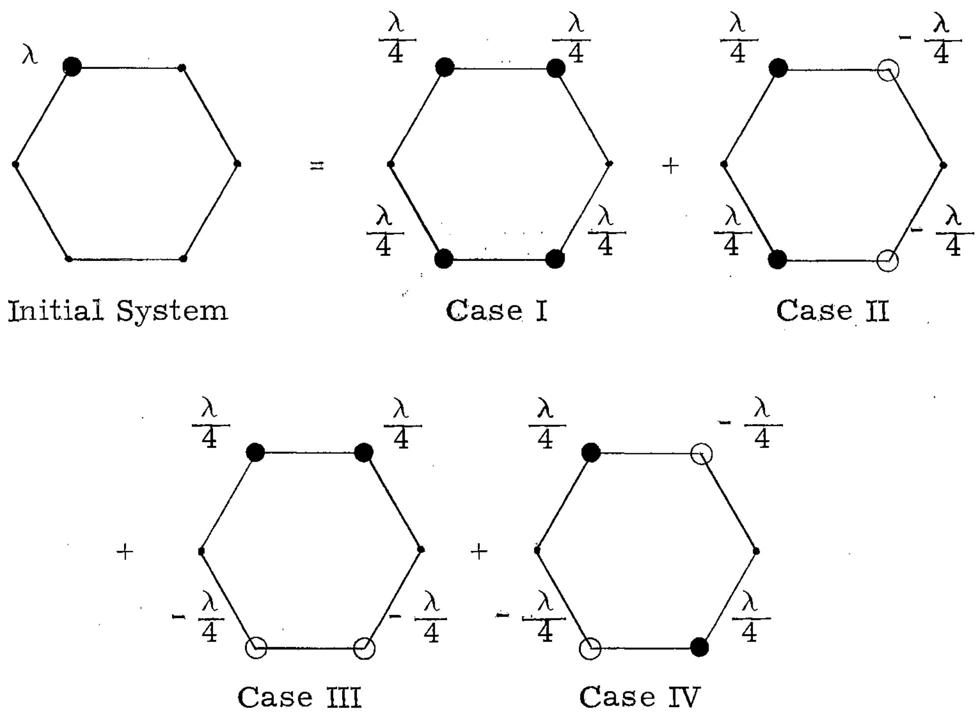


Fig. 82

Resolution of Circulatory System with Starting Value at Point 8 into Four Basic Cases

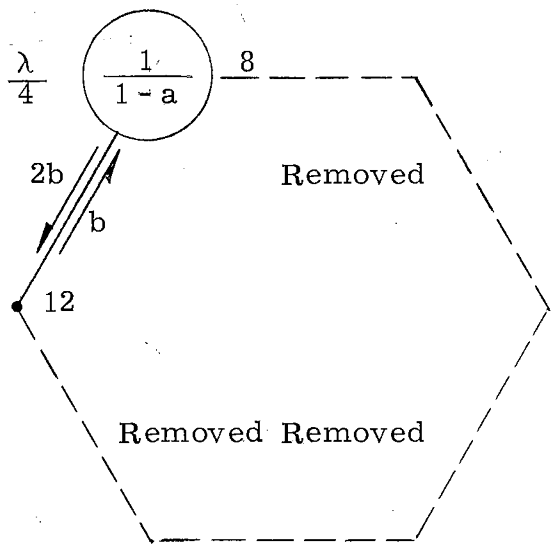


Fig. 83
Modified Circulatory System
Case I

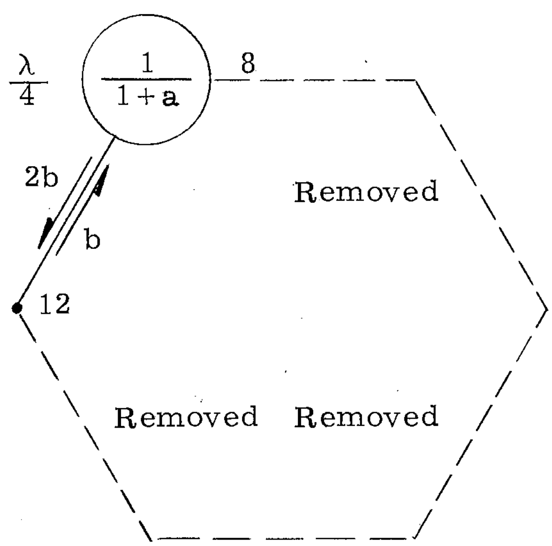


Fig. 84
Modified Circulatory System
Case II

<u>Case II.</u> From the symmetry and antisymmetry of this system with respect to the X- and Y-axes, respectively (Fig. 82), a reduction to the modified point set of Fig. 84 is possible. The constant $\frac{1}{1+a}$ is the over-relaxation factor for the series forming between the antisymmetric points 8 and 9.

The results of algebraic carry-over are

$$Q_8^{8(S)II} = \frac{1}{(1+a)\,X''_{02}}\,\frac{\lambda}{4} \qquad\Bigg|\qquad Q_{12}^{8(S)II} = \frac{2b}{(1+a)\,X''_{02}}\,\frac{\lambda}{4}$$

where

$$X''_{02} = 1 - \frac{2b^2}{1+a} = 1 - 2bb'' \quad .$$

<u>Case III.</u> This system can be reduced to the modified point set of Fig. 85. The final values are equal to the starting values multiplied by the over-relaxation factor $\frac{1}{1-a}$:

$$Q_8^{8(S)III} = \frac{1}{1-a}\,\frac{\lambda}{4} \qquad\Bigg|\qquad Q_{12}^{8(S)III} = 0 \quad .$$

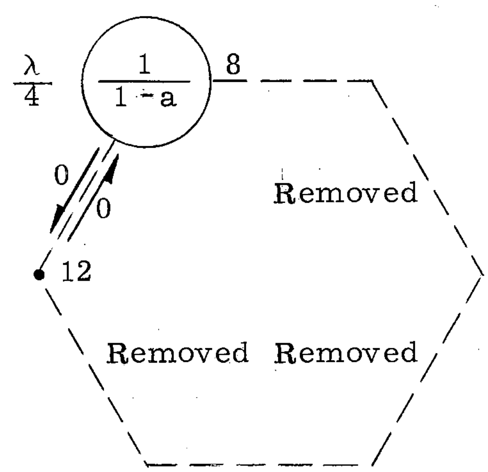


Fig. 85
Modified Circulatory System
Case III

Case IV. The reduction of this system is shown in Fig. 86. The final results are equal to the starting values multiplied by the over-relaxation factor $\frac{1}{1+a}$:

$$Q_8^{8(S)IV} = \frac{1}{1+a}\,\frac{\lambda}{4} \qquad \Big| \qquad Q_{12}^{8(S)IV} = 0 \ .$$

Superimposing Cases I through IV, it is seen that the circulatory series forming on the six point closed ring can be resolved into simple geometric series by the proper use of modified constants and over-relaxation factors.

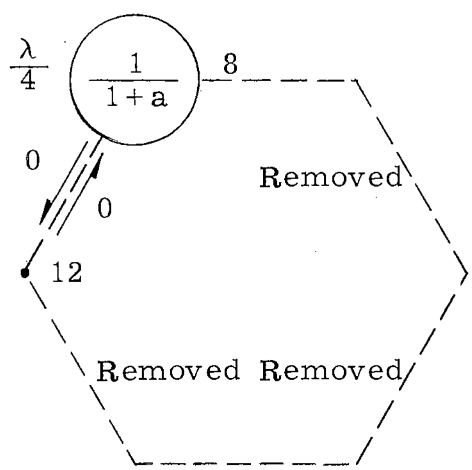


Fig. 86
Modified Circulatory System
Case IV

5-4 The Internal Series. Using the algebraic carry-over method to determine the flow of function values on an eight point set symmetrical with respect to the Y-axis (Fig. 87), the final results are sums of infinite geometric series all terms of which are infinite geometric series.

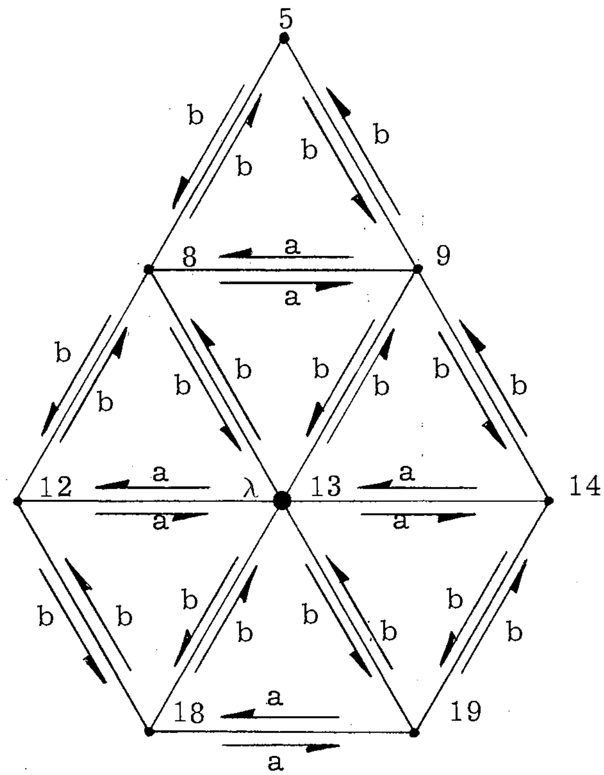


Fig. 87
Eight Point Set - Internal Series

A starting value λ at point 13, carried-over to the adjacent points 8, 9, 12, 14, 18, and 19, will return to 13 as well as begin to circulate through the closed rings 8, 9, 14, 19, 18, 12 and 8, 9, 5. In order to properly separate the resulting series and obtain finite algebric sums for the function coefficients, it is again convenient to introduce the concept of over-relaxation factors. The reduced point set is shown in Fig. 88, and the modification is accomplished as follows.

The over-relaxation factor for a symmetric series forming on the isolated three point set 8, 9, 5 is $\frac{1}{X'_{02}}$, from Art. 5-2. The over-relaxation factor for the symmetric series forming between points 8 and 9 is $\frac{1}{1-a}$. Thus the over-relaxation factor for a value carried into point 8 from either point 12 or point 13 is $\frac{1}{(1-a)X'_{02}}$, as given in Fig. 88.

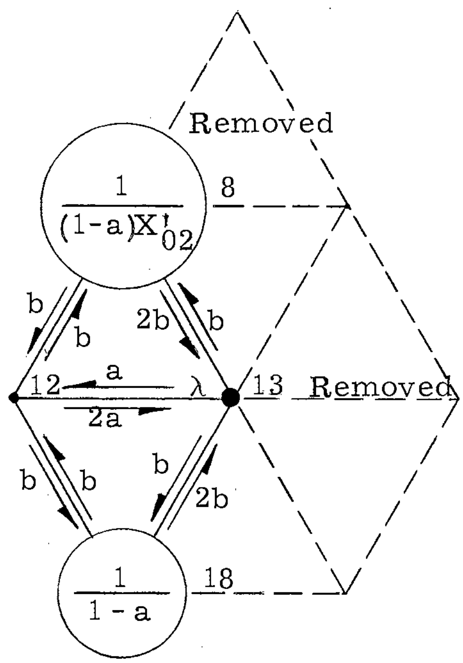


Fig. 88
Modified Point Set - Internal Series

The over-relaxation factor at point 18 is $\frac{1}{1-a}$, corresponding to the two point symmetric series forming between points 18 and 19. The modified carry-over values into the central point 13 are 2a and 2b, from the symmetry with respect to the Y-axis.

Having these modified constants, the algebraic carry-over is performed by alternately suppressing and releasing point 13. Carrying-over from that point to the adjacent points 8, 12, and 18, then temporarily introducing a zero point at 13, the results of the circulatory series can be determined. These values are

$$Q_{12}^{13(S)} = \frac{a}{Y'_{02}} \lambda + \left(\frac{1}{Y'_{02}} - 1 \right) \lambda$$

$$Q_8^{13(S)} = \frac{b}{(1-a)X'_{02}} \lambda + \frac{b}{(1-a)X'_{02}} Q_{12}^{13(S)} = \frac{(1+a)b}{(1-a)X'_{02}Y'_{02}} \lambda$$

$$Q_{18}^{13(S)} = \frac{b}{1-a} \lambda + \frac{b}{1-a} Q_{12}^{13(S)} = \frac{(1+a)\,b}{(1-a)Y'_{02}} \lambda$$

where

$$Y'_{02} = 1 - \frac{b^2}{1-a} \left(\frac{1}{X'_{02}} + 1 \right) .$$

Releasing point 13, the returned value is

$$Q_{13}^{13(CO)} = \left[\frac{2(1+a)^2}{Y'_{02}} - 2(1+2a) \right] \lambda .$$

This quantity, divided by the starting value λ, is the common ratio of the geometric carry-over series formed by repeating the procedure infinite times. Summing this series, which is called the internal series, the final function values are

$$Q_{13}^{13} = \frac{1}{Z_{02}} \lambda \qquad Q_{12}^{13} = \frac{A_{02}}{Z_{02}} \lambda$$

$$Q_{18}^{13} = \frac{B_{02}}{Z_{02}} \lambda \qquad Q_8^{13} = \frac{C_{02}}{Z_{02}} \lambda \tag{37}$$

$$Q_5^{13} = \frac{D_{02}}{Z_{02}} \lambda = \frac{2b\,C_{02}}{Z_{02}} \lambda .$$

The new equivalents used above are

$$A_{02} = \frac{1+a}{Y'_{02}} - 1 \qquad B_{02} = \frac{(1+a)\,b}{(1-a)\,Y'_{02}}$$

$$C_{02} = \frac{(1+a)b}{(1-a)X'_{02}Y'_{02}} = \frac{B_{02}}{X'_{02}} \qquad D_{02} = \frac{2(1+a)b^2}{(1-a)X'_{02}Y'_{02}} = \frac{2b\,B_{02}}{X'_{02}}$$

$$Z_{02} = 1 - \left[\frac{2(1+a)^2}{Y'_{02}} - 2(1+2a)\right] .$$

These constants are interpreted from the diagrammatic presentation of final results in Fig. 89. Thus

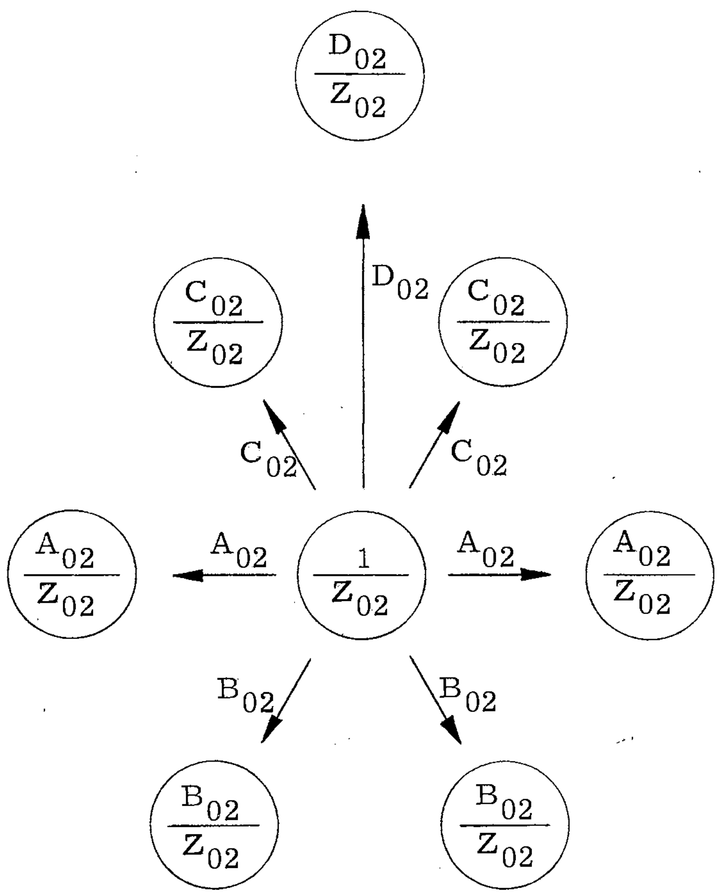


Fig. 89
Final Results - Internal Series

(a) The constant $\frac{1}{Z_{02}}$ is the over-relaxation factor for the internal carry-over series

(b) The final function value at the central point 13 is equal to the starting value multiplied by the over-relaxation factor.

(c) The final function value at any other point is equal to the final central value multiplied by the corresponding direct final carry-over factor A_{02}, B_{02}, C_{02}, or D_{02} .

5-5 The External Series. Considering a nine point triangular set with starting value λ at point 17 (Fig. 90), and applying algebraic carry-over to the computation of function coefficients, each final value

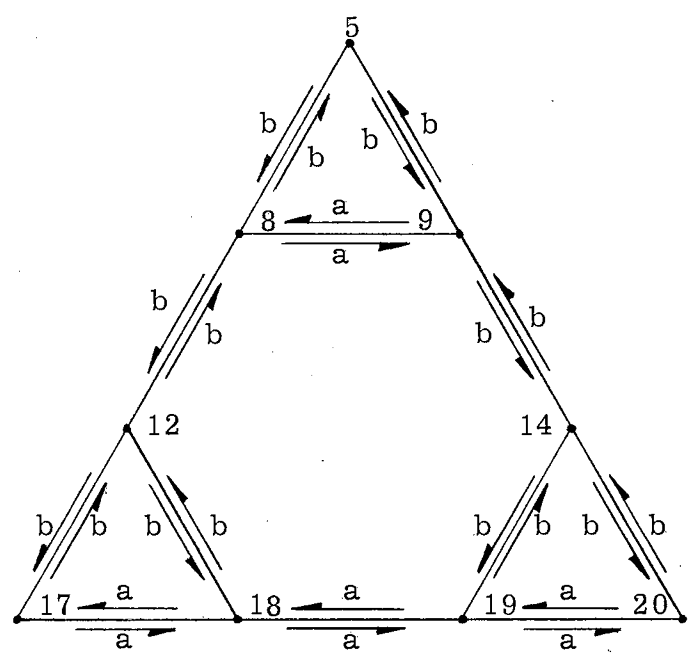


Fig. 90
Nine Point Set - External Series

is found to be the sum of two infinite geometric carry-over series, called the external series. To determine these series, the system is resolved into two basic cases, as shown in Fig. 91, and the results superimposed.

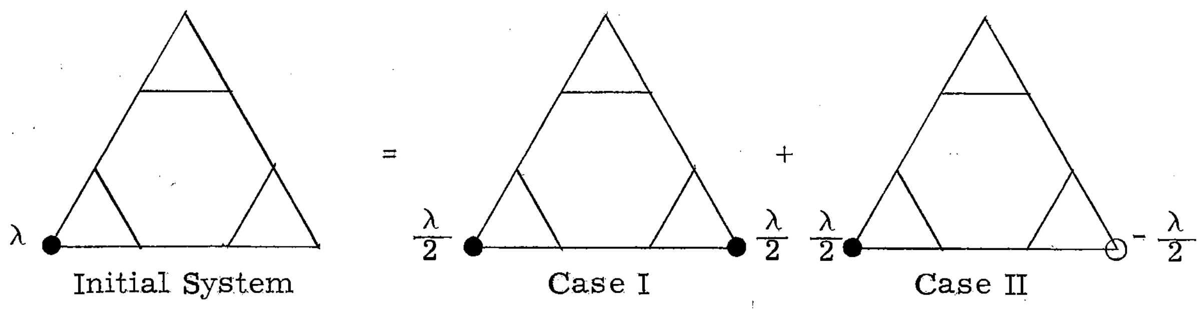


Fig. 91
Resolution of Triangular Circulatory System with Starting Value at Point 17 into Basic Cases

Case I. This system, symmetrical with respect to the Y-axis, can be reduced to the point set of Fig. 92a by introducing over-relaxation factors at points 8 and 18, as explained in Art. 5-4. Carrying-over from point 17, then suppressing that point, the results of series forming on the modified three point set 8, 12, 18 can be determined:

$$Q_{12}^{17(S)I} = \frac{b}{(1-a)Y'_{02}} \frac{\lambda}{2} \qquad \Bigg| \qquad Q_8^{17(S)I} = \frac{b^2}{(1-a)^2 X'_{02} Y'_{02}} \frac{\lambda}{2}$$

$$Q_{18}^{17(S)I} = \frac{a}{1-a} \frac{\lambda}{2} + \frac{b^2}{(1-a)^2 Y'_{02}} \frac{\lambda}{2} \quad .$$

Removing the zero point at 17, the returned value is

$$Q_{17}^{17(CO)} = \frac{a^2}{1-a} \frac{\lambda}{2} + \frac{b^2}{(1-a)^2 Y'_{02}} \frac{\lambda}{2} = \gamma_1 \quad .$$

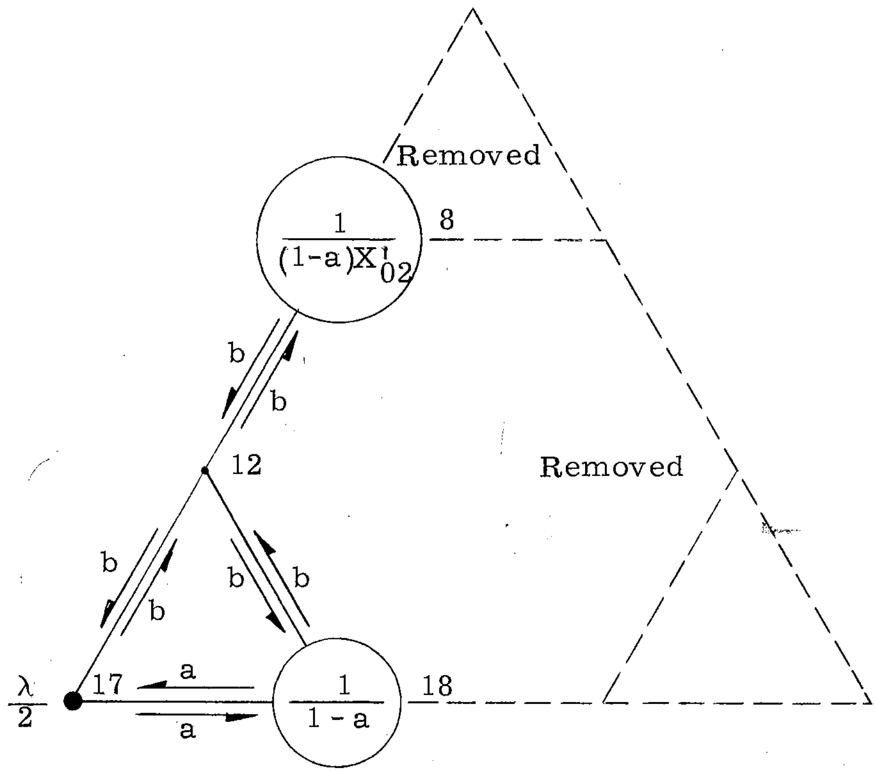


Fig. 92a
Modified Point Set - External Series
Case I

The ratio of the returned value γ_1 to the starting value $\frac{\lambda}{2}$ is the common ratio of the carry-over series developed by continuing this procedure. This series is called the external circulatory series and has the sum

$$\sum_{0}^{\infty} \gamma_n = \gamma_0 + \gamma_1 + \dots = \frac{1}{Z_{11}} \frac{\lambda}{2}$$

where

$$Z_{11} = 1 - \frac{a^2}{1 - a} - \frac{b^2}{(1 - a)^2 \, Y'_{02}} \quad .$$

The final function values, Case I, are

$$Q_{17}^{17(E)I} = \frac{1}{Z_{11}} \frac{\lambda}{2}$$

$$Q_{12}^{17(E)I} = \frac{b}{(1-a)Y'_{02}Z_{11}} \frac{\lambda}{2}$$

$$Q_{5}^{17(E)I} = \frac{2b^3}{(1-a)^2X'_{02}Y'_{02}Z_{11}} \frac{\lambda}{2}$$

$$Q_{8}^{17(E)I} = \frac{b^2}{(1-a)^2X'_{02}Y'_{02}Z_{11}} \frac{\lambda}{2}$$

$$Q_{18}^{17(E)I} = \frac{a}{(1-a)Z_{11}} \frac{\lambda}{2} + \frac{b^2}{(1-a)^2Y'_{02}Z_{11}} \frac{\lambda}{2} .$$

<u>Case II.</u> The modification of this system, antisymmetrical with respect to the Y-axis, is shown in Fig. 92b. The constant $\frac{1}{1+a}$ is the over-relaxation factor for the antisymmetric series which forms between both points 8 and 9 and 18 and 19.

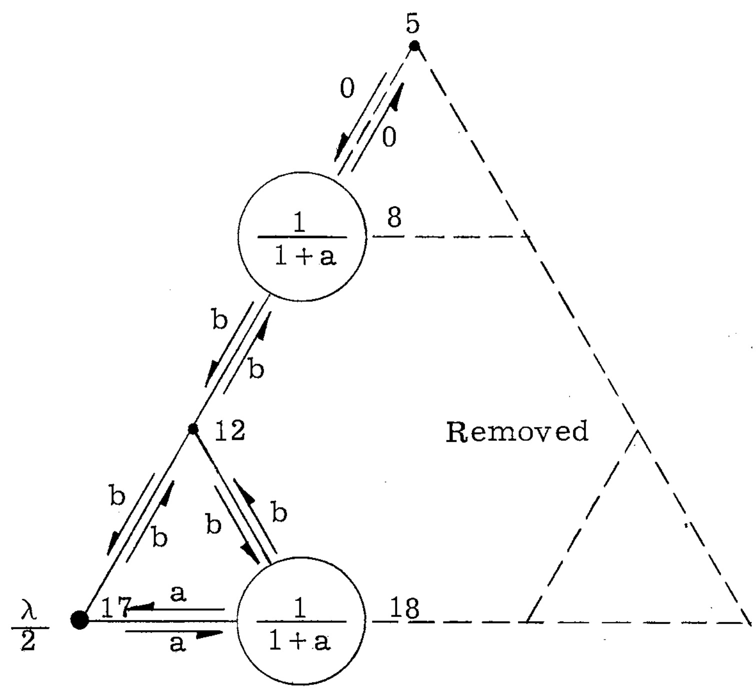


Fig. 92b

Modified Point Set - External Series
Case II

Carrying-over from point 17 to points 12 and 18, then introducing a zero at point 17, the three point set 8, 12, 18 is isolated. Performing algebraic carry-over on this reduced circulatory system, which is a modification of Case II of the hexagonal circulatory series, the results are

$$Q_{12}^{17(S)II} = \frac{(1+2a)b}{(1+a)X''_{02}} \frac{\lambda}{2} \qquad Q_{8}^{17(S)II} = \frac{(1+2a)b^2}{(1+a)^2 X''_{02}} \frac{\lambda}{2}$$

$$Q_{18}^{17(S)II} = \frac{a}{1+a} \frac{\lambda}{2} + \frac{(1+2a)b^2}{(1+a)^2 X''_{02}} \frac{\lambda}{2} .$$

Releasing point 17, the returned value is

$$Q_{17}^{17(CO)} = \frac{a^2}{1+a} \frac{\lambda}{2} + \frac{(1+2a)^2 b^2}{(1+a)^2 X''_{02}} \frac{\lambda}{2} = \delta_1 .$$

Repeating this operation infinite times, the carry-over series δ is formed having the geometric ratio

$$\frac{\delta_1}{\delta_0} = \frac{a^2}{1+a} + \frac{(1+2a)^2 b^2}{(1+a)^2 X''_{02}} .$$

The algebraic sum of this carry-over series is

$$\sum_0^{\infty} \delta_n = \delta_0 + \delta_1 + \cdots = \frac{1}{Z_{12}} \frac{\lambda}{2}$$

where

$$Z_{12} = 1 - \frac{a^2}{1+a} - \frac{(1+2a)^2 b^2}{(1+a)^2 X''_{02}} .$$

The final values for function coefficients, Case II, are thus

$$Q_{12}^{17(E)II} = \frac{(1+2a)b}{(1+a)X''_{02} Z_{12}} \frac{\lambda}{2} \qquad Q_{8}^{17(E)II} = \frac{(1+2a)b^2}{(1+a)^2 X''_{02} Z_{12}} \frac{\lambda}{2}$$

$$Q_{18}^{17(E)II} = \frac{a}{(1 + a)\, Z_{12}} \frac{\lambda}{2} + \frac{(1 + 2a)b^2}{(1 + a)^2\, X''_{02} Z_{12}} \frac{\lambda}{2}$$

$$Q_{17}^{17(E)II} = \frac{1}{Z_{12}} \frac{\lambda}{2} .$$

The final function values on the nine point triangular set (Fig. 90), due to a starting value λ at point 17, are obtained by superimposing the results from Cases I and II.

5-6 The Second Order Carry-Over Series. If now the carry-over procedure is applied to the analysis of a two dimensional twenty-eight point

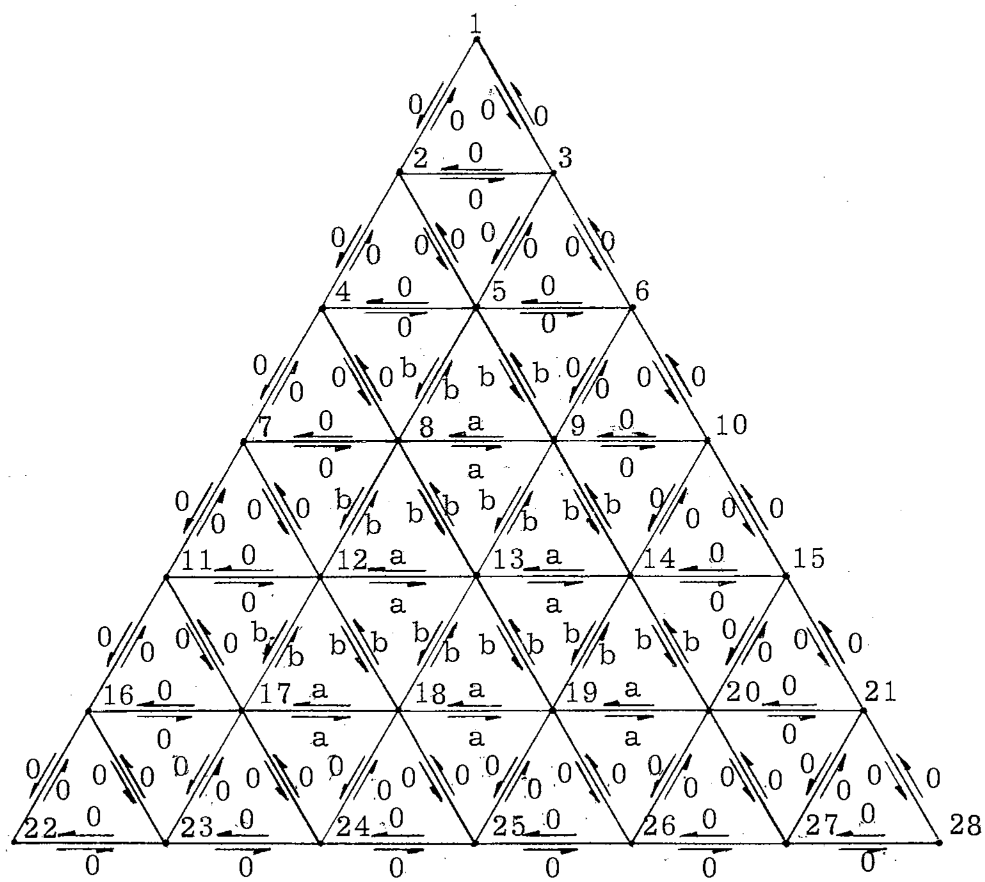


Fig. 93
Twenty-Eight Point Triangular Set

triangular set (Fig. 93), each final result is found to be the finite algebraic sum of an infinite geometric series of carry-over series. This new series, which interrelates the internal and the external series (Art's. 5-4, 5), is called the second order carry-over series.

It is evident from Fig. 93 that a starting value λ carried-over from point 13 to the adjacent points 8, 9, 12, 14, 18, and 19 will return to 13 as well as circulate in complex hexagonal and triangular patterns around point 13. The method of solution is again one of introducing suppressed points and utilizing the analyses of isolated systems which have already been achieved.

Thus, the internal series forming on the eight point set 5, 8, 9, 12, 13, 14, 18, and 19 can be isolated (Fig. 94a) by introducing zeros at the corner points 17 and 20. The function coefficients corresponding to the internal series are (Art. 5-4, Eq's. 37) :

$$Q_{13}^{13(I)} = \frac{1}{Z_{02}} \lambda \qquad Q_{12}^{13(I)} = \frac{A_{02}}{Z_{02}} \lambda = Q_{14}^{13(I)}$$

$$Q_{18}^{13(I)} = \frac{B_{02}}{Z_{02}} \lambda = Q_{19}^{13(I)} \qquad Q_{8}^{13(I)} = \frac{C_{02}}{Z_{02}} \lambda = Q_{9}^{13(I)}$$

$$Q_{5}^{13(I)} = \frac{D_{02}}{Z_{02}} \lambda \quad .$$

Removing the zeros at the corners and simultaneously suppressing the central point 13, a nine point triangular ring is isolated (Fig. 94 b) with the carried-over value

$$\alpha_0 = \frac{b\,A_{02}}{Z_{02}} \lambda + \frac{a\,B_{02}}{Z_{02}} \lambda = \frac{b}{(1-a)\,Z_{02}} \left[\frac{1+a}{Y'_{02}} - (1-a) \right] \lambda$$

at points 17 and 20. This isolated system is identical with Case I of the external series (Art. 5-5). The function values are

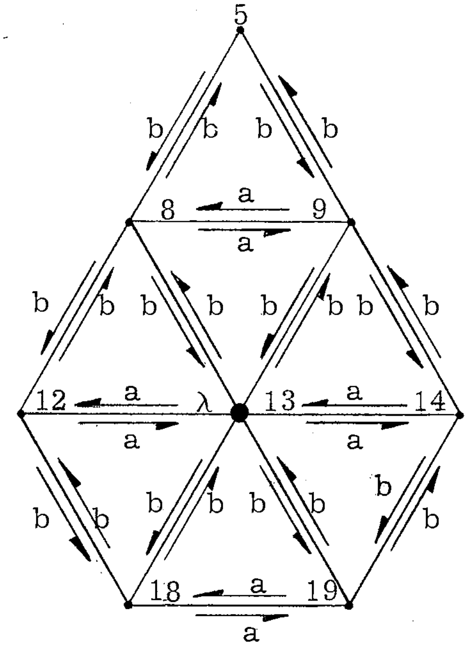


Fig. 94a
Isolated Eight Point Set - Internal Series

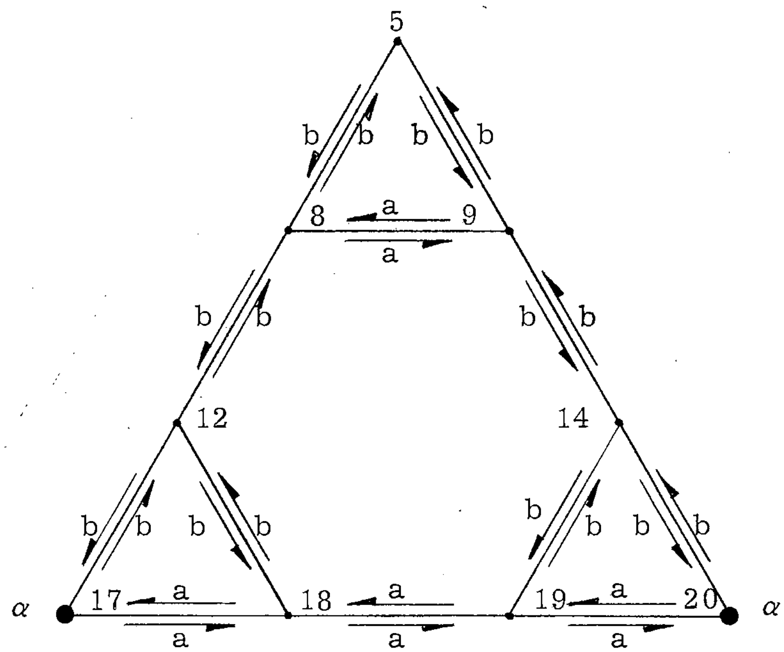


Fig. 94b
Isolated Nine Point Ring - External Series

$$Q_{17}^{13(E)} = \frac{1}{Z_{11}} \alpha_0 = Q_{20}^{13(E)} \quad \Big| \quad Q_{12}^{13(E)} = \frac{b}{(1-a)Y'_{02}Z_{11}} \alpha_0 = Q_{14}^{13(E)}$$

$$Q_8^{13(E)} = \frac{b^2}{(1-a)^2 X'_{02}Y'_{02}Z_{11}} \alpha_0 = Q_9^{13(E)}$$

$$Q_5^{13(E)} = \frac{2b^3}{(1-a)^2 X'_{02}Y'_{02}Z_{11}} \alpha_0$$

$$Q_{18}^{13(E)} = \frac{a}{(1-a)Z_{11}} \alpha_0 + \frac{b^2}{(1-a)^2 Y'_{02} Z_{11}} \alpha_0 = Q_{19}^{13(E)} .$$

The first cycle of carry-over is completed by releasing point 13 and finding the returned value:

$$Q_{13}^{13(CO)} = \frac{2b}{(1-a)Z_{11}} \left[\frac{1+a}{Y'_{02}} - (1-a) \right] \alpha_0$$

$$= \frac{2b^2}{(1-a)^2 Z_{02}Z_{11}} \left[\frac{1+a}{Y'_{02}} - (1-a) \right]^2 \lambda = \beta_1 .$$

The common ratio of the second order carry-over series interrelating the internal and the external series is therefore

$$\frac{\beta_1}{\beta_0} = \frac{\beta_1}{\lambda} = \frac{2b^2}{(1-a)^2 Z_{02} Z_{11}} \left[\frac{1+a}{Y'_{02}} - (1-a) \right]^2 .$$

Repeating this procedure infinite times, the carried-over values α and β form the infinite series of carry-over whose sums are

$$\sum_0^\infty \alpha_n = \alpha_0 + \alpha_1 + \dots = \frac{b}{(1-a)Z_{02}U_1} \left[\frac{1+a}{Y'_{02}} - (1-a) \right] \lambda$$

$$\sum_0^\infty \beta_n = \beta_0 + \beta_1 + \dots = \frac{1}{U_1} \lambda$$

where

$$U_1 = 1 - \frac{2b^2}{(1-a)^2 Z_{02} Z_{11}} \left[\frac{1+a}{Y'_{02}} - (1-a) \right]^2 .$$

Superimposing the α and β series, the final values for function coefficients on the twenty-eight point triangular set become

$$Q_{17}^{13} = \frac{1}{Z_{11}} \sum_0^\infty \alpha_n = \frac{E_1}{V_1} \lambda = Q_{20}^{13}$$

$$Q_5^{13} = \frac{2b^3}{(1-a)^2 X'_{02} Y'_{02} Z_{11}} \sum_0^\infty \alpha_n + \frac{D_{02}}{Z_{02}} \sum_0^\infty \beta_n = \frac{D_1}{V_1} \lambda$$

$$Q_8^{13} = \frac{b^2}{(1-a)^2 X'_{02} Y'_{02} Z_{11}} \sum_0^\infty \alpha_n + \frac{C_{02}}{Z_{02}} \sum_0^\infty \beta_n = \frac{C_1}{V_1} \lambda = Q_9^{13}$$

$$Q_{12}^{13} = \frac{b}{(1-a) Y'_{02} Z_{11}} \sum_0^\infty \alpha_n + \frac{A_{02}}{Z_{02}} \sum_0^\infty \beta_n = \frac{A_1}{V_1} \lambda = Q_{14}^{13} \qquad (38)$$

$$Q_{18}^{13} = \frac{1}{(1-a) Z_{11}} \left[a + \frac{b^2}{(1-a) Y'_{02}} \right] \sum_0^\infty \alpha_n + \frac{B_{02}}{Z_{02}} \sum_0^\infty \beta_n$$

$$= \frac{B_1}{V_1} \lambda = Q_{19}^{13}$$

$$Q_{13}^{13} = \frac{1}{Z_{02}} \sum_0^\infty \beta_n = \frac{1}{V_1} \lambda .$$

The new equivalents used in these equations are:

$$A_1 = A_{02} + \frac{b}{(1-a) Y'_{02}} E_1 \qquad B_1 = B_{02} + \frac{1}{1-a} \left[a + \frac{b^2}{(1-a) Y'_{02}} \right] E_1$$

$$C_1 = C_{02} + \frac{b^2}{(1-a)^2 X'_{02} Y'_{02}} E_1 \qquad D_1 = D_{02} + \frac{2b^3}{(1-a)^2 X'_{02} Y'_{02}} E_1$$

$$E_1 = \frac{b}{(1-a)\,Z_{11}}\left[\frac{1+a}{Y'_{02}} - (1-a)\right] \qquad \Bigg| \qquad V_1 = Z_{02}\,U_1 \quad .$$

A diagrammatic representation of the final results is given in Fig. 95. The constant $\frac{1}{V_1}$ is the over-relaxation factor for the second order carry-over series. The constants $A_1, B_1, C_1, D_1,$ and E_1 are the direct final carry-over factors.

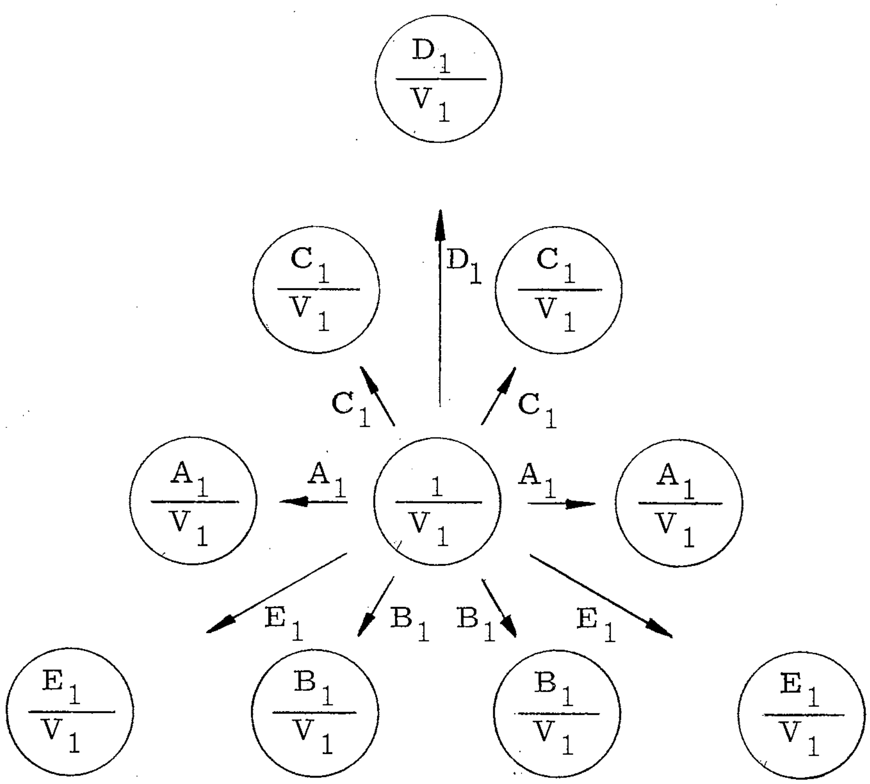
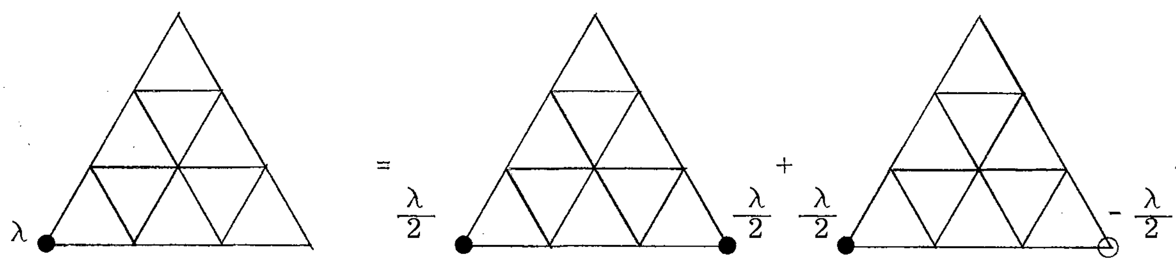


Fig. 95
Final Results - Second Order Carry-Over Series

<u>5-7 Resolution, Superposition, and Involution.</u> The analysis of the twenty-eight point set (Fig. 93) for a starting value λ at any other point can be accomplished by using the principles of resolution, superposition, and involution previously discussed (Art. 2-5).

For a starting value at point 17 the resolution is made as shown in Fig. 96. Case I is solved by superimposing the external series (Art. 5-5) and the second order carry-over series (Art. 5-6). Case II, from antisymmetry, reduces to the corresponding case of the external series (Fig. 91, Case II), as there is no carry-over into the central point 13.



Initial System — Case I — Case II

Fig. 96

Resolution of Twenty-Eight Point Triangular Set with Starting Value λ at Point 17 into Basic Cases

<u>Case I.</u> Temporarily suppressing the central point 13, a nine point triangular ring is isolated (Fig. 97) which is identical with Case I of the external series (Art. 5-5). The results are

$$Q_{17}^{17(E)I} = \frac{1}{Z_{11}} \frac{\lambda}{2} = Q_{20}^{17(E)I} \qquad Q_{12}^{17(E)I} = \frac{b}{(1-a)Y'_{02}Z_{11}} \frac{\lambda}{2} = Q_{14}^{17(E)I}$$

$$Q_{5}^{17(E)I} = \frac{2b^3}{(1-a)^2 X'_{02}Y'_{02}Z_{11}} \frac{\lambda}{2} \qquad Q_{8}^{17(E)I} = \frac{b^2}{(1-a)^2 X'_{02}Y'_{02}Z_{11}} \frac{\lambda}{2} = Q_{9}^{17(E)I}$$

$$Q_{18}^{17(E)I} = \frac{1}{(1-a)Z_{11}} \left[a + \frac{b^2}{(1-a)Y'_{02}} \right] \frac{\lambda}{2} = Q_{19}^{17(E)I} .$$

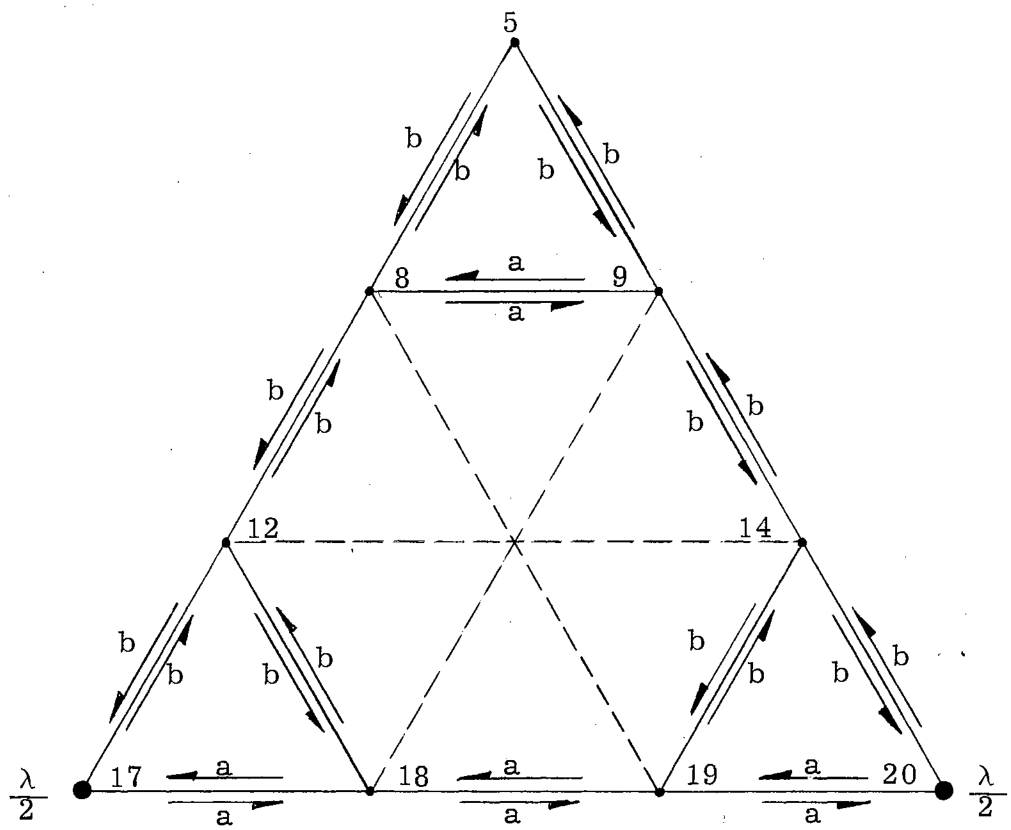


Fig. 97
Isolated Triangular Ring - Case I

Releasing point 13, the carried-over value

$$E_1 \lambda = \frac{2b}{(1-a)\,Z_{11}} \left[\frac{1+a}{Y'_{02}} - (1-a) \right] \frac{\lambda}{2}$$

becomes a new starting value which forms the central second order carry-over series (Art. 5-6). From Eq's. (38), the results due to this starting value are

$$Q_{17}^{17(C)I} = \frac{E_1^2}{V_1}\lambda = Q_{20}^{17(C)I}$$

$$Q_5^{17(C)I} = \frac{D_1 E_1}{V_1}\lambda$$

$$Q_8^{17(C)I} = \frac{C_1 E_1}{V_1}\lambda = Q_9^{17(C)I}$$

$$Q_{18}^{17(C)I} = \frac{B_1 E_1}{V_1}\lambda = Q_{19}^{17(C)I}$$

$$Q_{12}^{17(C)I} = \frac{A_1 E_1}{V_1}\lambda = Q_{14}^{17(C)I}$$

$$Q_{13}^{17(C)I} = \frac{E_1}{V_1}\lambda \quad .$$

The final results, starting value λ at point 7, are obtained by superimposing these values with those of the external series.

Considering now a starting value λ at point 12, the system is resolved into two basic cases as shown in Fig. 98 and the results superimposed.

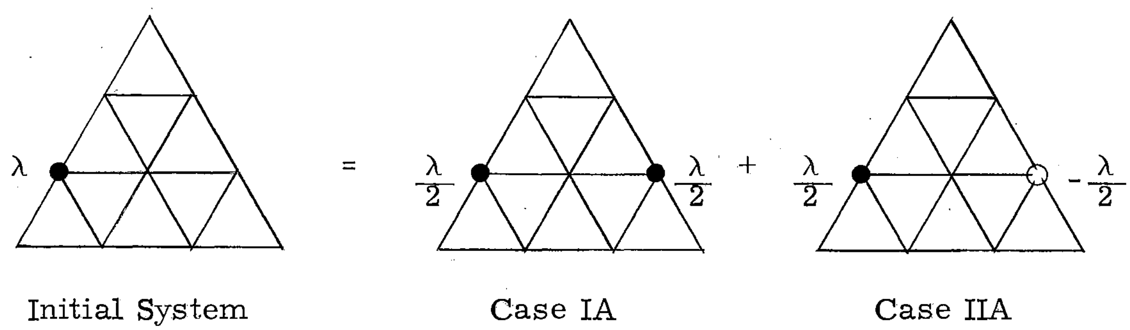


Fig. 98

Resolution of Twenty-Eight Point Triangular Set with Starting Value λ at Point 12 into Basic Cases

Case IA. This system is a modified form of Case I. Introducing over-relaxation factors and suppressing points 13, 17, and 20, the three point set (8, 12, 18) of Fig. 99 is isolated. The results of algebraic carry-over are

$$Q_{12}^{12(S)I} = \frac{1}{Y'_{02}} \frac{\lambda}{2} \qquad Q_8^{12(S)I} = \frac{b}{(1-a)X'_{02}Y'_{02}} \frac{\lambda}{2}$$

$$Q_{18}^{12(S)I} = \frac{b}{(1-a)Y'_{02}} \frac{\lambda}{2} \qquad Q_5^{12(S)I} = \frac{2b^2}{(1-a)X'_{02}Y'_{02}} \frac{\lambda}{2} \ .$$

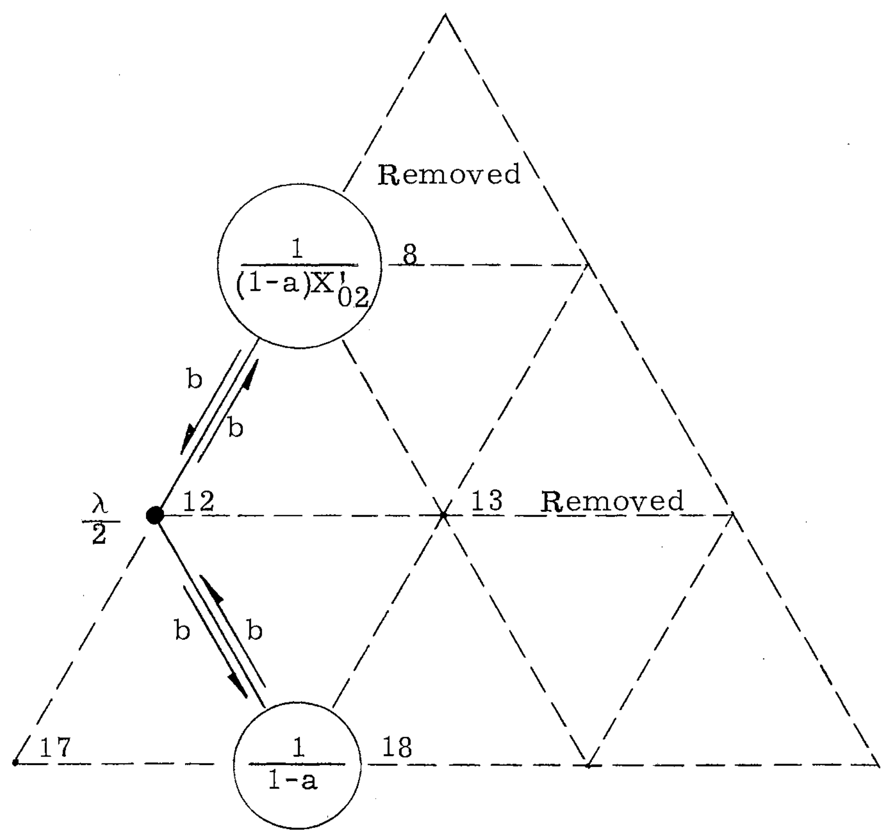


Fig. 99
Isolated Three Point Set
Case IA

Releasing points 13, 17, and 20, the carried-over values are

$$Q_{13}^{12(CO)} = 2\left(\frac{1+a}{Y'_{02}} - 1\right)\frac{\lambda}{2}$$

$$Q_{17}^{12(CO)} = \frac{b}{(1-a)Y'_{02}}\,\frac{\lambda}{2} = Q_{20}^{12(CO)} \quad .$$

The value at point 13 forms the second order carry-over series (Art. 5-6). The values at points 17 and 20 form series corresponding to Case I (starting value λ at point 17). Thus the function value at any point ij due to these involuted starting values is

$$Q_{ij} = \left(\frac{1+a}{Y'_{02}} - 1\right) Q_{ij}^{13} + \frac{b}{(1-a)Y'_{02}}\, Q_{ij}^{17(I)} \quad .$$

The superposition of values obtained from this equation with results of the isolated circulatory series above (Fig. 99) yields final values of function coefficients, Case IA.

Case IIA. From the antisymmetry of this system (Fig. 98), a modification of Case II, there is no carry-over into the central point 13. Introducing zero points at 17 and 20 and over-relaxation factors at 8 and 18, the three point set of Fig. 100 is isolated. Performing carry-over the results are

$$Q_{12}^{12(S)II} = \frac{1}{X''_{02}}\,\frac{\lambda}{2} \qquad \Big| \qquad Q_{8}^{12(S)II} = \frac{b}{(1+a)X''_{02}}\,\frac{\lambda}{2} = Q_{18}^{12(S)II} \quad .$$

Removing the zeros at points 17 and 20, the carried-over values

$$Q_{17}^{12(CO)} = \frac{(1+2a)\,b}{(1+a)X''_{02}}\,\frac{\lambda}{2} = Q_{20}^{12(CO)}$$

become involuted starting values and form series corresponding to Case II of the external series. The function coefficient at any point ij due to these starting values is

$$Q_{ij} = \frac{(1 + 2a)b}{(1 + a)X''_{02}} Q_{ij}^{17(II)} \quad .$$

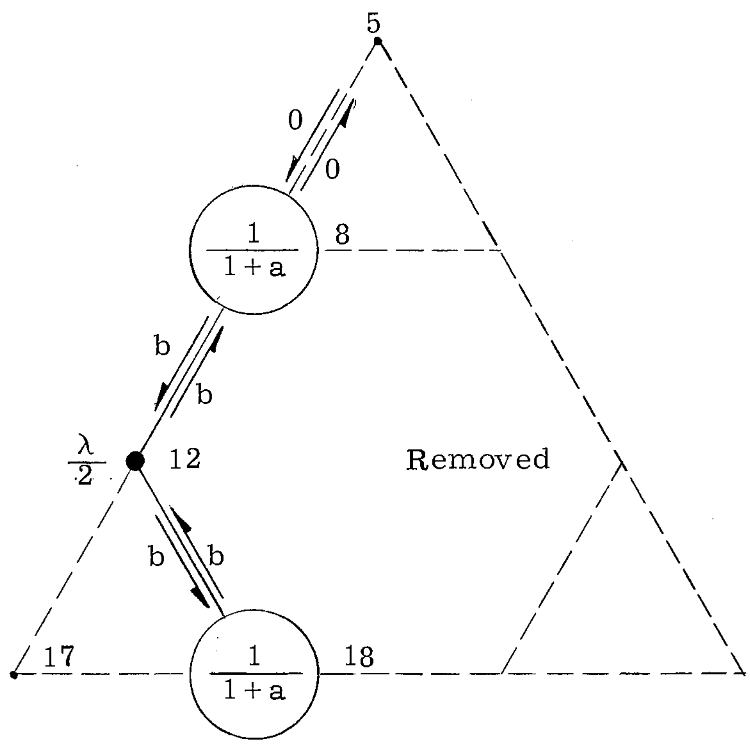


Fig. 100
Isolated Three Point Set - Case IIA

The final results, Case IIA, are thus

$$Q_{12}^{12(IIA)} = \frac{1}{X''_{02}} \frac{\lambda}{2} + \frac{(1 + 2a)b}{(1 + a)X''_{02}} Q_{12}^{17(II)} = - Q_{14}^{12(IIA)}$$

$$Q_{8}^{12(IIA)} = \frac{b}{(1 + a)X''_{02}} \frac{\lambda}{2} + \frac{(1 + 2a)b}{(1 + a)X''_{02}} Q_{8}^{17(II)} = - Q_{9}^{12(IIA)}$$

$$Q_{18}^{12(IIA)} = \frac{b}{(1+a)X''_{02}} \frac{\lambda}{2} + \frac{(1+2a)b}{(1+a)X''_{02}} Q_{18}^{17(II)} = -Q_{19}^{12(IIA)}$$

$$Q_{17}^{12(IIA)} = \frac{(1+2a)b}{(1+a)X''_{02}} Q_{17}^{17(II)} = -Q_{20}^{12(IIA)}$$

$$Q_{5}^{12(IIA)} = Q_{13}^{12(IIA)} = 0 \quad .$$

Final function values on the twenty-eight point set, starting value λ at 12, are obtained by superimposing the results of Cases IA and IIA.

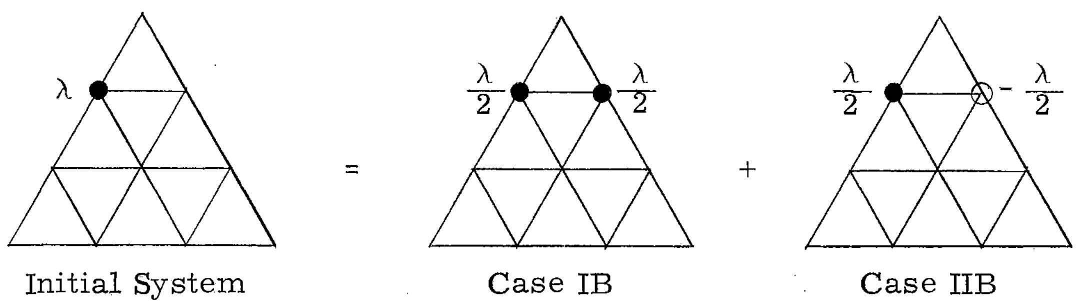


Fig. 101
Resolution of Twenty-Eight Point Triangular Set with Starting Value λ at Point 8 into Basic Cases

If now a starting value at point 8 is considered, a solution can be achieved by resolution (Fig. 101) followed by direct involution.

Case IB. This system is a modification of Case I. Introducing over-relaxation factors and modified carry-over factors as shown in Fig. 102, it is necessary to consider only those involuted starting values induced by the value $\frac{\lambda}{2}$ at point 8. The over-relaxation factor eliminates carry-over into points 5 and 9, and the values carried into points 12 and 13 develop series previously defined and determined.

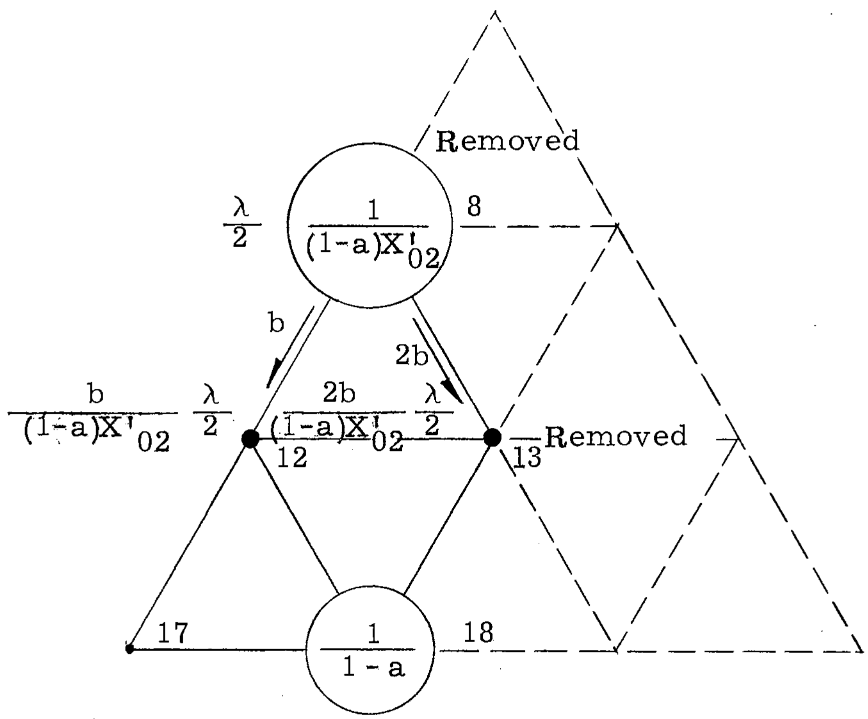


Fig. 102
Involuted Starting Values - Case IB

Superimposing these series and the over-relaxed starting values, the results are:

$$Q_5^{8(IB)} = \frac{2b}{(1-a)X'_{02}} \frac{\lambda}{2} + \frac{b}{(1-a)X'_{02}} Q_5^{13} + \frac{b}{(1-a)X'_{02}} Q_5^{12(IA)}$$

$$Q_8^{8(IB)} = \frac{1}{(1-a)X'_{02}} \frac{\lambda}{2} + \frac{b}{(1-a)X'_{02}} Q_8^{13} + \frac{b}{(1-a)X'_{02}} Q_8^{12(IA)}$$

$$= Q_9^{8(IB)}$$

and for any other point ij

$$Q_{ij}^{8(IB)} = \frac{b}{(1-a)X'_{02}} Q_{ij}^{13} + \frac{b}{(1-a)X'_{02}} Q_{ij}^{12(IA)} .$$

Case IIB. The modification of this system is shown in Fig. 103. The value at 8 carries-over to point 12, introducing an involuted starting value at that point which develops the series of Case IIA. From superposition the final values are

$$Q_8^{8(IIB)} = \frac{1}{1+a}\frac{\lambda}{2} + \frac{b}{1+a} Q_8^{12(IIA)} = -Q_9^{8(IIB)}$$

and for any other point

$$Q_{ij}^{8(IIB)} = \frac{b}{1+a} Q_{ij}^{12(IIA)} .$$

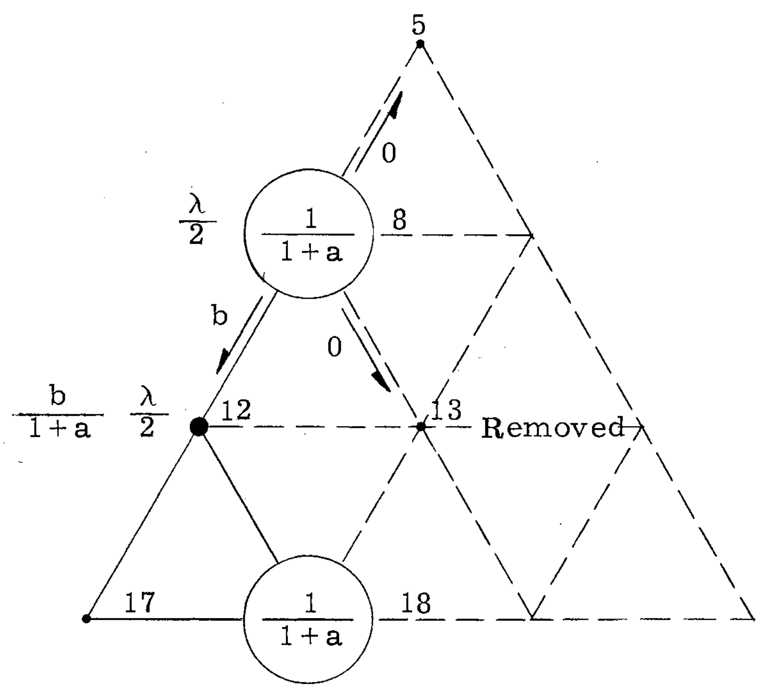


Fig. 103
Involuted Starting Values - Case IIB

Final results, starting value λ at point 8, are obtained by superimposing Cases IB and IIB.

For a starting value at some other point of the finite-difference net, the function values can be determined by resolution and involution in a similar manner.

5-8 The Laplace Equation. In triangular coordinates the Laplace equation has the form

$$\frac{\partial^2 Q}{\partial u^2} \operatorname{Sin} 2(\beta - \alpha) - \frac{\partial^2 Q}{\partial v^2} \operatorname{Sin} 2\beta + \frac{\partial^2 Q}{\partial w^2} \operatorname{Sin} 2\alpha = 0 \qquad (39)$$

with Q equal to a given function G(u, v, w) on the boundary (38).

For a symmetrical network ($\beta = \pi - \alpha$), the corresponding finite-difference equation written for an interior point ij is (Fig. 78)

$$Q_{ij} = \left\{ \begin{array}{c} a(Q_{i+1,j} + Q_{i-1,j}) \\ \\ b(Q_{i,j+1} + Q_{i,j-1} + Q_{i+1,j-1} + Q_{i-1,j+1} \end{array} \right\} \qquad (40)$$

At a boundary point kl, the function Q takes on the value of the given function G:

$$Q_{kl} = G_{kl} \quad . \qquad (41)$$

The boundary values are the starting values, and carry-over proceeds from the boundary into the interior of the net, as previously discussed in Art. 2-6. The finite-difference solution is again obtained from related solutions of the Poisson equation.

The interrelationship is established by considering the finite-difference equation written for a point ij adjacent to the boundary.

From Fig. 104 this equation is

$$Q_{ij} = \left\{ \begin{array}{c} a(Q_{i+1,j}) \\ \\ b(Q_{i,j+1} + Q_{i,j-1} + Q_{i+1,j-1}) \end{array} \right\} + \left\{ \begin{array}{l} a\,G_{i-1,j} \\ \\ b\,G_{i-1,j+1} \end{array} \right\} .$$

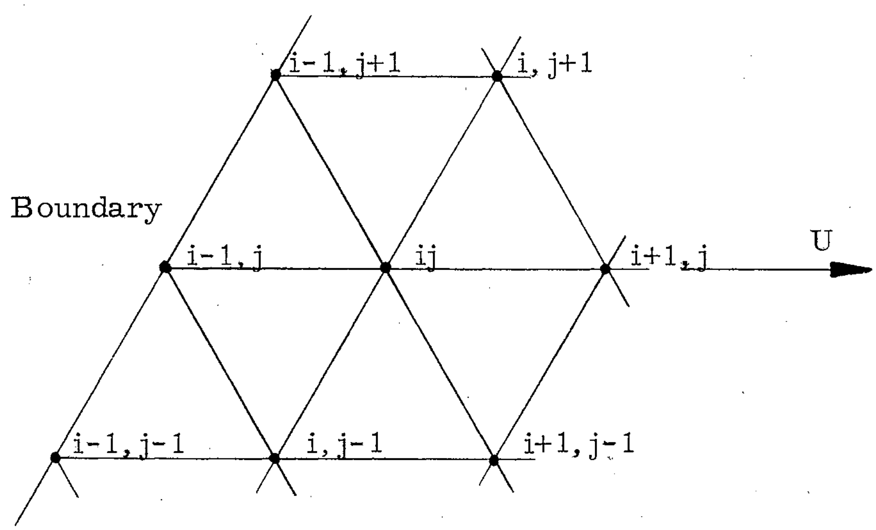


Fig. 104
Finite Difference Net Adjacent to the Boundary

Comparing this equation with the Poisson equation in finite-difference form (Eq. 35), it is evident that the sum of the carried-over values above ($a\,G_{i-1,j} + b\,G_{i-1,j+1}$) corresponds with the value Q_{ij}^*. This sum may thus be considered a new starting value at point ij and algebraic carry-over performed as before.

Final results for function coefficients due to a starting value λ at a boundary point of the triangular network are therefore equal to the sum of final results due to starting values λ at the adjacent interior points, multiplied by the corresponding carry-over factors.

CHAPTER VI

ANALYSIS OF ALGEBRAIC CARRY-OVER

6-1 Philosophy. The solution of simultaneous linear difference equations by the algebraic carry-over method is presented in this dissertation. From the literature survey it is concluded that algebraic carry-over is the only known approach to the problem which yields exact final values obtained by summing infinite geometric series.

The basic philosophy behind algebraic carry-over is the concept that certain types of matrices lend themselves to a solution by the series approach. These matrices need not be limited to finite-difference approximations of continuous systems, as considered here, but may represent the true sets of equations defining finite systems (eg. continuous beams or frames).

The method of procedure derives from a visualization of the problem on the finite network which serves as a model for the physical situation, as demonstrated in Chapters II through V, and the actual matrix is not directly investigated. It is nevertheless true that, as the carry-over proceeds, the matrix is divided into submatrices (corresponding to isolated point sets), and the submatrices are then interrelated (corresponding to carry-over series).

The steps of this procedure are:

(a) Selection of the point sets which can be isolated and solved by geometric series.

(b) Determination of the method of interrelating these isolated systems so as to again form geometric series, thus achieving the final solution of the problem.

6-2 Principles. The two fundamental principles of the algebraic carry-over method are the principle of the isolated system and the principle of interrelated systems.

I. The Principle of the Isolated System:

A portion of the network can be isolated and analyzed independently by surrounding it with suppressed points (points for which the function values are temporarily taken to be zero).

II. The Principle of Interrelated Systems:

Two or more isolated systems can be interrelated and analyzed by determining the carry-over values flowing between them.

These principles correspond with steps (a) and (b) above (Art. 6-1). The second principle, concerned with a final interrelation of portions of the network, may be considered the inverse of the first principle, which pertains to the initial isolation of these systems. Taken together, Principles I and II formulate the basic procedure of the algebraic carry-over method as presented in this dissertation.

There are in addition a number of other principles which facilitate the application of carry-over to the solution of finite-difference problems. The first of these are two broad and well known principles applicable to all linear systems: resolution and superposition. As used here these principles may be stated as follows.

(a) Resolution: The analysis of a geometrically arranged finite network with unsymmetrical starting values can be simplified by resolving the system into basic cases which take advantage of various combinations of symmetry and antisymmetry.

(b) Superposition: Final results for function values on the network due to the initial system of starting values are equal to the algebraic sums of the results corresponding to each of the resolved systems.

The second set of principles, corollaries of Principles I and II, are concerned with elimination of certain portions of the network from consideration in the carry-over procedure. The first of these, the concept of reduction, pertains to elimination by geometry, the second, that of over-relaxation, to elimination by removal.

(c) Reduction: Any finite network containing two symmetrical systems A, A', or two antisymmetrical systems A, A'', can be reduced to a modification of the single system A by incorporating properly modified carry-over factors.

(d) Over-relaxation: A portion of the network can be removed from the carry-over procedure by introducing the corresponding over-relaxation factor at the point connecting the removed system with the remainder of the network.

The final principle is that of involution, which enables the solution of one problem to be obtained from the solution of others.

(e) Involution: The function values on a finite network due to a starting value λ at some point kl are equal to the algebraic sums of the function values due to starting values

λ at points adjacent to kl multiplied by the corresponding carry-over factors.

6-3 Conclusions. The system of finite-difference equations corresponding to either the Poisson or the Laplace equation in rectangular, skew, polar, or triangular coordinates can be solved by summing infinite geometric series. Each final result is the algebraic sum of one or more infinite series whose terms are in themselves infinite series. The solution of Poisson's equation for a starting value at any point is achieved by resolving the corresponding network into simpler systems and superimposing the final results. Resolution is performed in such a way as to reduce each system to one solvable by interrelating its isolated parts. The solution of Laplace's equation for a prescribed boundary value is achieved by involving solutions of Poisson's equation through the method of carrying-over from the boundary.

Three classes of series are defined and used in the solutions: the basic series, the circulatory series, and the carry-over series. The basic and the circulatory series correspond with solutions of isolated point sets, step one of the carry-over procedure (Art. 6-1). The basic series is a simple geometric series forming on an internal set of points; the circulatory series is the sum of several series forming on one or more external closed rings. The carry-over series interrelates these two, and thereby corresponds with step two of the carry-over procedure (Art. 6-1). The internal and the external series, special higher order forms of the basic and circulatory series, are obtained in certain cases. All final values are expressed in terms of these various series.

6-4 Extensions and Applications. Having established the fact that solutions of finite-difference equations are obtainable and feasible by the algebraic carry-over method, the possible uses of these solutions can now be considered. Three important applications are immediately evident:

(a) The numerical error inherent in an approximate numerical solution of a finite-difference net is not present in an algebraic carry-over solution. Thus whenever an "exact" classical solution of the original differential equation is available for some special conditions, the true error involved in using networks of varying degrees of fineness can be determined. In this way, conclusions may be reached pertaining to the minimum number of points which should be considered in a given problem in order not to exceed the allowable range of error.

(b) In many cases, networks solvable by algebraic carry-over will yield solutions sufficiently accurate (in terms of the "exact" solution) that they can be evaluated for various values of the parameters involved (eg. length-width ratio and load position for rectangular plates) and useful tables prepared.

(c) Whenever function values on finer networks are required but complete solutions by algebraic carry-over are not available or feasible, the existing algebraic results can be used as an excellent set of initial approximations at the network points, and a rapidly convergent numerical iteration or relaxation procedure can be carried out.

These important applications are direct extensions of this dissertation and should be more fully investigated by research workers in the near future. The ideas presented in points (a) and (b) have already been adapted to the special problem of simply-supported rectangular plates.(42) A modification of point (c) was used by French (15) in applying a numerical carry-over procedure and incorporating over-relaxation factors and direct carry-over factors for basic point sets within the finite-difference network.

Finally, the results obtained in this dissertation suggest an even more important extension of the work that has been done: the concept that summing infinite geometric series may actually be the most natural approach to the solution of many classes of matrix equations. Selecting the network to fit a given system of equations and then visualizing the series which form on that network could well be the true physical-mathematical interpretation of this problem.

As only those types of matrices formed by five, seven, and nine term difference equations are considered here , it can not be concluded that the extension of algebraic carry-over to a general matrix would prove feasible. However, the relative simplicity with which solutions have been obtained and presented for the systems of equation investigated indicates the desirability of pursuing research along these lines. It might ultimately prove possible to classify matrices into basic types and demonstrate directly on each matrix the methods of isolation and interrelation which have been presented on the network.

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