BEAM CONSTANTS BY THE STRING POLYGON METHOD

By

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TABLE OF CONTENTS

LIST OF TABLES

INTRODUCTION

The purpose of the thesis is the derivation of the general formulas for beam constants for beams with parabolic haunches. The string polygon method is used for the derivation of these formulas, and the evaluation of the complicated integration function is made by means of Ritter's approximation. Tables of beam constants for the most important cases are included in this thesis.

The string polygon method mentioned above was presented by Professor J. J. Tuma in his course CE-620-Ph.D. Seminar in the Spring of 1959.

CHAPTER I

THE STRING POLYGON EQUATION

A simple beam of variable section loaded by a general system of forces is considered (Fig. 1-1).

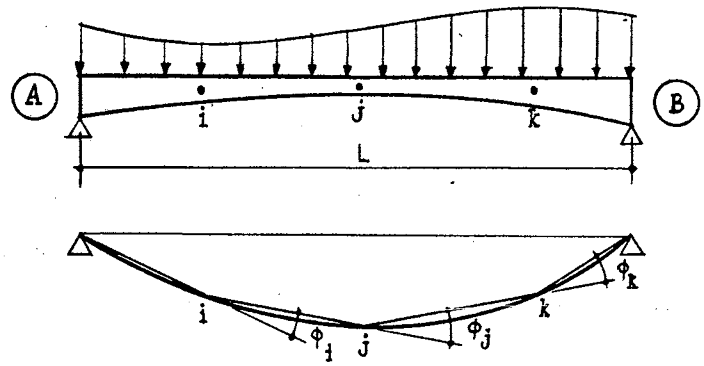


Fig. 1-1

The elastic curve of this beam is shown in an exaggerated form. Three arbitrarily selected points of the elastic curve are denoted as i, j, and k. The change in slope between the line ij and jk is designated by ϕ_j, (the change in slope of the string polygon). The algebraic expression for this change in slope is given by the equation (1-1)

$$\phi_j = G_{ij}M_i + (F_{ji} + F_{jk})M_j + G_{kj}M_k + \tau_{ji} + \tau_{jk} \quad . \qquad (1\text{-}1)$$

The notation of this equation follows:

M_i = the bending moment of the simple beam at i.

M_j = the bending moment of the simple beam at j.

M_k = the bending moment of the simple beam at k.

F_{ji} = the angular flexibility of the equivalent simple beam ij at j.

F_{jk} = the angular flexibility of the equivalent simple beam jk at j.

G_{ij} = the angular carry over value of the equivalent simple beam ij at i.

G_{kj} = the angular carry over value of the equivalent simple beam jk at k.

τ_{ji} = the angular load function of the equivalent simple beam ij at j.

τ_{jk} = the angular load function of the equivalent simple beam jk at j.

It may be easily proved that the changes in slope of the string polygon ϕ_i, ϕ_j, ϕ_k, when applied as elastic loads on the equivalent conjugate beam, develop shears and moments which, at given points i, j, k, are equal to the slopes and deflections of the real beam, respectively. A complete derivation of the equation (1-1) and the proof of the statement mentioned above can be found elsewhere (1).

CHAPTER II

RITTER'S FORMULA

A prismatic member of parabolic variation is considered (Fig. 2-1). The depths of the beam at the ends is denoted by h_0 and h_B respectively. The length of this member is $L\alpha$. The depth of the arbitrarily selected section given by position coordinate x is

$$h_x = h_0 + \frac{h_0\omega}{(L\alpha)^2}x^2 = h_0\left(1 + \frac{\omega}{(L\alpha)^2}x^2\right) = h_0 t_x \qquad (2\text{-}1)$$

where t_x = variable parameter.

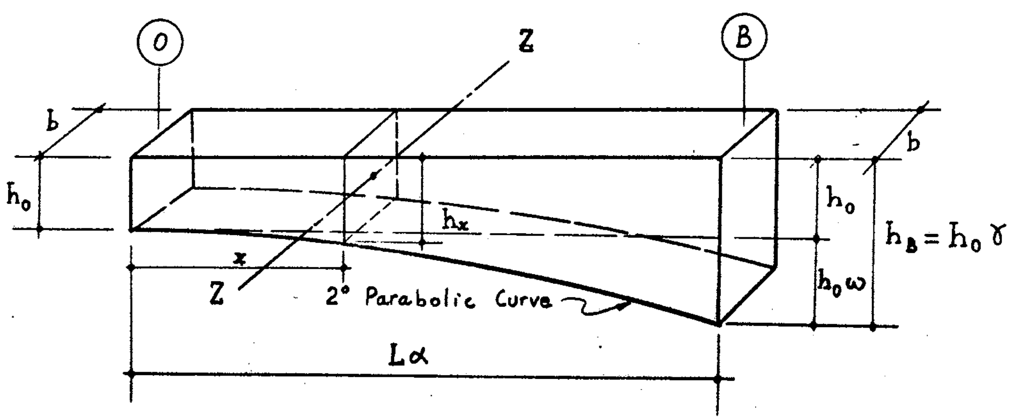


Fig. 2-1

The moment of inertia with respect to principal axis z-z is

$$I_z = \frac{bh_x^3}{12} = \frac{bh_o^3}{12} t_x^3 = I_o t_x^3 \qquad (2\text{-}2)$$

where

I_o = the moment of inertia of the section at left end.

t_x = the variable parameter defined by Eq. (2-1).

In the analysis of these members two typical integral expressions frequently occur:

$$\int_o^{L\alpha} \frac{x^n dx}{EI_x} = \frac{1}{EI_o}\int_o^{L\alpha} \frac{x^n dx}{t_x} = \frac{1}{EI_o}(L\alpha)^{n+1} Q_n^{(L\alpha)} \qquad (2\text{-}3a)$$

and

$$\int_o^{L\alpha k} \frac{x^n dx}{EI_x} = \frac{1}{EI_o}\int_o^{L\alpha k} \frac{x^n dx}{t_x} = \frac{1}{EI_o}(L\alpha k)^{n+1} Q_n^{(L\alpha k)} . \qquad (2\text{-}4a)$$

As the evaluation of these functions is laborious and time consuming, many approximate formulas for the solution of these expressions have been proposed. The most powerful approach has been suggested by Ritter (2). The application of the Ritter's formula to the evaluation of the Q's function is shown in the following part of this thesis.

With the notation

$$h_B = h_o + h_o\omega = h_o\gamma \qquad (2\text{-}5)$$

as shown in Fig. (2-1), the general Q functions of Eqs. (2-3a) and

(2-4a) become:

$$\int_0^{L\alpha} \frac{x^n dx}{EI_x} = \frac{1}{EI_o} \int_0^{L\alpha} x^n \left(\frac{h_o}{h_x}\right)^3 dx \qquad (2\text{-}3b)$$

and

$$\int_0^{L\alpha k} \frac{x^n dx}{EI_x} = \frac{1}{EI_o} \int_0^{L\alpha k} x^n \left(\frac{h_o}{h_x}\right)^3 dx \quad . \qquad (2\text{-}4b)$$

If the function $\left(\frac{h_o}{h_x}\right)^3$ is assumed to be a parabola of 2r degree, the following relationship can be stated:

$$\left(\frac{h_o}{h_x}\right)^3 = 1 - C_B \left(\frac{x}{L}\right)^{2r} \quad . \qquad (2\text{-}6)$$

The numerical constants C_B and r are unknown and must be computed from some special conditions. The graphical interpretation of this equation is shown in Fig. (2-2).

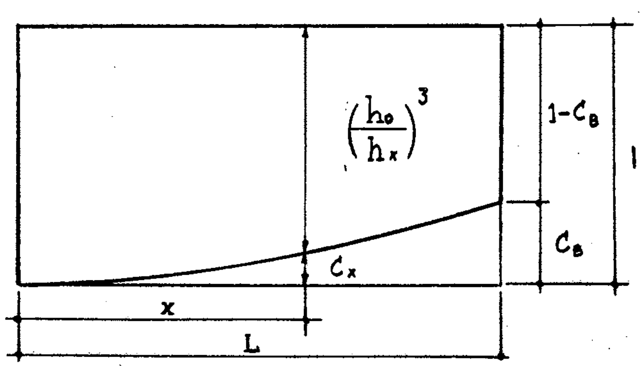


Fig. 2-2

The extreme values of this function are:

$$x = 0, \qquad \frac{h_o}{h_x} = 1 \tag{2-7}$$

$$x = L, \qquad \frac{h_o}{h_B} = \frac{1}{\gamma} \quad . \tag{2-8}$$

For the evaluation of the constants C_B and r, one additional condition is necessary. This condition may be selected arbitrarily. For example:

$$x = \frac{L}{2}, \qquad h_C = h_o \left(1 + \frac{\omega}{4}\right) \quad . \tag{2-9}$$

The meaning of the symbol h_C is explained by Fig. (2-3). The results of Eqs. (2-7, 8, 9) are substituted in the Eq. (2-5) and the constants C_B, C_C, and r are obtained:

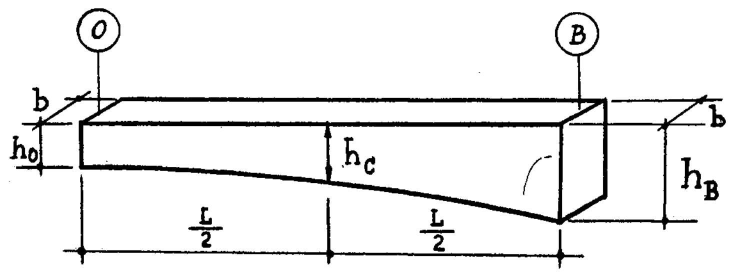


Fig. 2-3

$$C_B = 1 - \left(\frac{1}{1 + \omega}\right)^3 \tag{2-10}$$

$$C_C = 1 - \left(\frac{1}{1 + \frac{\omega}{4}}\right)^3 \tag{2-11}$$

$$r = -1.66 \log \frac{C_C}{C_B} \quad . \tag{2-12}$$

The relationship between ω and r is computed by means of the Eqs. (2-10, 11, 12) and recorded in Table I.

TABLE I

THE RELATIONSHIP BETWEEN ω AND r

ω	$\left(\frac{1}{1+\omega}\right)^3$	C_B	$\frac{1}{(1+\frac{\omega}{4})^3}$	C_C	$\log \frac{C_C}{C_B}$	r
0.1	.751315	.248685	.928576	.071424	-.541800	.899910
0.2	.578704	.421296	.863808	.136192	-.490394	.814527
0.3	.455166	.544834	.804992	.195008	-.446214	.741145
0.4	.364431	.635569	.751296	.248704	-.407479	.676808
0.5	.296296	.703704	.702336	.297664	-.373660	.620636
0.6	.244141	.755859	.657536	.342464	-.343825	.571081
0.7	.203542	.796458	.616448	.383552	-.317340	.527090
0.8	.171468	.828532	.578688	.421312	-.293701	.487827
0.9	.145794	.854206	.544000	.456000	-.272597	.452774
1.0	.125000	.875000	.512000	.488000	-.253592	.421207
1.1	.107980	.892020	.482496	.517504	-.236460	.392752
1.2	.093914	.906086	.455168	.544832	-.220909	.366922
1.3	.082190	.917810	.429888	.570112	-.206790	.343471
1.4	.072338	.927662	.406464	.593536	-.193942	.322131
1.5	.064000	.936000	.384704	.615296	-.182190	.302611
1.6	.056896	.943104	.364416	.635584	-.171385	.284664
1.7	.050805	.949195	.345600	.654400	-.161510	.268262
1.8	.045554	.954446	.328000	.672000	-.152384	.253104
1.9	.041002	.958998	.311680	.688320	-.144027	.239224
2.0	.037037	.962963	.296320	.703680	-.136237	.226285

CHAPTER III

BASIC FUNCTIONS

The algebraic solution of the basic functions Q's defined by Eqs. (2-3b,4b) for the most important cases is presented in this thesis.

The general solution of the Eq. (2-3b) follows

$$Q_n^{(L\alpha)} = \frac{1}{(L\alpha)^{n+1}} \int_0^{L\alpha} x^n \left(\frac{h_o}{h_x}\right)^3 dx$$

$$= \frac{1}{(L\alpha)^{n+1}} \int_0^{L\alpha} x^n \left[1 - C_B\left(\frac{x}{L\alpha}\right)^{2r}\right] dx$$

$$= \frac{1}{(L\alpha)^{n+1}} \left[\frac{x^{n+1}}{n+1} - C_B \frac{x^{2r+n+1}}{(L\alpha)^{2r}(2r+n+1)}\right]_0^{L\alpha}$$

$$= \frac{1}{n+1} - \frac{C_B}{2r+n+1} \quad . \tag{3-1}$$

If the exponent $n=0,1,2$, the $Q_n^{(L\alpha)}$ functions become:

$$\left.\begin{aligned} Q_0^{(L\alpha)} &= 1 - \frac{C_B}{2r+1} \\ Q_1^{(L\alpha)} &= 2 - \frac{C_B}{2r+2} \\ Q_2^{(L\alpha)} &= 3 - \frac{C_B}{2r+3} \end{aligned}\right\} \tag{3-2}$$

The general solution of the Eq. (2-4b) gives:

$$Q_n^{(L\alpha k)} = \frac{1}{(L\alpha k)^{n+1}} \int_0^{L\alpha k} x^n \left(\frac{h_o}{h_x}\right)^3 dx$$

$$= \frac{1}{(L\alpha k)^{n+1}} \int_0^{L\alpha k} \left[1 - C_B\left(\frac{x}{L\alpha}\right)^{2r}\right] x^n dx$$

$$= \frac{1}{(L\alpha k)^{n+1}} \left[\frac{x^{n+1}}{n+1} - C_B \frac{x^{2r+n+1}}{(L\alpha)^{2r}(2r+n+1)}\right]_0^{L\alpha k}$$

$$= \frac{1}{(L\alpha k)^{n+1}} \left[\frac{(L\alpha k)^{n+1}}{n+1} - \frac{C_B}{(L\alpha)^{2r}} \frac{(L\alpha k)^{2r+n+1}}{2r+n+1}\right]$$

$$= \frac{1}{n+1} - \frac{C_B\ k^{2r}}{2r+n+1} . \tag{3-3}$$

If the exponent $n = 0, 1, 2$, the $Q_n^{(L\alpha k)}$ functions become:

$$\left.\begin{aligned} Q_0^{(L\alpha k)} &= 1 - \frac{C_B\ k^{2r}}{2r+1} \\ Q_1^{(L\alpha k)} &= 2 - \frac{C_B\ k^{2r}}{2r+2} \\ Q_2^{(L\alpha k)} &= 3 - \frac{C_B\ k^{2r}}{2r+3} . \end{aligned}\right\} \tag{3-4}$$

CHAPTER IV

BEAM WITH ONE PARABOLIC HAUNCH

The angular constants for a prismatic member with a parabolic haunch at the right end, simply supported at both ends are derived in a general form (Fig. 4-1).

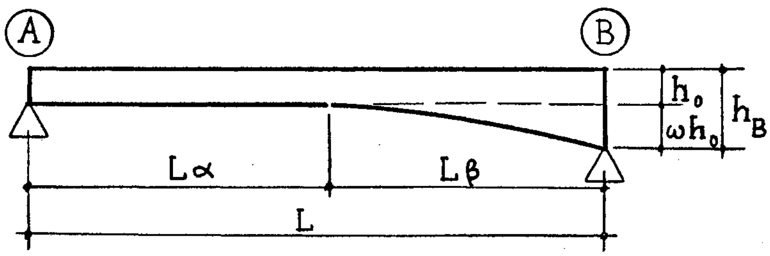


Fig. 4-1

a) Angular Flexibilities and Carry Over Values. If the beam shown in Fig. (4-1) is acted upon by a unit moment applied at A, the end slopes of the elastic curve at A and B are called the angular flexibility F_{AB} and the angular carry over value G_{BA}, respectively (Fig. 4-2). The bending moment, the string polygon, and the conjugate beam are shown in the same figure. The elastic weights ϕ_A, ϕ_D, ϕ_B shown in Fig. (4-2) and defined by Eq. (1-1) are:

$$\left.\begin{aligned} \phi_A &= F_{AD} + \beta G_{DA} \\ \phi_D &= G_{AD} + \beta (F_{DA} + F_{DB}) \\ \phi'_B &= \beta G_{DB} \,. \end{aligned}\right\} \qquad (4\text{-}1)$$

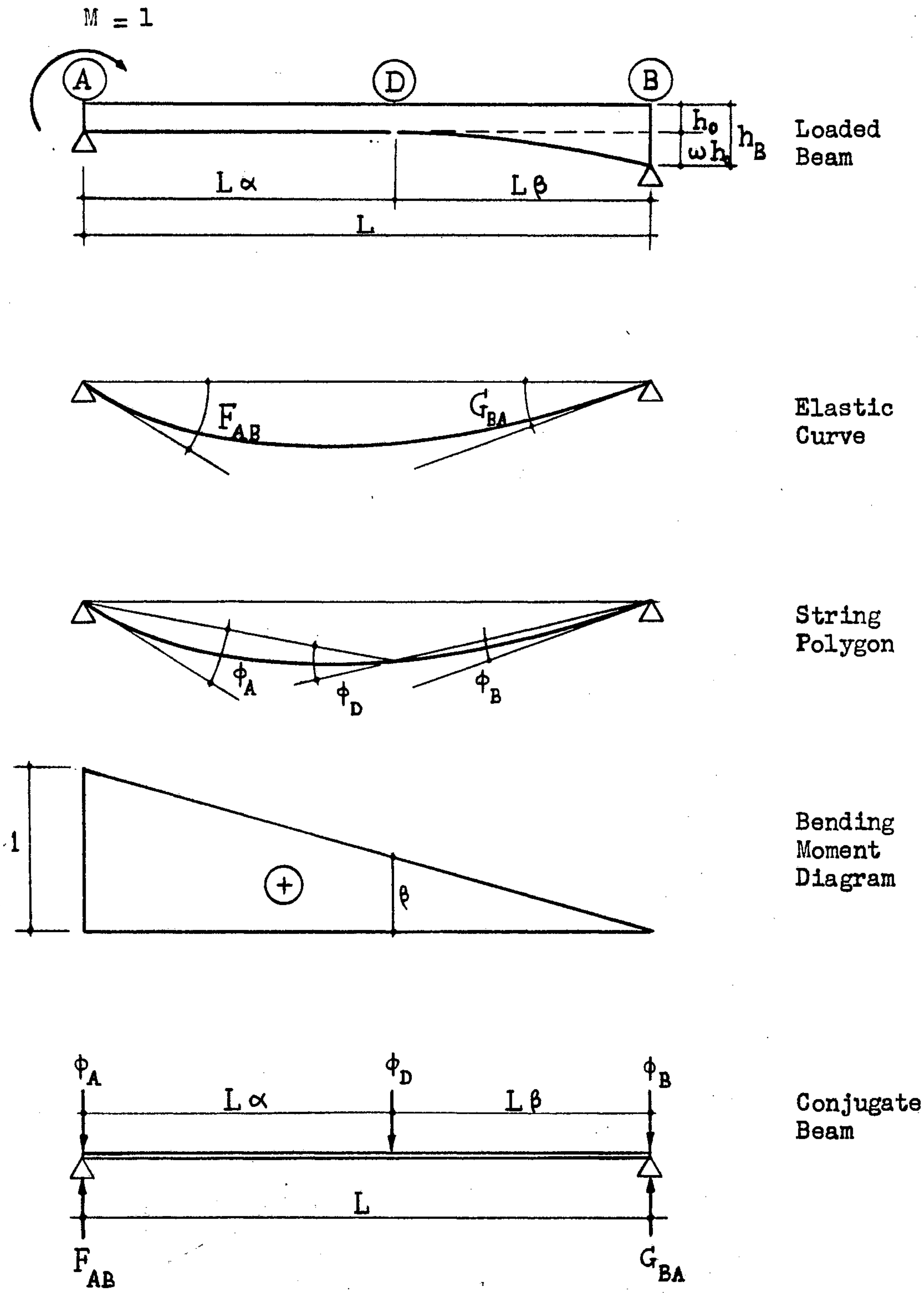

M = 1
A
D
B
h_o
ωh_o
h_B
Loaded Beam
$L\alpha$
$L\beta$
L
F_{AB}
G_{BA}
Elastic Curve
ϕ_A
ϕ_D
ϕ_B
String Polygon
1
+
β
Bending Moment Diagram
ϕ_A
ϕ_D
ϕ_B
$L\alpha$
$L\beta$
L
F_{AB}
G_{BA}
Conjugate Beam

Fig. 4-2

The angular flexibility F_{AB} is the left reaction of the conjugate beam (Fig. 4-2). From statics

$$F_{AB} = \phi_A + \phi_D \beta$$

$$= (F_{AD} + 2\beta G_{AD} + \beta^2 F_{DA}) + \beta^2 F_{DB} \qquad (4\text{-}2a)$$

or in terms of Q's (Eqs. 3-2)

$$F_{AB} = \frac{L\alpha}{3EI_0}(1 + \beta + \beta^2) + \frac{L\beta^3}{EI_0}\left(Q_0^{(L\beta)} - 2Q_1^{(L\beta)} + Q_2^{(L\beta)}\right). \qquad (4\text{-}2b)$$

The carry over value G_{BA} is the right reaction of the conjugate beam (Fig. 4-2):

$$G_{BA} = \phi_B + \phi_D \alpha$$

$$= \alpha(G_{AD} + \beta F_{DA}) + \beta(G_{DB} + \alpha F_{DB}) \qquad (4\text{-}3a)$$

or in terms of Q's (Eqs. 3-2)

$$G_{BA} = \frac{L\alpha^2}{6EI_0}(1 + 2\beta) + \frac{L\beta^2}{EI_0}\left[\alpha\left(Q_0^{(L\beta)} - Q_1^{(L\beta)}\right) + \beta\left(Q_1^{(L\beta)} - Q_2^{(L\beta)}\right)\right].$$

$$(4\text{-}3b)$$

If the beam shown in Fig. (4-1) is acted upon by a unit moment applied at B, the end slopes of the elastic curve at A and B are called the angular carry over value G_{AB} and the angular flexibility F_{BA}, respectively (Fig. 4-3). The elastic weights ϕ_A, ϕ_D, ϕ_B in Fig. (4-3) are:

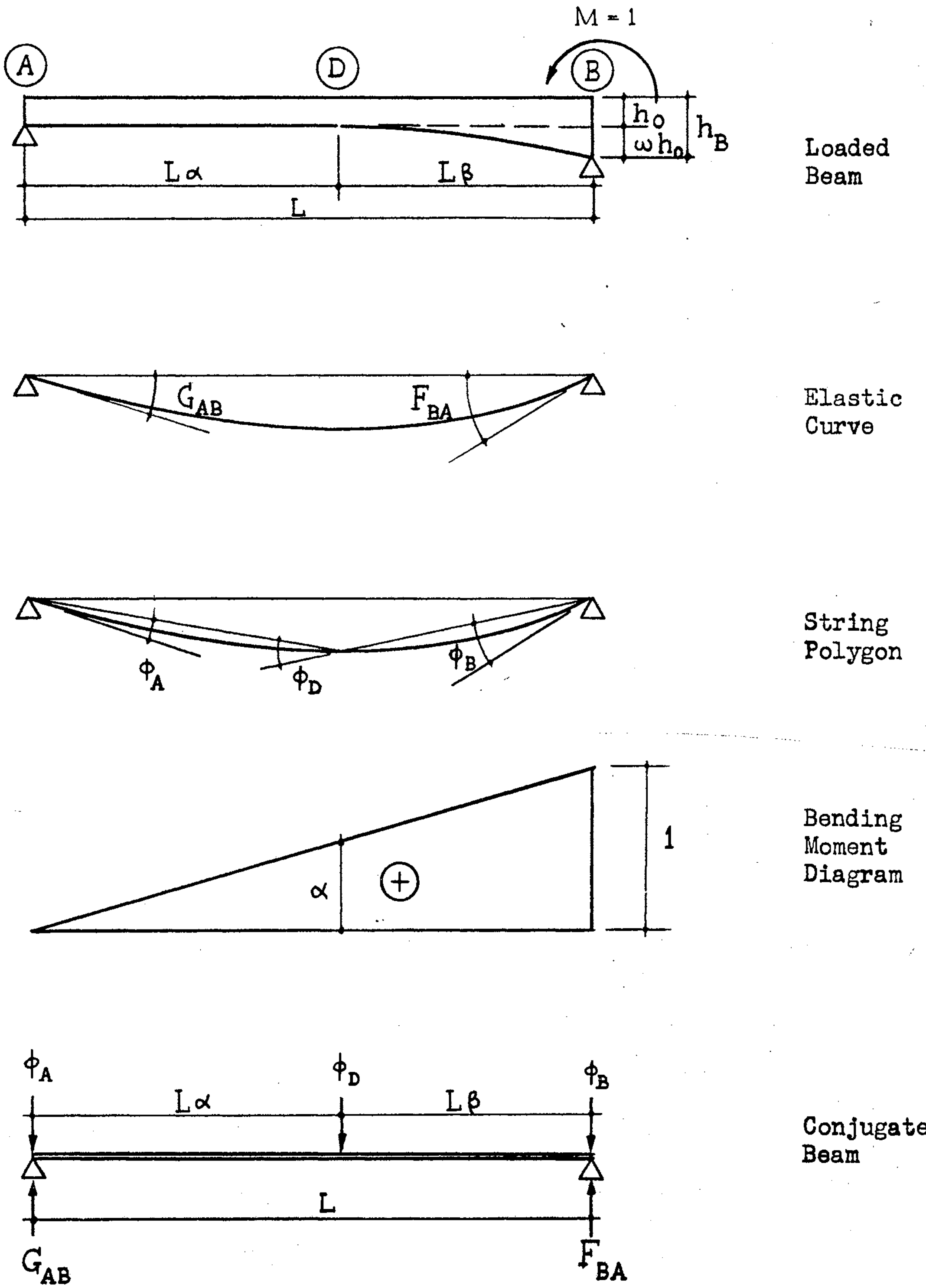

M = 1
A
D
B
h_0
ωh_0
h_B
$L\alpha$
$L\beta$
L
Loaded Beam
G_{AB}
F_{BA}
Elastic Curve
ϕ_A
ϕ_D
ϕ_B
String Polygon
1
α
+
Bending Moment Diagram
ϕ_A
ϕ_D
ϕ_B
$L\alpha$
$L\beta$
L
G_{AB}
F_{BA}
Conjugate Beam

Fig. 4-3

$$\left.\begin{aligned} \phi_A &= \alpha G_{DA} \\ \phi_D &= \alpha(F_{DA} + F_{DB}) + G_{BD} \\ \phi_B &= \alpha G_{DB} + F_{BD} \,. \end{aligned}\right\} \qquad (4\text{-}4)$$

The angular carry over value G_{AB} is the left reaction of the conjugate beam (Fig. 4-3).

$$G_{AB} = \phi_A + \phi_D \beta$$

$$= \alpha(G_{DA} + \beta F_{DA}) + \beta(G_{BD} + \alpha F_{DB}) \qquad (4\text{-}5a)$$

or in terms of Q's

$$G_{AB} = \frac{L\alpha^2}{6EI_0}(1 + 2\beta) + \frac{L\beta^2}{EI_0}\left[\alpha(Q_0^{(L\beta)} - Q_1^{(L\beta)}) + \beta(Q_1^{(L\beta)} - Q_2^{(L\beta)})\right]. \qquad (4\text{-}5b)$$

The angular flexibility F_{BA} is the right reaction of the conjugate beam (Fig. 4-3):

$$F_{BA} = \phi_B + \phi_D \alpha$$

$$= \alpha^2 F_{DA} + (F_{BD} + 2\alpha G_{BD} + \alpha^2 F_{DB}) \qquad (4\text{-}6a)$$

or in terms of Q's

$$F_{BA} = \frac{L\alpha^3}{3EI_0} + \frac{L\beta}{EI_0}(\alpha^2 Q_0^{(L\beta)} + 2\alpha\beta Q_1^{(L\beta)} + \beta^2 Q_2^{(L\beta)}). \qquad (4\text{-}6b)$$

b) <u>Angular Load Functions</u> If the beam shown in Fig. (4-1) is acted upon by a unit load moving gradually from A to B, the end slopes of the elastic curve at A and B are called the influence values of angular load function. The general formulas of the influence values will depend upon the position of the unit load, that is, whether it is located within the haunch or in the straight part of the member.

Fig. (4-4) shows a beam acted upon by a unit load applied in the straight part of the member. By method of superposition, the resultant effect can be obtained by adding together two partial effects (a) and (b). The angular load functions due to the bending moment of part (a) follow

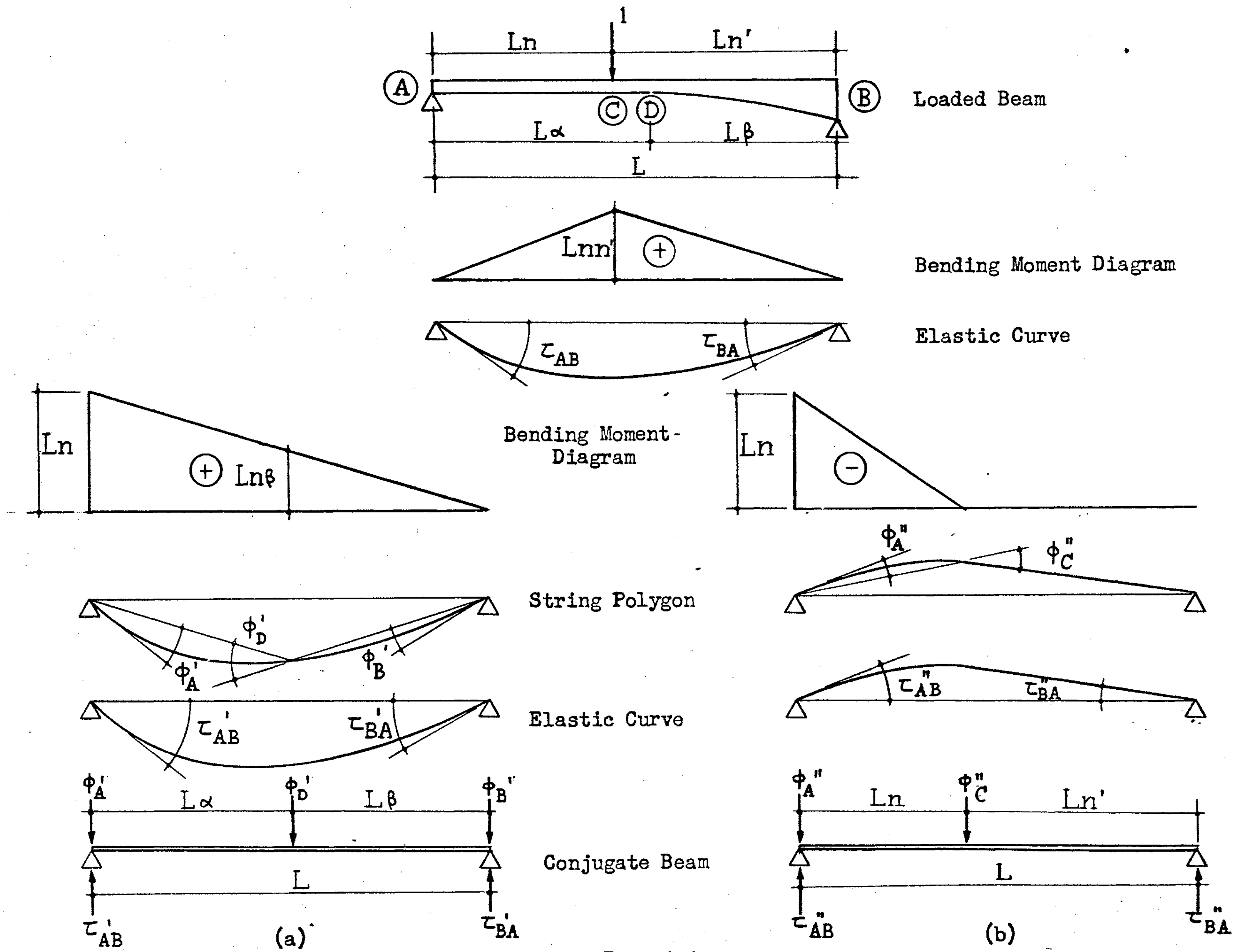
$$\left.\begin{aligned} \tau'_{AB} &= LnF_{AB} \\ \tau'_{BA} &= LnG_{BA}\,. \end{aligned}\right\} \qquad (4\text{-}7)$$

The elastic weights ϕ''_A, ϕ''_C due to the bending moment of part (b) are:

$$\left.\begin{aligned} \phi''_A &= -LnF_{AC} \\ \phi''_C &= -LnG_{AC}\,. \end{aligned}\right\} \qquad (4\text{-}8)$$

The angular load functions τ''_{AB}, τ''_{BA} are the reactions of the conjugate beam at left and right ends, respectively.

$$\left.\begin{aligned} \tau''_{AB} &= \phi_A + \phi_C n' = -Ln(F_{AC} + n'G_{AC}) = -\frac{(Ln)^2}{6EI_0}(2 + n') \\ \tau''_{BA} &= \phi_C n = -Ln^2 G_{AC} = -\frac{(Ln)^2 n}{6EI_0}\,. \end{aligned}\right\} \qquad (4\text{-}9)$$



1
Ln
Ln'
A
B
C
D
Lα
Lβ
L
Loaded Beam
Lnn'
Bending Moment Diagram
τAB
τBA
Elastic Curve
Ln
Lnβ
Bending Moment-Diagram
Ln
φ"A
φ"C
String Polygon
φ'D
φ'A
φ'B
τ"AB
τ"BA
Elastic Curve
τ'AB
τ'BA
φ'A
Lα
φ'D
Lβ
φ'B
Conjugate Beam
L
τ'AB
τ'BA
(a)
φ"A
Ln
φ"C
Ln'
L
τ"AB
τ"BA
(b)

Fig. 4-4

The angular load functions τ_{AB}, τ_{BA} may be obtained by adding the Eqs. (4-7) and (4-9)

$$\left.\begin{aligned}\tau_{AB} &= Ln\left[F_{AB} - \frac{Ln}{6EI_0}(2 + n')\right]\\ \tau_{BA} &= Ln\left[G_{BA} - \frac{Ln^2}{6EI_0}\right].\end{aligned}\right\} \qquad (4\text{-}10)$$

If the beam shown in Fig. (4-1) is acted upon by a unit load applied at a point within the haunch (Fig. 4-5), the angular load functions due to the bending moment of part (a) follow :

$$\left.\begin{aligned}\tau'_{AB} &= LnF_{AB}\\ \tau'_{BA} &= LnG_{BA}\ .\end{aligned}\right\} \qquad (4\text{-}11)$$

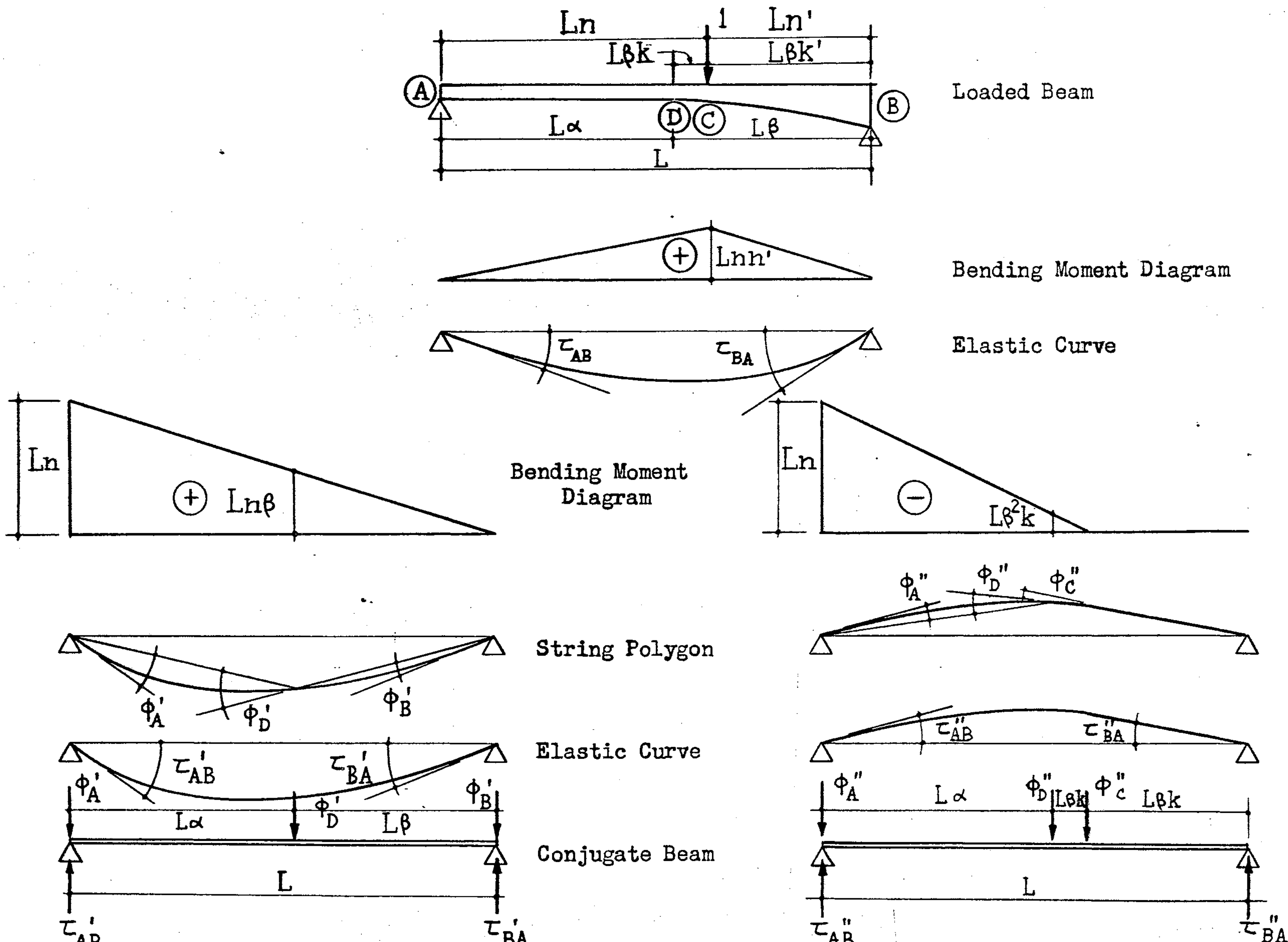
The elastic weights ϕ''_A, ϕ''_D, ϕ''_C shown in the Fig.(4-5b)are:

$$\left.\begin{aligned}\phi''_A &= -LnF_{AD} - LkG_{DA}\\ \phi''_D &= -LnG_{AD} - Lk(F_{DA} + F_{DC})\\ \phi''_C &= -LkG_{DC}\ .\end{aligned}\right\} \qquad (4\text{-}12)$$

The angular load functions τ''_{AB}, τ''_{BA} are the reactions of the conjugate beam at left and right ends, respectively.

$$\begin{aligned}\tau''_{AB} &= \phi''_A + \beta\phi''_D + \phi''_C\\ &= -\frac{L^2\alpha}{6EI_0}(6n - 3n\alpha - 3\alpha + 2\alpha^2) - Lk\beta^2F_{DC} - Lk\beta n'G_{DC}\end{aligned}$$

(4-13 a)


Ln
1
Ln'
Lβk
Lβk'
A
B
D
C
Lα
Lβ
L
Loaded Beam
Lnn'
Bending Moment Diagram
Elastic Curve
τ_{AB}
τ_{BA}
Ln
Lnβ
Bending Moment Diagram
Ln
$L\beta^2k$
ϕ_A''
ϕ_D''
ϕ_C''
String Polygon
ϕ_A'
ϕ_D'
ϕ_B'
Elastic Curve
τ_{AB}'
τ_{BA}'
τ_{AB}''
τ_{BA}''
ϕ_A'
ϕ_D'
ϕ_B'
Lα
Lβ
Conjugate Beam
L
τ_{AB}'
τ_{BA}'
ϕ_A''
Lα
ϕ_D''
Lβk
ϕ_C''
Lβk
L
τ_{AB}''
τ_{BA}''

(a) Fig. 4-5 (b)

$$\tau''_{BA} = \alpha \phi''_D + n \phi''_C$$

$$= -\frac{(L\alpha)^2}{6EI_0}(3n - 2\alpha) - Lk\beta\alpha F_{DC} - Lk\beta n G_{DC} \tag{4-13a}$$

or in terms of Q s

$$\left.\begin{aligned} \tau''_{AB} &= -\frac{L^2\alpha}{6EI_0}(6n-3n\alpha-3\alpha+2\alpha^2) - \frac{(Lk\beta)^2\beta}{EI_0}\left[Q_0^{(L\beta k)} - (k+1)\,Q_1^{(L\beta k)} + k\,Q_2^{(L\beta k)}\right] \\ \tau''_{BA} &= -\frac{(L\alpha)^2}{6EI_0}(3n-2\alpha) - \frac{(Lk\beta)^2}{EI_0}\left[\alpha Q_0^{(L\beta k)} + (k\beta-\alpha)\,Q_1^{(L\beta k)} - k\beta Q_2^{(L\beta k)}\right]. \end{aligned}\right\} \tag{4-13b}$$

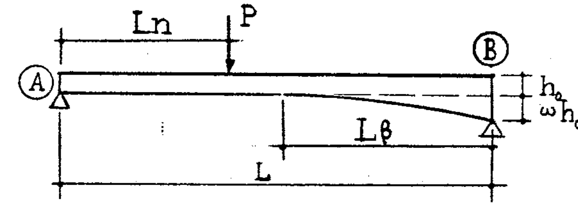
The angular load functions τ_{AB}, τ_{BA} may be obtained by adding the Eqs. (4-12) and (4-13b) :

$$\left.\begin{aligned} \tau_{AB} &= LnF_{AB} - \left\{\frac{L^2\alpha}{6EI_0}(6n-3n\alpha-3\alpha+2\alpha^2) + \frac{(Lk\beta)^2\beta}{EI_0}\left[Q_0^{(L\beta k)} - (k+1)\,Q_1^{(L\beta k)} + k\,Q_2^{(L\beta k)}\right]\right\} \\ \tau_{BA} &= LnG_{BA} - \left\{\frac{(L\alpha)^2}{6EI_0}(3n-2\alpha) + \frac{(Lk\beta)^2}{EI_0}\left[Q_0^{(L\beta k)} + (k\beta-\alpha)\,Q_1^{(L\beta k)} - k\beta\,Q_2^{(L\beta k)}\right]\right\}. \end{aligned}\right] \tag{4-14}$$

The table of beam constants for the beam with one parabolic haunch is computed and listed in Tables II and III.

TABLE II

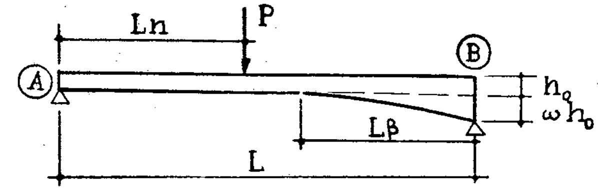
BEAM CONSTANTS FOR BEAM WITH
ONE PARABOLIC HAUNCH
(END A)



Right Haunch		Angular Flexibilities Coef.x $\frac{L}{EI_0}$	Angular Carry-Over Values Coef.x $\frac{L}{EI_0}$	Influence Values of Angular Load Functions; Coef.x $\frac{PL^2}{EI_0}$								
				n								
				0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
β	ω	F_{AB}	G_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}
0.3	1.0	.3310	.1540	.0283	.0475	.0588	.0731	.0613	.0546	.0539	.0303	.0155
	1.2	.3305	.1523	.0283	.0474	.0587	.0729	.0611	.0543	.0536	.0300	.0152
	1.4	.3301	.1507	.0282	.0473	.0585	.0727	.0609	.0541	.0533	.0297	.0150
	1.6	.3298	.1492	.0282	.0473	.0584	.0726	.0607	.0539	.0531	.0295	.0149
	1.8	.3294	.1478	.0281	.0472	.0583	.0725	.0605	.0536	.0538	.0293	.0148
	2.0	.3291	.1466	.0281	.0471	.0581	.0723	.0604	.0535	.0536	.0291	.0147
0.4	1.0	.3278	.1455	.0280	.0469	.0578	.0718	.0597	.0527	.0418	.0287	.0146
	1.2	.3268	.1426	.0279	.0467	.0575	.0714	.0592	.0521	.0412	.0282	.0143
	1.4	.3258	.1400	.0278	.0465	.0572	.0710	.0587	.0515	.0406	.0277	.0140
	1.6	.3249	.1376	.0277	.0463	.0570	.0707	.0583	.0509	.0400	.0272	.0137
	1.8	.3241	.1354	.0276	.0461	.0567	.0703	.0579	.0505	.0395	.0267	.0135
	2.0	.3233	.1334	.0275	.0459	.0565	.0700	.0574	.0500	.0390	.0262	.0133
0.5	1.0	.3225	.1357	.0275	.0458	.0563	.0697	.0571	.0496	.0391	.0277	.0135
	1.2	.3205	.1316	.0273	.0454	.0557	.0689	.0561	.0485	.0381	.0260	.0131
	1.4	.3187	.1279	.0271	.0450	.0551	.0682	.0552	.0475	.0372	.0253	.0128
	1.6	.3170	.1245	.0269	.0447	.0546	.0675	.0543	.0466	.0363	.0246	.0125
	1.8	.3153	.1214	.0267	.0444	.0539	.0668	.0530	.0457	.0354	.0240	.0122
	2.0	.3138	.1186	.0266	.0439	.0536	.0662	.0527	.0459	.0345	.0234	.0119

TABLE III

BEAM CONSTANTS FOR BEAM WITH
ONE PARABOLIC HAUNCH
(END B)



Right Haunch		Angular Flexi-bilities Coef.x $\frac{L}{EI_0}$	Angular Carry-Over Values Coef.x $\frac{L}{EI_0}$	Influence Values of Angular Load Functions; Coef.x $\frac{PL^2}{EI_0}$								
				n								
				0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
β	ω	F_{BA}	G_{BA}	τ_{BA}	τ_{BA}	τ_{BA}	τ_{BA}	τ_{BA}	τ_{BA}	τ_{BA}	τ_{BA}	τ_{BA}
0.3	1.0	.2186	.1540	.0152	.0295	.0417	.0509	.0562	.0564	.0506	.0382	.0207
	1.2	.2082	.1523	.0150	.0292	.0412	.0502	.0556	.0554	.0494	.0371	.0200
	1.4	.1993	.1507	.0149	.0288	.0407	.0496	.0546	.0545	.0483	.0361	.0193
	1.6	.1916	.1492	.0147	.0285	.0403	.0490	.0538	.0535	.0472	.0351	.0186
	1.8	.1848	.1478	.0146	.0283	.0398	.0484	.0531	.0527	.0463	.0341	.0180
	2.0	.1788	.1466	.0145	.0280	.0395	.0480	.0525	.0520	.0454	.0331	.0174
0.4	1.0	.1913	.1455	.0144	.0278	.0392	.0475	.0520	.0513	.0451	.0336	.0183
	1.2	.1789	.1426	.0141	.0272	.0383	.0463	.0513	.0496	.0431	.0319	.0170
	1.4	.1684	.1400	.0138	.0267	.0375	.0453	.0492	.0480	.0414	.0304	.0161
	1.6	.1594	.1376	.0136	.0262	.0368	.0443	.0480	.0466	.0399	.0291	.0154
	1.8	.1515	.1354	.0133	.0257	.0361	.0435	.0469	.0452	.0385	.0279	.0148
	2.0	.1446	.1334	.0131	.0254	.0355	.0427	.0459	.0440	.0372	.0267	.0143
0.5	1.0	.1686	.1357	.0134	.0258	.0362	.0436	.0469	.0456	.0397	.0294	.0160
	1.2	.1549	.1316	.0130	.0250	.0350	.0419	.0450	.0432	.0373	.0276	.0148
	1.4	.1433	.1279	.0126	.0243	.0339	.0406	.0432	.0411	.0352	.0258	.0137
	1.6	.1335	.1245	.0123	.0235	.0329	.0391	.0415	.0392	.0333	.0242	.0128
	1.8	.1249	.1214	.0119	.0230	.0319	.0379	.0399	.0375	.0315	.0228	.0120
	2.0	.1165	.1186	.0117	.0224	.0311	.0364	.0385	.0358	.0300	.0217	.0115

CHAPTER V

BEAM WITH TWO SYMMETRICAL PARABOLIC HAUNCHES

The angular constants for a prismatic member with two symmetrical parabolic haunches, simply supported at both ends, are derived in a general form (Fig. 5-1).

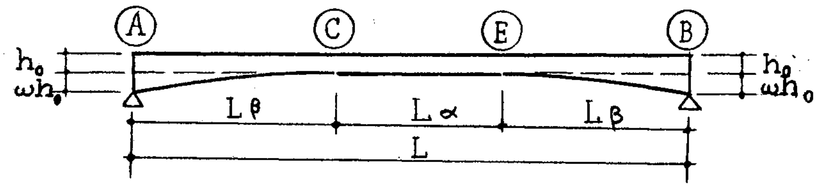


Fig. 5-1

a) Angular Flexibilities and Carry Over Values. If the beam shown in Fig. (5-1) is acted upon by a unit moment applied at A, the end slopes of the elastic curve at A and B are called the angular flexibility F_{AB} and the angular carry over value G_{BA}, respectively. Fig. 5-2 shows the bending moment, the elastic curve, the string polygon, and the conjugate beam. The elastic weights ϕ_A, ϕ_C, ϕ_D, ϕ_B shown in Fig. (5-2) and defined by Eq. (1-1) are:

$$\left.\begin{aligned} \phi_A &= F_{AC} + (1-\beta)\, G_{CA} \\ \phi_C &= G_{AC} + (1-\beta)\,(F_{CA} + F_{CE}) + \beta\, G_{EC} \\ \phi_E &= (1-\beta)\, G_{CE} + \beta\,(F_{EC} + F_{EB}) \\ \phi_B &= \beta\, G_{EB} \quad . \end{aligned}\right\} \qquad (5\text{-}1)$$

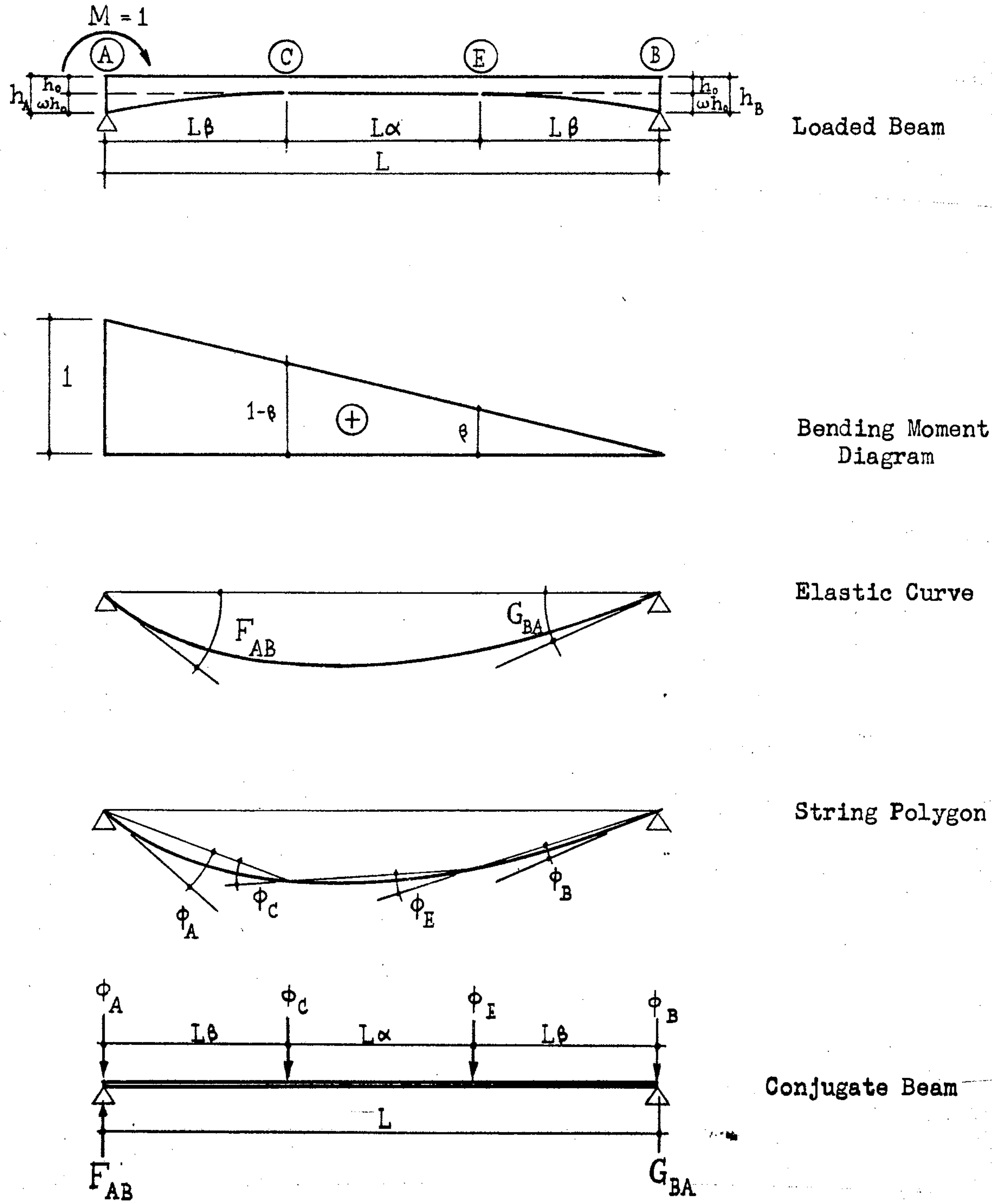

M = 1
A
C
E
B
Loaded Beam
1
1-β
β
Bending Moment Diagram
F_AB
G_BA
Elastic Curve
String Polygon
Conjugate Beam

Fig. 5-2

The angular flexibility F_{AB} is the left reaction of the conjugate beam (Fig. 5-2).

$$F_{AB} = \phi_A + (1-\beta)\,\phi_C + \beta\,\phi_E$$

$$= F_{AC} + 2(1-\beta)G_{AC} + (1-2\beta+2\beta^2)\,F_{CA} + \frac{L\alpha}{3EI_0}(1-\beta+\beta^2) \qquad (5\text{-}2a)$$

or in terms of Q s

$$F_{AB} = \frac{L\beta}{EI_0}\left[(1 - 2\beta + 2\beta^2)\,Q_0^{(L\beta)} + 2\beta(1 - 2\beta)\,Q_1^{(L\beta)} + 2\beta^2\,Q_2^{(L\beta)}\right]$$

$$+ \frac{L\alpha}{3EI_0}\,(1 - \beta + \beta^2). \qquad (5\text{-}2b)$$

The carry over value G_{BA} is the right reaction of the conjugate beam (Fig. 5-2).

$$G_{BA} = \beta\,\phi_C + (1-\beta)\,\phi_E + \phi_B$$

$$= 2\beta(1-\beta)\,F_{CA} + 2\beta G_{AC} + \frac{L\alpha}{6EI_0}\,(1 + 2\beta - 2\beta^2) \qquad (5\text{-}3a)$$

or in terms of Q s

$$G_{BA} = \frac{2L\beta^2}{EI_0}\left[(1-\beta)\,Q_0^{(L\beta)} - (1 - 2\beta)\,Q_1^{(L\beta)} - \beta Q_2^{(L\beta)}\right]$$

$$+ \frac{L\alpha}{6EI_0}\,(1 + 2\beta - 2\beta^2). \qquad (5\text{-}3b)$$

If the beam shown in Fig. (5-1) is acted upon by a unit moment applied at B, the end slopes of the elastic curve at A and B are called the angular carry over value G_{AB} and the angular flexibility F_{BA}, respectively. Because the beam shown in Fig. (5-1) is symmetrical,

$$F_{BA} = F_{AB} \tag{5-4}$$

and

$$G_{AB} = G_{BA} \ . \tag{5-5}$$

b) Angular Load Functions If the beam shown in Fig. (5-1) is acted upon by a unit load moving gradually from A to B, the end slopes of the elastic curve at A and B are called the influence values of angular load function. The procedure for determining these influence values is similar to that for a beam with one haunch. The general formulas of the influence values will depend upon the position of the load.

The beam shown in Fig. (4-3) is acted upon by a unit load applied in the intervening straight part of the member. The formula will be derived by adding together two partial effects, (a) and (b). The angular load functions due to the bending moment of part (a) follow:

$$\left.\begin{aligned} \tau'_{AB} &= Ln\, F_{AB} \\ \tau'_{BA} &= Ln\, G_{BA} \ . \end{aligned}\right\} \tag{5-6}$$

The elastic weights ϕ_A, ϕ_C, ϕ_P due to the bending moment of part (b) are:

$$\left.\begin{aligned} \phi_A &= -\, Ln\, F_{AC} - L\alpha k\, G_{CA} \\ \phi_C &= -\, Ln\, G_{AC} - L\alpha k\, (F_{CA} + F_{CP}) \\ \phi_P &= -\, L\alpha k\, G_{CP} \ . \end{aligned}\right\} \tag{5-7}$$

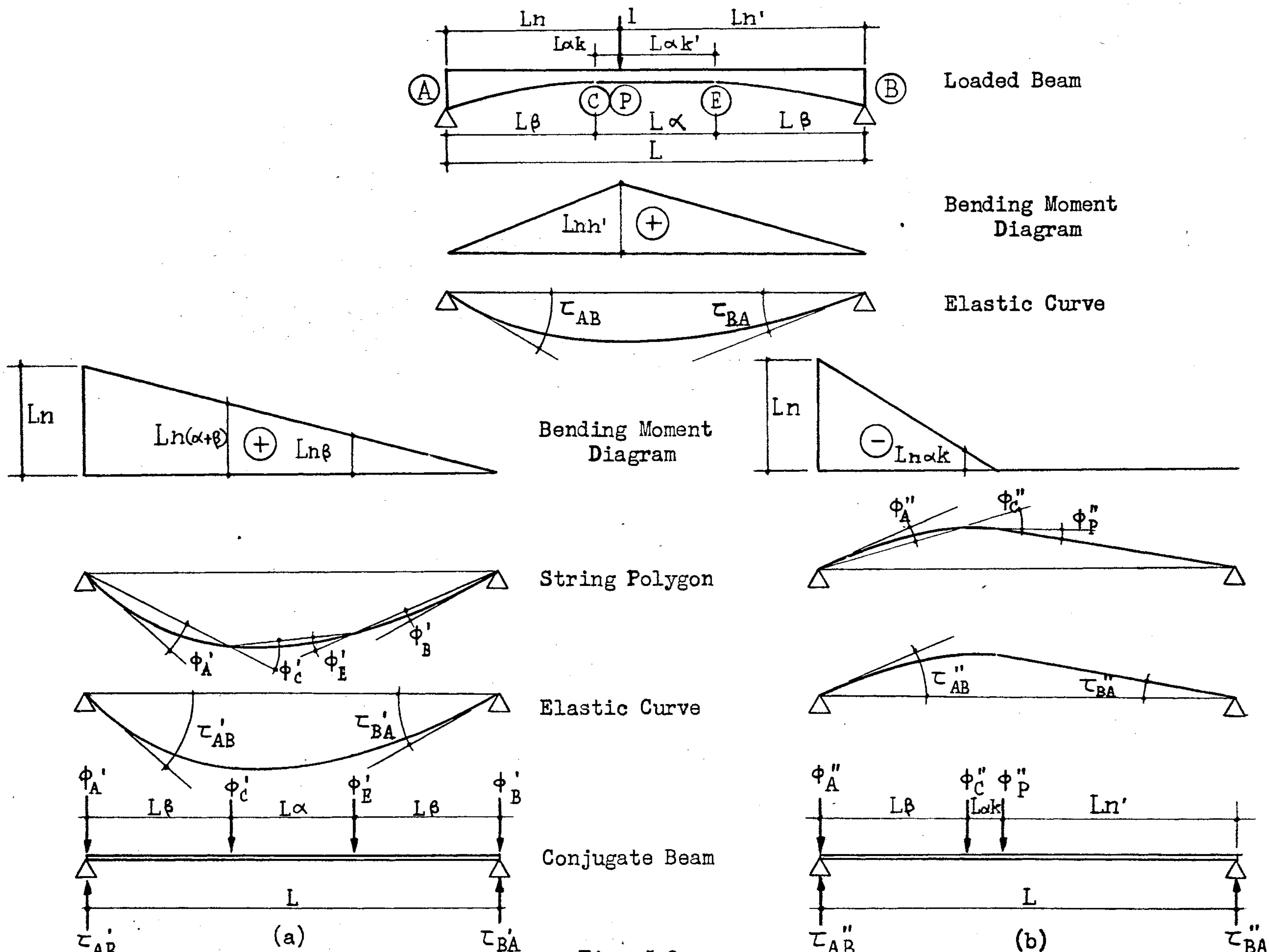

Loaded Beam
Bending Moment Diagram
Elastic Curve
Bending Moment Diagram
String Polygon
Elastic Curve
Conjugate Beam
(a)
(b)

Fig. 5-3

The angular load functions τ''_{AB}, τ''_{BA} are the reactions of the conjugate beam at left and right ends, respectively.

$$\left.\begin{aligned}
\tau''_{AC} &= \phi_A + (1-\beta)\,\phi_C + n'\,\phi_P \\
&= -\,Ln\,F_{AC} - L(2n-\beta-n\beta)\;G_{AC} - L\alpha k\,(1-\beta)\,F_{CA} \\
&\qquad - \frac{(L\alpha k)^2}{6EI_0}\,(3-2\beta-n) \\
\tau''_{BA} &= \beta\phi_C + n\,\phi_P \\
&= -\,L\beta n\,G_{AC} - L\beta\alpha k\,F_{CA} - \frac{(L\alpha k)^2}{6EI}\,(2\beta+n)\quad .
\end{aligned}\right\}\quad (5\text{-}8)$$

The angular load functions τ_{AB}, τ_{BA} may be obtained by adding the Eqs. (5-6) and(5-8) :

$$\left.\begin{aligned}
\tau_{AB} &= Ln\,F_{AB} - \left\{\frac{L\beta}{EI_0}\left[\alpha k(1-\beta)\;Q_0^{(L\beta)} + (n'+2\alpha k)\,\beta\,Q_1^{(L\beta)} + \beta^2\,Q_2^{(L\beta)}\right]\right. \\
&\qquad\left. + \frac{(L\alpha k)^2}{6EI_0}(3-2\beta-n)\right\} \\
\tau_{BA} &= Ln\,G_{BA} - \left\{\frac{(L\beta)^2}{EI_0}\left[\alpha k Q_0^{(L\beta)} + (\beta-\alpha k)\;Q_1^{(L\beta)} - \beta Q_2^{(L\beta)}\right]\right. \\
&\qquad\left. + \frac{(L\alpha k)^2}{6EI_0}(2\beta+n)\right\}\quad .
\end{aligned}\right]\quad (5\text{-}9)$$

If the beam shown in Fig. (5-1) is acted upon by a unit load applied at a point within one of the haunches (Fig. 5-4), the angular load functions due to the bending moment of part (a) follow:

$$\left.\begin{aligned}
\tau'_{AB} &= Ln\,F_{AB} \\
\tau'_{BA} &= Ln\,G_{BA}\quad .
\end{aligned}\right]\quad (5\text{-}10)$$

The elastic weights ϕ_A, ϕ_C, ϕ_E, ϕ_D due to the bending moment of part (b) are:

$$\left.\begin{aligned}
\phi_A &= - \; Ln\, F_{AC} - L\,(n-\beta)\, G_{CA} \\
\phi_C &= - \; Ln\, G_{AC} - L\,(n-\beta)\,(F_{CA} + F_{CE}) + L\beta k\, G_{EC} \\
\phi_E &= - \; Ln\,(n-\beta)\, G_{CE} - L\beta k\,(F_{EC} + F_{EP}) \\
\phi_P &= - \; L\beta k\, G_{EP} \quad .
\end{aligned}\right\} \qquad (5\text{-}11)$$

The angular load functions τ''_{AB}, τ''_{BA} are the reactions of the conjugate beam at left and right ends, respectively:

$$\left.\begin{aligned}
\tau''_{AB} &= \phi_A + \beta\,\phi_D + n'\phi_C \\
&= -\Big[Ln\, F_{AC} + L\,(2n - \beta n - \beta)\, G_{AC} + L\,(n-\beta)\,(1-\beta)\, F_{CA} \\
&\qquad + \frac{L^2}{EI_0}\,\frac{\alpha}{3}\,(3n - 1 - 2\beta + 2\beta^2) + L\beta^2 k\, F_{EP} + Ln'k\, G_{EP}\Big] \\
\tau''_{BA} &= \beta\phi_C + (1-\beta)\,\phi_E + n\,\phi_P \\
&= -\Big\{L\beta n\, G_{AC} + L\,\beta\,(n-\beta)\, F_{CA} + \frac{L^2}{EI_0}\,\frac{\alpha}{3}\,\big[3n - 2\,(1 - \beta + \beta^2)\big] \\
&\qquad + L\,(1-\beta)\,\beta k\, F_{EP} + Ln\beta k\, G_{EP}\Big\}.
\end{aligned}\right\} \qquad (5\text{-}12)$$

The angular load functions τ_{AB}, τ_{BA} may be obtained by adding the Eq. (5-10) and (5-12) together.

$$\begin{aligned}
\tau_{AB} = Ln\, F_{AB} - \Big\{ &\frac{L^2\beta}{EI_0}\Big[(n-\beta)(1-\beta)\, Q_0^{(L\beta)} + \beta(n+2\alpha)\, Q_1^{(L\beta)} + \beta^2 Q_2^{(L\beta)}\Big] \\
&+ \frac{L^2}{EI_0}\,\frac{\alpha}{3}\,(3n-1-2\beta+2\beta^2) + \frac{(L\beta k)^2}{EI_0}\Big[\beta k Q_0^{(L\beta k)} + (n'-2\beta k)\, Q_1^{(L\beta k)} \\
&+ (\beta k - n')\, Q_2^{(L\beta k)}
\end{aligned} \qquad (5\text{-}13)$$

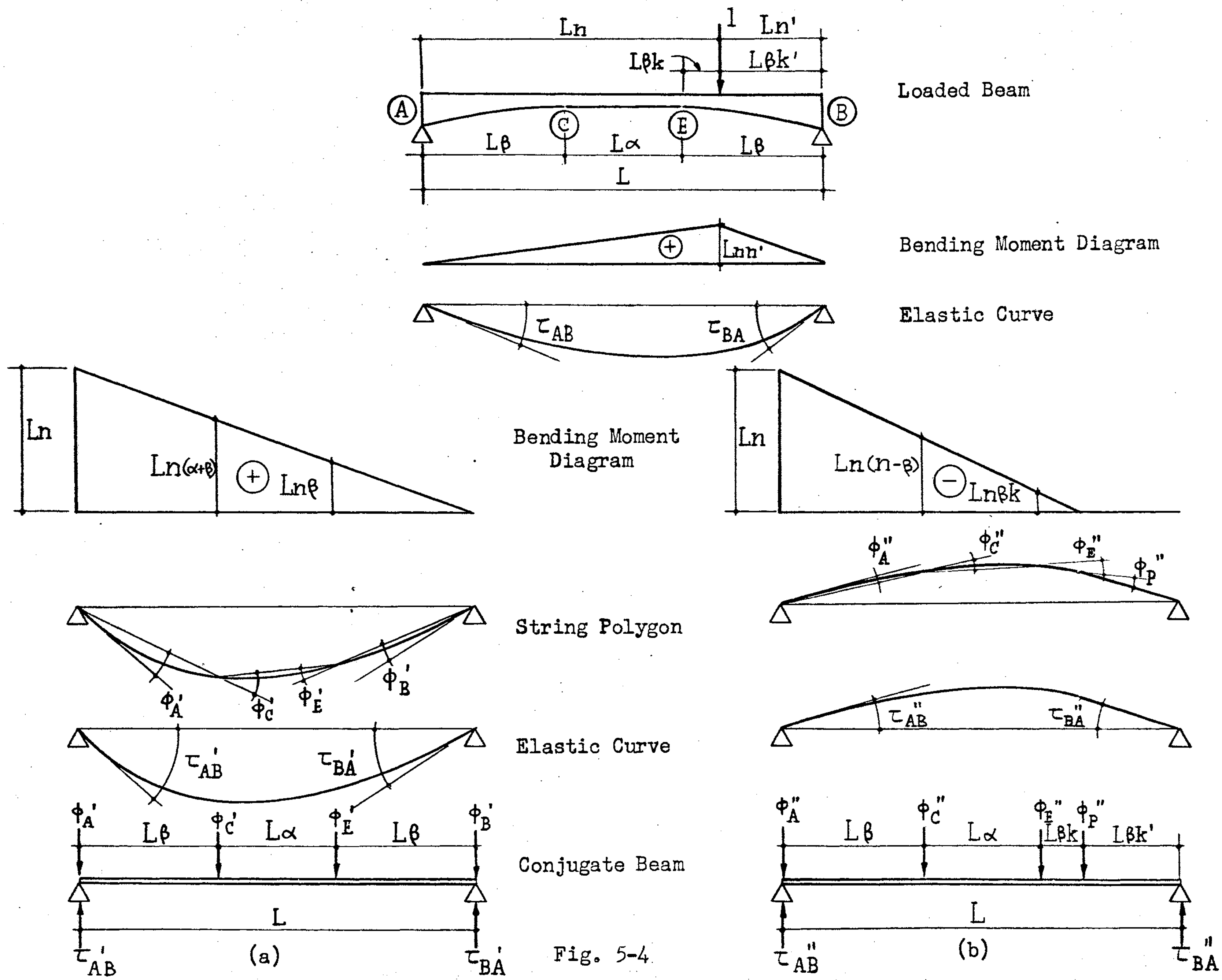

Ln
1
Ln'
Lβk
Lβk'
Loaded Beam
A
B
C
E
Lβ
Lα
Lβ
L
Bending Moment Diagram
+
Lnn'
Elastic Curve
τAB
τBA
Ln
Ln(α+β)
+
Lnβ
Bending Moment Diagram
Ln
Ln(n-β)
−
Lnβk
φ"A
φ"C
φ"E
φ"P
String Polygon
φ'A
φ'C
φ'E
φ'B
Elastic Curve
τ'AB
τ'BA
τ"AB
τ"BA
φ'A
Lβ
φ'C
Lα
φ'E
Lβ
φ'B
Conjugate Beam
L
τ'AB
(a)
τ'BA
φ"A
Lβ
φ"C
Lα
φ"E
φ"P
Lβk
Lβk'
L
τ"AB
(b)
τ"BA

Fig. 5-4

$$
\tau_{BA} = \text{Ln } G_{BA} - \Bigg\{ \frac{(L\beta)^2}{EI_0} \Big[(n-\beta)\, Q_0^{(L\beta)} + (2\beta-n)\, Q_1^{(L\beta)} - \beta\, Q_2^{(L\beta)}
$$

$$
+ \frac{L^2}{EI_0}\, \frac{\alpha}{3} \Big[3n-2\,(1-\beta+\beta^2) \Big]
$$

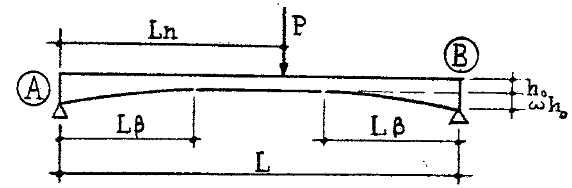
$$
+ \frac{(L\beta k)^2}{EI_0} \Big[(1-\beta)\, Q_0^{(L\beta k)} + (2\beta+n-2)\, Q_1^{(L\beta k)} - (n'-\beta)\, Q_2^{(L\beta k)} \Big] \Bigg\}. \qquad (5\text{-}13)
$$

(5-13)

The values of beam constants are computed and listed in Table IV.

TABLE IV

BEAM CONSTANTS FOR BEAM WITH TWO SYMMETRICAL PARABOLIC HAUNCHES (END A)



Haunch		Angular Flexibilities Coef. x $\frac{L}{EI_0}$	Angular Carry-Over Factors Coef. x $\frac{L}{EI_0}$	Influence Values of Angular Load Function, Coef. x $\frac{PL^2}{EI_0}$								
				n								
				0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
β	ω	F_{AB}	G_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}	τ_{AB}
0.2	1.0	.2503	.1547	.0238	.0431	.0552	.0601	.0592	.0539	.0437	.0302	.0154
	1.2	.2424	.1530	.0232	.0424	.0545	.0595	.0586	.0527	.0428	.0300	.0153
	1.4	.2357	.1514	.0226	.0416	.0539	.0589	.0582	.0524	.0426	.0297	.0151
	1.6	.2297	.1500	.0221	.0410	.0535	.0585	.0579	.0521	.0422	.0295	.0149
	1.8	.2245	.1487	.0217	.0408	.0529	.0581	.0575	.0518	.0420	.0292	.0148
	2.0	.2198	.1475	.0214	.0401	.0525	.0576	.0571	.0515	.0418	.0291	.0147
0.3	1.0	.2163	.1412	.0204	.0378	.0499	.0553	.0551	.0496	.0401	.0278	.0143
	1.2	.2054	.1378	.0196	.0365	.0485	.0543	.0540	.0486	.0392	.0271	.0138
	1.4	.1961	.1346	.0199	.0353	.0473	.0532	.0530	.0477	.0385	.0266	.0134
	1.6	.1880	.1313	.0182	.0343	.0462	.0521	.0520	.0469	.0378	.0260	.0131
	1.8	.1809	.1289	.0176	.0332	.0451	.0511	.0512	.0461	.0371	.0254	.0128
	2.0	.1746	.1265	.0171	.0323	.0442	.0502	.0504	.0455	.0366	.0250	.0126

CHAPTER VI

CONCLUSIONS

The general formulas for the angular flexibilities, carry over values and load functions for beams with parabolic haunches by means of string polygon are derived in this thesis.

The influence of parabolic haunches is expressed by means of Ritter's approximation. The final formulas are expressed in terms of the most common values of parameters β and ω , and the results are recorded in tables.

These tables are compared with results published elsewhere (3). The maximum error which occur obtained by using this approximation are 8 percent in the case of beams with one haunch and 7 percent in the case of beams with two symmetrical haunches.

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3. Tuma, Jan J., Lassley T., French S., "Analysis of Continuous Beam Bridges - Carry-over Procedure", Research Publication No. 3, School of Civil Engineering, Oklahoma State University, Stillwater, Oklahoma, Chapter 4, 1959.

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