THE CONCEPT OF A POTENTIAL WELL FOR THE SCATTERING OF NUCLEONS FROM He^4

By

JOHN ROBERT CHRISTY

Bachelor of Science

Oklahoma State University

Stillwater, Oklahoma

1957

Submitted to the faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
May, 1959

THE CONCEPT OF A POTENTIAL WELL FOR THE SCATTERING

OF NUCLEONS FROM He^4

Thesis Approved:

Benjamin Roth

Thesis Adviser

H. Harrington

Robert MacVicar

Dean of the Graduate School

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to all who have contributed to this investigation. It is only with the deepest gratitude that I thank Dr. Benjamin Roth, not only for many hours of discussion and advisement, but also for the patience and understanding expressed for my inexperience. I wish to thank Dr. H. E. Harrington and the Physics Department staff for their assistance and encouragement during the development of this investigation.

Also to the Special Services Department and the Physical Sciences Section of the Oklahoma State University Library go a special thanks for the aid rendered in securing research literature.

In closing, I wish to express my greatest thanks to my wife, Polly Anna, who has given me encouragement and understanding throughout the study.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	REVIEW OF THE LITERATURE	2
III.	NEUTRON AND PROTON SCATTERING FROM He^4	3
	Proton Scattering	3
	Neutron Scattering	5
IV.	THE SHAPE INDEPENDENT APPROXIMATION	6
	The Theory	6
	The Results	9
V.	THE WAVE EQUATION FOR SCATTERING USING A SIMPLE POTENTIAL WELL	13
	The Scattering Solution	13
	The Bound State	16
VI.	THE MODIFICATIONS TO THE WAVE EQUATION	19
	The Radial Wave Equation	19
	The Shape Independent Approximation Analog	22
	A Power Series Expansion	25
	The Bound State and Normalization	26
VII.	SUMMARY AND CONCLUSIONS	30
	The Problem Summarized	30
	The Findings and Conclusions	31
	Suggestions for Future Study	32
BIBLIOGRAPHY		34

LIST OF TABLES

Table		Page
I.	Proton Phase Shifts	4
II.	Neutron Phase Shifts	5
III.	Shape Independent Approximation for Neutron Scattering	7
IV.	Shape Independent Approximation for Proton Scattering	8
V.	Potential Well Width and Depth	15
VI.	Potential Well Width and Depth and Corresponding Bound State Energies	17

LIST OF FIGURES

Figure		Page
1.	The Fit of Neutron Scattering Data with the Shape Independent Approximation	10
2.	The Fit of Proton Scattering Data with the Shape Independent Approximation	11
3.	A Potential Well	14
4.	The Wave Function and the Scattering Length	18
5.	C_o as a Function of V_o for $a = 3.00\times10^{-13}$cm. and $b = 2.00\times10^{-13}$cm.	27

CHAPTER I

INTRODUCTION

The potential well as a scattering center is fundamental to the quantum mechanical theory. From such a concept, using known quantities, the results of an idealized scattering experiment may be predicted. Conversely, in the idealized problem, experimental scattering data may be used to calculate the quantities related to the potential well. Application of the concept of a potential well to He^4 to describe experimental nucleon scattering data is the problem under investigation in this thesis.

It will be shown in subsequent chapters as the ambiguities arise that the simplest quantum mechanical theory is inadequate to explain the experimental data from He^4.

The potential well computed using the experimental scattering length and derived from the simplest theory, as it will be shown, allows a bound energy state to exist. Such an energy state implies the existence of He^5 and Li^5, which do not appear in nature. The reconciling of the experimental scattering length and the nonexistent bound energy state will be considered in a later chapter.

Orbital angular momentum must be taken into account in the nucleon scattering analysis. This investigation will be limited to the case of orbital angular momentum equal to zero, or more commonly, the case of S-wave scattering. Incident nucleon energy will be limited to a maximum of 10 Mev. Further limitations will be noted as they appear.

CHAPTER II

REVIEW OF THE LITERATURE

The accuracy with which experimental data fits a theory is, in general, the criterion for the validity of the theory. Therefore, many scattering experiments at all incident nucleon energies have been performed on the elements. He^4 is no exception. The experimental investigations prior to 1955 are recorded by Adair (1) and Mather and Swan (2). More recently Miller and Phillips (3) have used He^4 as a scattering agent for protons at low energies. Their data analysis indicates the scattering nucleus to appear as a hard sphere of radius 2.00×10^{-13} cm. for the S-wave case. Previous experiments and data analysis by Dodder and Gammel (4) for protons scattered from He^4 indicate, again, hard sphere scattering with a radius of 2.60×10^{-13} cm. for the S-wave case. The analysis of data by these experimentors is a major aid to the development of theory.

The exact solution to the He^4 scattering problem has been derived by Hochberg, Massey and Underhill (5) using known forces. Using the Wheeler resonating group theory, Bransden and McKee (6) have developed a solution for the general case of neutron scattering by He^4. Others have approached the problem using complex potential wells (7). Kohn (8) has used the method of variational calculus in solving the problem of neutron scattering by He^4. It is readily apparent that an explanation of the experimental data by a more simple theory would be desirable.

CHAPTER III

NEUTRON AND PROTON SCATTERING FROM He^4

Proton Scattering

The scattering cross sections for a given incident nucleon energy is the quantity determined experimentally. The phase shifts, which are calculated from the experimental cross sections, provide the connection between experiment and theory. Hence, the phase shifts must be found.

The cross sections for protons scattered from He^4 as given by Dodder and Gammel (4) and Putnam, Brolley and Rosen (9) are in the differential cross section form:

$$d\sigma = \left| f(\theta) \right|^2 d\omega .$$

The $f(\theta)$ comes from the assymptotic form of the wave function which for neutron scattering is

$$\psi \sim \exp ikz + r^{-1} f(\theta) \exp ikr$$

Mott and Massey (10) give $f(\theta)$ as:

$$f(\theta) = (2ik)^{-1} \sum_{\ell=0}^{\infty} (2\ell+1)(\exp 2i\delta_\ell - 1)\, P_\ell(\cos\theta),$$

where the $P_\ell(\cos\theta)$ are Legendre polynomials and the δ_ℓ are the phase shifts corresponding to a certain orbital angular momentum, ℓ. The assymptotic form of the wave function for Coulomb scattering is

$$\psi_c \sim \exp i\,[kz + n \ln k(r-z)]\,[1 - n^2/ik(r-z)] + r^{-1} f_c(\theta) \exp i(kr - n \ln 2kr),$$

where

$$fc(\theta) = n/(2k \sin^2 \tfrac{1}{2} \theta) \exp i[n \ln(\sin^2 \tfrac{1}{2} \theta) - \pi - 2\eta_0]$$

$$\eta_0 = \arg \Gamma (1 + in)$$

and

$$n = ZZ'e^2/hv.$$

Hence,

$$f(\theta) = f_c(\theta) + \sum_{\ell=0} k^{-1}(2\ell + 1)\exp i(2\eta_\ell + \delta_\ell)\sin\delta_\ell P_\ell (\cos \theta)$$

is the scattered amplitude for protons (11). Critchfield and Dodder (12) use a numerical method of least mean squares applied to a digital computer to find the values of the phase shifts which best fit the experimental cross sections. Dodder and Gammel (4) and Putnam, Brolley and Rosen (9) also use this method in computing the phase shifts that they report. Table I gives the phase shifts and corresponding incident proton energy in laboratory coordinates, per cent error in phase shift and source (2).

TABLE I

PROTON PHASE SHIFTS

δ_0	E(Mev.)	Per cent error in δ_0	Source
168.0°	0.95	23.0	Critchfield and Dodder
132.6°	5.81	5.0	Dodder and Gammel
122.05°	7.50	3.0	Putnam, Brolley and Rosen
122.5°	9.48	5.0	Dodder and Gammel

Neutron Scattering

The cross sections for neutrons scattered from He^4 as reported by Bashkin, Mooring and Petree (13) are in the form of the total cross section.

$$\sigma = 4\pi \left| f(\theta) \right|^2$$

the $f(\theta)$ is the same expression as recorded on page 3 of this thesis. For small energies $\ell = 0$ is all that need be considered. The equation for the total cross section reduces to

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0$$

where $k = (2mEh^{-2})^{\frac{1}{2}}$ is the wave number for the incident nucleon. The phase shifts may now be solved for directly. Huber and Baldinger (14) have measured differential cross sections for neutrons scattered from He^4. Phase shifts were found using the numerical method of least mean squares for the best fit of experimental data. The phase shifts and corresponding incident neutron energies in laboratory coordinates, per cent error in phase shift and source are included in Table II. Per cent maximum probable error in phase shifts reported by Huber and Baldinger (14) is omitted since data was approximated from the graph.

TABLE II

NEUTRON PHASE SHIFTS

δ_0	E(Mev.)	Per cent error in δ_0	Source
172.90°	0.05	10.0	Bashkin, Mooring and Petree
166.83°	0.15	10.0	
160.16°	0.30	10.0	
158.1°	0.75	-	Huber and Baldinger
151.9°	1.00	-	
139.4°	2.00	-	
126.9°	3.00	-	
114.4°	4.00	-	

CHAPTER IV

THE SHAPE INDEPENDENT APPROXIMATION

The Theory

The general quantum mechanical theory of nuclear scattering indicates some dependence of the scattered neutron distribution upon the shape of the potential well representing the nucleus. Within certain limits, any reasonable approximation to a square potential well may be used. The approximation requires two parameters, "a" and "r_o", to describe the potential well. These conditions are known as the shape independent approximation and are discussed by Blatt and Weisskopf (15) on page 62. The shape independent approximation for neutron scattering is expressed in the formula:

$$k \cot \delta = -a^{-1} + \tfrac{1}{2} r_o k^2$$

carried only to the second power in k.

As shown by Bethe and Morrison (16) on page 57, higher powers of k are insignificant for low incident particle energies. Bethe (17) has derived the following similar expression of the shape independent approximation for proton scattering:

$$-a^{-1} + \tfrac{1}{2} r_o k^2 = \mu e^2 h^{-2} \left[2\pi \cot \delta_o (\exp 2\pi\eta - 1)^{-1} - 2 \ln \eta + 2\eta^2 \sum_{\nu=1}^{\infty} (\nu^3 + \nu\eta^2)^{-1} \right]$$

where $\eta = e^2 (h\, v)^{-1}$, e in e.s.u., v is velocity of relative motion, h is Plank's constant divided by 2π, and μ is the reduced mass of the system. Again, terms greater than the second power of k are negligible for energies of a few Mev. In both the neutron and proton scattering expression the "a" represents the scattering length, defined by Bethe and Morrison (16) on pages 54 and 55. Similarly, the r_o in both equations is the so-called effective range of the nuclear potential well as given by Bethe and Morrison (16) on pages 55 through 57.

While the shape independent approximation was proved for a nucleon - nucleon collision where the potential well is known to hold, it is interesting to see whether the same approximation is valid for scattering from He^4. The scattering data is now introduced into these expressions to test the validity of the shape independent approximation for He^4. Table III includes the computations for the neutron data.

TABLE III

SHAPE INDEPENDENT APPROXIMATION FOR NEUTRON SCATTERING

E(Mev.)	δ_o	$k_c x10^{13}$	$k_c^2 x10^{26}$	Error in cot δ_o	$k_c \cot \delta_o x10^{13}$
0.05	172.90°	0.0392	0.00154	±3.280	-0.319±0.129
0.15	166.83°	0.0680	0.00462	±0.968	-0.290±0.066
0.30	160.16°	0.0964	0.00926	±0.435	-0.266±0.042
0.75	158.1°	0.1524	0.0232	-	-0.379 -
1.00	151.9°	0.1755	0.0308	-	-0.330 -
2.00	139.4°	0.2486	0.0618	-	-0.290 -
3.00	126.9°	0.3043	0.0926	-	-0.229 -
4.00	114.4°	0.3513	0.1234	-	-0.159 -

The phase shifts as seen in Table III are in the form $\pi - \delta_o$ in agreement with the proton phase shifts (2). In the center of mass coordinates the wave number is represented by k_c, where

$$\frac{k_c \text{ (center of mass)}}{k \text{ (laboratory)}} = \frac{\text{m(reduced)}}{\text{m(nucleon)}} = 0.8 .$$

Consequently,

$$k_c^2 \text{ (center of mass)} = 0.64\ k^2 \text{(laboratory)}.$$

The "error in cot δ_0" as recorded in Table III was computed from the differential of cot δ,

$$d\ (\cot \delta) = \frac{1}{-\sin^2 \delta}\ d\,\delta .$$

Hence,

$$\text{error in } \cot \delta_0 = \pm \frac{1}{\sin^2 \delta_0} \times (\% \text{ error in } \delta_0) .$$

The phase shift, δ_0, was assumed to include the maximum probable error reported for the experiment.

The proton data was used to calculate Table IV. Representations are as in Table III.

TABLE IV

SHAPE INDEPENDENT APPROXIMATION FOR PROTON SCATTERING

E(Mev.)	δ_0	$k_c^2 x10^{26}$	Error in cot δ_0	$f(\eta, \delta) x10^{13}$
0.95	168.0°	0.0293	±5.336	-0.229±0.388
5.81	132.6°	0.1793	±0.093	-0.109±0.025
7.50	122.05°	0.2320	±0.042	-0.0589±0.0136
9.48	122.5°	0.2930	±0.070	-0.0781±0.0261

For the neutron case the k_c^2 were plotted against the k_c cot δ_0 of Table III to determine graphically the validity of the shape independent approximation. A straight line through all data points is the ideal case for validity. The intercept at k_c equal zero gives the value of $-a^{-1}$. The reciprocal of this value gives the scattering length for neutrons. Since there is no Coulomb field, the zero energy cross section can be given in terms of the scattering length as $\sigma_0 = 4\pi a^2$.

From Table IV, the k_c^2 were plotted against the $f(\eta, \delta)$ to determine if the shape independent approximation holds for the proton scattering case. As in the neutron case, the same criterion for validity is effective in the proton case. Similarly, the intercept at k_c equals zero is the reciprocal of the proton scattering length.

The Results

Figure 1 and Figure 2 are the graphs of the calculations for the neutron and proton shape independent approximations, respectively. The data given by Huber and Baldinger (14) was not used in computation of the scattering lengths since the error was not available and also since it was inconsistent with other data.

The k_c equal zero intercepts from the graphs yield as scattering lengths $a = 4.00 \times 10^{-13}$cm. and $a = 3.00 \times 10^{-13}$cm. for proton and neutron, respectively. At present, there is no published report of direct experimental measurement of either neutron or proton scattering lengths in He^4. Since the available data tends to agreement these scattering lengths are assumed correct within the range of experimental error. Within experimental error, the scattering lengths are seen to be the same for both neutron and proton scattering. This equality of scattering lengths then implies that nuclear forces are independent of the Coulomb field present in the vicinity of the nucleus. More briefly, the scattering of nucleons from the He^4 nucleus is charge independent.

From the equations for the shape independent approximation, it will

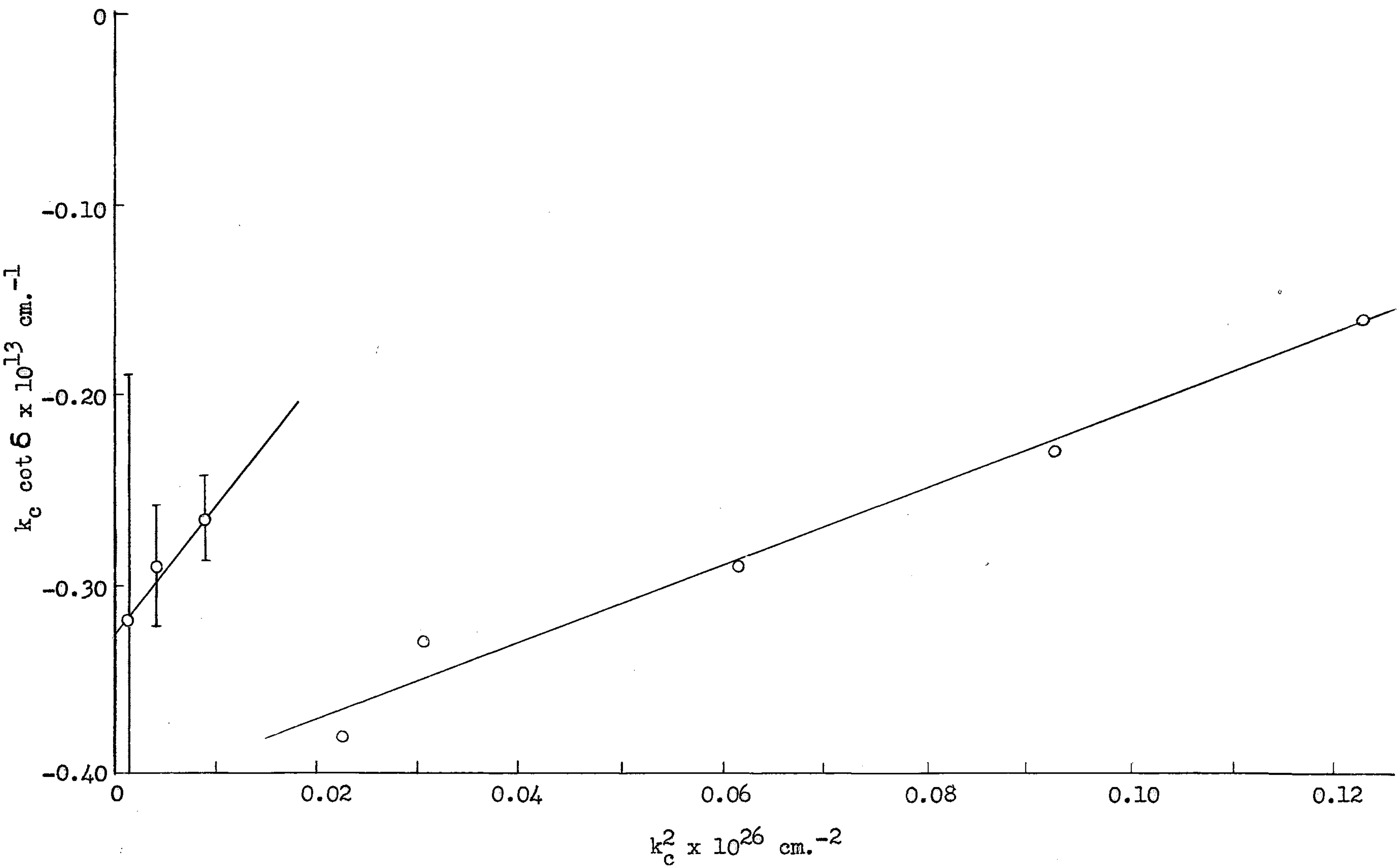

Fig. 1. The Fit of Neutron Scattering Data with the Shape Independent Approximation.

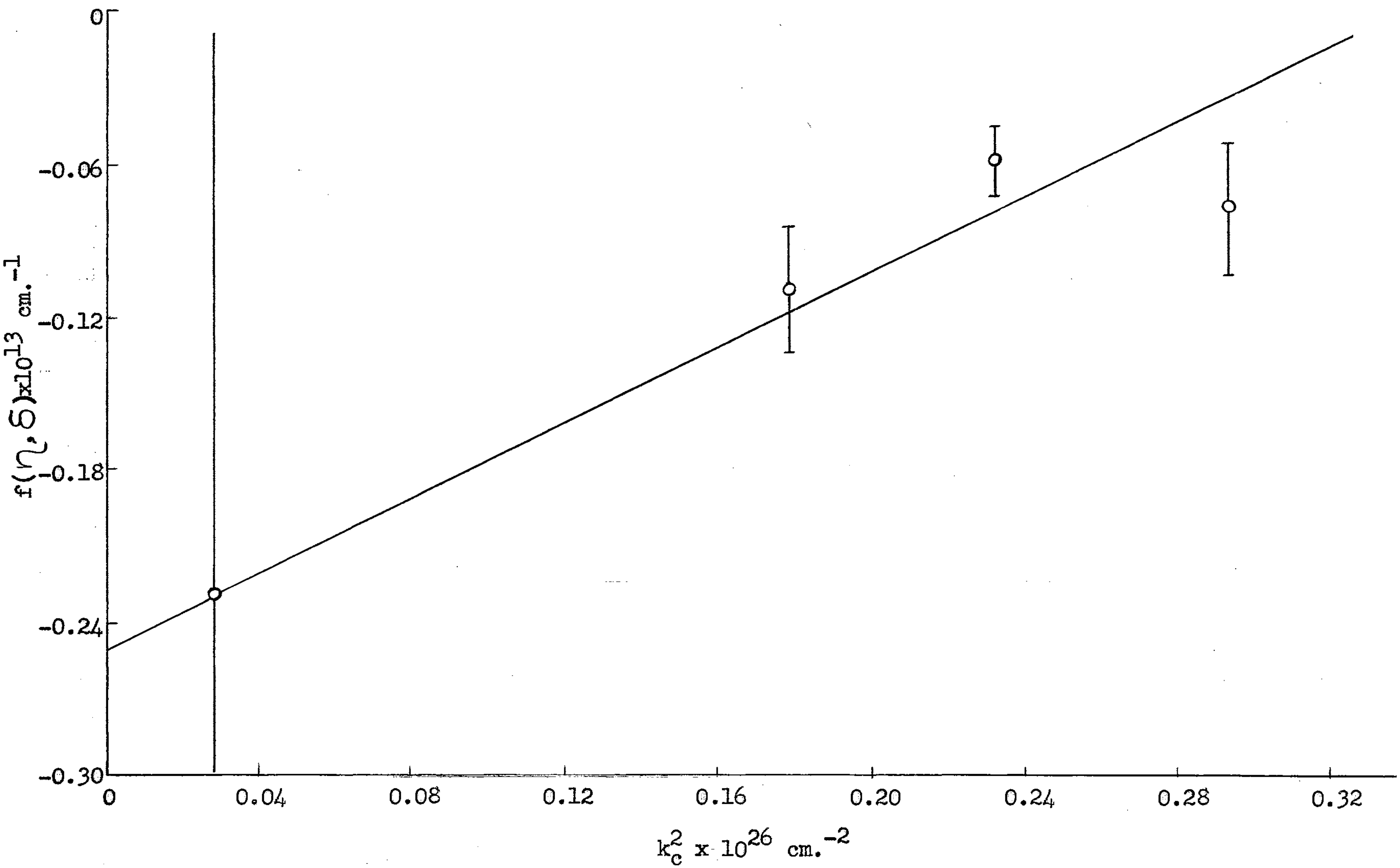

Fig. 2. The Fit of Proton Scattering Data With the Shape Independent Approximation.

be noted that the r_o, twice the coefficient of the k_c^2 term, is the slope of the line representing the plotted data. Hence, from the graph the slope of the line may be found. Twice this value gives r_o. The data used in this investigation, however, is inadequate to determine r_o and is left for future investigation.

Thus, a reasonably good fit of the experimental data to the shape independent approximation has led to a value of the scattering lengths for both neutrons and protons scattered from He^4. Also important is the fact that due to the validity of the shape independent approximation for nucleons scattered from He^4, the exact shape of the potential well representing the nucleus need not be taken into account in the scattering problem.

As a result, the square well potential may be used in the scattering problem of He^4 as a reasonable approximation to the actual shape of the scattering potential.

CHAPTER V

THE WAVE EQUATION FOR SCATTERING USING A SIMPLE POTENTIAL WELL

The Scattering Solution

From this point the investigation, for simplicity, will be limited to the situation of neutron scattering from He^4.

From quantum mechanical theory the scattering center is represented by a finite potential well. The wave equation representing the impinging particle must obey two restrictions (10). The wave equation must take on the assymptotic form

$$\psi \sim \exp ikz + r^{-1} f(\theta) \exp ikr$$

at large distances from the scatterer. Also the wave equation must be the solution to the radial portion of the Schroedinger time-independent equation (10) which is:

$$\frac{d^2u}{dr^2} + (k'^2 - \frac{\ell(\ell+1)}{r^2})u = 0.$$

For $\ell = 0$ this becomes

$$\frac{d^2u}{dr^2} + k'^2u = 0$$

where $u = \psi r$ and $k'^2 = 2mh^{-2}(E-V)$.

Assuming a square potential well for scattering as in Figure 3, V is set equal to $-V_0$ within the well and zero elsewhere. The boundary condition at $r = 0$ is that $u = 0$. Hence the solution of the differential equation is

$$u = A \sin k'r,$$

where A is an arbitrary constant. The solution outside the potential well fitting the boundary condition for $r \rightarrow \infty$ is then

$$u = B \sin (kr + \delta),$$

where $k = (2mh^{-2}E)^{\frac{1}{2}}$. At $r = b$ the wave function and its slope, or first derivative, must be continuous. Hence, the logarithmic derivation u'/u is continuous and

$$\frac{k'A \cos k'b}{A \sin k'b} = \frac{k B \cos (kb + \delta)}{B \sin (kb + \delta)} ,$$

or

$$k' \cot k'b = k \cot (kb + \delta).$$

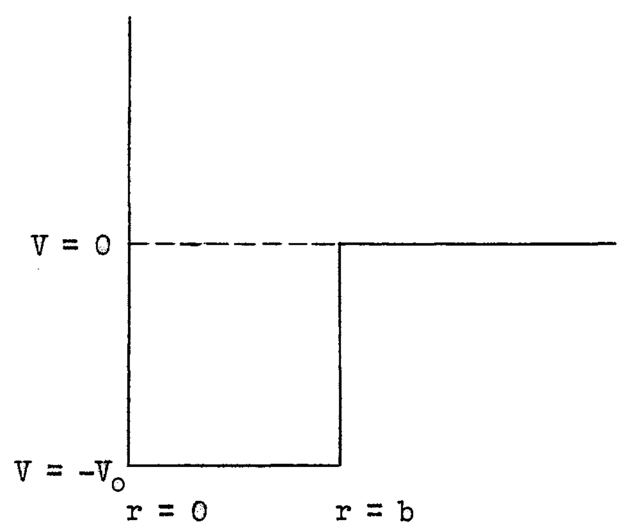

Fig. 3. A Potential Well.

For $E = 0$, the solution of the differential equation for r less than b again is

$$u_i = A \sin k'r$$

where the subscript i indicates the solution for the interior of the potential well. However, the differential equation for r greater than b becomes

$$\frac{d^2u_o}{dr^2} = 0$$

and has the solution

$$u_o = D\ (r - a),$$

where D is an arbitrary multiplying constant (16). Again applying the boundary condition at r = b, that the logarithmic derivation u'/u be continuous, gives

$$\frac{k'\ A\ \cos k'b}{A\ \sin k'b} = \frac{D}{D(b-a)}\ .$$

This reduces to

$$k'(b-a) = \tan k'b$$

and provides a relation between the scattering length, a, the potential well width, b, and the depth of the potential well, V_o.

Using the value of the scattering length for neutrons determined from experimental data by the shape independent approximation in Chapter IV, a relation between the potential well width, b, and depth, V_o, is found for the scattering from He^4. The solutions to the boundary condition equation were found numerically. Table V lists the potential well depth corresponding to the potential well widths suggested by Miller and Phillips (3) and Dodder and Gammel (4).

TABLE V

POTENTIAL WELL WIDTH AND DEPTH

$a = 3.00 \times 10^{-13}$cm

$b(\times 10^{-13})$cm.	$V_o(\times 10^{6})$ergs	V_o(Mev.)
2.00	54.3	34
2.60	46.1	29

The Bound State

It can be shown that a bound energy state will exist in the potential well of Figure 3 (10). The outside solution is

$$u_o = B \exp(-kr)$$

where $V = -V_o$ and $E = -W_b$. W_b is the binding energy of the bound state or the depth of the bound state in the potential well. As before, $k = (2mh^{-2}E)^{\frac{1}{2}}$. The interior solution is as before except that E is no longer zero;

$$u_o = A \sin k'r.$$

Again applying the boundary condition that the logarithmic derivative u'/u is continuous at $r = b$, the edge of the potential well, gives

$$\frac{k'A \cos k'b}{A \sin k'b} = \frac{-k\, B \exp(-kb)}{B \exp(-kb)} .$$

This becomes

$$k' \cot k'b = -k,$$

or

$$\tan k'b = -k'/k.$$

Here again a numerical method alone provides the solutions. Note also that the mass, m, used in the scattering equation is the reduced mass of the system. For neutron scattering from He^4 the reduced mass is 0.8 times the mass of a neutron.

The related values of Table V are inserted into the bound state equation to find the corresponding binding energies. The values of the potential well width and depth and corresponding bound state depths for two suggested potential well widths are compiled in Table VI.

For the potential well widths of interest a bound energy state, as derived by the preceeding simple theory, is available to an incident nucleon. Such an available bound energy state would allow the existence of He^5 for incident neutrons and the existence of Li^5 for incident protons. As is well known, no stable system composed of five nucleons exists in nature. The size of the scattering length and the nonexistence of a bound state are thus shown to be incompatible with this simple theory. It is, therefore, necessary to modify the simple theory used thus far in the investigation.

TABLE VI

POTENTIAL WELL WIDTH AND DEPTH AND CORRESPONDING BOUND STATE ENERGIES

$b(x10^{13})$cm.	V_o Mev.	W_b Mev.
2.00	34	8
2.60	29	11

For a potential well shallow enough to prohibit a reasonable bound state the scattering lengths are in excess of those calculated for He^4. The tangent to the interior wave function at the edge of the potential well determines the value of the scattering length. That is, the intercept at u equals zero of the tangent to the interior wave function evaluated at r equals b gives the zero energy scattering length. Thus, to vary the scattering length the curvature of the interior wave function within the potential well must be varied accordingly. To decrease the magnitude of the scattering length, the wave function must be adjusted so that it has a greater curvature within the width of the potential well. Figure 4 (a) and (b) illustrate the relation between the curvature of the wave function in the potential well and the scattering length. Figure 4 (b) is a picture of the wave function for neutrons scattered from He^4 as given by the simple theory. Figure 4 (a) presents the desired picture for neutron scattering from He^4. In Chapter VI this adjustment is considered.

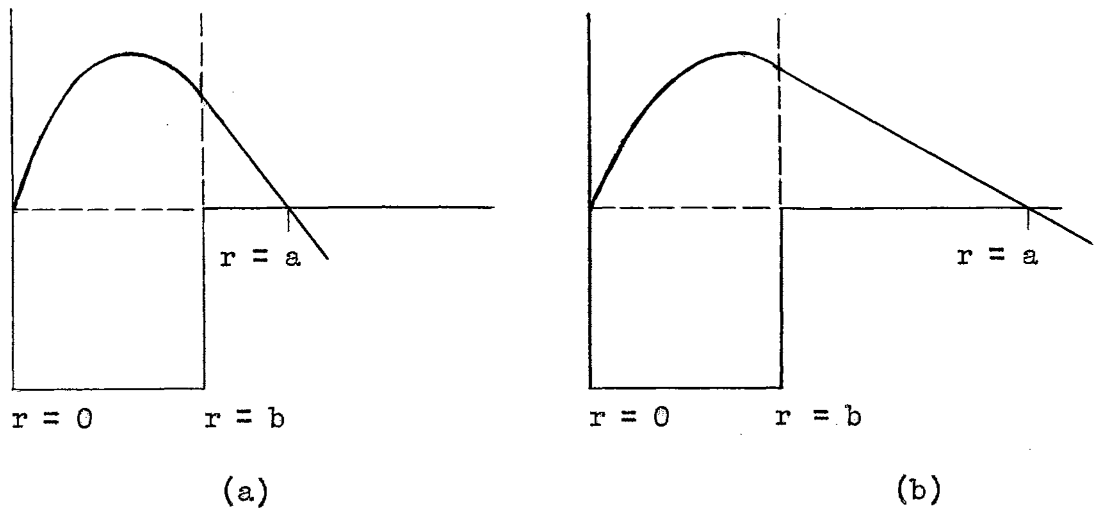

Fig. 4. The Wave Function and the Scattering Length.

CHAPTER VI

THE MODIFICATIONS TO THE WAVE EQUATION

The Radial Wave Equation

The exact solution to the problem of the scattering of nucleons by He^4 has been developed by Hochberg, Massey and Underhill (5). The Schroedinger equation was revised on the basis of the Pauli Exclusion Principle. To the Schroedinger time-independent equation,

$$\frac{d^2u}{dr_2} + (k'^2 - \frac{\ell(\ell+1)}{r^2})u = 0,$$

has been added the integral of a Kernel function. The Schroedinger time-independent equation as used by Hochberg, Massey and Underhill (5) appears as

$$\frac{d^2u}{dr^2} + k'^2u = \int K(r,r')\, u(r')\, dr'$$

for the case of orbital angular momentum equal zero. This is an integro-differential equation, the right-hand side of which implies a velocity dependent potential effect. The integro-differential equation is valid only within the range of nuclear forces. This is exactly where the change in the wave function is desired. To avoid the complexity of the solution of such an equation and to retain relatively simple theory, the right-hand side is assumed to be a constant dependent only on the energy of the incident nucleon. Hence, the trial form is

$$\frac{d^2u}{dr^2} + k'^2u = C$$

where $C = C_0 + C_1K + C_2k^2 + \cdots$. For k small such an approximation is reasonably well justified.

The solution to the modified wave equation must now be found. A solution of the form

$$u = A \cos k'r + B \sin k'r + J$$

is assumed where J is an added constant. Substituting the assumed solution into the modified wave equation yields,

$$-k'^2(A \cos k'r + B \sin k'r) + k'^2(A \cos k'r + B \sin k'r) + k'^2J = C$$

or

$$k'^2J = C.$$

Hence

$$J = Ck'^{-2}.$$

The assumed solution becomes

$$u = A \cos k'r + B \sin k'r + Ck'^{-2}.$$

The boundary condition at the origin requires u equal to zero. Applying this boundary condition gives

$$0 = A + Ck'^{-2}.$$

Hence

$$A = -Ck'^{-2}.$$

The assumed solution to the modified wave equation now is

$$u = Ck'^{-2}(1-\cos k'r) + B \sin k'r,$$

containing one arbitrary constant, B.

Outside the potential well the Schroedinger time-independent equation is unchanged,

$$\frac{d^2u}{dr^2} + k^2u = 0,$$

where $k^2 = 2mh^{-2}E$. The solution is of the form

$$u = A'' \sin(kr + \delta).$$

The δ appearing here is the phase shift of the scattered wave representing the scattered neutron and corresponds to the phase shifts discussed in Chapter III and Chapter IV. The A'' is not arbitrary and is given by Schiff (11) on page 105 as

$$A'' = \exp(i\delta)\, k^{-1}.$$

Therefore, the solution to the wave equation outside the potential well is completely specified in terms of the phase shift and the incident neutron energy and has the form

$$u = \exp(i\delta) k^{-1} \sin(kr + \delta).$$

The boundary condition at r = b, the edge of the potential well, requires that the interior and exterior wave functions and their first derivatives be continuous, or

$$u_i = u_o,\ u_i' = u_o'.$$

Hence for the first of the boundary conditions

$$Ck'^{-2}(1 - \cos k'b) + B \sin k'b = \exp(i\delta) k^{-1} \sin(kr + \delta)$$

and for the second of the boundary conditions

$$Ck'^{-1} \sin k'b + k'B \cos k'b = \exp(i\delta) \cos(kb + \delta).$$

From the two equations, B may be eliminated. Solving for B from the second equation yields

$$B = \frac{\exp(i\delta) \cos(kb + \delta) - Ck'^{-1} \sin k'b}{k' \cos k'b}.$$

Resubstituting this value of B into the first equation gives, upon simplification,

$$C = \frac{k'^2 \exp(i\delta) k^{-1} \cos k'b \sin(kb + \delta) - k' \exp(i\delta) \sin k'b \cos(kb + \delta)}{\cos k'b - 1}$$

Thus is developed an expression for C in terms of k, k' and δ. Such an expression, however, is rather uninformative as it is not easily analyzed. It would, therefore, be instructive to transform the expression to one that is more often seen and used.

The Shape Independent Approximation Analog

Analysis of the He^4 scattering data in Chapter IV made use of the shape independent approximation. Expressed analytically the shape independent approximation for neutron scattering is

$$k \cot \delta = G_o + G_1 k^2 + \cdots ,$$

where $G_o = -a^{-1}$ and $G_1 = \frac{1}{2} r_o$. The right-hand side is an even order power series in k. Thus, if the expression for C were expanded in a power series in the neighborhood of $k = 0$, a form similar to the shape independent approximation may be derived. By definition

$$k'^2 = 2mh^{-2}(E - V)$$

Now defining

$$U = 2mh^{-2} V_o,$$

gives

$$k'^2 = k^2 + U.$$

Hence k' is approximated for k small as

$$k' = (U)^{\frac{1}{2}} (1 + \tfrac{1}{2}k^2U^{-1}).$$

With these substitutions cos k'b and sin k'b become

$$\cos k'b = \cos (U^{\frac{1}{2}} + \tfrac{1}{2}k^2U^{-\frac{1}{2}})b \doteq \cos U^{\frac{1}{2}}b \left(1 - \frac{k^4b^2}{8U}\right) - \sin U^{\frac{1}{2}}b \left(\frac{k^2b}{2U^{\frac{1}{2}}}\right)$$

and

$$\sin k'b = \sin (U^{\frac{1}{2}} + \tfrac{1}{2}k^2U^{-\frac{1}{2}})b \doteq \sin U^{\frac{1}{2}}b \left(1 - \frac{k^4b^2}{8U}\right) + \cos U^{\frac{1}{2}}b \left(\frac{k^2b}{2U^{\frac{1}{2}}}\right).$$

By the same approximate expansions

$$\sin (kb + \delta) = kb \cos\delta + \left(1 - \frac{k^2b^2}{2}\right) \sin\delta$$

and

$$\cos (kb + \delta) = \left(1 - \frac{k^2b^2}{2}\right) \cos \delta - kb \sin\delta .$$

Rearranging the expression for C gives

$$(C/k')(\cos k'b-1)\exp(-i\delta)$$
$$= (k'/k) \cos k'b \sin(kb+\delta) - \sin k'b \cos(kb+\delta).$$

Upon inserting the above substitutions and expanding the product terms in powers of k the coefficients of $\sin\delta$ and $\cos\delta$ are found. Considering terms with powers of k less than or equal to two gives for the coefficient of $\cos\delta$,

$$[(C-Ub)\cos x - C + U^{\frac{1}{2}}\sin x]$$
$$+ [(1-bC)(4U)^{-\frac{1}{2}}\sin x - \frac{3b}{2}\cos x]k^2 .$$

where $x = U^{\frac{1}{2}}b$. Again including terms with powers of k not greater than one yields for the coefficient of $\sin\delta$,

$$(U\cos x)k^{-1} + iC(\cos x - 1)$$
$$+ [(1 - \tfrac{1}{2}Ub^2)\cos x + \tfrac{1}{2}bU^{\frac{1}{2}}\sin x]k.$$

Forming the quotient and multiplying by k gives the desired form upon simplification;

$$k\cot\delta = [U\cos x + iC(\cos x - 1)k + Dk^2]$$
$$\cdot K\left\{1 - K[(1-bC)(4U)^{-\frac{1}{2}}\sin x - \frac{3b}{2}\cos x]\right\}k^2$$

where

$$D = (1 - \tfrac{1}{2}Ub^2)\cos x + \tfrac{1}{2}bU^{\frac{1}{2}}\sin x$$

and

$$K = [(C - Ub)\cos x + U^{\frac{1}{2}}\sin x - C]^{-1}.$$

The shape independent approximation, as would be expected, may thus be derived from the modified wave equation.

As previously given for the case of neutron scattering the shape independent approximation is

$$k\cot\delta = -a^{-1} + \tfrac{1}{2}r_0k^2.$$

Equating the constant terms of the two above expressions for $k\cot\delta$ yields an expression for the scattering length in terms of C. Hence,

$$-a^{-1} = U \cos x \left[(C - Ub) \cos x + U^{\frac{1}{2}} \sin x - C \right]^{-1}.$$

When simplified, this becomes

$$a = b - CU^{-1} + (C - U^{\frac{1}{2}} \sin x)(U \cos x)^{-1}$$

where x is again equal to $U^{\frac{1}{2}}b$. The scattering length, "a", then is expressed as a function of the potential well width, b, and depth parameter, U and C. Note that the expression for C as a power series in k has been considered as a constant. Were C expanded, the expression for "a" would include only the constant term of the power series in k and thus would be included as C_o instead of C.

It will also be noted from the k cot δ expression as derived from the modified wave equation that a complex term in k is present. It is significant that the complex term also contains a factor of C. Such a term implies that the coefficients of the expansion for C or the phase shifts are complex. The fact that C may be complex is to be expected. The integro-differential equation given at the first of this chapter included a nonconservative term on the right-hand side. It will be recalled that the integral of the Kernel function represents a velocity dependent potential effect which may be an absorption or an emission phenomenon. The positive imaginary term in the k cot δ expression represents an absorption effect. Since C is an approximation to the integral of the Kernel function and appears in the analogous shape independent approximation expansion it is not at all unreasonable to expect to find C complex. Whether or not C has complex coefficients may be determined by the nature of δ. Conversely, if C is complex, a complex δ might be required. So far in this investigation there has been no restriction on the potential well to prohibit it from being either real or complex. Complex potential wells have been investigated and will not be discussed further in this thesis (7).

A Power Series Expansion

Returning now to the expression for C in terms of the boundary conditions at the potential well edge, a different procedure will be followed to relate C as a function of k to the scattering length, "a". The expression for C is

$$C_o + C_1k + C_2k^2 + \cdots$$

$$= \frac{(k'^2/k)\exp(i\delta)\cos k'b\ \sin(kb + \delta) - k'\ \exp(i\delta)\sin k'b\ \cos(kb+\delta)}{\cos k'b - 1}$$

The expansions approximating cos k'b, sin k'b, cos(kb + δ), sin(kb + δ) and k' are as before. The series representing exp (i δ) is approximated by

$$\exp(i\delta) = 1 + i\delta - \tfrac{1}{2}\delta^2 + \cdots .$$

It is further assumed that δ may be expanded as a power series in k of the form

$$\delta = D_o + D_1k + D_2k^2 + \cdots.$$

As indicated by Blatt and Weisskopf (15) on page 61 for the scattering with C equal zero, when k is small, δ is approximately equal to -ka. Hence from the above expression for δ, the zero energy cross section for scattering may be expressed in terms of $-D_1$ as

$$\sigma_o = 4\pi D_1^2 .$$

Substituting the above expansions into the expression for C and expanding product terms yields a power series in k. To determine the relation of the power series representing C to its equivalent, coefficients of like powers of k are equated yielding,

$$C_o = (X - 1)^{-1}\, UX(b + D_1) + U^{\frac{1}{2}}Y,$$

$$C_1 = (D_2UX)(X - 1)^{-1} + i\ D_1C_o,$$

$$C_2 = YbC_o\ (X-1)^{-1}\ (4U)^{-\frac{1}{2}} + i\ D_1 D_2 UX(X-1)^{-1} + C_o[(i\ D_2 - D_1{}^2) + (X - \tfrac{1}{2}UXbD_1 - \tfrac{1}{2}YbU^{\frac{1}{2}})(D_1 + b) + \tfrac{1}{2}(U^{\frac{1}{2}}Y + Xb^2 - b^2U^{\frac{1}{2}}Y - D_1{}^2U^{\frac{1}{2}}Y) - bD_1U^{\frac{1}{2}}Y],$$

where $X = \cos U^{\frac{1}{2}}b$ and $Y = \sin U^{\frac{1}{2}}b$.

As a type of first order correction to the wave function, the constant terms expression will be investigated. To test for the range of the correction factor C_o, it is necessary to numerically evaluate this expression. Figure 5 illustrates the relation of C_o and V_o for a potential well width of 2.00×10^{-13} cm. and the zero energy scattering length as determined from experimental data to be 3.00×10^{-13} cm. Note that for certain C_o, three solutions for V_o are available. It is then apparent that further restrictions on C_o would be desirable. Such limitations may be found in an investigation of the bound state solution for the modified wave equation.

The Bound State and Normalization

The conditions for the bound state are as in Chapter V. Again the wave function outside the potential well is taken as

$$u_o = D \exp(-kb).$$

Applying the boundary conditions at the edge of the potential well gives

$$Ck'^{-2}\ (1 - \cos k'b) + B \sin k'b = D \exp(-kb)$$

and

$$Ck'^{-1} \sin k'b + k'B \cos k'b = k\ D \exp(-kb).$$

From these two equations expressions for the coefficients B and D in terms of C are found as

$$B = -(Ck'^{-1})(\sin k'b + kk'^{-1} \cos k'b - kk'^{-1}) \cdot (k \sin k'b + k' \cos k'b)^{-1},$$

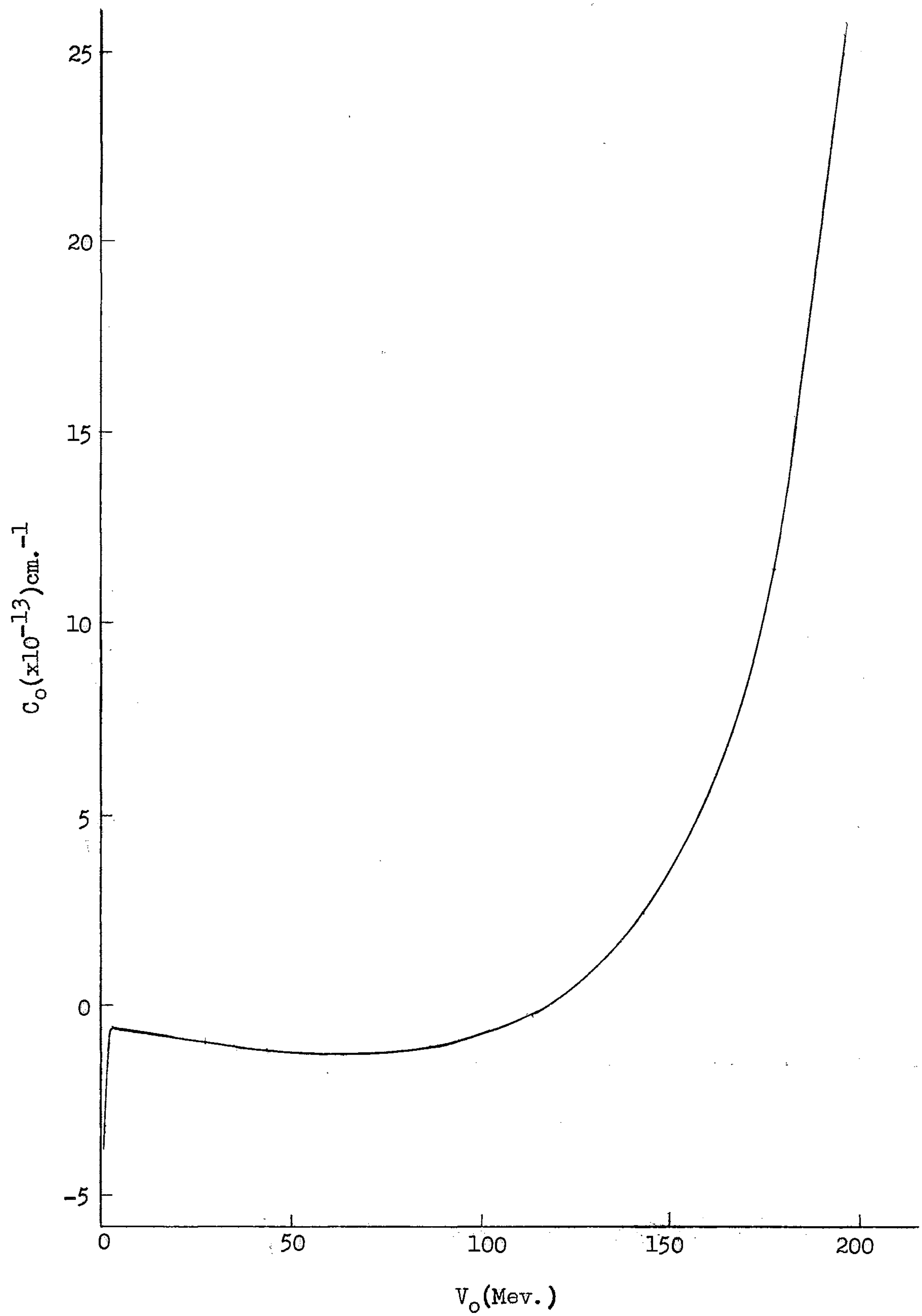

Fig. 5. C_o as a Function of V_o for a = 3.00x10^{-13}cm. and b = 2.00x10^{-13}cm.

and

$$D = Ck'^{-1}(\cos k'b^{-1})(k \sin k'b + k' \cos k'b)^{-1} \exp kb.$$

Thus the entire wave function may be expressed in terms of C.

The wave function is as yet unnormalized. A further restriction on C is provided by normalization of the wave function. The condition is

$$\int_o^b u_i^2\, dr + \int_b^\infty u_o^2\, dr = 1$$

where u_i and u_o are the inside and outside wave functions, respectively. Upon substitution the normalization equation becomes

$$1 = D^2(2k)^{-1} \exp(-2kb) + C^2k'^{-4} \left(\frac{3b}{2} - 2k'^{-1} \sin k'b + (2k')^{-1} \sin k'b \cos k'b\right) + B^2 \left[\tfrac{1}{2}b + (2k')^{-1}\sin k'b \cos k'b\right] + CBk'^{-3} (2 - 2 \cos k'b - \sin^2 k'b).$$

The coefficients B and D as derived from the boundary conditions may now be inserted into the normalization equation. Thus another restriction is placed on C as the normalization equation is now in terms of C, the potential well depth V_o, the binding energy W_b and the potential well b. The complete normalization equation is

$$1 = C^2k'^{-2}\left\{ \frac{(\cos k'b-1)^2}{2k\,(k \sin k'b + k' \cos k'b)^2} + k'^{-2} \left[\frac{3b}{2} - 2k'^{-1} \sin k'b + (2k')^{-1} \sin k'b \cos k'b\right] + \frac{\left[\sin k'b + kk'^{-1} (\cos k'b - 1)\right]^2}{(k \sin k'b + k' \cos k'b)^2} \left(\tfrac{1}{2}b + \tfrac{1}{2}k'^{-1}\sin k'b \cos k'b\right) - \frac{\left[\sin k'b + kk'^{-1} (\cos k'b -1)\right]}{k'^2 (k \sin k'b + k' \cos k'b)} (2 - 2 \cos k'b - \sin^2 k'b)\right\} .$$

When both sides of the above equation are multiplied by k sin k'b + k' cos k'b it can be readily seen that for C equal to zero the equation reduces to the form of the bound state equation used in Chapter V.

The values of C_o and corresponding V_o recorded earlier in this chapter were calculated from an equation assuming k equal to zero. It is, therefore, necessary to consider very small k when investigating the validity of the C_o values in the normalization equation. The case of k equal zero applied to the normalization equation is indeterminant for C not equal to zero. Thus for the wave function used in this chapter, zero binding energy has no meaning. In order to determine the effect of C on the binding energy, trial values of the binding energy and potential well depth are substituted into the normalization equation. Using a binding energy of 0.44 Mev., a potential well depth of 18.5 Mev. and a potential well width of 2.00×10^{-13}cm. the value for C is -5.23×10^{24}cm.$^{-1}$. From the scattering relation a potential well depth of 18.5 Mev. corresponds to a value of -0.830×10^{13} for C. The smaller value for C as computed from the scattering relation thus implies a bound state energy less than that used to find the C of the normalization equation. A bound state of sufficient depth to bind a fifth nucleon to the He^4 nucleus, therefore, is not allowed for a reasonable value of C as found from the scattering relation.

CHAPTER VII

SUMMARY AND CONCLUSIONS

The Problem Summarized

The simple quantum mechanical theory of elastic coherent scattering relates the zero energy scattering lengths and the depth of the bound energy state. The simple scattering theory is insufficient to explain the experimental data from nucleon scattering by He^4. A bound energy state is allowed by the simple scattering theory when applied to He^4. No stable five nucleon system as He^5 or Li^5 is found in nature. The question of reconciling the scattering lengths as derived from experimental data with the nonexistent bound state on the basis of a relatively simple scattering theory is raised. To determine the zero energy scattering lengths the shape independent approximation must be fitted with the experimental phase shifts and incident nucleon energies.

The wave equation is modified in an attempt to explain the experimental data. The integral of the Kernel function from an integro-differential equation is approximated by a power series in k. The solution is the trial wave function. A scattering equation with boundary conditions provides a restriction on the power series approximation, C. The bound energy state equations evaluated by the boundary conditions provide coefficients of the wave function in terms of the coefficients of the power series, C. The wave function can now be normalized to further restrict the power series approximation, C.

The Findings and Conclusions

When the shape independent approximation equations are fitted with the experimental data from the scattering of neutrons and protons on He^4 the zero energy scattering lengths are found to be 3.00×10^{-13}cm. and 4.00×10^{-13}cm. for neutrons and protons, respectively. These values are sufficiently close to consider them equal within range of experimental error. As a consequence of this "equality", it is reasonable to say that the scattering of nucleons by He^4 is charge independent. The shape independent approximation should determine the effective range, "r_o". The data available, however, is inadequate to calculate an acceptable value of "r_o". The satisfactory fitting of the data to the shape independent approximation also indicates that any potential well whose shape is reasonably close to a square potential well may be used to represent the He^4 nucleus in the scattering theory. Thus, a square potential well is quite acceptable in the theory developed in this investigation.

The approximation to an integro-differential by a power series in k, for k small, gives as a solution an acceptable wave function. From the boundary conditions at the edge of the potential well an equation relating C, the power series in k, and the potential well depth and width is found. From this equation a relation of C_o and V_o is found for k equal to zero by an analog to the shape independent approximation and by equating coefficients of like powers of k when both sides of the equation are expanded in a power series in k. This relation reconciles the zero energy scattering length and the zero energy binding. The wave equation is normalized and expressed in terms of C through the boundary conditions with the bound state equations. It expresses the relation of C with V_o, b and W_b. Thus, for small binding energies corresponding values of C can be calculated. Since it was found that the values of C_o computed from the scattering relations give binding energies less than what is obtained from the usual well, it may be concluded that the scattering lengths as derived from the experimental data can be reconciled with the lack of a bound state in a way that the well is not deep enough to be of significance. It is also of interest to note again that C may be complex.

Suggestions for Further Study

Many curious facets have made themselves manifest during this investigation. As it is not possible to explore all interests in a single investigation, several interests for further study will be listed here.

The bound state equation leaves much to be answered. What exactly are its trends? What happens if k is increased or if the well width is changed? Similarly, the effect of higher order terms of the C expansion is unknown.

In the case of the scattering equations, the coefficients of higher order terms might be investigated. The fact that some of the coefficents are complex may lead to informative results.

A complex C might provide interesting exploration. Which of the apparent implications are valid? Does the phase shift create the imaginary terms? This particular avenue of investigation would seem to be quite worthwhile.

On the experimental side of the nucleon scattering from He^4 is the lack of direct values. No published record of experimentally measured scattering lengths for He^4 is found in the literature. This information would be quite helpful, as is obvious, in determining the validity of the theory.

The answers to these and other questions about the relatively simple He^4 nucleus can make a profitable addition to the understanding and theoretical treatment of a nucleus represented by a potential well.

A SELECTED BIBLIOGRAPHY

1. Adair, R. K. "Neutron Cross Sections of the Elements." Reviews of Modern Physics, 22 (1950), 249-289.

2. Mather, K. B., and P. Swan. Nuclear Scattering. Cambridge: Cambridge University Press, 1958.

3. Miller, Philip D., and G. C. Phillips. "Scattering of Protons from Helium and Level Parameters in Li^5." Physical Review, 112 (1958), 2043-2047.

4. Dodder, D. C., and J. L. Gammel. "Elastic Scattering of Protons and Neutrons by Helium." Physical Review, 88 (1952), 520-526.

5. Hochberg, S., H. S. W. Massey and L. H. Underhill. "The scattering of Nucleons by Alpha Particles - The s-Phases." Physical Society Proceedings, (London), A67 (1954), 957-966.

6. Bransden, B. H., and J. S. C. McKee. "The Elastic Scattering of Neutrons by Alpha Particles." Philosophical Magazine, 45 (1954), 869-880.

7. Margolis, B., and E. S. Troubetzkoy. "Low Energy Neutron Scattering by a Spheroidal Complex Potential." Physical Review, 106 (1957), 105-109.

8. Kohn, W. "Variational Methods in Nuclear Collision Problems." Physical Review, 74 (1948), 1763-1772.

9. Putnam, T. M., J. E. Brolley, Jr. and Louis Rosen. "Scattering of 7.5 Mev. Protons by Helium." Physical Review, 104 (1956), 1303-1306.

10. Mott, N. F., and H. S. W. Massey. The Theory of Atomic Collisions. Oxford: Clarendon Press, 1949.

11. Schiff, Leonard I. Quantum Mechanics. 2nd ed. New York: McGraw-Hill Book Company, Inc., 1955.

12. Critchfield, C. L., and D. C. Dodder. "Phase Shifts in Proton-Alpha-Scattering." Physical Review, 76 (1949), 602-605.

13. Bashkin, S., F. P. Mooring and B. Petree. "Total Cross Section of Helium for Fast Neutrons." Physical Review, 82 (1951), 378-380.

14. Huber, von P., and E. Baldinger. "Winkelverteilung von gestreuten Neutronen an He^4." Helvetica Physica Acta, 25 (1952), 435-440.

15. Blatt, John M., and Victor F. Weisskopf. Theoretical Nuclear Physics. New York: John Wiley and Sons, Inc., 1952.

16. Bethe, Hans A., and Philip Morrison. Elementary Nuclear Theory. 2nd ed. New York: John Wiley and Sons, Inc., 1956.

17. Bethe, H. A. "Theory of the Effective Range in Nuclear Scattering." Physical Review, 76 (1949), 38-50.

VITA

John Robert Christy

Candidate for the Degree of

Master of Science

Thesis: THE CONCEPT OF A POTENTIAL WELL FOR THE SCATTERING OF NUCLEONS FROM He^4

Major Field: Physics

Biographical:

Personal Data: Born in Kansas City, Kansas, February 26, 1934, the son of Howard C. and Crittie Allegra Christy.

Education: Attended grade school in Kansas City, Kansas; graduated from Wyandotte High School, Kansas City, Kansas in 1952; received the Bachelor of Science degree from Oklahoma State University, with a major in Physics, in May, 1957; completed requirements for the Master of Science degree in May, 1959.

Organizations: Member of American Institute of Physics and American Physical Society.