ON ANALYTIC FUNCTIONS HAVING AS SINGULAR SETS CERTAIN CLOSED AND BOUNDED SETS

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PREFACE

The scope of this study is primarily concerned with the construction of analytic functions having, as singular sets, certain closed and bounded sets. In connection with the functions constructed, I show that they are: (1) analytic in the extended complex plane except at points of the given closed and bounded set, (2) single valued in the complement of this set, and (3) has each point of the given set as a singular point.

The ideas for this thesis evolved while I was a student in the Department of Mathematics at Oklahoma State University working mainly with Dr. O. H. Hamilton. I wish to express my gratitude to Dr. Hamilton for his sound and patient counsel, his helpful criticisms, and kind interest given me in the preparation of this thesis.

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CHAPTER I

INTRODUCTION

A. Statement of the Problem

V. V. Golubev, in his study, "Single Valued Analytic Functions with Perfect Singular Sets," (7, pp. 107-157),[1] constructed, by using definite integrals, single valued analytic functions having a perfect, nowhere dense set of singular points. In the attempt to extend his work to the problem of constructing, under very general conditions, analytic functions having a perfect, nowhere dense, singular set, he posed the following question: Given an arbitrary, perfect, nowhere dense point set E of positive Lebesgue one-dimensional measure in the complex plane; is it possible to construct, by passing a Jordan curve through E and by using definite integrals, a single valued function, analytic in the extended plane, which has E as its singular set? (7, pp. 128-129).

A more general problem with which this study is primarly concerned is that of constructing analytic functions having for their singular sets certain closed and bounded sets.

The present investigation is divided into three parts. In Chapter II, we shall require the set E to belong to the class of irregular

[1]Numbers in parentheses refer to the bibliography at the end of the paper.

point sets of finite (different from zero) Caratheodory linear measure.[2] We shall assume that E possesses Property A; that is, if p is any point of E, every neighborhood of p contains a subset of E whose Caratheodory linear measure is different from zero. Although point sets belonging to this class were not included in Golubev's investigation, from a set theoretic point of view, the present investigation and his are comparable. We shall obtain, by using definite integrals, a function $\phi(z)$ having the following properties:

(1) $\phi(z)$ is analytic in the extended complex plane except at points of E;

(2) $\phi(z)$ is single valued in the complement of E;

(3) Each point of E is an essential singularity of $\phi(z)$.

Such a function, as far as we have been able to determine, has not been constructed for any irregular set.

In Section 1 of Chapter II, we define, for functions bounded and measurable on E, an integral over E by using Caratheodory linear measure. In Section 2, employing the integral thus obtained, we define, in the complementary set, an analytic function by means of its integral over E. In Section 3, we generalize Golubev's technique of constructing a curvilinear integral of a function defined and continuous for

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[2]Point sets of finite (different from zero) linear measure are divided into two classes: the first, consisting of regular sets, and the second, irregular sets. Regular sets are analogous to rectifiable curves; irregular sets are dissimilar to regular sets in fundamental geometrical properties. (Cf. 1, pp. 424-426; 3, pp. 142-143). Throughout this study, the terms "measure" and "measurable" shall always be understood to mean "Caratheodory linear measure" and Caratheodory "linearly measurable" respectively.

a regular set E on a rectifiable curve, to the case where E is an irregular set, having Property A, on a non-rectifiable Jordan curve. We give, in Section 4(a), some properties of the curvilinear integral; and in 4(b), we establish the equivalence of the two types of integrals constructed on irregular sets.

In Chapter III, M is regarded as a bounded, non-degenerate, locally connected, plane continuum which does not separate the plane. We determine that there exists an analytic function $F(z)$ having M as its singular set by employing a new approach; that is, by making use of the mapping of the complement of M onto the interior of the unit circle by a simple analytic function. The analytic function $F(z)$ is thus defined without the help of integrals.

We summarize our findings and give recommendations for further study, in Chapter IV.

B. Definition of Terms

We give, in the following, definition of terms that are used in this study.

1. Let E be a plane set of points, and ρ an arbitrarily chosen positive number. Let $U_1(\rho,E)$, $U_2(\rho,E)$, ... , be a finite or denumerable sequence of open convex point sets which satisfies the following conditions:

   (a) Every point of E is an interior point of at least one of the sets U_1, U_2, ... ,

   (b) The diameter d_k of $U_k(\rho,E)$ is less than ρ for all values of k.

Denote by $U(\rho,E)$ the collection of points $U_1(\rho,E)$, $U_2(\rho,E)$, ... , and denote generally by d_i the diameter of the point set $U_i(\rho,E)$. Let L_ρ represent the greatest lower bound of the sum

$$\sum_{U(\rho,E)} d_i$$

for all possible coverings of E. As ρ decreases, L_ρ cannot decrease. Consequently,

$$\lim_{\rho \to 0} L_\rho = L^*(E)$$

always exists, finite or infinite. $L^*(E)$ will be called the Caratheodory exterior linear measure of E.

A set E will be called measurable if, for every set W of finite exterior linear measure, the relation

$$L^*(W) = L^*(E \cap W) + L^*(C(E) \cap W)$$

is satisfied. If the set E is measurable, we denote the number $L^*(E)$ by $L(E)$ and call it the Caratheodory linear measure of E.

2. Let E be a linearly measurable set, and let p be any point of the plane whether belonging to E or not. <u>The upper density</u> $D^*(p,E)$ and <u>the lower density</u> $D_*(p,E)$ of E at the point p will be defined as

$$\limsup_{r \to 0} \frac{L(E \cap c(p,r))}{2r}$$

and

$$\liminf_{r \to 0} \frac{L(E \cap c(p,r))}{2r}$$

respectively, where c(p,r) is a circle with center p and radius r. If $D^*(p,E)$ and $D_*(p,E)$ are equal, their common value will be denoted by D(p,E) and will be called the density of the set E at the point p.

3. A point p of a set will be called a regular point if the density, D(p,E), exists and is equal to unity. Otherwise, the point p will be called irregular (1, p. 424). If almost all points[3] of E are regular, the set itself will be called regular, (1, p. 424). If the subset of E consisting of irregular points is of positive Caratheodory linear measure, E will be said to be irregular.

4. A continuum is a compact,connected point set with at least two points.

5. A point set M is connected if and only if it cannot be represented as the sum $M_1 \cup M_2$ of two non-empty disjoint sets both of which are open relative to M or both of which are closed relative to M.

6. A non-null open connected set is called a domain.

7. A set of points M is bounded if the distances between pairs of points of M have a finite least upper bound.

8. A point set which contains all of its limit elements is closed.

9. An open curve is a locally compact continuum which is separated into two connected point sets by the omission of any of its points.

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[3]"Almost all" is used here to mean "except at points of a set of linear measure zero."

C. Review of the Literature

The current problem is one that has evolved as a result of investigations made by various authors. D. Pompeiu (13, pp. 914-915) was the first to exhibit an interest in constructing, with the help of definite integrals, an analytic function having a perfect, nowhere dense, bounded set of essential singular points. He proved that there exist a set E of two dimensional positive Lebesgue measure, and a function continuous and analytic in the extended plane with singular points in E.

Employing definite integrals, A. Denjoy (6, pp. 258-260) showed the existence of a single valued function, analytic in the extended plane, having a perfect, nowhere dense set E of essential singularities of one dimensional positive Lebesgue measure in the linear interval $0 \leq x \leq 1$.

Golubev (7, p. 122) extended Denjoy's result to the case in which E was a perfect, nowhere dense set of one dimensional positive Lebesgue measure on a rectifiable curve L. He formed the function

$$f(z) = \int_E \frac{dt}{t - z} = \int_a^b \frac{dt}{t - z} - \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \frac{dt}{t - z}$$

where (a_n, b_n) are interval components of L whose union is the complement of E on L, a_n an element of L, b_n an element of L, $a_n < b_n$ for each n, t being any point of E, and z a fixed point not on L, where " < " means " precedes " in a particular order. The line integrals

$$\int_a^b \frac{dt}{t - z} \quad \text{and} \quad \int_{a_n}^{b_n} \frac{dt}{t - z},$$

which are dependent upon the particular rectifiable curve L, are taken in the Lebesgue sense.

He investigated the case in which a perfect, nowhere dense point set E of positive one-dimensional Lebesgue measure is located on a Jordan arc C, $x = x(t)$ and $y = y(t)$, and established a correspondence between E and a perfect nowhere dense set E_t located on the t-axis. Golubev considered further the integral of a function $\phi(t)$, defined and continuous, for t in E_t and constructed a single valued analytic function having E as its singular set. Using the construction

$$f(z) = \int_{E_t} \frac{\phi(t)dt}{x(t) - z}$$

thus obtained, he disclosed that this representation of the function $f(z)$, in contrast with previous analyses, was burdened with one defect which considerably decreased the value of such a representation. The set E_t, located on the real t-axis, and upon which $\phi(t)$ depended, was not related closely enough to the set E on the Jordan arc C to permit one to infer significant properties of $f(z)$ from the analytic expression which represented it. (7, pp. 127-129).

CHAPTER II

ANALYTIC FUNCTIONS WITH AN IRREGULAR SET OF SINGULAR POINTS OF POSITIVE CARATHEODORY LINEAR MEASURE

Section 1. Integral Representation. We consider, in the real plane, an irregular, closed and bounded point set E. Let p denote any point of E, and f(p) a single valued, real valued function of a point defined, bounded and measurable on E with respect to Caratheodory linear measure. A function f(p) is said to be measurable if for each $\mu > 0$, the set $E(f > \mu)$ has Caratheodory linear measure.

We insert between the upper bound M and the lower bound m of f(p) the following numbers:

$$\mu_o \leq u_1 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \mu_n.$$
$$(\mu_o = m, \mu_n = M)$$

Let ϵ be greater than zero, and let these n divisions of the range of f(p) be such that the greatest of these parts $\mu_i - \mu_{i-1}$, for i = 1, 2, ... , n is less than ϵ.

Let E_i be the subset of E consisting of those points of E for which $\mu_{i-1} \leq f(p) < \mu_i$. Denote by $g_\epsilon(p)$ the function which has the value μ_{i-1} at all points of E_i, for i = 1, 2, ... , n; let $h_\epsilon(p)$ be the function which has the value μ_i at all points of E_i, for i = 1, 2, ... , n; where ΔE_i is the Caratheodory linear measure of E_i; and let

$$\int_E g_\epsilon(p) du = \sum_{i=1}^{n} \mu_{i-1} \Delta E_i \qquad (1.1)$$

and

$$\int_E h_\epsilon(p)du = \sum_{i=1}^{n} \mu_i \Delta E_i. \tag{1.2}$$

Then

$$0 \leq \int_E h_\epsilon(p)du - \int_E g_\epsilon(p)du < \epsilon \sum_{i=1}^{n} \Delta E_i = \epsilon L(E), \tag{1.3}$$

where L(E) is the Caratheodory linear measure of E, since E is linearly measurable.

Let, now, the range of f(p) be successively subdivided by introducing further points of division such that the corresponding values of ϵ form a sequence (ϵ_m) such that ϵ_m approaches zero as m approaches infinity. The set of numbers

$$\int_E g_{\epsilon_m}(p)du \quad \text{and} \quad \int_E h_{\epsilon_m}(p)du$$

are both bounded and monotone, the first being monotone increasing and the second, monotone decreasing. As a result, these two sets of numbers converge to a common limit as m increases without limit. This limit,

$$\lim_{m \to \infty} \int_E g_{\epsilon_m}(p)du = \lim_{m \to \infty} \int_E h_{\epsilon_m}(p)du \tag{1.4}$$

is defined to be the value of the integral

$$\int_E f(p)du$$

of f(p) taken over E.

We show that the value of the limit is independent of the particular mode in which the range of f(p) has been successively subdivided.

Let $\overline{g}_e(p)$ and $\overline{h}_e(p)$ be functions which correspond, in a second mode of subdivision, to $g_e(p)$ and $h_e(p)$. We superimpose these two subdivisions of the range of $f(p)$ and let $\overline{\overline{g}}_{e_m}(p)$ be the function defined with respect to this new subdivision as $g_{e_m}(p)$ is defined above. We have

$$0 \leq \int_E \overline{\overline{g}}_{e_m}(p)du - \int_E g_{e_m}(p)du < e_m L(E) \qquad (1.5)$$

$$0 \leq \int_E \overline{\overline{g}}_{e_m}(p)du - \int_E \overline{g}_{e_m}(p)du < e_m L(E) \qquad (1.6)$$

To show the first inequality, we note that in the finer subdivision of the range of $f(p)$ the difference between the maximum M and the minimum m of $f(p)$ in a given interval d of the subdivision is less than e_m since this difference was less than e_m on the larger intervals of which d is a subset. $\overline{\overline{g}}_{e_m}(p)$ and $g_e(p)$ are values of the function on the interval d and hence have a difference less than $|M - m|$ which is less than e_m. By reasoning similarly with $\overline{\overline{g}}_{e_m}(p)$ and $\overline{g}_{e_m}(p)$ we show the second inequality. Consequently,

$$\left| \int_E g_{e_m}(p)du - \int_E \overline{g}_{e_m}(p)du \right| < e_m L(E). \qquad (1.7)$$

As m approaches infinity, $e_m L(E)$ approaches zero. Therefore,

$$\lim_{m \to \infty} \int_E g_{e_m}(p)du = \lim_{m \to \infty} \int_E \overline{g}_{e_m}(p)du. \qquad (1.8)$$

The same reasoning applies to the functions $\overline{\overline{h}}_{e_m}(p)$, $h_{e_m}(p)$ and $\overline{h}_{e_m}(p)$.

We establish the following Lemmas. They will be used in connection with the proof of Theorems appearing later in this study.

Lemma I. *If $E = E_1 \cup E_2$ where E_1 and E_2 are two disjoint subsets of E, E, E_1 and E_2 being each Caratheodory linearly measurable, and, f(p) is measurable and bounded on E, E_1, and E_2, then,*

$$\int_E f(p)du = \int_{E_1} f(p)du_1 + \int_{E_2} f(p)du_2.$$

Proof: Each of the above integrals exist since f(p) is measurable and bounded on E, E_1, and E_2. From the definition of the integral,

$$\int_E f(p)du = \lim_{m \to \infty} \int_E g_{e_m}(p)du.$$

In like manner

$$\int_{E_1} f(p)du_1 + \int_{E_2} f(p)du_2 = \lim_{m \to \infty} \int_{E_1} g_{1e_m}(p)du_1 + \lim_{m \to \infty} \int_{E_2} g_{2e_m}(p)du_2, \quad (1.9)$$

where the limits are independent of the particular mode in which the range of f(p) has been successively subdivided. $g_{e_m}(p)$ is that function which has the value u_{i-1} on E_i, $i = 1, 2, \dots, m$: $g_{1e_m}(p)$ and $g_{2e_m}(p)$ are functions which have the value u_{i-1} on E_{1i} and E_{2i} respectively.

Consequently,

$$\int_{E_1} g_{1e_m}(p)du = \sum_{i=1}^{m} u_{i-1}\Delta E_{1i} \quad (1.10)$$

$$\int_{E_2} g_{2e_m}(p)du = \sum_{i=1}^{m} u_{i-1}\Delta E_{2i} \quad (1.11)$$

Now in defining

$$\int_E f(p)du, \quad \int_{E_1} f(p)du_1, \text{ and } \int_{E_2} f(p)du_2,$$

we use the same subdivisions of the range of f(p). Therefore, $g_{1e_m}(p)$

and $g_{2e_m}(p)$ differ only in notations from $g_{e_m}(p)$, that is, for every $g_{1e_m}(p)$ on E_1, there corresponds the same function $g_{e_m}(p)$ on E. In like manner, the same reasoning applies to $g_{2e_m}(p)$ on E_2. Correspondingly, $g_{1e_m}(p)$ and $g_{2e_m}(p)$ can be replaced by $g_{e_m}(p)$. Forming the sum of the integrals,

$$\int_{E_1} f(p)du_1 \quad \text{and} \quad \int_{E_2} f(p)du_2$$

and making use of the foregoing, we have,

$$\int_{E_1} f(p)du_1 + \int_{E_2} f(p)du_2 = \lim_{m \to \infty} \int_{E_1} g_{1e_m}(p)du_1 + \lim_{m \to \infty} \int_{E_2} g_{2e_m}(p)du_2$$

$$= \lim_{m \to \infty} \int_E g_{e_m}(p)du = \int_E f(p)du \qquad (1.12)$$

since $L(E) = L(E_1) + L(E_2)$, where $L(E)$, $L(E_1)$, and $L(E_2)$ denote the Caratheodory linear measure of E, E_1, and E_2 respectively.

A similar determination can be achieved using $\int_E h_{e_m}(p)$. But

$$\lim_{m \to \infty} \int_E h_{e_m}(p)du = \lim_{m \to \infty} \int_E g_{e_m}(p)du = \int_E f(p)du.$$

This Lemma may be extended, by induction, to the case where E is the sum of any finite number of disjoint point sets.

Lemma II. _If_ f(p) _is unity_, _then_ $\int_E du = L(E)$, _the Caratheodory linear measure of_ E.

Proof: Since f(p) is bounded and measurable on E having respectively M and m as its least upper and greatest lower bounds,

$$\int_E m\,du \le \int_E f(p)du \le \int_E M du. \qquad (1.13)$$

From the definition of the integral, we have

$$\sum_{i=1}^{m} m\Delta E_i \leq \int_E g_{e_m}(p)du \leq \int_E h_{e_m}(p)du \leq \sum_{i=1}^{m} M\Delta E_i, \qquad (1.14)$$

where ΔE_i denotes the Caratheodory linear measure of E_i. In the special case in which $m = M = 1$, $g_{e_m}(p) = h_{e_m}(p) = 1$ for every m. Upon passing to the limit as in (1.4), we have $f(p) = 1$, and consequently,

$$\int_E du = L(E)$$

Lemma III. $$\int_E kf(p)du = k\int_E f(p)du,$$

<u>that is, a constant factor may be placed before the integral sign.</u>

Proof: In relations (1.1) and (1.2), we replace the factors u_{i-1} and u_i by ku_{i-1} and ku_i respectively. For each m, $\int_E kg_{e_m}(p)du = k\int_E g_{e_m}(p)du$.

From the laws of operation with limits, we have

$$\lim_{m \to \infty} \int_E kg_{e_m}(p)du = k\lim_{m \to \infty} \int_E g_{e_m}(p)du \qquad (1.15)$$

A similar relation holds for $\lim_{m \to \infty} \int_E kh_{e_m}(p)du$. The final result now follows from the definition of the integral.

Lemma IV. $$\left|\int_E f(p)du\right| \leq \int_E |f(p)|du \leq ML(E),$$

<u>where M denotes the maximum value of f(p) over E and L(E) the Caratheodory linear measure of E.</u>

Proof: $f(p)$ is defined on E. Then $f_+(p)$ and $f_-(p)$ are defined on E as follows:

$$f_+(p) = \begin{cases} f(p) \text{ if } f(p) > 0. \\ 0 \text{ otherwise.} \end{cases} \qquad f_-(p) = \begin{cases} -f(p) \text{ if } f(p) < 0. \\ 0 \text{ otherwise.} \end{cases}$$

$f_+(p) = \max(f(p),0)$ and $f_-(p) = -\min(f(p),0)$. Since $f(p)$ is bounded and Caratheodory linearly measurable on E, $f_+(p)$ and $f_-(p)$ are likewise bounded and Caratheodory linearly measurable on E. The integral of f(p) over E, in terms of $f_+(p)$ and $f_-(p)$, is thus defined by

$$\int_E f(p)du = \int_E f_+(p)du - \int_E f_-(p)du. \qquad (1.151)$$

Now let $E_1 \subset E$ be the set on which $f(p) \geq 0$ and $E_2 \subset E$ be the set on which $f(p) < 0$. We have

$$\int_E f_+(p)du = \int_{E_1} f_+(p)du = \int_{E_1} |f(p)|du, \qquad (1.152)$$

and similarly,

$$\int_E f_-(p)du = \int_{E_2} f_-(p)du = \int_{E_2} |f(p)|du. \qquad (1.16)$$

From (1.152) and (1.16)

$$\int_E |f(p)|du = \int_E f_+(p)du + \int_E f_-(p)du, \qquad (1.17)$$

since $E = E_1 \cup E_2$. From (1.151) and (1.17), it follows that,

$$ML(E) \geq \int_E |f(p)|du \geq \left|\int_E f_+(p)du - \int_E f_-(p)du\right| = \left|\int_E f(p)du\right| \qquad (1.18)$$

Lemma V. <u>If</u> f(p) <u>and</u> g(p) <u>are two single valued, real valued, functions of a point defined, bounded, and measurable on</u> E, <u>then,</u>

$$\int_E (f(p) + g(p))du = \int_E f(p)du + \int_E g(p)du$$

Proof: Each of the given integrals exist; for since f(p) and g(p) are each bounded and measurable on E, then (f(p) + g(p)) is bounded and measurable on E. We show that the integral in the left member of the equality is equal to the two integrals in the right member.

First, we consider $\int_E (f(p) + A)du$ where A is a constant. From the definition of the integral,

$$\int_E (f(p) + A)du = \lim_{m \to \infty} \int_E g_{e_m}(p)du + \lim_{m \to \infty} \sum_{i=1}^{m} A\Delta E_i. \quad (1.19)$$

Upon passing to the limit as in (1.4),

$$\int_E (f(p) + A)du = \int_E f(p)du + \int_E A du = \int_E f(p)du + AL(E). \quad (1.20)$$

We next consider the integral of the sum of two functions, $(f(p) + g(p))$, each being bounded and measurable on E. Let E be decomposed into n disjoint measurable subsets E_i corresponding to functional values $u_{i-1} \leq f(p) < u_i$, where $i = 1, 2, \ldots, n$. We have,

$$\int_E (f(p) + g(p))du \geq \sum_{i=1}^{n} \int_{E_i} (u_{i-1} + g(p))du_i, \quad (1.21)$$

and,

$$\int_E (f(p) + g(p))du \leq \sum_{i=1}^{n} \int_{E_i} (u_i + g(p))du_i. \quad (1.22)$$

But by (1.20),

$$\int_{E_i} (u_{i-1} + g(p))du_i = \int_{E_i} u_{i-1}du_i + \int_{E_i} g(p)du_i,$$

and,

$$\int_{E_i} (u_i + g(p))du_i = \int_{E_i} u_i du_i + \int_{E_i} g(p)du_i,$$

for each i. Making use of Lemma I, we have, from (1.21), (1.22), and (1.1),

$$\int_E g(p)du + \int_E g_e(p)du \leq \int_E (f(p) + g(p))du \leq \int_E g(p)du + \int_E h_e(p)du \quad (1.23).$$

As the range of $f(p)$ becomes successively subdivided by introducing further points of division such that the corresponding values of e form a sequence $(e_m) \to 0$, the set of numbers $\int_E g_{e_m}(p)du$ and $\int_E h_{e_m}(p)du$ converge to the common limit $\int_E f(p)du$ as m increases without limit. It follows that $\int_E (f(p) + g(p))du = \int_E f(p)du + \int_E g(p)du$.

Section 2. Analytic Functions Defined by Means of An Integral Over its Singular Set. The integral of f(p) is a real number that depends upon the point set E. In order to extend the integral to the complex domain, we consider a complex valued

$$F(p) = f(p) + ig(p) \tag{2.1}$$

where f(p) and g(p) are real valued functions, defined, bounded, and measurable over E with respect to Caratheodory linear measure. Then,

$$\int_E F(p)du = \int_E f(p)du + i \int_E g(p)du \tag{2.2}$$

is a complex number that depends upon the set E.

We consider now the complex valued function

$$\phi(z) = \int_E F(p,z)du \tag{2.3}$$

defined by a definite integral which contains in the integrand a parameter z, if F(p,z) is a single valued function defined, bounded, and measurable when p lies in E and z is a fixed point in the complementary set, C(E). We now establish Lemma VI which will aid in proving I.

Lemma VI. If E has Caratheodory linear measure, and F(p,z) is continuous, bounded, and measurable for z in C(E), and F(p,z) possesses partial derivatives $F_x(p,z) = U_x + iV_y$ and $F_y(p,z) = U_y + iV_y$, continuous for every p in E and z in C(E), then,

$$\phi(z) = \int_E F(p,z)du$$

exists for each z in C(E) and possesses derivatives with respect to x and y continuous in C(E); namely,

$$\phi_x(z) = \int_E F_x(p,z)du$$

and

$$\phi_y(z) = \int_E F_y(p,z)du$$

Proof: Let $F(p,z) = U(p,x,y) + iV(p,x,y)$. Then

$$\frac{\Delta\phi}{\Delta x} = \frac{1}{\Delta x}\int_E (U(p,x+\Delta x,y) - U(p,x,y) + i(V(p,x+\Delta x,y) - V(p,x,y)))du. \quad (2.4)$$

$$\frac{d\phi}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_E (U(p,x+\Delta x,y) - U(p,x,y) + i(V(p,x+\Delta x,y) - V(p,x,y)))du.$$

(2.5) $U(p,x,y)$ and $iV(p,x,y)$ have continuous derivatives with respect to x and y by hypothesis. In order to apply the Theorem of Mean Value to the above equality, we consider a closed interval $a \leq x \leq b$ on the x axis. For any x in this closed interval,

$$\frac{d\phi}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_E (U_x(p,x+\theta_1\Delta x,y)\Delta x + iV_x(p,x+\theta_2\Delta x,y)\Delta x)du, \quad (2.6)$$

where θ_1 and θ_2 are positive real numbers each numerically less than one. Since, by assumption, $U_x(p,x,y)$ and $V_x(p,x,y)$ are jointly continuous in p, x, and y, the coefficients of Δx in (2.6) will approach $U_x(p,x,y)$ and $V_x(p,x,y)$ as limits when Δx approaches zero as a limit. Hence, if e_1 and e_2 are infinitesmals such that

$$\lim_{\Delta x \to 0} e_1 = \lim_{\Delta x \to 0} e_2 = 0,$$

we may write

$$U_x(p,x+\theta_1\Delta x,y) = U_x(p,x,y) + e_1,$$
$$V_x(p,x+\theta_2\Delta x,y) = V_x(p,x,y) + e_2 .$$

Hence,

$$\frac{d\phi}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_E (U_x(p,x,y)\Delta x + iV(p,x,y)\Delta x + e_1\Delta x + e_2\Delta x)du. \quad (2.7)$$

Therefore, given an e greater than zero, there exists a δ greater than zero, such that if $\Delta x < \delta$, then

$$e_1 < \frac{e}{2L(E)} \text{ and } e_2 < \frac{e}{2L(E)}.$$

We have

$$\frac{1}{\Delta x}\int_E ((U_x(p,x,y) + iV_x(p,x,y))\Delta x + e_1\Delta x + e_2\Delta x)du = \int_E (U_x(p,x,y) + iV_x(p,x,y) + e_1 + e_2)du. \quad (2.8)$$

Now for $\Delta x < \delta$,

$$\left|\frac{d\phi}{\Delta x} - \int_E F_x(p,z)du\right| = \left|\frac{1}{\Delta x}\int_E ((U_x(p,x,y) + iV_x(p,x,y))\Delta x + e_1\Delta x + e_2\Delta x)du - \int_E F_x(p,z)du\right| = \left|\int_E \left(\frac{(U_x(p,x,y) + iV_x(p,x,y))\Delta x}{\Delta x} + e_1 + e_2\right)du - \int_E (U_x(p,x,y) - iV_x(p,x,y))du\right| = \int_E (e_1 + e_2)du \leq \int_E \left(\frac{e}{2L(E)} + \frac{e}{2L(E)}\right)du = \frac{e}{2L(E)}\int_E du + \frac{e}{2L(E)}\int_E du = e/2 + e/2 = e,$$ by Lemmas I, II, and III. (2.9)

But e is arbitrary, and hence,

$$\phi_x(z) = \int_E F_x(p,z)du. \quad (2.10)$$

By reasoning similarly, we show that

$$\phi_y(z) = \int_E F_y(p,z)du. \quad (2.11)$$

This proves the Lemma.

We now establish the following fundamental integral theorem:

THEOREM I. <u>If</u> $F(p,z)$ <u>is a continuous, bounded, and measurable function of p for a fixed z and continuous in z, the function</u>

$$\phi(z) = \int_E F(p,z)du$$

is continuous in $C(E)$.

Moreover, if $F(p,z)$ has for each z a derivative $F_z(p,z)$ continuous in p and z together, the function $\phi(z)$ is analytic in $C(E)$; that is,

$$\phi'(z) = \int_E F_z(p,z)du$$

Proof: In order to prove the first part of the Theorem, we form the difference

$$\phi(z + \Delta z) - \phi(z) = \int_E (F(p, z + \Delta z) - F(p,z))du. \qquad (2.12)$$

Let ϵ be greater than zero. If $F(p,z)$ is continuous in p and z for every p in E and z in $C(E)$, there exists a δ such that for Δz sufficiently small,

$$|F(p,z + \Delta z) - F(p,z)| < \frac{\epsilon}{L(E)}$$

for $|\Delta z| < \delta$ and for all p and z. Therefore, by Lemmas II and III,

$$|\phi(z + \Delta z) - \phi(z)| = \int_E |F(p,z + \Delta z) - F(p,z)|du < \frac{\epsilon}{L(E)} \int_E du = \epsilon. \qquad (2.13)$$

Since $\epsilon/L(E)$ may be made as small as desired for ϵ sufficiently small, the continuity of $\phi(z)$ is assured under the conditions of the Theorem.

A necessary and sufficient condition for $F(p,z)$ to have a partial derivative $F_z(p,z)$ for each $z = x + iy$ is that $F_x(p,z)$ and $F_y(p,z)$ exist, be continuous, and satisfy the Cauchy-Riemann differential equations. Hence, from the existence and continuity of the partial derivative $F_z(p,z)$, there results the continuity of the partial derivatives $F_x(p,z)$ and $F_y(p,z)$.

Let $F(p,z) = U(p,x,y) + iV(p,x,y)$. Since $F(p,z)$ is analytic in the complementary set, the Cauchy-Riemann differential equations are satisfied; that is, $U_x(p,x,y) = V_y(p,x,y)$, and, $U_y(p,x,y) = -V_x(p,x,y)$. Also, let $\phi(z) = u(x,y) + iv(x,y)$. Then,

$$\phi(z) = \int_E F(p,z)du = \int_E U(p,x,y)du + i \int_E V(p,x,y)du,$$

where

$$u(x,y) = \int_E U(p,x,y)du \text{ and } v(x,y) = \int_E V(p,x,y)du.$$

Hence,

$$u_x(x,y) = \int_E U_x(p,x,y)du = \int_E V_y(p,x,y)du = v_y(x,y) \tag{2.14}$$

by Lemma VI. Also,

$$u_y(x,y) = \int_E U_x(p,x,y)du = -\int_E V_x(p,x,y)du = -v_x(x,y) \tag{2.15}$$

for the same reason. Therefore, the Cauchy-Riemann differential equations are satisfied for $\phi(z)$, and, this function is analytic in $C(E)$; that is,

$$\phi'(z) = \int_E F_z(p,z)du. \tag{2.16}$$

This proves the second part.

Let us now apply the theorem and the lemmas proved above to the construction of a function $\phi(z)$ having a given irregular set E of positive linear measure as its singular set. Accordingly, let

$$F(p,z) = \frac{1}{p - z}$$

where z is any element in $C(E)$, and construct the function

$$\phi(z) = \int_E \frac{du}{p - z}.$$

This integral exists and is analytic by Lemma VI and Theorem I. $\phi(z)$ is an analytic function. Its derivative is given by the formula

$$\phi'(z) = \int_E \frac{du}{(p - z)^2}; \tag{2.17}$$

for let z belong to $C(E)$, we form the difference quotent

$$\frac{\phi(z + \Delta z) - \phi(z)}{\Delta z} = \int_E \left(\frac{1}{p - z - \Delta z} - \frac{1}{p - z}\right)\frac{du}{\Delta z} = \int_E \frac{du}{(p - z - \Delta z)(p - z)}. \quad (2.18)$$

The limit of this integrand is

$$\frac{1}{(p - z)^2}.$$

We show that the limit of the integral is

$$\int_E \frac{du}{(p - z)^2}.$$

Consider,

$$\int_E \frac{du}{(p - z - \Delta z)(p - z)} = \int_E \frac{du}{(p - z)^2} + \Delta z \int_E \frac{du}{(p - z - \Delta z)(p - z)^2}. \quad (2.19)$$

The second integral in the right member of (2.19) is bounded; for let d be the minimum distance from z to E, then,

$$\left|\frac{1}{(p - z - \Delta z)(p - z)^2}\right| < \frac{1}{(d - h)d^2}$$

if h is chosen so that $0 < h < d$, and, $|\Delta z| < h$. Hence,

$$\Delta z\left|\int_E \frac{du}{(p - z - \Delta z)(p - z)^2}\right| \leq \Delta z \int_E \left|\frac{1}{(p - z - \Delta z)(p - z)^2}\right| du \leq \frac{L(E)\Delta z}{(d - h)d^2}. \quad (2.20)$$

Therefore, the second integral in the right member of (2.19) approaches zero as Δz approaches zero. Thus,

$$\int_E \frac{du}{(p - z - \Delta z)(p - z)} \text{ approaches } \int_E \frac{du}{(p - z)^2}.$$

Hence, $\dfrac{\phi(z + \Delta z) - \phi(z)}{\Delta z}$ approaches a limit; namely, $\phi'(z) = \int_E \dfrac{du}{(p - z)^2}$.

In a manner similar to that used above for $\phi(z)$, we can show that

$$\phi'(z) = \int_E \frac{du}{(p - z)^2}$$

is analytic at every point of the complementary set, C(E), and its derivative is given by the formula

$$\phi''(z) = 2! \int_E \frac{du}{(p - z)^3}. \quad (2.21)$$

In general, we can show that

$$\phi^{(n-1)}(z) = (n-1)! \int_E \frac{du}{(p - z)^n}$$

is analytic in C(E), and that its derivative is given by

$$\phi^{(n)}(z) = n! \int_E \frac{du}{(p - z)^{n+1}} \tag{2.22}$$

where n is any natural number.

From the well known fact that a function can be represented by a power series in the neighborhood of any point of a domain in which it is analytic, we have the following result:

COROLLARY I. *If $z = z_o$ be any fixed point in C(E),*

$$\phi(z) = \int_E \frac{du}{p - z}$$

can be represented, in a certain neighborhood of this point, by a Taylor series. This series will converge and represent the function throughout the largest circle, about $z = z_o$ as center, which contains in its interior no point of E.

We investigate the nature of the function $\phi(z)$ at $z = \infty$, and prove

THEOREM II. $\phi(z) = \int_E \frac{du}{p - z}$ *is analytic at* $z = \infty$ *and* $\phi(\infty) = 0$.

Proof: Let us begin by making the transformation $z' = 1/z$ and let $h(z') = \phi(1/z')$. That behavior is assigned to the function $\phi(z)$ at infinity, which $h(z')$ exhibits at $z' = 0$. Hence, we examine the function $h(z')$ at $z' = 0$.

For $h(z')$, we have

$$h(z') = \int_E \frac{z'du}{pz' - 1} \quad , \tag{2.23}$$

from which we find that $h(0) = 0$. We now take successive derivatives of $h(z')$ and then evaluate these derivatives at $z' = 0$ as follows:

$$h'(z') = -\int_E \frac{du}{(pz'-1)^2}, \qquad h'(0) = -\int_E du, \qquad (2.24)$$

$$h''(z') = 2! \int_E \frac{p\,du}{(pz'-1)^3}, \qquad h''(0) = -2!\int_E p\,du, \qquad (2.25)$$

and in general

$$h^{(n)}(z') = (-1)^n n! \int_E \frac{p^{n-1}du}{(pz'-1)^{n+1}}, \quad h^{(n)}(0) = -n! \int_E p^{n-1}du, \quad (2.26)$$

where n is any natural number.

The series

$$h(z') = h(0) + h'(0)z' + h''(0)\frac{z'^2}{2!} + \dots + h^{(n)}(0)\frac{z'^n}{n!} + \dots$$

becomes, by expressing h and its derivatives in terms of integrals,

$$h(z') = -\left(\int_E z'du + \int_E pz'^2du + \dots + \int_E p^{n-1}z'^n du + \dots \right). \quad (2.27)$$

This series is equivalent to the series

$$\phi(z) = -\left(\int_E z^{-1}du + \int_E pz^{-2}du + \dots + \int_E p^{n-1}z^{-n}du + \dots\right) \quad (2.28)$$

$$= -\left(z^{-1}\int_E du + z^{-2}\int_E p\,du + \dots + z^{-n}\int_E p^{n-1}du + \dots\dots\right), \quad (2.29)$$

by Lemma II. The coefficients $\int_E p^n du$ are finite because the coefficients p^{n-1} are bounded continuous functions of E, a set of finite Caratheodory linear measure. The last series converges uniformly for $|z| > R$, where R is the radius of a circle C about the origin enclosing E. Therefore, $\phi(z)$ is analytic in the neighborhood of $z = \infty$. The absence of the constant term indicates that $\phi(z)$ has a root at infinity; that is, $\phi(\infty) = 0$. These conditions are sufficient for the function to be analytic at $z = \infty$.

We consider the single valued character of $\phi(z)$ and the question arises: Does $\phi(z)$ return to its original value when z describes a continuous closed path around E? Let C be a simple closed curve which contains E in its interior. The domain exterior to C we designate by S. By Theorem I and II, $\phi(z)$ is analytic in S. Moreover, S is simply connected since any simple closed curve lying in S can be shrunk to a point without going outside the domain. Therefore, we have an analytic function in a simply connected domain, a fact which proves, according to the Monodromy Theorem, that $\phi(z)$ is single valued on any simple closed path about E which lies in S. If z describes a simple closed path through E, the above reasoning does not apply. The function $\phi(z)$ will not necessarily return to its original value for any such path through E. This can be seen by considering the following:

Let E' be a perfect nowhere dense set of positive Lebesgue one dimensional measure on the linear interval I: $a \leq x \leq b$. Consider

$$F(z) = \int_{E'} \frac{dt}{t - z} = \int_a^b \frac{dt}{t - z} - \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \frac{dt}{t - z}, \qquad (2.30)$$

where (a_n, b_n) are interval components whose union is the complement of E', $a_n < b_n$, for each n, t being any point of E', and z a fixed point not on I, where "<" means "precedes" in a particular order. a and b are the terminal points of I. The integrals

$$\int_a^b \frac{dt}{t - z} \quad \text{and} \quad \int_{a_n}^{b_n} \frac{dt}{t - z}$$

are taken in the Lebesgue sense.

Upon integrating, we have

$$F(z) = \log(b - z) - \log(a - z) - \sum_{n=1}^{\infty} (\log(b_n - z) - \log(a_n - z)). \quad (2.31)$$

The logarithmic functions are clearly multivalued with branch points at a, b, a_n and b_n. Hence as z describes a simple closed path through E, F(z) will be increased by some multiple of $2\pi i$, depending upon how many of the branch points are inclosed.

We can now state the following theorem:

THEOREM III. The analytic function $\phi(z)$ is single valued in every domain exterior to a simple closed curve whose bounded complementary domain contains E.

We come now to the problem of determining whether or not the given irregular set of positive Caratheodory linear measure is a singular set for $\phi(z)$. Related to this problem is the question of the existence of a function having a given irregular set of Caratheodory linear measure zero as its singular set.

Regarding sets of linear measure zero, W. Gross (8, p. 180) constructed an irregular set of positive linear measure of which the projections on two perpendicular directions are of linear measure zero. A. S. Besicovitch (1, pp. 455-458) formed a set of positive linear measure of which the projection on any direction is of linear measure zero. These two investigations are particularly significant here since they show the existence of sets of measure zero which are projections of irregular sets of measure greater than zero. The question arises: Can this procedure be reversed; that is, given a set of linear measure zero on a Jordan curve without double points, is it possible to construct, on another such Jordan curve, an irregular set of positive linear measure for which the given set appears as a projection? For some regular sets, this question has been answered affirmatively.[1] As far as we have been able to determine, the question remains unanswered for irregular sets. As a consequence of this limitation, we must restrict our discourse to irregular sets of positive linear measure having property A.[2]

The foregoing discussion brings us to

THEOREM IV. <u>The point set E is a singular set for $\phi(z)$.</u>

Proof: We first show that some points of E are singular for $\phi(z)$. Now the function $\phi(z)$ cannot be a constant. For, suppose $\phi(z)$ is a constant. By Theorem III, exterior to a simple closed curve which contains E in its interior, $\phi(z)$ is a single valued analytic function. According to Theorem II, $\phi(z) = 0$, for $z = \infty$. Therefore, if $\phi(z)$ is constant, $\phi(z) \equiv 0$, and, $z\phi(z) \equiv 0$. But

$$\lim_{z \to \infty} z\phi(z) = \lim_{z \to \infty} \int_E \frac{du}{p/z - 1} = -\int_E du = -L(E), \qquad (2.32)$$

by Lemma III. We thus have a contradiction since the Caratheodory linear measure L(E) is known to be nonzero. We conclude, from Liouville's Theorem,[3] that at least some points of E are singular points for $\phi(z)$.

Now E, by assumption, possesses Property A: that the Caratheodory linear measure of any subset included within any circle of radius $\rho > 0$, described about any one of its points, is different from zero. It follows immediately that all points of E are essential singular points for $\phi(z)$. For let Q be any circle of radius ρ described about any point p of E such

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[1] The existence of a function having a singular set of measure zero, in the case of some regular sets, has been shown by Golubev. (7, pp. 128-130).

[2] Cf. p. 2.

[3] A single valued analytic function which has no singularity either in the finite portion of the plane or at infinity reduces to a constant.

that $E \cap C(\overline{Q}) \neq 0$. Then E is divided into two Caratheodory linearly measurable subsets; namely, E_1 lying in the interior and on Q and E_2 lying in the exterior of Q. Then,

$$\phi(z) = \int_{E_1} \frac{du_1}{p - z} + \int_{E_2} \frac{du_2}{p - z} = \phi_1(z) + \phi_2(z). \qquad (2.32)$$

Now, applying the reasoning employed in the foregoing paragraph, we now show that E_1 has an essential singular point for $\phi_1(z)$ which is accordingly an essential singular point for $\phi(z)$. Hence, in the interior of any circle of radius $\rho > 0$, described about any point p of E, and satisfying the condition given above, there are singular points of $\phi(z)$. Thus, each point p of E is a limit point of points which are themselves limit points of E. Consequently, all points of E are essential singular points of $\phi(z)$.

Section 3. The Curvilinear Integral Defined on Irregular Sets.

We consider the construction of an analytic expression analogous to a curvilinear integral of F(p) on E.

R. L. Moore and J. R. Kline (12, pp. 218-223) have shown that if E is a closed and bounded , totally disconnected point set in the plane, there exists an open curve K (topological open one cell) which contains E.

The set complementary to E in K is open and consists of a sequence (h_n) of denumerably many disjoint open arcs. Generalizing the method of Golubev (7, p. 122), we obtain E by removing a denumerable sequence of open arcs in the following manner: first, we remove from E the complementary arc h_1 and thus decompose K into two sets K_1 and K_2 and E into two sets $E_1 = E \cap K_1$ and $E_2 = E \cap K_2$. Next, we remove from $K - h_1$ the complementary arc h_2. We obtain three sets

$$E_1^2, E_2^2, E_3^2 , \qquad (3.1)$$

where the set E_m^2, for each m is the intersection of a closed connected subset, an interval or a ray, of K and E.

In general, upon the removal from $K - (h_1 + h_2 + \dots + h_n)$ the arc h_{n+1}, complementary to E in K, we obtain a sequence

$$E_1^{n+1}, E_2^{n+1}, E_3^{n+1}, \quad \dots \quad , E_n^{n+1} , \qquad (3.2)$$

where E_m^{n+1}, for each m, is the intersection of a closed connected subset, an interval or ray of K and E. This operation continues, and we obtain the double sequence of point sets

$$\begin{array}{l} E_1^1 , \ E_2^1 \\ E_1^2 , \ E_2^2 , \ E_3^2 \\ E_1^3 , \ E_2^3 , \ E_3^3 , \ E_4^3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ E_1^n , \ E_2^n , \ E_3^n , \ E_4^n , \ E_5^n , \ \dots\dots , \ E_{n+1}^n \end{array} \qquad (3.3)$$

We denote by du_m^n the Caratheodory linear measure of E_m^n. Let F(p) be a continuous function of p in E and $F(p_m^n)$, the value of the function at some point p_m^n on E_m^n. We define the curvilinear integral of F(p) on E by the number J and denote it as follows:

$$J = \int_E F(p)du = \lim_{n \to \infty} \sum_{m=1}^{n+1} F(p_m^n)du_m^n \qquad (3.4)$$

For brevity, we call the sums

$$S_1 = \sum_{m=1}^{2} F(p_m^1)du_m^1 , \quad S_2 = \sum_{m=1}^{3} F(p_m^2)du_m^2 , \quad \ldots\ldots\ldots$$

$$S_n = \sum_{m=1}^{n+1} F(p_m^n)du_m^n , \quad \ldots\ldots\ldots \qquad (3.5)$$

sigma sums formed by removing from K, successively, the complementary arcs, $h_1, h_2, \ldots , h_n , \ldots$

We investigate the existence of the above limit, and show that J is independent of the rearrangement of the sequence (h_n). J_n designates some sigma sum.

Lemma VII. Let K' be a closed connected subset of K, and, $E' = K' \cap E$. Let $r_1, r_2, \ldots , r_n$ be a finite set of n disjoint open intervals each a subset of K. Let $K_1, K_2 , K_3 , \ldots , K_{n+1}$ be the n+1 disjoint connected subsets of $K - \bigcup_{i=1}^{n} r_i$; for each m, m = 1, 2, ... , n+1, let $p_m^{n'}$ be a point of $E' \cap K_m$, and $du_m^{n'}$ be the linear measure of $E' \cap K_m$. Let σ be a positive real number such that the oscillation of $F(p')$ on E' be less than σ. Let p' be an arbitrary point of E' and du' the linear measure of E'. Then,

$$\left| F(p')du' - \sum_{m=1}^{n+1} F(p_m^{n'})du_m^{n'} \right| < \sigma L(E')$$

Proof: The set complementary to E' in K' is open and consists of a sequence (h_n') of denumerably many disjoint open arcs.

The sums S' and S_n' are formed with respect to K' in a manner similar to that in which S_n is formed with respect to K, with the following exception: there is only one functional value of $F(p')$, for p' on E', involved in the sum S', whereas in S'_n there is a functional value of p' for a point p' in each of the sets $(E_m^{n'})$, $m = 1, 2, \dots, n+1$, whose linear measure is $du_m^{n'}$. We write S' in the form

$$S' = \sum_{m=1}^{n+1} F(p')du_m^{n'}.$$

Then

$$|S' - S'_n| = |F(p')du' - \sum_{m=1}^{n+1} F(p_m^{n'})du_m^{n'}| =$$

$$|\sum_{m=1}^{n+1} F(p')du_m^{n'} - \sum_{m=1}^{n+1} F(p_m^{n'})du_m^{n'}| \leq \sum_{m=1}^{n+1} |F(p') - F(p_m^{n'})|du_m^{n'} \quad (3.6)$$

$$< \sigma L(E'),$$

since $|F(p') - F(p_m^{n'})| < \sigma$, for $m = 1, 2, \dots, n+1$.

Lemma VIII. Let S_r be a fixed sigma sum consisting of r+1 parts obtained by the removal of r complementary arcs, successively, from K∩E. Let the oscillation of $F(p)$ on (E_m^r), $m = 1, 2, \dots, r+1$, be less than σ_1. Let S_t be a new sigma sum formed by the removal of the r complementary arcs of S_r together with a finite number of other complementary arcs. Then

$$|S_t - S_r| < \sigma_1 L(E)$$

Proof: Lemma VII is valid for each of the sets (E_m^r) of S_r, $m = 1, 2, \dots, r+1$, with respect to S_t, since E_m^r for each m is the intersection of a connected subset of K and E and, moreover, plays the role of S' in the previous Lemma.

Let

$$du_1^r,\ du_2^r,\ \dots\ ,\ du_{n+1}^r$$

denote the linear measures of the sets

$$E_1^r,\ E_2^r,\ \dots\ ,\ E_{n+1}^r$$

respectively. Then

$$E = E_1^r + E_2^r + \ \dots\ + E_{n+1}^r.$$

Thus, it follows, by repeated application of Lemma VII, that

$$|S_t - S_r| < \sigma_1 du_1^r + \sigma_1 du_2^r + \ \dots\ + \sigma_1 du_{r+1}^r = \sigma_1 L(E) \qquad (3.7)$$

Lemma IX. Let δ be a number greater than zero. Then there exists a number n_δ such that for $n > n_\delta$, E_m^n for every m is of diameter less than δ; that is, if $n > n_\delta$, every set E_m^n, $m = 1, 2, \dots, n+1$, involved in the sum S_n, is of diameter less than δ.

Proof: Suppose that the Lemma is false and E_m^n cannot be decomposed into subsets each of which is of diameter less than δ. Then there exists a monotonic descending sequence of sets $(E_m^{j_n})$ each of diameter greater than δ and such that

$$(1)\quad E_m^{j_{n+1}} \subset E_m^{j_n},$$

$$(2)\quad E_m^{j_n} \text{ is closed and compact.}$$

Hence $\prod_{n=1}^{\infty} E_m^{j_n} = E^*$, a subset of E of diameter greater than or equal to δ. Therefore, E^* contains at least two points, a and b, not separated in K by any arc of the set (h_n). This is impossible since E is totally disconnected. Thus, we have a contradiction. We conclude that the number n_δ, which satisfies the Lemma, exists.

Lemma X. *Let e be greater than zero. There exists a $\delta = \delta(e)$ such that if S_k and S_q are any two sigma sums defined by means of decompositions of E into subsets of diameter less than δ, then*

$$|S_k - S_q| < e/2,$$

where $k = 1, 2, \ldots, h, \ldots, q, \ldots$

Proof: A number δ exists such that if E_m^k is of diameter less than δ, and $p_m^{k_2}$ and $p_m^{k_1}$ are points of E_m^k, then

$$|F(p_m^{k_1}) - F(p_m^{k_2})| < e/4L(E) ,$$

since $F(p)$ is uniformly continuous on the closed and bounded set E.

Let S_k and S_q be any two sigma sums such that the sets E_m^k of E involved, each have diameter less than δ.

There exists an integer r such that if we remove a finite collection G of r arcs, $h_1, h_2, \ldots, h_r$, then G contains each of the arcs, which determines the sigma sum S_q and also each of the arcs, which determines S_k. We have,

$$|S_k - S_r| = \left| \sum_{m=1}^{k+1} F(p_m^k)du_m^k - \sum_{m=1}^{k+1} F(p_m^r)du_m^r \right|$$

$$\leq \sum_{m=1}^{r+1} |F(p_m^k) - F(p_m^r)|du_m^r < \frac{e}{4L(E)} L(E) = e/4 , \qquad (3.8)$$

by Lemma VIII.

In a similar manner, we can show that $|S_q - S_r| < e/4$.

But $S_k - S_q = (S_k - S_r) - (S_q - S_r)$. Therefore,

$$|S_k - S_r| = |(S_k - S_r) - (S_q - S_r)| \leq |S_k - S_r| + |S_q - S_r|$$

$$< e/4 + e/4 = e/2. \qquad (3.10)$$

THEOREM V. Let sigma sums S_k be formed for $k = 1, 2, \ldots, h, \ldots q, \ldots$ If $\lim_{k \to \infty} \Delta_m^k \to 0$, where Δ_m^k denotes the maximum diameter of E_m^k, for each k and every m, then, $\lim_{k \to \infty} S_k$ exists.

Proof: Let ϵ be greater than zero. Then, by Lemma X, there exists a $\delta = \delta(\epsilon)$; by Lemma IX, there exists a k_o such that, if $k > k_o$ and $p \geq 1$, then $|S_{k+p} - S_k| < \epsilon/2 < \epsilon$.

Since ϵ can be chosen as small as we please, the above relation shows, by Cauchy's Convergence Principle, that

$$\lim_{k \to \infty} S_k = \lim_{k \to \infty} \sum_{m=1}^{k+1} F(p_m^k) du_m^k \qquad (3.11)$$

exists. We define this limit to be J.

Lemma XI. Let $h_1, h_2, \ldots, h_n$ be a finite sequence of intervals complementary to E in K; let $g_1, g_2, \ldots, g_m$ be a second finite sequence of intervals complementary to E in K such that

$$\bigcup_{i=1}^{n} h_i \subset \bigcup_{i=1}^{m} g_i \,.$$

Let also S_1 and S_2 be the sums determined by the finite sequences (h_i) and (g_i), and let only the maximum value of the function $|F(p)|$ be used in these sums. Then, $S_2 \leq S_1$.

Proof: Since

$$\bigcup_{i=1}^{n} h_i \subset \bigcup_{i=1}^{n} g_i \qquad (3.12)$$

and only the maximum value of $|F(p)|$ is used in the sums S_1 and S_2, the relation

$$S_2 \leq S_1 \qquad (3.13)$$

follows directly.

THEOREM VI. Let $a_1 = h_1^1, h_2^1, \ldots, h_n^1, \ldots$ and $a_2 = h_1^2, h_2^2, \ldots, h_n^2, \ldots$ be different sequences of the set of open intervals, complementary to E in K, in any order. Let

$$J_1 = \lim_{n_1 \to \infty} \sum_{m=1}^{n_1+1} F(p_m^{n_1}) du_m^{n_1}$$

$$J_2 = \lim_{n_2 \to \infty} \sum_{m=1}^{n_2+1} F(p_m^{n_2}) du_m^{n_2}$$

where $du_m^{n_i}$ designates the measure of the intersection of E and a maximum connected subset of

$$K - \bigcup_{m=1}^{n_i} h_m^i \quad , i = 1, 2.$$

Then

$$|J_1 - J_2| < e$$

Proof: Let e be greater than zero. There exist integers n_1 and n_2 such that if $n > n_1$ and $k > n_2$

$$|J_1 - S_n^1| < e/2 \qquad (3.14)$$

$$|J_2 - S_n^2| < e/2 \qquad (3.15)$$

Now consider a third sequence

$$a_3 = h_1^3, h_2^3, \ldots, h_n^3, \ldots$$

of the set of open intervals complementary to E in K. A number w_1 can be found such that a subsequence of the original sequence consisting of the segments

$$h_1^3, h_2^3, \ldots, h_{w_1}^3 \qquad (3.16)$$

contains the first n terms of the sequence a_1; namely,

$$h_1^1, h_2^1, \ldots , h_n^1. \tag{3.17}$$

This means that

$$\bigcup_{i=1}^{n} h_i^1 \subset \bigcup_{i=1}^{w_1} h_i^3 \tag{3.18}$$

and, therefore,

$$S_{w_1}^3 \leq S_n^1 \tag{3.19}$$

by Lemma XI.

In like manner, there exists a number w_2 such that a subsequence of the sequence a_3, consisting of the segments

$$h_1^3, h_2^3, \ldots , h_{w_2}^3, \tag{3.20}$$

contains the first k terms of the sequence a_2; namely,

$$h_1^2, h_2^2, \ldots , h_k^2. \tag{3.21}$$

Consequently,

$$\bigcup_{i=1}^{n} h_i^2 \subset \bigcup_{i=1}^{w_2} h_i^3 \tag{3.22}$$

and therefore,

$$S_{w_2}^3 \leq S_h^2 \tag{3.23}$$

by Lemma XI.

Let $w = w_1 + w_2$. Then by Lemma X,

$$|J_1 - S_w^3| < \epsilon/2 , \tag{3.24}$$

$$|J_2 - S_w^3| < \epsilon/2 . \tag{3.25}$$

Therefore,

$$|J_1 - J_2| < \epsilon . \tag{3.26}$$

Since ϵ is arbitrary,

$$J_1 = J_2. \tag{3.27}$$

We have thus shown that

$$\lim_{n \to \infty} J_n = J \qquad (3.28)$$

exists and is independent of the arrangement of the intervals (h_i).

Section 4(a). Some Properties of the Curvilinear Integral.

We shall show some properties of the curvilinear integral which we have developed. We begin with an analogue of a well known integral theorem.

THEOREM VII. *If M denotes the maximum value of $|F(p)|$ for any p in E and L(E) denotes the linear measure of E, then for every p in E,*

$$\left|\int_E F(p)du\right| \leq ML(E)$$

Proof: From Section 3

$$\left|\int_E F(p)du\right| \leq \int_E |F(p)|du \leq M \lim_{n \to \infty} \sum_{m=1}^{n+1} du_m^n \qquad (4.1)$$

This summation represents for each n and every m, m = 1, 2, ... , n+1, the Caratheodory linear measure of the sets in the sequence (E_m^n) and equals, as n becomes infinite, the linear measure of E.

If the open curve K which contains E be divided into two half lines K_1 and K_2 having in common a single point p belonging to E and containing subsets E_1 and E_2 of E respectively, then because E has property A,[4] each subset has positive Caratheodory linear measure. We prove

Lemma XII. *Let K_1 and K_2 be half lines, subsets of K, having in common a single point of E. Let $E_1 = E \cap K_1$, and $E_2 = E \cap K_2$. Let $a = h_1, h_2, \ldots, h_n, \ldots$ be a sequence of open intervals, in any order, complementary to E in the open curve K. Of the first n intervals*

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[4] Cf. p. 2.

$h_1, h_2, \ldots, h_n$ belonging to a, let $a_1 = h_1^1, h_2^1, \ldots, h_{n_1}^1$ lie in K_1, and $a_2 = h_1^2, h_2^2, \ldots, h_{n_2}^2$ lie in K_2. Let $du_m^{n_i}$ denote the measure of the intersection of E and a maximum connected subset of

$$K_i - \bigcup_{m=1}^{n_i} h_m^i, \quad i = 1,2.$$

Let $F(p)$ be continuous on E. Then

$$\int_E F(p)du = \int_{E_1} F(p)du_1 + \int_{E_2} F(p)du_2$$

Proof: By virtue of the continuity of $F(p)$ on the closed and bounded set E and, according to Section 3, we can write

$$\int_E F(p)du = \lim_{n \to \infty} \sum_{m=1}^{n+1} F(p_m^n)du_m^n ,$$

where $F(p_m^n)$ is the value of the function $F(p)$ for p_m^n in E_m^n and du_m^n the Caratheodory linear measure of E_m^n. This limit is independent of the choice of the points p_m^n in E_m^n and the arrangement of the intervals a. But

$$\lim_{n \to \infty} \sum_{m=1}^{n+1} F(p_m^n)du_m^n = \lim_{n \to \infty} \left(\sum_{m=1}^{n_1+1} F(p_m^{n_1})\, du_m^{n_1} + \sum_{m=1}^{n_2+1} F(p_m^{n_2})du_m^{n_2} \right) =$$

$$\lim_{n_1 \to \infty} \sum_{m=1}^{n_1+1} F(p_m^{n_1})du_m^{n_1} + \lim_{n_2 \to \infty} \sum_{m=1}^{n_2+1} F(p_m^{n_2})du_m^{n_2} , \qquad (4.3)$$

where $F(p_m^{n_1})$ and $F(p_m^{n_2})$ are the functional values of $F(p)$ in $E_m^{n_1}$ and $E_m^{n_2}$ respectively. From the above determinations, it follows that

$$\int_E F(p)du = \int_{E_1} F(p)du_1 + \int_{E_2} F(p)du_2$$

as required.

Lemma XIII. Let K_1 and K_2 be half lines, subsets of K, having in common a single point of E. Let $E_1 = E \cap K_1$ and $E_2 = E \cap K_2$. Let $a = h_1, h_2, \ldots, h_n, \ldots$ be a sequence of open intervals, in any order, complementary to E in K. Of the first n intervals $h_1, h_2, \ldots, h_n$ belonging to a, let $a_1 = h_1^1, h_2^1, \ldots, h_{n_1}^1$ lie in K_1 and $a_2 = h_1^2, h_2^2, \ldots, h_{n_2}^2$ lie in K_2. Let $du_m^{n_i}$ denote the measure of the intersection of E and a maximum connected subset of

$$K_i - \bigcup_{m=1}^{n_i} h_m^i, \; i = 1, 2.$$

Let F(p) be continuous on E. Then,

$$\int_E |F(p)|du = \int_{E_1} |F(p)|du_1 + \int_{E_2} |F(p)|du_2$$

Proof: Because F(p) is continuous on E, we infer that $|F(p)|$ is continuous on this point set. The proof of this Lemma, then, follows directly from Lemma XII.

As a consequence of the definition of the integral, we observe another property of the integral; namely, there is no prescribed sense in which the integration is perform on E.

We note from Section 2 that a complex valued function F(p) may be written $f(p) + ig(p)$, where f(p) and g(p) are two real valued, single valued, functions of p. We therefore define

$$\int_E F(p)du$$

to be

$$\int_E f(p)du + i\int_E g(p)du \qquad (4.5)$$

We have established, thus far, two integrals: namely, an integral taken over an irregular closed and bounded set E of positive Caratheodory linear measure, and one which we have defined as a curvilinear integral taken on the same point set. For brevity, we shall designate the former integral as the (q)-integral and the latter as the (c)-integral. The preceeding remarks bring us to

Section 4(b). The Equivalence of the (c)-Integral of F(p) on E to the (q)-Integral of F(p) over E.

Let the open curve K which contains E be divided into half lines K_1 and K_2 having in common a single point p belonging to E and containing subsets E_1 and E_2 of E, respectively. Then, because E has property A,[5] each subset has positive Caratheodory linear measure. Preliminary to an investigation of the equivalence of the two integrals mentioned above, we establish two Lemmas which follow:

Lemma XIV. If K_1 and K_2 are half lines having in common a single point of E, if $E_1 = E \cap K_1$ and $E_2 = E \cap K_2$, then,

$$(c)\int_E F(p)du = (c)\int_{E_1} F(p)du_1 + (c)\int_{E_2} F(p)du_2$$

Proof: By definition, using the notations of Section 3 and making use of the results of Lemma XII,

$$(c)\int_E F(p)du = \lim_{n \to \infty} \sum_{m=1}^{n+1} F(p_m^n)du_m^n \qquad (4.6)$$

where p_m^n is a point in E_m^n and du_m^n is the Caratheodory linear measure of E_m^n. In like manner, by Lemma XII,

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[5]Cf. p. 2.

$$(c)\int_{E_1} F(p)du_1 = \lim_{n_1 \to \infty} \sum_{m=1}^{n_1+1} F(p_m^{n_1})du_m^{n_1} \qquad (4.7)$$

$$(c)\int_{E_2} F(p)du_2 = \lim_{n_2 \to \infty} \sum_{m=1}^{n_2+1} F(p_m^{n_2})du_m^{n_2} \qquad (4.8)$$

where the limits are independent of the order of the sequence of open intervals (h_n), and the choice of the points $p_m^{n_1}$ and $p_m^{n_2}$ of $E_m^{n_1}$ and $E_m^{n_2}$, respectively. An arc of the intervals (h_n) could not overlap a part of the half lines K_1 and K_2 since K_1 and K_2 have in common a single point p of E. Hence, in defining

$$(c)\int_{E} F(p)du, \quad (c)\int_{E_1} F(p)du_1, \quad (c)\int_{E_2} F(p)du_2 ,$$

we use the same order of the sequence of open intervals (h_n), and choose in the sets $E_m^{n_1}$ and $E_m^{n_2}$ a point $p_m^{n_1}$ and $p_m^{n_2}$, respectively, which is the same as the point p_m^n chosen in the corresponding E_m^n. The superscript n then replaces the superscript n_1 and n_2. We have,

$$(c)\int_{E_1} F(p)du_1 + (c)\int_{E_2} F(p)du_2 = \lim_{n_1 \to \infty} \sum_{m=1}^{n_1+1} F(p_m^{n_1})du_m^{n_1} + \lim_{n_2 \to \infty} \sum_{m=1}^{n_2+1} F(p_m^{n_2})du_m^{n_2} . \qquad (4.9)$$

But $p_m^{n_1}$ and $p_m^{n_2}$ in $E_m^{n_1}$ and $E_m^{n_2}$, respectively, are points which agree with p_m^n in the corresponding E_m^n. Thus each point $p_m^{n_1}$ and $p_m^{n_2}$ differ only in notation from the point p_m^n where n is replaced by the superscript n_1 and n_2; that is, for every $p_m^{n_1}$ on $E_m^{n_1}$ there corresponds the same point

p_m^n on E_m^n. In like manner, the same reasoning applies to the point $p_m^{n_2}$. Correspondingly, $F(p_m^{n_1})$ and $F(p_m^{n_2})$ can be replaced by $F(p_m^n)$, where m runs from 1 to n+1. It follows that

$$(c)\int_{E_1} F(p)du_1 + (c)\int_{E_2} F(p)du_2 = \lim_{n \to \infty} \sum_{m=1}^{n+1} F(p_m^n)du_m^n$$

$$= (c)\int_E F(p)du. \qquad (4.10)$$

Lemma XV. <u>Let E_1 and E_2 be two disjoint subsets of E such that</u> $E = E_1 \cup E_2$. <u>Then,</u>

$$(q)\int_E F(p)du = (q)\int_{E_1} F(p)du + (q)\int_{E_2} F(p)du$$

Proof: F(p) is continuous on the closed and bounded set E. Consequently, F(p) is measurable on E, and by Section 1, it is integrable over E in the (q)-sense. By Lemma I,

$$(q)\int_E f(p)du = (q)\int_{E_1} f(p)du + (q)\int_{E_2} f(p)du, \qquad (4.11)$$

and

$$(q)\int_E g(p)du = (q)\int_{E_1} g(p)du + (q)\int_{E_2} g(p)du. \qquad (4.12)$$

Therefore, in view of (4.5)

$$(q)\int_E F(p)du = (q)\int_E f(p)du + i(q)\int_E g(p)du. \qquad (4.13)$$

From this result we infer that

$$(q)\int_{E_1} F(p)du = (q)\int_{E_1} f(p)du + i(q)\int_{E_1} g(p)du \qquad (4.14)$$

and

$$(q)\int_{E_2} F(p)du = (q)\int_{E_2} f(p)du + i(q)\int_{E_2} g(p)du. \qquad (4.15)$$

From (4.13), (4.14) and (4.15), we conclude that

$$(q)\int_{E} F(p)du = (q)\int_{E_1} F(p)du + (q)\int_{E_2} F(p)du \qquad (4.16)$$

We now come to

THEOREM VIII. <u>The</u> (c)-<u>Integral</u> <u>of</u> F(p) <u>on</u> E <u>is</u> <u>Equivalent</u> <u>to</u> <u>the</u> (q)-<u>Integral</u> <u>of</u> F(p) <u>over</u> E.

Proof: Consider the open curve K containing E. We remove from K, successively, a finite collection G of arcs $h_1, h_2, \ldots, h_n$ complementary to E in K forming the sets

$$E_1^1, E_2^1$$

$$E_1^2, E_2^2, E_3^2$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E_1^n, E_2^n, E_3^n, E_4^n, \ldots, E_{n+1}^n$$

We have defined

$$(c)\int_{E} F(p)du = (c)\int_{E} f(p)du + i(c)\int_{E} g(p)du,$$[6]

where f(p) and g(p) are two single valued, real valued, functions of p. On the closed and bounded set E, let f(p) and g(p) be everywhere non-negative. Let M_m^n and m_m^n denote the least upper bound and the greatest lower bound respectively, of $f(p_m^n)$, for p_m^n on E_m^n. We have,

$$\sum_{m=1}^{n+1} m_m^n du_m^n \leq (c)\int_{E_m^n} f(p_m^n)du \leq \sum_{m=1}^{n+1} M_m^n du_m^n \qquad (4.18)$$

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[6] Cf. p. 38.

The sum

$$\sum_{m=1}^{n+1} m_m^n du_m^n$$

is nondecreasing, and the sum

$$\sum_{m=1}^{n+1} M_m^n du_m^n$$

is nonincreasing.

Consider now the least upper bound $\int_E Mdu$ and the greatest lower bound $\int_E mdu$ obtained when $n \to \infty$ in such a manner that

$$\lim_{n \to \infty} \Delta_m^n = 0$$

where Δ_m^n denotes the maximum diameter of E_m^n. From (4.18), we obtain

$$\int_E mdu \leq (c)\int_E f(p)du \leq \int_E Mdu \, . \qquad (4.19)$$

The (q)- integral of F(p) over E

$$(q)\int_E F(p)du = (q)\int_E f(p)du + i(q)\int_E g(p)du,$$

has already been shown to exist.[7] Let the range of f(p) be divided into subdivisions

$$a_o \leq a_1 \leq \dots \leq a_{n-1} \leq a_n.$$

Denote by M_m^n and m_m^n the maximum and minimum values of $f(p_m^n)$ in the respective n subdivisions of the range of f(p) and ΔE_m^n the linear measure of E_m^n where E_m^n is that subset of E for which $a_{m-1} \leq f(p) < a_m$. We have in this instance, as in the case of the (c)-integral,

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[7] Cf. Section 1, p. 8 ff., and Section 2, p. 16.

$$\sum_{m=1}^{n+1} m_m^n \Delta E_m^n \leq (q)\int_{E_m^n} f(p_m^n)\Delta E_m^n \leq \sum_{m=1}^{n+1} M_m^n \Delta E_m^n \qquad (4.20)$$

Now let the number n of the subdivisions of the range of f(p) increase indefinitely in such a way that

$$\lim_{n \to \infty} B_m^n = 0$$

where B_m^n is now the maximum diameter of E_m^n.

From (4.20), we obtain

$$\int_E m\,du \leq (q)\int_E f(p)du \leq \int_E M\,du \,. \qquad (4.21)$$

Consider now the double inequalities

$$\int_E m\,du \leq (c)\int_E f(p)du \leq \int_E M\,du \qquad (4.22)$$

and

$$\int_E m\,du \leq (q)\int_E f(p)du \leq \int_E M\,du \,. \qquad (4.23)$$

Since

$$\int_E M\,du = M\int_E du = ML(E)$$

and

$$\int_E m\,du = m\int_E du = mL(E)$$

by Lemma II, we have

$$mL(E) \leq (c)\int_E f(p)du \leq ML(E) \qquad (4.24)$$

and

$$mL(E) \leq (q)\int_E f(p)du \leq ML(E). \qquad (5.25)$$

Let

$$E = \sum_{i=1}^{n+1} E_i$$

where $E_i \cap E_j = 0$, for $i \neq j$, $E_i = E \cap K_i$, and $K_1, K_2, \ldots, K_{n+1}$ are the maximum connected subsets of

$$K - \bigcup_{i=1}^{n} h_i.$$

Then (4.24) and (4.25) applied to the set E_i become

$$m \sum_{i=1}^{n+1} L(E_i) \leq \sum_{i=1}^{n+1} (q)\int_{E_i} f(p)du \leq M \sum_{i=1}^{n+1} L(E_i) \qquad (4.26)$$

$$m \sum_{i=1}^{n+1} L(E_i) \leq \sum_{i=1}^{n+1} (c)\int_{E_i} f(p)du \leq M \sum_{i=1}^{n+1} L(E_i) \qquad (4.27)$$

Now let e be greater than zero. Then from the continuity of $f(p)$ on the closed and bounded set E and the inequalities (4.26) and (4.27), there exists in K some set H of segments $h_1, h_2, \ldots, h_n$ complementary to E such that

$$\left|(q)\int_{E_i} f(p)du - (c)\int_{E_i} f(p)du\right| < e/2L(E_i)$$

for each $E_i = E \cap K_i$.

Consider now the sum of all such inequalities for $i = 1, 2, \ldots, n+1$. Making use of the results obtained by Lemmas XII and XIII, we have,

$$\left|(q)\int_{E} f(p)du - (c)\int_{E} f(p)du\right| < e/2L(E). \qquad (4.28)$$

In a manner similar to that used above for $f(p)$, we can show that

$$\left|(q)\int_{E} g(p)du - (c)\int_{E} g(p)du\right| < e/2L(E). \qquad (4.29)$$

From inequalities (4.28) and (4.29), we conclude,

$$\left|(q)\int_{E} F(p)du - (c)\int_{E} F(p)du\right| < e. \qquad (4.30)$$

Since e is arbitrary, the inequality (4.30) proves the Theorem

CHAPTER III

AN ANALYTIC FUNCTION WITH A BOUNDED CONTINUUM AS A SINGULAR SET

In exploring methods of obtaining analytic functions having a prescribed singular set, we have constructed and employed integals in the process. We now construct a function having a given set M as its singular set by employing a new approach, that is, by making use of a mapping of the complement of M onto the interior of the unit circle.

Let M be a bounded, non degenerated, locally connected plane continuum which does not separate the plane. We first establish the following Lemmas:

Lemma I. *There exists a simple (1 - 1) analytic mapping H of I, the complement of M, onto E, the interior of the unit circle C. Moreover, the mapping H of I onto E can be extended to M in the sense that if p is a point of C, the boundary of E, $H^{-1}(p)$ is a prime end of M, the boundary of I.*

Proof: Because the boundary M of I is connected, I is a simple domain. Since the boundary of I consists of a bounded, non degenerate continuum M, and I is simple and simply connected, there exists, by the Riemann Mapping Theorem, a conformal mapping of I onto E, the interior of C. Furthermore, by the foregoing, there exists a mapping function H(z), single valued and analytic for z in I, the complement of M.

Further, as a result of the simple connectivity of I, in the mapping of I onto E, the prime ends of M, the boundary of I, and the points of the circle C, the boundary of E, correspond to one another in a (1 - 1) manner. This correspondence is in strict accordance with a known theorem on prime ends. (5, p. 350, Theorem XIII).

Consistent with (5, pp. 331-336), a prime end of an arbitrary simply connected region G is an equivalence class of chains of subregions of G. Although a prime end, as defined above, is actually an equivalence class of chains of subregions, we intend to refer to the closed point set associated with the prime end as *the prime end* employed in that which follows. The same closed point set, may, perhaps, be associated with one or more prime ends.

Lemma II. *If* $f(z) = \sum_{n=0}^{\infty} z^{n!}$, $|z| < 1$, *then* $f(z)$ *is an analytic function defined on* E, *the interior of the unit circle* C, *and has* C *for its natural boundary*.

Moreover, if $F(z) = f(H(z))$ *for* z *belonging to* I, *the complement of* M, *then* (1) $F(z)$ *is defined and analytic on* I, *the complement of* M, *and* , (2) *has* M *for its singular set*.

Proof: Let $f(z) = \sum_{n=0}^{\infty} z^{n!}$, $|z| < 1$, as described in the Lemma

The proof of the first part of this Lemma, then, follows from a known theorem (14, p. 163, Theorem 23.17).

Let $F(z) = f(H(z))$ for z belonging to I. (1) $H(z)$ is analytic for z in I and maps I onto E, the interior of the unit circle. By hypothesis, $f(z)$ is defined and analytic for z in E, the interior of the unit circle. Hence for z in I, the composite function $f(H(z))$ is an analytic function of an analytic function and consequently is itself analytic.[1] It follows that $F(z)$ defined by the functional equation $F(z) = f(H(z))$ is analytic for z and I. That $F(z)$ is defined on I follows from the definition of F and the foregoing.

(2) Since M is locally connected, M is locally connected at each of its points. We shall first show that a point set associated with a prime end of M, in this case, is a single point.

Let p be any point of M. We take as a neighborhood N_p of p the interior of a circle with center at p. Consequently, there exists for any circle K_1 with center at p, a concentric circle K_2 such that every point p' of M, interior to K_2, is joined to p by a connected subset of M lying wholly in K_1. Let p be a point of countable character. As a consequence of this property, there exists a sequence of concentric circles $K_1, K_2, \ldots, K_n, \ldots$ with common center p such that $K_{i+1} \subset K_i$ for each i, and such that $\bigcap_1^\infty K_n = p$.

We consider as a chain of cross cuts (q_n) those which lie on concentric circlar arcs (K_n) with end points on M. The end points of these cross cuts are different from one another unless p is a terminal point of M, in which case, the end points coincide. Now consider the

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[1] An analytic function of an analytic function is an analytic function.

subregions $g_1, g_2, \ldots, g_n, \ldots$ of I which are associated with the cross cuts $q_1, q_2, \ldots, q_n, \ldots$ and which define an end e_m of I. Because of the local connectivity of M, this chain of subregions can be taken so that they converge to the point p of M. Therefore, the end e_m is a prime end E_m, (5, p. 337, Theorem V), and, furthermore, the convergence of the chain of subregions (g_n) to the point p is a necessary and sufficient condition for a prime end E_m to contain a single point p (5, p. 352 Theorem XIV).

Now if p" is a point of C, the unit circle, $N_{p''}$ a neighborhood of p", and f(z) defined as in Lemma II, then f(z), according to Riemann's Theorem, cannot be bounded in $N_{p''}$. Denote $E \cap N_{p''}$ by G'. The inverse mapping H^{-1} of H is single valued and maps G' onto some region G of I carrying p" onto a prime end p of M, the inverse image of p" under the mapping $H^{-1}(p'')$.

f(z) is unbounded in G'. It follows that F(z) = f(H(z)) is unbounded for z in G and hence for z in a neighborhood of p, that is, p is a singular point for F(z). There exists a (1 - 1) correspondence between the points of C and the prime ends of M, the boundary of I. f(z), being defined as in Lemma II, is bounded in the neighborhood of every point of C. Therefore, F (z) by means of the functional equation F(z) = f(H(z)) is unbounded for z in the neighborhood of each prime end of M. Consequently, F(z) has M as its singular set, since the set of all prime ends of M constitutes the boundary of I.

Upon the validity of Lemmas I and II, we can state the following Theorem:

THEOREM I. _Let M be a bounded, non degenerate, locally connected, plane continuum which does not separate the plane. Then there exists_ *a single valued function F(z), analytic (but not necessarily bounded) in the extended plane, with M as its singular set.*

CHAPTER IV

SUMMARY

The problem with which this study is primarly concerned is that of constructing analytic functions having for their singular sets certain closed and bounded sets.

We have shown that if E is a bounded and closed point set, lying in the real plane, which is irregular and has positive Caratheodory linear measure, and which has property A,[1] there exists a function $\phi(z)$ with the following properties:

(1) $\phi(z)$ analytic in the extended z-plane except the points of E;

(2) $\phi(z)$ single valued in the complement of E;

(3) Each point of E is an essential singularity of $\phi(z)$.

We have also determined a single valued analytic function having for its singular set a nondegenerate, bounded, locally connected plane continuum M which does not separate the plane by making use of the mapping of the complement of M onto the interior of the unit circle. This analysis did not involve the use of integrals.

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[1]Cf. p. 2.

RECOMMENDATIONS FOR FURTHER STUDY

The following questions arise: (1) Does there exist an analytic function having a bounded nondegenerate arbitrary continuum for its singular sets? (2) If the answer to (1) is in the affirmative, then do the properties of an analytic function having a bounded, nondegenerate, locally connected continuum as a singular set differ from those of an analytic function having a bounded nondegenerate arbitrary continuum as a singular set? By properties we mean the following:

(a) If M denotes the arbitrary continuum, is each point of M a singular point for the function under consideration?

(b) Is the function single valued in the complement of M?

(c) Is the function bounded in the complement of M?

(d) Is the function analytic at $z = \infty$?

Although we have no propositions which bear on these situations, the answers might be a valuable complement to this study.

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